MARKOV DECISION PROCESS FORMULATIONS FOR MANAGEMENT OF PUMPED HYDRO ENERGY STORAGE SYSTEMS

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By

Parinaz Toufani

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

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Renewable energy sources have received much attention to mitigate the high dependence on fossil fuels and the resulting environmental impacts. Since the variability and intermittency of such renewable sources lower the reliability and security of energy systems, they should often be accompanied by efficient and flexible storage units. This dissertation focuses on pumped hydro energy storage (PHES) facilities, which are one of the most commonly used large-scale storage technologies. We study the energy generation and storage problem for PHES facilities with two connected reservoirs, where water is pumped from the lower reservoir to the upper reservoir to store energy during low-demand/low-electricity price periods, and released back to the lower reservoir to generate energy during high-demand/high-electricity price periods. The first part of this dissertation investigates the potential benefits of transforming conventional cascading hydropower stations into PHES facilities by replacing turbines with reversible ones. The second part compares the short-term cash flows obtained from different PHES configurations (cascading vs. non-cascading facilities, upstream vs. downstream inflows, and closed-loop facilities). We formulate both problems as Markov decision processes under uncertainty in the streamflow rate and electricity price. We include the streamflow rate and electricity price as exogenous state variables in our formulation. We analytically derive bounds on the profit improvement obtained from PHES transformation in the first part and bounds on the revenue differences obtained from different configurations in the second part. In the last part, we establish several structural properties of the optimal profit function for general two-reservoir PHES systems. We show the optimality of a state-dependent threshold policy for non-cascading PHES facilities when the electricity price is always positive. Leveraging our structural results, we construct a heuristic solution method for more general settings when the electricity price can also be negative. In this dissertation, we also conduct comprehensive numerical experiments with data-calibrated time series models to provide insights into the optimal operation of PHES facilities, considering distinct seasons with different streamflow rates, different negative electricity price occurrence frequencies, and different system parameters.

Keywords: Markov decision process, negative electricty price, pumped hydro energy storage, renewable energy sources, state-dependent treshold policy, uncertainty.

ÖZET

POMPAJ DEPOLAMALI HİDROELEKTRİK SANTRALLERİN YÖNETİMİNE İLİŞKİN MARKOV KARAR SÜRECİ FORMÜLASYONLARI

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Yenilenebilir enerji kaynakları, fosil yakıtlara olan yüksek bağımlılığı ve bunun sonucunda ortaya çıkan çevresel etkileri azaltmak için büyük ilgi görmektedir. Yenilenebilir kaynakların değişkenliği ve kesintili olması, enerji sistemlerinin güvenilirliğini azalttığından, bu kaynaklara genellikle verimli ve esnek depolama birimleri eşlik etmelidir. Bu çalışma, büyük ölçekli enerji depolama teknolojilerinden yaygın olarak kullanılan, pompaj depolamalı hidroelektrik santrallerine (PHES) odaklanmaktadır. Bu çalışmada, düşük talep yada düşük elektrik fiyatı dönemlerinde enerji depolamak için suyun alt rezervuardan üst rezervuara pompalandığı ve yüksek talep yada yüksek elektrik fiyatı dönemleri sırasında enerji üretmek için susyun alt rezervuara geri bırakıldığı bağlantılı iki rezervuara sahip PHES sistemleri için enerji üretim ve depolama problemi ele alınmıştır. Çalışmanın ilk kısmı, tersinir türbinler kullanarak konvensiyonel ardışık hidroelektrik santralleri PHES sistemlerine dönüştürmenin ve enerji depolamanın katma değerini araştırmaktadır. İkinci kısım, yukarı ve aşağı su akışlı, ardışık olan ve olmayan, açık döngülü ve kapalı döngülü PHES sistemler üzere olmak farklı PHES konfigürasyonlarından elde edilen kısa vadeli nakit akışlarını karşılaştırır. Her iki problem için, sisteme giren su miktarı ve elektrik fiyatındaki belirsizlikleri göz önünde bulundurularak, enerji üretim ve depolama kararlarını doğru ve etkin bir şekilde ele alan Markov karar süreçleri geliştirilmiştir. Su akışı ve elektrik fiyatı formülasyona dışsal durum değişkenleri olarak dahil edilmektedir. İlk çalışmada PHES dönüşümünden elde edilebilecek kâr artışının sınırı ve ikinci çalışmada ise farklı PHES konfigürasyonlarında elde edilen kazançlar arasındaki farkın sınırları analitik olarak sunulmaktadır. Son bölümde, farklı PHES konfigürasyonları için optimal kâr fonksiyonunun birkaç yapısal özelliği gösterilmektedir. Bu yapısal özellikler kullanılarak, elektrik fiyatı her zaman pozitif olduğunda ardışık olmayan PHES tesisleri için duruma bağlı bir eşik politikası ve ardışık olan PHES tesisler veya elektrik fiyatının negatif olabileceği daha genel durumlar için sezgisel bir çözüm yöntemi geliştirilmektedir. PHES tesislerinin optimum operasyonu hakkında fikir vermek için veri kalibreli zaman serisi modelleriyle farklı mevsimler, farklı su akış miktarları, farklı negatif elektrik fiyatı oluşum sıklığı ve farklı sistem parametreleri için kapsamlı sayısal deneyler yürütülmektedir.

Anahtar sözcükler: Markov karar süreci, negatif elektrik fiyat, pompaj depolamalı hidroelektrik santral, yenilenebilir enerji kaynaklar, duruma bağlı bir eşik politika, belirsizlik.

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Chapter 1

Introduction

In recent years, there has been a growing interest in operating electrical systems in a more efficient, reliable, and environmentally-friendly manner, which has increased the need for electrical energy storage. Prior to integrating a specific storage unit with electrical systems, several concerns should be addressed, such as whether it is better than alternatives in terms of costs and performance and how well energy sources are utilized in the presence of storage systems. In case energy storage systems are successfully integrated, they have an undeniable contribution in improving the use of existing generation and transmission sources by increasing the market value and availability of distributed sources [1]. Storage facilities provide flexibility to respond peak values of demand as well as the price shocks when there exist irregularities in the market. They also help address the environmental regulations by increasing the use of renewable energy sources. Although clean and renewable energy sources have received a lot of attention in today's modern world as a hedge against the significant dependence on fossil fuels and the resulting environmental issues, the variability and intermittent nature of these resources reduce the energy system's reliability. This is one of the primary reasons why they should be accompanied by mature, responsive, and flexible storage systems. Common storage techniques used for electricity generation are batteries, compressed-air energy storage, super-capacitor energy storage, hydrogen energy storage, and pumped hydro energy storage (PHES).

PHES is one of the most widely used bulk electrical energy storage (EES) technologies worldwide (accounting for 99% of the existing EES capacity) and has been used for over a century to assist in load balancing in the electricity industry. The installed capacity of PHES is expected to increase from 161 GW in 2018 to 300 GW in 2030 and 325 GW by 2050 [2]. However, the popularity of PHES stems not only from its historical use, but also from its technical, economic, and environmental benefits. PHES can provide cost-effective, large-scale, and long-term grid energy storage, as well as the added benefit of freshwater storage capacity. PHES uses the potential energy of water by exchanging the flow between two reservoirs located at different altitudes. Typically, PHES stores gravitational potential energy by pumping water from a lower reservoir to an upper reservoir (charging mode) during periods of low demand and excess power generation. The stored water in the upper reservoir is released (discharging mode) in order to produce power during the periods of high demand and supply shortfall or high prices. That is, PHES is one of the prominent storage types in smoothing the fluctuations arising from the demand side (e.g., demand variations and market price shocks) as well as from the supply side (e.g., uncertainties due to renewable energy generation). Currently, PHES is considered to be the most mature platform available with promising round-trip efficiencies (70% to 80%), operational flexibility to quickly respond to extreme load changes (minutes to seconds), long useful lifetimes (between 50 and 100 years), and low operation and maintenance costs [3, 4]. PHES facilities can be classified into two categories of closed-loop and open-loop, depending on whether they are connected to any streamflow like rivers. Closed-loop PHES facilities have natural or artificial reservoirs with no connection to any streamflow. On the other hand, the upper and/or lower reservoirs of the open-loop PHES facilities are fed by natural streamflows [5].

Driven by the significance role of PHES facilities in the electricity industry, in this dissertation, we study the energy generation and storage problem for a variety of realistic configurations of PHES facilities. The optimal energy generation and storage policies for PHES facilities are required to adequately capture the problem's stochastic nature. As a result, we employ Markov decision processes (MDP) in order to analyze this complex problem that includes sequential decision making under uncertainty arising from multiple sources such as renewable power plants and energy markets. We take a data-driven approach in order to model the multi-dimensional uncertainties and calibrate the problem parameters. We also examine the role geographical conditions and system configurations play in profitability of PHES facility.

The rest of this dissertation is organized as follows. In Chapter 2, we present the related literature on optimization of PHES facilities with an emphasis on uncertainty. In Chapter 3, we evaluate the potential benefit of retrofitting existing conventional cascading hydropower stations (CCHSs) with reversible turbines so as to operate them as PHES facilities. We examine the energy generation and storage problem for a CCHS with two connected reservoirs that can be transformed into a PHES facility in a market setting where the electricity price can be negative. We formulate this problem as an MDP under uncertainty in the streamflow rate and electricity price. We analytically derive an upper bound on the profit improvement that can be obtained from the PHES transformation. We also conduct numerical experiments with data-calibrated time series models and observe that the PHES facility provides a greater benefit under more limited streamflow conditions or more frequently observed negative electricity prices.

In Chapter 4, we study the energy generation and storage problem for various types of two-reservoir PHES facilities: cascading systems with the upstream or downstream flow, non-cascading systems with the upstream or downstream flow, and closed-loop facilitates. We provide the MDP formulation of these configurations including the streamflow rate and electricity price uncertainty in our formulation. We compare the short-term total cash flows obtained by running different PHES configurations in a market setting where the electricity price can be negative. We first derive theoretical bounds on the revenue gains obtained from different PHES configurations. We then conduct comprehensive numerical experiments by employing time-series models to formulate the evolution of our exogenous state variables (streamflow rate and electricity price). We consider three distinct seasons with different streamflow rates, different negative price occurrence frequencies, and different reservoir capacities. Our results show that: (1) The open-loop facility with the upstream flow can yield cash flows that are up to four times as large as those of the closed-loop facility; (2) The cash flow from operating a large closed-loop facility can be achieved by operating an open-loop facility with 10 times smaller reservoirs; and (3) The open-loop facility with the downstream flow can be more advantageous than the open-loop facility with the upstream flow (with an improvement of more than 10% in the cash flow) if the negative electricity price occurs more than 30% of the time.

In Chapter 5, we establish several structural properties of the optimal profit functions for different PHES configurations we consider in Chapter 4, when the electricity price is strictly positive throughout the finite horizon. For non-cascading PHES facilities, we show that the optimal energy generation and storage policy can be specified as a state-dependent threshold policy when the electricity price is always positive. Inspiring by the derived structural knowledge, we also develop a heuristic solution method that can be usefully employed in a more general setting where the configuration can also be cascading and the electricity price can be negative. We evaluate the performance of our heuristic method for settings with different negative electricity price frequencies and seasons. Based on our results, for non-cascading configuration the heuristic method yields near-optimal solutions with a maximum distance of only 1.12% from the optimal profit, and reduces the computation time of the optimal algorithm by 50.35% on average and by up to 56.47%. For the cascading configuration, we observe higher optimality gap percentages (with a maximum distance of 40.21%), but with a higher computation time reduction of 79.2% on average and up to 82.3%. We also observe that the performance of our heuristic method for the cascading configuration is positively impacted by an increase in the occurrence frequency of negative prices.

In Chapter 6, we provide a summary of concluding remarks and some future research directions and possible extensions of present work.

Chapter 2

Literature Review

There is a vast amount of literature on the modeling and optimization of PHES in exemplary or real-world power systems. There are also numerous review articles focusing on PHES. Several review articles provide an overview of energy storage technologies, including but not limited to PHES (see, for example, [6, 7] and [8]). Some PHES-review articles focus on the status of PHES development (see, for example, [9, 10]). Some other articles summarize and classify the drivers of and barriers to PHES development (see, for example, [11, 12]). Several studies examine the installed PHES generation capacity as well as the potential locations and timelines for new PHES development (see, for example, [13]). The results of such reviews can be a useful aid for researchers and practitioners seeking ways to better allocate available resources to promote PHES development in the future. Some articles consider PHES operations in various electricity markets (see, for example, [14, 15]). Some other articles focus on certain physical aspects of PHES facilities, such as the power converter topologies of variable-speed PHES and hydropower plants (see, for example, [16, 17]). Rehman et al. [18] provide a comprehensive review on the analysis of PHES conditions and operations. They examine the technical advances and capabilities of PHES as well as its potential integration with renewable sources. Javed et al. [19] review the key challenges with PHES implementation in hybrid power systems (HPS), taking into account the economic, environmental, and technical aspects of solar-wind-PHES systems.

In a recent study, Mahfoud et al. [20] provide a detailed review on the optimal operation of PHES-based energy systems.

Incorporating uncertainty into energy systems planning is needed to provide a secure, reliable, and affordable energy supply. The role of uncertainty is also critical for a variety of services PHES facilities can offer: (i) assisting in the integration of renewable energy into power systems by acting as a backup source that serves as a hedge against the intermittency of renewable outputs and by reducing the amount of renewable curtailment; (ii) providing energy arbitrage benefits by responding effectively to the price changes in electricity markets (i.e., storing electricity when the price is low and selling it back to the grid when the price is high); and (iii) providing ancillary services such as frequency control, black start, and power regulation. It is important to consider supply (i.e., renewable energy sources), demand, and price uncertainties in modeling and analyzing these services of PHES. This complex analysis does, in fact, involve optimization under uncertainty. There are valuable reviews of the uncertainty modeling techniques for decision-making in broadly defined energy systems in [21–23]. To our knowledge, however, there is no review of the optimization problems for PHES systems, as a specific energy storage option, with an emphasis on uncertainty. This chapter aims to fill this gap and reviews the literature dealing with the sizing and/or operational problems for PHES systems *under uncertainty*. Specifically, this review highlights the importance of considering uncertainty in PHES optimization and provides timely insights into this critical area. Although this review focuses on the uncertainty aspect of the problem, it also unravel several other major aspects relevant to the modeling of PHES facilities in the literature. At the end of this chapter, we highlight several research directions to draw the attention of the operations research community to the needs and challenges in PHES optimization yet to be addressed.

2.1 Review methodology

The primary source for our review process is the Web of Science database. We searched for "pumped storage*" AND optim* AND (stochastic OR uncertain* OR random) as well as "pumped hydro*" AND optim* AND (stochastic OR uncertain* OR random) to find relevant studies. We restricted our search to English-language journal articles during the review process, including all articles published between 2000 and January 2022. We also included additional articles based on the reference lists of the relevant studies found and our previous research experience to maximize the scope of our review. After a preliminary evaluation of 140 articles by title and abstract, we identified 125 articles to review in detail and classify according to problem type, energy sources paired with PHES, physical characteristics of PHES, uncertain system components, and solution methodology. Figure 2.1 illustrates the number of articles included in our review that were published in scientific journals each year. Figure 2.2 shows the percentage breakdown of the articles into the scientific journals, restricted to those journals in which at least four articles in our review were published (51% of the reviewed articles are from this list of journals).

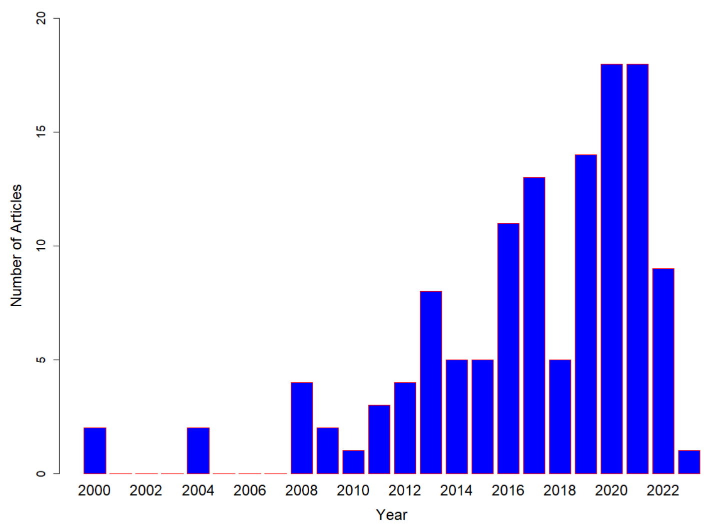

Figure 2.1: Number of articles per year.

The rest of this chapter is organized as follows: Chapter 2.2 classifies the articles based on the problem definition and objective. Chapter 2.3 classifies

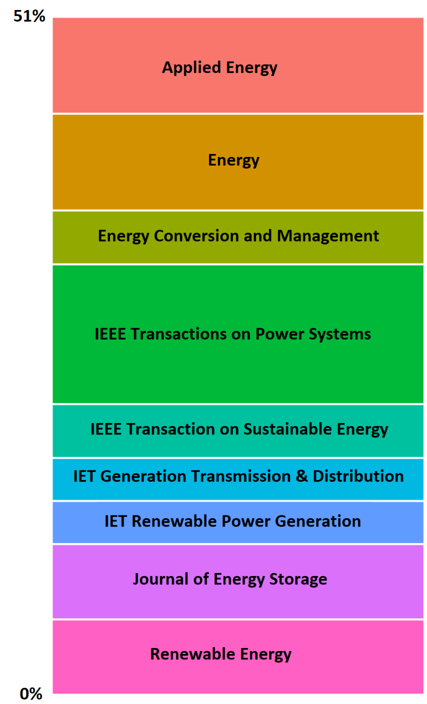

Figure 2.2: Percentage of articles per journal.

the articles based on PHES characteristics (configuration, pump/turbine, head, efficiency, and case study). Chapter 2.4 discusses the energy sources coupled with PHES facilities. Chapter 2.5 discusses the sources of uncertainty that typically arise in PHES optimization. Chapter 2.6 presents the approaches used to model such uncertainties. Chapter 2.7 presents the solution methodologies adopted in PHES optimization. Chapter 2.8 concludes by highlighting the key research gaps.

2.2 Classifications of the PHES-optimization problems

According to Frangopoulos [24], the optimization problems in the energy systems literature are broadly concerned with three stages: synthesis, design, and operation. Our detailed reading of the articles suggests that we can categorize the optimization problems for PHES systems under uncertainty based on (i) problem definition and (ii) problem objective.

2.2.1 Classification based on the problem definition

We first broadly categorize the PHES problems under uncertainty into two groups based on the problem definition: sizing problems and operational problems. The first group includes studies aiming to find the optimal sizing and configuration of PHES facilities, usually in HPS. The storage system should be configured throughout the design of HPS to ensure a reliable and cost-effective energy supply. The optimal sizing of a PHES facility is critical in recovering discarded generated energy for a balanced distribution while dealing with seasonal shortages or excesses of renewable sources. The second group includes studies aiming to find the optimal energy generation and storage policies for PHES facilities in HPS. In general, the joint optimization of other sources in HPS with PHES entails complex and sequential decision making over a planning horizon with an arbitrary number of periods. The typical operational decisions in each period involve the amount of energy that should be sold to or purchased from the market, the amount of energy that should be used for pumping water, the amount of energy that should be generated by releasing water, the amount of renewable energy that should be curtailed, and the amount of water that should be spilled from the PHES reservoirs. Figure 2.3 presents a Venn diagram with the numbers of studies that fall into these two categories. Our review reveals that only a few studies (10 out of 125 articles) focus on the sizing problems, the vast majority deal with the operational problems and only two studies address both problem types via a single optimization framework, as can be seen from the intersection area.

We can also classify the articles according to the length of planning horizon considered in optimization: short-term (from hours to days), medium-term (from months to a year), and long-term (over a year). Table 2.1 exhibits these categories. Two articles consider both short- and long-term planning horizons. This is the case in [25] and [26]. Only one article considers both medium- and long-term horizon [27]. This classification reveals that approximately 77% of the studies optimize their problems for short-term planning horizons.

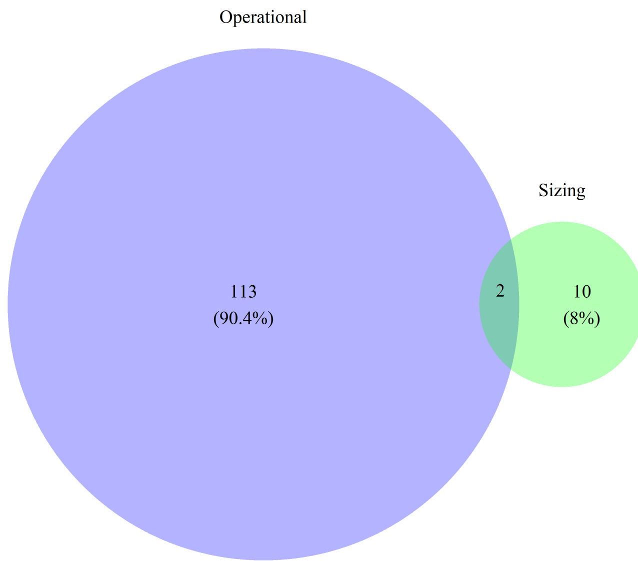

Figure 2.3: Classification based on the problem definition.

2.2.2 Classification based on the problem objective

The majority of the studies in the literature concentrated on energy generation/storage planning and reliability of hybrid systems, including PHES facilities. Some studies compared various storage options to demonstrate the validity of proposed hybrid configurations using techno-economic analysis. In particular, the objectives studied in the literature can be classified into the following [19]:

- Reliability: proposing different approaches to ensure system reliability by mitigating the variability of renewable energy sources.
- Techno-economic analysis: demonstrating the importance of proposed configurations technically and economically by comparing them with other options.
- Feasibility analysis: proposing feasibility studies for incorporating PHES into power generation systems.
- Performance evaluation: conducting comparative performance analysis of different HPS, including a PHES facility.

Table 2.1: Classification based on the length of the planning horizon used in optimization.

Time Horizon	References
Short-term	[28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], [56], [57], [58], [59], [60], [61], [62], [63], [64], [65], [66], [67], [68], [69], [70], [71], [72], [73], [74], [75], [76], [77], [78], [79], [80], [81], [82], [83], [84], [85], [86], [87], [88], [89], [90], [91], [92], [93], [94], [95], [96], [97], [98], [99], [100], [101], [102], [103], [104], [105], [106], [107], [108], [109], [110], [111], [112], [113], [114], [115], [116], [117], [118], [119], [120], [121], [122], [123]
Medium-term	[124], [125], [126], [127], [128], [129], [130], [131], [132], [133], [134], [135], [136], [137], [138], [139], [140], [141]
Long-term	[142], [143], [144], [145], [146], [147], [148], [149]

- Energy management and scheduling: proposing energy management strategies for the operation of HPS, including a PHES facility.

We can also categorize the problem objectives based on whether they are single-objective (involving only one objective function to be optimized) or multi-objective (involving more than one objective function to be optimized simultaneously) and whether they are minimization or maximization. Table 2.2 shows the global number of reviewed articles within each category. The majority of the articles consider single-objective optimization problems. Among them, the most commonly used objectives are maximizing the profit and minimizing the imbalance costs in electricity markets. 35 of 125 articles study energy commitment decisions in electricity markets. The majority of these articles optimize day-ahead market commitment decisions ([30–32], [35], [37], [38], [44], [49], [51–54], [57], [58], [61], [92], [93], [95], [102], [104], [110], [111], [113], [116], and [128]), four articles consider ancillary market operations ([63], [68], [71], and [130]), and only three articles optimize both day-ahead and intraday market commitment decisions ([66], [118], and [132]). PHES can provide several different services such as backup power for renewable sources and energy arbitrage [150]. PHES is mostly integrated with renewable sources, particularly wind energy, as a backup source to reduce possible energy imbalances in electricity markets. PHES can also be operated standalone in various electricity markets as a generator or an electricity consumer. This is the case in [63], [102], [104], [113], [118], [128], [130], and [132]. Only two of them ([128] and [113]) use PHES for arbitrage purposes.

We should note that some studies have objective structures that do not fall into any of the categories in Table 2.2. For example, Ntomaris and Bakirtzis [32] develop a bi-level optimization model in which the upper level identifies the optimal energy offers by maximizing the profit, while the lower level specifies how these energy offers should be dispatched by minimizing the production and wind spillage costs. Similarly, Alharbi and Bhattacharya [102] propose a bi-level model with the upper level maximizing the profit and the lower level maximizing social welfare. Ding et al. [66] propose a rolling optimization methodology for the operations of wind-PHES systems in day-ahead, intra-day and real-time markets. Castronuovo et al. [51] consider three sub-problems for a wind farm paired with a PHES facility: (i) the wind energy is stored in the PHES facility in low-price periods to sell it in high-price periods to maximize revenue; (ii) the PHES facility is used as a reserve option for deviations in wind generation to maximize profit; and (iii) unlike the previous two problems, the PHES facility is used to minimize regulation costs (thus increase revenue), but the wind energy trading is ignored. Ding et al. [39] construct three formulations for a wind farm paired with a PHES facility: a deterministic mixed integer programming formulation (where the wind uncertainty is ignored), a chance-constrained formulation, and a scenario-based formulation, all with the objective of maximizing the profit. Yahia and Pradhan [78] examine the PHES scheduling problem with simultaneous and sequential stochastic models. The simultaneous model minimizes the maintenance costs and demand-supply disparity, whereas the sequential model minimizes the demand-supply disparity and the total number of pump switches in the first and second stages, respectively. Finally, some articles have the objective of evaluating the reliability and feasibility of their proposed energy systems with PHES. This is the case in [76], [91], and [145].

Table 2.2: Classification based on the problem objective.

Type of Optimization Problem		Performance Measure	References
Single Objective	Max	Profit	[25], [35], [37], [38], [43], [46], [49], [52], [53], [54], [56], [57], [58], [60], [61], [63], [65], [68], [92], [93], [95], [96], [101], [104], [106], [110], [111], [112], [113], [118], [124], [127], [128], [129], [130], [131], [132], [134], [137], [139], [147]
		Revenue	[30], [36], [59], [100], [120], [121], [122], [133]
		Others	[31], [136], [142], [144]
	Min	Cost	[27], [28], [29], [33], [34], [40], [42], [44], [45], [47], [50], [55], [62], [64], [67], [71], [75], [77], [79], [80], [81], [82], [83], [84], [85], [86], [87], [88], [89], [90], [94], [97], [99], [103], [107], [108], [109], [114], [116], [117], [119], [125], [126], [135], [138], [140], [143], [146], [148]
		Others	[26], [69], [115]
Multi Objective	Max/Max/Max	Reliability/economic benefit/water power utilization	[70]
		Profit/available efficiency index	[73]
	Max/Min	Economic benefit/power output fluctuation	[41]
		Reliability/generation and storage technology investment	[149]
	Min/Min	Operating cost/energy not served	[48]
		Probability of load loss/operating cost	[72]
		Emissions/cost (generation, operation, purchase,...)	[74], [98], [141]
		Fluctuations/abandoned rate of wind and hydropower	[105]
		Operating cost/curtailment and water spillage	[123]

2.3 Classification based on the PHES characteristics

2.3.1 Configuration of the PHES facility

The design of the PHES systems is severely constrained by site characteristics, such as whether sufficient access to water sources exists and whether favorable topographical and geographical conditions exist. Their operational designs can be classified into two types: open-loop PHES facilities (58% of the articles), which are constantly connected to a flowing water source, and closed-loop PHES facilities (42% of the articles), which are isolated from any water source. Table 2.3 and Figure 2.4 illustrate the breakdown of the reviewed articles into these PHES configurations. In nearly half of the studies that consider the open-loop PHES facilities, the upper reservoir has a certain capacity limit, but the lower reservoir has no specified capacity limit. Since the lower reservoir is regarded as a sea with infinite capacity in these studies, their PHES facilities are considered open-loop. Streamflow is considered stochastic in 30.1% of the articles that include it as a component in the optimization of the open-loop PHES facilities, while it is assumed to be deterministic in 23.3%.

Table 2.3: Classification based on the PHES configuration.

PHES Configuration	References
Open-loop	[26], [28], [29], [37], [38], [39], [42], [43], [45], [46], [51], [52], [53], [56], [57], [59], [60], [62], [67], [69], [70], [73], [75], [77], [81], [82], [83], [85], [86], [87], [88], [89], [91], [92], [93], [94], [97], [98], [99], [100], [101], [104], [105], [106], [107], [108], [110], [113], [116], [117], [118], [121], [123], [125], [127], [128], [129], [131], [132], [133], [134], [136], [138], [139], [140], [141], [143], [144], [146], [148], [149]
Closed-loop	[25], [27], [30], [31], [32], [33], [34], [35], [36], [40], [41], [44], [47], [48], [49], [50], [54], [55], [58], [61], [63], [64], [65], [66], [68], [71], [72], [74], [76], [78], [79], [80], [84], [90], [95], [96], [102], [103], [109], [111], [112], [114], [115], [119], [124], [126], [130], [135], [137], [142], [145], [147]

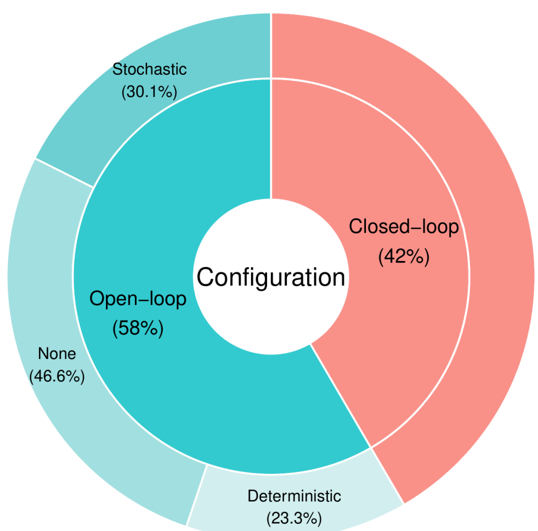

Figure 2.4: PHES configurations considered in the reviewed articles.

Although the energy stored in a closed-loop PHES facility is often potentially less than the energy stored in a comparable-size open-loop facility, the closed-loop PHES facilities account for a considerable fraction of the reviewed case studies since the potential locations are, in general, far away from an existing river or body of water. Furthermore, since there exists significant environmental and social opposition to damming or altering rivers, the closed-loop facilities seem to have a distinct advantage over the open-loop facilities. Nevertheless, constructing the open-loop facilities from the existing conventional hydropower systems by replacing their turbines with reversible ones has recently gained popularity [121]. Only two studies ([120] and [122]) considered both closed-loop and open-loop configurations in their optimization problems.

2.3.2 Type of the pump/turbine

A single pump-turbine can perform both water pumping and power generation in the PHES facilities. This is known as a reversible pump-turbine. Separate pumps (for water pumping) and turbines (for power generation) can also be utilized in the PHES facilities. The PHES facility can switch between generation and pumping modes in a matter of minutes. While the initial PHES facilities used separate pumps and turbines, the reversible pump-turbine has been widely adopted since the middle of the 20th century [18]. Among the reviewed articles, only [51] and [53] explicitly mention that they consider a reversible pump-turbine as well as a pump and a turbine separately.

Moreover, as an alternative to a traditional fixed-speed pump-turbine, a variable-speed one with an asynchronous motor-generator allows for power regulation during both pumping and generation modes, thereby increasing the system efficiency and flexibility. The variable-speed pump-turbine was introduced in the 1990s as a major advance in the operation of PHES facilities in Japan. Higher efficiencies under varying conditions, shorter switchover times between pumping and generation modes, higher ramp rates, and faster response times were all made possible by the pump-turbine runners' ability to change their rotation speed [151]. The majority of the reviewed articles discuss neither the type of pump-turbine nor the level of flexibility in its speed. Several papers, such as [70] and [94], state that the variable-speed pump-turbine is taken into account in their analyses. A fixed-speed pump-turbine is considered in [76] and both fixed-speed and variable-speed pump-turbines are considered in [79] and [102].

2.3.3 Head

The head is the difference in altitude between the water intake and water departure points. Although it typically varies between 100 and 800 meters, longer and shorter heads were also installed based on physical conditions [152]. One of the main challenges in modeling the PHES-optimization problems is the head variation (also known as the head effect), which directly affects the efficiency and operating limits of the PHES facilities. For computational flexibility, the head is assumed to be fixed in most PHES-optimization problems, regardless of the reservoir and pump/turbine characteristics. However, in the context of PHES optimization, disregarding these nonlinear relations may result in sub-optimal or even infeasible solutions. About 9% of the reviewed articles allow for a variable head in their problem settings. This is the case in [39], [59], [62], [69], [93], [106], [112], [123], [130], [133], and [136]. Most of these studies take into account the head dependency constraints via nonlinear or stochastic optimization frameworks. Several studies consider both the PHES facilities and the conventional hydropower stations, but they incorporate the variable head into their formulations for the conventional hydropower stations only; see, for example, [60] and [70].

2.3.4 Efficiency

The efficiency of a particular pump/turbine design is affected by the internal leakage and friction of its mechanical components with one another and with the water flowing through them. The pump/turbine efficiency improves as the flow rate of handled water increases, and the relative importance of losses decreases as the pump/turbine size increases [153]. The best pump/turbine efficiency levels typically range between 0.86 and 0.95, depending on the effective water head and flow rate between two reservoirs. One of the most difficult aspects of PHES optimization is the need to model the nonlinear pump/turbine performance curves that describe the relationship between output power, water discharge flow, and net head between the reservoirs. When choosing a pump-turbine for PHES installments, it is crucial to take into account the variable head and volume of water flowing through it to correctly calculate the efficiency dynamics of PHES. However, for the sake of simplicity, the vast majority of studies assume that the pump/turbine efficiency is a fixed constant free of such interactions. Only a few articles ([39], [59], [68], [71], [79], [112], [121], [130], and [133]) address the variability of pump/turbine efficiency. In addition, Afshari Igder et al. in [93] allow for a variable efficiency level for the turbine, but assume a fixed efficiency level for the pump.

2.3.5 Type of the case study

A large portion of today's installed PHES capacity was built in the 1960s, 1970s, and 1980s, with a huge growth rate in Europe, certain parts of Asia, and North America. Some of the initial PHES facilities were built in the Alpine regions of Switzerland and Austria, where there is an ample supply of water and a natural topography suitable for PHES construction [18]. The rise of PHES capacity in the United States can be attributed to nuclear power plant installments. Although the rise of PHES capacity in Europe is in line with the nuclear capacity growth rate, it is often linked to abundant hydropower resources [13]. Currently, China has the largest hydropower capacity, followed by the European Union, Brazil, Japan, and the United States [2]. Table 2.4 provides a detailed list of the case studies used in the reviewed literature. The articles with hypothetical cases constitute approximately 55% of the reviewed articles listed.

Table 2.4: Classification based on the case studies.

Case Study		References
Hypothetical Case Studies		[25], [28], [29], [30], [31], [33], [35], [40], [41], [42], [45], [46], [47], [48], [49], [50], [52], [53], [54], [55], [56], [59], [130], [131], [64], [65], [67], [68], [71], [72], [73], [74], [75], [76], [77], [80], [81], [82], [85], [86], [87], [90], [92], [93], [94], [95], [97], [98], [99], [101], [102], [103], [104], [107], [108], [109], [112], [113], [114], [115], [117], [119], [123], [129], [135], [137], [138], [139], [146]
Real Case Studies	Africa	[27], [36], [78], [141]
	Asia	[39], [62], [66], [69], [70], [84], [89], [96], [105], [116], [136], [140]
	Europe	[26], [32], [34], [37], [38], [44], [51], [57], [58], [60], [61], [79], [83], [88], [91], [106], [118], [110], [111], [125], [126], [128], [132], [133], [134], [139], [142], [143], [144], [145], [147]
	North America	[43], [63], [120], [121], [122], [124], [127], [148], [149]

2.4 Classification based on other sources integrated with PHES

PHES plays a crucial role in load balancing and supplementing intermittent renewable sources. Using multiple energy sources in conjunction with PHES can successfully overcome the drawbacks related to partial predictability and limited control of renewable sources and can increase their penetration into the grid. PHES is also recommended for use as a storage unit in remote areas to fill energy gaps caused by fluctuations in renewable energy supply. These power systems, which are frequently referred to as HPS, can be grouped into three major classes: wind-PHES, solar-PHES, and solar-wind-PHES power systems. Table 2.5 exhibits our classification of the articles that consider HPS. In addition to wind and solar, some HPS with PHES use other energy sources such as thermal power plants, gas power plants, and fuel-fired plants, which we classify as "other" in Table 2.5. Zheng et al. [124] compare storage dispatch strategies of various storage facilities, including the PHES facility, that are not integrated with other energy sources. Thus we do not include this study in Table 2.5.

