HUB LOCATION PROBLEMS UNDER POLYHEDRAL DEMAND UNCERTAINTY

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Hub Location Problems Under Polyhedral Demand Uncertainty
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Hubs are points of consolidation and transshipment in many-to-many distribution systems that benefit from economies of scale. In hub location problems, the aim is to locate hub facilities such that each pairwise demand is satisfied and the total cost is minimized. The problem usually arises in the strategic planning phase prior to observing actual demand values. Hence incorporating robustness into hub location decisions under data uncertainty is crucial for achieving a reliable hub network design. In this thesis, we study hub location problems under polyhedral demand uncertainty. We consider uncapacitated multiple allocation p-hub median problem under hose and hybrid demand uncertainty and capacitated multiple allocation hub location problem under hose demand uncertainty. We propose mixed integer linear programming formulations and devise several exact solution algorithms based on Benders decomposition in order to solve large-scale problem instances. Computational experiments are performed on instances of three benchmark data sets from the literature.

Keywords: hub location, multiple allocation, demand uncertainty, robustness, Benders decomposition.

ÖZET

ÇOKYÜZLÜ TALEP BELİRSİZLİĞİ ALTINDA ADÜ YER SEÇİMİ PROBLEMLERİ

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Ana dağıtım üsleri (ADÜ) ölçek ekonomilerinden faydalanan çoklu dağıtım sistemlerinde toplanma ve dağıtım noktalarıdır. ADÜ yer seçimi problemlerinde amaç ADÜ'leri noktalar arasındaki talebi en az maliyetle karşılayacak şekilde yerleştirmektir. ADÜ yer seçimi kararları genellikle noktalar arasındaki talep hakkında yeterli verinin olmadığı erken aşamalarda verilmektedir. Bu yüzden alınan kararlarda talep belirsizliğinin göz önünde bulundurulması taleplerdeki değişimlere dayanıklı ve uygulanabilir çözümler üretilmesi adına büyük önem taşımaktadır. Bu çalışmada, talep değerlerinin çokyüzlü belirsizliğe sahip olduğu durumlarda ADÜ yer seçimi problemleri incelenmiştir. Hose ve Hibrit belirsizlik modelleri altında kapasite kısıtsız çoklu atamalı p-ADÜ medyan problemleri ve Hose belirsizlik modeli altında kapasite kısıtlı çoklu atamalı ADÜ yer seçimi problemleri üzerinde çalışılmıştır. Bu problemler için doğrusal karışık tamsayılı matematiksel modeller önerilmiştir ve büyük ölçekli problemlerin çözülebilmesi için Benders ayrıştırma metodu kullanılarak farklı çözüm algoritmaları geliştirilmiştir. Önerilen tüm model ve algoritmalar, literatürde ölçüt olarak kullanılan üç veri seti üzerinde test edilmiştir.

Anahtar sözcükler: ADÜ yer seçimi, çoklu atama, talep belirsizliği, dayanıklı çözümler, Benders ayrıştırma metodu.

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Chapter 1

Introduction

Hubs are facilities that consolidate and distribute flow from many origins to many destinations. Hubbing is common in transportation networks that benefit from economies of scale such as airline and cargo delivery networks. Many variants of hub location problems have been studied in the last few decades. The p-hub median problem and the hub location problem with fixed costs are the most studied problems in the hub location literature. In the p-hub median problem, the aim is to locate p hubs and to route the flow between origin-destination pairs through these hubs so that the total transportation cost is minimized. Different from the p-hub median problem, the cost term in the hub location problem with fixed costs includes a fixed cost of hub openings. In this case, the number of hubs to be opened is not predetermined; it is a decision that depends on the trade-off between the total cost of hub openings and the transportation costs. Direct shipments between nonhub nodes are usually not allowed.

There are variants of these problems where a nonhub node can send and receive traffic through all hubs and others where there is a restriction on the number of hubs that a nonhub node can be connected to. The former is known as the multiple allocation setting. In some other variants, hub or edge capacities are imposed. In this thesis, we consider hub location problems with multiple allocation and no direct shipments. We study a p-hub median problem with no capacity constraints in Chapter 3 and a hub location problem with fixed costs and capacities on hubs in Chapter 4.

An important issue that arises while designing a hub network is coping with the uncertainty in the data. The hub location problem is solved in the strategic planning phase, usually before actual point-to-point demand values are realized and the network starts operating. The demand may have large variations depending on the seasons, holidays, prices, level of economic activities, population, service time and quality and the price and quality of the services provided by the competitors. A decision made based on a given realization of the data may be obsolete in time of operation.

The uncertainty in the demand values can be modeled in various forms: (i) the probability distribution of demand values may be known; (ii) the probability distribution may not be known but demands can take any value in a given set; (iii) a discrete set of possible scenarios may be identified. In this study, we model uncertainty with a polyhedral set. More precisely, we consider the hose model and its restriction with box constraints. The hose model has been introduced by Duffield et al. [1] and Fingerhut et al. [2] to model demand uncertainty in virtual private networks. In the hose model, the user specifies aggregate upper bounds on inbound and outbound traffic of each node. Modeling uncertainty with this model has several advantages. First, it is simpler to estimate a value for each node compared to for each node pair. Second, it has resource-sharing flexibility and is less conservative compared to a model in which each origin-destination demand is set to its worst case value. Still, it contains extreme scenarios in which few origin destination pairs may have large traffic demands and remaining pairs may have zero traffic. To consider more realistic situations, Altın et al. [3] propose to use a hybrid model where lower and upper bounds on individual traffic demands are added to the hose model. This requires estimation of bounds for each node pair but leads to less conservative solutions. Even though these uncertainty models are introduced for telecommunication applications, they can also be used for transportation applications where pairwise demands are often estimated based on the populations at origins and destinations. The hose model is a simple way of modeling correlations such as a person flying from Istanbul to Paris is not flying at the same time from London to Istanbul.

To hedge against uncertainty in the demand data, we adopt a minmax robustness criterion and minimize the cost of the network under the worst case scenario. In robust optimization, commonly, one does not make assumptions about the probability distributions, rather assumes that the data belongs to an uncertainty set. A robust solution is one whose worst case performance over all possible realizations in the uncertainty set is the best (see, e.g., Atamtürk [4]; Ben-Tal and Nemirovski [5, 6, 7]; Ben-Tal et al. [8]; Bertsimas and Sim [9, 10]; Mudchanatongsuk et al. [11]; Ordóñez and Zhao [12]; Yaman et al. [13, 14]).

In this study, we introduce two types of problems; namely the robust uncapacitated multiple allocation p-hub median problem under hose and hybrid demand uncertainty and the robust capacitated hub location problem with fixed costs under hose demand uncertainty. We derive mixed integer programming formulations and propose exact solution methods based on Benders decomposition. In our computational experiments, we first analyze the changes in cost and hub locations with different uncertainty sets. Then we test the limits of solving the model with an off-the-shelf solver and compare the performances of two decomposition approaches.

The rest of the thesis is organized as follows. In Chapter 2, we review the related studies in the literature. In Chapter 3, we introduce the robust multiple allocation p-hub median problem under hose and hybrid demand uncertainty and propose mixed integer programming formulations. We devise two different Benders decomposition based exact solution algorithms and report our computational findings. In Chapter 4, the robust capacitated hub location problem with fixed costs under hose demand uncertainty is introduced. We formulate the problem as a mixed integer linear programming problem and propose decomposition techniques to solve large-sized instances. We summarize our contributions and conclude in Section 5.

Chapter 2

Literature Review

Hub location has grown to be an important and well-studied area of network analysis. Detailed surveys of studies on hub location are given in Campbell [15], O'Kelly and Miller [16], Klincewicz [17], Campbell et al. [18], Alumur and Kara [19], Campbell and O'Kelly [20] and Farahani et al. [21]. Here we review first the studies on the uncapacitated multiple allocation p-hub median problem (UMApHMP) and the capacitated multiple allocation hub location problem with fixed costs (CMAHLP) and then, the studies on hub location problems under data uncertainty.

UMApHMP is first formulated by Campbell [22]. Alternative formulations with four index variables are given by Campbell [23] and Skorin-Kapov et al. [24]. Ernst and Krishnamoorthy [25] propose a three-indexed formulation based on aggregated flows. Various exact and heuristic solution algorithms are devised to solve UMApHMP efficiently (see, e.g., Campbell [26]; Ernst and Krishnamoorthy [25, 27]). Besides, the variant of the problem where the number of hubs is not fixed, namely the uncapacitated multiple allocation hub location problem with fixed costs (UMAHLP), is studied by Campbell [23], Klincewicz [28], Ernst and Krishnamoorthy [25], Ebery et al. [29], Mayer and Wagner [30], Boland et al. [31], Hamacher et al. [32], Marín [33], Cánovas et al. [34] and Contreras et al. [35]. Since this problem is analogous to the UMApHMP, most of the solution methods can be adapted to solve the UMApHMP.

Capacitated variants of the hub location problems received less attention in the literature compared to the uncapacitated versions. The first mixed integer linear programming formulation for the CMAHLP is proposed by Campbell [22] using four indexed variables. Ebery et al. [29] provide two formulations for the same problem with three indices and devise a heuristic algorithm to solve large instances. In order to strengthen these formulations, Boland et al. [31] propose preprocessing procedures and valid inequalities, which lead to a significant reduction in the computation times. Marín [36] also provides new formulations and resolution techniques to obtain better computational results and succeeds to solve instances with up to 75 nodes. Sasaki and Fukushima [37] consider a capacitated multiple allocation hub location problem where a capacity constraint is applied both on hubs and arcs and a flow can go through at most one hub on its way from origin to destination. They devise a branch and bound algorithm and perform computational studies on the CAB data set.

Several Benders decomposition based approaches have been proposed to solve hub location problems with multiple assignments and they proved to be effective. To the best of our knowledge, Camargo et al. [38] are the first ones to apply Benders decomposition to the uncapacitated multiple allocation hub location problem. They propose three different Benders approaches. The first one is the classical approach, which adds a single cut at each iteration, while the second is the multi-cut version in which Benders cuts are generated for each origin-destination pair. Another variant allows an error margin ϵ for the cuts added and the algorithm terminates when an ϵ-optimal solution is obtained. They solve instances with up to 200 nodes and conclude that the single-cut version of the algorithm shows the best computational performance. Contreras et al. [35] propose a Benders decomposition algorithm to solve UMAHLP. They generate cuts for each candidate hub location instead of each origin-destination pair. They construct pareto-optimal cuts in order to improve the convergence of the algorithm and offer elimination tests to reduce the size of the problem. Using the proposed approaches, they succeed to solve instances with up to 500 nodes.

There are also Benders decomposition applications for the capacitated multiple allocation hub location problems. Rodríguez-Martín and Salazar-González [39] consider a capacitated hub location problem with multiple assignments on an incomplete hub network. They provide a linear programming formulation and develop two exact solution algorithms. The first one utilizes classical Benders decomposition approach whereas the second employs a nested two level algorithm based on Benders decomposition. They show that the latter outperforms the classical Benders decomposition approach in terms of computation times. Contreras et al. [40] also study a related capacitated hub location problem in which the capacities installed on each hub is not a parameter but a decision variable. They devise a Benders decomposition algorithm such that the subproblem turns out to be a transportation problem which can be solved with a special algorithm. They apply pareto optimal Benders cuts and reduction tests to improve the convergence of the algorithm.

Benders decomposition is also used to solve other variants of the multiple allocation hub location problems. Camargo et al. [41] study UMAHLP where the discount factor for the connections between hub nodes is defined as a piecewise-linear concave function. They devise two Benders decomposition algorithms generating cuts for each origin-destination pair in each Benders iteration. Instances with up to 50 nodes from the Civil Aeronautics Board (CAB) data set and Australian Post (AP) data set are solved within six hours of CPU time. Gelareh and Nickel [42] work on UMAHLP for the urban transportation and liner shipping networks where the hub network is incomplete and the triangularity assumption does not hold. In order to solve this problem, they proposed a Benders decomposition algorithm such that cuts are generated for each node instead of each origin-destination pair. The algorithm is tested on the AP data set instances with up to 50 nodes and all the instances are solved within one hour.

Even though hub location problems are well studied over the years, the literature addressing data uncertainty in the context of hub location problems is rather limited. Marianov and Serra [43] investigate a hub location problem in an air transportation network in which hubs are assumed to behave as $M/D/c$ queues. The probability that the number of planes in the queue exceeds a certain number is bounded above. This restriction is later transformed into a capacity constraint for the hubs. The authors propose a tabu search based heuristic method and test it using the CAB data set and a randomly generated data set containing 900 instances with 30 nodes.

Yang [44] introduces demand uncertainty into the air freight hub location and flight routes planning problem in a two-stage stochastic programming setting. In the first stage, the number of hubs to be opened and the locations of these hubs are determined. The second stage deals with the flight routing decisions in response to different demand scenarios considering the hub locations determined in the first stage. Computational experiments are performed using real data from Taiwan-China air freight network. Comparison of the stochastic model with the deterministic model based on average demands shows that incorporating uncertainty into the problem leads to improvements in the total cost.

Sim et al. [45] study stochastic p-hub center problem with normally distributed travel times. They use a chance constraint to guarantee the desired service level. They propose several heuristic algorithms and test them on the CAB and the AP data sets.

Contreras et al. [46] consider the uncapacitated multiple allocation hub location problem under demand and transportation cost uncertainty. They show that the stochastic models for this problem with uncertain demands or transportation costs dependent to a single uncertain parameter are equivalent to the deterministic problem with mean values. This is not the case for the problem with stochastic independent transportation costs. This latter problem is solved using Benders decomposition and a sample average scheme. They use the AP data set to test the efficiency and effectiveness of the proposed models and algorithms.

Alumur et al. [47] study both multiple and single allocation hub location problems with setup costs and point-to-point demands as sources of uncertainty. The uncertainty in the setup costs is handled by a minimax regret formulation while demand uncertainty is modeled with a stochastic programming formulation. They integrate these two cases and propose a model considering both setup cost and demand uncertainty. Computational analysis of the proposed models is performed with more than 150 instances on the CAB data.

Most recently, Shahabi and Unnikrishnan [48] study the single and multiple allocation hub location problems with ellipsoidal demand uncertainty. They propose mixed integer conic quadratic programming formulations and a linear relaxation strategy. The proposed models are tested on the CAB data set with 25 nodes and it is concluded that opening more hubs reduces the effect of demand uncertainty on the total cost.

Different from the studies summarized above, in this study, we adopt two polyhedral uncertainty sets from the telecommunications literature, namely hose and hybrid models, to represent the uncertainty in the demand data. We propose mixed integer linear programming formulations for the UMApHMP under hose and hybrid demand uncertainty and the CMAHLP under hose demand uncertainty. Motivated by successful implementations of Benders decomposition to solve hub locations problems, we propose several exact decomposition algorithms to solve large-scale instances.

Chapter 3

Uncapacitated Multiple Allocation p-Hub Median Problem under Polyhedral Demand Uncertainty

In this chapter, we consider the robust uncapacitated multiple allocation p-hub median problem under polyhedral demand uncertainty. We model the demand uncertainty in two different ways. The hose model assumes that the only available information is the upper limit on the total flow adjacent at each node, while the hybrid model additionally imposes lower and upper bounds on each pairwise demand. We propose linear mixed integer programming formulations using a minmax criteria and devise two Benders decomposition based exact solution algorithms in order to solve large-scale problems. We report the results of our computational experiments on the effect of incorporating uncertainty and on the performance of our exact approaches.

3.1 Mathematical Models

In this section, we devise mathematical models for the multiple allocation p-hub median problem under different models of demand uncertainty. We consider the uncapacitated problem where the hub network is complete and there is no direct connection between nonhub nodes. Several formulations are developed for the deterministic UMApHMP. The model by Hamacher et al. [32] is the strongest among four index formulations.

We are given a set of demand points $N = \{1, ..., n\}$ and a set of possible hub locations $H = \{1, ..., h\}$. In the deterministic problem, we know the traffic demand w_{ij} from node i to node j for all distinct pairs i and j (we assume that $w_{ii} = 0$ for all nodes i). Let $C = \{(i, j) : i, j \in N, i \neq j\}$. We denote by d_{ij} the cost of transporting one unit of demand from node i to node j. We have cost multipliers χ, α and δ for collection, transfer between hubs and distribution, respectively. Hence the cost of transporting one unit of demand from node i to node j through hubs k and m is equal to $c_{ijkm} = \chi d_{ik} + \alpha d_{km} + \delta d_{mj}$.

For completeness, we first present the model of Hamacher et al. [32] for the deterministic problem. Let y_k be 1 if a hub is located at location k and be 0 otherwise and x_{ijkm} be the fraction of flow from node i to node j sent through hubs k and m in that order. The model is as follows:

(UMApHMP deterministic)

$$\min \sum_{(i,j)\in C} \sum_{k\in H} \sum_{m\in H} c_{ijkm} w_{ij} x_{ijkm} \tag{3.1}$$

$$\text{s.t.} \quad \sum_{k} y_k = p, \tag{3.2}$$

$$\sum_{k\in H} \sum_{m\in H} x_{ijkm} = 1 \qquad \forall (i,j) \in C, \tag{3.3}$$

$$\sum_{m\in H} x_{ijkm} + \sum_{\substack{m\in H: \\ m\neq k}} x_{ijmk} \leq y_k \qquad \forall (i,j) \in C,\ k \in H, \tag{3.4}$$

$$y_k \in \{0,1\} \qquad \forall k \in H, \tag{3.5}$$

$$x_{ijkm} \geq 0 \qquad \forall (i,j) \in C,\ \forall k,m \in H. \tag{3.6}$$

The objective is to minimize the total transportation cost. Constraint (3.2) ensures that p hubs are located in the network. Constraints (3.3) guarantee that the demand between each origin-destination pair is fully satisfied. Constraints (3.4) assure that the flow can go through only installed hub facilities. Constraints (3.5) and (3.6) are the domain constraints.

We consider two demand uncertainty models, the hose model and the hybrid model. In the telecommunications community, the hose model is a popular way to model demand uncertainty. It puts limitations on the total demand associated to demand nodes, rather than estimating pairwise demand values.

The total demand adjacent at each node $i \in N$ is required to be less than or equal to a finite and positive upper bound b_i. The uncertainty set under hose uncertainty model is

$$D_{hose} = \{w \in \mathbb{R}_+^{n(n-1)} : \sum_{j\in N\setminus\{i\}} w_{ij} + \sum_{j\in N\setminus\{i\}} w_{ji} \leq b_i,\ \forall i \in N\}.$$

The robust multiple allocation p-hub median problem under hose uncertainty asks to decide on the locations of hubs and the routes for origin-destination pairs so that the worst case cost over all possible demand realizations in set D_{hose} is minimized, i.e.,

$$\min_{(x,y)\in X} \max_{w\in D_{hose}} \sum_{(i,j)\in C} \sum_{k\in H} \sum_{m\in H} c_{ijkm} w_{ij} x_{ijkm},$$

where X is the set defined by constraints (3.2)-(3.6).

