THE HIROTA DIRECT METHOD

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Aslı Pekcan
July, 2005

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Metin Gürses (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Hüseyin Hüseyinov

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Azer Kerimov

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet B. Baray
Director of the Institute Engineering and Science

ABSTRACT

THE HIROTA DIRECT METHOD

Aslı Pekcan
M.S. in Mathematics
Supervisor: Prof. Dr. Metin Gürses
July, 2005

The search for integrability of nonlinear partial differential and difference equations includes the study on multi-soliton solutions. One of the most famous method to construct multi-soliton solutions is the Hirota direct method. In this thesis, we explain this method in detail and apply it to explicit examples.

Keywords: The Hirota direct method, integrable systems, solitons, exact solutions.

ÖZET

HİROTA METODU

Aslı Pekcan
Matematik, Yüksek Lisans
Tez Yöneticisi: Prof. Dr. Metin Gürses
Temmuz, 2005

İntegre edilebilir doğrusal olmayan kısmi türevli ve fark denklemlerinin bulunması multi-soliton çözümler üzerindeki çalışmaları içerir. Multi-soliton çözüm üretme metodlarından en ünlülerinden biri Hirota metodudur. Bu tezde, bu metodu ayrıntısıyla anlatıyor ve bu metodu bazı örneklere uyguluyoruz.

Anahtar sözcükler: Hirota metodu, integre edilebilir sistemler, solitonlar, kesin çözümler.

Acknowledgement

I would like to express my sincere gratitude to my supervisor Prof. Dr. Metin Gürses to whom to study with is an honor. His disciplinal teaching, encouragement and tolerance throughout my studies have been a major source of support.

I would like to thank Prof. Dr. Hüseyin Hüseyinov and Assoc. Prof. Azer Kerimov who accepted to review this thesis and commended on it.

My deepest gratitude further goes to my family for being with me in any situation, their encouragement, endless love and trust.

I am also so grateful to thank my old roommate Ansı who is with me in all the time with her patience, love and encouragement although we are not together.

Finally with my best feelings I would like to thank all my close friends whose names only a special chapter could encompass.

Contents

Chapter 1

Introduction

A soliton is a solitary wave which preserves its well-defined shape after it collides with another wave of the same kind. In the last 40 years there has been important developments in the soliton theory. Solitons have been studied by mathematicians, physicists and engineers for their applicability in physical applications (including plasmas, Josephson junctions, polyacetylene molecules etc.).

The first recorded observation of a solitary wave was made by J. Scott Russell in 1834 on the Edinburgh-Glasgow channel. Russell's experiments made him to discover,

(i) the existence of solitary waves,

(ii) the speed ν of these waves which are given by

$$\nu = \sqrt{g(h+\eta)} \tag{1.1}$$

where η is the amplitude of the wave measured from the plane of the water, h is the depth of the channel and g is the measure of gravity. He stated his observations to the British Association in 1844 [1]. But some mathematicians did not accept his results. In 1845, Airy wrote a formula for the speed of a wave relating its height and amplitude and concluded that a solitary wave could not exist in his article [2].

In 1895, Korteweg and de Vries [3] derived an equation, the so-called KdV equation which describes shallow water waves where the existence of solitary waves was verified mathematically.

In 1955, Fermi, Pasta and Ulam (FPU) decided to numerically solve Newton's equations of motion for a one-dimensional chain of identical masses attached by nonlinear springs [4]. Their studies inspired Zabusky and Kruskal [5] and they analyzed the KdV equation which had been arisen from the FPU problem. They observed that the localized waves preserve their shape and momentum in collisions. They called these waves 'solitons'.

By using the ideas of direct and inverse scattering, Gardner, Greene, Kruskal and Miura [6] derived a method of solution for the KdV equation in 1967. In 1968, the generalization of their results was made by Lax [7] and he introduced the concept of a Lax pair.

In 1971, Ryogo Hirota published an article giving a new method called 'the Hirota direct method' to find the exact solution of the KdV equation for multiple collisions of solitons [8]. In his successive articles, he dealt also with many other nonlinear evolution equations such as the modified Korteweg-de Vries (mKdV) [9], sine-Gordon (sG) [10], nonlinear Schrödinger (nlS) [11] and Toda lattice (Tl) [12] equations. The first step of this method is to make suitable transformations of nonlinear partial differential and difference equations which provide that the equations are in quadratic form in dependent variables. This new form is called 'bilinear form'. To find such a transformation is not easy for some equations and sometimes it requires the introduction of new dependent and sometimes even independent variables.

As a second step we introduce a special differential operator called Hirota D-operator which is used to write the bilinear form of the equation as a polynomial of D-operator which we call the Hirota bilinear form. Unfortunately there is no systematic way to construct the Hirota bilinear form for given nonlinear partial differential and difference equations. In fact, some equations may not be written in the Hirota bilinear form but perhaps in trilinear or multi-linear forms [13]. Here we can conjecture that all completely integrable nonlinear partial differential and difference equations can be put into the Hirota bilinear form. On the other hand, the converse is not true that is, there exist some equations which are not integrable but have Hirota bilinear forms. We will give an example to such an equation in this thesis.

The last step of the Hirota method is using the perturbation expansion, which becomes finite as we will see, in the Hirota bilinear form and analyzing the coefficients of the perturbation parameter and its powers separately. At that point the information we gained makes us to reach to multi-soliton solutions if the equation is integrable.

The Hirota direct method has taken an important role in the study of integrable systems. Most equations (even non-integrable ones) having Hirota bilinear form possess automatically one- and two-soliton solutions. When we come to the three-soliton solutions we come across a very restrictive condition. Actually this condition is not sufficient to search the integrability of an equation but it can be used as a powerful tool for this purpose [14]. This condition was also used to produce new integrable equations by Hietarinta in his articles, [15], [16], [17], [18].

The equations written in the Hirota bilinear form and having multi-soliton solutions are called Hirota integrable. These equations are very good candidates to be integrable. We know that another famous test for integrability is Painlevé test which is based on whether the solutions of the equation are free from movable critical singularities. For many years the Hirota direct method and Painlevé test have been used together. There is no need to write the equations in their usual nonlinear forms in order to test whether they have Painlevé property or not. We can perform Painlevé analysis under the Hirota bilinear form [19]. The equations that pass both tests are most probably integrable. Actually up to now, there is no counter example to this fact.

In this thesis, in Chapter 2, we explained the Hirota direct method in detail. We gave the necessary tools to apply this method. We introduced the Hirota D-operator and wrote the bilinear form of nonlinear partial differential and difference equations as polynomial of D-operator. We stated and proved the properties of this polynomial. We explained the Hirota perturbation. Finally we gave the theorems and their proofs in order to find one-, two- and three-soliton solutions of nonlinear partial differential and difference equations.

Starting from Chapter 3 to Chapter 8, we applied the Hirota direct method to the several examples which we may separate them in two parts: the equations written as a single Hirota bilinear equation which are the Korteweg-de Vries (KdV), Kadomtsev-Pethviashvili (KP), extendend Kadomtsev-Pethviashvili (eKP), Toda lattice (Tl) equations and the equations written as a pair of Hirota bilinear equations which are the modified Korteweg-de Vries (mKdV) and sine-Gordon (sG) equations. We constructed one-, two- and three-soliton solutions of all of these equations and additionally we gave N-soliton solutions of the KdV and mKdV equations. We gave also lists of KdV-, mKdV- and sG-type equations.

Chapter 2

The Hirota Direct Method

In this chapter we give an introduction to the Hirota direct method. Let $F[u] = F(u, u_x, u_t, ...) = 0$ be a nonlinear partial differential or difference equation. As the first step we transform $F[u]$ to a quadratic form in the dependent variables by using a transformation $u = T[f(x,t,...), g(x,t,...)]$. We call this new form as the bilinear form of $F[u]$. We should note that for some equations we may not find such a transformation. Another remark is that some integrable equations like the Korteweg-de Vries (KdV), Kadomtsev-Pethviashvili (KP) and Toda lattice (Tl) equations can be transformed to a single bilinear equation but many of them like the modified Korteweg-de Vries (mKdV), sine-Gordon (sG), and nonlinear Schrödinger (nlS) equations can only be written as combination of bilinear equations. Now we introduce the Hirota D-operator which makes the Hirota method very effective.

2.1 The Hirota D-Operator

Definition 2.1. *Let $S : \mathbb{C}^n \to \mathbb{C}$ be a space of differentiable functions. Then the Hirota D-operator $D : S \times S \to S$ is defined as*

$$[D_x^{m_1} D_t^{m_2} ...]\{f.g\} = [(\partial_x - \partial_{x'})^{m_1}(\partial_t - \partial_{t'})^{m_2} ...] f(x,t,...) \times g(x',t',...)|_{x'=x, t'=t,...} \quad (2.1)$$

where m_i, $i = 1, 2, ...$ are positive integers and $x, t, ...$ are independent variables.

We may also define the difference analogue of Hirota D-operator by the exponential identity,

$$\begin{aligned} e^{\delta D_z}\{a(z).b(z)\} =& e^{\delta \partial_y}\{a(z+y).b(z-y)\}|_{y=0} \\ =& a(z+\delta)b(z-\delta) \end{aligned} \tag{2.2}$$

where δ is a parameter. In this thesis this identity is used only for the Toda lattice (Tl) equation. By using some sort of combination of the Hirota D-operator, we try to write the bilinear form of $F[u]$ as a polynomial of D-operator. We call this polynomial $P(D)$.

Definition 2.2. *We say that nonlinear partial differential and difference equations can be written in the Hirota bilinear form if they are equivalent to*

$$\sum_{\alpha,\beta=1}^{m} P^{\eta}_{\alpha\beta}(D) f^{\alpha} f^{\beta} = 0, \quad \eta = 1, ..., r \tag{2.3}$$

for some m, r and linear operators $P^{\eta}_{\alpha\beta}(D)$, f^i's are new dependent variables.

Now let us state and prove some propositions and corollaries on $P(D)$.

Proposition 2.3. *Let $P(D)$ act on two differentiable functions f and g. Then we have*

$$P(D)\{f.g\} = P(-D)\{g.f\}. \tag{2.4}$$

Proof. We can simply take $P(D) = D_x^m$. The other combinations of D-operators follow in same manner. We can write

$$\begin{aligned} P(D)\{f.g\} &= D_x^m\{f.g\} \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} f_{(m-k)x} g_{kx} \\ &= f_{mx} g - m f_{(m-1)x} g_x + ... + (-1)^m f g_{mx} \end{aligned} \tag{2.5}$$

where the subscripts of the functions f and g define the order of the partial derivatives with respect to x. Indeed,

$$\begin{aligned} P(D)\{f.g\} &= f_{mx} g - m f_{(m-1)x} g_x + ... + (-1)^m f g_{mx} \\ &= (-1)^m [f g_{mx} - m f_x g_{(m-1)x} ... + (-1)^{m-1} m f_{(m-1)x} g_x + (-1)^m f_{mx} g] \end{aligned} \tag{2.6}$$

which is equal to $P(-D)\{g.f\}$. Hence $P(D)\{f.g\} = P(-D)\{g.f\}$. Note that if m is a positive even integer, interchanging the functions does not change the value of the Hirota bilinear equation.

Corollary 2.4. *Let $P(D)$ act on two differentiable functions f and $g = 1$, then we have*

$$P(D)\{f.1\} = P(\partial)f \quad , \quad P(D)\{1.f\} = P(-\partial)f. \tag{2.7}$$

Proposition 2.5. *Let P(D) act on two exponential functions e^{θ_1} and e^{θ_2} where $\theta_i = k_i x + ... + r_i z + l_i y + \alpha_i$ and $k_i, ..., r_i, l_i, \alpha_i$ are constants for $i = 1, 2$. Then we have*

$$P(D)\{e^{\theta_1}.e^{\theta_2}\} = P(k_1 - k_2, ..., r_1 - r_2, l_1 - l_2)e^{\theta_1+\theta_2}. \tag{2.8}$$

Proof. It is enough to consider $P(D) = [D_x^{m_1}...D_z^{m_{n-1}}D_y^{m_n}]$ where m_i, $i = 1, 2, ..., n$ are positive integers and $x, ..., z, y$ are the independent variables. When $P(D)$ acts on the product of the exponential functions e^{θ_1} and e^{θ_2} where $\theta_i = k_i x + ... + r_i z + l_i y + \alpha_i$, $i = 1, 2$, we have

$$\begin{aligned} P(D)\{e^{\theta_1}.e^{\theta_2}\} =& [D_x^{m_1}...D_z^{m_{n-1}}D_y^{m_n}]\{e^{\theta_1}.e^{\theta_2}\} \\ =& (l_1 - l_2)^{m_n}[D_x^{m_1}...D_z^{m_{n-1}}]\{e^{\theta_1}.e^{\theta_2}\} \\ =& (r_1 - r_2)^{m_{n-1}}(l_1 - l_2)^{m_n}[D_x^{m_1}...D_r^{m_{n-2}}]\{e^{\theta_1}.e^{\theta_2}\}. \end{aligned} \tag{2.9}$$

We continue this process until we apply all the Hirota D-operators to the exponential functions. Finally we have

$$\begin{aligned} P(D)\{e^{\theta_1}.e^{\theta_2}\} =& (k_1 - k_2)^{m_1}...(r_1 - r_2)^{m_{n-1}}(l_1 - l_2)^{m_n}e^{\theta_1+\theta_2} \\ =& P(k_1 - k_2, ..., r_1 - r_2, l_1 - l_2)e^{\theta_1+\theta_2}. \end{aligned} \tag{2.10}$$

This completes the proof. From now on, for a shorter notation we shall use $P(p_1 - p_2)$ instead of $P(k_1 - k_2, ..., r_1 - r_2, l_1 - l_2)$.

Corollary 2.6. *If we have a system such that $P(D)\{a.a\} = 0$ where a is any non-zero constant then by proposition 2.5 we have $P(0, 0, ..., 0) = 0$.*

Remark 2.7. *If we consider $P(D)\{f.f\}$, we may assume P is even since the odd terms cancel due to the antisymmetry of the D-operator i.e. we have $D_{x_1}^{m_1}D_{x_2}^{m_2}...D_{x_k}^{m_k}\{f.f\} = 0$ identically satisfied if $\sum_{i=1}^{k} m_i = odd$. For instance, as simple examples we clearly have*

$D_x\{f.f\} = f_x f - f f_x = 0,$

$D_t D_x^2\{f.f\} = f_{xxt}f - f_{xx}f_t - f_{xt}f_x + f_x f_{xt} - f_{xt}f_x + f_x f_{xt} + f_t f_{xx} - f f_{xxt} = 0.$

Let us now see the results of the application of the Hirota method to the following examples:

Example 2.1. The Kadomtsev-Petviashvili (KP) Equation

The KP equation is

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{2.11}$$

The bilinearizing transformation for KP is

$$u(x,t,y) = -2\partial_x^2 \log f. \tag{2.12}$$

The bilinear form of KP is

$$ff_{xt} - f_x f_t + 3f_{xx}^2 + ff_{xxxx} - 4f_x f_{xxx} + 3f_{yy}f - 3f_y^2 = 0. \tag{2.13}$$

The Hirota bilinear form of KP is

$$(D_x D_t + D_x^4 + 3D_y^2)\{f.f\} = 0. \tag{2.14}$$

For some equations, the Hirota bilinearization leads to more than one equation. As an example we can give the modified Korteweg-de Vries (mKdV) equation.

Example 2.2. The Modified KdV (MKdV) Equation

The mKdV equation is

$$u_t + 24u^2 u_x + u_{xxx} = 0. \tag{2.15}$$

The bilinearizing transformation for mKdV is

$$u(x,t) = \frac{g_x f - g f_x}{g^2 + f^2}. \tag{2.16}$$

The combination of bilinear equations of mKdV is

$$\begin{aligned} &-(g^2+f^2)\underline{(g_t f - g f_t + g_{xxx}f - 3g_{xx}f_x + 3g_x f_{xx} - g f_{xxx})} \\ &\qquad + 6(fg_x - gf_x)\underline{(ff_{xx} - f_x^2 + gg_{xx} - g_x^2)} = 0. \end{aligned} \tag{2.17}$$

The Hirota bilinear form of mKdV is the pair

$$\begin{cases} (D_x^3 + D_t)\{g.f\} = 0, \\ D_x^2\{f.f + g.g\} = 0. \end{cases} \tag{2.18}$$

2.2 The Hirota Perturbation and the Multi-Soliton Solutions

Here we consider the nonlinear partial differential or difference equation $F[u] = 0$ whose Hirota bilinear form is in the form $P(D)\{f.f\} = 0$ and we give the steps involved in finding exact solutions of $F[u] = 0$ by using its Hirota bilinear form. We shall use the perturbation expansions. For this purpose, let us write $f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + ...$ where f_0 is a constant, f_m, $m = 1, 2, ...$ are functions of $x, t, ...$ and so on. ε is a constant called the perturbation parameter. Without loss of generality, we take $f_0 = 1$. So the product $f.f$ becomes

$$f.f = 1.1 + \varepsilon(f_1.1 + 1.f_1) + \varepsilon^2(f_2.1 + f_1.f_1 + 1.f_2) + \varepsilon^3(f_3.1 + f_2.f_1 + f_1.f_2 + 1.f_3) + ... \tag{2.19}$$

Substituting this expression into $P(D)\{f.f\} = 0$ and using the linearity of the polynomial $P(D)$, we get

$$\begin{aligned} P(D)\{f.f\} = P(D)\{1.1\} + \varepsilon P(D)\{f_1.1 + 1.f_1\} + \varepsilon^2 P(D)\{f_2.1 + f_1.f_1 + 1.f_2\} \\ + \varepsilon^3 P(D)\{f_3.1 + f_2.f_1 + f_1.f_2 + 1.f_3\} + ... = 0. \end{aligned} \tag{2.20}$$

To satisfy this equation we make the coefficients of ε^m, $m = 0, 1, 2, ...$ to vanish. The coefficient of ε^0 is trivially zero. From the coefficient of ε^1 we have

$$P(D)\{f_1.1 + 1.f_1\} = 2P(\partial)f_1 = 0. \tag{2.21}$$

One of the solution of this equation is the exponential function. While we are applying the Hirota direct method we take f_1 as exponential function and so the other f_i's will also come as exponential functions. The effectiveness of the Hirota direct method reveals at this point. Since we will write f as a polynomial of exponential functions when we consider s-soliton solution of an equation $F[u] = 0$, f_j for all $j \geq s+1$ will be zero. So hereafter while we are constructing s-soliton solution of an equation we will assume that $f_j = 0$ for all $j \geq s+1$.