About 15% of the reviewed articles do not consider any HPS. The most commonly investigated HPS type is PHES integrated with wind generation: about 43% of the reviewed articles consider this HPS type. While about 29% of the reviewed articles consider PHES integrated with wind and solar generation, about 6% of the reviewed articles study PHES paired with an energy source other than wind and solar. The remaining few articles consider PHES integrated with solar generation.

Table 2.5: Classification based on the energy source coupled with PHES.

	Energy source	References
PHES-	Wind	[29], [30], [32], [36], [37], [39], [50], [51], [52], [53], [54], [56], [57], [58], [61], [65], [66], [69], [71], [76], [77], [90], [91], [93], [105], [111], [120], [127], [142], [143], [147]
	Wind-Other	[27], [33], [34], [35], [40], [47], [48], [55], [62], [64], [79], [81], [84], [92], [94], [98], [103], [108], [109], [110], [114], [125], [126]
	Solar	[70], [72], [119], [136], [139], [140]
	Solar-Other	[45]
	Solar-Wind	[49], [68], [73], [89], [95], [96], [115], [135], [138], [141], [145], [146], [148]
	Solar-Wind-Other	[28], [31], [38], [41], [42], [44], [74], [75], [80], [82], [83], [85], [86], [87], [97], [99], [101], [107], [116], [117], [123], [144], [149]
	Other	[25], [46], [63], [67], [88], [102], [130], [131]

2.5 Classification based on the source of uncertainty

As discussed in the previous sections, different system components are involved in PHES-related optimization studies. Incorporating the uncertain nature of these components improves the precision of optimization models, leading to more realistic results and stronger insights. Uncertain parameters in PHES optimizations often include operational parameters such as electricity demand, economic parameters such as electricity price, and generation sources such as natural inflow feeding the PHES facility, wind speed, and solar radiation. The variable and uncertain nature of renewable energy sources, demand, and prices create significant operational uncertainty, making it challenging to match supply and demand. In this regard, we provide a detailed table in Table 2.6 showing which system component is considered deterministic (denoted by D) and/or stochastic (denoted by S) in the reviewed articles. The wind is the most commonly studied component (72% of the articles), followed by the price (52.8% of the articles), while the streamflow is the least explored component (32% of the articles). A very large portion of the studies that include the wind component in their optimization problems take into account the wind uncertainty (95%). The stochasticity of wind speed has thus been widely studied in the reviewed articles. On the other hand, more than half of the studies that include the demand component in their optimization problems take it as a deterministic component (59%).

Table 2.6: Classification based on the source of uncertainty.

Paper	Wind	Solar	Streamflow	Demand	Price	Paper	Wind	Solar	Streamflow	Demand	Price
[28]	S	S	D	D	—	[52]	S	—	—	D	D
[29]	S	—	D	D	D	[53]	S	—	—	S	D
[30]	S	—	—	—	D	[143]	S	—	S	D	S
[142]	S	—	—	—	S	[129]	—	—	S	—	S
[31]	S	S	—	D	D	[54]	S	—	—	—	S
[32]	S	—	—	D	D	[55]	S	—	—	D	—
[33]	S	—	—	S	D	[56]	D&S	—	D	D	D
[25]	—	—	—	—	S	[57]	S	—	—	D	D
[34]	S	—	—	D	—	[58]	S	—	—	—	D
[35]	S	—	—	D	D	[59]	—	—	D&S	—	D
[124]	—	—	—	S	D	[60]	—	—	D	—	S
[125]	S	—	—	D	—	[130]	—	—	S	—	S
[36]	S	—	—	D	D	[61]	S	—	—	—	S
[126]	S	—	—	D	—	[62]	S	—	D	D	—
[127]	S	—	D	—	S	[63]	—	—	—	D	S
[37]	S	—	—	—	S	[131]	—	—	S	—	S
[38]	S	S	—	—	S	[64]	S	—	—	S	—
[39]	D&S	—	—	D	D	[65]	S	—	—	—	D
[128]	—	—	S	S	D	[66]	S	—	—	—	D
[40]	S	—	—	D	D	[26]	—	—	S	S	—
[41]	S	S	—	D	D	[132]	—	—	D	S	S
[42]	S	S	S	S	—	[67]	—	—	—	S	—
[43]	—	—	—	D	S	[68]	S	S	—	—	S
[44]	S	D	—	S	D	[69]	S	—	S	D	—
[45]	—	S	—	S	—	[70]	—	S	S	D	—
[46]	—	—	S	D	D	[71]	S	—	—	—	—
[47]	S	—	—	D	—	[72]	—	S	—	—	—
[48]	S	—	—	D	—	[73]	S	S	D	S	—
[49]	S	S	—	D	S	[74]	S	S	—	S	S
[50]	S	—	—	D	—	[75]	S	S	—	D	—
[51]	D&S	—	D	—	D	[76]	S	—	—	—	—

Paper	Wind	Solar	Streamflow	Demand	Price
[77]	S	—	—	—	—
[78]	—	—	—	S	—
[79]	S	—	—	S	—
[80]	S	S	—	D	—
[81]	S	—	D	S	—
[82]	S	S	—	D	—
[83]	S	S	D	S	—
[84]	S	—	—	—	—
[85]	S	S	—	D	—
[86]	S	S	—	D	—
[87]	S	S	—	D	—
[88]	—	—	S	D	—
[133]	—	—	S	—	S
[89]	S	S	D	S	D
[90]	S	—	—	—	—
[91]	S	—	—	—	—
[144]	S	S	S	—	D
[92]	S	—	—	—	S
[93]	S	—	D	—	S
[94]	S	—	S	S	—
[95]	S	S	—	S	—
[96]	S	S	—	—	D
[97]	S	S	—	D	—
[145]	S	S	—	D	—
[98]	S	—	—	S	—
[99]	S	S	—	D	—
[100]	—	—	D	—	S
[101]	S	S	D	D	S
[134]	—	—	—	—	S
[102]	—	—	—	D&S	—
[103]	D&S	—	—	D&S	—
[104]	—	—	—	—	S

Paper	Wind	Solar	Streamflow	Demand	Price
[105]	S	—	—	D	—
[106]	—	—	—	—	S
[107]	S	S	—	D	—
[108]	S	—	—	D	—
[135]	S	S	—	S	S
[109]	S	—	—	D	—
[110]	S	—	S	S	S
[111]	S	—	—	—	S
[112]	—	—	—	—	S
[113]	—	—	—	—	S
[114]	S	—	—	D	—
[136]	—	D	D	S	—
[115]	S	S	—	S	—
[116]	S	S	—	S	—
[146]	S	S	—	S	S
[117]	S	S	—	—	—
[137]	—	—	—	—	S
[118]	—	—	D	—	S
[119]	—	S	—	S	S
[147]	S	—	—	—	S
[138]	S	S	D	—	—
[139]	—	S	—	S	S
[140]	—	S	S	S	—
[141]	S	S	—	D	—
[148]	S	S	S	S	—
[27]	S	—	—	D	—
[149]	S	S	—	D	—
[120]	S	—	S	—	S
[121]	—	—	D&S	—	D&S
[122]	—	—	S	—	S
[123]	S	S	S	D	—

Figures 2.5 and 2.6 contain Venn diagrams that summarize our detailed observations in Table 2.6. Figure 2.5 shows the numbers of studies that include stochastic or deterministic components of demand, price, solar, streamflow, and wind. For example, 11 studies consider only price and wind components in their analysis. The most commonly observed set of components is the set of demand and wind, which appears in 16 studies, followed by 15 studies with the set of demand, solar, and wind. We then restrict our attention to the studies incorporating the inherently stochastic nature of these components into their analyses. Figure 2.6 shows the numbers of studies that include the stochasticities of demand, price, solar, streamflow, and wind. Consistent with our previous discussion, the most widely studied stochasticity is that of wind alone. The wind is the only stochastic component in 33 articles. The second most widely studied stochasticity is that of wind and solar together. Another important observation is that the combinations of stochastic components that are absent in the reviewed articles include the price component. These combinations are (i) price and solar; (ii) price, solar, and streamflow; (iii) price, wind, solar, and streamflow; (iv) price, wind, and demand; (v) price, demand, and streamflow; and (vi) price, demand, streamflow, and solar. Finally, none of the reviewed articles considers all of the five components together stochastic.

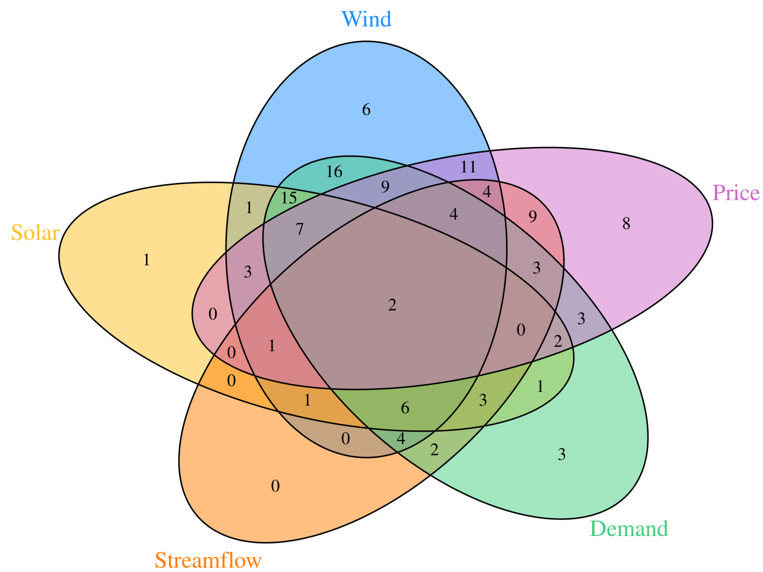

Figure 2.5: Numbers of studies that include the components of demand, price, solar, streamflow, and wind.

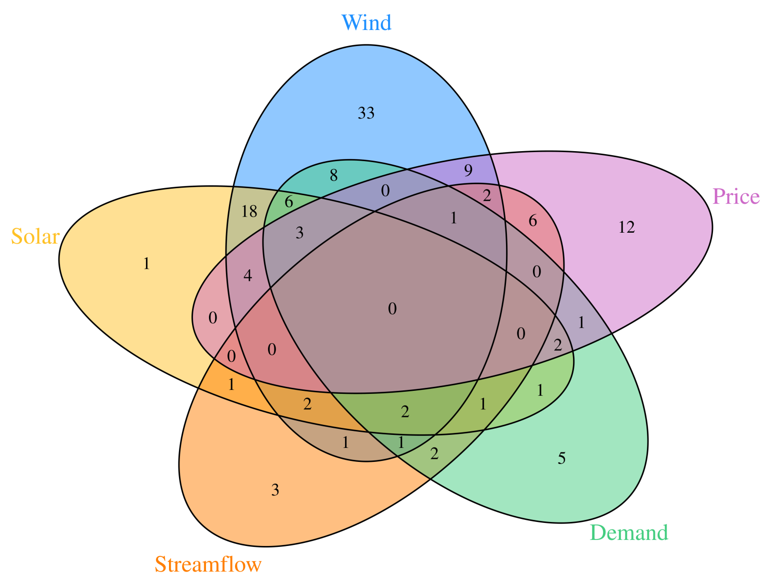

Figure 2.6: Numbers of studies that include the stochasticities of demand, price, solar, streamflow, and wind.

2.6 Classification based on the uncertainty modeling approaches

Optimization models can be divided into two broad classes: deterministic models that ignore uncertainty in the problem and models that consider uncertainty, partially or fully. It is vital to take into account uncertainties in decision-making in the energy sector because the partial predictability and limited controllability of the components involved in the energy systems are unavoidable. Systematically assessing the magnitude of uncertainties and incorporating the nonnegligible uncertainties into optimization models improves the robustness of the obtained results and provides more realistic insights. The impacts of uncertainties should be thoroughly investigated to propose effective strategies for dealing with complex system dynamics and develop quantitative analyses for correctly solving real-world problems.

To deal with uncertain parameters, various methods have been developed. The most commonly used methods are stochastic programming (SP), stochastic dynamic programming (SDP)/Markov decision process (MDP), robust optimization (RO), and chance-constrained programming (CCP). The level of available information about the model components and the assumptions about how it should be handled determine the appropriate modeling framework. Table 2.7 exhibits our classification of the articles based on the uncertainty modeling methods.

Table 2.7: Classification based on the uncertainty modeling approaches.

Modeling Framework	References
Stochastic Programming	[26], [30], [32], [34], [37], [43], [44], [49], [57], [61], [67], [78], [79], [81], [106], [118], [128], [129], [130], [135], [140], [143]
Stochastic Dynamic Programming/Markov Decision Process	[33], [36], [46], [59], [63], [77], [100], [115], [120], [121], [122], [131], [132], [147]
Robust Optimization	[27], [41], [55], [80], [84], [90], [94], [109], [113], [119], [123], [139]
Chance-Constrained Programming	[28], [29], [51], [66], [77], [112], [116], [127], [141]
Others	[25], [31], [35], [38], [40], [42], [45], [47], [48], [50], [52], [53], [54], [56], [58], [60], [62], [64], [65], [66], [68], [69], [70], [71], [72], [73], [74], [75], [76], [77], [82], [83], [85], [86], [87], [88], [89], [91], [92], [93], [95], [96], [97], [98], [99], [101], [102], [103], [104], [107], [108], [111], [114], [117], [124], [125], [126], [133], [134], [136], [137], [138], [142], [144], [145], [146], [148], [149]

The traditional approach of SP can be used if the probability distributions of the underlying uncertain components are known [154]. One can make probabilistic distributional assumptions, estimate the parameters using historical data, and then develop a stochastic optimization model. Typically, two- or multi-stage stochastic programs are considered, with the decisions of the first stage made before any uncertainty is resolved. In general, different scenarios are generated to capture uncertainties in the following stages, in which the decisions are made based on the occurrence scenario. Scenario analysis techniques are often needed when historical data is not informative enough. Stochastic programs may have intense computational requirements for large-scale problems, especially when there are a large number of scenarios. The studies capturing the uncertainties via two-stage stochastic programming are those of [30], [32], [37], [44], [49], [57], [61], [78], [79], [81], [106], [130], [135], and [140]. Alvarez et al. [110] propose a two-stage stochastic programming for the unit commitment problem along with a stochastic dual dynamic programming for the generic storage model. Energy offers, contractual agreements, and bidding decisions are commonly assumed to be first-stage decisions, particularly for the day-ahead scheduling problems, whereas operational decisions (energy generation, energy storage, curtailment) are mostly assumed to be second-stage decisions, especially for the intraday outlook. Among the reviewed articles, [26], [43], [67], [118], [128], [129], and [143] develop multi-stage stochastic programs.

Some other studies model the uncertainty via SDP. Murage and Anderson [36] develop an SDP model to find the optimal control strategy for PHES integrated with wind power generation to meet the committed dispatch. Zhao and Davison [59] employ SDP models to determine the best way toutilizethe available limited water for power generation. Reuter et al. [147] offer an SDP model for evaluating the economics of using a hybrid technology combining wind power generation and PHES. Picarelli and Vargiolu [100] consideran energy system consisting ofone or two hydropower plants with linked basins by introducing a stochastic optimal control problem with state constraints. Ni et al. [63] characterize the market offer curves by developing a stochastic optimization formulation that is separable in terms of individual units and taking a solution approach combining Lagrangian relaxation (LR) and SDP. Carmona and Ludkovski [131] investigate the valuation of energy storage facilities via a stochastic optimal control framework. Forouzandehmehr et al. [46] take a stochastic differential game-theoretical approach to analyze the competitive interactions between an autonomous PHES facility and a thermal power plant and to optimize power generation and storage decisions. Finally, Kiran and Kumari [33] model the wind speed as following a Weibull distribution, and penalize over- and under-estimation of wind generation in their objective function.

Some studies offer an MDP framework for sequential decision-making when outcomes are partially random and under partial control of the decision-maker. For example, Avci et al. [120] study the energy generation and storage problem of a hybrid energy system consisting of a wind farm and a PHES facility, modeling the problem as an MDP and characterizing the optimal policy structure. Toufani et al. [121] assess the potential benefits of transforming an existing cascading hydropower station into a PHES facility via an MDP framework that captures the price and streamflow uncertainties. In another study, Toufani et al. [122] compare the short-term cash-flow performance of various PHES configurations via an MDP framework. Löhndorf et al. [132] optimize the short-term intraday and long-term interday decisions of hydro storage systems with several connected reservoirs. They formulate the intraday problem as a stochastic program that takes into account bidding decisions as well as storage operations during the day. While they formulate the interday problem as an MDP, they propose a novel solution approach that integrates stochastic dual dynamic programming with approximate dynamic programming. Huang et al. [115] introduce a deep reinforcement learning agent for controlling the voltage of PHES-wind-solar systems, modeling the PHES voltage management problem as an MDP.

RO is a promising framework when the probability distributions of the underlying random components are unknown. This framework handles the unknown problem parameters by employing scenario sets or intervals [155]. It seeks a solution that can perform well under the majority of possible realizations of the uncertain inputs. Among the reviewed articles, [27], [55], [80], [84], [90], and [109] take into account the wind uncertainty via RO, and [113] and [139] take into account the price uncertainty via RO. Zhang et al. [41] propose a robust stochastic theory to model the uncertainties in wind power plants and solar generators. Zhou et al. [94] address the uncertainties in wind energy, water inflow, and power load via RO with intervals. Ju et al. [123] address the uncertainties in wind speed, solar radiation, and water inflow via RO. Finally, Ahmadi et al. [119] incorporate the uncertainties in renewable energy, electricity price, and local demand into an RO model.

CCP is a means to capture constraints that should hold with some prespecified probability [156]. This approach enables relaxation of hard constraints that may impose solutions influenced by extreme cases occurring with a small probability. CCP has been mostly used to ensure that the wind power output restriction is met at the decision maker's target probability level. As a chance constraint, Lu et al. [28] consider the forecast output errors of wind and solar power bounded by the reserved spare capacity. Zhang et al. [29] define the wind power uncertainty (curtailed power) as a chance constraint. Castronuovo et al. [51] introduce a chance constraint in which the available wind power is more than the fraction consumed by the PHES facility and the fraction delivered to the grid with a certain probability. Ding et al. [66] use a chance-constrained formulation in intraday market optimization to express the possibility of wind power generation in a specific time interval and scenario while not exceeding the forecast wind power. Toubeau et al. [112] use CCP for the day-ahead scheduling problem in the presence of approximation errors and endogenous model uncertainties. Hong et al. [116] study the unit commitment problem by considering chance constraints to address the uncertainties of load, wind, and PV power generation. Elnozahy et al. [141] consider a chance constraint to capture the weather uncertainty. In a few other studies, SP and CCP have been used together. This is the case in [39] and [105]. Finally, Lin et al. [77] develop a stochastic dynamic program to solve the energy dispatch problem of a power system with several wind farms and PHES facilities. They also provide scenario-based and CCP methods.

Scenario generation approaches have also been frequently used to capture uncertainties in the energy literature. Unlike the SP method discussed before, the scenarios here are not linked to each other through decision variables of the initial-stage problem; instead, they are treated individually. One stream of research in this category constructs a single model considering different scenarios (scenarios with the same or different probabilities of occurrence) and solves the problem once. However, the resulting models are usually decomposable into different scenarios. This is the case in [25], [38], [47], [48], [54], [56], [66], [68], [71], [77], [85], [89], [92], [93], [95], [97], [98], [102], [103], [104], [111], [117], [125], [126], [137], [142], and [144]. Another stream of research in this category similarly generates different scenarios in their experimental design and observes the effect of each scenario on their final solution. Unlike the previous stream, the studies in this stream do not include a scenario dimension in their optimization model; instead, they solve their model separately for each scenario in their experiments and simply compare the results or calculate the average of the results. This is the case in [31], [35], [40], [42], [45], [50], [52], [53], [58], [60], [62], [64], [65], [69], [70], [72], [73], [74], [75], [76], [82], [83], [86], [87], [88], [91], [96], [99], [101], [107], [108], [114], [124], [133], [134], [138], [136], [145], [146], [148], and [149]. We classify these two research streams as "others" in Table 2.7.

2.7 Classification based on the solution methodology

The solution methodologies proposed in the reviewed articles fall into three categories: exact optimization methods, heuristics, and metaheuristics. Table 2.8 classifies the reviewed articles according to the solution approach employed. Among the exact optimization methods, the majority of articles use linear programming (LP), nonlinear programming (NLP), mixed integer linear programming (MILP), mixed integer nonlinear programming (MINLP), and dynamic programming (DP). DP algorithms are among the most widely used optimization techniques for solving the operational planning problems, but their practical applications may suffer from the curse of dimensionality. While LP models can be very efficient for solving large-scale problems, these models fail to represent the possible nonlinear structure of the energy generation and storage problem, particularly the physical characteristics of PHES facilities.

Table 2.8: Classification based on the solution methodology.

Solution Methodology	Optimization Technique	References
Optimization Models	LP	[30], [51], [56], [58], [64], [65], [79], [96], [127], [129], [134], [137]
	MILP	[27], [32], [34], [37], [38], [41], [44], [48], [49], [54], [61], [66], [68], [71], [79], [83], [88], [89], [90], [92], [102], [104], [109], [110], [112], [116], [119], [123], [126], [128], [130], [135], [138], [140]
	DP	[33], [36], [46], [59], [62], [63], [69], [77], [100], [110], [115], [120], [121], [122], [125], [131], [143], [147]
	NLP	[25], [31], [35], [60], [95], [105], [133], [139]
	MINLP	[28], [39], [40], [57], [66], [80], [81], [84], [98], [103], [112]
	Others	[26], [113]
Heuristics	ADP	[77], [117], [120], [132]
	LR-based	[33], [67]
	Others	[32], [33], [43], [50], [55], [78], [106], [111], [117], [118], [144]
Metaheuristics	PSO	[29], [42], [47], [50], [62], [70], [72], [75], [77], [82], [87], [89], [93], [94], [107], [108], [114], [136], [141], [146]
	GA	[42], [47], [70], [74], [85], [97], [106], [146], [149]
	ABC	[35], [52], [53]
	ASA	[74], [85], [97]
	Others	[29], [42], [45], [47], [62], [70], [73], [74], [75], [82], [85], [86], [87], [93], [97], [99], [101], [106], [107], [108], [114], [115], [141]

NLP models can be employed to handle the nonlinear and nonconvex structure of hydro generation, but these models are, in general, difficult to solve for large-scale problems. If the problem's nonlinearity is approximated by piecewise linear functions along with discrete variables, the problem can be formulated as an MILP model [157].

Although exact optimization methods have been commonly utilized to obtain the optimal decisions and/or policies, various heuristics (common ones as approximate dynamic programming (ADP) and Lagrangian-based (LR-based) methods) and metaheuristics (common ones as particle swarm optimization (PSO), genetic algorithm (GA), artificial bee colony algorithm (ABC), and artificial sheep algorithm (ASA)) have also been constructed to solve large-scale or non-differentiable optimization problems. However, these approaches are often inefficient in the presence of binary decision variables and high-dimensional problem settings. They may also fail to provide sufficient information about the optimality of the obtained solution, which is in general highly dependent on the initial solution implemented into the solution algorithm. The most frequently employed heuristic technique is ADP. ADP algorithms provide approximations of the value function, the transition probabilities, or the system dynamics. The goal is to find an approximate solution that is close to the optimum in a computationally feasible way [158]. On the other hand, the most frequently employed metaheuristic technique is PSO. PSO is a population-based optimization algorithm that models the behavior of birds or fish to find the global optimum of a function in a multidimensional space. These algorithms consider a set of particles that move in the solution space and update their velocity and position based on their personal best and the global best [159].

The majority of studies compare the feasibility and computation efficiency of their solution methods to those of other approaches, all summarized in Table 2.8. For example, Abhindranath and Tiwari [50] take a heuristic approach based on interior point solution as well as a metaheuristic approach based on PSO. Alvarez et al. [110] solve a stochastic dual DP model for the generic storage problem, along with an MILP equivalence of a two-stage SP model for the unit commitment problem. Toubeau et al. [106] design a heuristic algorithm as well as a GA as a metaheuristic approach to optimize the operations of an underground PHES facility in day-ahead and reserve energy markets. Huang et al. [115] construct a deep deterministic policy gradient algorithm as a metaheuristic approach to obtain the optimal control policy for the PHES facility. They also solve their problem via DP to show the efficiency of this algorithm. Lee [62] compares the performance of different metaheuristic approaches to the standard DP algorithm. Avci et al. [120] solve their MDP model via MDP-based heuristics and several ADP approaches. Finally, several studies offer solutions that rely only on simulation results without explicitly specifying their solution methodologies. This is the case in [76], [91], [124], [145], and [148].

2.8 Conclusion and future research directions

We conclude by outlining possible directions for PHES-related optimization studies under uncertainty based on our literature review. Our review has revealed several research gaps in the PHES literature that can be divided into several different classes discussed below:

(1) Optimization modeling

The proper modeling of uncertainties is required to adequately capture the essence of most PHES facilities. The uncertainty modeling in PHES optimization has received a great deal of attention and various approaches have been used to incorporate the existing nonnegligible uncertainties. We observe that the majority of the studies employ stochastic programming approaches in which they generate scenarios in advance, which may fail to effectively capture the impact of events on sequential decision-making. Such approaches for a specific scenario violate the nonanticipativity condition by allowing the decision-maker to see the future outcomes. We also observe that MDPs have not been widely used as a modeling approach but have recently started to gain popularity. Alternative to in-advance scenario generation, MDPs allow the uncertainties to be resolved only when the decision-maker progresses in time, while enabling adaptive decisions based on the real-time realizations of uncertainties. MDPs also offer a precise mathematical framework to analytically derive structural results by capturing multidimensional dynamics and uncertainty inherent in the problem. We should note that scenario-based stochastic programming approaches may suffer from the computational burden when the number of scenarios generated is very large. Although MDPs may also suffer from the curse of dimensionality when the numbers of states and decisions are very large, ADP methods can be used to overcome this limitation. Structural properties derived from MDPs, in particular, can be incorporated into ADP algorithms to provide near-optimal policies in a computationally efficient way.

(2) Problem objective

The majority of studies develop single-objective optimization models, mostly focusing on either profit-maximization or cost-minimization. However, the sizing and operational problems for hybrid energy systems, particularly those including renewable energy sources, typically involve different aspects and multiple goals. In comparison to the single-objective models, the multi-objective models have advantages in making flexible decisions based on different priorities and evaluating the trade-offs between conflicting objectives. Since environmental issues are one of the primary drivers of growing interest in hybrid systems containing PHES, the environmental concerns, together with system profits/costs, deserve close attention and further investigations in the PHES literature.

(3) Scenario generation

Although scenarios are consistent representations of potential futures, it is not possible for scenarios to encompass the entire range of multivariate future possibilities. It can be challenging to identify all possible scenarios and their occurrence probabilities. Some articles use scenario-reduction techniques to reduce the computational burden. However, sensitivity analyses are required to ensure that an adequate number of scenarios are generated to capture the study environment. While the majority of the reviewed articles address the uncertainty via scenario generation, a limited number of studies discuss the sufficiency of the number of generated scenarios.

(4) System components

The wind-PHES power systems have received much attention in the literature. Only a few studies have looked into the solar-PHES power systems. Solar energy is less intermittent than wind energy [160], yet it should be combined with energy storage since it is available only during the day. Solar energy is currently the most accessible renewable source, and its installed capacity is predicted to surpass the installed wind capacity in the near future [161, 162]. Solar systems have also become a more cost-effective long-term investment option thanks to recent technological advances [19]. In addition, the solar-wind power systems have received considerable attention in recent years, partly because of their complementary nature [163]. Thus, given the increasing trend toward the use of solar energy, it is advisable for researchers to further explore the solar-PHES power systems.

(5) Nonlinear characteristics of PHES facilities

Variable head between the reservoirs and nonlinear efficiency curves of the pump/turbine are two of the most significant sources of complexity in the PHES-optimization studies. For the sake of simplicity, the head-dependence of water-power conversions is often ignored and/or the pump/turbine efficiency level is assumed to be constant. Although the nonlinear features of head and efficiency augment the problem complexity and dimension, accurate quantification of variable efficiency and head is required to precisely model PHES facilities. The efficiency curves, in particular, indicate the relationship between power, water flow rate, and water head, which should be carefully taken into account to model various losses during the water-power conversion process in each PHES cycle. This largely neglected aspect of the problem offers an area to explore, especially to provide realistic and reliable generation and storage plans for PHES facilities.

(6) Market operations

The PHES operations under uncertainty have been mostly optimized in a day-ahead electricity market setting. A PHES operator, on the other hand, can also participate in multiple electricity markets with different bidding strategies (e.g., sequential or coordinated) to make more profit and ancillary service auctions due to its high flexibility and quick response to market price variations. Developing optimal bidding strategies for the PHES operators by taking into account multi-settlement markets, particularly the market for reserve electricity, is still an open area to investigate.

(7) Sizing problems

Although there is a rich literature on the sizing problem for PHES facilities (stand-alone or paired with other energy sources), only 8% of the reviewed articles (on PHES optimization under uncertainty) fall into this category. The uncertainty modeling has thus been largely neglected in the PHES sizing literature. Incorporating uncertainties into the sizing problems will likely bring new challenges on how to optimally design the PHES facilities.

(8) Time horizon

Most studies deal with the PHES-optimization problems over short-term scheduling horizons, with the vast majority using hourly periods over a daily horizon. Only a few studies have focused on the long-term planning horizon, despite the fact that long-term optimization assists authorities with strategic-level generation and transmission planning. The short-term focus in optimization models can be partially attributed to the limited long-term accuracy of forecasting methods adopted for the system components. However, the intermittent characteristics of renewable energy sources as well as electricity demand and price may still be modeled as exogenous stochastic processes with acceptable accuracy levels over longer time horizons.

(9) Case studies

The vast majority of studies use hypothetical case studies to conduct numerical experiments. Using real-world case studies, on the other hand, enables researchers to validate their optimization models and insights obtained from their models in a more reasonable and realistic way. Moreover, a recent IRENA report [2] exemplifies innovative designs of PHES systems. Such designs include PHES coupled with floating solar PV technology, PHES coupled with the onshore wind where the foundations of the wind turbines are used as upper reservoirs of the PHES facility, and PHES with variable speed turbines and hydraulic short circuits. The academic literature, however, lacks studies on the optimization of these innovative designs.

Chapter 3

Operational Benefit of Transforming Cascading Hydropower Stations into Pumped Hydro Energy Storage Systems

3.1 Introduction

A typical PHES system stores and generates electricity by exchanging water flow between two reservoirs located at different altitudes. When the electricity price is low due to an excess energy supply, energy can be stored in the form of hydraulic potential energy by pumping water from the lower reservoir to the upper reservoir. When the electricity price is high due to a surge in energy demand, energy can be generated by releasing the water stored in the upper reservoir to

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the lower reservoir. PHES is an attractive storage option with promising round-trip efficiencies (70% to 80%), short response times (minutes to even seconds), and long useful lifetimes (50 to 100 years) [164–166].

As part of the International Renewable Energy Agency's global roadmap, the currently installed capacity of PHES needs to be doubled, reaching 325 GW by 2050 [167]. A major challenge in new PHES installations, however, is the need for quite specific site conditions such as water access and favorable topography [168]. This challenge draws the attention of the PHES investors to the existing CCHSs in which these conditions are already satisfied. Current trends show that it is often possible to retrofit the CCHSs with reversible turbines to endow them with pumping capability [169]. In this chapter, we evaluate the potential benefit of rehabilitating an existing CCHS to operate it as a PHES system in a market setting where the electricity price can be negative. We formulate the real-time decision making process of the PHES system as an MDP, by taking into account uncertainties in the streamflow incoming to the hydropower system as well as in the electricity price and by constructing data-calibrated time series models to include these uncertainties.

Previous research dealing with the optimization of PHES systems focus on finding the optimal size and configurations of the new systems ([64], [170], [140], [171], [172], and [173]) and the optimal operating rules of the existing systems ([174], [175], [61], [58], [132], [176], [43], [165], [133], [112], [177], and [120]). Most of these studies develop deterministic optimization models or scenario-based stochastic programming approaches. However, deterministic models do not take uncertainty into account. Stochastic programming approaches, on the other hand, require scenario generation in advance, which may fail to adequately capture the impact of exogenous events on sequential decision-making. Such approaches, for a particular scenario, violate the nonanticipativity condition by allowing the controller to see the future outcomes [178]. Alternatively, MDPs allow uncertainties to be resolved only when the controller progresses in time, while enabling adaptive decisions based on the real-time realizations of uncertainties. MDPs also offer a precise mathematical framework to analytically derive structural results by capturing multidimensional dynamics and uncertainty inherent in the problem [179].

To our knowledge, however, only a few studies have developed MDPs in the PHES literature. Löhndorf et al. [132] study the energy commitment and storage problem for a PHES system by formulating a multi-stage stochastic program for intraday decisions and an MDP for interday decisions. Avci et al. [120] examine the integrated operations of a hybrid energy system that includes a wind farm and a PHES facility (simpler than our PHES configuration in this chapter). They model the problem as an MDP and characterize the optimal policy structure. However, these papers present no result for the potential value of operating a CCHS as a PHES system.

Despite the widespread practice of rehabilitating CCHSs as PHES systems [180, 181], as far as we are aware, only a few studies quantify the value of such rehabilitation. Bozorg Haddad et al. [176] formulate nonlinear programs to examine the advantages of converting CCHSs to PHES systems by ignoring uncertainties in the electricity price and streamflow rate. Ak et al. [133] study the same problem by developing a nonlinear program and solving it as many times as the number of scenarios generated to take the price uncertainty into account and by ignoring the streamflow uncertainty. Finally, Ribeiro et al. [182] study a similar problem by formulating a deterministic dynamic program.

The main contributions of this chapter are as follows:

- Our study is the first to assess the potential benefit of transforming an existing CCHS into a PHES system via an MDP framework that captures the price and streamflow uncertainties as well as the nonlinear dynamics of the problem. With our MDP framework, we are the first to offer a theoretical upper bound on the profit improvement that can be achieved with the pumping capability in the CCHS.
- We conduct numerical experiments with data-calibrated time series models to provide insights into the optimal operations and profits of the CCHS as

well as the PHES system in different environments.

- We show the benefit of taking into account the price and streamflow uncertainties by comparing the results of our stochastic solution approach with a deterministic one.

The rest of this chapter is organized as follows: Chapter 3.2 formulates the energy generation and storage problem for a PHES facility. Chapter 3.3 presents our theoretical upper bound and numerical results for the benefit of pumping capability. Chapter 3.4 measures the value of including randomness in our formulation. Chapter 3.5 offers a summary and the conclusions. Proofs of the analytical results are contained in Appendix A.

