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COUNTING POSITIVE DEFECT IRREDUCIBLE CHARACTERS OF A FINITE GROUP

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Abstract. Let $z_+(G)$ be the number of ordinary irreducible characters of a finite group G which have positive defect with respect to a prime p. We express $z_+(G)$ as the p–adic limit of a sequence of enumerative parameters of G and p. When $p = 2$, and under a suitable hypothesis on the Sylow 2–subgroups of G, we give two local characterisations of the parity of $z_+(G)$, one of them compatible with Alperin's Weight Conjecture, the other apparently independent.

1. A Formula for the Number of Positive Defect Irreducible Characters

Richard Brauer's theory of finite group representations is an analysis of the representations of a finite group G (usually variable) in terms of a prime p (always fixed). Analysing the theory itself, three strands emerge. Firstly, — to borrow a phrase of Brauer — we may consider "arithmetic properties" of (ordinary absolutely) irreducible characters. For instance, can we express conveniently the number $z_+(G)$ of positive defect irreducible characters? (The *defect* of an irreducible character χ of G is the non–negative integer d such that $\chi(1)_p\, p^d = |G|_p$.) Secondly, we may consider the group algebra RG where R is a $\mathcal{P}$–adic completion of some ring of algebraic integers, and $\mathcal{P}$ is a prime divisor of (p). For instance, it is well–known that if the field of fractions of R splits for G, then $z_+(G)$ is determined by the isomorphism class of the centre ZRG. Thirdly, we may consider representations of "local" subgroups (by which we roughly mean non–trivial p–subgroups, their normalisers, and let us say, central extensions of factor groups of such normalisers.) For instance, Alperin's Weight Conjecture [**1**] asserts that $z_+(G)$ is determined by the "local" subgroups of G. A precise definition of the term "locally determined" is suggested in Thévenaz [**18**].

Since rings of algebraic integers, when they arise as ground rings, are often usefully replaced with $\mathcal{P}$–adic completions, might not even enumerative parameters of "arithmetic properties" of irreducible characters be usefully regarded as p–adic integers? But the entwinement of the first two strands is easy; the miracle of Brauer's theory (so it seems to me) is in the braiding of the third with the other two. We shall think of $z_+(G)$ as a p–adic integer, and see if this helps us obtain "local" information about it.

Another, entirely different angle on the material below is to deliberately ignore most of the content of Michel Broué's memorable remark (from a seminar in 1991) "Alperin's Conjecture must be the combinatorial shadow of some deeper algebraic structure". While considerable effort is being focused on searches for such "deeper

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algebraic structure" (see, for a small sample, Dade [**8**], Ellers [**9**], Robinson [**16**]), the bald fact is that Alperin's Conjecture is a "combinatorial" assertion. It is even enumerative.

We shall end this section with Theorem 1.1, a formula expressing $z_+(G)$ as the p-adic limit of a sequence of enumerative parameters of G. To discuss Sections 2 and 3, let us asume that $p = 2$ and that the Sylow 2-subgroups of G are abelian. (Actually, our hypothesis on the Sylow 2-subgroups will be weaker, but for now let us minimise technicalities.) Section 2 may be seen mainly as an application of Theorem 1.1 to prove that Alperin's Conjecture holds up to parity. In Section 3, Theorem 1.1 is used to deduce Theorem 3.4, a local characterisation of the parity of $z_+(G)$. My failure to derive Theorem 3.4 from Alperin's Conjecture is, to me, more interesting than success would have been. Speculation on improving Theorem 3.4 is left largely to the reader, but in connection with this, in Section 4, we point out a relationship between two formulas expressing $z_0(G)$ as the p-rank of an integer matrix.

Let K be an algebraically closed field of characteristic zero. The number $k(G)$ of conjugacy classes of G, and the number $z_0(G)$ of defect-zero irreducible characters of G are related by

$$k(G) \;=\; z_0(G) + z_+(G).$$

Given a positive integer n, let us write $\omega(n, G)|G|$ for the number of $(2+n)$-tuples $(x, y, g_1, \ldots, g_n)$ consisting of elements of G such that each g_i is a p-element, not all the g_i are trivial, and $[x, y] = g_1 \ldots g_n$. (By definition, $[x, y] := xyx^{-1}y^{-1}$.)