Theorem 2.8. *Let $u = T[f(x,t,...,y)]$ be a bilinearizing transformation of a nonlinear partial differential or difference equation $F[u] = 0$, which can be written in the Hirota bilinear form $P(D)\{f.f\} = 0$. Then one-soliton solution of this equation is*

$$u = T[f(x,t,...,y)] = T[1 + e^{\theta_1}] \tag{2.22}$$

where $\theta_1 = k_1x + \omega_1t + ... + l_1y + \alpha_1$ with the constants $k_1, \omega_1, ..., l_1$ satisfying $P(k_1, \omega_1, ..., l_1) = P(p_1) = 0$.

Proof. In order to construct one-soliton solution of $F[u] = 0$ we take $f = 1 + \varepsilon f_1$ where $f_1 = e^{\theta_1}$ with $\theta_1 = k_1x + \omega_1t + ... + l_1y + \alpha_1$. Note that we have $f_j = 0$ for all $j \geq 2$. After inserting f into the equation (2.20), we make the coefficients of ε^m, $m = 0, 1, 2$ to vanish. The coefficient of ε^0 is

$$P(D)\{1.1\} = P(0, 0, ..., 0)\{1\} \tag{2.23}$$

and it vanishes trivially by corollary 2.6. The corollary 2.4 makes the coefficient of ε^1 turns out to be

$$\begin{aligned} P(D)\{1.f_1 + f_1.1\} =& P(-\partial)f_1 + P(\partial)f_1 \\ =& 2P(\partial)e^{\theta_1}. \end{aligned} \tag{2.24}$$

Equating the above equation to zero and using the proposition 2.5 we obtain $P(k_1, \omega_1, ..., l_1) = P(p_1) = 0$. This relation is called as the dispersion relation. Since $f_2 = 0$, the coefficient of ε^2 becomes

$$\begin{aligned} P(D)\{1.f_2 + f_2.1\} + P(D)\{f_1.f_1\} =& P(D)\{e^{\theta_1}.e^{\theta_1}\} \\ =& P(p_1 - p_1)e^{2\theta_1} \end{aligned} \tag{2.25}$$

and it is identically zero. Without loss of generality, we may set $\varepsilon = 1$. So $f = 1 + e^{\theta_1}$ and one-soliton solution of $F[u] = 0$ is

$$u = T[f(x,t,...)] = T[1 + e^{\theta_1}] \tag{2.26}$$

where $\theta_1 = k_1x + \omega_1t + ... + l_1y + \alpha_1$ with the constants $k_1, \omega_1, ..., l_1$ satisfying $P(p_1) = 0$.

Theorem 2.9. *Let $u = T[f(x,t,...,y)]$ be a bilinearizing transformation of a nonlinear partial differential or difference equation $F[u] = 0$, which can be written in the Hirota bilinear form $P(D)\{f.f\} = 0$. Then two-soliton solution of this equation is*

$$u = T[f(x,t,...,y)] = T[1 + e^{\theta_1} + e^{\theta_2} + A(1,2)e^{\theta_1+\theta_2}] \tag{2.27}$$

where $\theta_i = k_ix + \omega_it + ... + l_iy + \alpha_i$ with the constants $k_i, \omega_i, ..., l_i$ satisfying $P(k_i, \omega_i, ..., l_i) = P(p_i) = 0$, $i = 1, 2$ and $A(1,2) = -\dfrac{P(p_1 - p_2)}{P(p_1 + p_2)}$.

Proof. To construct two-soliton solution of $F[u] = 0$ we take $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2$ where $f_1 = e^{\theta_1} + e^{\theta_2}$ for $\theta_i = k_ix + \omega_it + ... + l_iy + \alpha_i$, $i = 1, 2$ and $f_j = 0$ for all $j \geq 3$. We shall discover what f_2 is in the process of the method. After inserting f into the equation (2.20), we make the coefficients of ε^m, $m = 0, 1, ..., 4$ to vanish. The coefficient of ε^0

$$P(D)\{1.1\} = P(0,0,...,0)\{1\} = 0 \tag{2.28}$$

gives us no information. By the coefficient of ε^1 which is

$$P(D)\{1.f_1 + f_1.1\} = 2P(\partial)\{e^{\theta_1} + e^{\theta_2}\} = 0 \tag{2.29}$$

we get $P(p_i) = 0$ for $i = 1, 2$. From the coefficient of ε^2, we have

$$\begin{aligned} P(D)\{1.f_2 + f_2.1\} + P(D)\{f_1.f_1\} &= 2P(\partial)f_2 + P(D)\{(e^{\theta_1} + e^{\theta_2}).(e^{\theta_1} + e^{\theta_2})\} \\ &= 2P(\partial)f_2 + 2P(D)\{e^{\theta_1}.e^{\theta_2}\} \\ &= 2P(\partial)f_2 + 2P(p_1 - p_2)e^{\theta_1+\theta_2} = 0. \end{aligned} \tag{2.30}$$

Hence f_2 should be of the form $f_2 = A(1,2)e^{\theta_1+\theta_2}$. If we put f_2 in the above equation, we get $A(1,2)$ as

$$A(1,2) = -\frac{P(p_1 - p_2)}{P(p_1 + p_2)}. \tag{2.31}$$

Since $f_3 = 0$, the coefficient of ε^3 becomes

$$\begin{aligned} -P(D)\{f_1.f_2 + f_2.f_1\} &= A(1,2)[P(D)\{(e^{\theta_1} + e^{\theta_2}).e^{\theta_1+\theta_2}\} + P(D)\{e^{\theta_1+\theta_2}.(e^{\theta_1} + e^{\theta_2})\}] \\ &= A(1,2)[P(D)\{(e^{\theta_1}).(e^{\theta_1+\theta_2})\} + P(D)\{(e^{\theta_2}).(e^{\theta_1+\theta_2})\}] \\ &= A(1,2)[P(p_2)e^{2\theta_1+\theta_2} + P(p_1)e^{\theta_1+2\theta_2}] \end{aligned} \tag{2.32}$$

which is identically zero since $P(p_i) = 0,\ i = 1, 2$. The coefficient of ε^4 also vanishes trivially. Thus $f = 1 + e^{\theta_1} + e^{\theta_2} + A(1,2)e^{\theta_1+\theta_2}$ and two-soliton solution of $F[u] = 0$ is

$$u = T[f(x, t, ...)] = T[1 + e^{\theta_1} + e^{\theta_2} + A(1,2)e^{\theta_1+\theta_2}] \tag{2.33}$$

where $\theta_i = k_i x + \omega_i t + ... + l_i y + \alpha_i$ with the constants $k_i, \omega_i, ..., l_i$ satisfying $P(p_i) = 0,\ i = 1, 2$ and $A(1,2) = -\dfrac{P(p_1 - p_2)}{P(p_1 + p_2)}$.

Theorem 2.10. *Let $u = T[f(x, t, ..., y)]$ be a bilinearizing transformation of a nonlinear partial differential or difference equation $F[u] = 0$, which can be written in the Hirota bilinear form $P(D)\{f.f\} = 0$. Then if $F[u] = 0$ satisfies the three-soliton condition $(3SC)$ which is*

$$\sum_{\sigma_i=\pm1} P(\sigma_1 p_1+\sigma_2 p_2+\sigma_3 p_3)P(\sigma_1 p_1-\sigma_2 p_2)P(\sigma_2 p_2-\sigma_3 p_3)P(\sigma_3 p_3-\sigma_1 p_1) = 0 \tag{2.34}$$

with $P(p_i) = 0,\ i = 1, 2, 3$ then its three-soliton solution is

$$\begin{aligned} u &= T[f(x, t, ..., y)] \\ &= T[1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3}] \end{aligned} \tag{2.35}$$

where $\theta_i = k_i x + \omega_i t + ... + l_i y + \alpha_i,\ i = 1, 2, 3$. Here $A(i,j) = -\dfrac{P(p_i - p_j)}{P(p_i + p_j)}$ for $i, j = 1, 2, 3,\ i < j$ and $B = A(1,2)A(1,3)A(2,3)$.

Proof. To construct three-soliton solution of $F[u] = 0$ we take $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3$ where $f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}$ for $\theta_i = k_i x + \omega_i t + ... + l_i y + \alpha_i,\ i = 1, 2, 3$.

Note that $f_j = 0$ for all $j \geq 4$. Now we insert f into the the equation (2.20) and make the coefficients of ε^m, $m = 0, 1, 2, ..., 6$ to vanish. The coefficient of ε^0 is

$$P(D)\{1.1\} = P(0, 0, ..., 0)\{1\} \tag{2.36}$$

and it is trivially zero. From the coefficient of ε^1 which is

$$\begin{aligned} P(D)\{1.f_1 + f_1.1\} &= 2P(\partial)\{e^{\theta_1} + e^{\theta_2} + e^{\theta_3}\} \\ &= 2[P(\partial)e^{\theta_1} + P(\partial)e^{\theta_2} + P(\partial)e^{\theta_3}] = 0 \end{aligned} \tag{2.37}$$

we have the dispersion relation $P(p_i) = 0$, $i = 1, 2, 3$. From the coefficient of ε^2 we get

$$-2P(\partial)f_2 = P(D)\{f_1.f_1\} \tag{2.38}$$

where $f_1.f_1 = e^{\theta_1}.e^{\theta_1} + e^{\theta_2}.e^{\theta_2} + e^{\theta_3}.e^{\theta_3} + \sum\limits_{\substack{i,j=1,2,3 \\ i \neq j}} e^{\theta_i + \theta_j}$. Inserting this expression into the equation (2.38) we obtain

$$-2P(\partial)f_2 = 2[P(p_1 - p_2)e^{\theta_1+\theta_2} + P(p_1 - p_3)e^{\theta_1+\theta_3} + P(p_2 - p_3)e^{\theta_2+\theta_3}]. \tag{2.39}$$

Hence f_2 has the form $f_2 = A(1, 2)e^{\theta_1+\theta_2} + A(1, 3)e^{\theta_1+\theta_3} + A(2, 3)e^{\theta_2+\theta_3}$. After substituting f_2 into the equation (2.39), we find $A(i, j)$ as

$$A(i, j) = -\frac{P(p_i - p_j)}{P(p_i + p_j)} \tag{2.40}$$

for $i, j = 1, 2, 3$, $i < j$. The coefficient of ε^3 gives us

$$\begin{aligned} -2P(\partial)f_3 &= P(D)\{f_1.f_2 + f_2.f_1\} \\ &= 2P(D)\{f_1.f_2\} \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} P(D)\{f_1.f_2\} = A(1, 2)P(D)\{&e^{\theta_1}.e^{\theta_1+\theta_2} + e^{\theta_2}.e^{\theta_1+\theta_2} + e^{\theta_3}.e^{\theta_1+\theta_2}\} \\ +A(1, 3)P(D)\{&e^{\theta_1}.e^{\theta_1+\theta_3} + e^{\theta_2}.e^{\theta_1+\theta_3} + e^{\theta_3}.e^{\theta_1+\theta_3}\} \\ +A(2, 3)P(D)\{&e^{\theta_1}.e^{\theta_2+\theta_3} + e^{\theta_2}.e^{\theta_2+\theta_3} + e^{\theta_3}.e^{\theta_2+\theta_3}\}. \end{aligned} \tag{2.42}$$

Hence

$$\begin{aligned} -P(\partial)f_3 = e^{\theta_1+\theta_2+\theta_3}[&A(1, 2)P(p_3 - p_1 - p_2) + A(1, 3)P(p_2 - p_1 - p_3) \\ &+ A(2, 3)P(p_1 - p_2 - p_3)]. \end{aligned} \tag{2.43}$$

Note that f_3 should have the form $f_3 = Be^{\theta_1+\theta_2+\theta_3}$. We determine B from the above equation as

$$B = -\frac{A(1,2)P(p_3-p_1-p_2)+A(1,3)P(p_2-p_1-p_3)+A(2,3)P(p_1-p_2-p_3)}{P(p_1+p_2+p_3)}. \tag{2.44}$$

Since $f_4 = 0$, the coefficient of ε^4 becomes

$$P(D)\{f_1.f_3 + f_3.f_1 + f_2.f_2\} = 2P(D)\{f_1.f_3\} + P(D)\{f_2.f_2\} = 0 \tag{2.45}$$

where $P(D)\{f_1.f_3\}$ and $P(D)\{f_2.f_2\}$ are simplified as

$$P(D)\{f_1.f_3\} = B[P(p_2+p_3)e^{2\theta_1+\theta_2+\theta_3}+P(p_1+p_3)e^{\theta_1+2\theta_2+\theta_3}+P(p_1+p_2)e^{\theta_1+\theta_2+2\theta_3}], \tag{2.46}$$

$$\begin{aligned} P(D)\{f_2.f_2\} = 2[&A(1,2)A(1,3)P(p_2-p_3)e^{2\theta_1+\theta_2+\theta_3} \\ &+ A(1,2)A(2,3)P(p_1-p_3)e^{\theta_1+2\theta_2+\theta_3} \\ &+ A(1,3)A(2,3)P(p_1-p_2)e^{\theta_1+\theta_2+2\theta_3}]. \end{aligned} \tag{2.47}$$

Hence when we use these in the equation (2.45) we get

$$\begin{aligned} &e^{2\theta_1+\theta_2+\theta_3}[BP(p_2+p_3)+A(1,2)A(1,3)P(p_2-p_3)] \\ &\quad + e^{\theta_1+2\theta_2+\theta_3}[BP(p_1+p_3)+A(1,2)A(2,3)P(p_1-p_3)] \\ &\quad + e^{\theta_1+\theta_2+2\theta_3}[BP(p_1+p_2)+A(1,3)A(2,3)P(p_1-p_2)] = 0. \end{aligned} \tag{2.48}$$

To satisfy the above equation, the coefficients of the exponential terms should vanish. So we find that

$$B = A(1,2)A(1,3)A(2,3). \tag{2.49}$$

Remember that when we are analyzing the coefficient of ε^3, we have found another expression for the coefficient B. To be consistent these expressions for B should be equivalent i.e.

$$\begin{aligned} B = &-\frac{A(1,2)P(p_3-p_1-p_2)+A(1,3)P(p_2-p_1-p_3)+A(2,3)P(p_1-p_2-p_3)}{P(p_1+p_2+p_3)} \\ &= A(1,2)A(1,3)A(2,3). \end{aligned} \tag{2.50}$$

When we insert the formulas for $A(1,2)$, $A(1,3)$ and $A(2,3)$ in that equation, we obtain a relation that is

$$\begin{aligned} P(p_1-p_2)&P(p_1+p_3)P(p_1+p_2)P(p_3-p_1-p_2) \\ &+P(p_1-p_3)P(p_1+p_2)P(p_2+p_3)P(p_2-p_1-p_3) \\ &+P(p_2-p_3)P(p_1+p_2)P(p_1+p_3)P(p_1-p_2-p_3) \\ &=P(p_1-p_2)P(p_1-p_3)P(p_2-p_3)P(p_1+p_2+p_3). \end{aligned} \tag{2.51}$$

By writing the above equation in a more appropriate form we can conclude that to have three-soliton solution, nonlinear partial differential and difference equations which have the Hirota bilinear form $P(D)\{f.f\}=0$ should satisfy the condition which we call the three-soliton condition $(3SC)$:

$$\sum_{\sigma_i=\pm 1} P(\sigma_1 p_1+\sigma_2 p_2+\sigma_3 p_3)P(\sigma_1 p_1-\sigma_2 p_2)P(\sigma_2 p_2-\sigma_3 p_3)P(\sigma_3 p_3-\sigma_1 p_1)=0 \tag{2.52}$$

with the dispersion relation $P(p_i)=0$, $i=1,2,3$. An equation $F[u]=0$ satisfying $(3SC)$ possesses three-soliton solution given by

$$u=T[(1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+A(1,2)e^{\theta_1+\theta_2}+A(1,3)e^{\theta_1+\theta_3}+A(2,3)e^{\theta_2+\theta_3}+Be^{\theta_1+\theta_2+\theta_3})] \tag{2.53}$$

where $\theta_i=k_i x+\omega_i t+\ldots+l_i y+\alpha_i$, $i=1,2,3$. Here $A(i,j)=-\dfrac{P(p_i-p_j)}{P(p_i+p_j)}$ for $i,j=1,2,3$, $i<j$ and $B=A(1,2)A(1,3)A(2,3)$.