As such, this problem is a nonlinear problem. Next we apply the dual transformation used to linearize minmax type robust optimization problems (see, e.g., Bertsimas and Sim [9] and Altın et al. [49]). For given $(x,y) \in X$, the problem

$$\max_{w\in D_{hose}} \sum_{(i,j)\in C} \sum_{k\in H} \sum_{m\in H} c_{ijkm} w_{ij} x_{ijkm}$$

is a linear programming problem that is feasible and bounded. Hence, its optimal value is equal to the optimal value of its dual. Using this result, robust UMApHMP with hose demand uncertainty can be modeled as the following mixed integer program:

(UMApHMP Hose)

$$\min \sum_{i\in N} \lambda_i b_i \tag{3.7}$$

$$\text{s.t. } (3.2) - (3.6),$$

$$\lambda_i + \lambda_j \geq \sum_{k\in H} \sum_{m\in H} c_{ijkm} x_{ijkm} \qquad \forall (i,j) \in C, \tag{3.8}$$

$$\lambda_i \geq 0 \qquad \forall i \in N, \tag{3.9}$$

where λ_i is the dual variable associated with the constraint $\sum_{j\in N\setminus\{i\}} w_{ij} + \sum_{j\in N\setminus\{i\}} w_{ji} \leq b_i$ for $i \in N$.

The second uncertainty set we study is the hybrid set proposed by Altın et al. [49]:

$$D_{hybrid} = D_{hose} \cap \{w \in \mathbb{R}_+^{n(n-1)} : l_{ij} \leq w_{ij} \leq u_{ij},\ \forall (i,j) \in C\},$$

where l_{ij} and u_{ij} are lower and upper bounds for the traffic demand from node i to node j with $0 \leq l_{ij} \leq u_{ij}$. Note that when $l_{ij} = 0$ and $u_{ij} \geq \min\{b_i, b_j\}$ for all distinct pairs i and j, $D_{hybrid} = D_{hose}$. In addition, when $u_{ij} = l_{ij}$ for all $(i, j) \in C$ and $b_i \geq \sum_{j \in N \setminus \{i\}} (u_{ij} + u_{ji})$ for all i, we have the deterministic problem.

The robust multiple allocation p-hub median problem under hybrid uncertainty can be modeled as follows:

(UMApHMP Hybrid)

$$\min \sum_{i \in N} \lambda_i b_i + \sum_{(i,j) \in C} (u_{ij}\beta_{ij} - l_{ij}\mu_{ij}) \tag{3.10}$$

$$\text{s.t. } (3.2) - (3.6),$$

$$\lambda_i + \lambda_j + \beta_{ij} - \mu_{ij} \geq \sum_{k \in H} \sum_{m \in H} c_{ijkm} x_{ijkm} \quad \forall (i, j) \in C, \tag{3.11}$$

$$\lambda_i \geq 0 \quad \forall i \in N, \tag{3.12}$$

$$\beta_{ij}, \mu_{ij} \geq 0 \quad \forall (i, j) \in C. \tag{3.13}$$

where β_{ij} and μ_{ij} are the dual variables associated with the upper and lower bound constraints, respectively.

Both models UMApHMP Hose and UMApHMP Hybrid are compact mixed integer programming models that can be solved using a general purpose solver. However, as the number of nodes grows, the sizes of these formulations grow quickly. In the sequel, we propose decomposition algorithms to deal with these large mixed integer programs.

3.2 Benders Decomposition

Benders decomposition is a row generation based exact solution method that can be applied to solve large-scale mixed integer programming problems [50]. In this technique, the problem is reformulated using a smaller number of variables and a large number of constraints. Then this reformulation is solved using a cutting plane approach. The relaxation solved at each iteration is called as the master problem and the problem that finds a cutting plane is called as the subproblem.

Benders decomposition uses the fact that computational difficulty of a problem increases as the problem size increases and instead of solving a single large problem, solving smaller problems iteratively may be more efficient in terms of the computational effort required. With this motivation, we apply Benders decomposition to the robust UMApHMP under polyhedral demand uncertainty. In the classical Benders approach, the master problem is solved to optimality at each iteration. In our implementations, we use a branch-and-cut framework and separate Benders cuts each time a candidate integer solution is found.

We decompose UMApHMP with polyhedral demand uncertainty in two different ways. We present our approach for only the hybrid uncertainty model since the hose model is a special case with $l_{ij} = 0$ and $u_{ij} \geq min\{b_i, b_j\}$.

3.2.1 Decomposition with only location variables in the master

Consider the formulation UMApHMP Hybrid we provided in the previous section. For given hub locations represented with vector $\hat{y}$, the problem becomes

$$\text{(PS1)} \quad \min \sum_{i\in N} \lambda_i b_i + \sum_{(i,j)\in C} (u_{ij}\beta_{ij} - l_{ij}\mu_{ij}) \tag{3.14}$$

$$\text{s.t.} \quad \lambda_i + \lambda_j + \beta_{ij} - \mu_{ij} \geq \sum_{k\in H}\sum_{m\in H} c_{ijkm}x_{ijkm} \qquad \forall (i,j)\in C, \tag{3.15}$$

$$\sum_{k\in H}\sum_{m\in H} x_{ijkm} \geq 1 \qquad \forall (i,j)\in C, \tag{3.16}$$

$$\sum_{m\in H} x_{ijkm} + \sum_{m\in H\setminus\{k\}} x_{ijmk} \leq \hat{y}_k \qquad \forall (i,j)\in C, k\in H, \tag{3.17}$$

$$\lambda_i \geq 0 \qquad \forall i\in N, \tag{3.18}$$

$$\beta_{ij}, \mu_{ij} \geq 0 \qquad \forall (i,j)\in C, \tag{3.19}$$

$$x_{ijkm} \geq 0 \qquad \forall (i,j)\in C,\ \forall k,m\in H. \tag{3.20}$$

Note here that we modified constraints (3.16) as inequalities since the above model has an optimal solution where these inequalities are tight. Problem PS1 is a linear programming problem. It is feasible and bounded when $\sum_{k\in H}\hat{y}_k \geq 1$, $u_{ij} \geq l_{ij} \geq 0$ for all $(i,j)\in C$ and $b_i \geq \sum_{j\in N\setminus\{i\}}(l_{ij}+l_{ji})$ for all $i\in N$. We associate dual variables ω_{ij}, ρ_{ij} and ν_{ijk} to constraints (3.15)-(3.17), respectively. Then the dual subproblem is

$$\text{(DS1)} \quad \max \sum_{(i,j)\in C} \rho_{ij} - \sum_{(i,j)\in C}\sum_{k\in H} \hat{y}_k \nu_{ijk} \tag{3.21}$$

$$\text{s.t.} \quad \sum_{j\in N\setminus\{i\}} \omega_{ij} + \sum_{j\in N\setminus\{i\}} \omega_{ji} \leq b_i \qquad \forall i\in N, \tag{3.22}$$

$$l_{ij} \leq \omega_{ij} \leq u_{ij} \qquad \forall (i,j)\in C, \tag{3.23}$$

$$\rho_{ij} - \nu_{ijk} - \nu_{ijm} \leq c_{ijkm}\omega_{ij} \quad \forall (i,j)\in C,\ \forall k,m\in H: k\neq m, \tag{3.24}$$

$$\rho_{ij} - \nu_{ijk} \leq c_{ijkk}\omega_{ij} \qquad \forall (i,j)\in C,\ k\in H, \tag{3.25}$$

$$\rho_{ij} \geq 0 \qquad \forall (i,j)\in C, \tag{3.26}$$

$$\nu_{ijk} \geq 0 \qquad \forall (i,j)\in C,\ \forall k\in H, \tag{3.27}$$

and is also feasible and bounded by strong duality. Hence, the robust UMApHMP under hose demand uncertainty can be modeled as the master problem

$$
\begin{aligned}
\text{(MP1)} \quad & \min \; q && && (3.28)\\
& \text{s.t. } q \geq \sum_{(i,j)\in C} \rho^t_{ij} - \sum_{(i,j)\in C}\sum_{k\in H} y_k \nu^t_{ijk} && \forall t = 1,\ldots,T, && (3.29)\\
& \sum_{k} y_k = p, && && (3.30)\\
& y_k \in \{0,1\} && \forall k \in H, && (3.31)
\end{aligned}
$$

where $(\rho^t, \nu^t, \omega^t)$ is the t-th extreme point of the set defined by (3.22)-(3.27). We solve this master problem iteratively using constraints (3.29) as cutting planes. For a given $(\hat{q}, \hat{y})$, we check whether there exists an inequality (3.29) that is violated by solving the dual subproblem. Now, we investigate how the dual problem can be solved efficiently.

First, in order to eliminate the dependencies between the constraints, we let $\bar{\rho}_{ij} = \frac{\rho_{ij}}{\omega_{ij}}$ and $\bar{\nu}_{ijk} = \frac{\nu_{ijk}}{\omega_{ij}}$. Then the dual subproblem becomes

$$
\begin{aligned}
& \max \sum_{(i,j)\in C} \omega_{ij}(\bar{\rho}_{ij} - \sum_{k\in H} \hat{y}_k \bar{\nu}_{ijk}) && && (3.32)\\
& \text{s.t. (3.22) and (3.23)}, && && \\
& \quad \bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{\nu}_{ijm} \leq c_{ijkm} && \forall (i,j) \in C,\ \forall k, m \in H : k \neq m, && (3.33)\\
& \quad \bar{\rho}_{ij} - \bar{\nu}_{ijk} \leq c_{ijkk} && \forall (i,j) \in C,\ \forall k \in H, && (3.34)\\
& \quad \bar{\rho}_{ij} \geq 0 && \forall (i,j) \in C, && (3.35)\\
& \quad \bar{\nu}_{ijk} \geq 0 && \forall (i,j) \in C,\ \forall k \in H, && (3.36)
\end{aligned}
$$

which is equivalent to

$$\max_{\omega\in D_{hybrid}} \left(\max_{(\bar{\rho},\bar{\nu}):(3.33)-(4.112)} \sum_{(i,j)\in C} \omega_{ij}(\bar{\rho}_{ij} - \sum_{k\in H} \hat{y}_k \bar{\nu}_{ijk}) \right).$$

Now the inner problem decomposes into $n(n-1)$ problems:

$$\max_{\omega \in D_{hybrid}} \sum_{(i,j) \in C} \omega_{ij} \theta_{ij},$$

where for $(i,j) \in C$,

$$\begin{aligned}
\theta_{ij} = \max\ & \bar{\rho}_{ij} - \sum_{k \in H} \hat{y}_k \bar{\nu}_{ijk} && && (3.37)\\
\text{s.t. } & \bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{\nu}_{ijm} \leq c_{ijkm} && \forall k, m \in H : k \neq m, && (3.38)\\
& \bar{\rho}_{ij} - \bar{\nu}_{ijk} \leq c_{ijkk} && \forall k \in H, && (3.39)\\
& \bar{\rho}_{ij} \geq 0, && && (3.40)\\
& \bar{\nu}_{ijk} \geq 0 && \forall k \in H, && (3.41)
\end{aligned}$$

which is the dual of

$$\begin{aligned}
\theta_{ij} = \min\ & \sum_{k \in H} \sum_{m \in H} c_{ijkm} x_{ijkm} && && (3.42)\\
\text{s.t. } & \sum_{k \in H} \sum_{m \in H} x_{ijkm} \geq 1, && && (3.43)\\
& \sum_{m \in H} x_{ijkm} + \sum_{m \in H \setminus \{k\}} x_{ijmk} \leq \hat{y}_k && \forall k \in H, && (3.44)\\
& x_{ijkm} \geq 0 && \forall k, m \in H. && (3.45)
\end{aligned}$$

This problem can be solved by inspection and an optimal dual solution can be constructed using complementary slackness conditions as explained by Contreras et al. [35]. We note here that the dual problem computes the worst case cost for a given choice of hub locations and it uses the fact that each commodity is routed on a shortest path from its origin to its destination, independently of the demand realizations. Hence, we first compute the length of a shortest path for each origin-destination pair and then solve a linear problem to find the demand realization for which the routing cost is maximum.

Besides, different from the deterministic case, the cut (3.29) cannot be disaggregated into cuts for nodes or for node pairs since the problem DS1 does not decompose.

3.2.2 Decomposition by projecting out the routing variables

When we fix $(y, \lambda, \beta, \mu) = (\hat{y}, \hat{\lambda}, \hat{\beta}, \hat{\mu})$ in formulation UMApHMP Hybrid, we obtain the following problem

$$
\begin{aligned}
& \max\ 0\ x && && (3.46)\\
& \text{s.t.}\ \sum_{k\in H}\sum_{m\in H} c_{ijkm}x_{ijkm} \le \hat{\lambda}_i + \hat{\lambda}_j + \hat{\beta}_{ij} - \hat{\mu}_{ij} && \forall (i,j)\in C, && (3.47)\\
& \sum_{k\in H}\sum_{m\in H} x_{ijkm} \ge 1 && \forall (i,j)\in C, && (3.48)\\
& \sum_{m\in H} x_{ijkm} + \sum_{m\in H\setminus\{i\}} x_{ijmk} \le \hat{y}_k && \forall (i,j)\in C,\ k\in H, && (3.49)\\
& x_{ijkm} \ge 0 && \forall (i,j)\in C,\ \forall k,m\in H, && (3.50)
\end{aligned}
$$

which is a feasibility problem. For this problem to be feasible, we need its dual to be bounded. In other words, by Farkas' lemma, we need

$$
\sum_{(i,j)\in C} (\hat{\lambda}_i + \hat{\lambda}_j + \hat{\beta}_{ij} - \hat{\mu}_{ij})\gamma_{ij} - \sum_{(i,j)\in C} \rho_{ij} + \sum_{(i,j)\in C}\sum_{k\in H} \nu_{ijk}\hat{y}_k \ge 0 \qquad (3.51)
$$

for all $(\gamma,\ \rho,\ \nu)$ that satisfy

$$
\begin{aligned}
& \gamma_{ij}c_{ijkm} - \rho_{ij} + \nu_{ijk} + \nu_{ijm} \ge 0 && \forall (i,j)\in C,\ \forall k,m\in H:\ k\ne m, && (3.52)\\
& \gamma_{ij}c_{ijkk} - \rho_{ij} + \nu_{ijk} \ge 0 && \forall (i,j)\in C,\ \forall k\in H, && (3.53)\\
& \gamma_{ij} \ge 0, \rho_{ij} \ge 0 && \forall (i,j)\in C, && (3.54)\\
& \nu_{ijk} \ge 0 && \forall (i,j)\in C,\ \forall k,m\in H. && (3.55)
\end{aligned}
$$

First note that this system decomposes for each pair (i, j). In addition, since the vector can be scaled, we take γ_{ij} to be 0 or 1 without loss of generality. When

$\gamma_{ij} = 0$, we need $\sum_{k \in H} \nu_{ijk} \hat{y}_k \geq \rho_{ij}$ for all (ρ_{ij}, ν_{ij}) such that

$$\nu_{ijk} + \nu_{ijm} \geq \rho_{ij} \qquad \forall\ k, m \in H:\ k \neq m, \tag{3.56}$$

$$\nu_{ijk} \geq \rho_{ij} \qquad \forall\ k \in H, \tag{3.57}$$

$$\rho_{ij} \geq 0, \tag{3.58}$$

$$\nu_{ijk} \geq 0 \qquad \forall k, m \in H. \tag{3.59}$$

This is always satisfied when $\sum_{k \in H} \hat{y}_k \geq 1$. Hence, the only interesting case is $\gamma_{ij} = 1$. Consequently, we can conclude that the feasibility problem has a solution if for all $(i, j) \in C$ we have

$$\hat{\lambda}_i + \hat{\lambda}_j + \hat{\beta}_{ij} - \hat{\mu}_{ij} \geq \rho_{ij} - \sum_{k \in H} \nu_{ijk} \hat{y}_k \tag{3.60}$$

for all (ρ_{ij}, ν_{ij}) such that

$$c_{ijkm} + \nu_{ijk} + \nu_{ijm} \geq \rho_{ij} \qquad \forall\ k, m \in H:\ k \neq m, \tag{3.61}$$

$$c_{ijkk} + \nu_{ijk} \geq \rho_{ij} \qquad \forall\ k \in H, \tag{3.62}$$

$$\rho_{ij} \geq 0, \tag{3.63}$$

$$\nu_{ijk} \geq 0 \qquad \forall k, m \in H, \tag{3.64}$$

Let $M_{ij} = \{(\rho_{ij}, \nu_{ij}) \in \mathbb{R}_+ \times \mathbb{R}_+^h : (3.61) - (3.64)\}$ for $(i, j) \in C$. After projecting out the x variables, the model becomes

$$
\begin{aligned}
(\text{MP2}) \quad \min \quad & \sum_{i \in N} \lambda_i b_i + \sum_{(i,j) \in C} (u_{ij}\beta_{ij} - l_{ij}\mu_{ij}) && (3.65)\\
\text{s.t.} \quad & \lambda_i + \lambda_j + \beta_{ij} - \mu_{ij} \geq \rho_{ij}^t - \sum_{k \in H} y_k \nu_{ijk}^t \forall (i,j) \in C, \ \forall t = 1, \ldots, T_{ij}, && (3.66)\\
& \sum_{k} y_k = p, && (3.67)\\
& \lambda_i \geq 0 \qquad \forall i \in N, && (3.68)\\
& \beta_{ij}, \mu_{ij} \geq 0 \qquad \forall (i,j) \in C, && (3.69)\\
& y_k \in \{0,1\} \qquad \forall k \in H. && (3.70)
\end{aligned}
$$

where $(\rho_{ij}^t, \nu_{ij}^t)$ is the t-th extreme point of M_{ij}, which has T_{ij} extreme points. Hence the dual subproblem for each $(i,j) \in C$ can be stated as

$$
\max_{(\rho_{ij}, \nu_{ij}) \in M_{ij}} \left(\rho_{ij} - \sum_{k \in H} y_k \nu_{ijk} \right).
$$

which is the dual of a shortest path problem from i to j for each $(i,j) \in C$. Again the dual variables ρ and ν can be obtained using the algorithm provided in Contreras et al. [35].

Observe that keeping the dual variables λ_i's in the master problem enables us to disaggregate the cuts (3.29) into multiple cuts, one for each node pair.

3.3 Computational Analysis

For computational analysis, we used the Civil Aeronautics Board (CAB) data set with 25 nodes, the Turkish network (TR) data set with 81 nodes and the Australian Post (AP) data set with up to 200 nodes. All data sets are well-known and commonly used in the hub location literature (accessible from [51]). The CAB data set (Figure 3.1) was introduced by O'Kelly [52]. In this data set, the cost and demand values are symmetric and flow from one node to itself is not allowed. The unit collection and distribution cost factors are taken as $\chi = \delta = 1$ while the unit transfer cost factor α is allowed to be 0.2, 0.4, 0.6, 0.8 so that $c_{ijkm} = d_{ik} + \alpha\ d_{km} + d_{mj}$.