The essential origin of the following result is Strunkov [**17**]. We have also drawn an idea from Iizuka-Watanabe [**12**, page 58].

Theorem 1.1. *As a congruence of p-local integers,*

$$-z_+(G) \;\equiv_{p^n}\; \omega(n, G)$$

for all positive integers n. In particular, taking the p-adic limit,

$$-z_+(G) \;=\; \lim_{n\to\infty} \omega(n, G).$$

Proof. For each irreducible character χ of G, let e_χ denote the primitive idempotent of ZKG such that $\chi(e_\chi) = \chi(1)$. Let G_p^+ denote the sum in KG of the p-elements of G. Let ϕ be the class function on G such that $\phi(g)$ is the number of pairs (x, y) of elements of G such that $[x, y] = g$. We have

$$(G_p^+)^n \;=\; \sum_\chi \big(\chi(G_p^+)/\chi(1)\big)^n e_\chi$$

where χ runs over the irreducible characters of G. By calculating, in two different ways, the ordinary trace of each class sum acting by translation on ZKG — firstly with respect to the basis consisting of the class sums, and secondly with respect to the basis consisting of the idempotents e_χ — we deduce that

$$\phi \;=\; |G| \sum_\chi \chi/\chi(1).$$

A theorem of Frobenius in Curtis–Reiner [**7**, 41.10] implies that each $\chi(G_p^+)$ is divisible by $|G|_p$. If χ has defect zero, then $\chi(G_p^+) = \chi(1)$. Therefore, $|G|_p$ divides $\phi\big((G_p^+)^n\big)$, and

$$z_0(G) \equiv_{p^n} \phi\big((G_p^+)^n\big)/|G|.$$

We complete the argument by observing that

$$\omega(n, G) = \phi\big((G_p^+)^n - 1\big)/|G| = \phi\big((G_p^+)^n\big)/|G| - k(G).$$

□

2. Application to Alperin's Conjecture

We apply Theorem 1.1 to Alperin's Conjecture using techniques from Knörr–Robinson [**13**], and earlier related works by S. Bouc, K.S. Brown, D. Quillen, Thévenaz, and Webb. First, we need some notation, and must briefly discuss some results in [**13**].

Given a poset $\mathcal{P}$, we write $\mathrm{sd}(\mathcal{P})$ for the set of chains $\boldsymbol{x} = (x_0, \ldots, x_n)$ in $\mathcal{P}$, partially ordered by the subchain relation. (We disallow the empty chain.) The *length* of such a chain $\boldsymbol{x}$ is defined to be $n(\boldsymbol{x}) = n$. The simplicial complex with vertex set $\mathcal{P}$ and simplex set $\mathrm{sd}(\mathcal{P})$ will also be denoted as $\mathcal{P}$. (The rationale for our notation is that, as simplicial complexes, $\mathrm{sd}(\mathcal{P})$ is the barycentric subdivision of $\mathcal{P}$.) The Euler characteristic of $\mathcal{P}$ is

$$\chi(\mathcal{P}) = \sum_{\boldsymbol{x} \in \mathrm{sd}(\mathcal{P})} (-1)^{n(\boldsymbol{x})}.$$

For any G–set X, and $x \in X$, we write $N_G(x)$ for the stabiliser of x in G. When the notation $x \in_G X$ is used to index the terms of a sum, x is understood to run over representatives of the G–orbits of X.

Recall that a *G–poset* is a poset upon which G acts as automorphisms. We write $\mathcal{S}(G)$ for the G–poset consisting of the non–trivial p–subgroups of G, partially ordered by inclusion, and with G acting by conjugation. We write $\mathcal{A}(G)$ for the G–subposet of $\mathcal{S}(G)$ consisting of the non–trivial elementary abelian p–subgroups. A non–trivial p–subgroup P of G is said to be *radical* in G provided $P = O_p\big(N_G(P)\big)$; then we call $N_G(P)$ a *radical normaliser* in G. We say that a chain $(P_0 < \ldots < P_n)$ in $\mathrm{sd}\big(\mathcal{S}(G)\big)$ is *radical* in G provided each P_i is radical in $N_G(P_0 < \ldots < P_{i-1})$ $\big($interpreting the grotesque expression $N_G(P_0 < \ldots < P_{-1})$ to mean $G\big)$. As a weaker condition, we say that $(P_0 < \ldots < P_n)$ is *normal* provided each P_i is normal in P_n. Thus