Chapter 3

The Korteweg-de Vries (KdV) Equation

In this chapter we see the application of the Hirota direct method to the Korteweg de Vries (KdV) equation which is the first nonlinear partial differential equation shown to be integrable by Kruskal et al [6]. It is also the first equation studied by Hirota [8]. We construct one-, two-, three- and N-soliton solutions of KdV. Finally we will give a list of KdV-type equations. KdV is given by

$$u_t - 6uu_x + u_{xxx} = 0. \tag{3.1}$$

Step 1. *Bilinearization*: We use the transformation $u(x,t) = -2\partial_x^2 \log f$ to bilinearize KdV. So the bilinear form of KdV is

$$ff_{xt} - f_x f_t + ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2 = 0. \tag{3.2}$$

Step 2. *Transformation to the Hirota bilinear form*: By using the Hirota-D operator we try to write the bilinear form of KdV in the Hirota bilinear form. Let us consider $D_t D_x$ applied on the product $f.f$,

$$\begin{aligned} D_t D_x\{f.f\} &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)\{f(x,t).f(x',t')\}|_{x'=x,t'=t} \\ &= f_{xt}f + ff_{xt} - f_t f_x - f_x f_t \\ &= 2(ff_{xt} - f_x f_t). \end{aligned} \tag{3.3}$$

Note that these terms are the first two terms of the equation (3.2) multiplied by two. Now consider D_x^4.

$$\begin{aligned} D_x^4\{f.f\} = &\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^4 \{f(x,t).f(x',t')\}|_{x'=x,t'=t} \\ =& f_{xxxx}f - 4f_{xxx}f_x + 6f_{xx}f_{xx} - 4f_x f_{xxx} + f f_{xxxx} \\ =& 2(ff_{xxx} - 4f_{xxx}f_x + 3f_{xx}^2). \end{aligned} \tag{3.4}$$

Note that these terms are the last three terms of the equation (3.2) multiplied by two. Hence we can write the equation (3.2) in the Hirota bilinear form

$$P(D)\{f.f\} = (D_x D_t + D_x^4)\{f.f\} = 0. \tag{3.5}$$

Step 3. *Application of the Hirota perturbation*: Insert $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + ...$ into the equation (3.5) so we have

$$\begin{aligned} P(D)\{f.f\} = P(D)\{1.1\} + \varepsilon P(D)\{f_1.1 + 1.f_1\} + \varepsilon^2 P(D)\{f_2.1 + f_1.f_1 + 1.f_2\} \\ + \varepsilon^3 P(D)\{f_3.1 + f_2.f_1 + f_1.f_2 + 1.f_3\} + ... = 0. \end{aligned} \tag{3.6}$$

3.1 One-Soliton Solution of KdV

To construct one-soliton solution of KdV as we discussed in Chapter 2, we take $f = 1+\varepsilon f_1$ where $f_1 = e^{\theta_1}$ and $\theta_1 = k_1 x+\omega_1 t+\alpha_1$. Note that $f_j = 0$ for all $j \geq 2$. We insert f into the equation (3.6) and make the coefficients of ε^m, $m = 0, 1, 2$ to vanish. The coefficient of ε^0 is $P(D)\{1.1\} = 0$ since $P(0,0)\{1\} = 0$. By the coefficient of ε^1

$$\begin{aligned} P(D)\{1.f_1 + f_1.1\} =& P(\partial)e^{\theta_1} + P(-\partial)e^{\theta_1} \\ =& 2P(p_1)e^{\theta_1} = 0 \end{aligned} \tag{3.7}$$

we have the dispersion relation $P(p_1) = 0$ which implies $\omega_1 = -{k_1}^3$. The coefficient of ε^2 vanishes trivially since

$$\begin{aligned} P(D)\{f_1.f_1\} &= P(D)\{e^{\theta_1}.e^{\theta_1}\} \\ &= P(p_1 - p_1)e^{2\theta_1} = 0. \end{aligned} \tag{3.8}$$

Finally without loss of generality we may set $\varepsilon = 1$ so $f = 1 + e^{\theta_1}$ and one-soliton solution of KdV is

$$u(x,t) = -\frac{k_1^2}{2\cosh^2(\frac{\theta_1}{2})} \tag{3.9}$$

where $\theta_1 = k_1 x - k_1^3 t + \alpha_1$.

3.2 Two-Soliton Solution of KdV

In order to construct two-soliton solution of KdV we take $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2$ where $f_1 = e^{\theta_1} + e^{\theta_2}$ with $\theta_i = k_i x + \omega_i t + \alpha_i$ for $i = 1, 2$. We shall determine f_2 later. Note that $f_j = 0$ for all $j \geq 3$. Now we insert f into the equation (3.6) and make the coefficients of ε^m, $m = 0, 1, ..., 4$ to vanish. The coefficient of ε^0 is

$$P(D)\{1.1\} = P(0,0)\{1\} = 0. \tag{3.10}$$

From the coefficient of ε^1 we have

$$\begin{aligned} P(D)\{1.f_1 + f_1.1\} =& 2P(\partial)\{e^{\theta_1} + e^{\theta_2}\} \\ =& 2[P(\partial)e^{\theta_1} + P(\partial)e^{\theta_2}] = 0 \end{aligned} \tag{3.11}$$

which implies $P(p_i) = k_i^4 + k_i\omega_i = 0$ i.e. $\omega_i = -k_i^3$ for $i = 1, 2$. The coefficient of ε^2 becomes

$$\begin{aligned} P(D)\{1.f_2 + f_2.1\} + P(D)\{f_1.f_1\} =& 2P(\partial)f_2 + P(D)\{(e^{\theta_1} + e^{\theta_2}).(e^{\theta_1} + e^{\theta_2})\} \\ =& 2[P(\partial)f_2 + P(D)\{e^{\theta_1}.e^{\theta_2}\}] \\ =& 2[P(\partial)f_2 + P(p_1 - p_2)e^{\theta_1+\theta_2}] = 0. \end{aligned} \tag{3.12}$$

This makes f_2 to have the form $f_2 = A(1,2)e^{\theta_1+\theta_2}$. If we put f_2 in the above equation we obtain $A(1,2)$ as

$$A(1,2) = -\frac{P(p_1 - p_2)}{P(p_1 + p_2)} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \tag{3.13}$$

Since $f_3 = 0$, the coefficient of ε^3 turns out to be

$$\begin{aligned} P(D)\{f_1.f_2 + f_2.f_1\} =& 2A(1,2)[P(D)\{(e^{\theta_1}).(e^{\theta_1+\theta_2})\} + P(D)\{(e^{\theta_2}).(e^{\theta_1+\theta_2})\}] \\ =& 2A(1,2)[P(p_2)e^{2\theta_1+\theta_2} + P(p_1)e^{\theta_1+2\theta_2}] \end{aligned} \tag{3.14}$$

and this is already zero since $P(p_i) = 0$, $i = 1, 2$. The coefficient of ε^4 also vanishes trivially. At last we may set $\varepsilon = 1$, thus $f = 1 + e^{\theta_1} + e^{\theta_2} + A(1,2)e^{\theta_1+\theta_2}$ and two-soliton solution of KdV is

$$u(x,t) = -2\frac{\{k_1^2e^{\theta_1} + k_2^2e^{\theta_2} + A(1,2)(k_2^2e^{\theta_1} + k_1^2e^{\theta_2})e^{\theta_1+\theta_2} + 2(k_1 - k_2)^2e^{\theta_1+\theta_2}\}}{(1 + e^{\theta_1} + e^{\theta_2} + A(1,2)e^{\theta_1+\theta_2})^2}, \tag{3.15}$$

where $\theta_i = k_ix - k_i^3 + \alpha_i$, $i = 1, 2$ and $A(1,2) = \dfrac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$.

3.3 Three-Soliton Solution of KdV

Here we take, $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3$ where $f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}$ and $\theta_i = k_ix + \omega_it + \alpha_i$ for $i = 1, 2, 3$. Note that $f_j = 0$ for all $j \geq 4$. We insert f into the equation (3.6) and make the coefficients of ε^m for $m = 0, 1, ..., 6$ to vanish. The coefficient of ε^0 is identically zero. By the coefficient of ε^1 we have

$$P(D)\{1.f_1 + f_1.1\} = 2P(\partial)\{e^{\theta_1} + e^{\theta_2} + e^{\theta_3}\} = 0 \tag{3.16}$$

and this gives $P(p_i) = 0$ implying $\omega_i = -k_i^3$ for $i = 1, 2, 3$. From the coefficient of ε^2 we get the relation

$$\begin{aligned} -P(\partial)f_2 = &[(k_1 - k_2)(\omega_1 - \omega_2) + (k_1 - k_2)^4]e^{\theta_1+\theta_2} \\ &+[(k_1 - k_3)(\omega_1 - \omega_3) + (k_1 - k_3)^4]e^{\theta_1+\theta_3} \\ &+[(k_2 - k_3)(\omega_2 - \omega_3) + (k_2 - k_3)^4]e^{\theta_2+\theta_3}. \end{aligned} \tag{3.17}$$

We see that f_2 should be of the form $f_2 = A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}$. By inserting this form into the equation (3.17) we obtain $A(i,j)$ as

$$A(i,j) = -\frac{P(p_i - p_j)}{P(p_i + p_j)} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \tag{3.18}$$

for $i, j = 1, 2, 3$, $i < j$. The coefficient of ε^3 becomes

$$\begin{aligned} -P(\partial)\{f_3\} = e^{\theta_1+\theta_2+\theta_3}\{&A(1,2)P(p_3 - p_2 - p_1) + A(1,3)P(p_2 - p_1 - p_3) \\ &+ A(2,3)P(p_1 - p_2 - p_3)\}. \end{aligned} \tag{3.19}$$

Hence f_3 should be of the form $f_3 = Be^{\theta_1+\theta_2+\theta_3}$. So the equation (3.19) gives

$$B = -\frac{A(1,2)P(p_3-p_1-p_2)+A(1,3)P(p_2-p_1-p_3)+A(2,3)P(p_1-p_2-p_3)}{P(p_1+p_2+p_3)}. \tag{3.20}$$

If we make all the simplifications by using $\omega_i = -k_i^3$ for $i = 1,2,3$ we see that the above expression is equivalent to $B = A(1,2)A(1,3)A(2,3)$. Since $f_4 = 0$ from the coefficient of ε^4 we have

$$P(D)\{f_1.f_3 + f_3.f_1 + f_2.f_2\} = 0. \tag{3.21}$$

After some calculations the equation (3.21) turns out to be

$$\begin{aligned} &e^{2\theta_1+\theta_2+\theta_3}[BP(p_2+p_3)+A(1,2)A(1,3)P(p_2-p_3)] \\ &\quad + e^{\theta_1+2\theta_2+\theta_3}[BP(p_1+p_3)+A(1,2)A(2,3)P(p_1-p_3)] \\ &\quad + e^{\theta_1+\theta_2+2\theta_3}[BP(p_1+p_2)+A(1,3)A(2,3)P(p_1-p_2)] = 0. \end{aligned} \tag{3.22}$$

This is satisfied by $B = A(1,2)A(1,3)A(2,3)$. Finally the coefficients of ε^5 and ε^6 also vanish automatically. We may also set $\varepsilon = 1$, therefore $f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3}$ so three-soliton solution of KdV is

$$u(x,t) = -2\frac{L(x,t)}{M(x,t)} \tag{3.23}$$

where

$$\begin{aligned} L(x,t) &= e^{\theta_1+\theta_2}[2(k_1-k_2)^2 + 2(k_1-k_2)^2A(1,3)A(2,3)e^{2\theta_3} + A(1,2)k_1^2e^{\theta_2} + A(1,2)k_2^2e^{\theta_1}] \\ &+ e^{\theta_1+\theta_3}[2(k_1-k_3)^2 + 2(k_1-k_3)^2A(1,2)A(2,3)e^{2\theta_2} + A(1,3)k_1^2e^{\theta_3} + A(1,3)k_3^2e^{\theta_1}] \\ &+ e^{\theta_2+\theta_3}[2(k_2-k_3)^2 + 2(k_2-k_3)^2A(1,2)A(1,3)e^{2\theta_1} + A(2,3)k_2^2e^{\theta_3} + A(2,3)k_3^2e^{\theta_2}] \\ &+ k_1^2e^{\theta_1} + k_2^2e^{\theta_2} + k_3^2e^{\theta_3} + Be^{\theta_1+\theta_2+\theta_3}[A(1,2)k_3^2e^{\theta_1+\theta_2} + A(1,3)k_2^2e^{\theta_1+\theta_3} + A(2,3)k_1^2e^{\theta_2+\theta_3}] \\ &+ e^{\theta_1+\theta_2+\theta_3}[A(1,2)(k_1^2+k_2^2+k_3^2+2k_1k_2-2k_1k_3-2k_2k_3) \\ &+ A(1,3)(k_1^2+k_2^2+k_3^2+2k_1k_3-2k_1k_2-2k_2k_3) \\ &+ A(2,3)(k_1^2+k_2^2+k_3^2+2k_2k_3-2k_1k_2-2k_1k_3) \\ &+ B(k_1^2+k_2^2+k_3^2+2k_1k_2+2k_1k_3+2k_2k_3)] \end{aligned} \tag{3.24}$$

and

$$M(x,t) = [1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3}]^2 \quad (3.25)$$

for $\theta_i = k_i x - k_i^3 t + \alpha_i$, $A(i,j) = \dfrac{(k_i - k_j)^2}{(k_i + k_j)^2}$, $i,j = 1,2,3$, $i < j$ and $B = A(1,2)A(1,3)A(2,3)$.

3.4 N-Soliton Solution of KdV

The bilinear form of KdV is

$$ff_{xt} - f_x f_t + ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2 = 0. \quad (3.26)$$

For N-soliton solution of KdV, we claim that $f(x,t)$ takes the form

$$f(x,t) = 1 + \sum_{m=1}^{N} \sum_{{}_N C_m} A(i_1, ..., i_m) \exp(\theta_{i_1} + ... + \theta_{i_m}) \quad (3.27)$$

where

$$A(i_1, ..., i_m) = \prod_{l<j}^{(m)} A(l,j) \quad , \quad A(l,j) = \frac{(k_l - k_j)^2}{(k_l + k_j)^2}. \quad (3.28)$$

Here ${}_N C_m$ indicates the summation over all possible combinations of m elements from N and (m) indicates the product of all possible combinations of the m elements with $(l < j)$. Note that $A(i_m) = 1$ for $m = 1, 2, ..., N$. To prove our claim we substitute the expression for $f(x,t)$ into (3.26) and see whether it is satisfied. Substitution of $f(x,t)$ gives us some exponential terms. To satisfy the bilinear form of KdV the coefficients of the exponential terms should vanish. From these coefficients we get the relation

$$\sum_{r=0}^{m} \sum_{{}_m C_r} A(i_1, ..., i_r) A(i_{r+1}, ..., i_m) g(i_1, ..., i_r; i_{r+1}, ..., i_m), \quad m = 1, 2, ..., N, \quad (3.29)$$

where

$$g(i_1, ..., i_r; i_{r+1}, ..., i_m) = (-k_{i_1} - ... - k_{i_r} + k_{i_{r+1}} + ... + k_{i_m}) \times [(-k_{i_1} - ... - k_{i_r} + k_{i_{r+1}} + ... + k_{i_m})^3 - (-k_{i_1}^3 - ... - k_{i_r}^3 + k_{i_{r+1}}^3 + ... + k_{i_m}^3)]. \quad (3.30)$$

For fixed m the equation (3.29) becomes

$$D(k_1,...,k_m) = \sum_{\sigma_1,...,\sigma_m=\pm 1} b(\sigma_1 k_1,...,\sigma_m k_m) g(\sigma_1 k_1,...,\sigma_m k_m) = 0, \tag{3.31}$$

where

$$b(\sigma_1 k_1,...,\sigma_m k_m) = \prod_{l<j}^{(m)} (\sigma_l k_l - \sigma_j k_j)^2, \tag{3.32}$$

and

$$g(\sigma_1 k_1,...,\sigma_m k_m) = (\sigma_1 k_1 + ... + \sigma_m k_m) \times [(\sigma_1 k_1 + ... + \sigma_m k_m)^3 - ((\sigma_1 k_1)^3 + ... + (\sigma_m k_m)^3)]. \tag{3.33}$$

We will prove this identity by induction. Before that we state the following properties of $D(k_1,...,k_m)$ [8],

(i) D is a symmetric, homogeneous polynomial,

(ii) D is an even function of $k_1,...,k_m$,

(iii) If $k_l = k_j$ we have
$$D(k_1,...,k_m) = 2(2k_l)^2 D(k_1,...,k_{l-1},k_{l+1},...,k_{j-1},k_{j+1},...,k_n) \prod_{s=1}^{m\ \prime} (k_l^2 - k_s^2)^2.$$

Here the prime indicates that the product does not include $s = l$ and $s = j$. For $m = 1$, the identity clearly holds since

$$D(k_1) = (\sigma_1 k_1)^2 (\sigma_1 k_1)[(\sigma_1 k_1)^3 - (\sigma_1 k_1^3)] = 0. \tag{3.34}$$

To understand the behavior of D, let us look also for $m = 2$. We have

$$D(k_1,k_2) = \sum_{\sigma_1,\sigma_2=\pm 1} (\sigma_1 k_1 - \sigma_2 k_2)^2 (\sigma_1 k_1 + \sigma_2 k_2)[(\sigma_1 k_1 + \sigma_2 k_2)^3 - (\sigma_1 k_1^3 + \sigma_2 k_2^3)]. \tag{3.35}$$

Hence

$$D(k_1,k_2) = (k_1-k_2)^2(k_1+k_2)[3k_1^2k_2+3k_1k_2^2]+(k_1+k_2)^2(k_1-k_2)[-3k_1^2k_2+3k_1k_2^2] + (k_1-k_2)^2(-k_1-k_2)[-3k_1^2k_2-3k_1k_2^2]+(k_1+k_2)^2(-k_1+k_2)[3k_1^2k_2-3k_1k_2^2] = 0. \tag{3.36}$$

Now assume that the identity holds for $m-2$. By using the properties of $D(k_1, ..., k_m)$ we see that it can be factored by $\prod_{l<j}^{(m)}(k_l^2 - k_j^2)^2$ whose degree is $2m(m-1)$. But the equation (3.31) shows that the degree of D is $m(m-1)+4$ which is smaller than $2m(m-1)$ for $m > 2$. Since this is impossible, the identity should hold for m. This completes the proof.

3.5 The KdV-type Equations

Here we will give a list of equations which can be written in the Hirota bilinear form $P(D)\{f.f\} = 0$ [20]. These equations are called the KdV-type equations. This list also includes their bilinearizing transformations and Hirota bilinear forms.

(1) Lax fifth-order KdV equation

$$u_t + 10(u^3 + \frac{1}{2}u_x^2 + uu_{xx})_x + u_{xxxxx} = 0, \tag{3.37}$$

$$u = 2\partial_x^2 \log f, \tag{3.38}$$

$$[D_x(D_t + D_x^5) - \frac{5}{3}D_s(D_s + D_x^3)]\{f.f\} = 0, \tag{3.39}$$

where f also satisfies the bilinear equation

$$D_x(D_s + D_x^3)\{f.f\} = 0, \tag{3.40}$$

involving an auxiliary variable s.