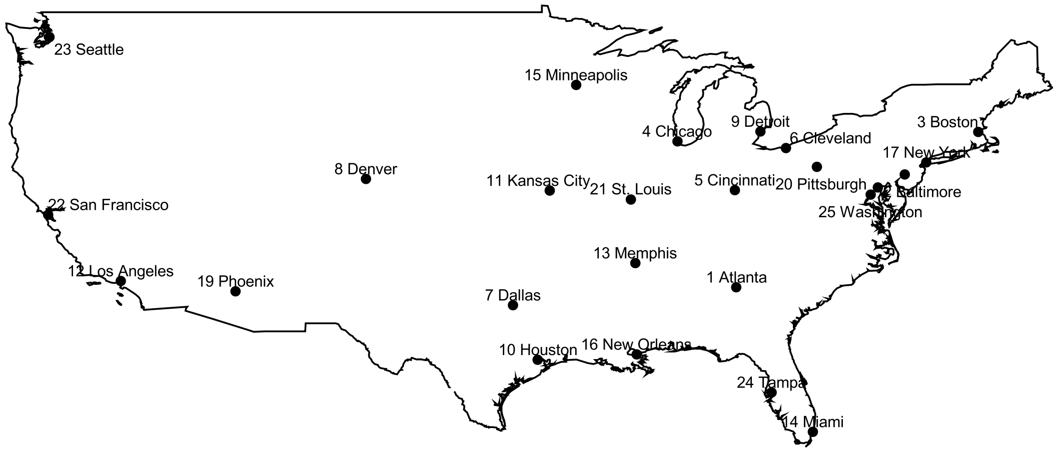

Figure 3.1: Locations of demand nodes for CAB data set

We also consider the TR data set (Figure 3.2) containing data for 81 cities of Turkey. The unit collection, distribution and transfer cost factors are taken as in the CAB data set. Different from the CAB data, the pairwise demand values are not symmetric in the TR data set. We use the original distance values and, for the ease of representation, scale the demand values by dividing with 1000.

Although the CAB and the TR data sets are small-to-medium size, the AP data set is available for larger instances. The AP data set is initially introduced by Ernst and Krishnamoorthy [53] and it consists of flow data for 200 postcode districts in Australia. The unit collection, transfer and distribution cost factors are taken as $\chi = 3$, $\alpha = 0.75$ and $\delta = 2$. In the AP data set, demand and flow values are not symmetric. For the uniformity of computation, we do not allow flow from a node to itself even though the AP data set contains such demand values.

In order to set the problem parameters, we use the nominal demand values of the deterministic problem instances. We generate our traffic bounds as $b_i = \sum_{j \in N \setminus \{i\}} (w_{ij} + w_{ji})$ for all $i \in N$. For the hybrid model, we let $l_{ij} = max\{0, (1-\psi)w_{ij}\}$ and $u_{ij} = (1+\psi)w_{ij}$ for all distinct pairs i and j, with $\psi \in \{0.2, 0.4, 0.6, 0.8, 1, 2\}$. All demand nodes are taken as candidate locations for

Figure 3.2: Locations of demand nodes for TR data set

hubs, i.e., $H = N$.

The experiments are done on a 64-bit machine with Intel Xeon E5-2630 v2 processor at 2.60 GHz and 96 GB of RAM using Java and Cplex 12.5.1 We set a time limit of ten hours. All solution times are given in seconds.

First we compare the hub location decisions made for each uncertainty set and their total transportation costs. In Table 3.1, we present results of different uncertainty sets using the CAB data set instances with 25 nodes, $p \in \{2, 3, 4, 5\}$ and $\alpha \in \{0.2, 0.4, 0.6, 0.8\}$. We obtained these results by solving our models using the solver CPLEX. For each p, α and uncertainty set, we report the optimal value and the locations of hubs in the optimal solutions.

When we compare the hub locations of the deterministic model, with those of the hose model, we see that there has been a change in the hub locations in 12 out of 16 instances. The hubs that are closed are usually replaced with a nearby alternative. For example, in the instance with $p = 3$ and $\alpha = 0.4$, the hubs are installed in Chicago (4), Los Angeles (12) and New York (17) in the deterministic model, whereas the hub at Chicago (4) is replaced with a hub at Cincinnati (5) in the hose model solution. The hub locations of some instances shift several times

Table 3.1: Results for the CAB data set (total transportation cost / hub locations)

p	α	**Deterministic**	**Hybrid ($\psi = 0.2$)**	**Hybrid ($\psi = 0.4$)**	**Hybrid ($\psi = 0.6$)**	**Hybrid ($\psi = 0.8$)**	**Hybrid ($\psi = 1$)**	**Hybrid ($\psi = 2$)**	**Hose**
		996.02	1007.72	1019.41	1031.10	1042.80	1054.49	1054.99	1054.99
2	0.2	12,20	12,20	12,20	12,20	12,20	12,20	12,20	12,20
		1072.49	1095.61	1118.73	1141.84	1164.96	1188.08	1190.79	1190.79
2	0.4	12,20	12,20	12,20	12,20	12,20	12,20	12,20	12,20
		1137.08	1172.03	1206.98	1241.87	1269.64	1297.42	1319.78	1319.78
2	0.6	12,20	12,20	12,20	8,20	8,20	8,20	12,20	12,20
		1180.02	1222.71	1256.55	1290.39	1318.08	1342.32	1417.49	1418.84
2	0.8	12,20	8,20	8,20	8,20	11,20	11,20	8,20	5,12
		752.91	770.59	788.02	805.36	822.70	839.44	845.12	845.26
3	0.2	12,17,21	12,17,21	4,12,17	4,12,17	4,12,17	5,12,17	5,12,17	5,12,17
		859.64	893.41	927.19	960.96	994.66	1024.40	1036.58	1037.64
3	0.4	4,12,17	4,12,17	4,12,17	4,12,17	4,12,18	5,12,17	5,12,17	5,12,17
		949.23	996.94	1044.22	1091.50	1136.48	1180.64	1209.00	1213.09
3	0.6	4,12,17	4,12,18	4,12,18	4,12,18	2,12,21	2,12,21	5,12,17	5,12,17
		1020.04	1079.13	1136.03	1190.64	1244.66	1293.22	1359.06	1367.93
3	0.8	4,12,17	12,18,21	2,12,21	2,12,21	12,21,25	12,20,21	5,8,17	5,12,17
		618.48	635.69	652.91	670.12	687.33	704.54	722.29	726.44
4	0.2	4,12,17,24	4,12,17,24	4,12,17,24	4,12,17,24	4,12,17,24	4,12,17,24	4,12,17,24	4,12,14,17
		754.49	788.62	821.96	854.22	886.47	918.73	954.92	967.16
4	0.4	4,12,17,24	4,12,17,24	1,4,12,17	1,4,12,17	1,4,12,17	1,4,12,17	1,4,12,17	5,12,14,17
		866.45	914.26	962.07	1009.88	1057.70	1105.51	1156.82	1170.07
4	0.6	1,4,12,17	1,4,12,17	1,4,12,17	1,4,12,17	1,4,12,17	1,4,12,17	1,4,12,17	5,12,17,24
		951.76	1013.03	1074.31	1135.59	1196.86	1251.39	1326.78	1343.21
4	0.8	1,4,12,17	1,4,12,17	1,4,12,17	1,4,12,17	1,4,12,17	1,4,8,17	4,5,12,17	5,12,17,22
		530.00	547.75	565.50	583.25	601.00	618.74	639.79	646.72
5	0.2	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17
		676.34	711.42	746.51	781.60	816.68	851.77	899.59	914.10
5	0.4	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,12,13,14,17	1,4,12,17,20
		804.70	855.24	905.78	956.32	1005.79	1055.19	1112.80	1129.91
5	0.6	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,7,12,14,17	4,7,12,14,18	4,7,12,14,18	1,4,12,17,20	5,8,12,17,24
		910.35	974.35	1037.38	1098.67	1158.20	1215.17	1298.23	1322.23
5	0.8	4,7,12,17,24	4,7,12,17,24	1,4,7,12,17	4,7,12,17,25	4,7,12,17,25	4,8,13,17,20	1,4,12,17,20	5,12,14,17,22

as the uncertainty set enlarges. Consider the instance with $p = 3$ and $\alpha = 0.2$. In the deterministic case, hubs are installed at Los Angeles (12), New York (17) and St. Louis (21). As we switch to hybrid uncertainty set with $\psi = 0.4$, Chicago (4) replaces St. Louis (21) in the optimal solution; whereas Chicago (4) is replaced with Cincinnati (5) in the hose model solution. Some instances are more sensitive to the demand model changes. The optimal hub locations of the instance with $p = 3$ and $\alpha = 0.8$ change for the hybrid models with $\psi = 0.2, 0.4, 0.8, 1, 2$ and the hose model. Moreover, the optimal hub locations for some instances change for the hybrid model, but not the hose model. In the instance with $p = 2$ and $\alpha = 0.6$, the hubs are located at Los Angeles (12) and Pittsburgh (20) for both deterministic and the hose models. However, considering the hybrid models with $\psi = 0.6, 0.8, 1$, the hub at Los Angeles (12) is moved to Denver (8).

We further observe that for larger values of transfer cost factor α, hub locations in the optimal solution are more likely to change for different demand uncertainty sets. The instances with no hub location change generally have smaller α values. For $p = 2$, none of the instances with $\alpha \in \{0.2, 0.4\}$ has a change in the hub locations. Considering $p = 5$, only the hub locations of the instance with the smallest α value, which is 0.2, remain unchanged. The possibility of a change in the optimal hub locations increases as α increases. On the other hand, the CAB data set instances do not display any patterns depending on the value of p. All instances with $p = 3, 4$ have a change in the hub locations while there are instances with no change with $p = 2, 5$. It is difficult to draw any conclusions about the effects of p value on the optimal hub locations for different uncertainty sets.

We observe, in Table 3.1, that the changes in the locations of hubs are not major. Another important aspect to be considered is the performance of deterministic hub location decisions under different demand realizations. In Table 3.2, we analyze, for the CAB data set, how the total cost will be affected if hubs are selected based on the deterministic model but the demand changes with one of the proposed uncertainty sets. We report the worst case costs using deterministic hub locations under different uncertainty sets and the percentage deviations from the optimal values. It can be observed that the deviation from the optimal value usually increases as α grows and the uncertainty set enlarges. However, there are some instances that does not follow this pattern. For example, the instance with $p = 2$ and $\alpha = 0.8$ has its largest deviation (4.11%) in the hybrid model with $\psi = 1$ which is significantly greater than the deviation in the corresponding hose model solution (0.81%). In addition, we observe that by incorporating uncertainty into the decision making process, we can make savings of up to 4.11% in the total cost.

We perform the same location and cost analysis also on the TR data set instances. Table 3.3 presents the optimal hub locations and corresponding total transportation costs under different demand uncertainty model settings. Considering the hub locations, it can be seen that the TR data set is more sensitive to the changes in the demand. For all 16 instances, there has been a change in the hub locations in response to the demand model changes. In six of them, the hub location change occurs in the least conservative model with the demand uncertainty (hybrid model with $\psi = 0.2$). 11 instances out of 16 are exposed to changes

Table 3.2: Cost analysis for the CAB data set

p	α	**Deterministic**	**Hybrid (ψ = 0.2)**	**Hybrid (ψ = 0.4)**	**Hybrid (ψ = 0.6)**	**Hybrid (ψ = 0.8)**	**Hybrid (ψ = 1)**	**Hybrid (ψ = 2)**	**Hose**
			Cost and Percentage deviation from the optimal solution						
2	0.2	12,20	1007.72 0.00	1019.41 0.00	1031.10 0.00	1042.80 0.00	1054.49 0.00	1054.99 0.00	1054.99 0.00
2	0.4	12,20	1095.61 0.00	1118.73 0.00	1141.84 0.00	1164.96 0.00	1188.08 0.00	1190.79 0.00	1190.79 0.00
2	0.6	12,20	1172.03 0.00	1206.98 0.00	1241.93 0.01	1276.88 0.57	1311.83 1.11	1319.78 0.00	1319.78 0.00
2	0.8	12,20	1223.51 0.07	1266.99 0.83	1310.48 1.56	1353.97 2.72	1397.45 4.11	1429.48 0.85	1430.32 0.81
3	0.2	12,17,21	770.59 0.00	788.27 0.03	805.94 0.07	823.62 0.11	841.30 0.22	859.58 1.71	863.10 2.11
3	0.4	4,12,17	893.41 0.00	927.19 0.00	960.96 0.00	994.74 0.01	1028.51 0.40	1055.69 1.84	1060.65 2.22
3	0.6	4,12,17	997.32 0.04	1045.41 0.11	1093.50 0.18	1141.60 0.45	1189.69 0.77	1239.84 2.55	1250.90 3.12
3	0.8	4,12,17	1080.04 0.08	1140.04 0.35	1200.04 0.79	1260.04 1.24	1320.05 2.07	1396.40 2.75	1414.78 3.43
4	0.2	4,12,17,24	635.69 0.00	652.91 0.00	670.12 0.00	687.33 0.00	704.54 0.00	722.29 0.00	730.25 0.52
4	0.4	4,12,17,24	788.62 0.00	822.75 0.10	856.88 0.31	891.00 0.51	925.13 0.70	961.00 0.64	972.51 0.55
4	0.6	1,4,12,17	914.26 0.00	962.07 0.00	1009.88 0.00	1057.70 0.00	1105.51 0.00	1156.82 0.00	1187.47 1.49
4	0.8	1,4,12,17	1013.03 0.00	1074.31 0.00	1135.59 0.00	1196.86 0.00	1258.14 0.54	1327.55 0.06	1372.05 2.15
5	0.2	4,7,12,14,17	547.75 0.00	565.50 0.00	583.25 0.00	601.00 0.00	618.75 0.00	639.79 0.00	646.72 0.00
5	0.4	4,7,12,14,17	711.42 0.00	746.51 0.00	781.60 0.00	816.68 0.00	851.77 0.00	900.93 0.15	915.26 0.13
5	0.6	4,7,12,14,17	855.24 0.00	905.78 0.00	956.32 0.00	1006.86 0.11	1057.40 0.21	1135.80 2.07	1160.08 2.67
5	0.8	4,7,12,17,24	974.35 0.00	1038.35 0.09	1102.35 0.34	1166.35 0.70	1230.35 1.25	1324.49 2.02	1367.82 3.45

Table 3.3: Results for the TR data set (total transportation cost / hub locations)

p	α	**Deterministic**	**Hybrid ($\psi = 0.2$)**	**Hybrid ($\psi = 0.4$)**	**Hybrid ($\psi = 0.6$)**	**Hybrid ($\psi = 0.8$)**	**Hybrid ($\psi = 1$)**	**Hybrid ($\psi = 2$)**	**Hose**
2	0.2	781669.72 44,54	786824.72 38,41	797134.72 38,41	802289.71 38,41	812599.71 38,41	822909.71 38,41	823485.27 38,41	826877.58 38,41
2	0.4	820586.50 38,41	840112.66 38,41	859638.82 38,41	879164.99 38,41	892040.38 6,44	902575.19 6,44	902575.19 6,44	902575.19 6,44
2	0.6	857219.51 38,41	883983.61 38,41	910717.95 38,54	926290.29 6,46	940622.96 6,46	954955.62 6,46	954955.62 6,46	954955.62 6,46
2	0.8	878672.80 38,41	909256.41 38,54	938955.05 38,54	959486.50 6,44	978472.84 6,44	996333.67 6,34	996504.70 6,34	996504.70 6,34
3	0.2	660218.05 12,41,68	669320.24 6,41,44	678208.41 6,41,44	687096.58 6,41,44	695984.75 6,41,44	704872.91 6,41,44	704872.91 6,41,44	704872.91 6,41,44
3	0.4	726196.77 6,41,44	743571.26 6,41,44	760945.74 6,41,44	778320.22 6,41,44	795694.70 6,41,44	812263.87 6,34,44	812263.87 6,34,44	812263.87 6,34,44
3	0.6	778077.05 6,41,44	802850.50 6,41,44	827328.13 6,41,46	851179.46 6,41,46	874652.79 6,34,46	896670.92 6,34,46	896670.92 6,34,46	896670.92 6,34,46
3	0.8	845601.96 6,41,44	861246.30 6,41,44	892534.96 6,41,44	908179.30 6,41,44	939110.71 6,34,44	963781.26 1,3,6	968747.12 6,34,44	968747.41 6,34,44
4	0.2	570217.55 6,34,44,45	580050.10 6,34,44,45	589882.64 6,34,44,45	598397.47 27,34,64,71	606822.73 27,34,64,71	615247.99 27,34,64,71	618170.78 27,34,64,71	619704.92 6,34,35,44
4	0.4	657662.12 6,34,44,45	676223.28 6,34,44,45	694784.44 6,34,44,45	713345.61 6,34,44,45	731857.44 6,34,35,44,	749377.80 3,34,71,80	751689.75 6,34,35,44	751736.11 6,34,35,44
4	0.6	729447.41 6,34,44,45	755449.94 6,34,45,46	780891.70 6,34,45,46	804676.28 3,6,34,46	828223.76 3,6,34,46	849488.45 1,3,6,34	856918.94 1,6,23,34	856956.89 1,6,23,34
4	0.8	777778.51 1,3,41,58	811709.80 1,6,23,41	843479.75 3,6,34,44	875182.54 3,6,34,44	906333.49 3,6,34,46	933544.51 1,3,6,34	947749.84 3,6,34,38	950994.70 1,6,34,44
5	0.2	492494.33 6,12,34,45,80	501839.91 6,12,34,45,80	511185.49 6,12,34,45,80	520391.93 1,6,12,34,35	529385.67 1,6,12,34,35	538379.41 1,6,12,34,35	540666.63 6,12,34,35,80	541609.30 6,12,34,35,80
5	0.4	595161.93 1,6,12,34,45	613491.23 1,6,12,34,45	631820.52 1,6,12,34,45	650149.82 1,6,12,34,45	668479.11 1,6,12,34,45	685959.90 1,6,12,34,64	691650.49 1,6,23,34,35	693039.20 1,6,23,34,35
5	0.6	678419.46 1,6,23,34,45	705452.52 1,6,23,34,45	732038.97 1,6,23,34,64	757265.98 1,3,6,23,34	782009.47 1,3,6,23,34	806752.95 1,3,6,23,34	816929.99 1,3,6,23,34	821577.20 1,3,6,23,34
5	0.8	744056.84 1,6,23,41,45	779668.60 1,3,6,23,41	812942.30 1,3,6,23,34	846138.12 1,3,6,23,34	879333.95 1,3,6,23,34	912157.71 1,3,6,34,44	928125.96 1,3,6,34,44	935014.05 1,3,6,23,34

in the hub locations under hybrid uncertainty models with ψ value up to 0.6. In the TR data, the cities Ankara (6), İstanbul (34) and İzmir (35) are the ones with the largest demand values. We observe that as the uncertainty set enlarges, these cities are more likely to be in the set of optimal hub locations. For example, with parameters $p = 2$ and $\alpha = 0.4, 0.6, 0.8$, the deterministic model chooses Kayseri (38) and Kocaeli (41) as hub locations while the hose model chooses Ankara (6) in all three instances and İstanbul (34) in one of them. Additionally, from Tables 3.1 and 3.3, it can be seen that the optimal value of the hybrid model converges to the optimal value of the hose model as ψ and consequently the upper bounds on the pairwise demands increases. Considering TR data set instance with $p = 2$

and $\alpha = 0.4$, the optimal solution value of the hybrid model increases as ψ grows and ultimately becomes equal to the optimal value of the hose model when $\psi = 1$.