3.2 Problem formulation

We consider a CCHS with two reservoirs at different altitudes where energy can be generated by releasing water from the upper reservoir to the lower reservoir or from the lower reservoir to the stream bed. There is a natural streamflow incoming to the upper reservoir. Any excess amount of the streamflow (and any excess amount of the water pumped in the PHES system) spills from the upper reservoir and feeds the lower reservoir. Utilizing a reversible turbine between the cascading reservoirs, one can transform this CCHS into a PHES system in which energy can be stored by pumping water from the lower reservoir to the upper reservoir. Figure 3.1 illustrates one such PHES system. Since the PHES system includes the CCHS as a special case when the pumping capability is disabled, we present below the MDP formulation of the PHES system. In order to quantify the value of transforming the CCHS into a PHES system, we compare the expected total cash flows that result from the transactions of these systems with an electricity market. We presume that the system operations are unable to influence the market price, that is, the operator is a price-taker. Such market settings appear in many related papers; see, for example, [43, 133, 176, 179, 183], and [184].

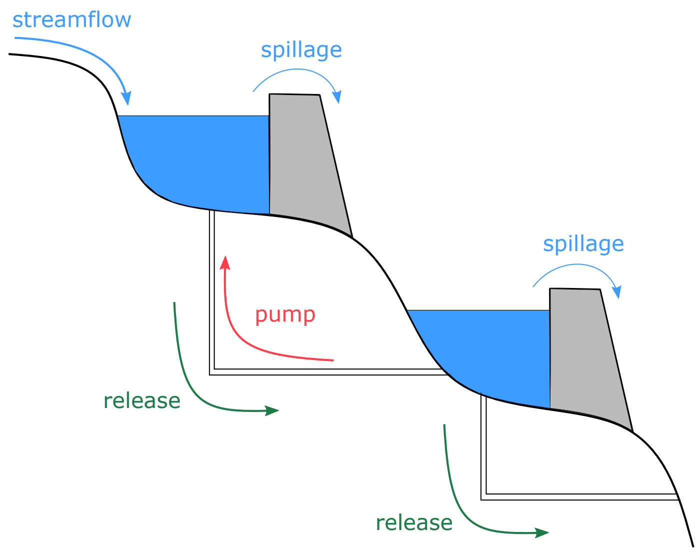


Figure 3.1: Illustration of the PHES system.

The amount of energy that can be generated by releasing a unit volume of water from any reservoir equals the multiplication of the water density (ρ), the gravitational constant (g), the potential head of the hydropower station, and the electricity conversion efficiency of the turbine. The efficiency depends on the water flow rate and the potential head available. The potential head is usually determined based on the constructed wall of the dam and the topography of the site. We assume that the vertical height of a waterfall is measured from the intake to the turbine, leading to constant heads for the upper and lower hydropower stations (h_1 and h_2). This assumption appears in many PHES papers; see, for example, [140, 185], and [186]. Nevertheless, our MDP formulation can be easily modified to incorporate a variable head as the water is drawn from the reservoir. Our efficiency calculations are based on an efficiency curve that is a function of the ratio of the water flow rate to the design charge or discharge amount of the pump or turbine. We denote the design discharge amounts of the upper and lower turbines by Q_{RU} and Q_{RL}, respectively, and the design charge amount of the pump by Q_P. We also denote the efficiency functions of the upper and lower turbines in the discharging mode by $\Phi_{RU}(\cdot)$ and $\Phi_{RL}(\cdot)$, respectively, and the efficiency function of the pump in the charging mode by $\Phi_P(\cdot)$. The energy generated by releasing a unit volume of water from the upper reservoir is $\rho g h_1 \Phi_{RU}$, the energy generated by releasing a unit volume of water from the lower reservoir is $\rho g h_2 \Phi_{RL}$, and the energy required to pump a unit volume of water from the lower reservoir is $\rho g h_1/\Phi_P$. We define C_U and C_L as the maximum amounts of water that can be stored in the upper and lower reservoirs, respectively. We assume no loss of water due to evaporation and no delay or efficiency loss due to vertical or horizontal distances between the reservoirs.

For the PHES system, we study the energy generation and storage problem via a dynamic model over a finite planning horizon of $\mathcal{T}$ periods. Let $\mathcal{T} := \{1, 2, \ldots, T\}$ denote the set of periods. We denote the accumulated amounts of water in the upper and lower reservoirs at the beginning of period t by x_{ut} and x_{lt}, respectively. Note that $x_{ut} \in [0, C_U]$ and $x_{lt} \in [0, C_L]$. We define r_t as the amount of streamflow incoming to the upper reservoir at the beginning of period t and p_t as the electricity price in period t. We also define $y_t := (r_\kappa, p_\kappa)_{\kappa \le t}$ as the history of the streamflow rate and electricity price, which evolves over time according to an exogenous stochastic process. We include x_{ut}, x_{lt}, and y_t in our state description. In each period t, after observing y_t as well as x_{ut} and x_{lt}, the operator determines the amount of water that will be released from or pumped to the upper reservoir $a_t \in \mathbb{R}$ and the amount of water that will be released from the lower reservoir $b_t \in \mathbb{R}_+$. If $a_t \geq 0$, the water is released from the upper reservoir to the lower reservoir. If $a_t < 0$ (in the PHES system only), the water is pumped from the lower reservoir to the upper reservoir. We thus assume that the reversible turbine can operate in only one mode (either pumping or releasing) within a time period, as widely recognized in the PHES literature dealing with short-term planning problems with hourly periods ([61], [34], [74]). We also assume that if $a_t > 0$ and $b_t > 0$, the water in the lower reservoir is released after the water in the upper reservoir is released. Finally, we assume that $b_t = 0$ if $a_t < 0$. Figure 3.2 illustrates the sequence of events in each period.

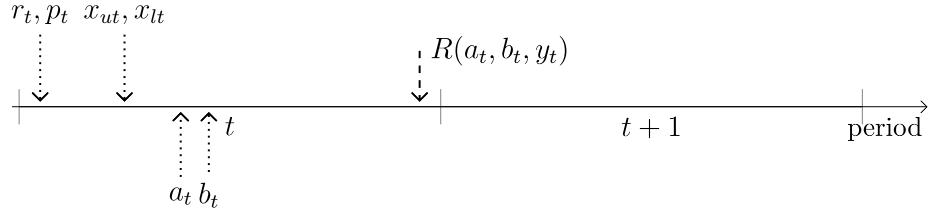


Figure 3.2: Sequence of events in each period.

Let $\mathbb{U}(x_{ut}, x_{lt}, y_t)$ denote the set of action pairs (a_t, b_t) that are admissible in state (x_{ut}, x_{lt}, y_t). For any action pair $(a_t, b_t) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, the following conditions must hold:

$$-\min\{x_{lt}, Q_P\} \leq a_t \leq \min\{x_{ut}, Q_{RU}\}$$

and

$$0 \leq b_t \leq \begin{cases} \min\{x_{lt} + a_t, C_L, Q_{RL}\} & \text{if } a_t \geq 0, \\ 0 & \text{if } a_t < 0. \end{cases}$$

The state variables x_{ut} and x_{lt} evolve over time as

$$x_{u(t+1)} = \min\{\min\{x_{ut} - a_t, C_U\} + r_{t+1}, C_U\} = \min\{x_{ut} - a_t + r_{t+1}, C_U\}.$$

Let $(x)^+ := \max\{x, 0\}$. If $a_t \geq 0$,

$$x_{l(t+1)} = \min\Big\{\min\{x_{lt} + a_t, C_L\} - b_t + (x_{ut} - a_t + r_{t+1} - C_U)^+, C_L\Big\}.$$

If $a_t < 0$,

$$x_{l(t+1)} = \min\Big\{x_{lt} + a_t + (x_{ut} - a_t - C_U)^+ + (\min\{x_{ut} - a_t, C_U\} + r_{t+1} - C_U)^+, C_L\Big\}.$$

In the above formulation, $\big(x_{ut} - a_t - C_U\big)^+$ is the amount of water spilling from the upper reservoir due to the water pumped, and $\big(x_{ut} - a_t + r_{t+1} - C_U\big)^+$ and $\big(\min\{x_{ut} - a_t, C_U\} + r_{t+1} - C_U\big)^+$ are the amounts of water spilling from the upper reservoir due to the water runoff.

The objective is to maximize the expected total cash flow from sales and purchases of energy over the finite horizon. In any period t, if some water is released from the upper or lower reservoir ($a_t \geq 0$ and $b_t \geq 0$), the energy generated is sold to the market. If some water is pumped to the upper reservoir to store energy ($a_t < 0$ and $b_t = 0$), the energy required is purchased from the market. Hence, given the action pair (a_t, b_t) and exogenous state pair y_t, the payoff in period t is formulated as

$$R(a_t, b_t, y_t) = \begin{cases} p_t \rho g \Big(a_t \Phi_{RU}\left(a_t/Q_{RU}\right) h_1 + b_t \Phi_{RL}\left(b_t/Q_{RL}\right) h_2 \Big) & \text{if } a_t \geq 0, \\ p_t \rho g h_1 \Big(a_t / \Phi_P\left(a_t/Q_P\right) \Big) & \text{if } a_t < 0. \end{cases}$$

A control policy π is the sequence of decision rules $(a_t^\pi(x_{ut}^\pi, x_{lt}^\pi, y_t), b_t^\pi(x_{ut}^\pi, x_{lt}^\pi, y_t))_{t\in\mathcal{T}}$, where x_{ut}^π and x_{lt}^π are the random state variables under policy π, $\forall t \in \mathcal{T}\backslash\{1\}$. We denote the set of all admissible control policies by Π. For any initial state (x_{u1}, x_{l1}, y_1), the optimal expected total cash flow over the finite horizon is

$$\max_{\pi\in\Pi} \mathbb{E}\Bigg[\sum_{t\in\mathcal{T}} R\Big(a_t^\pi\left(x_{ut}^\pi, x_{lt}^\pi, y_t\right), b_t^\pi\left(x_{ut}^\pi, x_{lt}^\pi, y_t\right) \Big) \Big| x_{u1}, x_{l1}, y_1 \Bigg].$$

For each state (x_{ut}, x_{lt}, y_t) in each period $t \in \mathcal{T}$, the optimal profit function $v_t^*(x_{ut}, x_{lt}, y_t)$ can be calculated with the following dynamic programming recursion:

$$\begin{aligned} v_t^*(x_{ut}, x_{lt}, y_t) = \max_{(a_t,b_t)\in\mathbb{U}(x_{ut},x_{lt},y_t)} & \Big\{ R(a_t, b_t, y_t) \\ & + \mathbb{E}_{y_{t+1}|y_t}\Big[v_{t+1}^*\Big(x_{u(t+1)}, x_{l(t+1)}, y_{t+1} \Big) \Big] \Big\} \end{aligned} \tag{3.1}$$

where $v_T^*(x_{uT}, x_{lT}, y_T) = 0$. This recursion proceeds backward from period T by calculating the optimal actions and profit for each state in each period via a complete enumeration method. For the initial state (x_{u1}, x_{l1}, y_1), $v_1^*(x_{u1}, x_{l1}, y_1)$ is the optimal expected total cash flow over the finite horizon.

3.3 Results and discussion

3.3.1 Theoretical bound on the value of PHES

In this section, we establish a theoretical upper bound on the profit improvement that can be achieved with the PHES transformation. Let $\widetilde{v}_t^*$ denote the optimal profit function in period t for the CCHS; this can be calculated via the recursion in (3.1) by restricting the action a_t to be nonnegative for all t. For our upper bound, we assume that the system efficiencies are constants in both charging and discharging modes, and are independent of the amount of water pumped or released throughout the entire planning horizon.

Assumption 3.3.1. *$\Phi_{RU}(a_t/Q_{RU}) = \phi_{RU} \in (0,1]$, $\Phi_P(a_t/Q_P) = \phi_P \in (0,1]$, and $\Phi_{RL}(b_t/Q_{RL}) = \phi_{RL} \in (0,1]$, $\forall a_t, b_t$.*

Theorem 3.3.1. *Suppose that Assumption 3.3.1 holds. Then,* $\widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) \leq v_t^*(x_{ut}, x_{lt}, y_t) \leq \widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) + Q_P \rho g h_1 (T-t)\left(\phi_{RU} \max_{t \leq \tau \leq T} \overline{p}_\tau - \min_{t \leq \tau \leq T} \underline{p}_\tau / \phi_P\right)^+$, *where $p_t \in [\underline{p}_t, \overline{p}_t]$ and $\overline{p}_t \geq 0$, for all $t \in \mathcal{T}$.*

Since the PHES system includes the CCHS as the special case when $a_t \geq 0$, $\forall t$, the optimal profit function of the PHES system is bounded below by that of the CCHS. The upper bound for the optimal profit difference between the PHES system and the CCHS can be viewed as the gap between the revenue obtained by generating and selling energy at full capacity at the maximum possible electricity price and the cost incurred by purchasing and storing energy at full capacity at the minimum possible electricity price, summed over all future periods except the last period (recall that the optimal profit function in period T is zero for both settings). This upper bound is affected by the electricity price and the physical features of the PHES system. A tight upper bound can be obtained in the markets with low volatility in the electricity price and in the systems with low efficiency in the installed turbine/pump. An important implication of this upper bound is that the PHES transformation brings no additional profit if $\min_{1 \leq \tau \leq T} \underline{p}_\tau \geq \phi_{RU} \phi_P \max_{1 \leq \tau \leq T} \overline{p}_\tau$. However, this condition never holds if the electricity price can be negative in some periods.

3.3.2 Experimental setup for the numerical study

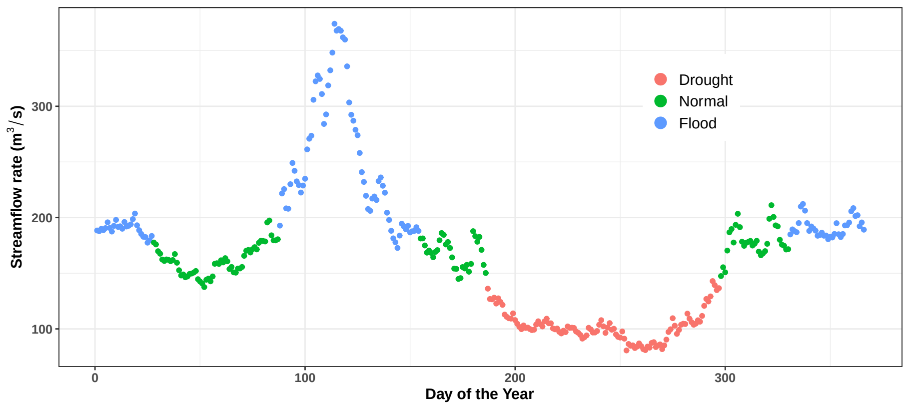
In our experiments we set the period length to be one hour, as in [140, 187], and [188]. For the streamflow rate, we consider two distinct data sets with substantially different rates in the State of New York, the Hudson River at Fort Edward and at North Creek, between the years 2000 and 2019 [189]. We restrict the intraday hourly streamflow rates to stay constant because the data sets display no substantial fluctuation within each day. However, a frequency spectrum analysis of the streamflow rates indicates the significance of daily frequency. Hence, we model the interday streamflow rates by fitting a periodic autoregressive (PAR) model to daily average streamflow rates. Specifically, we partition the 365 days of the year into disjoint clusters of normal, drought, and flood with the fuzzy c-means clustering approach in [190]. Figure 3.3 shows these three clusters at Fort Edward and North Creek.

Following [191], we characterize the streamflow rate in each cluster i as following a different autoregressive of order one, AR(1), process: $r_t = \delta_i + \phi_i r_{t-1} + \sigma_i \epsilon_t,\ \forall t \in \mathcal{T}_i$, where $\{\epsilon_t\}_{t \in \mathcal{T}_i}$ are independent standard normal error terms, δ_i is a constant, ϕ_i is the autoregressive coefficient, σ_i is the volatility of white noise, and $\mathcal{T}_i$ is the set of daily periods that belong to cluster i. Table 3.1 exhibits the parameter estimates of the PAR models at Fort Edward and North Creek.

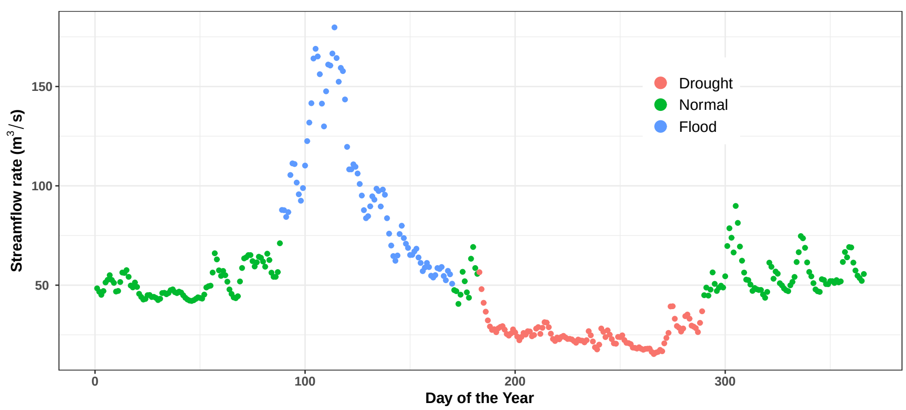
Table 3.1: Parameter estimates of the PAR model.

	Fort Edward			**North Creek**		
Parameters	Normal	Drought	Flood	Normal	Drought	Flood
$\hat{\phi}_i$	0.904	0.939	0.942	0.837	0.839	0.907
$\hat{\sigma}_i$	34.42	19.23	41.96	21.84	13.69	33.69
$\hat{\delta}_i$	168.17	102.68	223.10	52.05	25.87	91.48

To incorporate this parametric model into our MDP, we discretize the continuous space of the AR(1) processes: For Fort Edward and North Creek, we restrict



(a) Fort Edward



(b) North Creek

Figure 3.3: Three clusters for 365 days of the year and the daily average streamflow rates over the years 2000-2019.

the streamflow rate to take values from the sets $\mathcal{R}_F^n := \{75, 100, \dots, 250\}$ and $\mathcal{R}_N^n := \{20, 45, 70, 95\}$ in the normal flow cluster, the sets $\mathcal{R}_F^d := \{50, 75, \dots, 200\}$ and $\mathcal{R}_N^d := \{5, 30, 55\}$ in the drought flow cluster, and the sets $\mathcal{R}_F^f := \{100, 125, \dots, 350\}$ and $\mathcal{R}_N^f := \{25, 50, \dots, 200\}$ in the flood flow cluster, respectively. We then formulate the AR(1) process of the streamflow rate in each cluster as a different finite-state Markov chain for which we calculate the transition probabilities with the procedure in [192]. For Fort Edward, the transition matrix of the Markov chain in the normal flow cluster is

$$
P_F^n = \begin{array}{c} \\ 75 \\ 100 \\ 125 \\ 150 \\ 175 \\ 200 \\ 225 \\ 250 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 75 & 100 & 125 & 150 & 175 & 200 & 225 & 250 \end{array} \\
\left[\begin{array}{cccccccc}
.716 & .187 & .076 & .018 & .003 & .000 & .000 & .000 \\
.466 & .273 & .175 & .067 & .016 & .002 & .000 & .000 \\
.229 & .265 & .267 & .163 & .060 & .013 & .002 & .000 \\
.081 & .170 & .271 & .261 & .151 & .053 & .011 & .002 \\
.020 & .072 & .182 & .276 & .253 & .140 & .047 & .011 \\
.003 & .020 & .081 & .193 & .280 & .244 & .129 & .050 \\
.000 & .004 & .024 & .090 & .205 & .282 & .235 & .161 \\
.000 & .000 & .005 & .027 & .099 & .216 & .283 & .369
\end{array}\right]
\end{array},
$$

the transition matrix in the drought flow cluster is

$$
P_F^d = \begin{array}{c} \\ 50 \\ 75 \\ 100 \\ 125 \\ 150 \\ 175 \\ 200 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 50 & 75 & 100 & 125 & 150 & 175 & 200 \end{array} \\
\left[\begin{array}{ccccccc}
.791 & .192 & .017 & .000 & .000 & .000 & .000 \\
.340 & .473 & .173 & .014 & .000 & .000 & .000 \\
.051 & .318 & .464 & .155 & .012 & .000 & .000 \\
.002 & .058 & .340 & .452 & .138 & .009 & .000 \\
.000 & .003 & .067 & .361 & .439 & .122 & .008 \\
.000 & .000 & .003 & .078 & .381 & .424 & .114 \\
.000 & .000 & .000 & .004 & .090 & .400 & .506
\end{array}\right]
\end{array},
$$

and the transition matrix in the flood flow cluster is

$$P_F^f = \begin{array}{c} \\ 100 \\ 125 \\ 150 \\ 175 \\ 200 \\ 225 \\ 250 \\ 275 \\ 300 \\ 325 \\ 350 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 100 & 125 & 150 & 175 & 200 & 225 & 250 & 275 & 300 & 325 & 350 \end{array} \\ \begin{bmatrix} .669 & .180 & .099 & .039 & .011 & .002 & .000 & .000 & .000 & .000 & .000 \\ .450 & .231 & .176 & .095 & .036 & .010 & .002 & .000 & .000 & .000 & .000 \\ .246 & .218 & .229 & .171 & .091 & .034 & .009 & .002 & .000 & .000 & .000 \\ .106 & .151 & .220 & .228 & .167 & .086 & .032 & .008 & .002 & .000 & .000 \\ .035 & .077 & .156 & .223 & .226 & .162 & .082 & .030 & .008 & .001 & .000 \\ .009 & .029 & .081 & .161 & .225 & .223 & .157 & .078 & .028 & .007 & .001 \\ .002 & .008 & .031 & .085 & .165 & .227 & .221 & .152 & .074 & .026 & .008 \\ .000 & .002 & .009 & .033 & .090 & .170 & .229 & .218 & .148 & .071 & .031 \\ .000 & .000 & .002 & .010 & .036 & .094 & .175 & .231 & .215 & .143 & .095 \\ .000 & .000 & .000 & .002 & .010 & .038 & .098 & .179 & .232 & .212 & .227 \\ .000 & .000 & .000 & .000 & .002 & .011 & .041 & .103 & .184 & .233 & .426 \end{bmatrix} \end{array} .$$

For North Creek, the transition matrices of the Markov chain in the normal and drought flow clusters, respectively, are

$$P_N^n = \begin{array}{c} \\ 20 \\ 45 \\ 70 \\ 95 \end{array} \begin{array}{c} \begin{array}{cccc} 20 & 45 & 70 & 95 \end{array} \\ \begin{bmatrix} .765 & .204 & .030 & .001 \\ .407 & .412 & .162 & .020 \\ .116 & .364 & .383 & .137 \\ .016 & .141 & .398 & .446 \end{bmatrix} \end{array} \text{and} \quad P_N^d = \begin{array}{c} \\ 5 \\ 30 \\ 55 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 30 & 55 \end{array} \\ \begin{bmatrix} .834 & .163 & .003 \\ .288 & .610 & .103 \\ .018 & .377 & .605 \end{bmatrix} \end{array},$$

and the transition matrix in the flood flow cluster is

$$P_N^f = \begin{array}{c} \\ 25 \\ 50 \\ 75 \\ 100 \\ 125 \\ 150 \\ 175 \\ 200 \end{array} \begin{array}{c} \begin{array}{cccccccc} 25 & 50 & 75 & 100 & 125 & 150 & 175 & 200 \end{array} \\ \begin{bmatrix} .670 & .211 & .091 & .023 & .004 & .000 & .000 & .000 \\ .408 & .287 & .200 & .082 & .020 & .003 & .000 & .000 \\ .182 & .252 & .284 & .188 & .074 & .017 & .002 & .000 \\ .057 & .144 & .261 & .279 & .176 & .066 & .015 & .002 \\ .012 & .053 & .156 & .268 & .273 & .165 & .059 & .014 \\ .002 & .013 & .060 & .167 & .275 & .267 & .153 & .063 \\ .000 & .002 & .015 & .068 & .179 & .280 & .259 & .197 \\ .000 & .000 & .002 & .018 & .076 & .191 & .284 & .428 \end{bmatrix} \end{array}.$$

For our MDP, if the system moves to a different cluster in period t and the AR(1) process defined on the state space in period $t-1$ moves to a state in period t that is not in the state space in period t, we take the closest state in the state space in period t as the streamflow rate in period t.

For the electricity price, we consider the hourly average values of the real-time data available for New York City between the years 2007 and 2019; we retrieve this data from [193]. It is important to note that the electricity price can be negative according to our time series data. We model the hourly electricity prices as follows: First, we deseasonalize the price data to remove the effect of seasonal variation on our spike identification. Following [179], we construct a seasonality model by fitting a linear regression to the price data:

$$s_t = \gamma_1 + \sum_{i=1}^{11} \gamma_{2i} D_t^{2i} + \sum_{j=1}^{6} \gamma_{3j} D_t^{3j} + \sum_{h=1}^{23} \gamma_{4h} D_t^{4h},$$

where γ_1 is a constant and γ_{2i}, γ_{3j}, and γ_{4h} are the coefficients of the dummy variables D_t^{2i}, D_t^{3j}, and D_t^{4h}, that are equal to one if period t is in month i, week day j, and hour h, respectively. Then, we remove the seasonal effect from the observed prices and determine the spikes under the assumption that the highest 5% and lowest 5% of the deseasonalized prices are outliers. The spikes are the differences between these outliers and the mean of the remaining deseasonalized prices after these outliers are removed. Figure 3.4 illustrates the empirical distribution of the spikes. For a more refined seasonality model, we remove the spikes from the observed prices and fit the above linear regression to the despiked prices. Finally, we subtract the refined seasonal effect from the despiked prices and formulate the despiked and deseasonalized price, ρ_t, as an AR(1) process: $\rho_t = (1-\kappa)\,\rho_{t-1} + \sigma\epsilon_t, \ \forall t$, where κ is the speed of mean reversion and σ is the volatility of white noise. The parameter estimates of this AR(1) process are $\hat{\kappa} = 0.328$ and $\hat{\sigma} = 13.674$. Table 3.2 exhibits the parameter estimates of the seasonality model.

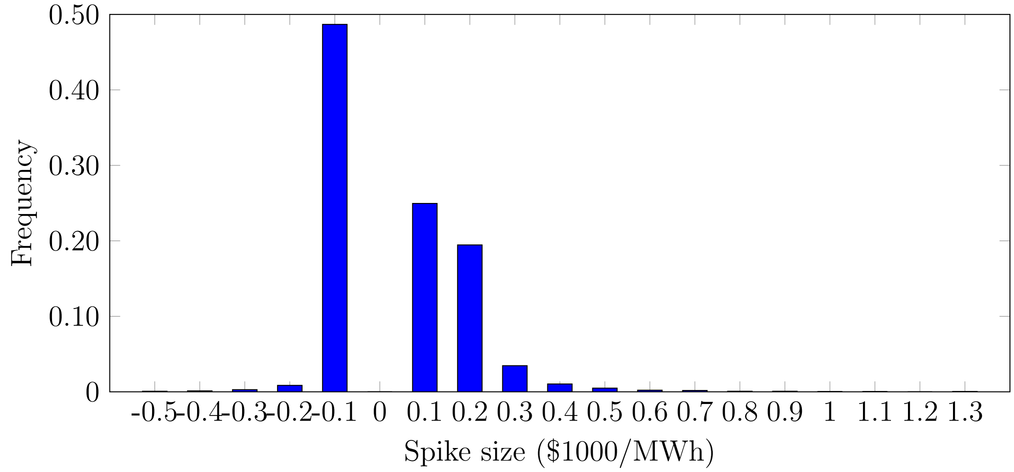


Figure 3.4: Empirical distribution of the spikes.

We employ the trinomial lattice method of [194] to characterize the AR(1) process of the electricity price as a finite-state Markov chain. Following the suggestions of [194] and [195] regarding the number of time steps that should be iterated, we construct a three-hour trinomial lattice for our AR(1) process (Figure 3.5).

The Markov chain obtained from this lattice has the state space $\mathcal{P} :=$

Table 3.2: Parameter estimates of the despiked price seasonality model.

$\hat{\gamma}_1$	34.50											
i	1	2	3	4	5	6	7	8	9	10	11	
$\hat{\gamma}_{2i}$	19.3	13.1	-1.5	-6.5	-9.5	-10.2	-5.5	-6.4	-9.0	-10.0	-6.4	
j	1	2	3	4	5	6						
$\hat{\gamma}_{3j}$	4.1	4.2	4.5	4.3	3.7	1.6						
h	1	2	3	4	5	6	7	8	9	10	11	12
$\hat{\gamma}_{4h}$	-1.9	-3.7	-3.4	-2.8	-1.6	2.6	5.0	7.3	9.4	10.5	10.4	10.4
h	13	14	15	16	17	18	19	20	21	22	23	
$\hat{\gamma}_{4h}$	9.6	9.0	9.3	11.6	15.4	15.6	13.2	9.9	6.4	3.0	0.6	

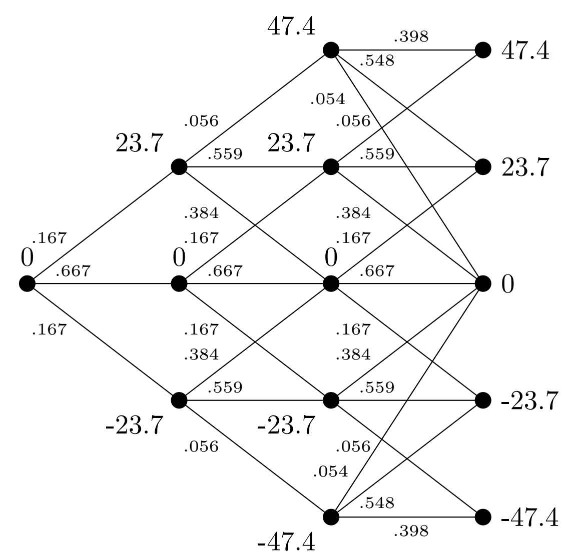


Figure 3.5: Trinomial lattice for the despiked and deseasonalized price. Nodes represent discrete states of the Markov chain and arcs represent transitions between these states.

$\{-47.4, -23.7, 0, 23.7, 47.4\}$ and the following transition matrix:

$$P = \begin{array}{c} \\ -47.4 \\ -23.7 \\ 0.0 \\ 23.7 \\ 47.4 \end{array} \begin{array}{c} \begin{array}{ccccc} -47.4 & -23.7 & 0.0 & 23.7 & 47.4 \end{array} \\ \begin{bmatrix} .398 & .548 & .054 & 0 & 0 \\ .056 & .559 & .384 & 0 & 0 \\ 0 & .167 & .667 & .167 & 0 \\ 0 & 0 & .384 & .559 & .056 \\ 0 & 0 & .054 & .548 & .398 \end{bmatrix} \end{array}.$$

We also restrict the spikes to take values from the set $\mathcal{J} :=$

$\{-500, -400, \ldots, 1200, 1300\}$. Note that the spike occurrence probability is 10% in each period.

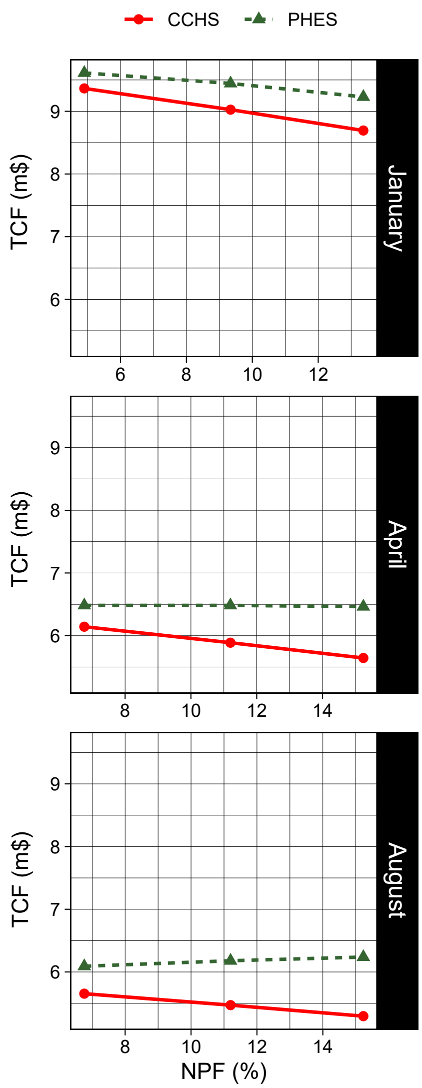
For our numerical study, we construct twelve different scenarios in which the pumping capability is turned on (PHES) or off (CCHS); the upper reservoir is fed by the Hudson River at Fort Edward or at North Creek; and the planning horizon spans the month of January, April, or August in 2019 ($T = 720$ hours). We employ the efficiency curve of Francis turbines [196] for our efficiency calculations; the same curve applies to the upper and lower turbines in the discharging mode as well as the pump (turbine in reverse mode) in the charging mode (do not require Assumption 3.3.1 in our experiments). We assume that $h_1 = h_2 = 100\ m$, $C_U = C_L = 10\ hm^3$, $Q_{RU} = Q_{RL} = Q_P = 0.4\ hm^3$, and $x_{u1} = x_{l1} = 5\ hm^3$ for our base case; we conduct our experiments by varying the values of C_U, C_L, h_1, h_2, and negative price occurrence frequency (NPF). The NPF is 4.90% in January, 6.76% in April, and 6.76% in August, according to our price model. For each month, we obtain two other values of NPF (in addition to the one observed in our price model) by multiplying the numbers of negative spike occurrences in our price model with certain constants. The initial exogenous states for Fort Edward and North Creek are $y_1 = (150, 0)$ and $y_1 = (45, 0)$ in January, $y_1 = (225, 0)$ and $y_1 = (100, 0)$ in April, and $y_1 = (125, 0)$ and $y_1 = (30, 0)$ in August, respectively. Finally, x_{ut} and x_{lt} take values from the set $\{0, 0.2, 0.4, \ldots, 10\}$, a_t takes values from the set $\{-0.4, -0.2, 0, 0.2, 0.4\}$, and b_t takes values from the set $\{0, 0.2, 0.4\}$. We solve the recursion of our MDP in each instance, with Figures 3.6 and 3.7 showing our results.

3.3.3 Discussion of the numerical results

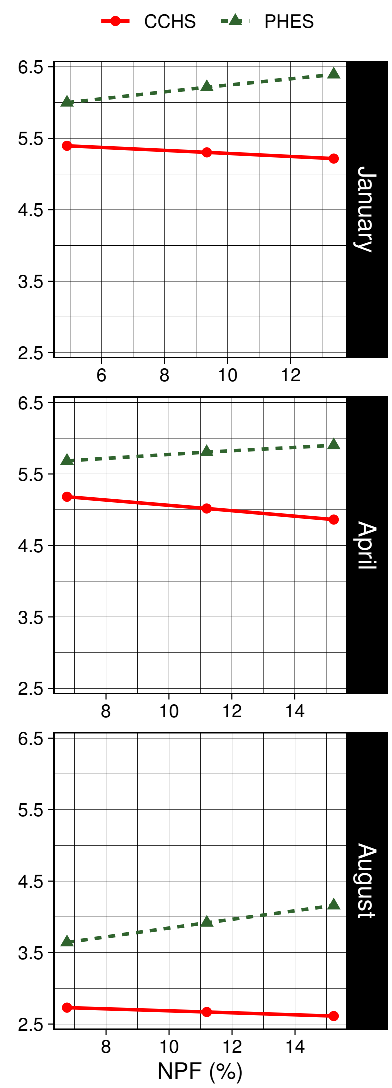
We observe from Figure 3.6 that, for both system configurations at Fort Edward, the total cash flow (TCF) decreases as the NPF grows, with the exception of the PHES system in August. However, in contrast to the CCHS, the PHES system is expected to benefit from an increase in the NPF. This counter-intuitive result can be explained by the availability of excess streamflow in January and April that dampens the incentive to pump water to purchase energy at low prices. The PHES system operator considers purchasing energy at only negative prices in January and April, leading to relatively small amounts of energy purchased from the market. Consequently, in the months of January and April with high streamflow rates, both system configurations yield similar and large amounts of water released from both reservoirs as well as energy sold to the market, thereby suffering from an increase in the NPF. Motivated by our observation in August with low streamflow rate, we repeat our experiments for North Creek with much lower streamflow rates compared to Fort Edward. We observe that the lower streamflow availability significantly drains the TCF for both system configurations, and the PHES system at North Creek always benefits from an increase in the NPF. The PHES transformation provides the greatest benefit (with an improvement of 59.2% in the TCF) at North Creek in August when the NPF is highest.