$$\mathrm{rad}\big(\mathcal{S}(G)\big) \subseteq \mathrm{nor}\big(\mathcal{S}(G)\big) \subseteq \mathrm{sd}\big(\mathcal{S}(G)\big)$$

where the elements of $\mathrm{rad}\big(\mathcal{S}(G)\big)$ and $\mathrm{nor}\big(\mathcal{S}(G)\big)$ are, respectively, the radical and normal chains. The G–sets $\mathrm{rad}\big(\mathcal{S}(G)\big)$ and $\mathrm{nor}\big(\mathcal{S}(G)\big)$ were introduced in Knörr–Robinson [**13**].

Proposition 2.1 (Knörr–Robinson). *Let f be a function defined on the set of subgroups of G, invariant on each conjugacy class of subgroups, and taking values in some abelian group. Then the value of*

$$\sum_{\boldsymbol{P} \in_G X} (-1)^{n(\boldsymbol{P})} f\big(N_G(\boldsymbol{P})\big)$$

is the same for all four choices of

$$X \in \{\mathrm{sd}(\mathcal{A}(G)), \mathrm{sd}(\mathcal{S}(G)), \mathrm{nor}(\mathcal{S}(G)), \mathrm{rad}(\mathcal{S}(G))\}.$$

Proposition 2.1 is given in Knörr–Robinson [**13**]. We must comment briefly on their proof because we shall need to adapt part of it below. Chain–pairing arguments in [**13**, 3.3] show that the three sums with $X \in \{\mathrm{sd}(\mathcal{A}(G)), \mathrm{sd}(\mathcal{S}(G)), \mathrm{nor}(\mathcal{S}(G))\}$ have the same value. As noted in [**13**, page 52], another chain–pairing argument shows that the two sums with $X \in \{\mathrm{nor}(\mathcal{S}(G)), \mathrm{rad}(\mathcal{S}(G))\}$ have the same value. (Details are spelt out in [**2**, 2.2].) Part of Proposition 2.1 was also discovered by Webb, and is illuminated in Thévenaz–Webb [**19**, Theorem 2].

Let $\ell(G)$ denote the number of irreducible p–modular characters of G.

Theorem 2.2 (Knörr–Robinson). *Let $\mathcal{X}$ be a family of isomorphism classes of finite groups which is closed under radical normaliser subgroups. If any one of the following equalities always holds whenever G belongs to $\mathcal{X}$, then all three do.*

$$\ell(G) = z_0(G) + \sum_{P \in_G \mathcal{S}(G)} z_0(N_G(P)/P),$$

$$z_0(G) = \ell(G) - \sum_{\boldsymbol{P} \in_G \mathrm{sd}(\mathcal{S}(G))} (-1)^{n(\boldsymbol{P})} \ell(N_G(\boldsymbol{P})),$$

$$z_+(G) = \sum_{\boldsymbol{P} \in_G \mathrm{sd}(\mathcal{S}(G))} (-1)^{n(\boldsymbol{P})} k(N_G(\boldsymbol{P})).$$

Furthermore, for any positive integer m, the assertion still holds when the equality relation $=$ is replaced by the congruence relation $\equiv_m$.

Theorem 2.2 is a slight generalisation of Knörr–Robinson [**13**, 3.8, 4.5]. The demonstration in [**13**] extends appropriately with little change. Alternatively, arguing as in [**2**, 3.2], Theorem 2.2 follows quickly from [**13**, 4.5] and [**2**, 3.1]. (To dispell any bemusement concerning the hypothesis on $\mathcal{X}$, it is worth pointing out that, in the first of the three sums, if $z_0(N_G(P)/P)$ is non–zero, then P is radical in G, whereupon $z_0(N_G(P)/P)$ is determined by the radical normaliser $N_G(P)$.)

Henceforth, S will always denote a Sylow p–subgroup of G. The proof of the following remark is routine.