Table 3.4: Cost analysis for the TR data set

			Cost and Percentage deviation from the optimal solution						
			Hybrid	**Hybrid**	**Hybrid**	**Hybrid**	**Hybrid**	**Hybrid**	
p	α	**Deterministic**	(ψ **= 0.2**)	(ψ **= 0.4**)	(ψ **= 0.6**)	(ψ **= 0.8**)	(ψ **= 1**)	(ψ **= 2**)	**Hose**
2	0.2	44,54	783544.78	796104.52	808664.27	821224.01	833783.75	833783.75	833783.75
			0.24	0.52	0.79	1.06	1.32	1.25	0.84
2	0.4	38,41	840112.66	859638.82	879164.99	898691.15	918217.31	921559.32	930866.70
			0.00	0.00	0.00	0.75	1.73	2.10	3.13
2	0.6	38,41	883983.61	910747.71	937511.82	964275.92	991040.02	1002321.11	1022210.03
			0.00	0.00	1.21	2.51	3.78	4.96	7.04
2	0.8	38,41	909727.56	940782.32	971837.08	1002891.84	1033946.60	1062003.87	1087904.35
			0.05	0.19	1.29	2.50	3.78	6.57	9.17
3	0.2	12,41,68	669853.39	679488.72	689124.06	698759.40	708394.74	709086.96	710106.96
			0.08	0.19	0.30	0.40	0.50	0.60	0.74
3	0.4	6,41,44	743571.26	760945.74	778320.22	795694.70	813069.18	813069.18	813069.18
			0.00	0.00	0.00	0.00	0.10	0.10	0.10
3	0.6	6,41,44	802850.50	827623.94	852397.39	877170.84	901944.28	901944.28	901944.28
			0.00	0.04	0.14	0.29	0.59	0.59	0.59
3	0.8	6,41,44	845601.96	876890.63	908179.30	939467.96	970756.63	972720.83	972972.74
			0.00	0.00	0.00	0.04	0.72	0.41	0.44
4	0.2	6,34,44,45	580050.10	589882.64	599715.19	609547.74	619380.28	620309.11	620511.25
			0.00	0.00	0.22	0.45	0.67	0.35	0.13
4	0.4	6,34,44,45	676223.28	694784.44	713345.61	731906.77	750467.93	752020.97	752067.33
			0.00	0.00	0.00	0.01	0.15	0.04	0.04
4	0.6	6,34,44,45	756075.41	782703.41	809331.41	835959.40	862587.40	864553.81	864563.66
			0.08	0.23	0.58	0.93	1.54	0.89	0.89
4	0.8	1,3,41,58	812107.62	846436.72	880765.83	915094.93	949424.03	973930.34	1011155.08
			0.05	0.35	0.64	0.97	1.70	2.76	6.33
5	0.2	6,12,34,45,80	501839.91	511185.49	520531.08	529876.66	539222.24	541268.35	541955.19
			0.00	0.00	0.03	0.09	0.16	0.11	0.06
5	0.4	1,6,12,34,45	613491.23	631820.52	650149.82	668479.11	686808.41	692054.29	694136.79
			0.00	0.00	0.00	0.00	0.12	0.06	0.16
5	0.6	1,6,23,34,45	705452.52	732485.59	759518.65	786551.71	813584.77	820365.80	822480.01
			0.00	0.06	0.30	0.58	0.85	0.42	0.11
5	0.8	1,6,23,41,45	780024.31	815991.77	851959.23	887926.70	923894.16	938903.81	941633.65
			0.05	0.38	0.69	0.98	1.29	1.16	0.71

We also investigate how the total transportation cost is affected as we change the demand uncertainty model using the TR data set instances. In Table 3.4, the deterministic model optimal hub locations, their total transportation costs under different uncertainty models and the percentage deviations from the optimal value of the corresponding model are presented. It can be seen that the deterministic model solutions perform well under the hybrid demand uncertainty with ψ value up to 0.6; the deviation from the optimal value is within less than 1.5%. However, for larger uncertainty sets, the total cost can be subject to an increase of up to 10%. Four of the instances under the hose model show percentage increase in the total transportation costs with 3.13%, 7.04%, 9.17% and 6.33%, respectively. An interesting observation is that for these instances, Ankara (6), İstanbul (34) and İzmir (35) are not selected as hub nodes in the deterministic model, unlike the hose model. It can be concluded that, in these instances, the cost of uncertainty may increase significantly when the nodes with large inbound and outbound traffic are not chosen as hubs.

We obtained similar results after performing cost and location analysis for the AP data set instances. In Table 3.5, we present the optimal transportation costs and hub locations under different demand models. There is a change in the optimal hub locations in 7 out of 12 AP data set instances. Again it can be seen that there is no pattern in the variations in the hub locations depending on the value of p. For the instances with 25 nodes, there exists a change in the optimal hub locations in all except the one with $p = 2$. On the other hand, considering the instances with 40 nodes, the only instance that shows a change in the hub locations is the one with $p = 2$. In Table 3.6, we also provide the analysis of how the optimal hub locations of the deterministic model performs under different demand uncertainty models. In view of our computational results, the AP data set instances turn out to be quite resilient to the uncertainty in the demand. It can be seen that the maximum percentage deviation from the optimal value is 1.37% and for many instances the percentage deviation is almost zero.

Table 3.5: Results for AP data set (total transportation cost / hub locations)

n	p	**Deterministic**	**Hybrid ($\psi = 0.2$)**	**Hybrid ($\psi = 0.4$)**	**Hybrid ($\psi = 0.6$)**	**Hybrid ($\psi = 0.8$)**	**Hybrid ($\psi = 1$)**	**Hybrid ($\psi = 2$)**	**Hose**
25	2	161302.58	165060.80	168819.02	172577.24	176335.46	180093.68	187247.20	203814.57
		8,18	8,18	8,18	8,18	8,18	8,18	8,18	8,18
	3	143324.89	147422.11	151519.33	155317.65	158889.51	162461.37	168723.49	182598.15
		2,8,18	2,8,18	2,8,18	7,14,18	7,14,18	7,14,18	7,14,18	7,14,18
	4	129326.76	133170.64	137000.41	140830.19	144659.96	148172.11	154566.97	166162.91
		2,8,18,20	2,8,15,18	2,8,15,18	2,8,15,18	2,8,15,18	7,14,17,18	2,12,14,18	2,8,15,18
	5	115391.48	119026.66	122661.84	126292.66	129914.99	133537.32	139672.40	152274.07
		2,8,17,18,20	2,8,17,18,20	2,8,17,18,20	2,8,15,17,18	2,8,15,17,18	2,8,15,17,18	2,8,17,18,20	2,8,15,16,18
40	2	167111.47	171404.41	175697.36	179990.31	184283.25	188576.20	196166.30	209111.18
		12,28	12,28	12,28	12,28	12,28	12,28	12,29	12,29
	3	149821.91	153747.16	157672.41	161597.66	165522.92	169448.17	176035.95	189952.43
		12,23,28	12,23,28	12,23,28	12,23,28	12,23,28	12,23,28	12,23,28	12,23,28
	4	135798.16	139463.84	143129.51	146795.19	150460.86	154126.54	160622.60	176189.84
		12,23,26,28	12,23,26,28	12,23,26,28	12,23,26,28	12,23,26,28	12,23,26,28	12,23,26,28	12,23,26,28
	5	126356.39	129982.82	133609.26	137235.70	140862.14	144488.57	150883.31	165649.37
		3,13,23,26,28	3,13,23,26,28	3,13,23,26,28	3,13,23,26,28	3,13,23,26,28	3,13,23,26,28	3,13,23,26,28	3,13,23,26,28
50	2	168991.03	173131.97	177272.91	181413.84	185554.78	189695.72	197309.05	211318.98
		15,35	15,35	15,35	15,35	15,35	15,35	15,36	14,36
	3	151329.99	155228.15	159126.30	163024.46	166922.61	170820.77	177595.47	191842.19
		14,28,35	14,28,35	14,28,35	14,28,35	14,28,35	14,28,35	14,28,35	14,29,35
	4	137087.13	140720.60	144354.06	147987.53	151620.99	155254.45	161910.24	177383.68
		14,28,32,35	14,28,32,35	14,28,32,35	14,28,32,35	14,28,32,35	14,28,32,35	14,28,32,35	14,28,32,35
	5	126236.27	130029.85	133816.84	137577.93	141339.02	145100.10	151722.01	166131.78
		4,14,28,32,35	4,14,28,32,35	4,15,28,32,35	4,15,28,32,35	4,15,28,32,35	4,15,28,32,35	4,15,28,32,35	4,15,28,32,35

Table 3.6: Cost analysis for the AP data set

			Cost and Percentage deviation from the optimal solution						
			Hybrid	**Hybrid**	**Hybrid**	**Hybrid**	**Hybrid**	**Hybrid**	
n	**p**	**Deterministic**	($\psi = \mathbf{0.2}$)	($\psi = 0.4$)	($\psi = \mathbf{0.6}$)	($\psi = \mathbf{0.8}$)	($\psi = \mathbf{1}$)	($\psi = \mathbf{2}$)	**Hose**
25	2	8,18	165060.80	168819.02	172577.24	176335.46	180093.68	187247.20	203814.57
			0.00	0.00	0.00	0.00	0.00	0.00	0.00
	3	2,8,18	147422.11	151519.33	155616.55	159713.77	163810.99	171026.91	184133.59
			0.00	0.00	0.19	0.52	0.83	1.37	0.84
	4	2,8,18,20	133184.51	137042.27	140900.02	144757.78	148615.54	154967.51	166906.79
			0.01	0.03	0.05	0.07	0.30	0.26	0.45
	5	2,8,17,18,20	119026.66	122661.84	126297.02	129932.19	133567.37	139672.40	153183.43
			0.00	0.00	0.00	0.01	0.02	0.00	0.60
40	2	12,28	171404.41	175697.36	179990.31	184283.25	188576.20	196172.81	210533.98
			0.00	0.00	0.00	0.00	0.00	0.00	0.68
	3	12,23,28	153747.16	157672.41	161597.66	165522.92	169448.17	176035.95	189952.43
			0.00	0.00	0.00	0.00	0.00	0.00	0.00
	4	12,23,26,28	139463.84	143129.51	146795.19	150460.86	154126.54	160622.60	176189.84
			0.00	0.00	0.00	0.00	0.00	0.00	0.00
	5	3,13,23,26,28	129982.82	133609.26	137235.70	140862.14	144488.57	150883.31	165649.37
			0.00	0.00	0.00	0.00	0.00	0.00	0.00
50	2	15,35	173131.97	177272.91	181413.84	185554.78	189695.72	197585.68	213080.60
			0.00	0.00	0.00	0.00	0.00	0.14	0.83
	3	14,28,35	155228.15	159126.30	163024.46	166922.61	170820.77	177595.47	191965.14
			0.00	0.00	0.00	0.00	0.00	0.00	0.06
	4	14,28,32,35	140720.60	144354.06	147987.53	151620.99	155254.45	161910.24	177383.68
			0.00	0.00	0.00	0.00	0.00	0.00	0.00
	5	4,14,28,32,35	130029.85	133823.44	137617.02	141410.60	145204.18	151832.60	166398.25
			0.00	0.00	0.03	0.05	0.07	0.07	0.16

Next we analyze the performance of the proposed exact solution methods using AP instances. In Table 3.7, we present the results obtained for the robust UMApHMP with hose demand uncertainty using the mathematical model, the first Benders decomposition proposed in Section 3.2.1 (Benders 1) and the Benders decomposition by projecting out flow variables as described in Section 3.2.2 (Benders 2). We compare the computational effectiveness of each approach in terms of solution times. We also present the number of Benders cuts added and the number of callbacks performed in Benders 1 and Benders 2 until the optimal solution or the best solution obtained within the time limit. Since we use the lazy constraint callback function of the CPLEX, the number of callbacks here implies how many times the lazy constraints are checked during the branch-and-bound process. At each time an incumbent solution is found, associated optimality cuts are added to a cut pool managed by the solver, but only a set of these cuts is active in the model. "# of Cuts Added" represents the number of optimality cuts added until optimality is achieved. Note that the first Benders approach adds at most a single cut in each iteration whereas in the second, at most $n(n-1)$ cuts can be added. The solutions marked with an asterisk are not the optimal solutions but the best of the solutions obtained within the ten hours of time limit. For the unsolved instances, instead of the solution time, the optimality gap is reported.

The mixed integer programming model can be solved for instances with at most 50 nodes while Benders decomposition based formulations succeed to solve instances with up to 200 nodes. Benders 1 can not solve three instances out of 32 whereas Benders 2 is not able to solve one of them. Although the number of iterations is much smaller in Benders 2, still the number of Benders cuts added is extremely high compared to Benders 1. It can be seen that for the model with hose demand uncertainty, the computational performance of Benders 2 is superior to Benders 1. Benders 2 has the shortest solution times for all the instances except two and the difference with the Benders 1 solution times for these two instances is less than one second. Benders 2 is able to solve two instances for which Benders 1 stopped with gaps of 2.20% and 2.18%. For the only instance for which both approaches failed to reach optimality, the finals gaps are 7.43% with Benders 1 and 0.49% for Benders 2. For these instances, adding multiple cuts clearly outperforms the approach where a single cut is added at each iteration. It is also interesting to note that decomposition approaches are faster than solving the compact formulation even for small instances.

Table 3.7: Comparison of exact approaches for hose demand uncertainty - AP instances

				Benders 1			**Benders 2**		
		Optimal	**MIP Model**	**CPU Time**	**# Cuts**	**#**	**CPU Time**	**# Cuts**	**#**
n	p	**Value**	**CPU Time**	**(gap)**	**Added**	**Callbacks**	**(gap)**	**Added**	**Callbacks**
25	2	203814.57	43.02	0.49	15	19	0.56	3148	10
	3	182598.15	124.45	1.29	69	73	0.41	3487	12
	4	166162.91	167.13	2.77	159	163	0.74	3373	12
	5	152274.07	111.91	5.00	244	255	0.44	3061	14
40	2	209111.18	706.84	1.05	17	21	1.19	6994	8
	3	189952.43	1696.29	5.37	90	94	1.53	7920	10
	4	176189.84	4901.37	20.93	332	340	2.91	7437	9
	5	165649.37	6490.57	69.41	972	985	5.14	9571	17
50	2	211318.98	4293.26	3.15	23	28	2.05	10583	8
	3	191842.19	22310.31	12.63	103	109	4.54	16932	12
	4	177383.68	(1.93)	52.27	421	434	6.14	16721	12
	5	166131.78	(1.70)	148.67	1034	1045	10.86	14418	14
75	2	215849.01	(100)	20.57	39	42	19.32	28048	9
	3	196368.51	(100)	89.57	170	175	29.08	25571	8
	4	181077.10	(100)	285.68	537	545	41.44	50558	18
	5	170306.35	(100)	999.65	1568	1581	54.04	35986	14
100	2	217300.63	*memory*	81.72	62	66	70.59	82426	15
	3	196754.67	*memory*	310.00	231	236	82.01	85402	17
	4	181884.09	*memory*	1109.29	791	804	196.37	82644	20
	5	172098.88	*memory*	6519.98	3122	3132	669.04	102888	25
125	2	217967.72	*memory*	177.87	59	63	99.92	124401	16
	3	197275.77	*memory*	731.11	247	255	257.16	141245	17
	4	182518.12	*memory*	2589.21	838	850	490.46	198014	31
	5	172420.17	*memory*	15116.85	3209	3225	945.73	191772	31
150	2	219010.32	*memory*	412.00	68	76	186.19	182517	18
	3	198361.42	*memory*	1755.15	293	299	715.35	257936	26
	4	183373.34	*memory*	6399.51	1050	1057	1470.29	222830	18
	5	173381.56	*memory*	(2.20)	3882	3896	4860.56	212098	22
200	2	219688.55	*memory*	1476.67	89	95	644.09	296487	17
	3	199944.64	*memory*	6951.20	430	437	4020.22	426417	22
	4	185433.91	*memory*	(2.18)	1830	1846	9332.57	490686	31
	5	176175.91*	*memory*	(7.43)	1783	1798	(0.49)	474147	26

Table 3.8: Comparison of exact approaches for hybrid demand uncertainty - small AP instances

					Benders 1			Benders 2		
n	p	ψ	**Optimal Value**	**MIP Model CPU Time**	**CPU Time**	**# Cuts Added**	**# Callbacks**	**CPU Time**	**# Cuts Added**	**# Callbacks**
25	2	0.2	165060.80	8.94	0.93	11	15	0.98	3011	10
	2	0.4	168819.02	7.11	0.58	13	17	0.61	3011	10
	2	0.6	172577.24	12.59	0.96	13	18	0.80	3218	10
	2	0.8	176335.46	15.10	0.61	15	20	0.54	2650	8
	2	1.0	180093.68	53.39	0.67	17	22	0.47	2650	8
	2	2.0	187247.20	54.70	0.88	15	20	0.99	3000	11
	3	0.2	147422.11	10.11	2.40	60	67	1.10	2775	10
	3	0.4	151519.33	13.92	2.48	61	67	0.95	3340	9
	3	0.6	155317.65	23.58	2.47	63	69	1.05	3119	10
	3	0.8	158889.51	23.94	2.27	67	71	1.29	3341	11
	3	1.0	162461.37	69.24	2.17	64	67	1.11	3369	13
	3	2.0	168723.49	88.71	2.30	75	80	0.93	3099	9
	4	0.2	133170.64	19.26	6.24	154	161	1.33	3786	15
	4	0.4	137000.41	29.90	6.78	169	180	1.68	3547	12
	4	0.6	140830.19	34.71	6.37	169	178	1.95	3874	14
	4	0.8	144659.96	37.91	7.09	195	201	2.35	4076	14
	4	1.0	148172.11	108.11	6.15	193	199	1.69	3772	16
	4	2.0	154566.97	113.37	5.69	209	213	3.30	3981	14
	5	0.2	119026.66	12.43	6.78	157	167	1.23	3333	12
	5	0.4	122661.84	14.62	7.00	173	180	1.29	3566	13
	5	0.6	126292.66	16.65	6.54	184	193	1.37	3164	16
	5	0.8	129914.99	23.45	7.41	198	206	1.63	2963	10
	5	1.0	133537.32	71.16	6.46	214	223	1.16	3243	14
	5	2.0	139672.40	92.62	6.80	235	242	1.34	3287	12
40	2	0.2	171404.41	104.02	2.98	12	16	3.07	7025	8
	2	0.4	175697.36	135.42	3.74	16	19	2.15	7025	7
	2	0.6	179990.31	138.01	3.48	17	21	3.25	8141	9
	2	0.8	184283.25	411.30	3.97	21	24	3.73	6240	6
	2	1.0	188576.20	934.60	3.44	21	24	3.12	7491	8
	2	2.0	196166.30	1035.25	3.14	20	24	4.10	7491	9
	3	0.2	153747.16	92.14	14.48	73	77	7.31	9734	13
	3	0.4	157672.41	121.64	19.82	100	104	6.36	8059	11
	3	0.6	161597.66	214.98	18.75	95	101	5.82	7067	9
	3	0.8	165522.92	350.33	20.20	108	116	8.96	8226	9
	3	1.0	169448.17	1128.39	15.77	97	103	7.59	9198	14
	3	2.0	176035.95	1924.03	13.85	97	104	10.86	9174	11
	4	0.2	139463.84	79.44	38.63	183	188	6.44	7371	8
	4	0.4	143129.51	81.20	44.15	216	225	10.24	8422	9
	4	0.6	146795.19	138.11	37.54	204	211	6.53	7904	8
	4	0.8	150460.86	167.99	36.81	211	220	12.23	9674	12
	4	1.0	154126.54	922.64	46.79	279	289	6.55	9037	10
	4	2.0	160622.60	1566.95	33.04	233	241	5.64	7863	9
	5	0.2	129982.82	75.20	90.13	440	448	5.92	9248	12
	5	0.4	133609.26	102.48	101.98	499	510	8.74	8657	8
	5	0.6	137235.70	176.89	114.02	559	569	12.86	10077	15
	5	0.8	140862.14	320.06	117.21	624	634	18.19	10437	14
	5	1.0	144488.57	1308.45	111.63	634	642	8.39	8193	10
	5	2.0	150883.31	2074.97	99.42	689	699	15.80	7764	8