For both Fort Edward and North Creek, our time series models indicate that the streamflow rates are the highest in April and the lowest in August, while the electricity prices are the highest in January and the lowest in April. For Fort Edward, the price effect is more dominant so that the amounts of energy sold and the TCFs are the highest in January. For North Creek, on the other hand, the streamflow effect is more dominant so that the amounts of energy sold are the highest in April. However, since the electricity price in January is significantly higher than in April, the large amounts of energy sold in April cannot lead to higher TCFs in April than in January.

We observe from Figure 3.7 that the PHES transformation tends to become more beneficial as the capacity levels of the upper and lower reservoirs grow only up to certain points. This increase in the value of PHES is more significant at North Creek in August. We also note that the benefit of PHES transformation increases as the head of the upper station increases, while it decreases as the head of the lower station increases. This is because increasing the head of the upper station improves the pumping capability of the PHES system, whereas increasing the head of the lower station improves the energy generation potential in the lower reservoir, reducing the need for energy storage by pumping water. This change in the value of PHES is again more significant at North Creek in August.

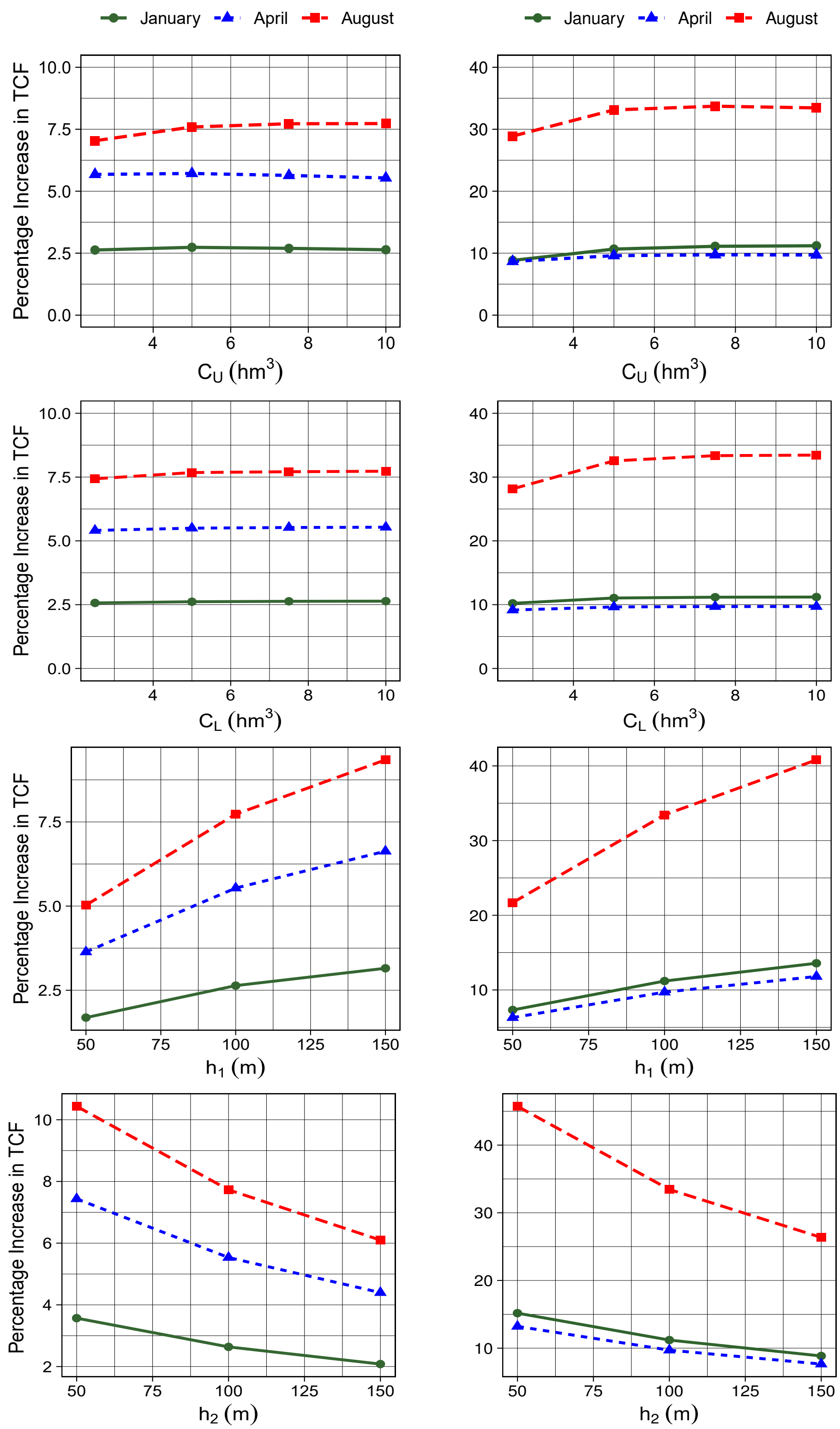


(a) Fort Edward



(b) North Creek

Figure 3.6: Numerical results for the value of PHES with respect to NPF.



(a) Fort Edward (b) North Creek

Figure 3.7: Numerical results for the value of PHES with respect to physical system characteristics.

The tightness of our upper bound in Theorem 3.3.1 is greatly influenced by the range in which the electricity prices vary. Our upper bound may not be tight enough in markets with high price spikes, particularly when the negative prices exist within the volatile structure. The price range is high in our experiments: In January, the maximum and minimum prices are \$300.99 and -\$62.7 per MWh, respectively, leading to an upper bound of 26 million dollars. In April, the maximum and minimum prices are \$379.76 and -\$79.41 per MWh, respectively, leading to an upper bound of 33 million dollars. In August, the maximum and minimum prices are \$163.91 and -\$6.4 per MWh, respectively, leading to an upper bound of 12 million dollars. Our upper bound is thus noticeably tighter at North Creek in August, when the PHES transformation provides the greatest benefit due to the limited streamflow availability. Our upper bound is expected to be much tighter in more stable markets like Nord Pool. For instance, Norway experienced low price ranges with the maximum and minimum prices of (€68.00, €43.81), (€60.70, €27.91), and (€52.79, €28.96) per MWh, in January, April, and August of 2019, respectively [197].

3.4 Comparison with deterministic solution approach

We compare our solution approach with a deterministic one and measure the value of including randomness in our formulation. The advantage of using a stochastic solution over a deterministic solution, i.e., the value of the stochastic solution (VSS), determines the cost of ignoring uncertainty in decision-making. Concentrating on the despiked prices in this section, we consider the deterministic version of our problem that ignores uncertainty by replacing the random components (r_t for the streamflow rate and ρ_t for the electricity price) with their expected values. Let π^d denote the optimal policy for this deterministic problem.

The TCF of the policy π^d can be calculated from the recursion in (3.1) (without optimization) when the actions are restricted to obey the policy π^d in each state and each period. Table 3.3 exhibits the VSS (i.e., the percentage improvement in the TCF via the stochastic solution) for the base case in each of our twelve scenarios. We have found that the VSS is maximum at North Creek in August (21.71% and 30.83% for CCHS and PHES, respectively).

Table 3.3: Numerical results for the VSS.

Type	Season	Fort Edward			North Creek		
		S	D	VSS (%)	S	D	VSS (%)
CCHS	January	8.473	8.221	2.97	4.314	3.812	11.64
	April	5.157	4.954	3.94	4.133	3.771	8.76
	August	4.607	4.253	8.07	1.704	1.334	21.71
PHES	January	8.490	8.230	3.06	4.555	3.890	14.60
	April	5.199	4.983	4.15	4.303	3.846	10.62
	August	4.724	4.305	8.87	2.199	1.521	30.83

S: TCF of the optimal policy. D: TCF of the policy π^d.

3.5 Conclusion

In this chapter, we consider a CCHS that can be converted to a PHES system by utilizing a reversible turbine between the reservoirs. In order to examine the value of this transformation, we formulate the energy generation and storage problem in both systems as an MDP by taking into account uncertainties in the streamflow rate and electricity price. With this formulation, we analytically derive an upper bound on the profit improvement that can be obtained from this transformation. Using data-calibrated time series models for the streamflow rate and electricity price, we solve the problem to optimality for various realistic cases of these systems in different seasons. Our numerical results imply that PHES becomes an important large scale storage option when the negative prices occur more frequently or the streamflow availability is more limited. Although adding a reversible turbine to a CCHS to obtain a PHES system can significantly increase the profits, a cost-benefit analysis is required in real-life cases to compare the cost of transformation with the profit improvement. Our MDP can help policymakers and investors to calculate the benefit in such an analysis, accurately taking into account the problems stochastic and nonlinear nature.

Chapter 4

Short-Term Assessment of Pumped Hydro Energy Storage Configurations: Up, Down, or Closed?

4.1 Introduction

Renewable energy sources have received much attention to mitigate the high dependence on fossil fuels and the resulting environmental impacts [198, 199]. Wind and solar account for roughly two-thirds of the global power capacity additions [152]. Since the variability and intermittency of such renewable sources lower the reliability and utilization of energy systems, they should often be accompanied by efficient and flexible storage units [19, 200]. In this regard, one of the most commonly used large-scale storage technologies is PHES [18]. This technology, which currently accounts for more than 99% of the global installed

This chapter is published in journal of *Renewable Energy* [122].

energy storage capacity, is among the best commercially available storage options in terms of environmental and economic performance, compared to other technologies like advanced batteries [201]. A PHES facility stores water by pumping it into an elevated reservoir and produces energy by releasing it into a lower reservoir. Depending on the local topographical conditions as well as the water runoff availability, these facilities can be configured as closed-loop or open-loop [202]. There is no natural inflow in a closed-loop PHES facility, whereas a natural inflow feeds either upper, lower, or both reservoirs in an open-loop PHES facility. The open-loop PHES facilities can be built as cascading or non-cascading. One can release water from multiple reservoirs in a cascading PHES facility, but from a single reservoir in a non-cascading PHES facility. The open-loop PHES facilities can be constructed from the existing conventional hydropower systems by replacing their turbines with reversible ones [121]. The closed-loop PHES facilities, on the other hand, have recently gained popularity since they can be developed artificially off-stream and are in general environmentally better than the open-loop PHES facilities [5, 152].

In the presence of suitable on-stream conditions, like mountainous areas with a flowing mid-altitude river, it is crucial to evaluate the potential revenues of different PHES configurations when installing a new PHES facility. Building an open-loop facility by constructing a new reservoir in addition to the existing one (for example in a conventional hydropower station) may be environmentally less severe than building an entirely new closed-loop facility. An important decision may then be whether to construct this new reservoir as the lower or upper reservoir (if both options are technically feasible). The answer to this question may depend nontrivially on the available infrastructure (cascading or non-cascading). This chapter aims to identify the most profitable option prior to PHES installation, by comparing the short-term total cash flows of different PHES configurations for both cascading and non-cascading infrastructures in a market setting where the electricity price can be negative. To this end, we model the real-time decision-making process of the PHES operator as an MDP. While the energy generation and storage problem of the PHES operator entails sequential decision-making over a multi-period planning horizon, the variable nature of streamflow rate and electricity price creates significant operational uncertainty. Our MDP framework enables us to capture this stochastic nature of the problem in a structured fashion.

The literature dealing with the optimization of PHES facilities (stand-alone or integrated with other energy sources in a hybrid system) can be divided into two major streams. The first research stream studies the optimal sizing problem for PHES facilities, which plays a critical role in ensuring a reliable and cost-effective energy supply as well as recovering discarded or excess energy from renewable sources [203]. In this research stream, Zheng et al. [124] optimize the sizing of various storage technologies including PHES for an average demand of a household. Brown et al. [64] utilize a scenario-based linear programming framework for the optimal sizing of wind-PHES systems. Kapsali et al. [204] take deterministic approaches for the optimal sizing of wind-PHES systems in remote islands. Ma et al. [170] propose a genetic algorithm approach for the optimal sizing of solar-PHES systems. Kocaman and Modi [205] and Kocaman [171] develop a two-stage stochastic mixed-integer programming model for the optimal sizing of solar-PHES systems. Abdalla et al. [27] apply a two-stage robust generation expansion planning model to the optimal sizing of wind-PHES systems. Bhayo et al. [173] propose a particle swarm optimization model for the optimal sizing of solar-PHES-battery systems.

The second research stream studies the energy generation and storage problem for PHES facilities. Foley et al. [125] formulate a deterministic dynamic program for the long-term operational planning problem for wind-PHES systems. Vojvodic et al. [206] develop a multi-stage stochastic optimization model for PHES facilities and propose a scatter search algorithm to solve their large-scale problems. Wang et al. [97] study the unit commitment problem for solar-wind-PHES systems and propose an artificial sheep algorithm as a meta-heuristic optimization method. Yıldıran and Kayahan [111] study the day-ahead market bidding decisions and real-time operations of a wind-PHES system via a risk-averse stochastic optimization model. Ak et al. [133] formulate a nonlinear program for PHES facilities obtained from conventional hydropower stations. Avci et al. [120] develop an MDP for wind-PHES systems to characterize the optimal policy structure.

Huang et al. [70] examine the short-term operations of a hybrid energy system including a PHES facility via a multi-objective stochastic optimization model. Lu et al. [28] study the short-term joint dispatch problem for a hybrid energy system including a PHES facility. Toufani et al. [121] demonstrate the short-term economical benefits of transforming conventional hydropower stations into PHES facilities via an MDP framework. Our study falls into this second research stream.

To our knowledge, no paper in the literature investigates the role that the existence and location of the streamflow play in the profitability of PHES facilities. We aim to fill this gap in the literature. Specifically, we provide a short-term assessment of the cash flows from selling and purchasing energy for each of the five different PHES configurations in Figure 4.1: cascading systems with the upstream or downstream flow, non-cascading systems with the upstream or downstream flow, and closed-loop systems. Short-term cash-flow comparisons of these configurations, if combined with upfront installation costs, may have important implications for energy investors, planners, and policy-makers while investing in new PHES systems as well as upgrading the appropriate existing energy systems. We also note that our MDP framework takes into account uncertainties in the streamflow rate and electricity price. Although the natural inflow is the most critical input to any typical PHES facility, only a limited number of papers integrate the streamflow rate into the optimization of PHES facilities, and most of these papers ignore the streamflow uncertainty and take the streamflow rate as a deterministic model component [51, 56, 73, 127].

The main contributions of this chapter are as follows:

- Our study is the first to compare different PHES configurations in which the natural inflow exists or not, the natural inflow exists and feeds either the upper or lower reservoir, and the infrastructure is either cascading or non-cascading.
- We develop an MDP to compare the short-term total cash flows from these different PHES configurations. We capture the nonlinear dynamics of the

problem as well as the streamflow and price uncertainties in our MDP, utilizing time-series models with random components.

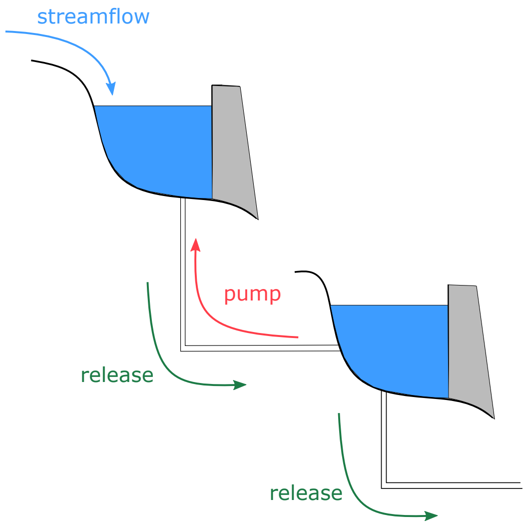
- With our MDP framework, we are able to offer theoretical bounds on the effects of different PHES configurations on the total cash flows.
- With data-calibrated time-series models, we conduct numerical experiments to provide insights into the optimal revenues of different PHES configurations in different environments.

The rest of this chapter is organized as follows: Chapter 4.2 formulates the energy generation and storage problem for a very general PHES configuration. Chapter 4.3 presents our analytical comparisons of various PHES configurations. Chapter 4.4 presents our numerical results. Chapter 4.5 offers a summary and concludes. Proofs of the analytical results are contained in Appendix B.

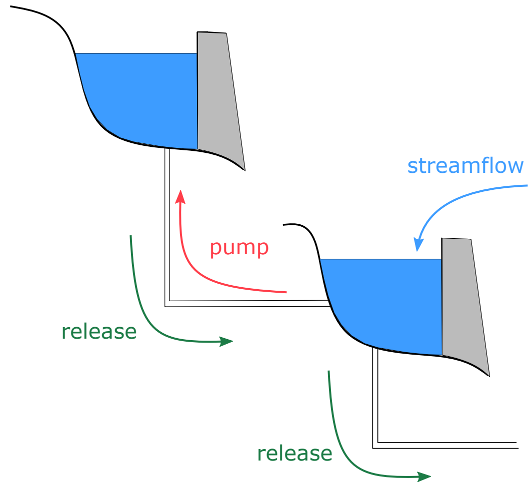
4.2 Problem formulation

Figure 4.1 illustrates the five different PHES configurations that we consider in this study: cascading facilities (see Figures 4.1a and 4.1b), non-cascading facilities (see Figures 4.1c and 4.1d), and closed-loop facilities (see Figure 4.1e). In this section, we present the MDP formulation of a very general PHES configuration that includes each configuration in Figure 4.1 as a special case. This general configuration has two connected reservoirs located at different altitudes. The natural streamflows feed both the upper and lower reservoirs. Energy can be generated by releasing water from the upper reservoir to the lower reservoir and from the lower reservoir to the stream bed, while energy can be stored by pumping water from the lower reservoir to the upper reservoir. The excess streamflow into the reservoirs as well as the excess water flow between the reservoirs lead to a water spillage from the facility. The water spilling from any reservoir is lost. We assume that the market accepts the dispatch and purchase amounts set by the PHES operator. The system's contribution to the market's overall energy supply is very small, making it benign to assume that the operator is merely a price-taker [120, 133, 176, 183].

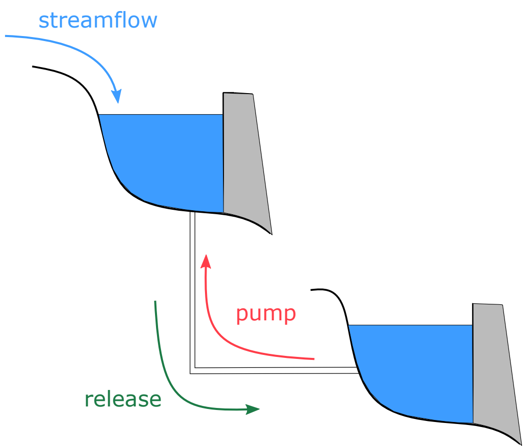
We study the energy generation and storage problem of the PHES operator via an MDP defined over a finite planning horizon of T periods. Let $\mathcal{T} := \{1, 2, \ldots, T\}$ denote the set of periods. The amount of energy that can be generated by releasing a unit volume of water from any reservoir equals the multiplication of the water density, the gravitational constant, the potential vertical distance of the hydropower station, and the efficiency of the system. We assume constant and equal vertical distances for both reservoirs. We denote the maximum amount of water (in energy units) that can be stored in the upper and lower reservoirs by C_U and C_L, respectively. We denote the maximum amounts of water (in energy units) that can be released from the upper (or lower) reservoir and pumped from the lower reservoir in any period by C_R and C_P, respectively. We assume that $C_R \leq \min\{C_L, C_U\}$ and $C_P \leq C_L$. We denote the efficiency level of the PHES facility by $\theta \in (0, 1]$. We define x_{ut} and x_{lt} as the amounts of water accumulated (in energy units) at the beginning of period t in the upper and lower reservoirs, respectively. We include the variables $x_{ut} \in [0, C_U]$ and $x_{lt} \in [0, C_L]$ in our state description. These state variables evolve over time according to the energy generation and storage decisions as well as the streamflow rate. We define p_t as the electricity price in period t, r_{ut} as the amount of water runoff to the upper reservoir in period t (in energy units), and r_{lt} as the amount of water runoff to the lower reservoir in period t (in energy units). We also include the tuple $y_t := (p_t, r_{ut}, r_{lt})$ in our state description.



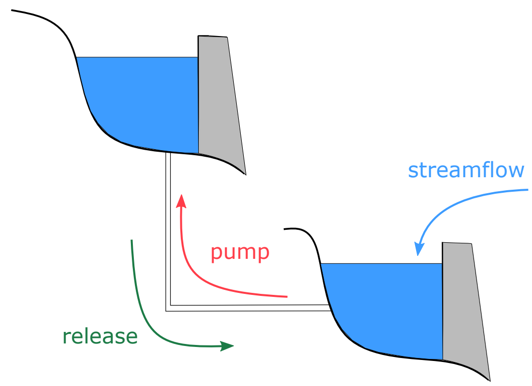
(a) Cascading system with upstream flow



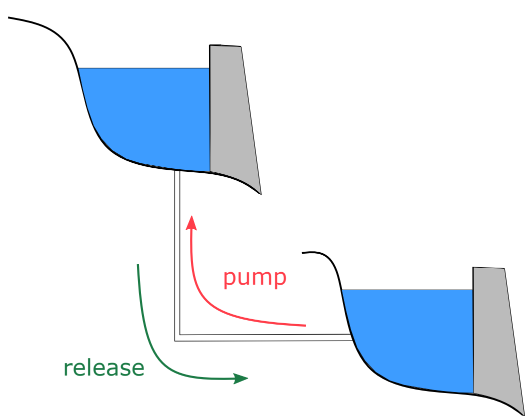
(b) Cascading system with downstream flow



(c) Non-cascading system with upstream flow



(d) Non-cascading system with downstream flow



(e) Closed-loop system

Figure 4.1: Illustration of various PHES configurations.

Figure 4.2 illustrates the sequence of events in each period. In any period $t \in \mathcal{T}$, the operator observes first the exogenous state variables (p_t, r_{ut}, and r_{lt}) and then the accumulated amounts of water in the reservoirs (x_{ut} and x_{lt}). With these observations, the operator determines first the amount of water that will be released from the lower reservoir $b_t \in \mathbb{R}_+$ (in energy units) and then the amount of water that will be released from or pumped to the upper reservoir $a_t \in \mathbb{R}$ (in energy units). Non-negative values of a_t represent the action of releasing water from the upper reservoir, whereas negative values of a_t represent the action of pumping water from the lower reservoir.

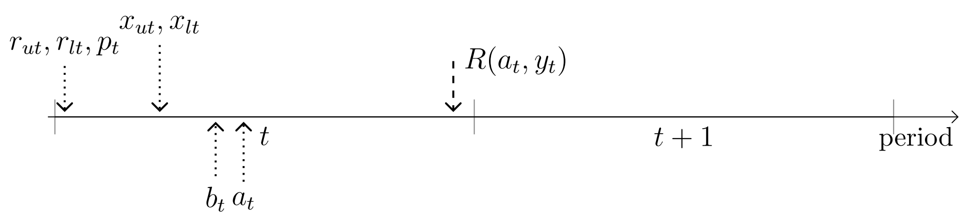


Figure 4.2: Sequence of events in each period.

Let $\mathbb{U}(x_{ut}, x_{lt}, y_t)$ denote the set of action pairs (a_t, b_t) that are admissible in state (x_{ut}, x_{lt}, y_t). For any action pair $(a_t, b_t) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, the energy and power capacities of the PHES facility imply that $-\min\{x_{lt} - b_t, C_P\} \leq a_t \leq \min\{x_{ut}, C_R\}$ and $0 \leq b_t \leq \min\{x_{lt}, C_R\}$. Since the excess amount of water spilling from each reservoir is lost, the state variables x_{ut} and x_{lt} evolve over time as

$$x_{u(t+1)} = \min\Big\{\min\{x_{ut} - a_t, C_U\} + r_{u(t+1)}, C_U\Big\} = \min\{x_{ut} - a_t + r_{u(t+1)}, C_U\}$$

and

$$x_{l(t+1)} = \min\Big\{\min\{x_{lt} + a_t - b_t, C_L\} + r_{l(t+1)}, C_L\Big\} = \min\{x_{lt} + a_t - b_t + r_{l(t+1)}, C_L\}.$$

The objective is to maximize the expected total cash flow that accrues from selling or purchasing energy over the finite horizon. There are two different types of decisions that we need to consider in our payoff formulation in any period t: A certain amount of water is (i) released from the upper reservoir to generate energy ($a_t > 0$) or (ii) pumped from the lower reservoir to store energy ($a_t \leq 0$). The amount of energy generated from or stored in the upper reservoir in period t as a function of action a_t can be formulated as $E(a_t) = \min\{\theta a_t, a_t/\theta\}$. Considering this energy together with the energy generated by releasing water from the lower reservoir, the amount of energy sold or purchased in period t is given by $E(a_t) + \theta b_t$. Hence, the payoff in period t as a function of action pair (a_t, b_t) and exogenous state tuple y_t can be formulated as $R(a_t, b_t, y_t) = (E(a_t) + \theta b_t)\, p_t$.

A control policy π is the sequence of decision rules $(a_t^\pi(x_{ut}^\pi, x_{lt}^\pi, y_t), b_t^\pi(x_{ut}^\pi, x_{lt}^\pi, y_t))_{t\in\mathcal{T}}$, where x_{ut}^π and x_{lt}^π denote the random state variables governed by policy π, $\forall t \in \mathcal{T}\backslash\{1\}$. We denote the set of all admissible control policies by Π. For any initial state (x_{u1}, x_{l1}, y_1), the optimal expected total cash flow over the finite horizon is

$$\max_{\pi\in\Pi} \mathbb{E}\left[\sum_{t\in\mathcal{T}} R(a_t^\pi(x_{ut}^\pi, x_{lt}^\pi, y_t), b_t^\pi(x_{ut}^\pi, x_{lt}^\pi, y_t), y_t)\middle| x_{u1}, x_{l1}, y_1\right].$$

For each period $t \in \mathcal{T}$ and each state (x_{ut}, x_{lt}, y_t), the optimal value function $v_t^*(x_{ut}, x_{lt}, y_t)$ can be calculated with the following dynamic programming recursion:

$$\begin{aligned} v_t^*(x_{ut}, x_{lt}, y_t) = \max_{(a_t,b_t)\in\mathbb{U}(x_{ut},x_{lt},y_t)} \Big\{ & R(a_t, b_t, y_t) \\ & + \mathbb{E}_{y_{t+1}|y_t}\Big[v_{t+1}^*\Big(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}\Big)\Big]\Big\} \end{aligned} \tag{4.1}$$

where $v_T^*(x_{uT}, x_{lT}, y_T) = 0$. Note that $v_1^*(x_{u1}, x_{l1}, y_1)$ is the optimal expected total cash flow for the initial state (x_{u1}, x_{l1}, y_1) over the finite horizon. We denote by $(a_t^*(x_{ut}, x_{lt}, y_t)$,$b_t^*(x_{ut}, x_{lt}, y_t))$ the optimal action pair for the optimization problem in equation (4.1).

4.3 Analytical comparisons of different configurations

In this section, we establish several theoretical bounds on the revenue gains and losses that can be observed by switching from one configuration to another. For all $t \in \mathcal{T}$, let $v^*_{ct}(x_{ut}, x_{lt}, y_t)$ denote the value function for the closed-loop PHES facility, $v^*_{lt}(x_{ut}, x_{lt}, y_t)$ denote the value function for the open-loop PHES facility with the downstream flow, and $v^*_{ut}(x_{ut}, x_{lt}, y_t)$ denote the value function for the open-loop PHES facility with the upstream flow. The value function $v^*_{ct}(x_{ut}, x_{lt}, y_t)$ can be calculated from the recursion in (4.1) by taking $r_{ut} = r_{lt} = 0, \ \forall t$; the value function $v^*_{lt}(x_{ut}, x_{lt}, y_t)$ can be calculated from the recursion in (4.1) by taking $r_{ut} = 0$ and $r_{lt} = r_t, \ \forall t$; and $v^*_{ut}(x_{ut}, x_{lt}, y_t)$ can be calculated from the recursion in (4.1) by taking $r_{lt} = 0$ and $r_{ut} = r_t, \ \forall t$. Let $p_t \in \left[\underline{p}_t, \overline{p}_t\right]$ and $r_t \in [0, \overline{r}_t]$, for all $t \in \mathcal{T}$. With this notation we obtain the following theorem:

Theorem 4.3.1. *For a given infrastructure (cascading or non-cascading), the following relationships hold for each $t \in \mathcal{T}$.*

(a) $v^*_t(x_{ut}, x_{lt}, y_t) \le v^*_t(x_{ut}, x_{lt} + \alpha, y_t)$ *and* $v^*_t(x_{ut}, x_{lt}, y_t) \le v^*_t(x_{ut} + \alpha, x_{lt}, y_t)$, *where* $\alpha > 0$.

(b) Suppose that $r_t \le C_R$. *Then:*

$$
\begin{aligned}
&v^*_{ct}(x_{ut}, x_{lt}, y_t)\\
&\le v^*_{lt}(x_{ut}, x_{lt}, y_t)\\
&\le v^*_{ct}(x_{ut}, x_{lt}, y_t) + \frac{1}{\theta} C_R \sum_{i=t}^{T} (T-i) \max\left\{\left(\max_{i \le \tau \le T} \overline{p}_\tau\right)^+, \left(\max_{i \le \tau \le T} -\underline{p}_\tau\right)^+\right\}\\
&\le v^*_{ut}(x_{ut}, x_{lt}, y_t) + \frac{1}{\theta} C_R \sum_{i=t}^{T} (T-i) \max\left\{\left(\max_{i \le \tau \le T} \overline{p}_\tau\right)^+, \left(\max_{i \le \tau \le T} -\underline{p}_\tau\right)^+\right\}\\
&\le v^*_{ct}(x_{ut}, x_{lt}, y_t) + \frac{2}{\theta} C_R \sum_{i=t}^{T} (T-i) \max\left\{\left(\max_{i \le \tau \le T} \overline{p}_\tau\right)^+, \left(\max_{i \le \tau \le T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}
$$

(c) Suppose that the upper and lower reservoirs have sufficiently large capacities to prevent any water spillage. Then:

$$
\begin{aligned}
&v_{lt}^*(x_{ut}, x_{lt}, y_t) - (T-t)\frac{1}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}\left(\max_{t\leq\tau\leq T}\overline{r}_\tau\right) \\
&\leq\ v_{ut}^*(x_{ut}, x_{lt}, y_t) \\
&\leq v_{lt}^*(x_{ut}, x_{lt}, y_t) + (T-t)\theta\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+\left(\max_{t\leq\tau\leq T}\overline{r}_\tau\right).
\end{aligned}
$$

Point (a) of Theorem 4.3.1 states that the PHES facility becomes more profitable as the amount of water in the upper or lower reservoir grows. This is because a larger amount of water in the upper reservoir induces greater energy generation capacity and a larger amount of water in the lower reservoir induces greater energy storage and generation capacity.

The first and third inequalities in point (b) of Theorem 4.3.1 state that the open-loop PHES facility with the downstream or upstream flow is more profitable than the closed-loop PHES facility. This is because the PHES facility with downstream or upstream flow includes the closed-loop PHES facility as a special case when $r_t = 0$ and $b_t = 0$, $\forall t$. The second inequality in point (b) presents an upper bound on the revenue gain from switching from the closed-loop configuration to the open-loop configuration with the downstream flow. This bound can be viewed as the cumulative revenue potential from utilizing the downstream flow at full capacity at the best possible electricity price, summed over all future periods. The last inequality in point (b) implies that the same bound also applies to the revenue gain from switching from the closed-loop configuration to the open-loop configuration with the upstream flow.

Point (c) of Theorem 4.3.1 presents lower and upper bounds on the cash flow difference caused by changing the downstream flow to the upstream flow in a special case where water spillage is not possible in any period. Such a case may indeed arise in dry seasons or regions. The bound that we found in point (b) is not tight enough due to the extreme cases where the excess amount of downstream flow spills from the lower reservoir or the excess amount of upstream flow spills from the upper reservoir. Therefore, we constructed tighter bounds in point (c) by disregarding the water spillage possibility in the PHES facility. Finally, we note that our bounds in points (b) and (c) indicate no clear advantage of one of these open-loop configurations over the other.

4.4 Numerical results and discussions

In this section, we conduct numerical experiments to gain further insights into the short-term economical comparisons of different PHES configurations in different environments, by varying the values of reservoir capacity ($C_U = C_L$), negative price occurrence frequency (NPF), and streamflow rate. Using the historical data available from the State of New York, we develop distinct time-series models for the electricity price and streamflow rate that involve stochastic autoregressive processes of order one. We incorporate these parametric models into our MDP by utilizing the exogenous state variables. We then discretize the continuous space of the exogenous state variables for numerical calculations. We refer the reader to Chapter 3 (or [121]) for the time-series models as well as the parameter estimates and Markov chain representations of these time-series models. The planning horizon spans the months of January, April, or August (T = 720 hours); the upper and lower vertical distances are the same and equal to 30 meters; the power capacity of the turbine and pump is 100 MW ($C_R = C_P = 100$ MWh); and the round trip efficiency of the PHES facility is 0.88. We assume that the initial water levels x_{u1} and x_{l1} are the closest states to $C_U/2$ and $C_L/2$, respectively, for all instances. We solve the recursion of our MDP to optimality in each of these instances. In order to evaluate the impact of streamflow availability, we repeat these experiments by calibrating the streamflow model for different fractions of the observed values (25%, 50%, or 75% of the original streamflow rate). Figures 4.3, 4.4, 4.5, and 4.6 exhibit our numerical results.