Remark 2.3. Given a chain $\boldsymbol{P} \in \mathrm{nor}(\mathcal{S}(G))$, and writing $m_G(\boldsymbol{P})$ for the number of G–conjugates of $\boldsymbol{P}$ belonging to $\mathrm{nor}(\mathcal{S}(S))$, then as p–local integers,

$$m_G(\boldsymbol{P})/|G : N_G(\boldsymbol{P})| \equiv_p 1.$$

Let $\omega(G) := \omega(1, G)$, and let $\Omega(G)$ be the set of pairs (x, y) of elements of G such that $[x, y]$ is a non–trivial p–element. Thus $|G|\omega(G) = |\Omega(G)|$. We seek some "local" way of counting the elements of $\Omega(G)$ (up to congruence modulo $p|G|_p$).

Let $\Gamma(S, G)$ be the subset of $\Omega(G)$ consisting of the pairs (x, y) such that no non–trivial subgroup of S is normalised by both x and y. Let

$$\Delta(S, G) := \Omega(G) - \Gamma(S, G) = \bigcup_{P \in \mathcal{S}(S)} \Omega(N_G(P)).$$

We define $\gamma(G) := |\Gamma(S,G)|/|G|$ and $\delta(G) := |\Delta(S,G)|/|G|$. Then

$$\omega(G) \;=\; \gamma(G) + \delta(G).$$

Note that $\gamma(G)$ and $\delta(G)$ are independent of the choice of S. Since S acts freely on $\Gamma(S,G)$, the rational numbers $\gamma(G)$ and $\delta(G)$ are p–local integers.

Theorem 2.4. *As p–local integers,*

$$z_+(G) + \gamma(G) \;\equiv_p\; \sum_{\boldsymbol{P}\in \mathrm{sd}(\mathcal{S}(G))} (-1)^{n(\boldsymbol{P})} k\big(N_G(\boldsymbol{P})\big).$$

Proof. By Theorem 1.1, the assertion is equivalent to the congruence

$$\delta(G) \;\equiv_p\; -\sum_{\boldsymbol{P}\in \mathrm{sd}(\mathcal{S}(G))} (-1)^{n(\boldsymbol{P})} k\big(N_G(\boldsymbol{P})\big).$$

Given $(x,y)\in\Delta(S,G)$, let $S(x,y)$ denote the unique largest subgroup of S normalised by x and y. The construction of $\Delta(S,G)$ ensures that $S(x,y)$ is non–trivial. We have

$$\big\{\boldsymbol{P}\in \mathrm{sd}\big(\mathcal{S}(S)\big) : \{x,y\}\subseteq N_G(\boldsymbol{P})\big\} \;=\; \mathrm{sd}\big(\mathcal{S}\big(S(x,y)\big)\big).$$

Also, $\mathcal{S}\big(S(x,y)\big)$ has a unique maximal element, so as a simplicial complex, $\mathcal{S}\big(S(x,y)\big)$ is contractible. Therefore

$$\delta(G) = \sum_{(x,y)\in\Delta(S,G)} \chi\Big(\mathcal{S}\big(S(x,y)\big)\Big)/|G| = \sum_{\boldsymbol{P}\in \mathrm{sd}(\mathcal{S}(S))} (-1)^{n(\boldsymbol{P})}\omega\big(N_G(\boldsymbol{P})\big)|N_G(\boldsymbol{P})|/|G|.$$

Adaptations of chain–pairing arguments used to prove Proposition 2.1 show that the indexing set $\mathrm{sd}\big(\mathcal{S}(S)\big)$ may be replaced with $\mathrm{sd}\big(\mathcal{S}(S)\big)\cap \mathrm{nor}\big(\mathcal{S}(G)\big)$. Therefore,

$$\delta(G) \;=\; \sum_{\boldsymbol{P}\in_G \mathrm{nor}(\mathcal{S}(G))} (-1)^{n(\boldsymbol{P})}\omega\big(N_G(\boldsymbol{P})\big)\,|N_G(\boldsymbol{P})|\, m_G(\boldsymbol{P})/|G|.$$

By Theorem 1.1, each $\omega\big(N_G(\boldsymbol{P})\big) \equiv_p -z_+\big(N_G(\boldsymbol{P})\big) = -k\big(N_G(\boldsymbol{P})\big)$. Proposition 2.1 and Remark 2.3 now finish the proof. $\square$

Adapting another chain–pairing argument used to prove Proposition 2.1 gives

$$\delta(G) \;=\; \sum_{\boldsymbol{P}\in_G \mathrm{rad}(\mathcal{S}(G))} (-1)^{n(\boldsymbol{P})}\omega\big(N_G(\boldsymbol{P})\big)\,|N_G(\boldsymbol{P})|\, m_G(\boldsymbol{P})/|G|.$$