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(Table 3.8 Continued) Comparison of exact approaches for hybrid demand uncertainty - small AP instances

					Benders 1			Benders 2		
			Optimal	**MIP Model**	**CPU**	**# Cuts**	**#**	**CPU**	**# Cuts**	**#**
n	p	ψ	**Value**	**CPU Time**	**Time**	**Added**	**Callbacks**	**Time**	**Added**	**Callbacks**
50	2	0.2	173131.97	514.61	5.70	17	22	7.76	11155	7
	2	0.4	177272.91	943.06	6.46	19	24	6.60	12598	9
	2	0.6	181413.84	1859.99	7.01	21	27	7.21	13489	8
	2	0.8	185554.78	1885.02	7.07	21	26	9.02	13943	11
	2	1.0	189695.72	3841.30	8.38	26	31	7.50	13469	10
	2	2.0	197309.05	5515.32	7.27	24	28	8.63	12532	9
	3	0.2	155228.15	495.82	30.15	104	112	16.03	15462	14
	3	0.4	159126.30	680.51	31.94	115	121	23.80	14787	14
	3	0.6	163024.46	1308.55	36.52	124	133	25.26	13948	10
	3	0.8	166922.61	1635.23	36.04	125	133	24.21	14554	11
	3	1.0	170820.77	11089.81	33.96	119	125	30.52	16151	11
	3	2.0	177595.47	23037.79	41.90	154	164	43.42	20128	16
	4	0.2	140720.60	350.84	72.92	258	265	20.88	15936	13
	4	0.4	144354.06	430.25	74.70	259	268	21.64	16615	12
	4	0.6	147987.53	499.01	75.89	262	272	23.42	16757	14
	4	0.8	151620.99	1132.57	83.17	287	295	32.97	13563	12
	4	1.0	155254.45	4467.07	84.23	290	302	21.12	15511	11
	4	2.0	161910.24	5585.95	82.31	300	310	34.95	14725	10
	5	0.2	130029.85	440.22	145.28	482	493	29.94	16715	18
	5	0.4	133816.84	604.71	155.34	536	543	64.05	15333	12
	5	0.6	137577.93	651.43	179.25	588	598	48.70	14081	11
	5	0.8	141339.02	1212.00	207.58	672	684	61.76	15423	16
	5	1.0	145100.10	5370.77	222.32	722	733	62.97	14736	13
	5	2.0	151722.01	6910.13	246.59	843	852	63.50	17597	16

Tables 3.8 and 3.9 show the results of comparison between exact solution methods for the robust UMApHMP under hybrid demand uncertainty. The results obtained from small instances with up to 50 nodes are presented in Table 3.8 and the results for the larger ones with up to 150 nodes are in Table 3.9. Among the instances with more than 50 nodes, the MIP formulation is able to solve only the instance with $n = 75$, $p = 2$ and $\psi = 0.2$. For the others, it fails to find lower bounds within the time limit, hence MIP formulation results are not included in Table 3.9. All instances in Table 3.8 are solved to optimality by all three exact methods proposed. For the small instances presented in Table 3.8, Benders 2 outperforms the others in terms of computational times. Benders 2 has the shortest solution times for 61 instances out of 72. For the instances which Benders 1 performs better, the difference between the solution times of the two Benders

algorithms is less than two seconds. Again the number of cuts added in Benders 2 is higher than the number of cuts added in Benders 1 even though less number of iterations is performed by Benders 2.

The results of UMApHMP under hybrid demand uncertainty for instances with 75, 100, 125 and 150 nodes are provided in Table 3.9. These results indicate that Benders 1 outperforms Benders 2 in terms of solution times for large instances with hybrid demand uncertainty. Benders 1 is able to solve all of 96 instances whereas Benders 2 can not solve 11 of them. Considering all the results for the robust UMApHMP under hybrid demand uncertainty, we observe that Benders 1 tends to perform better as n increases and p decreases. Even though the number of callbacks performed is much smaller for Benders 2 compared to Benders 1; for the instances with large n and small p values, the computational effort required at each node of the branch-and-cut tree is too high to be compensated by the decrease in the number of callbacks. It can be seen that for instances with up to 50 nodes, although Benders 2 outperforms Benders 1 in the overall, Benders 1 has shorter solution times for some instances with small p values. Considering the large instances presented in Table 3.9, Benders 2 has shorter solution times for some instances with large p values even though Benders 1 performs better in general.

Table 3.9: Comparison of exact approaches for hybrid demand uncertainty - large AP instances

			Benders 1				Benders 2			
n	p	ψ	Best Upper Bound	CPU Time (gap)	# Cuts Added	# Callbacks	Best Upper Bound	CPU Time (gap)	# Cuts Added	# Callbacks
75	2	0.2	176713.81	18.99	22	26	176713.81	73.43	35648	12
	2	0.4	181152.69	22.76	26	32	181152.69	49.58	26079	6
	2	0.6	185591.56	24.47	29	35	185591.56	132.48	37573	12
	2	0.8	189777.48	21.28	27	31	189777.48	85.84	42447	12
	2	1.0	193811.57	21.36	27	31	193811.57	32.77	24805	7
	2	2.0	201768.16	23.46	34	38	201768.16	68.81	49102	16
	3	0.2	158616.03	87.51	116	122	158616.03	132.58	36343	11
	3	0.4	162688.30	96.27	128	136	162688.30	286.56	55479	18
	3	0.6	166760.57	119.75	157	169	166760.57	374.71	48110	18
	3	0.8	170832.83	108.09	144	152	170832.83	268.94	36734	10
	3	1.0	174905.10	104.82	146	151	174905.10	391.39	61704	28
	3	2.0	181884.48	115.25	176	180	181884.48	262.15	50051	13
	4	0.2	143378.36	217.57	289	299	143378.36	280.31	55631	18
	4	0.4	147155.22	229.61	302	316	147155.22	303.98	50264	19
	4	0.6	150932.07	238.42	317	326	150932.07	364.40	56162	23
	4	0.8	154708.93	253.93	340	351	154708.93	461.78	49790	21
	4	1.0	158485.78	252.40	344	351	158485.78	300.92	44829	12
	4	2.0	165109.53	244.89	366	376	165109.53	309.91	48873	18
	5	0.2	133621.16	650.01	813	827	133621.16	919.75	47096	19
	5	0.4	137394.13	695.39	874	883	137394.13	254.12	41254	13
	5	0.6	141167.10	753.99	934	947	141167.10	549.18	43507	17
	5	0.8	144940.08	833.32	1022	1032	144940.08	570.64	53242	17
	5	1.0	148713.05	894.67	1097	1111	148713.05	1192.06	55431	24
	5	2.0	155507.19	1014.41	1295	1304	155507.19	605.26	45319	19
100	2	0.2	177186.48	53.63	30	34	177186.48	301.60	66051	12
	2	0.4	181474.44	61.17	35	39	181474.44	291.48	77118	15
	2	0.6	185762.40	67.16	38	42	185762.40	327.72	86229	16
	2	0.8	190050.37	74.32	42	46	190050.37	615.40	117601	26
	2	1.0	194338.33	79.53	44	51	194338.33	1202.99	129983	22
	2	2.0	202290.52	83.85	52	58	202290.52	429.12	96230	19
	3	0.2	158994.51	266.28	158	164	158994.51	613.84	95231	18
	3	0.4	163006.54	286.83	171	177	163006.54	794.49	63053	11
	3	0.6	167018.57	363.24	208	222	167018.57	956.68	58406	12
	3	0.8	171030.60	329.37	194	201	171030.60	1810.63	84255	15
	3	1.0	175042.62	335.73	202	207	175042.62	1543.70	90341	17
	3	2.0	181994.20	331.32	217	225	181994.20	1253.26	84163	13
	4	0.2	144217.26	775.50	451	460	144217.26	2421.80	94983	17
	4	0.4	147958.05	777.60	452	461	147958.05	1481.09	105415	25
	4	0.6	151698.84	838.58	483	495	151698.84	1428.31	88840	20
	4	0.8	155439.64	902.40	518	530	155439.64	2241.51	132278	39
	4	1.0	159180.43	869.02	507	517	159180.43	1932.22	101617	20
	4	2.0	166020.92	895.31	568	583	166020.92	1354.11	90833	21
	5	0.2	135171.84	3335.92	1716	1725	135171.84	2680.25	102190	22
	5	0.4	138973.88	3769.12	1897	1906	138973.88	6385.78	100316	21
	5	0.6	142764.70	4227.88	2026	2039	142764.70	3482.66	86521	18
	5	0.8	146518.15	4689.46	2210	2222	146518.15	3844.03	105163	23
	5	1.0	150271.61	5794.66	2380	2396	150271.61	5963.45	99574	19
	5	2.0	157031.30	5717.20	2732	2743	157031.30	6479.52	134770	38

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(Table 3.9 Continued) Comparison of exact approaches for hybrid demand uncertainty - large AP instances

			Benders 1				**Benders 2**			
			Best	**CPU Time**	**# Cuts**	**#**	**Best**	**CPU Time**	**# Cuts**	**#**
n	p	ψ	**Upper Bound**	**(gap)**	**Added**	**Callbacks**	**Upper Bound**	**(gap)**	**Added**	**Callbacks**
125	2	0.2	177656.21	101.37	27	31	177656.21	792.85	110365	9
	2	0.4	181911.00	123.76	33	37	181911.00	368.67	97826	11
	2	0.6	186165.79	129.38	35	38	186165.79	511.98	105653	11
	2	0.8	190420.58	158.12	43	47	190420.58	763.25	125546	13
	2	1.0	194675.38	159.73	44	47	194675.38	1036.49	132165	15
	2	2.0	202645.51	186.27	55	60	202645.51	1189.37	149437	18
	3	0.2	159417.26	602.77	168	176	159417.26	3864.58	117924	12
	3	0.4	163452.67	624.23	177	184	163452.67	2535.40	136126	15
	3	0.6	167488.07	685.71	190	198	167488.07	3504.79	150912	16
	3	0.8	171523.48	700.98	200	202	171523.48	4964.69	176865	24
	3	1.0	175538.13	779.58	221	227	175538.13	9866.77	180790	27
	3	2.0	182590.08	767.05	237	245	182590.08	4695.79	195335	24
	4	0.2	144632.27	1807.48	504	515	144632.27	4242.95	145345	17
	4	0.4	148396.25	1827.44	506	521	148396.25	4193.69	207417	28
	4	0.6	152160.22	1931.55	541	551	152160.22	4711.28	168262	25
	4	0.8	155924.20	1977.29	551	563	155924.20	12664.45	207851	34
	4	1.0	159681.40	2078.27	571	584	159681.40	13821.47	150987	24
	4	2.0	166538.70	2130.83	649	661	166538.70	10433.65	212734	35
	5	0.2	135542.63	9681.35	1900	1912	135542.63	22826.78	178238	29
	5	0.4	139343.37	8937.65	2001	2012	139343.37	24590.05	146195	18
	5	0.6	143144.12	9801.77	2205	2222	143144.12	16461.26	169122	22
	5	0.8	146944.86	10986.91	2395	2401	146944.86	24385.10	138443	19
	5	1.0	150745.61	12202.34	2643	2656	150745.61	29901.92	224433	40
	5	2.0	157452.82	13971.64	3059	3073	157481.18*	(0.97)	227450	35
150	2	0.2	178241.48	226.10	31	34	178241.48	2402.19	202748	18
	2	0.4	182556.63	268.77	37	40	182556.63	1803.00	195567	15
	2	0.6	186871.77	314.94	43	46	186871.77	3691.49	221884	17
	2	0.8	191186.91	294.34	42	45	191186.91	1859.89	162276	12
	2	1.0	195487.73	351.05	49	54	195487.73	5731.63	242812	18
	2	2.0	203638.50	421.42	61	69	203638.50	3228.16	188447	12
	3	0.2	159983.60	1244.12	171	182	159983.60	5269.90	193319	15
	3	0.4	164032.32	1412.94	199	208	164032.32	22681.38	257271	23
	3	0.6	168081.04	1522.25	218	221	168081.04	15230.17	196443	15
	3	0.8	172129.76	1701.13	240	249	172129.76	20277.20	216270	17
	3	1.0	176178.48	1756.99	251	259	176178.48*	(1.06)	210705	15
	3	2.0	183335.89	1674.74	269	276	183335.89*	(0.72)	306489	27
	4	0.2	145198.98	4088.74	585	596	145198.98	19923.08	219854	17
	4	0.4	148975.27	4572.64	648	663	148975.27	20134.93	221218	22
	4	0.6	152751.56	4361.46	607	620	152751.56*	(16.22)	300320	33
	4	0.8	156527.85	4651.64	650	665	156527.85	35125.56	316005	33
	4	1.0	160304.14	4900.01	695	709	160304.14	35840.99	292375	25
	4	2.0	167163.01	5001.49	784	796	167163.01*	(0.54)	266554	26
	5	0.2	136037.51	20716.03	2169	2186	137205.61*	(11.11)	259562	17
	5	0.4	139833.66	21068.13	2300	2309	140087.65*	(1.36)	315885	29
	5	0.6	143629.82	22197.72	2508	2520	143629.82*	(0.40)	278246	28
	5	0.8	147425.97	25575.96	2720	2729	148260.76*	(1.50)	314151	27
	5	1.0	151222.12	28342.47	2963	2975	151222.12*	(2.96)	292740	31
	5	2.0	158001.70	33081.89	3667	3681	158001.70*	(11.52)	278170	26

3.4 Conclusions

In this chapter, we introduced the robust multiple allocation p-hub median problems under hose and hybrid demand uncertainty sets. We presented compact mixed integer programming formulations and two Benders decomposition approaches to solve these problems. The results showed that keeping the dual variables in the master problem and adding multiple cuts works better with the hose model whereas moving the dual variables to the subproblem and adding a single cut at each iteration works better for large size instances with hybrid uncertainty.

Chapter 4

Capacitated Multiple Allocation Hub Location Problem under Hose Demand Uncertainty

In this chapter we consider the capacitated multiple allocation hub location problem (CMAHLP) where the aim is to locate a set of hub facilities on a given network and route flows through these hubs so that pairwise demands are satisfied at minimum cost. Unlike the p-hub median problem, the cost term consists of a fixed cost of opening hubs and the transportation costs. The number of hubs to be opened is a decision that depends on the trade-off between these two cost components. We assume that the hub network is complete and there is no direct connection between nonhub nodes. The total flow coming to a hub node is restricted by a capacity level and flows can be split among different paths. Classical approach in the literature is to assume that the pairwise demands are deterministic. In this study, we assume that demands take value from a polyhedral uncertainty set, namely the hose set. We propose a mixed integer programming (MIP) formulation for the robust CMAHLP under hose demand uncertainty and devise different Benders decomposition based exact solution algorithms. We test our mathematical model and solution algorithms on the AP data set instances.

4.1 MIP Formulation

In this section we formulate the robust CMAHLP under hose demand uncertainty. We consider a hub location problem where nonhub nodes can be connected to multiple hubs and a capacity constraint on the incoming flow at each hub from nonhub nodes is applied. The deterministic version of this problem is well-studied in the literature and has been formulated in several ways. Since the formulation proposed by Hamacher et. al. [32] is the strongest one for the hub location problems among path based models, we will use it as a starting point. This formulation is devised for the uncapacitated version of the problem, hence we will adjust it by adding a set of capacity constraints as proposed in [29].

Assume that we are given a complete graph $G = (N, A)$ where N is the set of demand points and A is the set of directed connections. Let $H \subseteq N$ be the set of possible hub locations and C be the set of commodities such that $C = \{(i, j) : i, j \in N, i \neq j\}$, meaning there is no demand from a node to itself. The demand from node i to node j is assumed to be known in the deterministic problem and denoted by w_{ij}. We define the remaining problem parameters as follows: f_k is the fixed cost of opening a hub facility at node k, a_k is the capacity of the hub at node k, d_{ij} is the unit cost of transshipment from node i to node j and χ, α and δ are the cost multipliers of collection, transfer between hubs and distribution, respectively. The cost of sending one unit of flow from node i to node j through hubs k and m in this order is expressed as $c_{ijkm} = \chi d_{ik} + \alpha d_{km} + \delta d_{mj}$.

First we present the MIP formulation for the deterministic CMAHLP. The decision variables of this model are y_k, the binary variable taking value of 1 if there is a hub located at node k and 0 otherwise, and x_{ijkm}, the fraction of flow sent from node i to node j through hubs k and m in that order. Then the deterministic problem is

(CMAHLP deterministic)

$$\min \sum_{k\in H} f_k y_k + \sum_{(i,j)\in C} \sum_{k\in H} \sum_{m\in H} c_{ijkm} w_{ij} x_{ijkm} \tag{4.1}$$

$$\text{s.t.} \quad \sum_{k\in H} \sum_{m\in H} x_{ijkm} = 1 \qquad \forall (i,j) \in C, \tag{4.2}$$

$$\sum_{m\in H} x_{ijkm} + \sum_{\substack{m\in H:\\ m\neq k}} x_{ijmk} \leq y_k \qquad \forall (i,j) \in C,\ k \in H, \tag{4.3}$$

$$\sum_{(i,j)\in C} \sum_{m\in N} w_{ij} x_{ijkm} \leq a_k y_k \qquad \forall k \in H, \tag{4.4}$$

$$y_k \in \{0,1\} \qquad \forall k \in H, \tag{4.5}$$

$$x_{ijkm} \geq 0 \qquad \forall (i,j) \in C,\ \forall k,m \in H. \tag{4.6}$$

The objective is to minimize the total cost of opening hubs and transportation costs. Constraints (4.2) quarantee that pairwise demands are fully satisfied. With constraints (4.3), direct flow between nonhub nodes is prevented. Constraints (4.4) are the capacity constraints that limit the total incoming flow at each hub. Constraints (4.5) and (4.6) are the domain constraints.

Different from previous studies in the literature, we assume that demand is not known in advance but can be modelled with a hose uncertainty set. In this model, as explained in Chapter 3, instead of estimating pairwise demands, we limit the total flow associated with each demand node. The demand uncertainty set under hose model can be stated as

$$D_{hose} = \{w \in \mathbb{R}_+^{n(n-1)} : \sum_{j\in N\setminus\{i\}} w_{ij} + \sum_{j\in N\setminus\{i\}} w_{ji} \leq b_i,\ \forall i \in N\}.$$

The robust CMAHLP under hose demand uncertainty aims to build a hub network which is viable under any demand realization while minimizing the worst case total cost over all possible demand realizations in the set D_{hose}. Hence the robust problem can be represented as:

$$\min \sum_{k \in H} f_k y_k + \max_{w \in D_{hose}} \sum_{(i,j) \in C} \sum_{k \in H} \sum_{m \in H} w_{ij} c_{ijkm} x_{ijkm} \tag{4.7}$$

$$\text{s.t.} \quad (4.2), (4.3), (4.5), (4.6)$$

$$\max_{w \in D_{hose}} \sum_{(i,j) \in C} \sum_{m \in H} w_{ij} x_{ijkm} \leq a_k y_k \qquad \forall k \in H \tag{4.8}$$

Here the capacity constraints (4.4) of the deterministic model is altered so that each hub facility opened is ensured to have enough capacity to serve under the worst case demand realization in the set D_{hose}.