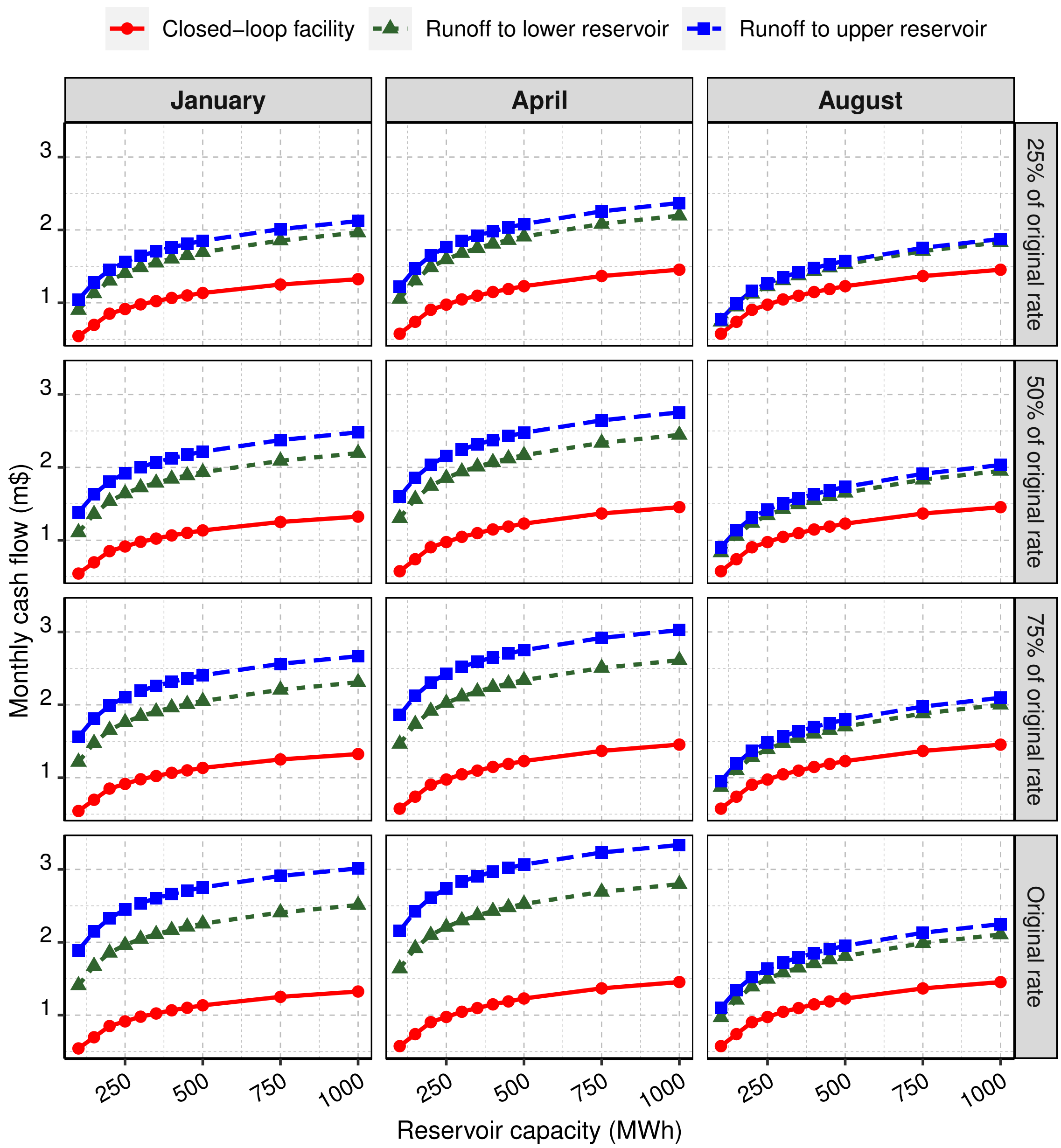


Figure 4.3: Numerical results for various values of the reservoir capacity ($C_U = C_L$) in cascading systems. The NPF is 4.90% in January and 6.76% in April and August.

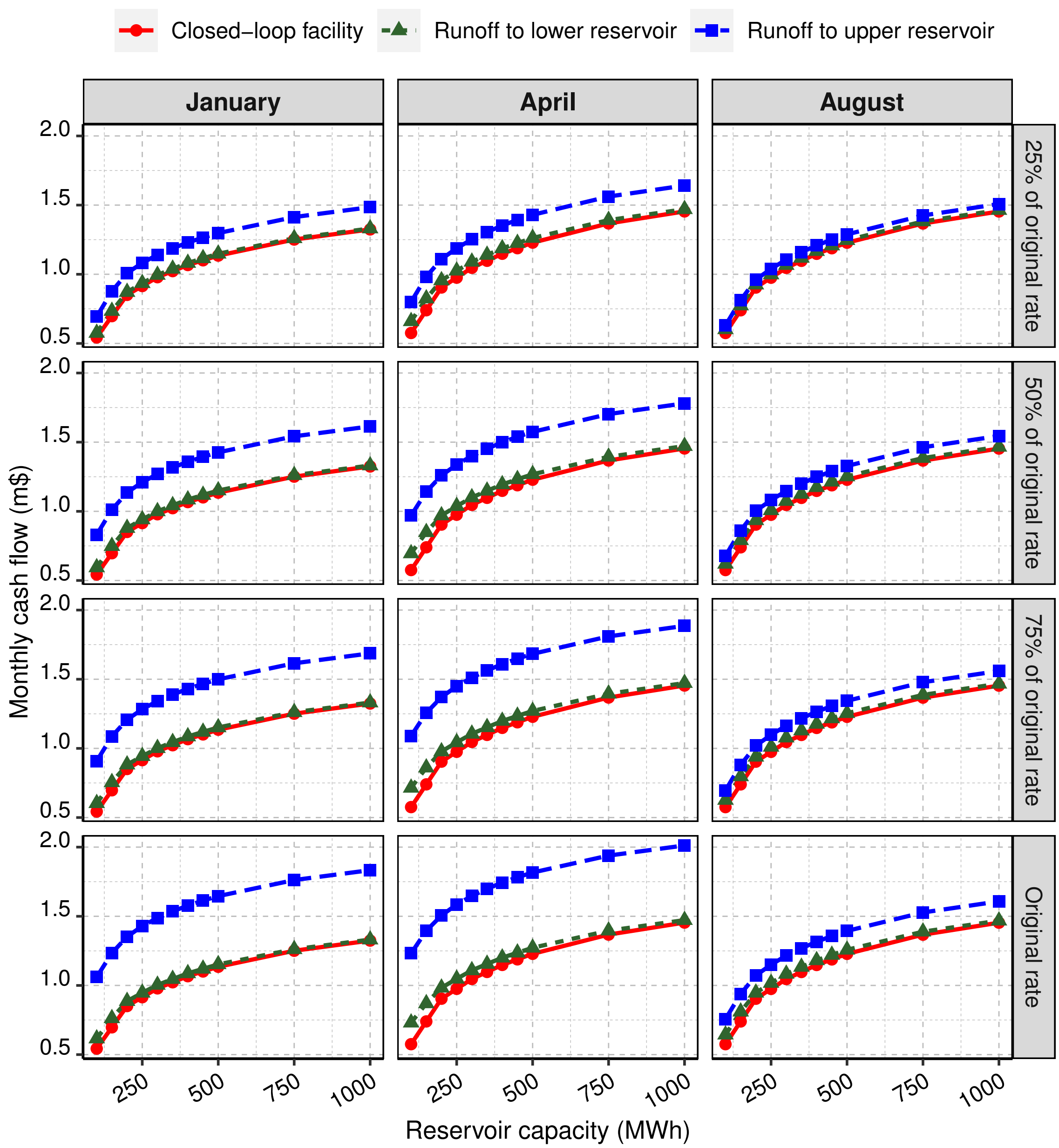


Figure 4.4: Numerical results for various values of the reservoir capacity ($C_U = C_L$) in non-cascading systems. The NPF is 4.90% in January and 6.76% in April and August.

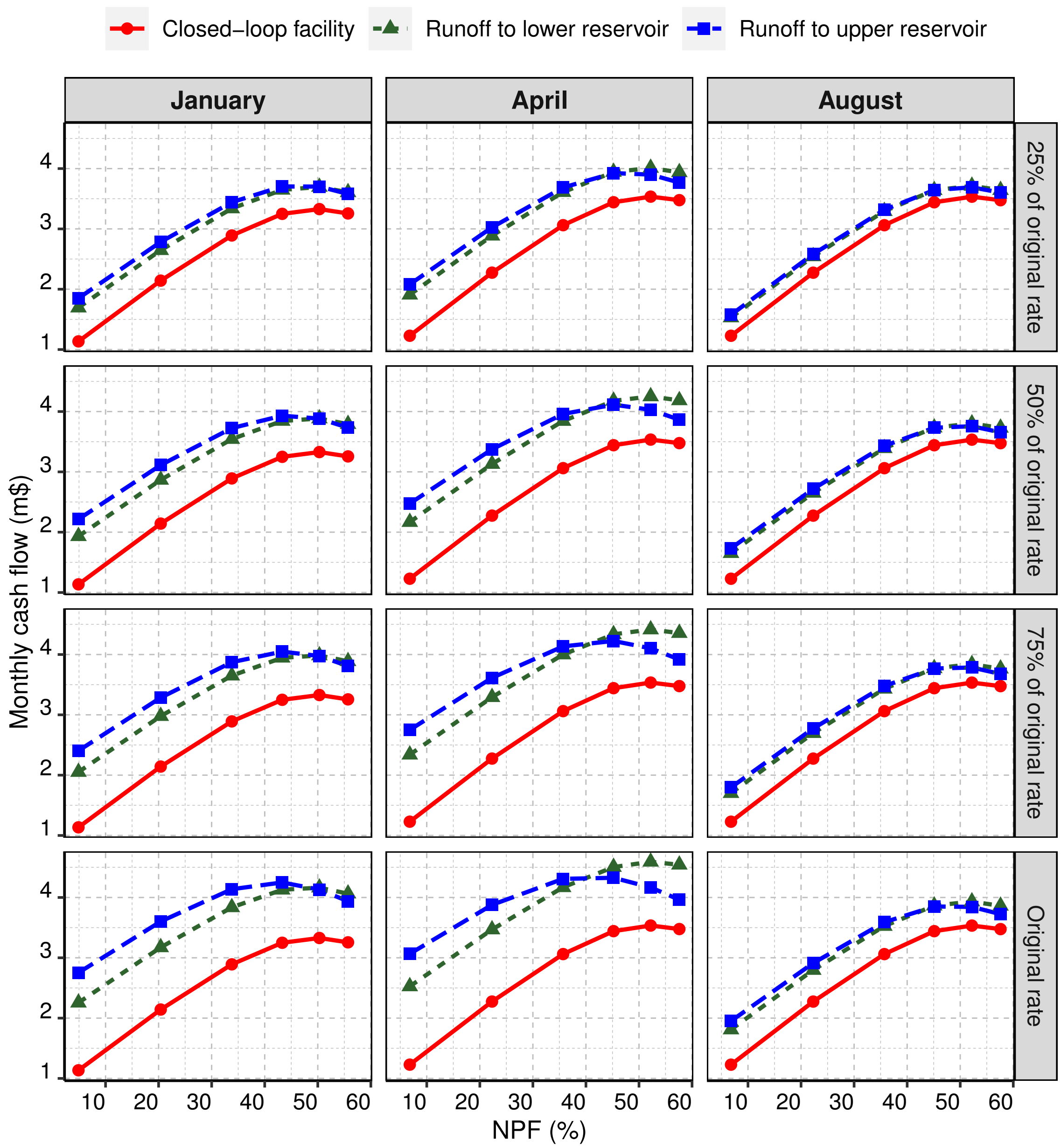


Figure 4.5: Numerical results for various values of the NPF in cascading systems. $C_U = C_L = 500$ MWh.

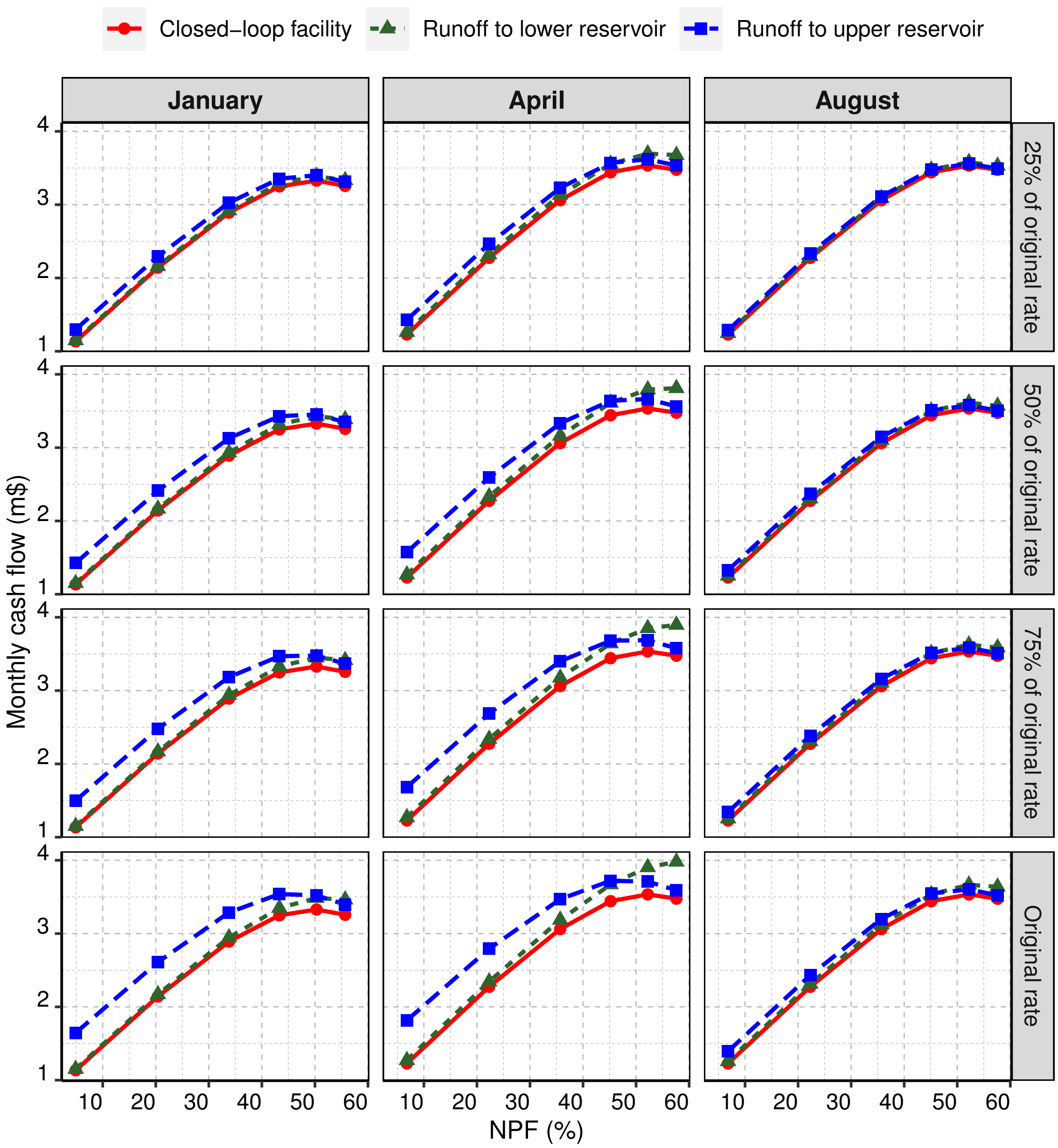


Figure 4.6: Numerical results for various values of the NPF in non-cascading systems. $C_U = C_L = 500$ MWh.

We observe from Figures 4.3 and 4.4 that the monthly cash flow is the highest for the open-loop configurations with the upstream flow and the lowest for the closed-loop configurations. The open-loop configuration has the greatest benefit over the closed-loop configuration when the original streamflow is considered in

April and $C_U = C_L = 100$ MWh (with an improvement of 274.85% in cascading facilities and 114.16% in non-cascading facilities). We also observe that the monthly cash flow decreases for both cascading and non-cascading facilities as the streamflow availability becomes more limited. Likewise, the monthly cash flows in January and April are higher than in August (the driest season of the year) for both cascading and non-cascading facilities.

We observe from Figure 4.3 that, in cascading systems (where the water in the lower reservoir can be released to generate energy), the monthly cash flows of the open-loop configurations with upstream and downstream flows approach each other as the streamflow availability becomes more severely limited so that the location of water runoff is less critical. In August, for example, the cash flows differ by only 2.45% when the streamflow rate is 25% of the original rate and $C_U = C_L = 1000$ MWh. On the other hand, in non-cascading systems, the monthly cash flows of the closed-loop configuration are in general very close to those of the open-loop configuration with the downstream flow. This is because the inability to release water from the lower reservoir significantly restricts the potential gain from utilizing the downstream flow and thus the open-loop configuration with downstream flow becomes similar to the closed-loop configuration. All these results may assist the investors in their choice of the PHES configuration with both economical and environmental concerns, especially in areas with limited streamflow availability.

We also note that the monthly cash flow of operating a large closed-loop PHES facility can be achieved by operating an open-loop PHES facility with much smaller reservoirs. For instance, in cascading systems with the original streamflow rate in April (see Figure 4.3), the monthly cash flow of the closed-loop facility with $C_U = C_L = 1000$ MWh is similar to that of the open-loop facility with the downstream flow and $C_U = C_L = 100$ MWh. In non-cascading systems with the original streamflow rate in April (see Figure 4.4), the monthly cash flow of the closed-loop facility with $C_U = C_L = 1000$ MWh is similar to that of the open-loop facility with the upstream flow and $C_U = C_L = 200$ MWh.

Our time-series model for the electricity price implies that the NPF is 4.90%

in January and 6.76% in April and August. For each month, we obtain five other values of the NPF (in addition to the one implied by our price model) by multiplying the number of negative spike occurrences in our price model with prespecified constants. We observe from Figures 4.5 and 4.6 that the open-loop configurations with downstream flow benefit more (or suffer less) from increased NPF than the open-loop configurations with the upstream flow. This is because the existence of the downstream flow increases the amount of water that can be pumped from the lower reservoir to the upper reservoir, allowing the PHES operator to purchase more energy at negative prices. It thus helps better exploit the benefit of larger NPF values. When the NPF is sufficiently large, for both cascading and non-cascading systems, the open-loop configuration with downstream flow outperforms the open-loop configuration with the upstream flow. This effect of the NPF is the most noticeable in April when the streamflow availability is at its peak. For example, in April with the original streamflow rate, switching from the open-loop configuration with the upstream flow to the open-loop configuration with downstream flow increases the monthly cash flow by 14.63% in cascading systems (see Figure 4.5) and 10.81% in non-cascading systems (see Figure 4.6). We also note from Figure 4.6 that, for non-cascading systems in August, the open-loop configurations of both types perform similarly to the closed-loop configuration. This is because the limited streamflow availability in August provides no significant advantage to open-loop configurations of any type.

The bounds we found in Theorem 4.3.1 vary with the efficiency level and power capacity of the pump/turbine as well as the stochastic components of the system. The tightness of our bounds in point (c) of Theorem 4.3.1 is greatly influenced by the ranges of the electricity prices and streamflow rates. These bounds may not be tight enough in markets with very large price spikes or in environments with very large streamflow rates. For example, for our experiments in April, the maximum and minimum prices are \$379.76 and -\$79.41 per MWh, respectively, and the maximum streamflow rate is 539.7 m^3/s, leading to bounds of 49.3 and 38.1 million dollars, respectively, in point (c) of Theorem 4.3.1. However, for our experiments in August, the maximum and minimum prices are \$163.91 and -\$6.4 per MWh, respectively, and the maximum streamflow rate is 19.25 m^3/s, leading to bounds of only 0.7 and 0.5 million dollars, respectively, in point (c) of Theorem 4.3.1.

4.5 Conclusion

In this chapter, we have compared the short-term cash flow performance of different realistic configurations of the PHES technology. In order to make such a comparison, we formulate the energy generation and storage problem of the PHES operator as an MDP by taking into account uncertainties in the streamflow rate and electricity price. Using this formulation, we analytically derive bounds on the revenue gains and losses from operating various PHES configurations. We then conduct numerical experiments for a wide variety of realistic scenarios involving different PHES configurations with different capacity levels, various seasons of the year, and large ranges of streamflow availability and negative price occurrence frequency. Our numerical results indicate that the open-loop facility with the upstream flow outperforms the open-loop facility with the downstream flow as well as the closed-loop facility if the negative electricity prices are rarely observed (as in our time-series model). However, the open-loop facility with the downstream flow can outperform the open-loop facility with the upstream flow if the negative prices are observed more frequently. Our results also imply that operating a small open-loop facility can yield the same revenue as operating a relatively large closed-loop facility.

The state and/or decision space of the problem grows if the time-series models require more distant past observations of the streamflow rate and electricity price, if the PHES facility involves a larger number of connected reservoirs, and/or if the PHES facility is integrated with other energy sources in a hybrid setting. The practical application of our MDP framework may then suffer from the curse of dimensionality. For such large-size problems, however, approximate dynamic programming techniques may be employed to overcome this drawback [158].

Chapter 5

Structural Results for General Pumped Hydro Energy Storage Systems with Two Reservoirs

5.1 Analytical results

In this chapter, we consider the energy generation and storage problem for PHES systems presented in Chapter 4. Adopting the problem formulation and notation of Chapter 4, we establish below several structural properties of the optimal profit function for different PHES configurations (see Figure 4.1). For our analysis, we assume that the electricity price is strictly positive throughout the finite planning horizon.

Assumption 5.1.1. *$p_t > 0$, $\forall t \in \mathcal{T}$.*

We require Assumption 5.1.1 to show that the payoff in period t is jointly concave in action pair (a_t, b_t):

Lemma 5.1.1. *Under Assumption 5.1.1, $R(a_t, b_t, y_t)$ is jointly concave in a_t and b_t for all y_t.*

The proofs of all analytical results in this chapter are provided in Appendix C.

Lemma 5.1.2. *Under Assumption 5.1.1,* $v_t^*(x_{ut}, x_{lt}, y_t) \leq v_t^*(x_{ut}, x_{lt} + \alpha, y_t)$ *and* $v_t^*(x_{ut}, x_{lt}, y_t) \leq v_t^*(x_{ut} + \alpha, x_{lt}, y_t)$*, where* $\alpha > 0$*,* $\forall t \in \mathcal{T}$*.*

Lemma 5.1.2 states that the system becomes more profitable as the amount of water in the PHES facility grows. This is because the system is capable of generating and storing more energy when the amount of water in the PHES facility is larger. Using Lemma 5.1.2, we introduce an upper bound on the optimal amount of water that should be pumped:

Lemma 5.1.3. *Under Assumption 5.1.1, for each* $t \in \mathcal{T}$*,* $x_{ut} - C_U \leq a_t^*(x_{ut}, x_{lt}, y_t)$*.*

Lemma 5.1.3 states that it is never optimal to pump water to the upper reservoir by leading to a water spillage from the PHES facility.

Lemma 5.1.4. *Under Assumption 5.1.1, for each* $t \in \mathcal{T}$*,* $a_t^*(x_{ut}, x_{lt}, y_t) - b_t^*(x_{ut}, x_{lt}, y_t) \leq C_L - x_{lt}$*.*

Lemma 5.1.4 states that it is never optimal to release water from the upper reservoir by leading to a water spillage from the PHES facility. Using Lemmas 5.1.1–5.1.4, we establish several other structural properties of our optimal profit function:

Lemma 5.1.5. *Under Assumption 5.1.1, the following structural properties hold for* $\alpha > 0$ *and* $\beta > 0$*:*

(a) $v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t) \leq v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha+\beta, y_t)$*,* $\forall t$*.*

(b) $v_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t) - v_t^*(x_{ut}+\beta, x_{lt}+\alpha, y_t) \leq v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t)$*,* $\forall t$*.*

(c) $v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_t^*(x_{ut}+\alpha, x_{lt}, y_t) \leq v_t^*(x_{ut}, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}, y_t)$*,* $\forall t$*.*

We discuss below the implications of Lemma 5.1.5:

- Point (a) of Lemma 5.1.5 says that it becomes less desirable to release a certain amount of water from the upper reservoir as the amount of water in the lower reservoir grows. The intuition behind this result can be seen more easily when the controller wants to minimize the loss of water in the PHES facility. In this case, having too much water in the lower reservoir may have the follwoing implications: Holding a large amount of water in the lower reservoir in any period increases the risk of losing some water in the PHES facility in future periods with high electricity prices in which it is beneficial to sell energy by releasing too much water from the upper reservoir. On the other hand, pumping a large amount of water from the lower reservoir in any period limits the capacity of the PHES facility to sell energy to the market in this period. This may even entail purchasing energy from the market. Hence, increasing the amount of water in the lower reservoir exhibits diminishing returns.

- Point (b) of Lemma 5.1.5 says that it becomes more desirable to release a certain amount of water from the upper reservoir as the amount of water in the upper reservoir grows. This is because holding some amount of water in the upper reservoir may be beneficial in anticipation of high electricity prices in future periods while holding a large amount of water in the upper reservoir increases the risk of underutilizing the water runoff in elevating the total amount of water in the PHES facility. Hence, increasing the amount of water in the upper reservoir exhibits diminishing returns. The summation of the properties in part (a) and (b) implies that $v_t^*(x_{ut} + \alpha + \beta, x_{lt}, y_t) - v_t^*(x_{ut} + \beta, x_{lt} + \alpha, y_t) \leq v_t^*(x_{ut} + \alpha, x_{lt} + \beta, y_t) - v_t^*(x_{ut}, x_{lt} + \alpha + \beta, y_t)$, $\forall y_t$. Thus, if $\alpha = \beta$, it becomes less desirable to release water as the water flows from the upper reservoir to the lower reservoir at any particular rate.

- Point (c) of Lemma 5.1.5 says that it is more desirable to have an extra amount of water in the upper reservoir when the amount of water in the lower reservoir is smaller. Likewise, it is more desirable to have an extra amount of water in the lower reservoir when the amount of water in the upper reservoir is smaller. It is important to note that the summation of the properties in parts (a) and (c) implies the concavity of $v_t^*(x_{ut}, \cdot, y_t)$, i.e., $v_t^*(x_{ut}, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t) \leq v_t^*(x_{ut}, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha+\beta, y_t)$, $\forall x_{ut}, y_t$. Similarly, the summation of the properties in parts (b) and (c) implies the concavity of $v_t^*(\cdot, x_{lt}, y_t)$, i.e., $v_t^*(x_{ut}, x_{lt}, y_t) - v_t^*(x_{ut}+\alpha, x_{lt}, y_t) \leq v_t^*(x_{ut}+\beta, x_{lt}, y_t) - v_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t)$, $\forall x_{lt}, y_t$. Finally, we note that the property in part (c) can be viewed as [207] submodularity property in x_{ut} and x_{lt}, $\forall y_t$.

Let $Z_t(x_{ut}, x_{lt}, y_t)$ and $Z_{ut}(Z_t, x_{ut}, y_t)$ denote the optimal state-dependent target levels that are associated with the total amount of water in the PHES facility and the amount of water in the upper reservoir, respectively, i.e., $Z_t(x_{ut}, x_{lt}, y_t) = x_{ut} + x_{lt} - b_t^*(x_{ut}, x_{lt}, y_t)$ and $Z_{ut}(Z_t, x_{ut}, y_t) = x_{ut} - a_t^*(x_{ut}, x_{lt}, y_t)$. The target levels $Z_t(.)$ and $Z_{ut}(.)$ can be calculated as follows:

$$\begin{aligned} Z_t(x_{ut}, x_{lt}, y_t) = \underset{z_t \in [\underline{Z}_t, \overline{Z}_t]}{\arg\max} \Big\{ &\theta(x_{ut} + x_{lt} - z_t)p_t + E(x_{ut} - Z_{ut}(z_t, x_{ut}, y_t))p_t \\ &+ \mathbb{E}\Big[v_{t+1}^* \Big(\min\{Z_{ut}(z_t, x_{ut}, y_t) + r_{u(t+1)}, C_U\}, \\ &\min\{z_t - Z_{ut}(z_t, x_{ut}, y_t) + r_{l(t+1)}, C_L\}, y_{t+1} \Big) \Big] \Big\}, \end{aligned} \tag{5.1}$$

where

$$Z_{ut}(z_t, x_{ut}, y_t) = \underset{z_{ut} \in [\underline{Z}_{ut}, \overline{Z}_{ut}]}{\arg\max} \Big\{ E(x_{ut} - z_{ut})p_t + V_t(z_t, z_{ut}, y_t) \Big\}, \tag{5.2}$$

$$\begin{aligned} V_t(z_t, z_{ut}, y_t) := \mathbb{E}\Big[v_{t+1}^* \Big(&\min\{z_{ut} + r_{u(t+1)}, C_U\}, \\ &\min\{z_t - z_{ut} + r_{l(t+1)}, C_L\}, y_{t+1} \Big) \Big], \end{aligned}$$

$\underline{Z}_t = \max\{x_{ut}, x_{ut} + x_{lt} - C_R\}$, $\overline{Z}_t = x_{ut} + x_{lt}$, $\underline{Z}_{ut} = \max\{0, x_{ut} - C_R\}$, and $\overline{Z}_{ut} = \min\{z_t, C_U, x_{ut} + C_P\}$.

Using Lemma 5.1.2 and Lemma 5.1.5, Lemma 5.1.6 proves that $V_t(z_t, z_{ut}, y_t)$ is concave in z_{ut}. Since the payoff function is linear in z_{ut}, the expression in the argmax function of (5.2) is concave in z_{ut}.

Lemma 5.1.6. *Under Assumption 5.1.1, $V_t(z_t, z_{ut}, y_t)$ is concave in z_{ut}.*

Equations (5.1) and (5.2) are interdependent, with Z_{ut} appearing multiple times in (5.1). This interdependency makes it challenging to characterize the optimal policy structure via a simple threshold policy. However, by restricting our analysis to the non-cascading PHES configuration presented in Figure 4.1c, where there is no water release from the lower reservoir, we are able to characterize the optimal policy structure. Therefore, in the remainder of this chapter, we focus on the non-cascading configuration in Figure 4.1c.

Assumption 5.1.2. $b_t = r_{lt} = 0$, $\forall t \in \mathcal{T}$.

Letting $r_t = r_{ut}$, $\forall t \in \mathcal{T}$, we now introduce optimal state-dependent target levels that are associated with the water level of the upper reservoir for each of the two different action types: For $\nu \in \{\mathsf{PP}, \mathsf{RS}\}$,

$$S_t^{(\nu)}(x_{ut}, x_{lt}, y_t) := \operatorname*{arg\,max}_{z_{ut} \in [0, C_U]} \{V_t(x_{ut} + x_{lt}, z_{ut}, y_t) + R_t^{(\nu)}(x_{ut} - z_{ut}, y_t)\},$$

where

$$\begin{aligned} V_t(x_{ut} + x_{lt}, z_{ut}, y_t) := \mathbb{E}_{y_{t+1}|y_t}\Big[v_{t+1}^*\Big(&\min\left\{z_{ut} + r_{t+1}, C_U\right\}, \\ &\min\{x_{ut} + x_{lt} - z_{ut}, C_L\}, y_{t+1}\Big)\Big], \end{aligned}$$

$$R_t^{(\nu)}(x_{ut} - z_{ut}, y_t) = \begin{cases} p_t(x_{ut} - z_{ut})/\theta & \text{if } \nu = \mathsf{PP}, \\ p_t\theta(x_{ut} - z_{ut}) & \text{if } \nu = \mathsf{RS}, \end{cases}$$

and $z_{ut} := x_{ut} - a_t$ is the water level of the upper reservoir at the end of period t if the action a_t is taken in period t. Recall that x_{ut} is the water level of the upper reservoir at the beginning of period t, including the amount of water runoff r_t. For notational convenience, we often suppress the dependency of $S_t^{(\nu)}$ on (x_{ut}, x_{lt}, y_t) in the remainder of the chapter. Recall from Lemma 5.1.3 that the amount of water pumped can be limited to prevent water spillage from the upper reservoir. Hence, in any period, one can easily determine the optimal amount of water that should remain in the lower reservoir at the end of this period by bringing the amount of water in the upper reservoir to the optimal target level in this period.

It is thus sufficient to define the optimal target level for only the upper reservoir in our optimal policy characterization.

Let Ω denote the domain of (x_{ut}, x_{lt}), i.e., $\Omega := [0, C_U] \times [0, C_L]$. With this notation, and leveraging the above analytical results, we are now ready to state the main result of this chapter:

Theorem 5.1.1. *Under Assumptions 5.1.1 and 5.1.2, the structure of the optimal energy generation and storage policy in the PHES facility can be specified as follows. In any period t, for $(x_{ut}, x_{lt}) \in \Omega$, it is optimal to*

- *pump up the water to get as close as possible to $S_t^{(\mathsf{PP})}$ if $x_{ut} \leq S_t^{(\mathsf{PP})}$,*
- *keep the water level unchanged if $S_t^{(\mathsf{PP})} < x_{ut} \leq S_t^{(\mathsf{RS})}$,*
- *release down the water to get as close as possible to $S_t^{(\mathsf{RS})}$ if $S_t^{(\mathsf{RS})} < x_{ut}$.*

Furthermore, the optimal state-dependent target levels obey (i) $S_t^{(\mathsf{PP})}(x_{ut}, x_{lt}, y_t) \leq S_t^{(\mathsf{RS})}(x_{ut}, x_{lt}, y_t)$, (ii) $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t) = S_t^{(\nu)}(x_{ut}, x_{lt} + \alpha, y_t)$ if $x_{ut} + x_{lt} \geq C_U$, and (iii) $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t) \leq S_t^{(\nu)}(x_{ut}, x_{lt} + \alpha, y_t) = S_t^{(\nu)}(x_{ut} + \alpha, x_{lt}, y_t)$, for each $\nu \in \{\mathsf{PP}, \mathsf{RS}\}$ and $\alpha > 0$.

Notice that the target water level is highest if the optimal action type is RS and lowest if it is PP: The PHES facility inefficiency leads to different marginal payoffs in the energy generation and storage modes. Thus $S_t^{(\mathsf{PP})} \leq S_t^{(\mathsf{RS})}$. Note that $S_t^{(\mathsf{PP})} = S_t^{(\mathsf{RS})}$ if $\theta = 1$. Finally, for each action type, the target level increases with the total amount of water in the PHES facility as long as the total amount of water is no larger than the upper reservoir capacity. However, the target level is independent of how the available water is distributed between the two reservoirs.

5.2 A policy approximation algorithm

Theorem 5.1.1 establishes the optimality of a state-dependent threshold policy when the electricity price is always positive and the PHES configuration is non-cascading. In this section, we implement this policy structure into a heuristic solution method for more general settings with possibly negative electricity prices and cascading PHES configurations. Our heuristic method determines the action pair in each state with a positive price using the target water levels described in Theorem 5.1.1, while it implements the myopically optimal action pair in each state with a negative price. Although this approach need not be optimal in general, it yields instantaneous decisions without a significant drain on the total profit, since the occurrence frequency of negative prices is quite small in our time series data.

Let $y_t = (r_{ut}, \rho_t, j_t) \in \mathcal{Y}_t := \mathcal{R}_t \times \mathcal{P} \times \mathcal{J}$ and $\bar{y}_t = (r_{ut}, \rho_t) \in \overline{\mathcal{Y}}_t := \mathcal{R}_t \times \mathcal{P}$. We restrict the water levels in the upper and lower reservoirs to take values from the sets $\mathcal{X}_u := \{n\zeta_a \in [0, C_U] : n \in \mathbb{Z}\}$ and $\mathcal{X}_l := \{n\zeta_a \in [0, C_L] : n \in \mathbb{Z}\}$, respectively, where ζ_a is a prespecified constant. In this method, for each period $t \in \mathcal{T}$ and each state (x_{ut}, x_{lt}, y_t), we define $v_t^{\mathsf{PA}}(x_{ut}, x_{lt}, y_t)$ as the profit function and $S_t^{(\nu),\mathsf{PA}}(x_{ut}, x_{lt}, y_t)$ as the state-dependent target level for each $\nu \in \{\mathsf{PP}, \mathsf{RS}\}$. We compute these profit functions and target levels, as well as the corresponding action a_t^{PA}, as outlined by Theorem 5.1.1. See Algorithm 1 for the resulting backward induction algorithm. We incorporate the properties of the target levels of Theorem 5.1.1 into this algorithm: the target levels increase with the total amount of water in the PHES facility (if less than C_U) and with the decision type from PP to RS, and the variables $S_t^{(\nu),lower}$ and S_t^{lower} reduce the search space for the target levels in each iteration. See steps 9-16 of Algorithm 1. We label this method PA (the initials of policy approximation).

Algorithm 1 Policy approximation based on noncascading-PHES operations.