(2) Sawada-Kotera equation

$$u_t + 15(u^3 + uu_{xx}) + u_{xxxxx} = 0, \tag{3.41}$$

$$u = 2\partial_x^2 \log f, \tag{3.42}$$

$$D_x(D_t + D_x^5)\{f.f\} = 0. \tag{3.43}$$

(3) Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0, \tag{3.44}$$

$$u = 2\partial_x^2 \log f, \tag{3.45}$$

$$(D_t^2 - D_x^2 - D_x^4)\{f.f\} = 0. \tag{3.46}$$

(4) Model equations for shallow water waves

$$(i) \qquad u_t - u_{xx} - 4uu_t + 2u_x \int_x^\infty u_x dx' + u_x = 0, \tag{3.47}$$

$$u = 2\partial_x^2 \log f, \tag{3.48}$$

$$[D_x(D_t - D_t D_x^2 + D_x) + \frac{1}{3} D_t(D_s + D_x^3)]\{f.f\} = 0, \tag{3.49}$$

where f also satisfies the bilinear equation

$$D_x(D_s + D_x^3)\{f.f\} = 0, \tag{3.50}$$

involving an auxiliary variable s.

$$(ii) \qquad u_t - u_{xxt} - 3uu_t + 3u_x \int_x^\infty u_t dx' + u_x = 0, \tag{3.51}$$

$$u = 2\partial_x^2 \log f, \tag{3.52}$$

$$D_x(D_t - D_t D_x^2 + D_x)\{f.f\} = 0. \tag{3.53}$$

There are also the Kadomtsev-Petviashvili (KP) and Toda lattice (Tl) equations in this list but we will analyze them separately in the following chapters.

Chapter 4

The Kadomtsev-Petviashvili (KP) Equation

In this chapter, we apply the Hirota method to the Kadomtsev-Petviashvili (KP) equation, which is a KdV-type equation, in order to find one-, two- and three-soliton solutions of it. We also consider the extended Kadomtsev-Petviashvili (eKP) equation which is constructed by adding some terms to the KP equation. The eKP equation shows the applicability of Hirota's method to the non-integrable partial differential equations. KP is given by

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{4.1}$$

Step 1. *Bilinearization*: We use the transformation $u(x,t,y) = -2\partial_x^2 \log f$ to bilinearize the KP equation. The bilinear form of KP is

$$ff_{xt} - f_x f_t + ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2 + 3f_{yy}f - 3f_y^2 = 0. \tag{4.2}$$

Step 2. *Transformation to the Hirota bilinear form*: The Hirota bilinear form of KP is

$$P(D)\{f.f\} = (D_t D_x + D_x^4 + 3D_y^2)\{f.f\} = 0. \tag{4.3}$$

Step 3. *Application of the Hirota perturbation*: We insert $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + ...$

into the equation (4.3) so we have

$$P(D)\{f.f\} = P(D)\{1.1\} + \varepsilon P(D)\{f_1.1 + 1.f_1\} + \varepsilon^2 P(D)\{f_2.1 + f_1.f_1 + 1.f_2\} + \varepsilon^3 P(D)\{f_3.1 + f_2.f_1 + f_1.f_2 + 1.f_3\} + ... = 0. \quad (4.4)$$

4.1 One-Soliton Solution of KP

To construct one-soliton solution of KP we take $f = 1 + \varepsilon f_1$ where $f_1 = e^{\theta_1}$ and $\theta_1 = k_1 x + \omega_1 t + l_1 y + \alpha_1$. Note that $f_j = 0$ for all $j \geq 2$. We insert f into the equation (4.4) and make the coefficients of the ε^m, $m = 0, 1, 2$ to vanish. Here let us only consider ε^1 since the others vanish trivially. By the coefficient of ε^1

$$P(D)\{1.f_1 + f_1.1\} = 2P(\partial)e^{\theta_1} = 0 \quad (4.5)$$

we have $P(p_1) = 0$ which implies $\omega_1 = -\dfrac{k_1^4 + 3l_1^2}{k_1}$. Without loss of generality we may set $\varepsilon = 1$ so $f = 1 + e^{\theta_1}$ and one-soliton solution of KP is

$$u(x, t, y) = -\frac{k_1^2}{2\cosh^2(\frac{\theta_1}{2})} \quad (4.6)$$

where $\theta_1 = k_1 x - (\dfrac{k_1^4 + 3l_1^2}{k_1})t + l_1 y + \alpha_1$.

4.2 Two-Soliton Solution of KP

In order to construct two-soliton solution of KP we take $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2$ where $f_1 = e^{\theta_1} + e^{\theta_2}$ with $\theta_i = k_i x + \omega_i t + l_i y + \alpha_i$ for $i = 1, 2$. Note that $f_j = 0$ for all $j \geq 3$. We insert f into the equation (4.4) and make the coefficients of ε^m, $m = 0, 1, ..., 4$ to vanish. We shall only examine the nontrivial ones which are the coefficients of ε^1 and ε^2. From the coefficient of ε^1

$$P(D)\{1.f_1 + f_1.1\} = 2P(\partial)\{e^{\theta_1} + e^{\theta_2}\} = 0 \quad (4.7)$$

we get $P(p_i) = k_i^4 + k_i\omega_i + 3l_i^2 = 0$ which implies $\omega_i = -\dfrac{k_i^4 + 3l_i^2}{k_i}$ for $i = 1, 2$. The coefficient of ε^2 is

$$\begin{aligned} P(D)\{1.f_2 + f_2.1\} + P(D)\{f_1.f_1\} =& 2P(\partial)f_2 + P(D)\{e^{\theta_1}.e^{\theta_2} + e^{\theta_2}.e^{\theta_1}\} \\ =& 2P(\partial)f_2 + 2P(p_1 - p_2)e^{\theta_1+\theta_2} = 0. \end{aligned} \tag{4.8}$$

Hence we obtain

$$P(\partial)f_2 = -P(p_1 - p_2)e^{\theta_1+\theta_2} \tag{4.9}$$

which makes f_2 to have the form $f_2 = A(1, 2)e^{\theta_1+\theta_2}$. If we put f_2 in the above equation and use $k_i^4 + k_i\omega_i + 3l_i^2 = 0$ for $i = 1, 2$, we obtain $A(1, 2)$ as

$$A(1, 2) = -\frac{P(p_1 - p_2)}{P(p_1 + p_2)} = \frac{k_1\omega_2 + k_2\omega_1 + 4k_1^3k_2 - 6k_1^2k_2^2 + 4k_1k_2^3 + 6l_1l_2}{k_1\omega_2 + k_2\omega_1 + 4k_1^3k_2 + 6k_1^2k_2^2 + 4k_1k_2^3 + 6l_1l_2}. \tag{4.10}$$

We may set $\varepsilon = 1$, thus $f = 1 + e^{\theta_1} + e^{\theta_2} + A(1, 2)e^{\theta_1+\theta_2}$ and two-soliton solution of KP is

$$u = \frac{-2\{k_1^2e^{\theta_1} + k_2^2e^{\theta_2} + [(k_1 - k_2)^2 + A(1, 2)((k_1 + k_2)^2 + k_1^2e^{\theta_2} + k_2^2e^{\theta_1})]e^{\theta_1+\theta_2}\}}{(1 + e^{\theta_1} + e^{\theta_2} + A(1, 2)e^{\theta_1+\theta_2})^2} \tag{4.11}$$

where $\theta_i = k_ix - \dfrac{k_i^4 + 3l_i^2}{k_i}t + l_iy + \alpha_i$, $i = 1, 2$ and $A(1, 2)$ is as given in (4.10).

4.3 Three-Soliton Solution of KP

Now in a similar way we construct three-soliton solution of KP. We take $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3$ where $f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}$ with $\theta_i = k_ix + \omega_it + l_iy + \alpha_i$ for $i = 1, 2, 3$ and insert it into (4.4). Note that $f_j = 0$ for all $j \geq 4$. Here let us consider only the coefficients of ε^m, $m = 1, 2, 3, 4$, since others vanish automatically. From the coefficient of ε^1

$$P(D)\{1.f_1 + f_1.1\} = 2P(\partial)\{e^{\theta_1} + e^{\theta_2} + e^{\theta_3}\} = 0 \tag{4.12}$$

we obtain $P(p_i) = 0$ which implies $\omega_i = -\dfrac{k_i^4 + 3l_i^2}{k_i}$ for $i = 1, 2, 3$. By the coefficient of ε^2 we have

$$\begin{aligned} -P(\partial)f_2 = &[(k_1-k_2)^4 + (k_1-k_2)(\omega_1-\omega_2) + 3(l_1-l_2)^2]e^{\theta_1+\theta_2} \\ &+[(k_1-k_3)^4 + (k_1-k_3)(\omega_1-\omega_3) + 3(l_1-l_3)^2]e^{\theta_1+\theta_3} \\ &+[(k_2-k_3)^4 + (k_2-k_3)(\omega_2-\omega_3) + 3(l_2-l_3)^2]e^{\theta_2+\theta_3}. \end{aligned} \tag{4.13}$$

We see that f_2 should be of the form $f_2 = A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}$. We put f_2 into the equation (4.13) and use $k_i^4 + k_i\omega_i + 3l_i^2 = 0$ for $i = 1, 2, 3$ we get $A(i,j)$ where $i, j = 1, 2, 3$, $i < j$ as

$$A(i,j) = -\frac{P(p_i - p_j)}{P(p_i + p_j)} = \frac{k_i\omega_j + k_j\omega_i + 4k_i^3k_j - 6k_i^2k_j^2 + 4k_ik_j^3 + 6l_il_j}{k_i\omega_j + k_j\omega_i + 4k_i^3k_j + 6k_i^2k_j^2 + 4k_ik_j^3 + 6l_il_j}. \tag{4.14}$$

From the coefficient of ε^3 we get

$$\begin{aligned} P(\partial)\{f_3\} = -[&A(1,2)P(p_3-p_2-p_1) + A(1,3)P(p_2-p_1-p_3) \\ &+ A(2,3)P(p_1-p_2-p_3)]e^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{4.15}$$

Hence f_3 is in the form $f_3 = Be^{\theta_1+\theta_2+\theta_3}$. If we insert f_3 into the above equation we find that

$$B = -\frac{A(1,2)P(p_3-p_1-p_2) + A(1,3)P(p_2-p_1-p_3) + A(2,3)P(p_1-p_2-p_3)}{P(p_1+p_2+p_3)}. \tag{4.16}$$

Since $f_4 = 0$ the coefficient of ε^4 gives us

$$\begin{aligned} &e^{2\theta_1+\theta_2+\theta_3}[BP(p_2+p_3) + A(1,2)A(1,3)P(p_2-p_3)] \\ &\quad + e^{\theta_1+2\theta_2+\theta_3}[BP(p_1+p_3) + A(1,2)A(2,3)P(p_1-p_3)] \\ &\quad + e^{\theta_1+\theta_2+2\theta_3}[BP(p_1+p_2) + A(1,3)A(2,3)P(p_1-p_2)] = 0 \end{aligned} \tag{4.17}$$

which is satisfied when

$$B = A(1,2)A(1,3)A(2,3). \tag{4.18}$$

The consistency is not destroyed since after some calculations we see that the equations (4.16) and (4.18) are equal to each other. We may set $\varepsilon = 1$, hence

$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3}$ and three-soliton solution of KP is

$$u(x,t,y) = -2\frac{L(x,t,y)}{M(x,t,y)} \tag{4.19}$$

where

$$\begin{aligned}
L(x,t,y) &= k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + k_3^2 e^{\theta_3} + e^{2\theta_1+\theta_2+\theta_3}[A(1,2)A(1,3)(k_2-k_3)^2 + B(k_2+k_3)^2] \\
&+ e^{\theta_1+\theta_2+2\theta_3}[A(1,3)A(2,3)(k_1-k_2)^2 + B(k_1+k_2)^2] \\
&+ e^{\theta_1+2\theta_2+\theta_3}[A(1,2)A(2,3)(k_1-k_3)^2 + B(k_1+k_3)^2] \\
&+ e^{\theta_1+\theta_2}[(k_1-k_2)^2 + A(1,2)(k_1^2 e^{\theta_2} + k_2^2 e^{\theta_1} + (k_1+k_2)^2)] \\
&+ e^{\theta_1+\theta_3}[(k_1-k_3)^2 + A(1,3)(k_1^2 e^{\theta_3} + k_3^2 e^{\theta_1} + (k_1+k_3)^2)] \\
&+ e^{\theta_2+\theta_3}[(k_2-k_3)^2 + A(2,3)(k_2^2 e^{\theta_3} + k_3^2 e^{\theta_2} + (k_2+k_3)^2)] \\
&+ e^{\theta_1+\theta_2+\theta_3}[A(1,2)(k_1^2 + k_2^2 + k_3^2 + 2k_1k_2 - 2k_1k_3 - 2k_2k_3) \\
&+ A(1,3)(k_1^2 + k_2^2 + k_3^2 + 2k_1k_3 - 2k_1k_2 - 2k_2k_3) \\
&+ A(2,3)(k_1^2 + k_2^2 + k_3^2 + 2k_2k_3 - 2k_1k_2 - 2k_1k_3) \\
&+ B(k_1^2 + k_2^2 + k_3^2 + 2k_1k_2 + 2k_1k_3 + 2k_2k_3)] \\
&+ Be^{\theta_1+\theta_2+\theta_3}[A(1,2)k_3^2 e^{\theta_1+\theta_2} + A(1,3)k_2^2 e^{\theta_1+\theta_3} + A(2,3)k_1^2 e^{\theta_2+\theta_3}]
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
M(x,t,y) = [1 &+ e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} \\
&+ A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3}]^2
\end{aligned} \tag{4.21}$$

for $\theta_i = k_i x - \dfrac{k_i^4 + 3l_i^2}{k_i} t + l_i y + \alpha_i,\ i = 1,2,3,\ A(i,j),\ i,j = 1,2,3,\ i < j$ as in (4.14) and $B = A(1,2)A(1,3)A(2,3)$.

4.4 The Extended Kadomtsev-Petviashvili (EKP) Equation (A non-integrable case)

Theorem 4.1. *The extended Kadomtsev-Pethviashvili (eKP) equation is*

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} + \gamma u_{tt} + \beta u_{ty} = 0 \tag{4.22}$$

which is constructed by adding two terms γu_{tt} and βu_{ty} to the KP equation where γ and β are non-zero constants, is integrable (transformable to the KP equation) if the relation $\gamma = \beta^2/12$ holds. Otherwise it is not integrable.

Proof. We know that for a nonlinear partial differential equation, satisfying the three-soliton condition given in (2.34) is not sufficient but necessary to be integrable. As we will see while we are searching for three-soliton solution of eKP, it should satisfy the condition $\gamma = \beta^2/12$. Indeed eKP is equivalent to KP under this condition since by the transformation

$$\begin{aligned} \tilde{u} =& u, \\ \tilde{t} =& t + ay, \\ \tilde{x} =& x, \\ \tilde{y} =& y, \end{aligned} \tag{4.23}$$

where $a = -\dfrac{\beta}{6} = \sqrt{\dfrac{\gamma}{3}}$ we reach to KP, which is an integrable equation. Now we will apply the Hirota direct method to eKP.

Step 1. *Bilinearization*: We use the bilinearizing transformation $u(x,t,y) = -2\partial_x^2 \log f$ so the bilinear form of eKP is

$$\begin{aligned} f_{tx}f - f_t f_x + f_{xxxx}f - 4f_x f_{xxx} + 3f_{xx}^2 + 3f_{yy}f - 3f_y^2 + \\ \gamma f f_{tt} - \gamma f_t^2 + \beta f_{ty} f - \beta f_t f_y = 0. \end{aligned} \tag{4.24}$$

Step 2. *Transformation to the Hirota bilinear form*: The Hirota bilinear form of eKP is

$$P(D)\{f.f\} = (D_t D_x + D_x^4 + 3D_y^2 + \gamma D_t^2 + \beta D_t D_y)\{f.f\} = 0. \tag{4.25}$$

Step 3. *Application of the Hirota perturbation*: Insert $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + ...$ into the equation (4.25) so we have

$$\begin{aligned} P(D)\{f.f\} = P(D)\{1.1\} + \varepsilon P(D)\{f_1.1 + 1.f_1\} + \varepsilon^2 P(D)\{f_2.1 + f_1.f_1 + 1.f_2\} \\ + \varepsilon^3 P(D)\{f_3.1 + f_2.f_1 + f_1.f_2 + 1.f_3\} + ... = 0. \end{aligned} \tag{4.26}$$

4.4.1 One-Soliton Solution of EKP

To construct one-soliton solution of eKP, we take $f = 1+\varepsilon f_1$ where $f_1 = e^{\theta_1}$ with $\theta_1 = k_1x+\omega_1t+l_1y+\alpha_1$ and insert f into the equation (4.26). Note that $f_j = 0$ for all $j \geq 2$. Now let us consider only the coefficient of ε^1 since the others are trivially zero. From the coefficient of ε^1

$$P(D)\{1.f_1 + f_1.1\} = 2P(\partial)e^{\theta_1} = 0 \tag{4.27}$$

we have $P(p_1) = k_1^4 + k_1\omega_1 + 3l_1^2 + \gamma\omega_1^2 + \beta\omega_1 l_1 = 0$. We may set $\varepsilon = 1$ so $f = 1 + e^{\theta_1}$. Thus one-soliton solution of eKP is

$$u(x,t,y) = -\frac{k_1^2}{2\cosh^2(\frac{\theta_1}{2})} \tag{4.28}$$

where $\theta_1 = k_1x + \omega_1t + l_1y + \alpha_1$ with the constants k_1, ω_1 and l_1 satisfying $k_1^4 + k_1\omega_1 + 3l_1^2 + \gamma\omega_1^2 + \beta\omega_1 l_1 = 0$.