Observe that this formulation is nonlinear since the demand is a variable. To linearise it, we use a dual transformation, which is widely used in the robust optimization literature. For a feasible flow vector $\hat{x}$, the inner maximization problem of the objective function

$$\max_{w \in D_{hose}} \sum_{(i,j) \in C} \sum_{k \in H} \sum_{m \in H} w_{ij} c_{ijkm} \hat{x}_{ijkm} \tag{4.9}$$

and the maximization problem at the left hand side of the capacity constraint

$$\max_{w \in D_{hose}} \sum_{(i,j) \in C} \sum_{m \in H} w_{ij} \hat{x}_{ijkm} \tag{4.10}$$

are both linear programming problems that are feasible and bounded. Therefore the optimal value of these problems are equal to the optimal value of their corresponding duals. Using this property, we obtain the following MILP formulation for the robust CMAHLP under hose demand uncertainty:

(CMAHLP hose)

$$\min \sum_{k\in H} f_k y_k + \sum_{i\in N} \lambda_i b_i \tag{4.11}$$

s.t. (4.2), (4.3), (4.5), (4.6)

$$\sum_{k\in H} a_k y_k \geq \min\left\{\left(\sum_{i\in N} b_i - \max_{i\in N} b_i\right), \sum_{i\in N} b_i/2\right\} \tag{4.12}$$

$$\lambda_i + \lambda_j \geq \sum_{k\in H}\sum_{m\in H} c_{ijkm} x_{ijkm} \qquad \forall (i,j)\in C, \tag{4.13}$$

$$\sum_{i\in N} \beta_i^k b_i \leq a_k y_k \qquad \forall k\in H, \tag{4.14}$$

$$\beta_i^k + \beta_j^k \geq \sum_{m\in H} x_{ijkm} \qquad \forall (i,j)\in C,\ \forall k\in H, \tag{4.15}$$

$$\lambda_i \geq 0 \qquad \forall i\in N, \tag{4.16}$$

$$\beta_i^k \geq 0 \qquad \forall i\in N, \forall k\in H, \tag{4.17}$$

where the dual variables λ and β are associated with the hose model constraint in problems (4.9) and (4.10), respectively.

Here constraint (4.12) actually corresponds to the following inequality.

$$\sum_{k\in H} a_k y_k \geq \max_{w\in D_{hose}} \sum_{(i,j)\in C} w_{ij} \tag{4.18}$$

With this inequality, we ensure that we open hubs with sufficient capacity to route the worst case traffic. However right-hand-side of the inequality is also an optimization problem. In the following proposition we compute the optimal value of this maximization problem.

Proposition 1.

$$\max_{w\in D_{hose}} \sum_{(i,j)\in C} w_{ij} = \min\left\{\left(\sum_{i\in N} b_i - \max_{i\in N} b_i\right), \sum_{i\in N} b_i/2\right\}.$$

Proof. The problem $\max_{w \in D_{hose}} \sum_{(i,j)\in C} w_{ij}$ is

$$\max \sum_{(i,j)\in C} w_{ij} \tag{4.19}$$

$$\text{s.t.} \sum_{j\in N\setminus\{i\}} w_{ij} + \sum_{j\in N\setminus\{i\}} w_{ji} \le b_i \qquad \forall i \in N, \tag{4.20}$$

$$w_{ij} \ge 0 \qquad \forall (i,j) \in C. \tag{4.21}$$

Taking the dual of this problem, we obtain the following LP:

$$\min \sum_{i\in N} \vartheta_i b_i \tag{4.22}$$

$$\text{s.t. } \vartheta_i + \vartheta_j \ge 1 \quad \forall (i,j) \in C, \tag{4.23}$$

$$\vartheta_i \ge 0 \qquad \forall i \in N. \tag{4.24}$$

Observe that the dual problem is equivalent to the LP relaxation of a weighted vertex covering problem. Nemhauser and Trotter [54] show that any extreme point ϑ of this LP satisfies $\vartheta_i \in \{0, 1/2, 1\}$ for all $i \in N$. Since we have a complete graph, we can further characterize the optimal solution.

The vector of all ones $(1, 1, .., 1)$ is clearly not an optimal solution as none of the constraints strictly holds and one can obtain a better objective function value by decreasing $\vartheta_{i'}$ with $\epsilon > 0$ for an arbitrary $i' \in N$ since $b_{i'}$ is positive. In the case that we know $\vartheta_{i'} = 1/2$ for a node $i' \in N$, $\vartheta_i \ge 1/2$ for all $i \in N\setminus\{i'\}$ for feasibility. Hence the solution with the smallest objective value is the vector $(1/2, 1/2, .., 1/2)$ with the objective function value equal to $\sum_{i\in N} b_i/2$. Finally, if there exists a node $i' \in N$ such that $\vartheta_{i'} = 0$, then we must have $\vartheta_i = 1$ for all $i \in N \setminus \{i'\}$ to ensure feasibility. The objective function value of this solution is $\sum_{i\in N} b_i - b_{i'}$. To minimize this value, we set $\vartheta_i = 0$ for a node i with the largest b_i value. Therefore the minimum objective value in this case is $\sum_{i\in N} b_i - \max_{i\in N} b_i$. Hence, the dual optimal value is $\min\left\{\left(\sum_{i\in N} b_i - \max_{i\in N} b_i\right), \sum_{i\in N} b_i/2\right\}$. By strong duality, this is also the optimal value of the primal. $\square$

Even though the model (CMAHLP hose) is linear, its size increases rapidly as the number of demand points increases, which makes it difficult to solve large instances using this formulation. In the next section, we devise several Benders decomposition algorithms as an attempt to solve large problem instances.

4.2 Benders Reformulation

In the previous section, we provide a mixed integer programming formulation for the capacitated multiple allocation hub location problem under hose demand uncertainty. Next we propose different Benders decomposition approaches to benefit from the computational efficiency of solving many small-sized problems iteratively instead of a single large model.

Benders decomposition is an exact solution method proposed by Benders [50] which is effectively used to solve various mixed integer programming problems in the literature. In this method, the original problem is reformulated by projecting out some of the variables and hence obtaining a formulation with a smaller number of variables and a large number of constraints. Afterwards, the reformulation is solved using a cutting plane approach such that each time a candidate solution is found, related cuts are added to the relaxed formulation. The relaxation solved at each iteration is called the master problem and the separation problem solved at each time a candidate solution is found is called the subproblem.

The effectiveness of a Benders decomposition algorithm depends on various factors; the number of times the subproblem is solved until optimality is achieved, the computational effort required to solve the master problem and the subproblem etc. In this study, we propose several Benders reformulations for the CMAHLP under hose demand uncertainty by considering these factors in order to obtain an effective decomposition scheme.

4.2.1 Decomposition by fixing location variables y (Benders1)

Consider the mixed integer formulation CMAHLP hose as presented in Section 4.1. Assume that only the hub location decisions are handled in the master problem and the rest is left to the subproblem. For fixed hub location vector y, the subproblem becomes the following:

$$
\begin{aligned}
\text{(PS1)} \quad & \min \sum_{i \in N} \lambda_i b_i && && (4.25)\\
\text{s.t.} \quad & \sum_{k \in H} \sum_{m \in H} x_{ijkm} \geq 1 && \forall (i,j) \in C, && (4.26)\\
& \sum_{m \in H} x_{ijkm} + \sum_{\substack{m \in H: \\ m \neq k}} x_{ijmk} \leq \hat{y}_k && \forall (i,j) \in C,\ k \in H, && (4.27)\\
& \lambda_i + \lambda_j \geq \sum_{k \in H} \sum_{m \in H} c_{ijkm} x_{ijkm} && \forall (i,j) \in C, && (4.28)\\
& \sum_{i \in N} \beta_i^k b_i \leq a_k \hat{y}_k && \forall k \in H, && (4.29)\\
& \beta_i^k + \beta_j^k \geq \sum_{m \in H} x_{ijkm} && \forall (i,j) \in C,\ \forall k \in H, && (4.30)\\
& \lambda_i \geq 0 && \forall i \in N, && (4.31)\\
& \beta_i^k \geq 0 && \forall i \in N, \forall k \in H, && (4.32)\\
& x_{ijkm} \geq 0 && \forall (i,j) \in C,\ \forall k,m \in H. && (4.33)
\end{aligned}
$$

Note that even though we modify constraints (4.26) here as inequalities, there exists an optimal solution where they hold as equalities. Taking the dual of PS1, we obtain the dual subproblem

$$
\begin{aligned}
\text{(DS1)} \quad \max \; & \sum_{(i,j)\in C} \rho_{ij} - \sum_{(i,j)\in C} \sum_{k\in H} \hat{y}_k \nu_{ijk} - \sum_{k\in H} a_k \hat{y}_k \gamma_k && (4.34)\\
\text{s.t.} \; & \sum_{j\in N\setminus\{i\}} \omega_{ij} + \sum_{j\in N\setminus\{i\}} \omega_{ji} \le b_i && \forall i \in N, \quad (4.35)\\
& \rho_{ij} - \nu_{ijk} - \nu_{ijm} - u_{ijk} \le c_{ijkm}\omega_{ij} \forall (i,j) \in C, \; \forall k, m \in H : k \neq m, && (4.36)\\
& \rho_{ij} - \nu_{ijk} - u_{ijk} \le c_{ijkk}\omega_{ij} && \forall (i,j) \in C, \; \forall k \in H, \quad (4.37)\\
& \sum_{j\in N\setminus\{i\}} u_{ijk} + \sum_{j\in N\setminus\{i\}} u_{jik} \le b_i \gamma_k && \forall i \in N, \; \forall k \in H, \quad (4.38)\\
& \gamma_k \ge 0 && \forall k \in H, \quad (4.39)\\
& \rho_{ij} \ge 0, \omega_{ij} \ge 0, && \forall (i,j) \in C, \quad (4.40)\\
& u_{ijk}, \nu_{ijk} \ge 0 && \forall (i,j) \in C, \; \forall k \in H. \quad (4.41)
\end{aligned}
$$

where dual variables ρ, ν, ω, γ and u correspond to constraints (4.26) - (4.30), respectively.

Note that both PS1 and DS1 are feasible and bounded when

$$
\sum_{k\in H} a_k y_k \ge \min \left\{ \left(\sum_{i\in N} b_i - \max_{i\in N} b_i \right), \sum_{i\in N} b_i/2 \right\}
$$

As primal and dual subproblems are both feasible and bounded, we need only the Benders optimality cuts in the master problem.

Let S_1 be the set of extreme points $(\rho, \nu, \omega, \gamma, u)$ of the dual subproblem. Then the master problem can be formulated as;

$$
\begin{aligned}
\text{(MP1)} \quad \min \; & \sum_{k\in H} f_k y_k + q && (4.42)\\
\text{s.t.} \; & q \ge \sum_{(i,j)\in C} \rho_{ij} - \sum_{(i,j)\in C} \sum_{k\in H} y_k \nu_{ijk} \\
& \qquad - \sum_{k\in H} a_k y_k \gamma_k \quad \forall (\rho, \omega, \nu, u) \in S_1, && (4.43)\\
& (4.5), (4.12).
\end{aligned}
$$

Since we are not able to decompose the subproblem, we add a single cut at each iteration.

4.2.2 Decomposition by fixing variables y and λ (Benders 2)

After fixing variables y and λ as $\bar{y}$ and $\bar{\lambda}$ in the master problem, the remaining becomes a feasibility problem as presented below,

$$
\begin{aligned}
\text{(PS2)} \quad & \min 0 && && (4.44)\\
& \text{s.t.} \sum_{k \in H} \sum_{m \in H} x_{ijkm} \geq 1 && \forall (i,j) \in C, && (4.45)\\
& \sum_{m \in H} x_{ijkm} + \sum_{\substack{m \in H: \\ m \neq k}} x_{ijmk} \leq \hat{y}_k && \forall (i,j) \in C,\ k \in H, && (4.46)\\
& \hat{\lambda}_i + \hat{\lambda}_j \geq \sum_{k \in H} \sum_{m \in H} c_{ijkm} x_{ijkm} && \forall (i,j) \in C, && (4.47)\\
& \sum_{i \in N} \beta_i^k b_i \leq a_k \hat{y}_k && \forall k \in H, && (4.48)\\
& \beta_i^k + \beta_j^k \geq \sum_{m \in H} x_{ijkm} && \forall (i,j) \in C,\ \forall k \in H, && (4.49)\\
& \beta_i^k \geq 0 && \forall i \in N, \forall k \in H, && (4.50)\\
& x_{ijkm} \geq 0 && \forall (i,j) \in C,\ \forall k,m \in H. && (4.51)
\end{aligned}
$$

Its dual is

$$
\text{(DS2)} \quad \max \sum_{(i,j) \in C} \rho_{ij} - \sum_{(i,j) \in C} \sum_{k \in H} \hat{y}_k \nu_{ijk} - \sum_{k \in H} a_k \hat{y}_k \gamma_k - \sum_{(i,j) \in C} (\hat{\lambda}_i + \hat{\lambda}_j) \omega_{ij} \qquad (4.52)
$$

$$
\begin{aligned}
\text{s.t. } & \rho_{ij} - \nu_{ijk} - \nu_{ijm} - u_{ijk} \leq c_{ijkm}\omega_{ij} && \forall (i,j) \in C, \ \forall k, m \in H : k \neq m, && (4.53)\\
& \rho_{ij} - \nu_{ijk} - u_{ijk} \leq c_{ijkk}\omega_{ij} && \forall (i,j) \in C, \ \forall k \in H, && (4.54)\\
& \sum_{j \in N \setminus \{i\}} u_{ijk} + \sum_{j \in N \setminus \{i\}} u_{jik} \leq b_i \gamma_k && \forall i \in N, \ k \in H, && (4.55)\\
& \gamma_k \geq 0 && \forall k \in H, && (4.56)\\
& \rho_{ij} \geq 0, \omega_{ij} \geq 0 && \forall (i,j) \in C, && (4.57)\\
& u_{ijk}, \nu_{ijk} \geq 0 && \forall (i,j) \in C, \ \forall k \in H. && (4.58)
\end{aligned}
$$

where dual variables ρ, ν, ω, γ and u are associated with the constraints (4.45) - (4.49), respectively.

Clearly we only need Benders feasibility cuts in the master problem since the subproblem is a feasibility problem. Hence the master problem becomes

$$
\begin{aligned}
\text{(MP2)} \quad \min \ & \sum_{k \in H} f_k y_k + \sum_{i \in N} \lambda_i b_i && (4.59)\\
\text{s.t. } \ & 0 \geq \sum_{(i,j) \in C} \rho_{ij} - \sum_{(i,j) \in C} \sum_{k \in H} \hat{y}_k \nu_{ijk} \\
& \quad - \sum_{k \in H} a_k y_k \gamma_k - \sum_{(i,j) \in C} (\lambda_i + \lambda_j)\omega_{ij} \ \forall (\rho, \omega, \nu, u) \in S_2, && (4.60)\\
& (4.5), (4.12).
\end{aligned}
$$

where S_2 is the set of extreme rays $(\rho, \nu, \omega, \gamma)$ of the dual subproblem. Again, we are not able to decompose the subproblem because of the dependencies between variables. Hence in each iteration of the Benders algorithm a single feasibility cut is added.

4.2.3 Decomposition by fixing variables y and β (Benders 3)

By fixing variables y and β as $\hat{y}$ and $\hat{\beta}$, we obtain the following primal subproblem:

$$\text{(PS3)} \quad \min \sum_{i \in N} \lambda_i b_i \tag{4.61}$$

$$\text{s.t.} \sum_{k \in H} \sum_{m \in H} x_{ijkm} \geq 1 \qquad \forall (i,j) \in C, \tag{4.62}$$

$$\sum_{m \in H} x_{ijkm} + \sum_{\substack{m \in H: \\ m \neq k}} x_{ijmk} \leq \hat{y}_k \qquad \forall (i,j) \in C,\ k \in H, \tag{4.63}$$

$$\lambda_i + \lambda_j \geq \sum_{k \in H} \sum_{m \in H} c_{ijkm} x_{ijkm} \qquad \forall (i,j) \in C, \tag{4.64}$$

$$\hat{\beta}_i^k + \hat{\beta}_j^k \geq \sum_{m \in H} x_{ijkm} \qquad \forall (i,j) \in C,\ \forall k \in H, \tag{4.65}$$

$$\lambda_i \geq 0 \qquad \forall i \in N, \tag{4.66}$$

$$x_{ijkm} \geq 0 \qquad \forall (i,j) \in C,\ \forall k, m \in H. \tag{4.67}$$

The dual of the subproblem is

$$\text{(DS3)} \quad \max \sum_{(i,j) \in C} \rho_{ij} - \sum_{(i,j) \in C} \sum_{k \in H} \hat{y}_k v_{ijk} - \sum_{(i,j) \in C} \sum_{k \in H} (\hat{\beta}_i^k + \hat{\beta}_j^k) u_{ijk} \tag{4.68}$$

$$\text{s.t.} \sum_{j \in N \setminus \{i\}} w_{ij} + \sum_{j \in N \setminus \{i\}} \omega_{ji} \leq b_i \qquad \forall i \in N, \tag{4.69}$$

$$\rho_{ij} - \nu_{ijk} - \nu_{ijm} - u_{ijk} \leq c_{ijkm} \omega_{ij} \forall (i,j) \in C,\ \forall k, m \in H : k \neq m, \tag{4.70}$$

$$\rho_{ij} - \nu_{ijk} - u_{ijk} \leq c_{ijkk} \omega_{ij} \qquad \forall (i,j) \in C,\ \forall k \in H, \tag{4.71}$$

$$\omega_{ij}, \rho_{ij} \geq 0 \qquad \forall (i,j) \in C, \tag{4.72}$$

$$u_{ijk}, \nu_{ijk} \geq 0 \qquad \forall (i,j) \in C,\ \forall k \in H. \tag{4.73}$$

Dual variables ρ, ν, ω and u correspond to constraints (4.62) - (4.65), respectively. Note that for any vector $\hat{\beta} \in \mathbb{R}_+^{|N||H|}$ such that $\sum_{k \in H}(\hat{\beta}_i^k + \hat{\beta}_j^k) \geq 1$ for all $(i,j) \in C$, both the primal and dual subproblems will be feasible and bounded. This result follows from constraints (4.62) and (4.65).