$\bar{v}_T^{\mathsf{PA}}(x_{uT}, x_{lT}, \bar{y}_T) \leftarrow 0$, $\forall (x_{uT}, x_{lT}, \bar{y}_T) \in \mathcal{X}_u \times \mathcal{X}_l \times \overline{\mathcal{Y}}_T$.
for $t = T-1, \ldots, 1$ **do**
  for $y_t \in \mathcal{Y}_t$ such that $p_t > 0$ **do**
    $S_t^{(\nu),lower} \leftarrow 0, \ \forall \nu.$
    for $x = x_{ut} + x_{lt} \in \{0, \zeta_a, ..., \lfloor C_U/\zeta_a \rfloor \zeta_a, C_U\}$ **do**
      $S_t^{lower} \leftarrow 0.$
      for $\nu \in \{\mathsf{PP}, \mathsf{RS}\}$ **do**
        $S_t \leftarrow \arg\max_{z_{ut} \in [\max\{S_t^{(\nu),lower}, S_t^{lower}\}, C_U]} \Big\{ R_t^{(\nu)}(x_{ut} - z_{ut}, y_t) + \mathbb{E}_{\bar{y}_{t+1}|\bar{y}_t} \Big[\bar{v}_{t+1}^{\mathsf{PA}}(\min\{z_{ut} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - z_{ut}, C_L\}, \bar{y}_{t+1}) \Big] \Big\}.$
        $S_t^{lower} \leftarrow S_t.$ ▷ See point (i) of Theorem 5.1.1.
        $S_t^{(\nu),lower} \leftarrow S_t.$ ▷ See point (iii) of Theorem 5.1.1.
        for $(x_{ut}, x_{lt}) \in \mathcal{X}_u \times \mathcal{X}_l$ such that $x_{ut} + x_{lt} = x$ **do**
          $S_t^{(\nu),\mathsf{PA}}(x_{ut}, x_{lt}, y_t) \leftarrow S_t.$ ▷ See point (iii) of Theorem 5.1.1.
        end for
        if $x = C_U$ **then**
          for $(x_{ut}, x_{lt}) \in \mathcal{X}_u \times \mathcal{X}_l$ such that $x_{ut} + x_{lt} \geq C_U$ **do**
            $S_t^{(\nu),\mathsf{PA}}(x_{ut}, x_{lt}, y_t) \leftarrow S_t.$ ▷ See point (ii) of Theorem 5.1.1.
          end for
        end if
      end for
    end for
  end for
  for $(x_{ut}, x_{lt}, y_t) \in \mathcal{X}_u \times \mathcal{X}_l \times \mathcal{Y}_t$ **do**
    if $p_t > 0$ **then** ▷ The price is positive.
      Compute a_t^{PA} from Theorem 5.1.1 with $S_t^{(\nu)}$ replaced by $S_t^{(\nu),\mathsf{PA}}$, $\forall \nu$.
    else ▷ The price is negative.
      $a_t^{\mathsf{PA}} = -\min\{x_{lt}, C_P\}.$
    end if
    $v_t^{\mathsf{PA}}(x_{ut}, x_{lt}, y_t) \leftarrow R(a_t^{\mathsf{PA}}, 0, y_t) + \mathbb{E}_{\bar{y}_{t+1}|\bar{y}_t} \left[\bar{v}_{t+1}^{\mathsf{PA}}(x_{u(t+1)}, x_{l(t+1)}, \bar{y}_{t+1}) \right].$
  end for
  for $(x_{ut}, x_{lt}, \bar{y}_t) \in \mathcal{X}_u \times \mathcal{X}_l \times \overline{\mathcal{Y}}_t$ **do**
    $\bar{v}_t^{\mathsf{PA}}(x_{ut}, x_{lt}, \bar{y}_t) \leftarrow \mathbb{E}_{j_t} \left[v_t^{\mathsf{PA}}(x_{ut}, x_{lt}, y_t) \right].$
  end for
end for

5.3 Performance of the heuristic

Using the experimental setup in Chapter 4, we evaluate the performances of our heuristic solution method in 18 instances in which the PHES facility is cascading or non-cascading. All computations were executed on a 13th Gen Intel(R) Core(TM) i7-1360P 2.20 GHz CPU computer with 16 GB of RAM. Tables 5.1 and 5.2 exhibit the optimality gaps and computation times of our heuristic method for the cascading and non-cascading configurations, respectively. For the non-cascading configuration, our heuristic method yields near-optimal solutions with a maximum distance of only 1.12% from the optimal profit, and reduces the computation time of the optimal algorithm by 50.35% on average and by up to 56.47%. For the cascading configuration, as expected, we observe higher optimality gap percentages (with a maximum distance of 40.21%), but with a higher computation time reduction of 79.2% on average and up to 82.3%. In the case of non-cascading configuration, our heuristic method may perform worse when the negative prices are observed more frequently. In the case of cascading configuration, however, an increase in NPF results in a noticeable decrease in the optimality gaps. This is because when NPF is larger, there is a greater incentive to keep more water in the facility and pump more water from the lower reservoir to the upper reservoir, and thus the cascading configuration performs more similarly to the non-cascading one.

Table 5.1: Optimality gaps and computation times (in CPU seconds) for cascading configuration.

Season	NPF	Optimality gap	Computation times	
			Optimal policy	Heuristic method
January	4.90%	39.40%	1262.8	247.1
	9.34%	35.46%	1235.1	219.8
	13.37%	32.43%	1183.2	269.9
April	6.76%	40.21%	2126.2	488.7
	11.20%	36.52%	2284.2	533.6
	15.24%	33.67%	2133.6	499.2
August	6.76%	29.17%	859.3	151.9
	11.20%	25.23%	860.3	162.9
	15.23%	22.40%	874.7	182.2

Table 5.2: Optimality gaps and computation times (in CPU seconds) for non-cascading configuration.

Season	NPF	Optimality gap	Computation times	
			Optimal policy	Heuristic method
January	4.90%	0.39%	497.9	247.1
	9.34%	0.41%	490.0	219.8
	13.37%	0.41%	511.4	269.9
April	6.76%	0.89%	914.5	488.7
	11.20%	0.99%	1109.4	533.6
	15.24%	1.12%	944.0	499.2
August	6.76%	0.91%	348.9	151.9
	11.20%	0.90%	368.9	162.9
	15.23%	0.91%	364.4	182.2

5.4 Conclusion

In this chapter, we establish several structural properties of the optimal profit function for general PHES systems. We characterize the optimal policy structure for non-cascading PHES configurations under the assumption of positive electricity prices. With this structural knowledge, we develop the policy-approximation algorithm as a heuristic solution method to tackle the more complex setting where the configuration can be cascading and the prices can also be negative. Our numerical experiments demonstrate that this algorithm can significantly reduce the computation times while maintaining the profits at acceptable levels.

Chapter 6

Conclusion

The goal of this dissertation is to improve the optimization of PHES facilities in different settings by addressing existing uncertainties through a Markov decision process. In Chapter 2, we provide a comprehensive literature review of previous studies on the sizing and operational problems for PHES facilities under uncertainty. In Chapter 3, we explore the benefits of transforming a conventional cascading hydropower station into a PHES facility and provide an analytical upper bound on the profit improvement achievable through such a transformation. We also conduct numerical experiments with data-calibrated time series models and observe that the PHES facility provides a greater benefit under more limited streamflow conditions or more frequently observed negative prices. In Chapter 4, we conduct a comparative analysis of different realistic configurations of PHES technology and identify the configurations that outperform the others under various scenarios, both analytically and numerically. We observe that the open-loop PHES facility with upstream flow generates higher cash flows compared to the other configurations, while the open-loop PHES facility with downstream flow becomes more advantageous as negative electricity prices occur more frequently. Finally, in Chapter 5, we establish several structural properties of the optimal profit function for general PHES systems and characterize the optimal policy structure for non-cascading PHES configurations under the assumption of positive electricity prices. Using the obtained structural knowledge, we devise a policy-approximation algorithm as a heuristic solution approach to address the challenges posed by complex settings involving cascading configurations and negative prices. Our numerical experiments show that this algorithm achieves substantial reductions in computation times while ensuring profits remain at satisfactory levels.

Moving forward, there are several directions for future research. One promising avenue would be to extend the analysis to PHES facilities integrated with other renewable energy sources, such as solar and wind, to assess their overall performance and potential synergies. Another direction would be to consider the joint optimization of sizing and operational planning decisions for PHES facilities, incorporating upfront installation costs via employing multi-stage stochastic programming frameworks. Additionally, future research may integrate the economic analysis with environmental and social factors, including considerations of the potential impact of different PHES configurations on communities and ecosystems. Finally, there is a need to explore the use of the structural properties to develop approximate dynamic programming methods that offer greater computational advantages. Taken together, these future research directions have the potential to advance our understanding of PHES and contribute to the development of sustainable and efficient energy systems.

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Appendix A

Proofs of the Analytical Results in Chapter 3

Proof of Theorem 3.3.1

First, we will prove that $\widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) \leq v_t^*(x_{ut}, x_{lt}, y_t)$, $\forall t \in \mathcal{T}$. Note that $\widetilde{v}_T^*(x_{uT}, x_{lT}, y_T) = v_T^*(x_{uT}, x_{lT}, y_T) = 0$. Assuming $\widetilde{v}_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}) \leq v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{t+1})$, we show $\widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) \leq v_t^*(x_{ut}, x_{lt}, y_t)$. Let $\widetilde{a} = \widetilde{a}_t^*(x_{ut}, x_{lt}, y_t) \geq 0$ and $\widetilde{b} = \widetilde{b}_t^*(x_{ut}, x_{lt}, y_t)$ denote the optimal actions in state (x_{ut}, x_{lt}, y_t) for the CCHS. Also, let $\widetilde{\mathbb{U}}(x_{ut}, x_{lt}, y_t)$ denote the set of admissible action pairs (a_t, b_t) in state (x_{ut}, x_{lt}, y_t) for the CCHS. Since $\widetilde{\mathbb{U}}(x_{ut}, x_{lt}, y_t) \subseteq \mathbb{U}(x_{ut}, x_{lt}, y_t)$, $(\widetilde{a}, \widetilde{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus:

$$\begin{aligned}
\widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) &= R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[\widetilde{v}_{t+1}^*\big(\min\{x_{ut} - \widetilde{a} + r_{t+1}, C_U\}, \\
&\qquad \min\big\{\min\{x_{lt} + \widetilde{a}, C_L\} - \widetilde{b} + (x_{ut} - \widetilde{a} - C_U + r_{t+1})^+, C_L\big\}, y_{t+1}\big)\Big] \\
&\leq R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[v_{t+1}^*\big(\min\{x_{ut} - \widetilde{a} + r_{t+1}, C_U\}, \\
&\qquad \min\big\{\min\{x_{lt} + \widetilde{a}, C_L\} - \widetilde{b} + (x_{ut} - \widetilde{a} - C_U + r_{t+1})^+, C_L\big\}, y_{t+1}\big)\Big] \\
&\leq v_t^*(x_{ut}, x_{lt}, y_t).
\end{aligned}$$

Next, we will prove that: $v_t^*(x_{ut}, x_{lt}, y_t) \leq \widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) + Q_P \rho g h_1 (T - t)\left(\phi_{RU} \max_{t \leq \tau \leq T} \overline{p}_\tau - \min_{t \leq \tau \leq T} \underline{p}_\tau / \phi_P\right)^+$, $\forall t \in \mathcal{T}$. Note that $v_T^*(x_{uT}, x_{lT}, y_T) = \widetilde{v}_T^*(x_{uT}, x_{lT}, y_T) + Q_P \rho g h_1 (T - T)(\phi_{RU} \overline{p}_T - \underline{p}_T / \phi_P)^+ = 0$. To this end, we first show that $\widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) - \widetilde{v}_t^*(x_{ut} - \alpha, x_{lt} + \alpha, y_t) \leq \rho g h_1 \alpha \phi_{RU} \max_{t \leq \tau \leq T} \overline{p}_\tau$ where $0 \leq \alpha \leq \min\{x_{ut}, C_L - x_{lt}\}$, $\forall t \in \mathcal{T}$. Note that $\widetilde{v}_T^*(x_{uT}, x_{lT}, y_T) = \widetilde{v}_T^*(x_{uT} - \alpha, x_{lT} + \alpha, y_T) = 0$. Assuming $\widetilde{v}_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) - \widetilde{v}_{t+1}^*(x_{u(t+1)} - \alpha, x_{l(t+1)} + \alpha, y_{(t+1)}) \leq \rho g h_1 \alpha \phi_{RU} \max_{t+1 \leq \tau \leq T} \overline{p}_\tau$, we show $\widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) - \widetilde{v}_t^*(x_{ut} - \alpha, x_{lt} + \alpha, y_t) \leq \rho g h_1 \alpha \phi_{RU} \max_{t \leq \tau \leq T} \overline{p}_\tau$. Suppose that $\widetilde{a} = \widetilde{a}_t^*(x_{ut}, x_{lt}, y_t) \geq \alpha$. Recall that $\widetilde{b} = \widetilde{b}_t^*(x_{ut}, x_{lt}, y_t)$. Notice that $(\widetilde{a} - \alpha, \widetilde{b}) \in \widetilde{\mathbb{U}}(x_{ut} - \alpha, x_{lt} + \alpha, y_t)$. Thus:

$$
\begin{aligned}
&\widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) - \widetilde{v}_t^*(x_{ut} - \alpha, x_{lt} + \alpha, y_t) \\
&\leq R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[\widetilde{v}_{t+1}^*\big(\min\{x_{ut} - \widetilde{a} + r_{t+1}, C_U\}, \min\{\min\{x_{lt} + \widetilde{a}, C_L\} \\
&\quad - \widetilde{b} + (x_{ut} - \widetilde{a} - C_U + r_{t+1})^+, C_L\}, y_{(t+1)}\big)\Big] - R(\widetilde{a} - \alpha, \widetilde{b}, y_t) \\
&\quad - \mathbb{E}\Big[\widetilde{v}_{t+1}^*\big(\min\{x_{ut} - \alpha - (\widetilde{a} - \alpha) + r_{t+1}, C_U\}, \\
&\quad \min\{\min\{x_{lt} + \alpha + (\widetilde{a} - \alpha), C_L\} - \widetilde{b} \\
&\quad + (x_{ut} - \alpha - (\widetilde{a} - \alpha) - C_U + r_{t+1})^+, C_L\}, y_{(t+1)}\big)\Big] \\
&= p_t \rho g h_1 \phi_{RU} (\widetilde{a} - \widetilde{a} + \alpha) + p_t \rho g h_2 \phi_{RL} (\widetilde{b} - \widetilde{b}) \\
&\leq \rho g h_1 \alpha \phi_{RU} \max_{t \leq \tau \leq T} \overline{p}_\tau.
\end{aligned}
$$

Now suppose that $\widetilde{a} < \alpha$. If the optimal actions $(\widetilde{a}, \widetilde{b})$ are taken in state (x_{ut}, x_{lt}, y_t):

$$
\begin{aligned}
x_{u(t+1)} &= \min\{x_{ut} - \widetilde{a} + r_{t+1}, C_U\} \text{ and} \\
x_{l(t+1)} &= \min\{x_{lt} + \widetilde{a} - \widetilde{b} + (x_{ut} - \widetilde{a} - C_U + r_{t+1})^+, C_L\}.
\end{aligned}
$$

If the actions $(0, \widetilde{b})$ are taken in state $(x_{ut} - \alpha, x_{lt} + \alpha, y_t)$:

$$
\begin{aligned}
x'_{u(t+1)} &= \min\{x_{ut} - \alpha + r_{t+1}, C_U\} \text{ and} \\
x'_{l(t+1)} &= \min\{x_{lt} + \alpha - \widetilde{b} + (x_{ut} - \alpha - C_U + r_{t+1})^+, C_L\}.
\end{aligned}
$$

For these state variables, we make the following observations:

(1) If $x_{ut} - \widetilde{a} + r_{t+1} \leq C_U$, $x'_{u(t+1)} = x_{ut} - \alpha + r_{t+1} = x_{u(t+1)} - (\alpha - \widetilde{a})$ and $x'_{l(t+1)} = x_{lt} + \alpha - \widetilde{b} = x_{l(t+1)} + (\alpha - \widetilde{a})$.

(2) If $x_{ut} - \alpha + r_{t+1} \leq C_U \leq x_{ut} - \widetilde{a} + r_{t+1}$ and $x_{ut} + x_{lt} - \widetilde{b} - C_U + r_{t+1} \leq C_L$, $x'_{u(t+1)} = x_{ut} - \alpha + r_{t+1} = x_{u(t+1)} - (C_U - r_{t+1} - x_{ut} + \alpha)$ and $x'_{l(t+1)} = x_{lt} + \alpha - \widetilde{b} = x_{l(t+1)} + (C_U - r_{t+1} - x_{ut} + \alpha)$. Note that $0 \leq C_U - r_{t+1} - x_{ut} + \alpha \leq \alpha - \widetilde{a}$.

(3) If $x_{ut} - \alpha + r_{t+1} \leq C_U \leq x_{ut} - \widetilde{a} + r_{t+1}$ and $x_{ut} + x_{lt} - \widetilde{b} - C_U + r_{t+1} > C_L$, note that $x_{lt} + \alpha - \widetilde{b} > C_L$. This scenario is not possible since $x_{lt} + \alpha \leq C_L$.

(4) If $C_U \leq x_{ut} - \alpha + r_{t+1}$, $x'_{u(t+1)} = x_{u(t+1)} = C_U$ and $x'_{l(t+1)} = x_{l(t+1)} = \min\{x_{lt} - \widetilde{b} + x_{ut} - C_U + r_{t+1}, C_L\}$.

Thus, there exists β such that $0 \leq \beta \leq \alpha - \widetilde{a}$ and

$$
\begin{aligned}
&\widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) - \widetilde{v}_t^*(x_{ut} - \alpha, x_{lt} + \alpha, y_t) \\
&\leq R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[\widetilde{v}_{t+1}^*\big(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}\big)\Big] \\
&\quad - R(0, \widetilde{b}, y_t) - \mathbb{E}\Big[\widetilde{v}_{t+1}^*\big(x'_{u(t+1)}, x'_{l(t+1)}, y_{(t+1)}\big)\Big] \\
&= p_t \rho g h_1 \phi_{RU} \widetilde{a} + \mathbb{E}\Big[\widetilde{v}_{t+1}^*\big(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}\big)\Big] \\
&\quad - \mathbb{E}\Big[\widetilde{v}_{t+1}^*\big(x_{u(t+1)} - \beta, x_{l(t+1)} + \beta, y_{(t+1)}\big)\Big] \\
&\leq \widetilde{a} \rho g h_1 \phi_{RU} p_t + \beta \rho g h_1 \phi_{RU} \max_{t+1 \leq \tau \leq T} \overline{p}_\tau \\
&\leq \widetilde{a} \rho g h_1 \phi_{RU} \max_{t \leq \tau \leq T} \overline{p}_\tau + \beta \rho g h_1 \phi_{RU} \max_{t \leq \tau \leq T} \overline{p}_\tau \\
&\leq \alpha \rho g h_1 \phi_{RU} \max_{t \leq \tau \leq T} \overline{p}_\tau.
\end{aligned}
$$

Using the above result and assuming $v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) - \widetilde{v}_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) \leq Q_P \rho g h_1 (T - t - 1) \left(\phi_{RU} \max_{t+1 \leq \tau \leq T} \overline{p}_\tau - \min_{t+1 \leq \tau \leq T} \underline{p}_\tau / \phi_P\right)^+$, we show $v_t^*(x_{ut}, x_{lt}, y_t) - \widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) \leq Q_P \rho g h_1 (T -$

$t)\left(\phi_{RU}\max_{t\le\tau\le T}\overline{p}_\tau-\min_{t\le\tau\le T}\underline{p}_\tau/\phi_P\right)^+$. Let $a=a^*(x_{ut},x_{lt},y_t)$ and $b=b^*(x_{ut},x_{lt},y_t)$ denote the optimal actions in state (x_{ut},x_{lt},y_t) for the PHES system. Suppose that $a\ge 0$. Note $(a,b)\in\widetilde{\mathbb{U}}(x_{ut},x_{lt},y_t)$. Thus:

$$\begin{aligned}
&v_t^*(x_{ut},x_{lt},y_t)-\widetilde{v}_t^*(x_{ut},x_{lt},y_t)\\
&\le R(a,b,y_t)+\mathbb{E}\Big[v_{t+1}^*\big(\min\{x_{ut}-a+r_{t+1},C_U\},\min\{\min\{x_{lt}+a,C_L\}\\
&\quad -b+(x_{ut}-a-C_U+r_{t+1})^+,C_L\},y_{(t+1)}\big)\Big]-R(a,b,y_t)\\
&\quad -\mathbb{E}\Big[\widetilde{v}_{t+1}^*\big(\min\{x_{ut}-a+r_{t+1},C_U\},\min\{\min\{x_{lt}+a,C_L\}\\
&\quad -b+(x_{ut}-a-C_U+r_{t+1})^+,C_L\},y_{(t+1)}\big)\Big]\\
&\le Q_P\rho g h_1(T-t-1)\left(\phi_{RU}\max_{t+1\le\tau\le T}\overline{p}_\tau-\min_{t+1\le\tau\le T}\underline{p}_\tau/\phi_P\right)^+\\
&\le Q_P\rho g h_1(T-t)\left(\phi_{RU}\max_{t\le\tau\le T}\overline{p}_\tau-\min_{t\le\tau\le T}\underline{p}_\tau/\phi_P\right)^+.
\end{aligned}$$

Now suppose that $a<0$. Note that $b=0$ in this case. If the optimal actions $(a,0)$ are taken in state (x_{ut},x_{lt},y_t) for the PHES system:

$$\begin{aligned}
x_{u(t+1)}&=\min\{x_{ut}-a+r_{t+1},C_U\}\text{ and}\\
x_{l(t+1)}&=\min\Big\{x_{lt}+a+(x_{ut}-a-C_U)^+\\
&\qquad+\left(\min\{x_{ut}-a,C_U\}+r_{t+1}-C_U\right)^+,C_L\Big\}.
\end{aligned}$$

If the actions $(0,0)$ are taken in state (x_{ut},x_{lt},y_t) for the CCHS:

$$\begin{aligned}
x'_{u(t+1)}&=\min\{x_{ut}+r_{t+1},C_U\}\text{ and}\\
x'_{l(t+1)}&=\min\left\{x_{lt}+(x_{ut}+r_{t+1}-C_U)^+,C_L\right\}.
\end{aligned}$$

For these state variables, we make the following observations:

(1) If $x_{ut}-a+r_{t+1}\le C_U$, $x'_{u(t+1)}=x_{ut}+r_{t+1}=x_{u(t+1)}-(-a)$ and $x'_{l(t+1)}=x_{lt}=x_{l(t+1)}+(-a)$.

(2) If $\max\{x_{ut}+r_{t+1}, x_{ut}-a\} \leq C_U \leq x_{ut}-a+r_{t+1}$, $x'_{u(t+1)} = x_{ut}+r_{t+1} = x_{u(t+1)} - (C_U - r_{t+1} - x_{ut})$ and $x'_{l(t+1)} = x_{lt} = x_{l(t+1)} + (C_U - r_{t+1} - x_{ut})$. Note that $0 \leq C_U - r_{t+1} - x_{ut} \leq -a$.

(3) If $x_{ut} - a \leq C_U \leq x_{ut} + r_{t+1}$ and $x_{ut} + x_{lt} + r_{t+1} - C_U \leq C_L$, $x'_{u(t+1)} = x_{u(t+1)} = C_U$ and $x'_{l(t+1)} = x_{l(t+1)} = x_{ut} + x_{lt} + r_{t+1} - C_U$.

(4) If $x_{ut} - a \leq C_U \leq x_{ut} + r_{t+1}$ and $x_{ut} + x_{lt} + r_{t+1} - C_U > C_L$, $x'_{u(t+1)} = x_{u(t+1)} = C_U$ and $x'_{l(t+1)} = x_{l(t+1)} = C_L$.

(5) If $x_{ut}+r_{t+1} \leq C_U \leq x_{ut}-a$, $x'_{u(t+1)} = x_{ut}+r_{t+1} = x_{u(t+1)} - (C_U - r_{t+1} - x_{ut})$ and $x'_{l(t+1)} = x_{lt} = x_{l(t+1)} + (C_U - r_{t+1} - x_{ut})$. Note that $0 \leq C_U - r_{t+1} - x_{ut} \leq -a$.

(6) If $C_U \leq \min\{x_{ut}-a, x_{ut}+r_{t+1}\}$ and $x_{ut} + x_{lt} + r_{t+1} - C_U \leq C_L$, $x'_{u(t+1)} = x_{u(t+1)} = C_U$ and $x'_{l(t+1)} = x_{l(t+1)} = x_{ut} + x_{lt} + r_{t+1} - C_U$.

(7) If $C_U \leq \min\{x_{ut}-a, x_{ut}+r_{t+1}\}$ and $x_{ut} + x_{lt} + r_{t+1} - C_U > C_L$, $x'_{u(t+1)} = x_{u(t+1)} = C_U$ and $x'_{l(t+1)} = x_{l(t+1)} = C_L$.

Thus, there exists α such that $0 \leq \alpha \leq -a$ and

$$
\begin{aligned}
& v^*_t(x_{ut}, x_{lt}, y_t) - \widetilde{v}^*_t(x_{ut}, x_{lt}, y_t) \\
& \leq R(a, 0, y_t) + \mathbb{E}\left[v^*_{t+1}\left(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}\right)\right] \\
& \quad - R(0, 0, y_t) - \mathbb{E}\left[\widetilde{v}^*_{t+1}\left(x'_{u(t+1)}, x'_{l(t+1)}, y_{(t+1)}\right)\right] \\
& = p_t a \rho g h_1/\phi_P + \mathbb{E}\left[v^*_{t+1}\left(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}\right)\right. \\
& \quad \left. -\widetilde{v}^*_{t+1}\left(x_{u(t+1)} - \alpha, x_{l(t+1)} + \alpha, y_{(t+1)}\right)\right] \\
& \leq p_t a \rho g h_1/\phi_P + \mathbb{E}\left[v^*_{t+1}\left(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}\right)\right. \\
& \quad \left. -\widetilde{v}^*_{t+1}\left(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}\right)\right] + \alpha \rho g h_1 \phi_{RU} \max_{t+1 \leq \tau \leq T} \overline{p}_\tau \\
& \leq p_t a \rho g h_1/\phi_P + \alpha \rho g h_1 \phi_{RU} \max_{t+1 \leq \tau \leq T} \overline{p}_\tau \\
& \quad + Q_P \rho g h_1 (T-t-1) \left(\phi_{RU} \max_{t+1 \leq \tau \leq T} \overline{p}_\tau - \min_{t+1 \leq \tau \leq T} \underline{p}_\tau/\phi_P\right)^+ .
\end{aligned}
$$

If $p_t \geq 0$:

$$
\begin{aligned}
& p_t a\rho g h_1/\phi_P + \alpha\rho g h_1 \phi_{RU} \max_{t+1\le\tau\le T} \overline{p}_\tau \\
& \le -p_t \alpha\rho g h_1/\phi_P + \alpha\rho g h_1 \phi_{RU} \max_{t+1\le\tau\le T} \overline{p}_\tau \\
& \le -\alpha\rho g h_1/\phi_P \min_{t\le\tau\le T} \underline{p}_\tau + \alpha\rho g h_1 \phi_{RU} \max_{t\le\tau\le T} \overline{p}_\tau \\
& \le \alpha\rho g h_1 \left(\phi_{RU} \max_{t\le\tau\le T} \overline{p}_\tau - \min_{t\le\tau\le T} \underline{p}_\tau/\phi_P\right)^+ \\
& \le Q_P\rho g h_1 \left(\phi_{RU} \max_{t\le\tau\le T} \overline{p}_\tau - \min_{t\le\tau\le T} \underline{p}_\tau/\phi_P\right)^+ .
\end{aligned}
$$

If $p_t < 0$:

$$
\begin{aligned}
& p_t a\rho g h_1/\phi_P + \alpha\rho g h_1 \phi_{RU} \max_{t+1\le\tau\le T} \overline{p}_\tau \\
& \le -Q_P\rho g h_1 p_t/\phi_P + \alpha\rho g h_1 \phi_{RU} \max_{t+1\le\tau\le T} \overline{p}_\tau \\
& \le -Q_P\rho g h_1/\phi_P \min_{t\le\tau\le T} \underline{p}_\tau + Q_P\rho g h_1 \phi_{RU} \max_{t\le\tau\le T} \overline{p}_\tau \\
& \le Q_P\rho g h_1 \left(\phi_{RU} \max_{t\le\tau\le T} \overline{p}_\tau - \min_{t\le\tau\le T} \underline{p}_\tau/\phi_P\right)^+ .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& v_t^*(x_{ut}, x_{lt}, y_t) - \widetilde{v}_t^*(x_{ut}, x_{lt}, y_t) \\
& \le Q_P\rho g h_1 \left(\phi_{RU} \max_{t\le\tau\le T} \overline{p}_\tau - \min_{t\le\tau\le T} \underline{p}_\tau/\phi_P\right)^+ \\
& \quad + Q_P\rho g h_1 (T-t-1) \left(\phi_{RU} \max_{t+1\le\tau\le T} \overline{p}_\tau - \min_{t+1\le\tau\le T} \underline{p}_\tau/\phi_P\right)^+ \\
& \le Q_P\rho g h_1 (T-t) \left(\phi_{RU} \max_{t\le\tau\le T} \overline{p}_\tau - \min_{t\le\tau\le T} \underline{p}_\tau/\phi_P\right)^+ .
\end{aligned}
$$

Appendix B

Proofs of the Analytical Results in Chapter 4

Proof of Theorem 4.3.1

(a) Note that $v_T^*(x_{uT}, x_{lT}, y_T) = v_T^*(x_{uT}, x_{lT} + \alpha, y_T) = v_T^*(x_{uT} + \alpha, x_{lT}, y_T) = 0$. Assuming $v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}) \leq v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)} + \alpha, y_{t+1})$, we show $v_t^*(x_{ut}, x_{lt}, y_t) \leq v_t^*(x_{ut}, x_{lt} + \alpha, y_t)$. Let $a = a_t^*(x_{ut}, x_{lt}, y_t)$ and $b = b_t^*(x_{ut}, x_{lt}, y_t)$. Note that $(a, b) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$. Thus:

$$\begin{aligned}
v_t^*(x_{ut}, x_{lt}, y_t) &= R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\},\\
&\qquad \min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\leq R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\},\\
&\qquad \min\{x_{lt} + a - b + \alpha + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\leq v_t^*(x_{ut}, x_{lt} + \alpha, y_t).
\end{aligned}$$

Also, assuming $v^*_{t+1}(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}) \le v^*_{t+1}(x_{u(t+1)} + \alpha, x_{l(t+1)}, y_{t+1})$, we show $v^*_t(x_{ut}, x_{lt}, y_t) \le v^*_t(x_{ut} + \alpha, x_{lt}, y_t)$. Note that $(a, b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$. Thus:

$$\begin{aligned}
v^*_t(x_{ut}, x_{lt}, y_t) &= R(a, b, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\le R(a, b, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - a + \alpha + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\le v^*_t(x_{ut} + \alpha, x_{lt}, y_t).
\end{aligned}$$

(b) Note that $v^*_{cT}(x_{uT}, x_{lT}, y_T) = v^*_{lT}(x_{uT}, x_{lT}, y_T) = v^*_{uT}(x_{uT}, x_{lT}, y_T) = 0$. Assuming $v^*_{c(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}) \le v^*_{l(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{t+1})$, we show $v^*_{ct}(x_{ut}, x_{lt}, y_t) \le v^*_{lt}(x_{ut}, x_{lt}, y_t)$. Let $a = a^*_{ct}(x_{ut}, x_{lt}, y_t)$. Note that $(a, 0) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus:

$$\begin{aligned}
v^*_{ct}(x_{ut}, x_{lt}, y_t) &= R(a, 0, y_t) + \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut} - a, C_U\},\\
&\quad \min\{x_{lt} + a, C_L\}, y_{t+1}\big)\Big]\\
&\le R(a, 0, y_t) + \mathbb{E}\Big[v^*_{l(t+1)}\big(\min\{x_{ut} - a, C_U\},\\
&\quad \min\{x_{lt} + a, C_L\}, y_{t+1}\big)\Big]\\
&\le R(a, 0, y_t) + \mathbb{E}\Big[v^*_{l(t+1)}\big(\min\{x_{ut} - a, C_U\},\\
&\quad \min\{x_{lt} + a + r_{t+1}, C_L\}, y_{t+1}\big)\Big]\\
&\le v^*_{lt}(x_{ut}, x_{lt}, y_t).
\end{aligned}$$

The first inequality follows from the induction assumption and the second inequality follows from point (a).