4.4.2 Two-Soliton Solution of EKP

In order to construct two-soliton solution of eKP we take $f = 1+\varepsilon f_1+\varepsilon^2 f_2$ where $f_1 = e^{\theta_1} + e^{\theta_2}$ for $\theta_i = k_ix + \omega_it + l_iy + \alpha_i$, $i = 1,2$ and $f_j = 0$ for all $j \geq 3$. The function f_2 shall be determined later. We insert f into (4.26) and analyze only the coefficients of ε^m, $m = 1,2$ since the others vanish automatically. From the coefficient of ε^1, we have

$$P(D)\{1.f_1 + f_1.1\} = P(\partial)\{e^{\theta_1} + e^{\theta_2}\} = 0 \tag{4.29}$$

which implies $P(p_i) = k_i\omega_i + k_i^4 + 3l_i^2 + \gamma\omega_i^2 + \beta\omega_i l_i = 0$ for $i = 1,2$. From the coefficient of ε^2 we get

$$P(\partial)f_2 + P(D)\{e^{\theta_1}.e^{\theta_2}\} = P(\partial)f_2 + P(p_1 - p_2)e^{\theta_1+\theta_2} = 0. \tag{4.30}$$

Hence f_2 should have the form $f_2 = A(1,2)e^{\theta_1+\theta_2}$. By substituting f_2 into (4.30), we obtain $A(1,2)$ as

$$A(1,2) = \frac{\beta(\omega_1l_2+\omega_2l_1) + 2\gamma\omega_1\omega_2 + k_1(\omega_2+4k_2^3) + k_2(\omega_1+4k_1^3) - 6k_1^2k_2^2 + 6l_1l_2}{\beta(\omega_1l_2+\omega_2l_1) + 2\gamma\omega_1\omega_2 + k_1(\omega_2+4k_2^3) + k_2(\omega_1+4k_1^3) + 6k_1^2k_2^2 + 6l_1l_2}. \tag{4.31}$$

Finally, we may set $\varepsilon = 1$ so $f = 1 + e^{\theta_1} + e^{\theta_2} + A(1,2)e^{\theta_1+\theta_2}$ and two-soliton solution of eKP is

$$u = \frac{-2\{k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + [(k_1-k_2)^2 + A(1,2)((k_1+k_2)^2 + k_1^2 e^{\theta_2} + k_2^2 e^{\theta_1})]e^{\theta_1+\theta_2}\}}{(1+e^{\theta_1}+e^{\theta_2}+A(1,2)e^{\theta_1+\theta_2})^2} \tag{4.32}$$

where $\theta_i = k_i x + \omega_i t + l_i y + \alpha_i$ satisfying $k_i\omega_i + k_i^4 + 3l_i^2 + \gamma\omega_i^2 + \beta\omega_i l_i = 0$ $i = 1,2$ and $A(1,2)$ is as given in (4.31).

4.4.3 Three-Soliton-like Solution of EKP

Trying to construct three-soliton solution of eKP we take $f = 1+\varepsilon f_1+\varepsilon^2 f_2+\varepsilon^3 f_3$ where $f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}$ with $\theta_i = k_i x + \omega_i t + l_i y + \alpha_i$ for $i = 1,2,3$ and insert it into (4.26). Note that $f_j = 0$ for all $j \geq 4$. Now we will only consider the coefficients of ε^m, $m = 1,2,3,4$. By the coefficient of ε^1

$$P(D)\{1.f_1 + f_1.1\} = 2P(\partial)\{e^{\theta_1} + e^{\theta_2} + e^{\theta_3}\} = 0 \tag{4.33}$$

we have

$$P(p_i) = k_i\omega_i + k_i^4 + 3l_i^2 + \gamma\omega_i^2 + \beta\omega_i l_i = 0 \tag{4.34}$$

for $i = 1,2,3$. From the coefficient of ε^2 we get

$$\begin{aligned} -P(\partial)f_2 = \sum_{i<j}^{(3)} [&(k_i-k_j)(\omega_i-\omega_j) + (k_i-k_j)^4 + 3(l_i-l_j)^2 \\ &+ \gamma(\omega_i-\omega_j)^2 + \beta(\omega_i-\omega_j)(l_i-l_j)e^{\theta_i+\theta_j}] \end{aligned} \tag{4.35}$$

where (3) indicates the summation of all possible combinations of the three elements with $(i < j)$. Thus f_2 should be in the form $f_2 = A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}$ to satisfy the equation. We insert f_2 into the equation (4.35) and use $k_i\omega_i + k_i^4 + 3l_i^2 + \gamma\omega_i^2 + \beta\omega_i l_i = 0$ for $i = 1,2,3$ we get $A(i,j)$ where $i,j = 1,2,3$, $i < j$ as

$$A(i,j) = \frac{\beta(\omega_i l_j + \omega_j l_i) + 2\gamma\omega_i\omega_j + k_i(\omega_j + 4k_j^3) + k_j(\omega_i + 4k_i^3) - 6k_i^2k_j^2 + 6l_il_j}{\beta(\omega_i l_j + \omega_j l_i) + 2\gamma\omega_i\omega_j + k_i(\omega_j + 4k_j^3) + k_j(\omega_i + 4k_i^3) + 6k_i^2k_j^2 + 6l_il_j}. \tag{4.36}$$

From the coefficient of ε^3 we obtain

$$P(\partial)\{f_3\} = -[A(1,2)P(p_3 - p_2 - p_1) + A(1,3)P(p_2 - p_1 - p_3) \\ + A(2,3)P(p_1 - p_2 - p_3)]e^{\theta_1+\theta_2+\theta_3}. \quad (4.37)$$

Hence f_3 is in the form $f_3 = Be^{\theta_1+\theta_2+\theta_3}$ where B is found as

$$B = -\frac{A(1,2)P(p_3 - p_1 - p_2) + A(1,3)P(p_2 - p_1 - p_3) + A(2,3)P(p_1 - p_2 - p_3)}{P(p_1 + p_2 + p_3)}. \quad (4.38)$$

Since $f_4 = 0$ the coefficient of ε^4 gives us

$$e^{2\theta_1+\theta_2+\theta_3}[BP(p_2 + p_3) + A(1,2)A(1,3)P(p_2 - p_3)] \\ + e^{\theta_1+2\theta_2+\theta_3}[BP(p_1 + p_3) + A(1,2)A(2,3)P(p_1 - p_3)] \\ + e^{\theta_1+\theta_2+2\theta_3}[BP(p_1 + p_2) + A(1,3)A(2,3)P(p_1 - p_2)] = 0 \quad (4.39)$$

which is satisfied when

$$B = A(1,2)A(1,3)A(2,3). \quad (4.40)$$

The two expressions (4.38) and (4.40) should be equivalent

$$B = -\frac{A(1,2)P(p_3 - p_1 - p_2) + A(1,3)P(p_2 - p_1 - p_3) + A(2,3)P(p_1 - p_2 - p_3)}{P(p_1 + p_2 + p_3)} \\ = A(1,2)A(1,3)A(2,3), \quad (4.41)$$

in fact which means eKP should satisfy the three-soliton condition for KdV-type equations given in (2.34). After long calculations we see that this three-soliton solution condition is satisfied when $\gamma = \beta^2/12$ as it is stated in the theorem 4.1 and this makes eKP transformable to KP. Even if we do not have this relation between γ and β, we construct exact solution of eKP. The equation (4.41) can be considered as a constraint on the arbitrary constants γ, β, k_i, l_i and ω_i, $i = 1, 2, 3$ which are satisfying the relation (4.34). Hence by using these relations, among eleven variables we have seven independent left. Due to this last condition the solutions we obtain are not called 'solitonic' solutions, but they constitute exact solutions of eKP.

Chapter 5

The Toda Lattice (TL) Equation

In this chapter, we see the application of the Hirota direct method to a nonlinear partial difference equation. We give the construction of one-, two- and three-soliton solutions of the Toda lattice (Tl) equation which is also a KdV-type equation. The Tl equation is given by

$$\frac{d^2}{dt^2}\log(1+V_n(t)) = V_{n+1}(t) + V_{n-1}(t) - 2V_n(t). \tag{5.1}$$

Step 1. *Bilinearization*: By using the transformation

$$V_n(t) = \frac{d^2}{dt^2}\log f_n \tag{5.2}$$

in (5.1) we get the bilinear form of Tl as

$$\ddot{f}_n f_n - 2(\dot{f}_n)^2 - f_{n-1}f_{n+1} + f_n^2 = 0 \tag{5.3}$$

where $\dot{f}_n = \dfrac{d}{dt}f_n$ and $\ddot{f}_n = \dfrac{d^2}{dt^2}f_n$.

Step 2. *Transformation to the Hirota bilinear form*: The Hirota bilinear form of Tl is

$$(D_t^2 - 4\sinh^2(\frac{1}{2}D_n))\{f_n.f_n\} = 0, \tag{5.4}$$

where

$$D_t^2\{f_n.f_n\} = 2(\ddot{f}_n f_n - 2(\dot{f}_n)^2), \tag{5.5}$$

and

$$
\begin{aligned}
4\sinh^2(\frac{1}{2}D_n)\{f_n.f_n\} =&4(\frac{e^{\frac{D_n}{2}}-e^{-\frac{D_n}{2}}}{2})^2\{f_n.f_{n'}\}|_{n=n'} \\
=&(e^{D_n}-2+e^{-D_n})\{f_n.f_{n'}\}|_{n=n'} \\
=&2(f_{n-1}f_{n+1}-f_n^2).
\end{aligned}
\tag{5.6}
$$

The above form of the Hirota D-operator is new to us. But when we analyze $Q(D) = \sinh^2(\delta D_n)$, δ is a parameter, by using the exponential form of the function sinh and the identity (2.2), we see that it is an even polynomial of D-operator satisfying all properties given in section 2.1.

Step 3. *Application of the Hirota perturbation*: Similar to before we insert $f_n = 1 + \varepsilon f_n^{(1)} + \varepsilon^2 f_n^{(2)} + ...$ into the equation (5.4) so we have

$$
\begin{aligned}
P(D)\{f_n.f_n\} =&P(D)\{1.1\} + \varepsilon P(D)\{f_n^{(1)}.1 + 1.f_n^{(1)}\} \\
&+ \varepsilon^2 P(D)\{f_n^{(2)}.1 + f_n^{(1)}.f_n^{(1)} + 1.f_n^{(2)}\} \\
&+ \varepsilon^3 P(D)\{f_n^{(3)}.1 + f_n^{(2)}.f_n^{(1)} + f_n^{(1)}.f_n^{(2)} + 1.f_n^{(3)}\} + ... = 0.
\end{aligned}
\tag{5.7}
$$

5.1 One-Soliton Solution of TL

In order to construct one-soliton solution of Tl we take $f_n = 1 + \varepsilon f_n^{(1)}$ where $f_n^{(1)} = e^{\theta_n^{(1)}}$ for $\theta_n^{(1)} = \omega_1 t + \kappa_1 n + \alpha_1$ and $f_n^{(j)} = 0$ for all $j \geq 2$. We insert f into the equation (5.7) and provide that the coefficients of ε^m, $m = 0, 1, 2$ are zero. The coefficient of ε^0 vanishes trivially since

$$
\begin{aligned}
P(D)\{1.1\} =&(D_t^2 - e^{D_n} + 2 - e^{-D_n})\{1.1\} \\
=&0 - 1 + 2 - 1 = 0.
\end{aligned}
\tag{5.8}
$$

From the coefficient of ε^1, we have

$$
\begin{aligned}
P(D)\{f_n^{(1)}.1 + 1.f_n^{(1)}\} =&2P(\partial)f_n^{(1)} \\
=&2[\omega_1^2 - 4\sinh^2(\frac{\kappa_1}{2})]e^{\theta_n^{(1)}} = 0.
\end{aligned}
\tag{5.9}
$$

Hence we obtain $P(p_1) = 0$ which implies $\omega_1 = \xi_l 2\sinh(\frac{\kappa_1}{2})$ where $\xi_l = 1$ or -1. The coefficient of ε^2 becomes zero since

$$
P(D)\{e^{\theta_n^{(1)}}.e^{\theta_n^{(1)}}\} =(\omega_1 - \omega_1)^2 - 4\sinh^2(\frac{\kappa_1 - \kappa_1}{2}) = 0. \tag{5.10}
$$

Without loss of generality we may take $\varepsilon = 1$, hence $f_n = 1 + e^{\theta_n^{(1)}}$ and so one-soliton solution of Tl is

$$V_n(t) = \frac{4\sinh^2(\frac{\kappa_1}{2})e^{\theta_n^{(1)}}}{(1+e^{\theta_n^{(1)}})^2} \tag{5.11}$$

where $\theta_n^{(1)} = \xi_l 2\sinh(\frac{\kappa_1}{2})t + \kappa_1 n + \alpha_1$ for $\xi_l = 1$ or -1.

5.2 Two-Soliton Solution of TL

Here we take $f_n = 1+\varepsilon f_n^{(1)}+\varepsilon^2 f_n^{(2)}$ where $f_n^{(1)} = e^{\theta_n^{(1)}}+e^{\theta_n^{(2)}}$ for $\theta_n^{(i)} = \omega_i t+\kappa_i n+\alpha_i$, $i = 1, 2$. The function $f_n^{(2)}$ shall be determined later. Note that $f_n^{(j)} = 0$ for all $j \geq 3$. We insert f_n into (5.7) and make the coefficients of ε^m, $m = 0, 1, ..., 4$ to vanish appeared in the Hirota perturbation. The coefficient of ε^0 is identically zero. From the coefficient of ε^1 we have

$$\begin{aligned} P(D)\{f_n^{(1)}.1 + 1.f_n^{(1)}\} =& 2P(\partial)\{e^{\theta_n^{(1)}} + e^{\theta_n^{(2)}}\} \\ =& 2[\sum_{i=1}^{2}(\omega_i^2 - 4\sinh^2(\frac{\kappa_i}{2}))e^{2\theta_n^{(i)}}] = 0 \end{aligned} \tag{5.12}$$

so we get the dispersion relation $P(p_i) = 0$ implying $\omega_i = \xi_l 2\sinh(\frac{\kappa_i}{2})$, $i = 1, 2$ for $\xi_l = 1$ or -1. The coefficient of ε^2 becomes

$$\begin{aligned} P(D)\{f_n^{(2)}.1 + 1.f_n^{(2)} + f_n^{(1)}.f_n^{(1)}\} =& 2P(\partial)f_n^{(2)} + 2P(D)\{e^{\theta_n^{(1)}}.e^{\theta_n^{(2)}}\} \\ =& 2P(\partial)f_n^{(2)} + 2P(p_1 - p_2)e^{\theta_n^{(1)}+\theta_n^{(2)}} = 0. \end{aligned} \tag{5.13}$$

To satisfy the above equation $f_n^{(2)}$ should have the form $f_n^{(2)} = A(1, 2)e^{\theta_n^{(1)}+\theta_n^{(2)}}$. If we insert $f_n^{(2)}$ into (5.13), we obtain $A(1, 2)$ as

$$A(1,2) = -\frac{P(p_1 - p_2)}{P(p_1 + p_2)} = -\frac{(\omega_1 - \omega_2)^2 - 4\sinh^2(\frac{\kappa_1-\kappa_2}{2})}{(\omega_1 + \omega_2)^2 - 4\sinh^2(\frac{\kappa_1+\kappa_2}{2})}. \tag{5.14}$$

Since $f_n^{(3)} = 0$, the coefficient of ε^3 turns out to be

$$\begin{aligned} P(D)\{f_n^{(1)}.f_n^{(2)} + f_n^{(2)}.f_n^{(1)}\} =& 2A(1,2)[P(D)\{e^{\theta_n^{(1)}}.e^{\theta_n^{(1)}+\theta_n^{(2)}} + e^{\theta_n^{(2)}}.e^{\theta_n^{(1)}+\theta_n^{(2)}}\}] \\ =& 2A(1,2)[P(p_2)e^{2\theta_n^{(1)}+\theta_n^{(2)}} + P(p_1)e^{\theta_n^{(1)}+2\theta_n^{(2)}}] \end{aligned} \tag{5.15}$$

and this is already zero since $P(p_i) = 0$, $i = 1, 2$. The coefficient of ε^4 also vanishes automatically. Hence by setting $\varepsilon = 1$ we have $f_n = 1+e^{\theta_n^{(1)}}+e^{\theta_n^{(2)}}+A(1,2)e^{\theta_n^{(1)}+\theta_n^{(2)}}$ and two-soliton solution of Tl is

$$V_n(t) = \frac{L(t)}{M(t)}, \tag{5.16}$$

where

$$L(t) = 4\{[A(1,2)(\sinh(\frac{\kappa_1}{2}) + \sinh(\frac{\kappa_2}{2}))^2 + (\sinh(\frac{\kappa_1}{2}) - \sinh(\frac{\kappa_2}{2}))^2]e^{\theta_n^{(1)}+\theta_n^{(2)}}$$
$$+\sinh^2(\frac{\kappa_1}{2})e^{\theta_n^{(1)}} + \sinh^2(\frac{\kappa_2}{2})e^{\theta_n^{(2)}} + A(1,2)[\sinh^2(\frac{\kappa_1}{2})e^{\theta_n^{(1)}+2\theta_n^{(2)}} + \sinh^2(\frac{\kappa_2}{2})e^{2\theta_n^{(1)}+\theta_n^{(2)}}]\} \tag{5.17}$$

and

$$M(t) = (1 + e^{\theta_n^{(1)}} + e^{\theta_n^{(2)}} + A(1,2)e^{\theta_n^{(1)}+\theta_n^{(2)}})^2 \tag{5.18}$$

for $\theta_n^{(i)} = \xi_l 2\sinh(\frac{\kappa_i}{2})t + \kappa_i n + \alpha_i$, $i = 1, 2$ with $\xi_l = 1$ or -1 and $A(1,2)$ is as given in (5.14).