Let S_3 be the set of extreme points (ρ, ω, ν, u) of the dual subproblem. Then the master problem can be formulated as follows;

$$\text{(MP3)} \quad \min \sum_{k \in H} f_k y_k + q \tag{4.74}$$

$$\text{s.t. } q \geq \sum_{(i,j) \in C} \rho_{ij} - \sum_{(i,j) \in C} \sum_{k \in H} y_k \nu_{ijk} - \sum_{(i,j) \in C} \sum_{k \in H} (\beta_i^k + \beta_j^k) u_{ijk} \quad \forall (\rho, \omega, \nu, u) \in S_3, \tag{4.75}$$

$$\sum_{k \in H} (\beta_i^k + \beta_j^k) \geq 1 \quad \forall (i,j) \in C, \tag{4.76}$$

$$(4.5), (4.12), (4.14), (4.17).$$

Constraints (4.75) are the Benders optimality cuts, which enable algorithm to converge to an optimal solution while constraints (4.76) are added into the formulation to ensure feasibility. In the next subsections we describe how to solve the subproblem efficiently.

4.2.3.1 The Subproblem

In the dual subproblem, constraints (4.69) and (4.70)-(4.71) are interdependent due to the variables ω. In order to eleminate these dependencies, we let $\bar{\rho}_{ij} = \frac{\rho_{ij}}{\omega_{ij}}$, $\bar{\nu}_{ijk} = \frac{\nu_{ijk}}{\omega_{ij}}$ and $\bar{u}_{ijk} = \frac{u_{ijk}}{\omega_{ij}}$. Hence the dual subproblem can be presented as

$$\max_{\omega \in D_{hose}} \sum_{(i,j) \in C} w_{ij} \theta_{ij}.$$

where for $(i,j) \in C$,

$$(D_{ij}) \quad \theta_{ij} = \max \; \bar{\rho}_{ij} - \sum_{k \in H} \hat{y}_k \bar{\nu}_{ijk} - \sum_{k \in H} (\hat{\beta}_i^k + \hat{\beta}_j^k) \bar{u}_{ijk} \tag{4.77}$$

$$\text{s.t. } \bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{\nu}_{ijm} - \bar{u}_{ijk} \leq c_{ijkm} \qquad \forall k, m \in H : k \neq m, \tag{4.78}$$

$$\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{u}_{ijk} \leq c_{ijkk} \qquad \forall k \in H, \tag{4.79}$$

$$\bar{\rho}_{ij} \geq 0, \tag{4.80}$$

$$\bar{\nu}_{ijk}, \bar{u}_{ijk} \geq 0 \qquad \forall k \in H. \tag{4.81}$$

which is the dual of

$$(P_{ij}) \quad \theta_{ij} = \min \; \sum_{k \in H} \sum_{m \in H} c_{ijkm} x_{ijkm} \tag{4.82}$$

$$\text{s.t. } \sum_{k \in H} \sum_{m \in H} x_{ijkm} \geq 1, \tag{4.83}$$

$$\sum_{m \in H} x_{ijkm} + \sum_{m \in H \setminus \{k\}} x_{ijmk} \leq \hat{y}_k \qquad \forall k \in H, \tag{4.84}$$

$$\hat{\beta}_i^k + \hat{\beta}_j^k \geq \sum_{m \in H} x_{ijkm} \qquad \forall k \in H, \tag{4.85}$$

$$x_{ijkm} \geq 0 \qquad \forall k, m \in H. \tag{4.86}$$

Note that there exists an optimal solution of P_{ij} such that constraints (4.83) strictly hold. Next we devise an algorithm to compute the dual variables of D_{ij} for any distinct origin destination pair (i,j).

4.2.3.2 Computing the Dual Variables

For given $\hat{y}$ and $\hat{\beta}$ vectors, the optimal solution of problem P_{ij} can be found by inspection. Notice that when all hub capacities are large enough, each flow is routed through the shortest path. In the case that capacities are tightly imposed, the flow sent through a path only affects the capacity of the first hub on that path. Therefore the flow from i to j using hub k first will go through only the path $i - k - m_{(k)} - j$ which is a shortest path from i to j using hub k as the first hub. Besides, $(\hat{\beta}_i^k + \hat{\beta}_j^k)$ value sets a bound on the amount of flow from node i to node j that can be sent through hub k. As the capacity of hub k reserved for commodity (i, j) is known, the routing decision for each commodity becomes independent from each other. Hence, for commodity $(i, j) \in C$, sequencing shortest paths $i - k - m_{(k)} - j$ for each hub k in a nondecreasing order of cost and sending flow from i to j using these paths in a greedy manner provides an optimal solution for our problem.

Algorithm 1 Compute optimal solution of P_{ij}

Set $x_{ijkm} \leftarrow 0 \ \forall k, m \in H$

Set $residual \leftarrow 1$ and $p \leftarrow \sum_{k \in H} \hat{y}_k$

Sequence hubs as $k_1, k_2, ..., k_p$ such that $c_{ijk_1m(k_1)} \leq c_{ijk_2m(k_2)} \leq ... \leq c_{ijk_pm(k_p)}$

for $h = 1$ to p **do**

  if $residual > 0$ **and** $(\hat{\beta}_i^{k_h} + \hat{\beta}_j^{k_h}) > 0$ **then**

    Set $x_{ijk_hm(k_h)} \leftarrow \min\{residual, (\hat{\beta}_i^{k_h} + \hat{\beta}_j^{k_h})\}$

    Set $residual \leftarrow residual - x_{ijk_hm(k_h)}$

  end if

end for

Algorithm 1 describes how an optimal solution of P_{ij} is computed for $(i, j) \in C$. Here $residual$ represents the fraction of remaining flow to be sent from node i to node j. Since there exists an optimal solution in which the total fraction of flow sent from i to j is equal to 1, we initially set $residual$ to 1. Afterwards, the remaining flow from i to j is routed through hub k with the shortest $i - k - m_{(k)} - j$ path among the hubs that have available capacity.

With the optimal primal solution obtained above, an optimal solution for the dual problem D_{ij} can be constructed using the complementary slackness conditions. An optimal dual solution should satisfy both the constraints (4.78)-(4.81) and the complementary slackness conditions given below:

$$\bar{\rho}_{ij}(\sum_{k\in H}\sum_{m\in H} x_{ijkm} - 1) = 0 \tag{4.87}$$

$$\bar{\nu}_{ijk}(\sum_{m\in H} x_{ijkm} + \sum_{m\in H\setminus\{k\}} x_{ijmk} - \hat{y}_k) = 0 \qquad \forall k \in H, \tag{4.88}$$

$$\bar{u}_{ijk}(\hat{\beta}_i^k + \hat{\beta}_j^k - \sum_{m\in H} x_{ijkm}) = 0 \qquad \forall k \in H, \tag{4.89}$$

$$x_{ijkm}(\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{\nu}_{ijm} - \bar{u}_{ijk} - c_{ijkm}) = 0 \qquad \forall k, m \in H : k \neq m, \tag{4.90}$$

$$x_{ijkk}(\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{u}_{ijk} - c_{ijkk}) = 0 \qquad \forall k \in H. \tag{4.91}$$

We compute the dual variables in two steps. First, we fix a set of variables to some feasible values and hence drop the constraints related with them. In the second step, we compute the values of the remaining variables by solving a reduced system of inequalities.

Since $\sum_{k\in H}\sum_{m\in H} x_{ijkm} = 1$ in our optimal solution of P_{ij}, conditions (4.87) are always satisfied. Let $H_0 = \{k \in H : y_k = 0\}$, $H_1 = \{k \in H : y_k = 1\}$ and $F_{ij} = \{(k, m) \in H_1 \times H_1 : x_{ijkm} > 0\}$. Then, conditions (4.90) and (4.91) imply

$$\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{\nu}_{ijm} - \bar{u}_{ijk} = c_{ijkm} \qquad \forall (k, m) \in F_{ij} : k \neq m, \tag{4.92}$$

$$\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{u}_{ijk} = c_{ijkk} \qquad \forall (k, k) \in F_{ij}. \tag{4.93}$$

In order to find a solution satisfying these conditions, we fix $\bar{\rho}_{ij} = max_{(k,m)\in F_{ij}} c_{ijkm}$ and $\bar{\nu}_{ijk} = 0$ for all $k \in H_1$. We set $\bar{u}_{ijk} = \bar{\rho}_{ij} - c_{ijkm_{(k)}}$ for all $k \in H_1$ such that $\sum_{m\in H} x_{ijkm} > 0$. When $\sum_{m\in H} x_{ijkm} = 0$, we need to evaluate two cases in order to compute the value of $\bar{u}_{ijk}$. In the case that $(\hat{\beta}_i^k + \hat{\beta}_j^k) > 0$, we need to have $\bar{u}_{ijk} = 0$ for condition (4.89) to hold. When $(\hat{\beta}_i^k + \hat{\beta}_j^k) = 0$ we need to check dual feasibility. We set $\bar{u}_{ijk} = \max\{0, \bar{\rho}_{ij} - c_{ijkm_{(k)}}\}$ for all $k \in H_1$ such that $\sum_{m\in H} x_{ijkm} = 0$.

Then we consider the variables $\bar{\nu}_{ijk}$ and $\bar{u}_{ijk}$ for $k \in H_0$. Since we fixed the values of $\bar{\rho}_{ij}$, $\bar{\nu}_{ijk}$ and $\bar{u}_{ijk}$ for $k \in H_1$, we can use them to obtain bounds on $\bar{\nu}_{ijk}$ and $\bar{u}_{ijk}$ for $k \in H_0$. Using constraints (4.78), we obtain the following bounds for each $k \in H_0$,

$$\bar{\nu}_{ijk} \geq \max_{m \in H_1} \{\bar{\rho}_{ij} - c_{ijmk} - \bar{u}_{ijm}\} = l^1_{ijk} \tag{4.94}$$

$$\bar{\nu}_{ijk} + \bar{u}_{ijk} \geq \max_{m \in H_1} \{\bar{\rho}_{ij} - c_{ijkm}\} = l^2_{ijk} \tag{4.95}$$

$$\bar{\nu}_{ijk} + \bar{\nu}_{ijm} + \bar{u}_{ijk} \geq \bar{\rho}_{ij} - c_{ijkm} \qquad \forall m \in H_0 \tag{4.96}$$

Similarly, using constraints (4.79), we obtain

$$\bar{\nu}_{ijk} + \bar{u}_{ijk} \geq \bar{\rho}_{ij} - c_{ijkk} = l^4_{ijk} \tag{4.97}$$

Due to constraints (4.89), for $k \in H_0$, in the case that $(\hat{\beta}^k_i + \hat{\beta}^k_j) > 0$, variable $\bar{u}_{ijk}$ must be equal to zero. Hence, we fix $\bar{u}_{ijk} = 0$ for all $k \in H_0$. Next, considering the bounds we obtained so far on variables $\bar{\nu}_{ijk}$, we set $\bar{\nu}_{ijk} = \max\{0, l^1_{ijk}, l^2_{ijk}, l^4_{ijk}\}$ and then adjust these variables so that constraint (4.96) is satisfied. $\kappa \in [0, 1]$ is the scaling parameter used in this adjustment.

The algorithm for computing $\bar{\rho}_{ij}, \bar{\nu}_{ij}$ and $\bar{u}_{ij}$ for any $(i, j) \in C$ can be seen in Algorithm 2.

Proposition 2. *The dual solution computed using Algorithm 2 is optimal.*

Proof. We first check the complementary slackness conditions and then the dual feasibility.

The dual solution computed using this algorithm satisfies the complementary slackness conditions with the primal solution computed using Algorithm 1. As mentioned before, conditions (4.87) are already satisfied since $\sum_{k \in H} \sum_{m \in H} x_{ijkm} = 1$ for all $(i, j) \in C$. Considering conditions (4.88), we know that $\sum_{m \in H} x_{ijkm} + \sum_{m \in H \setminus \{k\}} x_{ijmk} = \hat{y}_k = 0$ when $k \in H_0$ and otherwise $\bar{\nu}_{ijk}$ is set to zero, hence they hold. Conditions (4.89) are also satisfied. We know that $\bar{u}_{ijk} > 0$ when $\hat{\beta}^k_i + \hat{\beta}^k_j = 0$ or when path $i - k - m_{(k)} - j$ is used but it is shorter than the longest path among the ones used to send flow from i to j. In both cases the capacity bound on hub k is tight. Therefore if $\bar{u}_{ijk} > 0$ then $\hat{\beta}^k_i + \hat{\beta}^k_j = \sum_{m \in H} x_{ijkm}$. Finally, conditions

Algorithm 2 Compute $(\bar{\rho}_{ij}, \bar{\nu}_{ij}, \bar{u}_{ij})$

Compute optimal solution of P_{ij}
Set $\bar{\rho}_{ij} = max_{(k,m) \in F_{ij}} c_{ijkm}$
for $k \in H_1$ **do**
    Set $\bar{\nu}_{ijk} = 0$
    if $\sum_{m \in H} x_{ijkm} > 0$ **then**
        Set $\bar{u}_{ijk} = \bar{\rho}_{ij} - c_{ijkm_{(k)}}$
    else if $\sum_{m \in H} x_{ijkm} = 0$ and $(\hat{\beta}_i^k + \hat{\beta}_j^k) = 0$ **then**
        Set $\bar{u}_{ijk} = \max\{0, \bar{\rho}_{ij} - c_{ijkm_{(k)}}\}$
    else
        Set $\bar{u}_{ijk} = 0$
    end if
end for
for $k \in H_0$ **do**
    Set $\bar{u}_{ijk} = 0$
    Set $\bar{\nu}_{ijk} = \max\{0, l_{ijk}^1, l_{ijk}^2, l_{ijk}^4\}$
end for
for $k, m \in H_0$ such that $\bar{\rho}_{ij} - c_{ijkm} > 0$ **do**
    Define $\Delta = (\bar{\rho}_{ij} - c_{ijkm}) - \bar{\nu}_{ijk} - \bar{\nu}_{ijm}$
    if $\Delta > 0$ **then**
        Update $\bar{\nu}_{ijk} \leftarrow \bar{\nu}_{ijk} + \kappa\Delta$
        Update $\bar{\nu}_{ijm} \leftarrow \bar{\nu}_{ijm} + (1-\kappa)\Delta$
    end if
end for

(4.90) hold since when $x_{ijkm} > 0$ for any $(i,j) \in C$, that means $k, m \in H_1$ and thus $\bar{\nu}_{ijk} = \bar{\nu}_{ijm} = 0$. Consequently $\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{\nu}_{ijm} - \bar{u}_{ijk} - c_{ijkm} = \bar{\rho}_{ij} - \bar{u}_{ijk} - c_{ijkm} = \bar{\rho}_{ij} - (\bar{\rho}_{ij} - c_{ijkm_{(k)}}) - c_{ijkm} = 0$. Validity of conditions (4.91) can be shown similarly.

Next we check the dual feasibility of a solution constructed with our algorithm. First we consider the constraints (4.78). For each commodity $(i,j) \in C$ there are four cases to be evaluated:

- Case 1: $k \in H_1$, $m \in H_1$
  Since $k, m \in H_1$ we know that $\bar{\nu}_{ijk} = \bar{\nu}_{ijm} = 0$. Hence $\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{\nu}_{ijm} - \bar{u}_{ijk} = \bar{\rho}_{ij} - \bar{u}_{ijk}$. All possible values of $\bar{u}_{ijk}$ should be considered.
  If $x_{ijkm} > 0$ then $\bar{\rho}_{ij} - \bar{u}_{ijk} = \bar{\rho}_{ij} - (\bar{\rho}_{ij} - c_{ijkm_{(k)}}) = c_{ijkm_{(k)}} = c_{ijkm}$. When $x_{ijkm} = 0$ and $\hat{\beta}_i^k + \hat{\beta}_j^k = 0$, the value of $\bar{u}_{ijk}$ is set to $\max\{0, \bar{\rho}_{ij} - c_{ijkm_{(k)}}\}$.

If $\bar{u}_{ijk} = 0$, then the proof is as in the previous case. Otherwise, $\bar{u}_{ijk} = \bar{\rho}_{ij} - c_{ijkm_{(k)}}$ and $\bar{\rho}_{ij} - \bar{u}_{ijk} = \bar{\rho}_{ij} - (\bar{\rho}_{ij} - c_{ijkm_{(k)}}) = c_{ijkm_{(k)}} \leq c_{ijkm}$ by definition. When $x_{ijkm} = 0$ and $\hat{\beta}_i^k + \hat{\beta}_j^k > 0$, $\bar{\rho}_{ij} - \bar{u}_{ijk} \leq \bar{\rho}_{ij} = max_{(k',m') \in F_{ij}} c_{ijk'm'} \leq c_{ijkm}$. The last relation holds since any path not chosen although it has available capacity should be not shorter than those which are chosen to send flow.

- Case 2: $k \in H_1$, $m \in H_0$
  Since $\bar{\nu}_{ijk'} = 0 \; \forall k' \in H_1$ and our solution satisfies the inequality (4.94), we have
  $\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{\nu}_{ijm} - \bar{u}_{ijk} \leq \bar{\rho}_{ij} - \max_{k' \in H_1}\{\bar{\rho}_{ij} - c_{ijk'm} - \bar{u}_{ijk'}\} - u_{ijk} \leq c_{ijkm}$.

- Case 3: $k \in H_0$, $m \in H_1$
  In this case, we know that $\bar{\nu}_{ijm} = \bar{u}_{ijk} = 0$. Using the inequality (4.95),
  $\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{\nu}_{ijm} - \bar{u}_{ijk} = \bar{\rho}_{ij} - \bar{\nu}_{ijk} \leq \bar{\rho}_{ij} - \max_{m' \in H_1}\{\bar{\rho}_{ij} - c_{ijkm'}\} \leq c_{ijkm}$.

- Case 4: $k \in H_0$, $m \in H_0$
  Due to the inequality (4.96), we have $\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{\nu}_{ijm} - \bar{u}_{ijk} \leq \bar{\rho}_{ij} - (\bar{\rho}_{ij} - c_{ijkm}) = c_{ijkm}$.

In order to check constraints (4.79), we evaluate two cases.

- Case 1: $k \in H_1$
  For $k \in H_1$, the value of $\bar{\nu}_{ijk}$ is set to zero in our algorithm. Thus, $\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{u}_{ijk} = \bar{\rho}_{ij} - \bar{u}_{ijk}$. Again we need to consider all possible values of $\bar{u}_{ijk}$. If $x_{ijkk} > 0$ then $\bar{\rho}_{ij} - \bar{u}_{ijk} = \bar{\rho}_{ij} - (\bar{\rho}_{ij} - c_{ijkm_{(k)}}) = c_{ijkk}$ since we are sending flow through path $i - k - j$. When $x_{ijkk} = 0$ and $\hat{\beta}_i^k + \hat{\beta}_j^k = 0$, the value of $\bar{u}_{ijk}$ is set to $\max\{0, \bar{\rho}_{ij} - c_{ijkm_{(k)}}\}$. If $\bar{u}_{ijk} = 0$, then the proof is the same as the former. Otherwise, $\bar{\rho}_{ij} - \bar{u}_{ijk} = \bar{\rho}_{ij} - (\bar{\rho}_{ij} - c_{ijkm_{(k)}}) \leq c_{ijkk}$. If $x_{ijkk} = 0$ and $\hat{\beta}_i^k + \hat{\beta}_j^k > 0$, then $\bar{u}_{ijk}$ is also set to zero. Hence $\bar{\rho}_{ij} - \bar{u}_{ijk} = \bar{\rho}_{ij} = max_{(k',m') \in F_{ij}} c_{ijk'm'} \leq c_{ijkk}$.