For each radical chain $\boldsymbol{P}$, the term $\omega\big(N_G(\boldsymbol{P})\big)\,|N_G(\boldsymbol{P})|\,m_G(\boldsymbol{P})$ depends only on $N_G(\boldsymbol{P})$ and G. It can thence be shown that $\delta(G)$ depends only on the Lefschetz invariant of $\mathcal{S}(G)$, and so depends only on the G–homotopy class of $\mathcal{S}(G)$; see Benson [**4**, Chapter 6] for the terminology and techniques.

Concerning the final result in this section, let us note that a sufficient condition for a finite 2–group P to have a unique maximal elementary abelian subgroup is that P is a direct product of abelian, generalised quaternion, and modular groups Mod_{2^n} with $n \neq 3$. Further examples can be constructed using semidirect products.

Theorem 2.5. *Suppose that $p = 2$, and that a Sylow 2–subgroup of G has a unique maximal elementary abelian subgroup. Then the three equalities in Theorem 2.2 all hold up to parity.*

Proof. Let $\Gamma := \Gamma(S, G)$. By Theorem 2.4, it suffices to show that $2|S|$ divides $|\Gamma|$. Let α and β be involutions in D_8 such that $\alpha\beta$ has order 4, and let D_8 act on Γ such that $\alpha(x, y) = (y, x)$ and $\beta(x, y) = (x, y^{-1})$ for all $(x, y) \in \Gamma$. Since the conjugation action of S on Γ is free, we may assume, for a contradiction, that some element $(x, y) \in \Gamma$ lies in an $S \times D_8$–orbit of order $|S|$. By the hypothesis on G, the stabiliser of (x, y) in $S \times D_8$ must intersect non-trivially with $1 \times D_8$. But $x \notin \{y, y^{-1}\}$ because $[x, y] \neq 1$, hence $(x, y) \in \{(x^{-1}, y), (x, y^{-1})\}$. Also, each image of (x, y) under D_8 is S–conjugate to (x, y). Therefore x and y are both involutions. Writing 2^n for the order of the element $(xy)^2 = [x, y]$, then x and y generate a dihedral group of order 2^{n+2}. This contradicts the hypothesis on G. □

3. Möbius Inversion

The proof in Knörr–Robinson [**13**] of Theorem 2.1 is an inversion argument based on the G–poset $\mathcal{S}(G)$. Some of the proof is recast in [**2**] as a Möbius inversion argument on a directed multigraph. However, it is not a Möbius inversion argument on $\mathcal{S}(G)$. Here, we combine Theorem 1.1 with a Möbius inversion argument on $\mathcal{S}(G)$, obtaining local information about $z_+(G)$ which appears to be independent of the results and conjectures in [**13**]. The following explicit formula for Möbius inversion in a family of p–groups is an immediate consequence of Kratzer–Thévenaz [**14**, 2.4]; see also Hall [**10**, 1.4], [**11**, Section 3].

Theorem 3.1 (Kratzer–Thévenaz). *Let $\mathcal{P}$ be a family of p–groups such that all chains (of inclusions) in $\mathcal{P}$ are of finite length, and whenever $P \leq R \leq Q$ with $P, Q \in \mathcal{P}$, then $R \in \mathcal{P}$. Let α and β be two functions on $\mathcal{P}$ taking values in some abelian group.*

(1) *The following two identities, taken over all $P \in \mathcal{P}$, are equivalent:*

$$\alpha(P) = \sum_{Q \in \mathcal{P} : Q \leq P} \beta(Q),$$

$$\beta(P) = \sum_{r \geq 0} (-1)^r p^{r(r-1)/2} \sum_{Q \in \mathcal{P} : P/Q \cong C_p^r} \alpha(Q),$$

where the notation $Q \in \mathcal{P} : P/Q \cong C_p^r$ indicates that Q runs over the normal subgroups of P in $\mathcal{P}$ such that P/Q is elementary abelian of rank r.