5.3 Three-Soliton Solution of TL

To construct three-soliton solution of Tl we take $f = 1 + \varepsilon f_n^{(1)} + \varepsilon^2 f_n^{(2)} + \varepsilon^3 f_n^{(3)}$ where $f_n^{(1)} = e^{\theta_n^{(1)}} + e^{\theta_n^{(2)}} + e^{\theta_n^{(3)}}$, for $\theta_n^{(i)} = \omega_i t + \kappa_i n + \alpha_i$, $i = 1, 2, 3$ and substitute it into (5.7). Note that $f_n^{(j)} = 0$ for all $j \geq 4$. As we did before we provide that the coefficients of ε^m, $m = 0, 1, ..., 6$ are zero. The coefficient of ε^0 is identically zero. From the coefficient of ε^1 we have

$$P(D)\{f_n^{(1)}.1 + 1.f_n^{(1)}\} = 2P(D)\{e^{\theta_n^{(1)}} + e^{\theta_n^{(2)}} + e^{\theta_n^{(3)}}\} = 0 \tag{5.19}$$

which implies $P(p_i) = 0$ so $\omega_i = \xi_l 2\sinh(\frac{\kappa_i}{2})$, $i = 1, 2, 3$ for $\xi_l = 1$ or -1. By the coefficient of ε^2 we get

$$\begin{aligned}-P(\partial)f_n^{(2)} &= [(\omega_1-\omega_2)^2 - 4\sinh^2(\frac{\kappa_1-\kappa_2}{2})]e^{\theta_n^{(1)}+\theta_n^{(2)}} \\ &+[(\omega_1-\omega_3)^2 - 4\sinh^2(\frac{\kappa_1-\kappa_3}{2})]e^{\theta_n^{(1)}+\theta_n^{(3)}} \\ &+[(\omega_2-\omega_3)^2 - 4\sinh^2(\frac{\kappa_2-\kappa_3}{2})]e^{\theta_n^{(2)}+\theta_n^{(3)}}.\end{aligned} \tag{5.20}$$

To satisfy the above equation $f_n^{(2)}$ has to be in the form $f_n^{(2)} = A(1,2)e^{\theta_n^{(1)}+\theta_n^{(2)}} + A(1,3)e^{\theta_n^{(1)}+\theta_n^{(3)}} + A(2,3)e^{\theta_n^{(2)}+\theta_n^{(3)}}$. We insert $f_n^{(2)}$ into the equation (5.20) and we obtain $A(i,j)$ as

$$A(i,j) = -\frac{P(p_i - p_j)}{P(p_i + p_j)} = -\frac{(\omega_i - \omega_j)^2 - 4\sinh^2(\frac{\kappa_i - \kappa_j}{2})}{(\omega_i + \omega_j)^2 - 4\sinh^2(\frac{\kappa_i + \kappa_j}{2})} \tag{5.21}$$

for $i, j = 1, 2, 3,\ i < j$. The coefficient of ε^3 gives

$$\begin{aligned} -P(\partial)\{f_n^{(3)}\} =& P(D)\{f_n^{(1)}.f_n^{(2)}\} \\ =& \{A(1,2)P(p_3 - p_2 - p_1) + A(1,3)P(p_2 - p_1 - p_3) \\ &+ A(2,3)P(p_1 - p_2 - p_3)\}e^{\theta_n^{(1)}+\theta_n^{(2)}+\theta_n^{(3)}}. \end{aligned} \tag{5.22}$$

Thus $f_n^{(3)}$ is in the form $f_n^{(3)} = Be^{\theta_n^{(1)}+\theta_n^{(2)}+\theta_n^{(3)}}$. So the equation (5.22) gives

$$B = -\frac{A(1,2)P(p_3 - p_1 - p_2) + A(1,3)P(p_2 - p_1 - p_3) + A(2,3)P(p_1 - p_2 - p_3)}{P(p_1 + p_2 + p_3)}. \tag{5.23}$$

If we make all the simplifications by using $\omega_i = \xi_l 2\sinh(\dfrac{\kappa_i}{2})$, $i = 1, 2, 3$ for $\xi_l = 1$ or -1, we see that $B = A(1,2)A(1,3)A(2,3)$. Since $f_n^{(4)} = 0$ from the coefficient of ε^4 we have

$$P(D)\{f_n^{(1)}.f_n^{(3)} + f_n^{(3)}.f_n^{(1)} + f_n^{(2)}.f_n^{(2)}\} = 0. \tag{5.24}$$

After some calculations we get

$$\begin{aligned} &e^{2\theta_1+\theta_2+\theta_3}[BP(p_2 + p_3) + A(1,2)A(1,3)P(p_2 - p_3)] \\ &\quad + e^{\theta_1+2\theta_2+\theta_3}[BP(p_1 + p_3) + A(1,2)A(2,3)P(p_1 - p_3)] \\ &\quad + e^{\theta_1+\theta_2+2\theta_3}[BP(p_1 + p_2) + A(1,3)A(2,3)P(p_1 - p_2)] = 0. \end{aligned} \tag{5.25}$$

This is satisfied by $B = A(1,2)A(1,3)A(2,3)$. Finally the coefficients of ε^5 and ε^6 also vanish automatically. We may set $\varepsilon = 1$, therefore

$$\begin{aligned} f =& 1 + e^{\theta_n^{(1)}} + e^{\theta_n^{(2)}} + e^{\theta_n^{(3)}} + A(1,2)e^{\theta_n^{(1)}+\theta_n^{(2)}} \\ &+ A(1,3)e^{\theta_n^{(1)}+\theta_n^{(3)}} + A(2,3)e^{\theta_n^{(2)}+\theta_n^{(3)}} + Be^{\theta_n^{(1)}+\theta_n^{(2)}+\theta_n^{(3)}} \end{aligned} \tag{5.26}$$

and three-soliton solution of Tl is

$$V_n(t) = \frac{L(t)}{M(t)} \tag{5.27}$$

where

$$
\begin{aligned}
L(t) &= k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + k_3^2 e^{\theta_3} + e^{2\theta_1+\theta_2+\theta_3}[A(1,2)A(1,3)(k_2-k_3)^2 + B(k_2+k_3)^2] \\
&+ e^{\theta_1+\theta_2+2\theta_3}[A(1,3)A(2,3)(k_1-k_2)^2 + B(k_1+k_2)^2] \\
&+ e^{\theta_1+2\theta_2+\theta_3}[A(1,2)A(2,3)(k_1-k_3)^2 + B(k_1+k_3)^2] \\
&+ e^{\theta_1+\theta_2}[(k_1-k_2)^2 + A(1,2)(k_1^2 e^{\theta_2} + k_2^2 e^{\theta_1} + (k_1+k_2)^2)] \\
&+ e^{\theta_1+\theta_3}[(k_1-k_3)^2 + A(1,3)(k_1^2 e^{\theta_3} + k_3^2 e^{\theta_1} + (k_1+k_3)^2)] \\
&+ e^{\theta_2+\theta_3}[(k_2-k_3)^2 + A(2,3)(k_2^2 e^{\theta_3} + k_3^2 e^{\theta_2} + (k_2+k_3)^2)] \\
&+ e^{\theta_1+\theta_2+\theta_3}[A(1,2)(k_1^2+k_2^2+k_3^2+2k_1k_2-2k_1k_3-2k_2k_3) \\
&+ A(1,3)(k_1^2+k_2^2+k_3^2+2k_1k_3-2k_1k_2-2k_2k_3) \\
&+ A(2,3)(k_1^2+k_2^2+k_3^2+2k_2k_3-2k_1k_2-2k_1k_3) \\
&+ B(k_1^2+k_2^2+k_3^2+2k_1k_2+2k_1k_3+2k_2k_3)] \\
&+ Be^{\theta_1+\theta_2+\theta_3}[A(1,2)k_3^2 e^{\theta_1+\theta_2} + A(1,3)k_2^2 e^{\theta_1+\theta_3} + A(2,3)k_1^2 e^{\theta_2+\theta_3}]
\end{aligned}
\tag{5.28}
$$

and

$$
\begin{aligned}
M(t) = [1 &+ e^{\theta_n^{(1)}} + e^{\theta_n^{(2)}} + e^{\theta_n^{(3)}} + A(1,2)e^{\theta_n^{(1)}+\theta_n^{(2)}} \\
&+ A(1,3)e^{\theta_n^{(1)}+\theta_n^{(3)}} + A(2,3)e^{\theta_n^{(2)}+\theta_n^{(3)}} + Be^{\theta_n^{(1)}+\theta_n^{(2)}+\theta_n^{(3)}}]^2
\end{aligned}
\tag{5.29}
$$

for $\theta_n^{(i)} = \xi_l 2\sinh(\frac{\kappa_i}{2})t+\alpha_i$ for $\xi_l = 1$ or -1, $A(i,j)$ is as given in (5.21) $i,j = 1,2,3$, $i < j$ and $B = A(1,2)A(1,3)A(2,3)$.

Chapter 6

The Modified Korteweg-de Vries (MKdV) Equation

In this chapter, we analyze the modified Korteweg-de Vries (mKdV) equation and construct one-, two-, three- and N-soliton solutions of it. The mKdV equation is different than the KdV-type equations since it can only be written as a pair of Hirota bilinear equations. At last, we give a list of the mKdV-type equations. The mKdV equation is given by

$$u_t + 24u^2u_x + u_{xxx} = 0. \tag{6.1}$$

Step 1. *Bilinearization* By the transformation $u(x,t) = \dfrac{g_x f - g f_x}{g^2 + f^2}$ the mKdV equation can be written as a combination of bilinear equations,

$$\begin{aligned} &-(g^2+f^2)(g_t f - g f_t + g_{xxx} f - 3g_{xx} f_x + 3g_x f_{xx} - g f_{xxx}) \\ &\qquad + 6(f g_x - g f_x)(f f_{xx} - f_x^2 + g g_{xx} - g_x^2) = 0. \end{aligned} \tag{6.2}$$

Step 2. *Transformation to the Hirota bilinear form*: MKdV can only be expressed as the following pair of the Hirota bilinear equations:

$$\begin{cases} P_1(D)\{f.f + g.g\} = D_x^2\{f.f + g.g\} = 0 \\ P_2(D)\{g.f\} = (D_x^3 + D_t)\{g.f\} = 0. \end{cases} \tag{6.3}$$

Step 3. *Application of the Hirota perturbation*: We have two different differentiable functions g and f which may have different forms than we use for the KdV-type equations. Let $f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + ...$ and $g = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + ...$ where f_0, g_0 are constants with $(f_0, g_0) \neq (0, 0)$ to avoid the trivial solution and f_m, g_m, $m = 1, 2, ..., N$ are exponential functions. We insert f and g into the Hirota bilinear form of mKdV so we have

$$\begin{aligned} P_1(D)\{f.f + g.g\} = & P_1(D)\{f_0.f_0 + g_0.g_0\} \\ & + \varepsilon P_1(D)\{f_1.f_0 + f_0.f_1 + g_1.g_0 + g_0.g_1\} \\ & + \varepsilon^2 P_1(D)\{f_2.f_0 + f_1.f_1 + f_0.f_2 + g_2.g_0 + g_1.g_1 + g_0.g_2\} + ... = 0, \end{aligned} \tag{6.4}$$

$$\begin{aligned} P_2(D)\{g.f\} = & P_2(D)\{g_0.f_0\} + \varepsilon P_2(D)\{g_0.f_1 + g_1.f_0\} \\ & + \varepsilon^2 P_2(D)\{g_1.f_1 + g_0.f_2 + g_2.f_0\} + ... = 0. \end{aligned} \tag{6.5}$$

6.1 One-Soliton Solution of MKdV

We take $f = f_0 + \varepsilon f_1$ and $g_1 = g_0 + \varepsilon g_1$ where $f_1 = F_1 e^{\phi_1}$ and $g_1 = G_1 e^{\theta_1}$. Here $\phi_1 = \hat{k}_1 x + \hat{\omega}_1 t + \hat{\alpha}_1$, $\theta_1 = k_1 x + \omega_1 t + \alpha_1$ and F_1, G_1 are constants. Note that $f_j = g_j = 0$ for all $j \geq 2$. For nontrivial solution, f_0 and g_0 should not vanish at the same time. Both $P_1(D)$ and $P_2(D)$ should be considered to find the functions f and g. At first let us examine the Hirota perturbation on $P_1(D)\{f.f + g.g\} = 0$ by inserting f and g into the equation (6.4), and make the coefficients of ε^m, $m = 0, 1, 2$ to vanish. The coefficient of ε^0 is identically zero because both f_0 and g_0 are constants. The coefficient of ε^1 gives us

$$\begin{aligned} P_1(D)\{f_0.f_1 + f_1.f_0 + g_0.g_1 + g_1.g_0\} = & 2f_0 P_1(\partial) f_1 + 2g_0 P_1(\partial) g_1 \\ = & 2f_0 F_1 \hat{k}_1^{\,2} e^{\phi_1} + 2g_0 G_1 k_1^2 e^{\theta_1} = 0. \end{aligned} \tag{6.6}$$

We have two possibilities to satisfy this equality:

1. $g_0 = 0$ and $F_1 = 0$ so $f_1 = 0$,
2. $f_0 = 0$ and $G_1 = 0$ so $g_1 = 0$.

We shall take the first one. Choosing the other one changes only the sign of solutions which are also one-soliton solution of mKdV. By this choice and since $f_2 = 0$, the coefficient of ε^2 turns out to be

$$\begin{aligned} P_1(D)\{f_1.f_1 + f_2.f_0 + f_0.f_2 + g_1.g_1 + g_2.g_0 + g_0.g_2\} =& P_1(D)\{g_1.g_1\} \\ =& P_1(p_1 - p_1)e^{2\theta_1} \end{aligned} \tag{6.7}$$

which is zero. Now we shall examine $P_2(D)\{g.f\} = 0$ by inserting f and g into the equation (6.5) and going through in the same way. The coefficient of ε^0 is again equal to zero because f_0 is a constant and $g_0 = 0$. By the coefficient of ε^1 we have

$$P_2(D)\{g_0.f_1 + g_1.f_0\} = f_0 G_1(k_1^3 + \omega_1)e^{\theta_1} = 0. \tag{6.8}$$

For nontrivial solution, f_0 and G_1 do not vanish at the same time with g_0 and F_1 so $\omega_1 = -k_1^3$ (dispersion relation). The coefficient of ε^2

$$P_2(D)\{g_1.f_1 + g_0.f_2 + g_2.f_0\} = 0 \tag{6.9}$$

since $f_1 = g_0 = g_2 = 0$. Finally without loss of generality we may set $f_0 = G_1 = \varepsilon = 1$ and so we get $f = 1$, $g = e^{\theta_1}$ and therefore one-soliton solution of mKdV is

$$u(x,t) = \frac{g_x f - g f_x}{g^2 + f^2} = \frac{k_1}{2\cosh(\theta_1)}. \tag{6.10}$$

where $\theta_1 = k_1 x - k_1^3 t + \alpha_1$.

6.2 Two-Soliton Solution of MKdV

In the process of finding one-soliton solution of mKdV, it has been found that g_0 and f_1 are zero. Hence to find two-soliton solution, we can take $f = f_0 + \varepsilon^2 f_2$ and $g = \varepsilon g_1 + \varepsilon^2 g_2$ where $g_1 = e^{\theta_1} + e^{\theta_2}$ for $\theta_i = k_i x + \omega_i t + \alpha_i$, $i = 1, 2$. Note that here $f_j = g_j = 0$ for all $j \geq 3$. At first we substitute f and g into the equation (6.4) and we try to make the coefficients of ε^m, $m = 0, 1, ..., 4$ to vanish. The coefficient of ε^0 is trivially zero. The coefficient of ε^1

$$P_1(D)\{f_0.f_1 + f_1.f_0 + g_0.g_1 + g_1.g_0\} = 0 \tag{6.11}$$

also vanishes automatically since $g_0 = f_1 = 0$. From the coefficient of ε^2 we have

$$\begin{aligned} 2f_0P_1(\partial)f_2 + P_1(D)\{g_1.g_1\} =& 2f_0P_1(\partial)f_2 + 2P_1(D)e^{\theta_1}.e^{\theta_2} \\ =& 2f_0P_1(\partial)f_2 + 2P_1(p_1 - p_2)e^{\theta_1+\theta_2} = 0. \end{aligned} \tag{6.12}$$

Hence f_2 should have the form $f_2 = A(1,2)e^{\theta_1+\theta_2}$. By putting f_2 into the above equation we obtain $A(1,2)$ as

$$A(1,2) = -\frac{P_1(p_1 - p_2)}{f_0P_1(p_1 + p_2)} = -\frac{(k_1 - k_2)^2}{f_0(k_1 + k_2)^2}. \tag{6.13}$$

Since $f_3 = 0$, from the coefficient of ε^3 we have

$$P_1(D)\{g_1.g_2\} = 0. \tag{6.14}$$

We do not know g_2 so we stop here and apply same procedure to $P_2(D)\{g.f\} = 0$. The coefficient of ε^0 is trivially zero. From the coefficient of ε^1 we get

$$P_2(\partial)g_1 = P_2(p_1)e^{\theta_1} + P_2(p_2)e^{\theta_2} = 0 \tag{6.15}$$

which implies $P_2(p_i) = 0$ so $\omega_i = -k_i^3$ for $i = 1, 2$. The coefficient of ε^2 gives us

$$P_2(D)\{g_1.f_1 + g_0.f_2 + g_2.f_0\} = f_0P_2(\partial)g_2 = 0. \tag{6.16}$$

Thus we may choose $g_2 = 0$. So the equation (6.14) is also satisfied. The coefficient of ε^3 becomes

$$\begin{aligned} P_2(D)\{g_1.f_2 + g_2.f_1 + g_0.f_3 + g_3.f_0\} =& P_2(D)\{g_1.f_2\} \\ =& -A(1,2)[P_2(p_2)e^{2\theta_1+\theta_2} + P_2(p_1)e^{\theta_1+2\theta_2}] \end{aligned} \tag{6.17}$$

and it is zero since $P_2(p_i) = 0$, $i = 1, 2$. The coefficient of ε^4 also becomes zero. Without loss of generality we take $f_0 = \varepsilon = 1$ so finally we get $g = e^{\theta_1} + e^{\theta_2}$ and $f = 1 + A(1,2)e^{\theta_1+\theta_2}$. Hence two-soliton solution of mKdV is

$$u(x,t) = \frac{k_1e^{\theta_1} + k_2e^{\theta_2} - A(1,2)e^{\theta_1+\theta_2}(k_1e^{\theta_2} + k_2e^{\theta_1})}{1 + e^{2\theta_1} + e^{2\theta_2} + 2e^{\theta_1+\theta_2}(1 + A(1,2)) + A(1,2)^2e^{2\theta_1+2\theta_2}} \tag{6.18}$$

where $\theta_i = k_ix - k_i^3t + \alpha_i$ for $i = 1, 2$ and $A(1,2) = -\dfrac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$.