- Case 2: $k \in H_0$
  Using inequality (4.97), $\bar{\rho}_{ij} - \bar{\nu}_{ijk} - \bar{u}_{ijk} \leq \bar{\rho}_{ij} - (\bar{\rho}_{ij} - c_{ijkk}) = c_{ijkk}$.

Since the solution computed using Algorithm 2 is dual feasible and it satisfies complementary slackness conditions with solution x, it is an optimal dual solution. □

4.2.4 Decomposition by projecting out the flow variables (Benders 4)

In this section, we aim to find a decomposition scheme such that the subproblem can be further decomposed for each commodity. For fixed vectors $\hat{y}$, $\hat{\lambda}$ and $\hat{\beta}$, the subproblem becomes the following feasibility problem;

$$\min \ 0 \tag{4.98}$$

$$\text{s.t.} \sum_{k\in H}\sum_{m\in H} x_{ijkm} \geq 1 \qquad \forall (i,j)\in C, \tag{4.99}$$

$$\sum_{m\in H} x_{ijkm} + \sum_{\substack{m\in H:\\ m\neq k}} x_{ijmk} \leq \hat{y}_k \qquad \forall (i,j)\in C,\ k\in H, \tag{4.100}$$

$$\hat{\lambda}_i + \hat{\lambda}_j \geq \sum_{k\in H}\sum_{m\in H} c_{ijkm}x_{ijkm} \qquad \forall (i,j)\in C, \tag{4.101}$$

$$\hat{\beta}_i^k + \hat{\beta}_j^k \geq \sum_{m\in H} x_{ijkm} \qquad \forall (i,j)\in C,\ \forall k\in H, \tag{4.102}$$

$$\lambda_i \geq 0 \qquad \forall i\in N, \tag{4.103}$$

$$x_{ijkm} \geq 0 \qquad \forall (i,j)\in C,\ \forall k,m\in H. \tag{4.104}$$

For this problem to be feasible, its dual needs to be bounded, meaning that

$$\sum_{(i,j)\in C}\rho_{ij} - \sum_{(i,j)\in C}\sum_{k\in H}\hat{y}_k\nu_{ijk} - \sum_{(i,j)\in C}\sum_{k\in H}(\hat{\beta}_i^k+\hat{\beta}_j^k)u_{ijk} - \sum_{(i,j)\in C}\omega_{ij}(\hat{\lambda}_i+\hat{\lambda}_j) \leq 0 \tag{4.105}$$

for all (ρ,ν,ω,u) that satisfy

$$\rho_{ij} - \nu_{ijk} - \nu_{ijm} - u_{ijk} \leq c_{ijkm}\omega_{ij} \qquad \forall (i,j)\in C,\ \forall k,m\in H : k\neq m, \tag{4.106}$$

$$\rho_{ij} - \nu_{ijk} - u_{ijk} \leq c_{ijkk}\omega_{ij} \qquad \forall (i,j)\in C,\ \forall k\in H, \tag{4.107}$$

$$\rho_{ij}, \omega_{ij} \geq 0 \qquad \forall (i,j)\in C \tag{4.108}$$

$$u_{ijk}, \nu_{ijk} \geq 0 \qquad \forall (i,j)\in C, \forall k\in H \tag{4.109}$$

This system decomposes for each $(i,j)\in C$. Without loss of generality, we can take $\omega_{ij} = 0$ or $\omega_{ij} = 1$ for $(i, j) \in C$. When $\omega_{ij} = 0$, the system becomes,

$$\rho_{ij} - \sum_{k \in H} \hat{y}_k \nu_{ijk} - \sum_{k \in H} (\hat{\beta}_i^k + \hat{\beta}_j^k) u_{ijk} \leq 0 \tag{4.110}$$

for all (ρ_{ij},ν_{ij},u_{ij}) such that

$$\rho_{ij} - \nu_{ijk} - \nu_{ijm} - u_{ijk} \leq 0 \qquad \forall k, m \in H : k \neq m, \tag{4.111}$$

$$\rho_{ij} - \nu_{ijk} - u_{ijk} \leq 0 \qquad \forall k \in H, \tag{4.112}$$

$$\rho_{ij} \geq 0 \tag{4.113}$$

$$u_{ijk}, \nu_{ijk} \geq 0 \qquad \forall k \in H \tag{4.114}$$

It can be seen that this system of inequalities always holds when $\sum_{k \in H} \hat{y}_k \geq 1$ and $\sum_{k \in H} (\hat{\beta}_i^k + \hat{\beta}_j^k) \geq 1$ and the former inequality is already implied by constraint 4.12. Hence we only need to consider the case $\omega_{ij} = 1$. When we fix $\omega_{ij} = 1$, the system becomes,

$$\rho_{ij} - \sum_{k \in H} \hat{y}_k \nu_{ijk} - \sum_{k \in H} (\hat{\beta}_i^k + \hat{\beta}_j^k) u_{ijk} \leq \hat{\lambda}_i + \hat{\lambda}_j \tag{4.115}$$

for all (ρ_{ij},ν_{ij},u_{ij}) satisfying

$$\rho_{ij} - \nu_{ijk} - \nu_{ijm} - u_{ijk} \leq c_{ijkm} \qquad \forall (i, j) \in C,\ \forall k, m \in H : k \neq m, \tag{4.116}$$

$$\rho_{ij} - \nu_{ijk} - u_{ijk} \leq c_{ijkm} \qquad \forall k \in H, \tag{4.117}$$

$$\rho_{ij} \geq 0 \tag{4.118}$$

$$u_{ijk}, \nu_{ijk} \geq 0 \qquad \forall k \in H \tag{4.119}$$

Hence, after projecting out x variables, the problem can be reformulated as follows

$$\min \quad \sum_{k\in H} f_k y_k + \sum_{i\in N} \lambda_i b_i \tag{4.120}$$

$$\text{s.t. } \lambda_i + \lambda_j \geq \rho_{ij}^t - \sum_{k\in H} y_k \nu_{ijk}^t - \sum_{k\in H}(\beta_i^k + \beta_j^k) u_{ijk}^t \ \forall (i,j) \in C, \forall t = 1,..,T_{ij}, \tag{4.121}$$

$$\sum_{k\in H}(\beta_i^k + \beta_j^k) \geq 1 \qquad \forall (i,j) \in C, \tag{4.122}$$

$$(4.5), (4.12), (4.14), (4.16), (4.17).$$

where $(\rho_{ij}^t, \omega_{ij}^t, \nu_{ij}^t, u_{ij}^t)$ is the t-th extreme point of $S_{ij} = \{(\rho_{ij}, \omega_{ij}, \nu_{ij}, u_{ij}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^h \times \mathbb{R}_+^h : (4.116) - (4.119)\}$ for $(i,j) \in C$ and T_{ij} is the number of extreme points of S_{ij}. The variables corresponding to an extreme point of S_{ij} maximizing the right-hand-side of constraint (4.121) can be computed as in Section 4.2.3.2. In this reformulation, we are able to add multiple cuts at each iteration of the Benders decomposition algorithm instead of a single cut.

4.3 Computational Analysis

We test our mathematical model and solution algorithms on well-known Australian Post (AP) data set instances with up to 50 nodes. To the extend of our knowledge, the AP data set is the only data set with fixed costs of hub opening and node capacities (accessible from OR-Library [51]). For both fixed costs and capacities, two settings are available. Instances with tight (T) fixed costs have larger costs of hub opening compared to the instances with loose (L) fixed costs. Similarly, the instances with tight (T) capacities have smaller available capacities in comparison with the instances with loose (L) capacities. For each problem size n, we consider four cases: LL, LT, TL, TT where the first letter corresponds to the fixed cost setting and the second to the capacity setting.

In our experiments we consider AP instances with $n = 25, 40, 50$. The traffic bounds for the hose model are generated as $b_i = \sum_{j\in N\setminus\{i\}}(w_{ij} + w_{ji})$ for all $i \in N$.

All nodes are taken as possible hub locations, i.e., $H = N$. We perform our computational experiments on a 64-bit machine with Intel Xeon E5-2630 v2 processor at 2.60 GHz and 96 GB of RAM using Java and Cplex 12.5.1 We set a time limit of three hours. All solution times are given in seconds. For the Benders decomposition algorithm implementations, we use the lazy constraint callback function of CPLEX.

Table 4.1: Computational results for the MIP formulation of deterministic CMAHLP

# nodes	Instance	Objective	CPU time	Hubs
25	LL	222411.23	4.26	8,18
25	LT	248713.51	54.56	9,16,19
25	TL	293850.21	26.08	9,23
25	TT	312743.36	227.73	6,14,24
40	LL	230495.10	46.64	14,29
40	LT	252982.48	909.59	14,26,30
40	TL	284821.33	518.83	14,19
40	TT	326827.27	2637.96	14,25,38

For completeness of analysis, we first present the results of CMAHLP with deterministic demand values in Table 4.1. We present the optimal value, CPU time until the optmality is achieved and the hub locations in the optimal solution for each instance with $n = 25, 40$. Larger instances can not be solved to optimality within the time limit. It can be seen that computational difficulty of the problem rapidly increases as the problem size n increases and as the fixed cost-capacity settings become tight (T).

In Table 4.2, the optimal objective value, CPU time and hub locations in the optimal solution of the mathematical model (CMAHLP hose) are reported. For instances with 40 nodes, the mathematical model was not able to obtain the optimal solution within the time limit of three hours so that only the instances with 25 nodes are included in our analysis. As we compare these results with the ones for the deterministic case represented in Table 4.1, it can be seen that the computational effort required to solve the problems to optimality significantly increases as the demand vector becomes a variable taking value from a hose uncertainty set. The deterministic model instances can be solved to optimality within five minutes, while it may take up to three hours for the instances with hose demand uncertainty. Considering the hub locations, it can be seen that there is a change in the optimal hub locations in all instances. In some of them, as in $25LL$, only one hub location is changed whereas in some others like $25TL$ all hubs of the deterministic model are replaced in the solution of the problem with the hose model.

Table 4.2: Computational results for the MIP formulation with hose demand uncertainty

# nodes	Instance	Objective	CPU time	Hubs
25	LL	269373.49	5896.78	8,19
25	LT	299613.46	4883.77	9,12,19
25	TL	330504.18	9765.26	11,14
25	TT	361699.66	3372.66	9,12,14

As expected, the optimal total costs increase in the CMAHLP with hose demand uncertainty in comparison with the deterministic case. We observe an average increase of 17.43% in the objective value of the optimal solutions. The maximum percentage increase is for the instance $25LL$ with 21.12% whereas the minimum is 12.47% for the instance $25TL$.

Next we analyse the computational effectiveness of proposed Benders reformulations. For each reformulation, we present the optimal objective value, CPU time until achieving optimality, the number of Benders cuts added and hub locations in the optimal solutions. In Table 4.3, results of Benders 1, in which we only fix the location variables in the master problem, are shown. Benders 1 is able to solve eight instances out of 12 to optimality within the time limit. For the instances that can not be solved, the run is terminated due to inadequate memory. Benders 1 solves four additional instances that can not be solved by the MIP formulation. Besides, this Benders decomposition approach clearly outperforms the mathematical model in terms of computation times. All eight instances with no memory errors are solved to optimality within 1.5 hours. The reason behind the memory error may be the size of the subproblem; in this decomposition scheme, most of the decisions are left to the subproblem.

Table 4.3: Computational results for Benders 1

# nodes	Instance	Objective	CPU time	# Cuts	Hubs
25	LL	269373.49	215.81	69	8,19
25	LT	299613.46	1339.73	411	9,12,19
25	TL	330504.18	104.22	34	11,14
25	TT	361699.66	399.52	126	9,12,14
40	LL	271656.49	1502.42	66	14,29
40	LT	*memory*	-	-	-
40	TL	314904.30	479.48	19	14,19
40	TT	385661.54	3880.55	185	14,19,25
50	LL	*memory*	-	-	-
50	LT	*memory*	-	-	-
50	TL	340552.90	1729.11	32	24,27
50	TT	*memory*	-	-	-

Table 4.4 displays results of Benders 2 in which only the Benders feasibility cuts are added to the formulation. Since the algorithms give a memory error for instances with more than 25 nodes, only the results for the instances with 25 nodes are included. In view of our results, it can be said that fixing $\hat{y}$ and $\hat{\lambda}$ in the master problem is not an efficient decomposition approach for the CMAHLP under hose demand uncertainty. Out of 12 AP instances only one can be solved to optimality within 3 hours. For the instance $25TL$ we obtain a feasible solution with a large optimality gap. The other two instances marked as *time* do not yield any feasible solutions within the time limit. The failure of this decomposition approach may be due to the excessive number of callbacks until optimality is guaranteed and the computational burden of finding extreme rays of the dual subproblem. If we compare it with the number of callbacks in Benders 1, it can be seen that Benders 2 performs almost 10 times more number of callbacks than Benders 1.

Table 4.4: Computational results for Benders 2

# nodes	Instance	Objective	CPU time	# Cuts	Hubs
25	LL	269373.49	8921.29	649	8,19
25	LT	*time*	-	-	-
25	TL	566200.91	(67.30)	558	21
25	TT	*time*	-	-	-

We also report the results of Benders 3 where y and β are fixed as $\hat{y}$ and $\hat{\beta}$ in the master problem in Table 4.5. Out of 12 AP data set instances, Benders 3 is able to solve six of them within three hours. Among the instances with 50 nodes, it obtains a feasible solution only for instance $50LT$. For the others, it cannot find a feasible solution within the time limit. It can be seen that Benders 3 works better than Bender 2 and the MIP formulation but it is still outperformed by Benders 1. An interesting observation here is the large number of cuts added until the algorithm converges to an optimal solution. As we add at most one cut at each time the callback is invoked, the number of times callback is performed is at least as many as the number of cuts. Hence it can be concluded that the quality of the cuts added needs to be improved to obtain better results using this decomposition scheme.

Table 4.5: Computational results for Benders 3

# nodes	Instance	Objective	CPU time	# Cuts	Hubs
25	LL	269373.49	929.93	10657	8,19
25	LT	299613.46	2923.96	7002	9,12,19
25	TL	330504.18	40.52	1284	11,14
25	TT	361699.66	400.15	5180	9,12,14
40	LL	*time*	-	-	-
40	LT	*time*	-	-	-
40	TL	314904.29	285.99	2362	14,19
40	TT	*time*	-	-	-
50	LL	*time*	-	-	-
50	LT	343860.19	(43.69)	19164	14,32,35
50	TL	340552.89	914.53	3362	24,27
50	TT	*time*	-	-	-

Finally, we evaluate the performance of reformulation Benders 4, which employs a multicut approach, in Table 4.6. It is clear that Benders 4 is the best exact solution algorithm for the CMAHLP under hose demand uncertainty among the ones we proposed in this study. It succeeds to solve nine instances out of 12 to optimality within the time limit and for the others it is able to obtain feasible solutions with relatively lower gap values. The maximum CPU time for the solved instances is approximately nine minutes. We here note that Benders 4 spends a considerable amount of time to improve the lower bound after an optimal solution is found. Therefore its computational performance can be further improved by the use of stronger cuts that could yield better lower bounds.

Table 4.6: Computational results for Benders 4

# nodes	Instance	Objective	CPU time	# Cuts	Hubs
25	LL	269373.49	5.39	5334	8,19
25	LT	299613.46	121.20	10987	9,12,19
25	TL	330504.18	4.33	5058	11,14
25	TT	361699.66	29.78	8806	9,12,14
40	LL	271656.49	53.27	13446	14,29
40	LT	315624.51	(3.22)	63883	14,26,30
40	TL	314904.30	4.74	3852	14,19
40	TT	385661.54	524.30	34195	14,19,25
50	LL	276091.56	285.91	26848	15,35
50	LT	315039.04	(6.57)	92034	6,26,32,46
50	TL	340552.90	109.71	21632	24,27
50	TT	452151.20	(11.50)	108806	25,26,41,48

4.4 Conclusions

In this chapter, we studied a capacitated multiple allocation hub location problem where the demand takes value from a hose uncertainty set. We proposed a mixed integer programming formulation and devised four different Benders decomposition based exact solution algorithms. In view of our computational results, Benders 4, which utilizes a multicut approach outperformed all the others in terms of computational efficiency. It succeeded to solve AP data set instances with up to 50 nodes and obtained feasible solutions with relatively lower optimality gaps for the instances that can not be solved within the time limit.

Chapter 5

Conclusion

In this thesis, we studied hub location problems under polyhedral demand uncertainty. Hub location decisions are a part of the strategic planning process in airline, cargo and telecommunications networks. They are usually made long before system starts operating so that any decision based on the estimated pairwise demands may become obsolete in the time of operation. Hence incorporating uncertainty into our decisions is an important aspect to be considered. However, even though hub location problems are well-studied in the literature, uncertainty in the hub location context is rather an unexplored area. Most of the existing works assume that the probability distribution for demand is known. In our study, demand is assumed to take value from a polyhedral uncertainty set. We used two different models from the telecommunications literature to represent the demand uncertainty; namely hose and hybrid models. For the hose model, only an estimation of the total demand associated with each node is required while the hybrid model additionally puts bounds on the pairwise demands.

First we introduced demand uncertainty into the uncapacitated multiple allocation p-hub median problem. In this problem, the aim is to find hub locations and routes for the commodities that minimize the total transportation cost under the worst case demand realization. We provided mixed integer linear programming formulations for the problem with hose and hybrid uncertainty sets. In order to solve large-scale instances, we devised two Benders decomposition based exact solution algorithms such that the first one is a classical Benders decomposition application while the second utilizes a multicut approach. For the problem with the hose model, the second decomposition algorithm succeeded to solve instances with up to 200 nodes whereas the mathematical model was able to solve instances with up to 50 nodes. In the case of the hybrid model, the first Benders decomposition algorithm outperformed the others for large instances while the second algorithm has smaller computational times for small instances.

Next we considered a capacitated multiple allocation hub location problem. Since capacity constraints are imposed, demand uncertainty has an impact on the total cost and the feasibility of the solutions. With this motivation, we studied the capacitated multiple allocation hub location problem under hose demand uncertainty. We presented a mathematical formulation of the problem and devised four different Benders decomposition based exact solution algorithms. We also developed an algorithm for the subproblem to accelerate the convergence of the proposed solution algorithms. With the last Benders decomposition approach, we were able to solve instances with up to 50 nodes to optimality while the MIP formulation could solve instances with up to 25 nodes. Besides, three of our algorithms are superior to the MIP formulation in terms of solution times even for small instances.

As future research, demand uncertainty can be incorporated into different variants of the hub location problem using hose and hybrid models. In particular, problems with single allocation would be interesting to study. Note that when each demand node is required to be assigned to a single hub facility, routing decisions become interdependent. We expect these problems to be more challenging. Moreover, considering the sequential nature of the decisions in the hub location problems, another research direction may be to use an adjustable robust optimization approach. As hub location decisions are usually handled before system starts operating, it is possible to take recourse actions in the routing phase based on the demand realizations. This approach may help balancing the conservatism of solutions and thus reducing the price of robustness. One last extension could be to consider other sources of uncertainty in the hub location problems such as fixed cost of opening hubs and transportation costs. These cost terms are also subject to uncertainty due to prices of property, raw materials etc. and it would be worthwhile to investigate their effects on location and routing decisions in hub networks.

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