We next prove that $v^*_{lt}(x_{ut}, x_{lt}, y_t) \le v^*_{ct}(x_{ut}, x_{lt}, y_t) + \frac{1}{\theta} C_R \sum_{i=t}^{T} (T - i) \max\left\{\Big(\max\limits_{i \le \tau \le T} \overline{p}_\tau\Big)^+, \Big(\max\limits_{i \le \tau \le T} -\underline{p}_\tau\Big)^+\right\}$, $\forall t \in \mathcal{T}$. Note that $v^*_{lT}(x_{uT}, x_{lT}, y_T) = v^*_{cT}(x_{uT}, x_{lT}, y_T) + \frac{1}{\theta} C_R \sum_{i=T}^{T} (T - i) \max\left\{\Big(\max\limits_{i \le \tau \le T} \overline{p}_\tau\Big)^+, \Big(\max\limits_{i \le \tau \le T} -\underline{p}_\tau\Big)^+\right\} = 0$. To this end, we first show that $v^*_{ct}(x_{ut} + \alpha, x_{lt} + \beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \le \frac{1}{\theta}(\alpha + |\beta|)(T - t) \max\left\{\Big(\max\limits_{t \le \tau \le T} \overline{p}_\tau\Big)^+, \Big(\max\limits_{t \le \tau \le T} -\underline{p}_\tau\Big)^+\right\}$ where $\alpha \ge 0$. Note that

$v^*_{cT}(x_{uT}+\alpha, x_{lT}+\beta, y_T) = v^*_{cT}(x_{uT}, x_{lT}, y_T) = 0$. Assuming $v^*_{c(t+1)}(x_{u(t+1)} + \alpha, x_{l(t+1)} + \beta, y_{(t+1)}) - v^*_{c(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) \le \frac{1}{\theta}(\alpha + |\beta|)(T-t-1)\max\left\{\left(\max\limits_{t+1\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max\limits_{t+1\le\tau\le T} -\underline{p}_\tau\right)^+\right\}$, we show $v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \le \frac{1}{\theta}(\alpha+|\beta|)(T-t)\max\left\{\left(\max\limits_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max\limits_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}$. Suppose that $\widetilde{a} = a^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) \ge \alpha$. Notice that $(\widetilde{a}-\alpha, 0) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus:

$$
\begin{aligned}
&v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \\
&\le R(\widetilde{a}, 0, y_t) + \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut} - \widetilde{a} + \alpha, C_U\}, \\
&\quad \min\{x_{lt} + \widetilde{a} + \beta, C_L\}, y_{(t+1)}\big)\Big] \\
&\quad - R(\widetilde{a}-\alpha, 0, y_t) - \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut} - \widetilde{a} + \alpha, C_U\}, \\
&\quad \min\{x_{lt} + \widetilde{a} - \alpha, C_L\}, y_{(t+1)}\big)\Big].
\end{aligned}
$$

If $p_t \ge 0$, $R(\widetilde{a}, 0, y_t) - R(\widetilde{a}-\alpha, 0, y_t) = p_t\theta\alpha \le p_t\frac{\alpha}{\theta} \le \frac{\alpha}{\theta}\max\left\{\left(\max\limits_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max\limits_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}$. If $p_t < 0$, $R(\widetilde{a}, 0, y_t) - R(\widetilde{a}-\alpha, 0, y_t) = p_t\theta\alpha \le 0 \le \frac{\alpha}{\theta}\max\left\{\left(\max\limits_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max\limits_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}$. Thus:

$$
\begin{aligned}
&v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \\
&\le \frac{\alpha}{\theta}\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{1}{\theta}(|\alpha+\beta|)(T-t-1)\max\left\{\left(\max_{t+1\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t+1\le\tau\le T} -\underline{p}_\tau\right)^+\right\} \\
&\le \frac{1}{\theta}(\alpha+|\beta|)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{1}{\theta}(\alpha+|\beta|)(T-t-1)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\} \\
&\le \frac{1}{\theta}(\alpha+|\beta|)(T-t)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}
$$

Now suppose that $-\beta \le \widetilde{a} < \alpha$ and $\widetilde{a} \ge 0$. If the action $(0,0)$ is taken in state (x_{ut}, x_{lt}, y_t),

$$
\begin{aligned}
&v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t)\\
&\le R(\widetilde{a}, 0, y_t) + \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut}-\widetilde{a}+\alpha, C_U\},\\
&\quad \min\{x_{lt}+\widetilde{a}+\beta, C_L\}, y_{(t+1)}\big)\Big]\\
&\quad - R(0,0,y_t) - \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut}, C_U\}, \min\{x_{lt}, C_L\}, y_{(t+1)}\big)\Big].
\end{aligned}
$$

If $p_t \ge 0$, $R(\widetilde{a},0,y_t) - R(0,0,y_t) = p_t\theta\widetilde{a} \le p_t\frac{\widetilde{a}}{\theta} \le \frac{\alpha}{\theta}\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}$. If $p_t < 0$, $R(\widetilde{a},0,y_t) - R(0,0,y_t) = p_t\theta\widetilde{a} \le 0 \le \frac{\alpha}{\theta}\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}$. Thus:

$$
\begin{aligned}
&v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t)\\
&\le \frac{\alpha}{\theta}\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}\\
&\quad + \frac{1}{\theta}(\alpha-\widetilde{a}+\widetilde{a}+\beta)(T-t-1)\max\left\{\left(\max_{t+1\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t+1\le\tau\le T} -\underline{p}_\tau\right)^+\right\}\\
&\le \frac{1}{\theta}(\alpha+|\beta|)(T-t)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}
$$

Now suppose that $-\beta \le \widetilde{a} < \alpha$ and $\widetilde{a} < 0$. If the action $(0,0)$ is taken in state (x_{ut}, x_{lt}, y_t),

$$
\begin{aligned}
&v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t)\\
&\le R(\widetilde{a}, 0, y_t) + \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut}-\widetilde{a}+\alpha, C_U\},\\
&\quad \min\{x_{lt}+\widetilde{a}+\beta, C_L\}, y_{(t+1)}\big)\Big]\\
&\quad - R(0,0,y_t) - \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut}, C_U\}, \min\{x_{lt}, C_L\}, y_{(t+1)}\big)\Big].
\end{aligned}
$$

If $p_t \ge 0$, $R(\widetilde{a},0,y_t) - R(0,0,y_t) = p_t\frac{\widetilde{a}}{\theta} \le 0 \le \frac{1}{\theta}(\alpha + |\beta|)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}$. If $p_t < 0$, $R(\widetilde{a},0,y_t) - R(0,0,y_t) = p_t\frac{\widetilde{a}}{\theta} \le -p_t\frac{\beta}{\theta} \le \frac{\beta}{\theta}\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\} \le \frac{1}{\theta}(\alpha +$

$|\beta|) \max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}$. Thus:

$$\begin{aligned}
&v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t)\\
&\le \frac{1}{\theta}(\alpha+|\beta|)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}\\
&\quad + \frac{1}{\theta}(\alpha-\widetilde{a}+\widetilde{a}+\beta)(T-t-1)\max\left\{\left(\max_{t+1\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t+1\le\tau\le T} -\underline{p}_\tau\right)^+\right\}\\
&\le \frac{1}{\theta}(\alpha+|\beta|)(T-t)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}$$

Now suppose that $0 \le \widetilde{a} < -\beta < \alpha$. If the action (0,0) is taken in state (x_{ut}, x_{lt}, y_t),

$$\begin{aligned}
&v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t)\\
&\le R(\widetilde{a},0,y_t) + \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut}-\widetilde{a}+\alpha, C_U\},\\
&\quad \min\{x_{lt}+\widetilde{a}+\beta, C_L\}, y_{(t+1)}\big)\Big]\\
&\quad - R(0,0,y_t) - \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut}, C_U\}, \min\{x_{lt}, C_L\}, y_{(t+1)}\big)\Big].
\end{aligned}$$

If $p_t \ge 0$, $R(\widetilde{a},0,y_t) - R(0,0,y_t) = p_t\theta\widetilde{a} \le \frac{\widetilde{a}}{\theta}\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\} \le \frac{1}{\theta}(\alpha-\beta)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}$. If $p_t < 0$, $R(\widetilde{a},0,y_t) - R(0,0,y_t) = p_t\theta\widetilde{a} \le 0 \le \frac{1}{\theta}(\alpha-\beta)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}$. Thus:

$$\begin{aligned}
&v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t)\\
&\le \frac{1}{\theta}(\alpha-\beta)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}\\
&\quad + \frac{1}{\theta}(\alpha-\widetilde{a}-\widetilde{a}-\beta)(T-t-1)\max\left\{\left(\max_{t+1\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t+1\le\tau\le T} -\underline{p}_\tau\right)^+\right\}\\
&\le \frac{1}{\theta}(\alpha-\beta)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}\\
&\quad + \frac{1}{\theta}(\alpha-\beta)(T-t-1)\max\left\{\left(\max_{t+1\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t+1\le\tau\le T} -\underline{p}_\tau\right)^+\right\}\\
&\le \frac{1}{\theta}(\alpha+|\beta|)(T-t)\max\left\{\left(\max_{t\le\tau\le T}\overline{p}_\tau\right)^+, \left(\max_{t\le\tau\le T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}$$

Now suppose that $\widetilde{a} < 0 < -\beta < \alpha$. If the action $(\widetilde{a}, 0)$ is taken in state (x_{ut}, x_{lt}, y_t),

$$
\begin{aligned}
& v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \\
& \leq R(\widetilde{a}, 0, y_t) + \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut} - \widetilde{a} + \alpha, C_U\}, \\
& \quad \min\{x_{lt} + \widetilde{a} + \beta, C_L\}, y_{(t+1)}\big)\Big] \\
& \quad - R(\widetilde{a}, 0, y_t) - \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut} - \widetilde{a}, C_U\}, \min\{x_{lt} + \widetilde{a}, C_L\}, y_{(t+1)}\big)\Big] \\
& \leq \frac{1}{\theta}(\alpha + |\beta|)(T-t-1)\max\left\{\Big(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\Big)^+, \Big(\max_{t+1\leq\tau\leq T} -\underline{p}_\tau\Big)^+\right\} \\
& \leq \frac{1}{\theta}(\alpha + |\beta|)(T-t)\max\left\{\Big(\max_{t\leq\tau\leq T}\overline{p}_\tau\Big)^+, \Big(\max_{t\leq\tau\leq T} -\underline{p}_\tau\Big)^+\right\}.
\end{aligned}
$$

Now suppose that $\widetilde{a} < -\beta < 0 \leq \alpha$. If the action $(\widetilde{a}+\beta,\ 0)$ is taken in state (x_{ut}, x_{lt}, y_t),

$$
\begin{aligned}
& v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \\
& \leq R(\widetilde{a}, 0, y_t) + \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut} - \widetilde{a} + \alpha, C_U\}, \\
& \quad \min\{x_{lt} + \widetilde{a} + \beta, C_L\}, y_{(t+1)}\big)\Big] \\
& \quad - R(\widetilde{a}+\beta, 0, y_t) - \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut} - \widetilde{a} - \beta, C_U\}, \\
& \quad \min\{x_{lt} + \widetilde{a} + \beta, C_L\}, y_{(t+1)}\big)\Big].
\end{aligned}
$$

If $p_t \geq 0$, $R(\widetilde{a}, 0, y_t) - R(\widetilde{a}+\beta, 0, y_t) = -p_t\frac{\beta}{\theta} \leq 0 \leq \frac{1}{\theta}(\alpha + \beta)\max\left\{\Big(\max_{t\leq\tau\leq T}\overline{p}_\tau\Big)^+, \Big(\max_{t\leq\tau\leq T} -\underline{p}_\tau\Big)^+\right\}$. If $p_t < 0$, $R(\widetilde{a}, 0, y_t) - R(\widetilde{a}+\beta, 0, y_t) = -p_t\frac{\beta}{\theta} \leq \frac{1}{\theta}(\alpha + \beta)\max\left\{\Big(\max_{t\leq\tau\leq T}\overline{p}_\tau\Big)^+, \Big(\max_{t\leq\tau\leq T} -\underline{p}_\tau\Big)^+\right\}$. Thus:

$$
\begin{aligned}
& v^*_{ct}(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \\
& \leq \frac{1}{\theta}(\alpha + \beta)\max\left\{\Big(\max_{t\leq\tau\leq T}\overline{p}_\tau\Big)^+, \Big(\max_{t\leq\tau\leq T} -\underline{p}_\tau\Big)^+\right\} \\
& \quad + \frac{1}{\theta}(\alpha + \beta)(T-t-1)\max\left\{\Big(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\Big)^+, \Big(\max_{t+1\leq\tau\leq T} -\underline{p}_\tau\Big)^+\right\} \\
& \leq \frac{1}{\theta}(\alpha + |\beta|)(T-t)\max\left\{\Big(\max_{t\leq\tau\leq T}\overline{p}_\tau\Big)^+, \Big(\max_{t\leq\tau\leq T} -\underline{p}_\tau\Big)^+\right\}.
\end{aligned}
$$

Now suppose that $0 \leq \widetilde{a} < \alpha \leq -\beta$. If the action (0,0) is taken in state (x_{ut}, x_{lt}, y_t),

$$
\begin{aligned}
& v_{ct}^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_{ct}^*(x_{ut}, x_{lt}, y_t) \\
& \leq R(\widetilde{a}, 0, y_t) + \mathbb{E}\Big[v_{c(t+1)}^*\big(\min\{x_{ut}-\widetilde{a}+\alpha, C_U\}, \\
& \quad \min\{x_{lt}+\widetilde{a}+\beta, C_L\}, y_{(t+1)}\big)\Big] \\
& \quad - R(0,0,y_t) - \mathbb{E}\Big[v_{c(t+1)}^*\big(\min\{x_{ut}, C_U\}, \min\{x_{lt}, C_L\}, y_{(t+1)}\big)\Big].
\end{aligned}
$$

If $p_t \geq 0$, $R(\widetilde{a}, 0, y_t) - R(0,0,y_t) = p_t\theta\widetilde{a} \leq p_t\frac{\alpha}{\theta} \leq \frac{\alpha}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}$. If $p_t < 0$, $R(\widetilde{a}, 0, y_t) - R(0,0,y_t) = p_t\theta\widetilde{a} \leq 0 \leq \frac{\alpha}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}$. Thus:

$$
\begin{aligned}
& v_{ct}^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_{ct}^*(x_{ut}, x_{lt}, y_t) \\
& \leq \frac{\alpha}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
& \quad + \frac{1}{\theta}(\alpha-\widetilde{a}-\widetilde{a}-\beta)(T-t-1)\max\left\{\left(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t+1\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
& \leq \frac{\alpha}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
& \quad + \frac{1}{\theta}(\alpha+|\beta|)(T-t-1)\max\left\{\left(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t+1\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
& \leq \frac{1}{\theta}(\alpha+|\beta|)(T-t)\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}
$$

Finally suppose that $\widetilde{a} < 0 \leq \alpha \leq -\beta$. If the action $(\widetilde{a},\ 0)$ is taken in state (x_{ut}, x_{lt}, y_t),

$$
\begin{aligned}
& v_{ct}^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_{ct}^*(x_{ut}, x_{lt}, y_t) \\
& \leq R(\widetilde{a}, 0, y_t) + \mathbb{E}\Big[v_{c(t+1)}^*\big(\min\{x_{ut}-\widetilde{a}+\alpha, C_U\}, \\
& \quad \min\{x_{lt}+\widetilde{a}+\beta, C_L\}, y_{(t+1)}\big)\Big] \\
& \quad - R(\widetilde{a},0,y_t) - \mathbb{E}\Big[v_{c(t+1)}^*\big(\min\{x_{ut}-\widetilde{a}, C_U\}, \min\{x_{lt}+\widetilde{a}, C_L\}, y_{(t+1)}\big)\Big] \\
& \leq \frac{1}{\theta}(\alpha+|\beta|)(T-t)\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}
$$

Using the above result and assuming $v^*_{l(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) - v^*_{c(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) \leq \frac{1}{\theta} C_R \sum_{i=t+1}^{T} (T-i) \max\left\{\left(\max_{i\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{i\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}$, we show $v^*_{lt}(x_{ut}, x_{lt}, y_t) \leq v^*_{ct}(x_{ut}, x_{lt}, y_t) + \frac{1}{\theta} C_R \sum_{i=t}^{T} (T - i) \max\left\{\left(\max_{i\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{i\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}$. Let $a = a^*_{lt}(x_{ut}, x_{lt}, y_t)$ and $b = b^*_{lt}(x_{ut}, x_{lt}, y_t)$ denote the optimal actions in state (x_{ut}, x_{lt}, y_t) for the system with streamflow coming to the lower reservoir. Note that $(a, 0) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus:

$$
\begin{aligned}
& v^*_{lt}(x_{ut}, x_{lt}, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \\
& \leq R(a, b, y_t) + \mathbb{E}\Big[v^*_{l(t+1)}\big(\min\{x_{ut} - a, C_U\}, \\
& \quad \min\{x_{lt} + a - b + r_{t+1}, C_L\}, y_{(t+1)}\big)\Big] \\
& \quad - R(a, 0, y_t) - \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut} - a, C_U\}, \min\{x_{lt} + a, C_L\}, y_{(t+1)}\big)\Big] \\
& = R(a, b, y_t) - R(a, 0, y_t) \\
& \quad + \mathbb{E}\Big[v^*_{l(t+1)}\big(\min\{x_{ut} - a, C_U\}, \min\{x_{lt} + a - b + r_{t+1}, C_L\}, y_{(t+1)}\big) \\
& \quad - v^*_{c(t+1)}\big(\min\{x_{ut} - a, C_U\}, \min\{x_{lt} + a - b + r_{t+1}, C_L\}, y_{(t+1)}\big) \\
& \quad + v^*_{c(t+1)}\big(\min\{x_{ut} - a, C_U\}, \min\{x_{lt} + a - b + r_{t+1}, C_L\}, y_{(t+1)}\big) \\
& \quad - v^*_{c(t+1)}\big(\min\{x_{ut} - a, C_U\}, \min\{x_{lt} + a, C_L\}, y_{(t+1)}\big)\Big].
\end{aligned}
$$

If $p_t \geq 0$, $R(a, b, y_t) - R(a, 0, y_t) = p_t \theta b \leq p_t \frac{b}{\theta} \leq \frac{b}{\theta} \max\left\{\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}$. If $p_t < 0$, $R(a, b, y_t) - R(a, 0, y_t) = p_t \theta b \leq 0 \leq \frac{b}{\theta} \max\left\{\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}$. Thus:

$$
\begin{aligned}
&v^*_{lt}(x_{ut}, x_{lt}, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \\
&\leq \frac{b}{\theta} \max\left\{\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{C_R}{\theta} \sum_{i=t+1}^{T} (T-i)\max\left\{\left(\max_{i\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{i\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{1}{\theta}|r_{t+1} - b|(T-t-1)\max\left\{\left(\max_{t+1\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t+1\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\leq \frac{C_R}{\theta} \max\left\{\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{C_R}{\theta} \sum_{i=t+1}^{T} (T-i)\max\left\{\left(\max_{i\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{i\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{C_R}{\theta}(T-t-1)\max\left\{\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&= \frac{C_R}{\theta} \sum_{i=t+1}^{T} (T-i)\max\left\{\left(\max_{i\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{i\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{C_R}{\theta}(T-t)\max\left\{\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&= \frac{C_R}{\theta} \sum_{i=t}^{T}(T-i)\max\left\{\left(\max_{i\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{i\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}
$$

Moreover, assuming $v^*_{c(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}) \leq v^*_{u(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{t+1})$, we show $v^*_{ct}(x_{ut}, x_{lt}, y_t) \leq v^*_{ut}(x_{ut}, x_{lt}, y_t)$, which implies the third inequality in point (b). Let $a = a^*_{ct}(x_{ut}, x_{lt}, y_t)$. Note that $(a, 0) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus:

$$
\begin{aligned}
v^*_{ct}(x_{ut}, x_{lt}, y_t) &= R(a, 0, y_t) + \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut} - a, C_U\}, \\
&\quad \min\{x_{lt} + a, C_L\}, y_{t+1}\big)\Big] \\
&\leq R(a, 0, y_t) + \mathbb{E}\Big[v^*_{u(t+1)}\big(\min\{x_{ut} - a, C_U\}, \\
&\quad \min\{x_{lt} + a, C_L\}, y_{t+1}\big)\Big] \\
&\leq R(a, 0, y_t) + \mathbb{E}\Big[v^*_{u(t+1)}\big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \\
&\quad \min\{x_{lt} + a, C_L\}, y_{t+1}\big)\Big] \\
&\leq v^*_{ut}(x_{ut}, x_{lt}, y_t).
\end{aligned}
$$

The first inequality follows from the induction assumption and the second inequality follows from point (a).

Finally we prove that $v^*_{ut}(x_{ut}, x_{lt}, y_t) \leq v^*_{ct}(x_{ut}, x_{lt}, y_t) + \frac{1}{\theta} C_R \sum_{i=t}^{T} (T - i) \max\left\{ \left(\max_{i \leq \tau \leq T} \overline{p}_\tau \right)^+, \left(\max_{i \leq \tau \leq T} -\underline{p}_\tau \right)^+ \right\}$, $\forall t \in \mathcal{T}$. Note that $v^*_{uT}(x_{uT}, x_{lT}, y_T) = v^*_{cT}(x_{uT}, x_{lT}, y_T) + \frac{1}{\theta} C_R \sum_{i=T}^{T} (T - i) \max\left\{ \left(\max_{i \leq \tau \leq T} \overline{p}_\tau \right)^+, \left(\max_{i \leq \tau \leq T} -\underline{p}_\tau \right)^+ \right\} = 0$. To this end, we will use the inequality $v^*_{ct}(x_{ut} + \alpha, x_{lt} + \beta, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \leq \frac{1}{\theta}(\alpha + |\beta|)(T - t) \max\left\{ \left(\max_{t \leq \tau \leq T} \overline{p}_\tau \right)^+, \left(\max_{t \leq \tau \leq T} -\underline{p}_\tau \right)^+ \right\}$ where $\alpha \geq 0$, which we showed before. We assume $v^*_{u(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) - v^*_{c(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) \leq \frac{1}{\theta} C_R \sum_{i=t+1}^{T} (T - i) \max\left\{ \left(\max_{i \leq \tau \leq T} \overline{p}_\tau \right)^+, \left(\max_{i \leq \tau \leq T} -\underline{p}_\tau \right)^+ \right\}$. Also, let $a = a^*_{ut}(x_{ut}, x_{lt}, y_t)$ and $b = b^*_{ut}(x_{ut}, x_{lt}, y_t)$ denote the optimal actions in state (x_{ut}, x_{lt}, y_t) for the system with streamflow coming to the upper reservoir. Note that $(a, 0) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus:

$$
\begin{aligned}
&v^*_{ut}(x_{ut}, x_{lt}, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \\
&\leq R(a, b, y_t) + \mathbb{E}\Big[v^*_{u(t+1)}\big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \\
&\quad \min\{x_{lt} + a - b, C_L\}, y_{(t+1)}\big)\Big] \\
&\quad - R(a, 0, y_t) - \mathbb{E}\Big[v^*_{c(t+1)}\big(\min\{x_{ut} - a, C_U\}, \\
&\quad \min\{x_{lt} + a, C_L\}, y_{(t+1)}\big)\Big] \\
&= R(a, b, y_t) - R(a, 0, y_t) \\
&\quad + \mathbb{E}\Big[v^*_{u(t+1)}\big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + a - b, C_L\}, y_{(t+1)}\big) \\
&\quad - v^*_{c(t+1)}\big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + a - b, C_L\}, y_{(t+1)}\big) \\
&\quad + v^*_{c(t+1)}\big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + a - b, C_L\}, y_{(t+1)}\big) \\
&\quad - v^*_{c(t+1)}\big(\min\{x_{ut} - a, C_U\}, \min\{x_{lt} + a, C_L\}, y_{(t+1)}\big)\Big] \\
&\leq R(a, b, y_t) - R(a, 0, y_t) \\
&\quad + \mathbb{E}\Big[v^*_{u(t+1)}\big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + a - b, C_L\}, y_{(t+1)}\big) \\
&\quad - v^*_{c(t+1)}\big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + a - b, C_L\}, y_{(t+1)}\big) \\
&\quad + v^*_{c(t+1)}\big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + a, C_L\}, y_{(t+1)}\big) \\
&\quad - v^*_{c(t+1)}\big(\min\{x_{ut} - a, C_U\}, \min\{x_{lt} + a, C_L\}, y_{(t+1)}\big)\Big].
\end{aligned}
$$

If $p_t \geq 0$, $R(a, b, y_t) - R(a, 0, y_t) = p_t\theta b \leq p_t\frac{b}{\theta} \leq \frac{b}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}$. If $p_t < 0$, $R(a, b, y_t) - R(a, 0, y_t) = p_t\theta b \leq 0 \leq \frac{b}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}$. Thus:

$$
\begin{aligned}
&v^*_{ut}(x_{ut}, x_{lt}, y_t) - v^*_{ct}(x_{ut}, x_{lt}, y_t) \\
&\leq \frac{b}{\theta} \max\left\{\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{C_R}{\theta} \sum_{i=t+1}^{T} (T-i)\max\left\{\left(\max_{i\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{i\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{1}{\theta} r_{t+1}(T-t-1)\max\left\{\left(\max_{t+1\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t+1\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\leq \frac{C_R}{\theta} \max\left\{\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{C_R}{\theta} \sum_{i=t+1}^{T} (T-i)\max\left\{\left(\max_{i\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{i\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{C_R}{\theta}(T-t-1)\max\left\{\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&= \frac{C_R}{\theta} \sum_{i=t+1}^{T} (T-i)\max\left\{\left(\max_{i\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{i\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&\quad + \frac{C_R}{\theta}(T-t)\max\left\{\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
&= \frac{C_R}{\theta} \sum_{i=t}^{T}(T-i)\max\left\{\left(\max_{i\leq\tau\leq T} \overline{p}_\tau\right)^+, \left(\max_{i\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}
$$

(c) First, we will prove that $v^*_{ut}(x_{ut}, x_{lt}, y_t) \leq v^*_{lt}(x_{ut}, x_{lt}, y_t) + (T-t)\theta\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+\left(\max_{t\leq\tau\leq T} \overline{r}_\tau\right)$, $\forall t \in \mathcal{T}$. Note that $v^*_{uT}(x_{uT}, x_{lT}, y_T) = v^*_{lT}(x_{uT}, x_{lT}, y_T) = 0$. To this end, we first show that $v^*_{lt}(x_{ut}+\alpha, x_{lt}-\alpha, y_t) - v^*_{lt}(x_{ut}, x_{lt}, y_t) \leq \alpha\theta\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+$ where $\alpha \geq 0$. Note that $v^*_{lT}(x_{uT}+\alpha, x_{lT}-\alpha, y_T) = v^*_{lT}(x_{uT}, x_{lT}, y_T) = 0$. Assuming $v^*_{l(t+1)}(x_{u(t+1)}+\alpha, x_{l(t+1)}-\alpha, y_{(t+1)}) - v^*_{l(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) \leq \alpha\theta\left(\max_{t+1\leq\tau\leq T} \overline{p}_\tau\right)^+$, we show $v^*_{lt}(x_{ut}+\alpha, x_{lt}-\alpha, y_t) - v^*_{lt}(x_{ut}, x_{lt}, y_t) \leq \alpha\theta\left(\max_{t\leq\tau\leq T} \overline{p}_\tau\right)^+$. Suppose that $\widetilde{a} = a^*_{lt}(x_{ut}+\alpha, x_{lt}-\alpha, y_t) < 0$

and $\widetilde{b} = b^*_{lt}(x_{ut} + \alpha, x_{lt} - \alpha, y_t)$. Notice that $(\widetilde{a}, \widetilde{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus:

$$\begin{aligned}
&v^*_{lt}(x_{ut} + \alpha, x_{lt} - \alpha, y_t) - v^*_{lt}(x_{ut}, x_{lt}, y_t) \\
&\le R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[v^*_{l(t+1)}\big(x_{ut} - \widetilde{a} + \alpha, x_{lt} + \widetilde{a} - \widetilde{b} - \alpha + r_{t+1}, y_{(t+1)}\big)\Big] \\
&\quad - R(\widetilde{a}, \widetilde{b}, y_t) - \mathbb{E}\Big[v^*_{l(t+1)}\big(x_{ut} - \widetilde{a}, x_{lt} + \widetilde{a} - \widetilde{b} + r_{t+1}, y_{(t+1)}\big)\Big] \\
&\le \alpha\theta \left(\max_{t+1 \le \tau \le T} \overline{p}_\tau\right)^+ \le \alpha\theta \left(\max_{t \le \tau \le T} \overline{p}_\tau\right)^+ .
\end{aligned}$$

Now suppose that $0 \le \widetilde{a} < \alpha$. If the action $(0, \widetilde{b})$ is taken in state (x_{ut}, x_{lt}, y_t),

$$\begin{aligned}
&v^*_{lt}(x_{ut} + \alpha, x_{lt} - \alpha, y_t) - v^*_{lt}(x_{ut}, x_{lt}, y_t) \\
&\le R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[v^*_{l(t+1)}\big(x_{ut} - \widetilde{a} + \alpha, x_{lt} + \widetilde{a} - \widetilde{b} - \alpha + r_{t+1}, y_{(t+1)}\big)\Big] \\
&\quad - R(0, \widetilde{b}, y_t) - \mathbb{E}\Big[v^*_{l(t+1)}\big(x_{ut}, x_{lt} - \widetilde{b} + r_{t+1}, y_{(t+1)}\big)\Big].
\end{aligned}$$

If $p_t \ge 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(0, \widetilde{b}, y_t) = p_t\theta\widetilde{a} \le \theta\widetilde{a} \left(\max_{t \le \tau \le T} \overline{p}_\tau\right)^+$. If $p_t < 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(0, \widetilde{b}, y_t) = p_t\theta\widetilde{a} \le 0 \le \theta\widetilde{a} \left(\max_{t \le \tau \le T} \overline{p}_\tau\right)^+$. Thus:

$$\begin{aligned}
&v^*_{lt}(x_{ut} + \alpha, x_{lt} - \alpha, y_t) - v^*_{lt}(x_{ut}, x_{lt}, y_t) \\
&\le \widetilde{a}\theta \left(\max_{t \le \tau \le T} \overline{p}_\tau\right)^+ + (\alpha - \widetilde{a})\theta \left(\max_{t \le \tau \le T} \overline{p}_\tau\right)^+ \\
&= \alpha\theta \left(\max_{t \le \tau \le T} \overline{p}_\tau\right)^+ .
\end{aligned}$$

Finally suppose that $\widetilde{a} \ge \alpha$. If the action $(\widetilde{a} - \alpha, \widetilde{b})$ is taken in state (x_{ut}, x_{lt}, y_t),

$$\begin{aligned}
&v^*_{lt}(x_{ut} + \alpha, x_{lt} - \alpha, y_t) - v^*_{lt}(x_{ut}, x_{lt}, y_t) \\
&\le R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[v^*_{l(t+1)}\big(x_{ut} - \widetilde{a} + \alpha, x_{lt} + \widetilde{a} - \widetilde{b} - \alpha + r_{t+1}, y_{(t+1)}\big)\Big] \\
&\quad - R(\widetilde{a} - \alpha, \widetilde{b}, y_t) - \mathbb{E}\Big[v^*_{l(t+1)}\big(x_{ut} - \widetilde{a} + \alpha, x_{lt} + \widetilde{a} - \widetilde{b} - \alpha + r_{t+1}, y_{(t+1)}\big)\Big].
\end{aligned}$$

If $p_t \ge 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(\widetilde{a} - \alpha, \widetilde{b}, y_t) = p_t\theta\alpha \le \alpha\theta \left(\max_{t \le \tau \le T} \overline{p}_\tau\right)^+$. If $p_t < 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(\widetilde{a} - \alpha, \widetilde{b}, y_t) = p_t\theta\alpha \le 0 \le \alpha\theta \left(\max_{t \le \tau \le T} \overline{p}_\tau\right)^+$. Thus:

$$v^*_{lt}(x_{ut} + \alpha, x_{lt} - \alpha, y_t) - v^*_{lt}(x_{ut}, x_{lt}, y_t) \le \alpha\theta \left(\max_{t \le \tau \le T} \overline{p}_\tau\right)^+ .$$

Using the above result and assuming $v^*_{u(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) - v^*_{l(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) \leq (T-t-1)\theta\left(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\right)^+\left(\max_{t+1\leq\tau\leq T}\overline{r}_\tau\right)$, we show $v^*_{ut}(x_{ut}, x_{lt}, y_t) \leq v^*_{lt}(x_{ut}, x_{lt}, y_t) + (T-t)\theta\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+\left(\max_{t\leq\tau\leq T}\overline{r}_\tau\right)$. Let $a = a^*_{ut}(x_{ut}, x_{lt}, y_t)$ and $b = b^*_{ut}(x_{ut}, x_{lt}, y_t)$ denote the optimal actions in state (x_{ut}, x_{lt}, y_t) for the system with streamflow coming to the upper reservoir. Note that $(a, b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus:

$$
\begin{aligned}
&v^*_{ut}(x_{ut}, x_{lt}, y_t) - v^*_{lt}(x_{ut}, x_{lt}, y_t)\\
&\leq R(a,b,y_t) + \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - a + r_{t+1}, x_{lt} + a - b, y_{(t+1)}\big)\Big]\\
&\quad - R(a,b,y_t) - \mathbb{E}\Big[v^*_{l(t+1)}\big(x_{ut} - a, x_{lt} + a - b + r_{t+1}, y_{(t+1)}\big)\Big]\\
&= \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - a + r_{t+1}, x_{lt} + a - b, y_{(t+1)}\big)\\
&\quad - v^*_{l(t+1)}\big(x_{ut} - a + r_{t+1}, x_{lt} + a - b, y_{(t+1)}\big)\\
&\qquad + v^*_{l(t+1)}\big(x_{ut} - a + r_{t+1}, x_{lt} + a - b, y_{(t+1)}\big)\\
&\quad - v^*_{l(t+1)}\big(x_{ut} - a, x_{lt} + a - b + r_{t+1}, y_{(t+1)}\big)\Big]\\
&\leq (T-t-1)\theta\left(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\right)^+\left(\max_{t+1\leq\tau\leq T}\overline{r}_\tau\right) + \overline{r}_{t+1}\theta\left(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\right)^+\\
&\leq (T-t-1)\theta\left(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\right)^+\left(\max_{t+1\leq\tau\leq T}\overline{r}_\tau\right) + \theta\left(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\right)^+\left(\max_{t+1\leq\tau\leq T}\overline{r}_\tau\right)\\
&\leq (T-t)\theta\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+\left(\max_{t\leq\tau\leq T}\overline{r}_\tau\right).
\end{aligned}
$$

Next, we will prove that $v^*_{lt}(x_{ut}, x_{lt}, y_t) \leq v^*_{ut}(x_{ut}, x_{lt}, y_t) + (T-t)\frac{1}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T}-\underline{p}_\tau\right)^+\right\}\left(\max_{t\leq\tau\leq T}\overline{r}_\tau\right)$, $\forall t \in \mathcal{T}$. Note that $v^*_{lT}(x_{uT}, x_{lT}, y_T) = v^*_{uT}(x_{uT}, x_{lT}, y_T) = 0$. To this end, we first show that $v^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) - v^*_{ut}(x_{ut}, x_{lt}, y_t) \leq \frac{\alpha}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T}-\underline{p}_\tau\right)^+\right\}$ where $\alpha \geq 0$. Note that $v^*_{uT}(x_{uT} - \alpha, x_{lT} + \alpha, y_T) = v^*_{uT}(x_{uT}, x_{lT}, y_T) = 0$. Assuming $v^*_{u(t+1)}(x_{u(t+1)} - \alpha, x_{l(t+1)} + \alpha, y_{(t+1)}) - v^*_{u(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) \leq \frac{\alpha}{\theta}\max\left\{\left(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t+1\leq\tau\leq T}-\underline{p}_\tau\right)^+\right\}$, we show $v^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) - v^*_{ut}(x_{ut}, x_{lt}, y_t) \leq \frac{\alpha}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T}-\underline{p}_\tau\right)^+\right\}$. Suppose that $\widetilde{a} = a^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t)$ and $\widetilde{b} = b^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) > \alpha$. Notice that

$(\widetilde{a}, \widetilde{b} - \alpha) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus:

$$\begin{aligned}
&v^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) - v^*_{ut}(x_{ut}, x_{lt}, y_t) \\
&\leq R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - \widetilde{a} - \alpha + r_{t+1}, x_{lt} + \widetilde{a} - \widetilde{b} + \alpha, y_{(t+1)}\big)\Big] \\
&\quad - R(\widetilde{a}, \widetilde{b} - \alpha, y_t) - \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - \widetilde{a} + r_{t+1}, x_{lt} + \widetilde{a} - \widetilde{b} + \alpha, y_{(t+1)}\big)\Big].
\end{aligned}$$

If $p_t \geq 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(\widetilde{a}, \widetilde{b} - \alpha, y_t) = p_t\theta\alpha \leq \frac{\alpha}{\theta} \max\left\{\Big(\max_{t \leq \tau \leq T} \overline{p}_\tau\Big)^+, \Big(\max_{t \leq \tau \leq T} -\underline{p}_\tau\Big)^+\right\}$. If $p_t < 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(\widetilde{a}, \widetilde{b} - \alpha, y_t) = p_t\theta\alpha \leq 0 \leq \frac{\alpha}{\theta} \max\left\{\Big(\max_{t \leq \tau \leq T} \overline{p}_\tau\Big)^+, \Big(\max_{t \leq \tau \leq T} -\underline{p}_\tau\Big)^+\right\}$. Thus:

$$\begin{aligned}
&v^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) - v^*_{ut}(x_{ut}, x_{lt}, y_t) \\
&\leq \frac{\alpha}{\theta} \max\left\{\Big(\max_{t \leq \tau \leq T} \overline{p}_\tau\Big)^+, \Big(\max_{t \leq \tau \leq T} -\underline{p}_\tau\Big)^+\right\}.
\end{aligned}$$

Now suppose that $\widetilde{a} \geq 0$ and $\widetilde{b} \leq \alpha$. Notice that $(\widetilde{a}, 0) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$.

$$\begin{aligned}
&v^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) - v^*_{ut}(x_{ut}, x_{lt}, y_t) \\
&\leq R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - \widetilde{a} - \alpha + r_{t+1}, x_{lt} + \widetilde{a} - \widetilde{b} + \alpha, y_{(t+1)}\big)\Big] \\
&\quad - R(\widetilde{a}, 0, y_t) - \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - \widetilde{a} + r_{t+1}, x_{lt} + \widetilde{a}, y_{(t+1)}\big)\Big] \\
&\leq R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - \widetilde{a} - \alpha + \widetilde{b} + r_{t+1}, x_{lt} + \widetilde{a} - \widetilde{b} + \alpha, y_{(t+1)}\big)\Big] \\
&\quad - R(\widetilde{a}, 0, y_t) - \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - \widetilde{a} + r_{t+1}, x_{lt} + \widetilde{a}, y_{(t+1)}\big)\Big].
\end{aligned}$$

If $p_t \geq 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(\widetilde{a}, 0, y_t) = p_t\theta\widetilde{b} \leq \frac{\widetilde{b}}{\theta} \max\left\{\Big(\max_{t \leq \tau \leq T} \overline{p}_\tau\Big)^+, \Big(\max_{t \leq \tau \leq T} -\underline{p}_\tau\Big)^+\right\}$. If $p_t < 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(\widetilde{a}, 0, y_t) = p_t\theta\widetilde{b} \leq 0 \leq \frac{\widetilde{b}}{\theta} \max\left\{\Big(\max_{t \leq \tau \leq T} \overline{p}_\tau\Big)^+, \Big(\max_{t \leq \tau \leq T} -\underline{p}_\tau\Big)^+\right\}$. Thus:

$$\begin{aligned}
&v^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) - v^*_{ut}(x_{ut}, x_{lt}, y_t) \\
&\leq \frac{\widetilde{b}}{\theta} \max\left\{\Big(\max_{t \leq \tau \leq T} \overline{p}_\tau\Big)^+, \Big(\max_{t \leq \tau \leq T} -\underline{p}_\tau\Big)^+\right\} \\
&\quad + (\alpha - \widetilde{b})\, \frac{1}{\theta} \max\left\{\Big(\max_{t \leq \tau \leq T} \overline{p}_\tau\Big)^+, \Big(\max_{t \leq \tau \leq T} -\underline{p}_\tau\Big)^+\right\} \\
&\leq \frac{\alpha}{\theta} \max\left\{\Big(\max_{t \leq \tau \leq T} \overline{p}_\tau\Big)^+, \Big(\max_{t \leq \tau \leq T} -\underline{p}_\tau\Big)^+\right\}.
\end{aligned}$$

Now suppose that $-\alpha \leq \widetilde{a} < 0$ and $\widetilde{b} \leq \alpha$. If the action $(0,0)$ is taken in state (x_{ut}, x_{lt}, y_t),

$$
\begin{aligned}
& v^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) - v^*_{ut}(x_{ut}, x_{lt}, y_t) \\
& \leq R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - \widetilde{a} - \alpha + r_{t+1}, x_{lt} + \widetilde{a} - \widetilde{b} + \alpha, y_{(t+1)}\big)\Big] \\
& \quad - R(0,0,y_t) - \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} + r_{t+1}, x_{lt}, y_{(t+1)}\big)\Big] \\
& \leq R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - \widetilde{a} - \alpha + \widetilde{b} + r_{t+1}, x_{lt} + \widetilde{a} - \widetilde{b} + \alpha, y_{(t+1)}\big)\Big] \\
& \quad - R(0,0,y_t) - \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} + r_{t+1}, x_{lt}, y_{(t+1)}\big)\Big].
\end{aligned}
$$

If $p_t \geq 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(0,0,y_t) = p_t\frac{\widetilde{a}}{\theta} + p_t\theta\widetilde{b} \leq p_t\theta\widetilde{b} \leq p_t\frac{\widetilde{b}}{\theta} \leq \frac{\widetilde{b}}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \leq -\frac{\widetilde{a}}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} + \frac{\widetilde{b}}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}$. If $p_t < 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(0,0,y_t) = p_t\frac{\widetilde{a}}{\theta} + p_t\theta\widetilde{b} \leq p_t\frac{\widetilde{a}}{\theta} \leq -\frac{\widetilde{a}}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \leq -\frac{\widetilde{a}}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} + \frac{\widetilde{b}}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}$.