(2) *The following two identities, for $P \in \mathcal{P}$, are equivalent:*

$$\alpha(P) = \sum_{Q \in \mathcal{P} : Q \geq P} \beta(Q),$$

$$\beta(P) = \sum_{r \geq 0} (-1)^r p^{r(r-1)/2} \sum_{Q \in \mathcal{P} : Q/P \cong C_p^r} \alpha(Q).$$

I cannot resist briefly digressing to show how Theorem 3.1 provides a very short proof of Virág's recent generalisation [**20**] of Sylow's Theorem.

Theorem 3.2 (Virág [**20**]). *Let $P \le G$ with $|P| = p^b$, and let $a \ge b$ with p^a dividing $|G|$. Then the number of subgroups Q of G containing P with $|Q| = p^a$ is congruent to unity modulo p.*

Proof. The case where $b = 0$ and $a = 1$ has long been known, indeed, it was essentially known to Cauchy, and anyway, is an immediate consequence of Frobenius' theorem (again) in Curtis–Reiner [**7**, 41.11]. In general, for a p–subgroup U of G, define $\beta(U) = 1$ if $|U| = p^a$, and $\beta(U) = 0$ otherwise. Let α be as in Theorem 3.1 (2). By induction on $p^a/|U|$, and Cauchy's case above, $\alpha(U) \equiv_p 1$ when $|U| \le p^a$. □

Let $\Gamma'(S, G)$ be the subset of $\Omega(G)$ consisting of the pairs (x, y) such that no non–trivial element of S is centralised by both x and y. Let

$$\Delta'(S,G) := \Omega(G) - \Gamma'(S,G) = \bigcup_{P \in \mathcal{S}(S)} \{(x,y) \in \Omega(G) : P = C_S(x) \cap C_S(y)\}.$$

Much as in Section 2, we define p–local integers $\gamma'(G) := |\Gamma'(S,G)|/|G|$ and $\delta'(G) := |\Delta'(S,G)|/|G|$. Of course,

$$\omega(G) = \gamma'(G) + \delta'(G).$$

Given a finite elementary abelian p–group P, let $r(P)$ denote the rank of P. Given a subgroup $H \le G$, let $\mathrm{out}_G(H)$ denote the index of the canonical image of $N_G(H)$ in the automorphism group $\mathrm{Aut}(H)$. Thus

$$\mathrm{out}_G(H)\,|N_G(H)| = |\mathrm{Aut}(H)|\,|C_G(H)|.$$

Proposition 3.3. *As p–local integers,*

$$z_+(G) + \gamma'(G) \equiv_p \sum_Q (-1)^{r(Q)+1} k\big(C_G(Q)\big)/|N_G(Q) : C_G(Q)|_{p'}$$

where Q runs over representatives of the G–conjugacy classes of non–trivial elementary abelian p–subgroups of G such that $\mathrm{out}_G(Q)$ is coprime to p.

Proof. For any subgroup $P \le S$ put

$$\alpha(P) := \omega\big(C_G(P)\big)|C_G(P)| = |\{(x,y) \in \Omega(G) : P \le C_S(x) \cap C_S(y)\}|,$$
$$\beta(P) := |\{(x,y) \in \Omega(G) : P = C_S(x) \cap C_S(y)\}|.$$

By Theorem 3.1 (2),

$$\delta'(G)|G| = \alpha(1) - \beta(1) = \sum_{Q \in \mathcal{A}(S)} (-1)^{r(Q)+1} p^{r(Q)\left(r(Q)-1\right)/2} \alpha(Q)$$
$$= \sum_{Q \in \mathcal{A}(G)} (-1)^{r(Q)+1} \mathrm{out}_G(Q)_p |N_G(Q)|_p |C_G(Q)|_{p'}\, m_G(Q)\, \omega\big(C_G(Q)\big).$$

Theorem 1.1 and Remark 2.3 give

$$\delta'(G) \equiv_p \sum_Q (-1)^{r(Q)} k\big(C_G(Q)\big)/|N_G(Q) : C_G(Q)|_{p'}$$

where Q runs as in the assertion. Applying Theorem 1.1 again completes the argument. □

A sufficient condition for all the involutions in a finite 2–group P to be central is that P is the direct product of abelian and generalised quaternion groups. Again, further examples may be constructed using semidirect products.