6.3 Three-Soliton Solution of MKdV

By our previous experiments we have discovered that $g_0 = f_1 = g_2 = 0$. So for three-soliton solution we may take $f = f_0 + \varepsilon^2 f_2 + \varepsilon^3 f_3$ and $g = \varepsilon g_1 + \varepsilon^3 g_3$ where $g_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}$ for $\theta_i = k_i x + \omega_i t + \alpha_i$, $i = 1, 2, 3$. Note that $f_j = g_j = 0$ for all $j \geq 4$. Now we insert f and g into the equation (6.4) and then to (6.5). Similar to before we make the coefficients of the ε^m, $m = 0, 1, ..., 6$ to be zero. The coefficients of ε^0 and ε^1 are automatically zero since f_0 is constant and $g_0 = f_1 = 0$. From the coefficient of ε^2 we have

$$\begin{aligned} -2f_0 P_1(\partial) f_2 =& P_1(D)\{g_1.g_1\} \\ =& 2[P_1(D)e^{\theta_1}.e^{\theta_2} + P_1(D)e^{\theta_1}.e^{\theta_3} + P_1(D)e^{\theta_2}e^{\theta_3}] \\ =& 2[P_1(p_1 - p_2)e^{\theta_1+\theta_2} + P_1(p_1 - p_3)e^{\theta_1+\theta_3} + P_1(p_2 - p_3)e^{\theta_2+\theta_3}]. \end{aligned} \tag{6.19}$$

We see that f_2 is in the form $f_2 = A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}$. When we substitute f_2 into the above equation we get

$$A(i,j) = -\frac{P_1(p_i - p_j)}{f_0 P_1(p_i + p_j)} = -\frac{(k_i - k_j)^2}{f_0(k_i + k_j)^2} \tag{6.20}$$

for $i, j = 1, 2, 3$, $i < j$. From the coefficient of ε^3 we get $P_1(\partial)\{f_3\} = 0$ so we may choose $f_3 = 0$. Since $f_4 = 0$, the coefficient of ε^4 gives us

$$2P_1(D)\{g_1.g_3\} + P_1(D)\{f_2.f_2\} = 0. \tag{6.21}$$

We do not know g_3 so we cannot go further. Thus let us now consider the Hirota perturbation on $P_2(D)\{g.f\} = 0$. The coefficient of ε^0 is trivially zero. The coefficient of ε^1

$$P_2(D)\{g_0.f_1 + g_1.f_0\} = f_0 P_2(\partial)\{e^{\theta_1} + e^{\theta_2} + e^{\theta_3}\} = 0 \tag{6.22}$$

gives us the dispersion relation $P_2(p_i) = 0$ so $\omega_i = -k_i^3$ for $i = 1, 2, 3$. From the coefficient of ε^3 we have

$$\begin{aligned} -f_0 P_2(\partial) g_3 =& P_2(D)\{g_1.f_2\} \\ =& e^{\theta_1+\theta_2+\theta_3}[A(1,2)P_2(p_3 - p_1 - p_2) + A(1,3)P_2(p_2 - p_1 - p_3) \\ &+ A(2,3)P_2(p_1 - p_2 - p_3)]. \end{aligned} \tag{6.23}$$

Hence g_3 is in the form $g_3 = Be^{\theta_1+\theta_2+\theta_3}$ where after some calculations by using the dispersion relation the coefficient B is found as

$$B = f_0 A(1,2)A(1,3)A(2,3). \tag{6.24}$$

Since g_3 is known now, we can replace it in the equation (6.21) and we see that the equation is satisfied. The coefficient of ε^4 is $P_2(D)\{g_1.f_3\} + P_2(\partial)g_4 = 0$ since $f_3 = g_4 = 0$. The coefficients of ε^5 and ε^6 vanish trivially. Finally by setting $f_0 = \varepsilon = 1$ we have $f = 1 + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}$, $g = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + Be^{\theta_1+\theta_2+\theta_3}$ and so three-soliton solution of mKdV is

$$u(x,t) = \frac{L(x,t)}{M(x,t)} \tag{6.25}$$

where

$$\begin{aligned} L(x,t) &= B[(k_1+k_2+k_3)+k_3A(1,2)e^{\theta_1+\theta_2}+k_2A(1,3)e^{\theta_1+\theta_3}+k_1A(2,3)e^{\theta_2+\theta_3}]\}e^{\theta_1+\theta_2+\theta_3} \\ &+k_1e^{\theta_1}+k_2e^{\theta_2}+k_3e^{\theta_3}+\{A(1,2)(k_3-k_1-k_2)+A(1,3)(k_2-k_1-k_3)+A(2,3)(k_1-k_2-k_3) \\ &-A(1,2)e^{\theta_1+\theta_2}(k_2e^{\theta_1}+k_1e^{\theta_2})-A(1,3)e^{\theta_1+\theta_3}(k_3e^{\theta_1}+k_1e^{\theta_3})-A(2,3)e^{\theta_2+\theta_3}(k_2e^{\theta_3}+k_3e^{\theta_2}) \end{aligned} \tag{6.26}$$

and

$$\begin{aligned} M(x,t) &= 1 + e^{2\theta_1} + e^{2\theta_2} + e^{2\theta_3} + 2(1 + A(1,2))e^{\theta_1+\theta_2} + 2(1 + A(2,3))e^{\theta_2+\theta_3} \\ &+2(1+A(1,3))e^{\theta_1+\theta_3}+A(1,2)^2e^{2\theta_1+2\theta_2}+A(2,3)^2e^{2\theta_2+2\theta_3}+A(1,3)^2e^{2\theta_1+2\theta_3}+B^2e^{2\theta_1+2\theta_2+2\theta_3} \\ &+2e^{\theta_1+\theta_2+\theta_3}[(B+A(1,2)A(1,3))e^{\theta_1}+(B+A(1,2)A(2,3))e^{\theta_2}+(B+A(1,3)A(2,3))e^{\theta_3}] \end{aligned} \tag{6.27}$$

with $\theta_i = k_i x - k_i^3 t + \alpha_i$, $A(i,j) = -\dfrac{(k_i - k_j)^2}{(k_i + k_j)^2}$ for $i,j = 1,2,3,\ i < j$ and $B = A(1,2)A(1,3)A(2,3)$.

6.4 N-Soliton Solution of MKdV

MKdV is written as a combination of Hirota bilinear equations as

$$-(g^2+f^2)\underbrace{(g_t f - g f_t + g_{xxx} f - 3g_{xx} f_x + 3g_x f_{xx} - g f_{xxx})}_{I} + 6(f g_x - g f_x)\underbrace{(f f_{xx} - f_x^2 + g g_{xx} - g_x^2)}_{II} = 0 \tag{6.28}$$

and we have seen that to construct soliton-solution of mKdV, we find f and g satisfying the equations I and II separately. For N-soliton solution of mKdV, we claim that $f(x,t)$ and $g(x,t)$ take the form

$$f(x,t) = \sum_{m=0}^{[N/2]} \sum_{{}_N C_{2m}} A(i_1, ..., i_{2m}) exp(\theta_{i_1} + ... + \theta_{i_{2m}}), \tag{6.29}$$

and

$$g(x,t) = \sum_{m=0}^{[(N-1)/2]} \sum_{{}_N C_{2m+1}} A(i_1, ..., i_{2m+1}) exp(\theta_{i_1} + ... + \theta_{i_{2m+1}}) \tag{6.30}$$

where

$$A(i_1, ..., i_m) = \prod_{l<j}^{(m)} A(l,j) \quad , \quad A(l,j) = -\frac{(k_l - k_j)^2}{(k_l + k_j)^2}. \tag{6.31}$$

Here $[N/2]$ denotes the maximum integer which does not exceed $N/2$, ${}_N C_m$ indicates the summation over all possible combinations of m elements from N and (m) indicates the product of all possible combinations of the m elements with $(l<j)$. Note that $A(i_m) = 1$ for $m = 1, 2, ..., N$. To prove our claim we substitute the expression for $f(x,t)$ and $g(x,t)$ into the equations I and II. Then we will check that these equations are satisfied. Substitution of $f(x,t)$ and $g(x,t)$ give us some exponential terms. To satisfy the equations I and II, the coefficients of the exponential terms should vanish. From these coefficients we get the relations

$$\sum_{r=0}^{m} \sum_{{}_m C_r} A(i_1, ..., i_r) A(i_{r+1}, ..., i_m) g_1(i_1, ..., i_r; i_{(r+1)}, ..., i_m), \quad m = 1, 3, 5, ... \le N, \tag{6.32}$$

and

$$\sum_{r=0}^{m}\sum_{{}_mC_r}(-1)^r A(i_1,...,i_r)A(i_{r+1},...,i_m)g_2(i_1,...,i_r;i_{(r+1)},...,i_m), \quad m=2,4,6,...\leq N, \tag{6.33}$$

where

$$g_1(i_1,...,i_r;i_{(r+1)},...,i_m) = (k_{i_1}+...+k_{i_r}-k_{i_{(r+1)}}-...-k_{i_m})^3 \\ -(k_{i_1}^3+...+k_{i_r}^3-k_{i_{(r+1)}}^3-...-k_{i_m}^3), \tag{6.34}$$

and

$$g_2(i_1,...,i_r;i_{(r+1)},...,i_m) = (k_{i_1}+...+k_{i_r}-k_{i_{(r+1)}}-...-k_{i_m})^2. \tag{6.35}$$

For fixed m the equations (6.34) and (6.35) become

$$D_1(k_1,...,k_m) = \sum_{\sigma_1,...,\sigma_m=\pm1} b(\sigma_1k_1,...,\sigma_mk_m)g_1(\sigma_1k_1,...,\sigma_mk_m)=0, \tag{6.36}$$

for m is odd and

$$D_2(k_1,...,k_m) = \sum_{\sigma_1,...,\sigma_m=\pm1} (\prod_{i=1}^{m}\sigma_i) b(\sigma_1k_1,...,\sigma_mk_m)g_2(\sigma_1k_1,...,\sigma_mk_m)=0, \tag{6.37}$$

for m is even, where

$$b(\sigma_1k_1,...,\sigma_mk_m) = \prod_{l<j}^{(m)}(\sigma_lk_l-\sigma_jk_j)^2, \tag{6.38}$$

and

$$g_1(\sigma_1k_1,...,\sigma_mk_m) = (\sigma_1k_1+...+\sigma_mk_m)^3 - ((\sigma_1k_1)^3+...+(\sigma_mk_m)^3), \tag{6.39}$$

and

$$g_2(\sigma_1k_1,...,\sigma_mk_m) = (\sigma_1k_1+...+\sigma_mk_m)^2. \tag{6.40}$$

We will prove these identities by induction. Before that we state the following properties of $D_1(k_1,...,k_m)$ and $D_2(k_1,...,k_m)$ [9],

(i) D_1 and D_2 are symmetric, homogeneous polynomials,

(ii) D_1 is an even function of $k_1,...,k_m$,

(iii) If $k_1 = 0$, then $D_2 = 0$,

(iv) If $k_l = k_j$ we have

$$D_1(k_1, ..., k_m) = 2(2k_l)^2 D(k_1, ..., k_{l-1}, k_{l+1}, ..., k_{j-1}, k_{j+1}, ..., k_n) \prod_{s=1}^{m\,\prime} (k_l^2 - k_s^2)^2,$$
$$D_2(k_1, ..., k_m) = -2(2k_l)^2 D(k_1, ..., k_{l-1}, k_{l+1}, ..., k_{j-1}, k_{j+1}, ..., k_n) \prod_{s=1}^{m\,\prime} (k_l^2 - k_s^2)^2.$$

Here the primes indicate that the products do not include $s = l$ and $s = j$. Now let us first consider $D_1(k_1, ..., k_m)$. For $m = 1$, the identity clearly holds since

$$D_1(k_1) = (\sigma_1 k_1)^2 (\sigma_1 k_1)[-(\sigma_1 k_1)^3 + (\sigma_1 k_1)^3] = 0. \tag{6.41}$$

Now assume that the identity holds for $m - 2$. By using the properties of $D_1(k_1, ..., k_m)$ we see that it can be factored by $\displaystyle\prod_{l<j}^{(m)} (k_l^2 - k_j^2)^2$ whose degree is $2m(m-1)$. But the equation (6.36) shows that the degree of D_1 is $m(m-1)+3$ which is smaller than $2m(m-1)$ for $m > 1$. Since this is impossible, the identity should hold for m. In a similar way we will analyze $D_2(k_1, ..., k_m)$. It is easily seen that for $m = 2$ the identity holds since

$$\begin{aligned} D_2(k_1, k_2) &= \sum_{\sigma_1, \sigma_2 = \pm 1} \left(\prod_{i=1}^{2} \sigma_i\right)(\sigma_1 k_1 - \sigma_2 k_2)^2 (\sigma_1 k_1 + \sigma_2 k_2)^2 \\ &= \sum_{\sigma_1, \sigma_2 = \pm 1} (\sigma_1 (k_1^2 - k_2^2)^2 + \sigma_2 (k_1^2 - k_2^2)^2) = 0. \end{aligned} \tag{6.42}$$

Now we assume that the identity holds for $m - 2$. By the help of the properties of $D_2(k_1, ..., k_m)$ we obtain that D_2 can be factored by $\prod_{i=1}^{m} k_i \prod_{l<j}^{(m)} (k_l - k_j)^2$ of degree m^2. On the other hand by the equation (6.37), we have known that D_2 is of degree $m(m-1)+2$ which is smaller than m^2 for $m > 2$. But this is impossible so the identity holds for m.

6.5 The MKdV-type Equations

Here we will give a list of equations [21] which can be written as a pair of Hirota bilinear equations,

$$\begin{cases} P_1(D)\{g.f\} &= 0, \\ P_2(D)\{f.f + g.g\} &= 0. \end{cases} \tag{6.43}$$

where $P_1(D)$ is an even and $P_2(D)$ is odd. We call the equations written in this form the mKdV-type equations. For mKdV-type Hirota bilinear equations we can make a rotation $f = F + G$ and $g = i(F - G)$ so the pair (6.43) becomes $P_1(D)\{G.F\} = 0$ and $P_2(D)\{G.F\} = 0$. The mKdV-type equations are

$$\begin{cases} (aD_x^7 + bD_x^5 + D_x^2 D_t + D_y)\{G.F\} &= 0, \\ D_x^2\{G.F\} &= 0, \end{cases} \tag{6.44}$$

$$\begin{cases} (aD_x^3 + bD_x^3 + D_y)\{G.F\} &= 0, \\ D_x D_t\{G.F\} &= 0, \end{cases} \tag{6.45}$$

$$\begin{cases} (D_x D_y D_t + aD_x + bD_t)\{G.F\} &= 0, \\ D_x D_t\{G.F\} &= 0, \end{cases} \tag{6.46}$$

$$\begin{cases} (D_x^3 + D_y)\{G.F\} &= 0, \\ (D_x^3 D_t + aD_x^2 + D_t D_y)\{G.F\} &= 0, \end{cases} \tag{6.47}$$

$$\begin{cases} (D_x^3 + D_y)\{G.F\} &= 0, \\ (D_x^6 + 5D_x^3 D_y - 5D_y^2 + D_t D_x)\{G.F\} &= 0. \end{cases} \tag{6.48}$$

Here a and b are arbitrary constants.

Chapter 7

The Sine-Gordon (SG) Equation

In this chapter, we consider the sine-Gordon (sG) equation, which describes motion of dislocations in crystals, some models of elementary particles, self-transparency due to nonlinear effects of optical pulses, motion magnetic flux in Josephson junctions, and so forth. We construct one-, two-, three-soliton solutions of it. Similar to the mKdV equation, the sG equation is different than the KdV-type equations since it can only be written as a pair of Hirota bilinear equations. The form of the sG equation also differs from the mKdV equation by the parity of Hirota bilinear equations. Finally, we give a list of the sG-type equations. The sG equation is given by

$$\phi_{xx} - \phi_{tt} = \sin\phi. \tag{7.1}$$

Step 1. *Bilinearization*: Here we use the transformation $\phi = 4\arctan(g/f)$ which provides sG to be written as a combination of bilinear equations,

$$\begin{aligned}(f^2-g^2)(f_{xx}g - 2f_xg_x + fg_{xx} - f_{tt}g + 2f_tg_t - fg_{tt} - fg)\\ - 2fg(ff_{xx} - f_x^2 - gg_{xx} + g_x^2 - ff_{tt} + f_t^2 + gg_{tt} - g_t^2) = 0.\end{aligned} \tag{7.2}$$

Step 2. *Transformation to the Hirota bilinear form*: Similar to mKdV, sG is not exactly bilinearizable. We can only express it as a pair of the Hirota bilinear equations

$$\begin{cases} P_1(D)\{f.g\} = (D_x^2 - D_t^2 - 1)\{f.g\} = 0 \\ P_2(D)\{f.f - g.g\} = (D_x^2 - D_t^2)\{f.f - g.g\} = 0. \end{cases} \tag{7.3}$$

Step 3. *Application of the Hirota perturbation*: When we insert the finite perturbation expansions $f(x,t) = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + ...$ and $g(x,t) = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + ...$ into $P_1(D)\{f.g\} = 0$ and $P_2(D)\{f.f - g.g\} = 0$, to satisfy these Hirota bilinear equations $g(x,t)$ and $f(x,t)$ take different forms as in mKdV. We do not give the details but we use the forms given in [10]. The general form of $g(x,t)$ and $f(x,t)$ are

$$g(x,t) = \sum_{m=0}^{[(N-1)/2]} \sum_{{}_N C_{2m+1}} A(i_1, ..., i_{2m+1}) exp(\theta_{i_1} + ... + \theta_{i_{2m+1}}), \tag{7.4}$$

$$f(x,t) = \sum_{m=0}^{[N/2]} \sum_{{}_N C_{2m}} A(i_1, ..., i_{2m}) exp(\theta_{i_1} + \theta_{i_2} + ... + \theta_{i_{2m}}), \tag{7.5}$$

where

$$A(i_1, ..., i_m) = \prod_{l<j}^{(m)} A(l,j) \quad , \quad A(l,j) = \frac{(k_l - k_j)^2 - (\omega_l - \omega_j)^2}{(k_l + k_j)^2 + (\omega_l + \omega_j)^2}. \tag{7.6}$$

Here $[N/2]$ denotes the maximum integer which does not exceed $N/2$. ${}_N C_m$ indicates the summation over all possible combinations of m elements taken from N and (m) indicates the product of all possible combinations of m elements with $(l < j)$. Note that $A(i_m) = 1$ for $m = 1, 2, ..., N$.