Theorem 3.4. *Suppose that $p = 2$, and that a Sylow 2–subgroup S of G has a unique maximal elementary abelian subgroup. Then $z_+(G)$ has the same parity as the number of conjugacy classes of elementary abelian 2–subgroups Q of G such that $k\big(C_G(Q)\big)$ and $\mathrm{out}_G(Q)$ are both odd. In particular, if all the involutions in S are central, then $z_+(G)$ has the same parity as the number of conjugacy classes of involutions g in G such that $k\big(C_G(g)\big)$ is odd.*

Proof. We are to show that

$$z_+(G) \equiv_2 \sum_Q k\big(C_G(Q)\big)$$

where Q runs as in Proposition 3.3. Let $\Gamma := \Gamma'(S, G)$. We must show that $2|S|$ divides $|\Gamma|$. The demonstration now proceeds exactly as in the proof of Theorem 2.5. □

Examples for which $\gamma'(G)$ is coprime to p abound when $p \geq 5$. Thus the arguments in this section seem less susceptible to improvement than those in Section 2. However, I have been unable to find any such examples when $p = 2$ or $p = 3$, and in view of Proposition 4.1 below, propose that Theorem 3.4 is a "combinatorial shadow" of some general equality of sums of squares.

4. Formulas for the Number of Defect–Zero Irreducible Characters

This appendant section, based on material in [**3**], Broué [**5**], Broué–Robinson [**6**], Robinson [**15**] bears on the material above only in the suggestive sense discussed in the previous paragraph. Following Broué [**5**], we define a function ψ_p on $G \times G$ such that $\psi_p(g, h)$ is the number of elements $z \in G$ for which $gzh^{-1}z^{-1}$ is a p–element. Let $\psi(g, h)$ be the number of solutions in $x, y, z \in G$ to the equation $gzh^{-1}z^{-1} = [x, y]$. We define $\Psi_p(G) := \big(\psi_p(g, h)\big)_{g,h}$ and $\Psi(G) := \big(\psi(g, h)\big)_{g,h}$ as matrices indexed by representatives of the conjugacy classes of G. Parts of the following result — as indicated in the proof — are due to Broué and Robinson.

Proposition 4.1.

(a) *$z_0(G)$ is the p–rank of $\Psi_p(G)$.*

(b) *$z_0(G)$ is the p–rank of $\Psi(G)$.*

(c) *$\psi_p(g, h) \equiv_p \sum_\chi \chi(g)\chi(h^{-1})$ where $g, h \in G$, and χ runs over the defect–zero irreducible characters of G.*

(d) $\psi(g, h) = \sum_{\chi} \big(|G|/\chi(1)\big)^2 \chi(g)\chi(h^{-1})$ *where* χ *runs over all the irreducible characters of* G.

(e) *If* $p = 2$ *or* $p = 3$, *then* $\Psi_p(G) \equiv_p \Psi(G)$.

Proof. Parts (a), (b), (d) are special cases of Broué–Robinson [**6**, 1.15], [**3**, 6(a)], [**3**, 3(b)], respectively. Part (e) will be immediate from (c) and (d). (We also note that part (a) may quickly be recovered from (c) by considering the basis of ZKG consisting of all the primitive idempotents e_χ.) Finally, to prove part (c), we persue a line of reasoning from Broué–Robinson [**6**, page 385]. Let χ_p be the permutation character of G afforded by the cosets of the Sylow p–subgroup S. Then

$$|G : S|\psi_p(g, h) \equiv_p \chi_p(\sum_{z \in G} gzh^{-1}z^{-1})$$

because $\chi_p/|G : S|$ is congruent to the characteristic function on the subset of G consisting of the p–elements. By considering the central character associated with any irreducible character χ of G, we obtain

$$\chi_p(\sum_{z \in G} gzh^{-1}z^{-1}) = |G| \sum_{\chi} \langle \chi, \chi_p \rangle \chi(g)\chi(h^{-1})/\chi(1)$$

where χ runs over all the irreducible characters of G. If χ has defect–zero, then it vanishes on the non–trivial p–elements and therefore has multiplicity $\chi(1)/|S|$ in χ_p. Therefore

$$\chi_p(\sum_{z \in G} gzh^{-1}z^{-1}) \equiv_p |G : S| \sum_{\chi} \chi(g)\chi(h^{-1})$$

where χ now runs over all the defect-zero irreducible characters of G. □

When the congruence in (e) holds, the formulas (a) and (b) for $z_0(G)$ of course coincide; the congruence rarely holds when $p \geq 5$.

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