7.1 One-Soliton Solution of SG

For one-soliton solution of sG we take $g = \varepsilon g_1$ where $g_1 = e^{\theta_1}$, $\theta_1 = k_1 x + \omega_1 t + \alpha_1$ and $f = 1$. At first we insert them into $P_1(D)\{f.g\} = 0$ and consider the coefficients of ε^1 since there is no other power of ε comes from the Hirota perturbation. By the coefficient of ε^1 we have

$$P_1(D)\{1.g_1\} = P_1(p_1) e^{\theta_1} = 0 \tag{7.7}$$

which is satisfied when $P_1(p_1)$=0 and this gives us the dispersion relation $\omega_1 = \xi_l\sqrt{k_1^2-1}$ for $\xi_l = 1$ or -1. Now we analyze the Hirota perturbation on $P_2(D)\{f.f - g.g\} = 0$. Here only ε^0 and ε^2 appear and we make their coefficients to vanish. The coefficient of ε^0 is $P_2(D)\{1.1\}$ and it is identically zero. The coefficient of ε^2 also vanishes trivially since

$$P_2(D)\{e^{\theta_1}.e^{\theta_1}\} = P_2(0,0)e^{2\theta_1} = 0. \tag{7.8}$$

Finally we set $\varepsilon = 1$ and so $g = e^{\theta_1}$, $f = 1$. Thus one-soliton solution of sG is

$$\phi(x,t) = 4\arctan(e^{\theta_1}), \tag{7.9}$$

where $\theta_1 = k_1x + \xi_l\sqrt{k_1^2-1}t + \alpha_1$ for $\xi_l = 1$ or -1.

7.2 Two-Soliton Solution of SG

To construct two-soliton solution of sG we take $g = \varepsilon g_1$ where $g_1 = e^{\theta_1} + e^{\theta_2}$ with $\theta_i = k_ix + \omega_it + \alpha_i$, $i = 1,2$ and $f = 1 + \varepsilon^2 f_2$ where f_2 shall be determined later. Now we examine the Hirota perturbation on $P_1(D)\{f.g\} = 0$. The coefficient of ε^1 gives

$$P_1(D)\{1.g_1\} = P_1(p_1)e^{\theta_1} + P_1(p_2)e^{\theta_2} = 0 \tag{7.10}$$

which implies $P_1(p_i) = 0$, $i = 1,2$ and so $\omega_i = \xi_l\sqrt{k_i^2-1}$, $i = 1,2$ for $\xi_l = 1$ or -1. The coefficient of ε^3 becomes

$$P_1(D)\{f_2.g_1\} = P_1(D)\{f_2.e^{\theta_1}\} + P_1(D)\{f_2.e^{\theta_2}\} \tag{7.11}$$

but since we do not know f_2 we stop here and go to $P_2(D)\{f.f - g.g\} = 0$. The coefficient of ε^0 is identically zero. The coefficient of ε^2 turns out to be

$$\begin{aligned} P_2(D)\{f_2.1 + 1.f_2 - g_1.g_1\} &= 2P_2(\partial)f_2 - 2P_2(D)\{e^{\theta_1}.e^{\theta_2}\} \\ &= 2P_2(\partial)f_2 - 2P_2(p_1 - p_2)e^{\theta_1+\theta_2} = 0. \end{aligned} \tag{7.12}$$

To satisfy the above equation f_2 has to be in the form $f_2 = A(1,2)e^{\theta_1+\theta_2}$. Inserting this form gives us $A(1,2)$ as

$$A(1,2) = \frac{P_2(p_1 - p_2)}{P_2(p_1 + p_2)} = \frac{(k_1 - k_2)^2 - (\omega_1 - \omega_2)^2}{(k_1 + k_2)^2 - (\omega_1 + \omega_2)^2}. \tag{7.13}$$

The coefficient of ε^4 vanishes trivially. Now since we know f_2 we go back to the equation (7.11)

$$\begin{aligned} P_1(D)\{f_2.g_1\} =& A(1,2)[P_1(D)e^{\theta_1+\theta_2}.e^{\theta_1} + P_1(D)e^{\theta_1+\theta_2}.e^{\theta_2}] \\ =& A(1,2)[P_1(p_2)e^{2\theta_1+\theta_2} + P_1(p_1)e^{\theta_1+2\theta_2}]. \end{aligned} \tag{7.14}$$

This equation vanishes automatically since $P_1(p_i) = 0$, $i = 1, 2$. Finally we may set $\varepsilon = 1$, so $g = e^{\theta_1} + e^{\theta_2}$ and $f = 1 + A(1,2)e^{\theta_1+\theta_2}$. Thus two-soliton solution of sG is

$$\phi(x,t) = 4\arctan\left(\frac{e^{\theta_1} + e^{\theta_2}}{1 + A(1,2)e^{\theta_1+\theta_2}}\right) \tag{7.15}$$

where $\theta_i = k_i x + \xi_l\sqrt{k_i^2 - 1}t + \alpha_i$, $i = 1, 2$ for $\xi_l = 1$ or -1 and $A(1,2)$ is as given in (7.13).

7.3 Three-Soliton Solution of SG

In order to construct three-soliton solution of sG, we take $g = \varepsilon g_1 + \varepsilon^3 g_3$ where $g_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}$ with $\theta_i = k_i x + \omega_i t + \alpha_i$, $i = 1, 2$ and $f = 1 + \varepsilon^2 f_2$. Here we shall decide what are g_3 and f_2 while we are applying the method. Now firstly we insert g and f into $P_1(D)\{f.g\} = 0$ and make the coefficients of ε^{2m+1}, $m = 0, 1, 2$ to be zero. From the coefficient of ε^1 we have

$$P_1(D)\{1.g_1\} = P_1(p_1)e^{\theta_1} + P_1(p_2)e^{\theta_2} + P_1(p_3)e^{\theta_3} = 0. \tag{7.16}$$

Hence $P_1(p_i) = 0$, $i = 1, 2, 3$ and so $\omega_i = \xi_l\sqrt{k_i^2 - 1}$, $i = 1, 2, 3$ for $\xi_l = 1$ or -1. The coefficients of ε^3 and ε^5 give $P_1(D)\{1.g_3 + f_2.g_1\} = 0$ and $P_1(D)\{f_2.g_3\} = 0$ respectively. But since we do not know f_2 and g_3, we cannot go further. So now we analyze the Hirota perturbation on $P_2(D)\{f.f - g.g\} = 0$. We provide that the coefficients of ε^{2m}, $m = 0, 1, 2, 3$ vanish. The coefficient of ε^0 is identically zero. By the coefficient of ε^2 we have

$$P_2(D)\{f_2.1 + 1.f_2 - g_1.g_1\} = 2P(\partial)f_2 - 2P(D)\{e^{\theta_1}.e^{\theta_2} + e^{\theta_1}.e^{\theta_3} + e^{\theta_2}.e^{\theta_3}\} = 0, \tag{7.17}$$

hence

$$P_2(\partial)f_2 = [P_2(p_1 - p_2)e^{\theta_1+\theta_2} + P_2(p_1 - p_3)e^{\theta_1+\theta_3} + P_2(p_2 - p_3)e^{\theta_2+\theta_3}]. \tag{7.18}$$

We see that f_2 is in the form $f_2 = A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}$. If we insert this expression for f_2 into the equation (7.18) we obtain

$$A(i,j) = \frac{P_2(p_i - p_j)}{P_2(p_i + p_j)} = \frac{(k_i - k_j)^2 - (\omega_i - \omega_j)^2}{(k_i + k_j)^2 - (\omega_i + \omega_j)^2} \tag{7.19}$$

for $i, j = 1, 2, 3,\ i < j$. Now we again turn to the Hirota perturbation on $P_1(D)\{f.g\} = 0$ to find g_3 from the coefficient of ε^3. The coefficient of ε^3 becomes

$$\begin{aligned} -P_1(\partial)g_3 =& P_1(D)\{f_2.g_1\} \\ =& [A(1,2)P_1(p_3 - p_1 - p_2) + A(1,3)P_1(p_2 - p_1 - p_3) \\ &+ A(2,3)P_1(p_1 - p_2 - p_3)]e^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{7.20}$$

We see that $g_3 = Be^{\theta_1+\theta_2+\theta_3}$. Inserting g_3 into the above equation gives us B as

$$B = -\frac{A(1,2)P_1(p_3 - p_1 - p_2) + A(1,3)P_1(p_2 - p_1 - p_3) + A(2,3)P_1(p_1 - p_2 - p_3)}{P_1(p_1 + p_2 + p_3)}. \tag{7.21}$$

The coefficient of ε^5 appeared in the first Hirota bilinear equation of sG turns out to be

$$\begin{aligned} P_1(D)\{f_2.g_3\} = A(1,2)P_1(p_3)e^{2\theta_1+2\theta_2+\theta_3} &+ A(1,3)P_1(p_2)e^{2\theta_1+\theta_2+2\theta_3} \\ &+ A(2,3)P_1(p_1)e^{\theta_1+2\theta_2+2\theta_3} \end{aligned} \tag{7.22}$$

and this is trivially zero since $P_1(p_i) = 0,\ i = 1, 2, 3$. Thus we are done with $P_1(D)\{f.g\} = 0$. Now we turn back to the second Hirota bilinear equation of sG where only the coefficients of ε^4 and ε^6 are remained. The coefficient of ε^4 is

$$P_2(D)\{f_2.f_2 - g_1.g_3 - g_3.g_1\} = 0 \tag{7.23}$$

and after some calculation it becomes

$$\begin{aligned} e^{2\theta_1+\theta_2+\theta_3}&[A(1,2)A(1,3)P(p_2 - p_3) - BP(p_2 + p_3)] \\ &+ e^{\theta_1+2\theta_2+\theta_3}[A(1,2)A(2,3)P(p_1 - p_3) - BP(p_1 + p_3)] \\ &+ e^{\theta_1+\theta_2+2\theta_3}[A(1,3)A(2,3)P(p_1 - p_2)] - BP(p_1 + p_2) = 0. \end{aligned} \tag{7.24}$$

Hence we obtain that $B = A(1,2)A(1,3)A(2,3)$. We should check that this equation for B is same with the previous expression found for B in (7.21). Indeed we see that they are same after some simplifications. The coefficient of ε^6 clearly vanishes. We may set $\varepsilon = 1$ thus $g = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + Be^{\theta_1+\theta_2+\theta_3}$, $f = 1 + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}$ and so three-soliton solution of sG is

$$\phi(x,t) = 4\arctan\left(\frac{e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + Be^{\theta_1+\theta_2+\theta_3}}{1 + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}}\right) \tag{7.25}$$

where $\theta_i = k_i x + \xi_l\sqrt{k_i^2 - 1}t + \alpha_i$ for $\xi_l = 1$ or -1, $A(i,j)$ is as given in (7.19), $i,j = 1,2,3$, $i < j$ and $B = A(1,2)A(1,3)A(2,3)$.

7.4 The SG-type Equations

Here we will give a list of equations [21] which can be written as a pair of Hirota bilinear equations,

$$\begin{cases} P_1(D)\{g.f\} &= 0, \\ P_2(D)\{f.f + g.g\} &= 0. \end{cases} \tag{7.26}$$

where both $P_1(D)$ and $P_2(D)$ are even. We call the equations written in this form the sG-type equations. Note that the bilinear form of sG that we have analyzed is different than the above one but under a suitable transformation we can get this form. The sG-type equations are

$$\begin{cases} (D_xD_t + b)\{g.f\} &= 0, \\ (D_x^3D_t + 3bD_x^2 + D_tD_y)\{f.f + g.g\} &= 0, \end{cases} \tag{7.27}$$

$$\begin{cases} (aD_x^3D_t + D_tD_y + b)\{g.f\} &= 0, \\ D_xD_t\{f.f + g.g\} &= 0, \end{cases} \tag{7.28}$$

where a and b are arbitrary constants.

Chapter 8

Conclusion

In this thesis we have studied the Hirota direct method. It is used to construct multi-soliton solutions of integrable nonlinear partial differential and difference equations. We have applied the Hirota method to the Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP), extended Kadomtsev-Petviashvili (eKP) and Toda lattice (Tl) equations. We have constructed one-,two- and three-soliton solutions of them. Additionally we have given N-soliton solution of KdV. Since the others are KdV-type equations, in a similar way, we can also construct their N-soliton solutions.

Different than KdV, KP, and Tl, the extended Kadomtsev-Petviashvili equation is a non-integrable equation unless it satisfies a condition which makes it transformable to KP. By using eKP, we have seen that even an equation is not integrable it may possess one- and two-soliton solutions but to possess three-soliton solution it should satisfy a special condition. In spite of this fact, we obtain infinitely many solutions of this equation by using the Hirota method.

We have also studied the equations written as a pair of Hirota bilinear equations such that the modified Korteweg-de Vries (mKdV) and sine-Gordon (sG) equations. We have constructed their one-, two- and three-soliton solutions. We have also given N-soliton solution of mKdV. Similarly we can also construct N-soliton solution of sG since sG has similar Hirota bilinear form as mKdV.

Bibliography

[1] Russell, S. J., *Fourteenth meeting of the British association of the advancement of science,* Report on waves, $311 - 390$, eds. John Murray, London (1844).

[2] Airy, G. B., *Tides and waves,* Encyclopaedia Metropolitana, **5**, $241-396$, London (1845).

[3] Korteweg D. J. and de Vries G., *On the change of form of long waves advancing in rectangular canal and on a new type of long stationary waves,* Philos. Mag. Ser. 5, **39**, 422-443 (1895).

[4] Fermi A., Pasta J. and Ulam S., *Studies of non-linear problems,* I. Los. Alamos Report LA 1940 (1955).

[5] Zabusky N. J. and Kruskal M. D., *Interaction of solitons in a collisionless plasma and the recurrence of initial states,* Phys. Rev. Lett., **15**, 240-243 (1965).

[6] Gardner C. S., Greene J. M., Kruskal M. D. and Miura R. M., *Method for solving the Korteweg-de Vries equation,* Phys. Rev. Lett., **19**, 1095-1097 (1967).

[7] Lax P. D., *Integrals of non-linear equations of evolution and solitary waves,* Commun. Pure Appl. Math., **21**, 467-490 (1968).

[8] Hirota R., *Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons,* Phys. Rev. Lett., **27**, 1192 (1971).

[9] Hirota R., *Exact solution of the modified Korteweg-de Vries equation for multiple collisions of solitons,* J. Phys. Soc. Japan, **33**, 1456 (1972).

[10] Hirota R., *Exact solution of the sine-Gordon equation for multiple collisions of solitons,* J. Phys. Soc. Japan, **33**, 1459 (1972).

[11] Hirota R., *Exact envelope-soliton solutions of a nonlinear wave equation,* J. Math. Phys, **14**, 805-809 (1973).

[12] Hirota R., *Exact N-soliton solution of a nonlinear lumped network equation,* J. Phys. Soc. Japan, **35**, 286-288 (1973).

[13] Grammaticos B., Ramani A. and Hietarinta J., *Multilinear operators: the natural extension of Hirota's bilinear formalism,* Phys. Lett. A, **190**, 65-70 (1994).

[14] Hietarinta J., *Searching for integrable PDE's by testing Hirota's three-soliton condition,* Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation, ISSAC'91", Stephen M. Watt, (Association for Computing Machinery, 1991), 295-300.

[15] Hietarinta J., *A search of bilinear equations passing Hirota's three-soliton condition: I. KdV-type bilinear equations,* J. Math. Phys. **28**, 1732-1742 (1987).

[16] Hietarinta J., *A search of bilinear equations passing Hirota's three-soliton condition: II. mKdV-type bilinear equations,* J. Math. Phys. **28**, 2094-2101 (1987).

[17] Hietarinta J., *A search of bilinear equations passing Hirota's three-soliton condition: III. sine-Gordon-type bilinear equations,* J. Math. Phys. **28**, 2586-2592 (1987).

[18] Hietarinta J., *A search of bilinear equations passing Hirota's three-soliton condition: IV. Complex bilinear equations,* J. Math. Phys. **29**, 628-635 (1988).

[19] Grammaticos B., Ramani A. and Hietarinta J., *A search for integrable bilinear equations: the Painleve approach,* J. Math. Phys., **31**, 2572-2578 (1990).

[20] Hirota R., *The direct method in soliton theory,* Iwanami Shoten (1992) [Japanese], English transl. Cambridge University Press, Cambridge (2004).

[21] Hietarinta J., *Introduction to the bilinear method,* in "Integrability of Nonlinear Systems", eds. Y. Kosman-Schwarzbach, B. Grammaticos and K.M. Tamizhmani, Springer Lecture Notes in Physics **495**, 95-103 (1997) `arxiv.org/abs/solv-int/9708006`.