Thus:

$$
\begin{aligned}
& v^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) - v^*_{ut}(x_{ut}, x_{lt}, y_t) \\
& \leq -\frac{\widetilde{a}}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
& \quad + \frac{\widetilde{b}}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
& \quad + (\widetilde{a} + \alpha - \widetilde{b})\frac{1}{\theta}\max\left\{\left(\max_{t+1\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t+1\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\} \\
& \leq \frac{\alpha}{\theta}\max\left\{\left(\max_{t\leq\tau\leq T}\overline{p}_\tau\right)^+, \left(\max_{t\leq\tau\leq T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}
$$

Finally suppose that $\widetilde{a} < -\alpha$ and $\widetilde{b} \leq \alpha$. If the action $(\widetilde{a} + \alpha, 0)$ is taken in state

(x_{ut}, x_{lt}, y_t),

$$\begin{aligned}
&v^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) - v^*_{ut}(x_{ut}, x_{lt}, y_t)\\
&\le R(\widetilde{a}, \widetilde{b}, y_t) + \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - \widetilde{a} - \alpha + r_{t+1}, x_{lt} + \widetilde{a} - \widetilde{b} + \alpha, y_{(t+1)}\big)\Big]\\
&\quad - R(\widetilde{a} + \alpha, 0, y_t) - \mathbb{E}\Big[v^*_{u(t+1)}\big(x_{ut} - \widetilde{a} - \alpha + r_{t+1}, x_{lt} + \widetilde{a} + \alpha, y_{(t+1)}\big)\Big].
\end{aligned}$$

If $p_t \ge 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(\widetilde{a} + \alpha, 0, y_t) = \left(-\frac{\alpha}{\theta} + \theta\widetilde{b}\right) p_t \le 0 \le \frac{\alpha}{\theta} \max\left\{\left(\max\limits_{t \le \tau \le T} \overline{p}_\tau\right)^+, \left(\max\limits_{t \le \tau \le T} -\underline{p}_\tau\right)^+\right\}$. If $p_t < 0$, $R(\widetilde{a}, \widetilde{b}, y_t) - R(\widetilde{a} + \alpha, 0, y_t) = \left(-\frac{\alpha}{\theta} + \theta\widetilde{b}\right) p_t \le -p_t\frac{\alpha}{\theta} \le \frac{\alpha}{\theta} \max\left\{\left(\max\limits_{t \le \tau \le T} \overline{p}_\tau\right)^+, \left(\max\limits_{t \le \tau \le T} -\underline{p}_\tau\right)^+\right\}$. Thus:

$$\begin{aligned}
&v^*_{ut}(x_{ut} - \alpha, x_{lt} + \alpha, y_t) - v^*_{ut}(x_{ut}, x_{lt}, y_t)\\
&\le \frac{\alpha}{\theta} \max\left\{\left(\max_{t \le \tau \le T} \overline{p}_\tau\right)^+, \left(\max_{t \le \tau \le T} -\underline{p}_\tau\right)^+\right\}.
\end{aligned}$$

Using the above result and assuming $v^*_{l(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) - v^*_{u(t+1)}(x_{u(t+1)}, x_{l(t+1)}, y_{(t+1)}) \le (T - t - 1)\frac{1}{\theta} \max\left\{\left(\max\limits_{t+1 \le \tau \le T} \overline{p}_\tau\right)^+, \left(\max\limits_{t+1 \le \tau \le T} -\underline{p}_\tau\right)^+\right\} \left(\max\limits_{t+1 \le \tau \le T} \overline{r}_\tau\right)$, we show $v^*_{lt}(x_{ut}, x_{lt}, y_t) \le v^*_{ut}(x_{ut}, x_{lt}, y_t) + (T - t)\frac{1}{\theta} \max\left\{\left(\max\limits_{t \le \tau \le T} \overline{p}_\tau\right)^+, \left(\max\limits_{t \le \tau \le T} -\underline{p}_\tau\right)^+\right\}\left(\max\limits_{t \le \tau \le T} \overline{r}_\tau\right)$. Let $a = a^*_{lt}(x_{ut}, x_{lt}, y_t)$ and $b = b^*_{lt}(x_{ut}, x_{lt}, y_t)$ denote the optimal actions in state (x_{ut}, x_{lt}, y_t) for the system with streamflow coming to the lower reservoir. Note that $(a, b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$.

Thus:

$$
\begin{aligned}
& v_{lt}^*(x_{ut}, x_{lt}, y_t) - v_{ut}^*(x_{ut}, x_{lt}, y_t) \\
& \leq R(a, b, y_t) + \mathbb{E}\Big[v_{l(t+1)}^*\big(x_{ut} - a, x_{lt} + a - b + r_{t+1}, y_{(t+1)}\big)\Big] \\
& \quad - R(a, b, y_t) - \mathbb{E}\Big[v_{u(t+1)}^*\big(x_{ut} - a + r_{t+1}, x_{lt} + a - b, y_{(t+1)}\big)\Big] \\
& = \mathbb{E}\Big[v_{l(t+1)}^*\big(x_{ut} - a, x_{lt} + a - b + r_{t+1}, y_{(t+1)}\big) \\
& \quad - v_{u(t+1)}^*\big(x_{ut} - a, x_{lt} + a - b + r_{t+1}, y_{(t+1)}\big) \\
& \quad + v_{u(t+1)}^*\big(x_{ut} - a, x_{lt} + a - b + r_{t+1}, y_{(t+1)}\big) \\
& \quad - v_{u(t+1)}^*\big(x_{ut} - a + r_{t+1}, x_{lt} + a - b, y_{(t+1)}\big)\Big] \\
& \leq (T - t - 1)\frac{1}{\theta} \max\left\{\left(\max_{t+1 \leq \tau \leq T} \overline{p}_\tau\right)^+, \left(\max_{t+1 \leq \tau \leq T} -\underline{p}_\tau\right)^+\right\}\left(\max_{t+1 \leq \tau \leq T} \overline{r}_\tau\right) \\
& \quad + \frac{\overline{r}_{t+1}}{\theta} \max\left\{\left(\max_{t+1 \leq \tau \leq T} \overline{p}_\tau\right)^+, \left(\max_{t+1 \leq \tau \leq T} -\underline{p}_\tau\right)^+\right\} \\
& \leq (T - t - 1)\frac{1}{\theta} \max\left\{\left(\max_{t+1 \leq \tau \leq T} \overline{p}_\tau\right)^+, \left(\max_{t+1 \leq \tau \leq T} -\underline{p}_\tau\right)^+\right\}\left(\max_{t+1 \leq \tau \leq T} \overline{r}_\tau\right) \\
& \quad + \frac{1}{\theta} \max\left\{\left(\max_{t+1 \leq \tau \leq T} \overline{p}_\tau\right)^+, \left(\max_{t+1 \leq \tau \leq T} -\underline{p}_\tau\right)^+\right\}\left(\max_{t+1 \leq \tau \leq T} \overline{r}_\tau\right) \\
& \leq (T - t)\frac{1}{\theta} \max\left\{\left(\max_{t \leq \tau \leq T} \overline{p}_\tau\right)^+, \left(\max_{t \leq \tau \leq T} -\underline{p}_\tau\right)^+\right\}\left(\max_{t \leq \tau \leq T} \overline{r}_\tau\right).
\end{aligned}
$$

Appendix C

Proofs of the Analytical Results in Chapter 5

Proof of Lemma 5.1.1. Under Assumption 5.1.1, $p_t > 0$, $\forall t \in \mathcal{T}$. Note that $R(a_t, b_t, y_t) = \min\{(\theta a_t + \theta b_t)p_t, (a_t/\theta + \theta b_t)p_t\}$. Since the minimum of affine functions is concave, $R(a_t, b_t, y_t)$ is jointly concave in a_t and b_t. $\square$

Proof of Lemma 5.1.2. Note that $v_T^*(x_{uT}, x_{lT}, y_T) = v_T^*(x_{uT}, x_{lT} + \alpha, y_T) = v_T^*(x_{uT} + \alpha, x_{lT}, y_T) = 0$. Assuming $v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}) \leq v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)} + \alpha, y_{t+1})$, we show $v_t^*(x_{ut}, x_{lt}, y_t) \leq v_t^*(x_{ut}, x_{lt} + \alpha, y_t)$. Let $a = a_t^*(x_{ut}, x_{lt}, y_t)$ and $b = b_t^*(x_{ut}, x_{lt}, y_t)$. Note that $(a, b) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$. Thus:

$$\begin{aligned}
v_t^*(x_{ut}, x_{lt}, y_t) &= R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\},\\
&\qquad \min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\leq R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\},\\
&\qquad \min\{x_{lt} + a - b + r_{l(t+1)} + \alpha, C_L\}, y_{t+1}\Big)\Big]\\
&\leq v_t^*(x_{ut}, x_{lt} + \alpha, y_t).
\end{aligned}$$

Now, assuming $v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}) \leq v_{t+1}^*(x_{u(t+1)} + \alpha, x_{l(t+1)}, y_{t+1})$, we show $v_t^*(x_{ut}, x_{lt}, y_t) \leq v_t^*(x_{ut} + \alpha, x_{lt}, y_t)$. Let $a = a_t^*(x_{ut}, x_{lt}, y_t)$ and $b = b_t^*(x_{ut}, x_{lt}, y_t)$. Note that $(a, b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$. Thus:

$$\begin{aligned}
v_t^*(x_{ut}, x_{lt}, y_t) = R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(&\min\{x_{ut} - a + r_{u(t+1)}, C_U\}, \\
&\min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
\leq R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(&\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\}, \\
&\min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
\leq v_t^*(x_{ut} + \alpha, x_{lt}, y_t).&
\end{aligned}$$

□

Proof of Lemma 5.1.3. Let $a = a_t^*(x_{ut}, x_{lt}, y_t)$ and $b = b_t^*(x_{ut}, x_{lt}, y_t)$. Pick an arbitrary y_t. Assume to the contrary that $a < x_{ut} - C_U \leq 0$. Note that $-C_P \leq a < x_{ut} - C_U \leq 0 \leq \min\{x_{ut}, C_R\}$ and $b + C_U - x_{ut} < b - a \leq x_{lt}$. Hence $(x_{ut} - C_U, b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Since $R(\cdot, b, y_t)$ is an increasing function, $R(x_{ut} - C_U, b, y_t) > R(a, b, y_t)$. Thus, the first inequality in Lemma 5.1.2 implies that

$$\begin{aligned}
v_t^*(x_{ut}, x_{lt}, y_t) &= R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(C_U, \min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&< R(x_{ut} - C_U, b, y_t) \\
&\quad + \mathbb{E}\Big[v_{t+1}^*\Big(C_U, \min\{x_{lt} + x_{ut} - C_U - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big].
\end{aligned}$$

Since the action pair $(x_{ut} - C_U, b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$ leads to a larger profit function, we have a contradiction. Thus $a = a_t^*(x_{ut}, x_{lt}, y_t) \geq x_{ut} - C_U$. □

Proof of Lemma 5.1.4. Let $a = a_t^*(x_{ut}, x_{lt}, y_t)$ and $b = b_t^*(x_{ut}, x_{lt}, y_t)$. Pick an arbitrary y_t. Assume to the contrary that $b - a < x_{lt} - C_L \leq 0$. Note that $0 \leq b < a \leq C_R$. Since $C_R \leq C_L$, $b < x_{lt} + a - C_L \leq x_{lt} + C_R - C_L \leq x_{lt}$. Let $\Delta = \min\{x_{lt}, C_R, x_{lt} + a - C_L\} - b > 0$. Note that $0 \leq b < b + \Delta \leq \min\{x_{lt}, C_R\}$ and $b + \Delta - a \leq x_{lt} - C_L \leq x_{lt}$. Hence $(a, b + \Delta) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Since $R(a, \cdot, y_t)$ is an increasing function, $R(a, b, y_t) < R(a, b + \Delta, y_t)$. Note that $x_{lt} + a - b - \Delta \geq C_L$.

Thus:

$$\begin{aligned}
v_t^*(x_{ut}, x_{lt}, y_t) &= R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\}, C_L, y_{t+1}\Big)\Big] \\
&< R(a, b + \Delta, y_t) \\
&\quad + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\}, C_L, y_{t+1}\Big)\Big] \\
&= R(a, b + \Delta, y_t) \\
&\quad + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + a - b + r_{l(t+1)} - \Delta, C_L\}, y_{t+1}\Big)\Big].
\end{aligned}$$

Since the action pair $(a, b + \Delta) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$ leads to a larger profit function, we have a contradiction. Thus $b - a = b_t^*(x_{ut}, x_{lt}, y_t) - a_t^*(x_{ut}, x_{lt}, y_t) \geq x_{lt} - C_L$. □

Proof of Lemma 5.1.5. Note that $v_T^*(.)$ satisfies properties (a)–(c). Pick an arbitrary $t < T$. Assuming $v_{t+1}^*(.)$ satisfies properties (a)–(c), we first prove that $v_{t+1}^*(.)$ satisfies the following properties:

(i)

$$\begin{aligned}
&v_{t+1}^*(\min\{\bar{x}_{ut} + \alpha, C_U\}, \min\{\bar{x}_{lt}, C_L\}, y_{t+1}) \\
&\quad - v_{t+1}^*(\min\{\bar{x}_{ut}, C_U\}, \min\{\bar{x}_{lt} + \alpha, C_L\}, y_{t+1}) \\
&\leq v_{t+1}^*(\min\{\bar{x}_{ut} + \alpha, C_U\}, \min\{\bar{x}_{lt} + \beta, C_L\}, y_{t+1}) \\
&\quad - v_{t+1}^*(\min\{\bar{x}_{ut}, C_U\}, \min\{\bar{x}_{lt} + \alpha + \beta, C_L\}, y_{t+1}),
\end{aligned}$$

(ii)

$$\begin{aligned}
&v_{t+1}^*(\min\{\bar{x}_{ut} + \alpha + \beta, C_U\}, \min\{\bar{x}_{lt}, C_L\}, y_{t+1}) \\
&\quad - v_{t+1}^*(\min\{\bar{x}_{ut} + \beta, C_U\}, \min\{\bar{x}_{lt} + \alpha, C_L\}, y_{t+1}) \\
&\leq v_{t+1}^*(\min\{\bar{x}_{ut} + \alpha, C_U\}, \min\{\bar{x}_{lt}, C_L\}, y_{t+1}) \\
&\quad - v_{t+1}^*(\min\{\bar{x}_{ut}, C_U\}, \min\{\bar{x}_{lt} + \alpha, C_L\}, y_{t+1}),
\end{aligned}$$

(iii)

$$\begin{aligned}
&v^*_{t+1}(\min\{\bar{x}_{ut}+\alpha, C_U\}, \min\{\bar{x}_{lt}+\beta, C_L\}, y_{t+1}) \\
&\quad - v^*_{t+1}(\min\{\bar{x}_{ut}, C_U\}, \min\{\bar{x}_{lt}+\beta, C_L\}, y_{t+1}) \\
&\le v^*_{t+1}(\min\{\bar{x}_{ut}+\alpha, C_U\}, \min\{\bar{x}_{lt}, C_L\}, y_{t+1}) \\
&\quad - v^*_{t+1}(\min\{\bar{x}_{ut}, C_U\}, \min\{\bar{x}_{lt}, C_L\}, y_{t+1}),
\end{aligned}$$

where $\bar{x}_{ut} = x_{ut}+\delta_u$, $\bar{x}_{lt} = x_{lt}+\delta_l$, $\delta_u \ge 0$, and $\delta_l \ge 0$. With the above properties, we will prove that $v^*_t(.)$ satisfies properties (a)–(c).

In order to prove property (i), we consider the following fourteen cases:

(1) If $\delta_u \le C_U - x_{ut} - \alpha < C_U - x_{ut}$ and $\delta_l \le C_L - x_{lt} - \alpha - \beta$, as we assume $v^*_{t+1}(.)$ satisfies property (a),

$$\begin{aligned}
&v^*_{t+1}(x_{ut}+\delta_u+\alpha, x_{lt}+\delta_l, y_{t+1}) - v^*_{t+1}(x_{ut}+\delta_u, x_{lt}+\delta_l+\alpha, y_{t+1}) \\
&\le v^*_{t+1}(x_{ut}+\delta_u+\alpha, x_{lt}+\delta_l+\beta, y_{t+1}) \\
&\quad - v^*_{t+1}(x_{ut}+\delta_u, x_{lt}+\delta_l+\alpha+\beta, y_{t+1}).
\end{aligned}$$

(2) If $\delta_u \le C_U - x_{ut} - \alpha < C_U - x_{ut}$ and $C_L - x_{lt} - \alpha - \beta < \delta_l \le C_L - x_{lt} - \alpha$, as we assume $v^*_{t+1}(.)$ satisfies property (a), and by Lemma 5.1.2,

$$\begin{aligned}
&v^*_{t+1}(x_{ut}+\delta_u+\alpha, x_{lt}+\delta_l, y_{t+1}) - v^*_{t+1}(x_{ut}+\delta_u, x_{lt}+\delta_l+\alpha, y_{t+1}) \\
&\le v^*_{t+1}(x_{ut}+\delta_u+\alpha, C_L-\alpha, y_{t+1}) - v^*_{t+1}(x_{ut}+\delta_u, C_L, y_{t+1}) \\
&\le v^*_{t+1}(x_{ut}+\delta_u+\alpha, \min\{x_{lt}+\delta_l+\beta, C_L\}, y_{t+1}) \\
&\quad - v^*_{t+1}(x_{ut}+\delta_u, C_L, y_{t+1}).
\end{aligned}$$

(3) If $\delta_u \le C_U - x_{ut} - \alpha < C_U - x_{ut}$ and $C_L - x_{lt} - \alpha < \delta_l \le C_L - x_{lt}$, by Lemma 5.1.2,

$$\begin{aligned}
&v^*_{t+1}(x_{ut}+\delta_u+\alpha, x_{lt}+\delta_l, y_{t+1}) - v^*_{t+1}(x_{ut}+\delta_u, C_L, y_{t+1}) \\
&\le v^*_{t+1}(x_{ut}+\delta_u+\alpha, \min\{x_{lt}+\delta_l+\beta, C_L\}, y_{t+1}) \\
&\quad - v^*_{t+1}(x_{ut}+\delta_u, C_L, y_{t+1}).
\end{aligned}$$

(4) If $\delta_u \leq C_U - x_{ut} - \alpha < C_U - x_{ut}$ and $C_L - x_{lt} < \delta_l$, both sides of the inequality reduce to $v^*_{t+1}(x_{ut} + \delta_u + \alpha, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1})$.

(5) If $C_U - x_{ut} - \alpha < \delta_u \leq C_U - x_{ut}$ and $\delta_l \leq C_L - x_{lt} - \alpha - \beta$, as we assume $v^*_{t+1}(.)$ satisfies properties (a) and (c) (which together imply the concavity of $v^*_{t+1}(x_{ut} + \delta_u, \cdot, y_{t+1})$),

$$\begin{aligned}
&v^*_{t+1}(C_U, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(C_U, x_{lt} + \delta_l + \beta, y_{t+1}) \\
&\leq v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + C_U - x_{ut} - \delta_u, y_{t+1}) \\
&\quad - v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + \beta + C_U - x_{ut} - \delta_u, y_{t+1}) \\
&\leq v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + \alpha, y_{t+1}) \\
&\quad - v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + \alpha + \beta, y_{t+1}).
\end{aligned}$$

(6) If $C_U - x_{ut} - \alpha < \delta_u \leq C_U - x_{ut}$ and $C_L - x_{lt} - \alpha - \beta < \delta_l \leq \min\{C_L - x_{lt} - \beta, C_L - x_{lt} - \alpha\}$, as we assume $v^*_{t+1}(.)$ satisfies properties (a) and (c) (which together imply the concavity of $v^*_{t+1}(x_{ut} + \delta_u, \cdot, y_{t+1})$), and by Lemma 5.1.2,

$$\begin{aligned}
&v^*_{t+1}(C_U, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(C_U, x_{lt} + \delta_l + \beta, y_{t+1}) \\
&\leq v^*_{t+1}(C_U, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(C_U, C_L - \alpha, y_{t+1}) \\
&\leq v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + C_U - x_{ut} - \delta_u, y_{t+1}) \\
&\quad - v^*_{t+1}(x_{ut} + \delta_u, C_L - \alpha + C_U - x_{ut} - \delta_u, y_{t+1}) \\
&\leq v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + \alpha, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1}).
\end{aligned}$$

(7) If $C_U - x_{ut} - \alpha < \delta_u \leq C_U - x_{ut}$, $C_L - x_{lt} - \beta < \delta_l \leq C_L - x_{lt}$, and $\delta_l \leq C_L - x_{lt} - \alpha$, as we assume $v^*_{t+1}(.)$ satisfies property (a), and by Lemma 5.1.2,

$$\begin{aligned}
&v^*_{t+1}(C_U, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \\
&\leq v^*_{t+1}(C_U, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(C_U, C_L - C_U + x_{ut} + \delta_u, y_{t+1}) \\
&\leq v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + C_U - x_{ut} - \delta_u, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1}) \\
&\leq v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + \alpha, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1}).
\end{aligned}$$

(8) If $C_U - x_{ut} - \alpha < \delta_u \leq C_U - x_{ut}$, $C_L - x_{lt} - \beta < \delta_l \leq C_L - x_{lt}$, and $C_L - x_{lt} - \alpha < \delta_l$, by Lemma 5.1.2, $v^*_{t+1}(C_U, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \leq 0 = v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1})$.

(9) If $C_U - x_{ut} - \alpha < \delta_u \leq C_U - x_{ut}$ and $C_L - x_{lt} - \alpha < \delta_l \leq C_L - x_{lt}$, by Lemma 5.1.2, $v^*_{t+1}(C_U, x_{lt}+\delta_l, y_{t+1}) - v^*_{t+1}(C_U, \min\{x_{lt}+\delta_l+\beta, C_L\}, y_{t+1}) \leq 0 = v^*_{t+1}(x_{ut}+\delta_u, C_L, y_{t+1}) - v^*_{t+1}(x_{ut}+\delta_u, C_L, y_{t+1})$.

(10) If $C_U - x_{ut} - \alpha < \delta_u \leq C_U - x_{ut}$ and $C_L - x_{lt} < \delta_l$, both sides of the inequality reduce to zero.

(11) If $C_U - x_{ut} - \alpha < C_U - x_{ut} < \delta_u$ and $\delta_l \leq C_L - x_{lt} - \alpha - \beta$, as we assume $v^*_{t+1}(.)$ satisfies properties (a) and (c) (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$),

$$\begin{aligned}
&v^*_{t+1}(C_U, x_{lt}+\delta_l, y_{t+1}) - v^*_{t+1}(C_U, x_{lt}+\delta_l+\alpha, y_{t+1}) \\
&\leq v^*_{t+1}(C_U, x_{lt}+\delta_l+\beta, y_{t+1}) - v^*_{t+1}(C_U, x_{lt}+\delta_l+\alpha+\beta, y_{t+1}).
\end{aligned}$$

(12) If $C_U - x_{ut} - \alpha < C_U - x_{ut} < \delta_u$ and $C_L - x_{lt} - \alpha - \beta < \delta_l \leq C_L - x_{lt} - \alpha$, as we assume $v^*_{t+1}(.)$ satisfies properties (a) and (c) (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), and by Lemma 5.1.2,

$$\begin{aligned}
&v^*_{t+1}(C_U, x_{lt}+\delta_l, y_{t+1}) - v^*_{t+1}(C_U, x_{lt}+\delta_l+\alpha, y_{t+1}) \\
&\leq v^*_{t+1}(C_U, C_L-\alpha, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \\
&\leq v^*_{t+1}(C_U, \min\{x_{lt}+\delta_l+\beta, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}).
\end{aligned}$$

(13) If $C_U - x_{ut} - \alpha < C_U - x_{ut} < \delta_u$ and $C_L - x_{lt} - \alpha < \delta_l \leq C_L - x_{lt}$, by Lemma 5.1.2,

$$\begin{aligned}
&v^*_{t+1}(C_U, x_{lt}+\delta_l, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \\
&\leq v^*_{t+1}(C_U, \min\{x_{lt}+\delta_l+\beta, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}).
\end{aligned}$$

(14) If $C_U - x_{ut} - \alpha < C_U - x_{ut} < \delta_u$ and $C_L - x_{lt} < \delta_l$, both sides of the inequality reduce to zero.

In order to prove property (ii), we consider the following nine cases:

(1) If $\delta_u \leq C_U - x_{ut} - \alpha - \beta$ and $\delta_l \leq C_L - x_{lt} - \alpha$, as we assume $v^*_{t+1}(.)$ satisfies property (b),

$$\begin{aligned}
&v^*_{t+1}(x_{ut} + \delta_u + \alpha + \beta, x_{lt} + \delta_l, y_{t+1}) \\
&\quad - v^*_{t+1}(x_{ut} + \delta_u + \alpha, x_{lt} + \delta_l, y_{t+1}) \\
&\leq v^*_{t+1}(x_{ut} + \delta_u + \beta, x_{lt} + \delta_l + \alpha, y_{t+1}) \\
&\quad - v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + \alpha, y_{t+1}).
\end{aligned}$$

(2) If $\delta_u \leq C_U - x_{ut} - \alpha - \beta$ and $C_L - x_{lt} - \alpha < \delta_l \leq C_L - x_{lt}$, as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, x_{lt} + \delta_l, y_{t+1})$),

$$\begin{aligned}
&v^*_{t+1}(x_{ut} + \delta_u + \alpha + \beta, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u + \alpha, x_{lt} + \delta_l, y_{t+1}) \\
&\leq v^*_{t+1}(x_{ut} + \delta_u + \beta + C_L - x_{lt} - \delta_l, x_{lt} + \delta_l, y_{t+1}) \\
&\quad - v^*_{t+1}(x_{ut} + \delta_u + C_L - x_{lt} - \delta_l, x_{lt} + \delta_l, y_{t+1}) \\
&\leq v^*_{t+1}(x_{ut} + \delta_u + \beta, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1}).
\end{aligned}$$

(3) If $\delta_u \leq C_U - x_{ut} - \alpha - \beta$ and $C_L - x_{lt} < \delta_l$, as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, C_L, y_{t+1})$),

$$\begin{aligned}
&v^*_{t+1}(x_{ut} + \delta_u + \alpha + \beta, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_l + \alpha, C_L, y_{t+1}) \\
&\leq v^*_{t+1}(x_{ut} + \delta_u + \beta, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1}).
\end{aligned}$$

(4) If $C_U - x_{ut} - \alpha - \beta < \delta_u \leq C_U - x_{ut} - \alpha$ and $\delta_l \leq C_L - x_{lt} - \alpha$, as we assume $v^*_{t+1}(.)$ satisfies property (b), and by Lemma 5.1.2,

$$\begin{aligned}
&v^*_{t+1}(C_U, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u + \alpha, x_{lt} + \delta_l, y_{t+1}) \\
&\leq v^*_{t+1}(C_U - \alpha, x_{lt} + \delta_l + \alpha, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + \alpha, y_{t+1}) \\
&\leq v^*_{t+1}(\min\{x_{ut} + \delta_u + \beta, C_U\}, x_{lt} + \delta_l + \alpha, y_{t+1}) \\
&\quad - v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + \alpha, y_{t+1}).
\end{aligned}$$

(5) If $C_U - x_{ut} - \alpha - \beta < \delta_u \leq C_U - x_{ut} - \alpha$ and $C_L - x_{lt} - \alpha < \delta_l \leq C_L - x_{lt}$, as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, x_{lt} + \delta_l, y_{t+1})$), and by Lemma 5.1.2,

$$
\begin{aligned}
&v_{t+1}^*(C_U, x_{lt}+\delta_l, y_{t+1}) - v_{t+1}^*(x_{ut}+\delta_u+\alpha, x_{lt}+\delta_l, y_{t+1})\\
&\le v_{t+1}^*(C_U + C_L - x_{lt} - \delta_l - \alpha, x_{lt}+\delta_l, y_{t+1})\\
&\quad - v_{t+1}^*(x_{ut}+\delta_u + C_L - x_{lt} - \delta_l, x_{lt}+\delta_l, y_{t+1})\\
&\le v_{t+1}^*(C_U - \alpha, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}+\delta_u, C_L, y_{t+1})\\
&\le v_{t+1}^*(\min\{x_{ut}+\delta_u+\beta, C_U\}, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}+\delta_u, C_L, y_{t+1}).
\end{aligned}
$$

(6) If $C_U - x_{ut} - \alpha - \beta < \delta_u \le C_U - x_{ut} - \alpha$ and $C_L - x_{lt} < \delta_l$, as we assume $v_{t+1}^*(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v_{t+1}^*(\cdot, C_L, y_{t+1})$), and by Lemma 5.1.2,

$$
\begin{aligned}
&v_{t+1}^*(C_U, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}+\delta_u+\alpha, C_L, y_{t+1})\\
&\le v_{t+1}^*(C_U - \alpha, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}+\delta_u, C_L, y_{t+1})\\
&\le v_{t+1}^*(\min\{x_{ut}+\delta_u+\beta, C_U\}, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}+\delta_u, C_L, y_{t+1}).
\end{aligned}
$$

(7) If $C_U - x_{ut} - \alpha < \delta_u \le C_U - x_{ut}$ and $\delta_l \le C_L - x_{lt} - \alpha$, by Lemma 5.1.2, $v_{t+1}^*(C_U, x_{lt}+\delta_l, y_{t+1}) - v_{t+1}^*(C_U, x_{lt}+\delta_l, y_{t+1}) = 0 \le v_{t+1}^*(\min\{x_{ut}+\delta_u+\beta, C_U\}, x_{lt}+\delta_l+\alpha, y_{t+1}) - v_{t+1}^*(x_{ut}+\delta_u, x_{lt}+\delta_l+\alpha, y_{t+1})$.

(8) If $C_U - x_{ut} - \alpha < \delta_u \le C_U - x_{ut}$ and $C_L - x_{lt} - \alpha < \delta_l$, by Lemma 5.1.2, $v_{t+1}^*(C_U, \min\{x_{lt}+\delta_l, C_L\}, y_{t+1}) - v_{t+1}^*(C_U, \min\{x_{lt}+\delta_l, C_L\}, y_{t+1}) = 0 \le v_{t+1}^*(\min\{x_{ut}+\delta_u+\beta, C_U\}, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}+\delta_u, C_L, y_{t+1})$.

(9) If $C_U - x_{ut} < \delta_u$, both sides of the inequality reduce to zero.

In order to prove property (iii), we consider the following seven cases:

(1) If $\delta_u \le C_U - x_{ut} - \alpha$ and $\delta_l \le C_L - x_{lt} - \beta$, as we assume $v_{t+1}^*(.)$ satisfies property (c),

$$
\begin{aligned}
&v_{t+1}^*(x_{ut}+\delta_u+\alpha, x_{lt}+\delta_l+\beta, y_{t+1}) - v_{t+1}^*(x_{ut}+\delta_u, x_{lt}+\delta_l+\beta, y_{t+1})\\
&\le v_{t+1}^*(x_{ut}+\delta_u+\alpha, x_{lt}+\delta_l, y_{t+1}) - v_{t+1}^*(x_{ut}+\delta_u, x_{lt}+\delta_l, y_{t+1}).
\end{aligned}
$$

(2) If $\delta_u \leq C_U - x_{ut} - \alpha$ and $C_L - x_{lt} - \beta < \delta_l \leq C_L - x_{lt}$, as we assume $v^*_{t+1}(.)$ satisfies property (c),

$$v^*_{t+1}(x_{ut} + \delta_u + \alpha, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1})$$
$$\leq v^*_{t+1}(x_{ut} + \delta_u + \alpha, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l, y_{t+1}).$$

(3) If $\delta_u \leq C_U - x_{ut} - \alpha$ and $C_L - x_{lt} < \delta_l$, both sides of the inequality reduce to $v^*_{t+1}(x_{ut} + \delta_u + \alpha, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1})$.

(4) If $C_U - x_{ut} - \alpha < \delta_u \leq C_U - x_{ut}$ and $\delta_l \leq C_L - x_{lt} - \beta$, as we assume $v^*_{t+1}(.)$ satisfies property (c),

$$v^*_{t+1}(C_U, x_{lt} + \delta_l + \beta, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l + \beta, y_{t+1})$$
$$\leq v^*_{t+1}(C_U, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l, y_{t+1}).$$

(5) If $C_U - x_{ut} - \alpha < \delta_u \leq C_U - x_{ut}$ and $C_L - x_{lt} - \beta < \delta_l \leq C_L - x_{lt}$, as we assume $v^*_{t+1}(.)$ satisfies property (c),

$$v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1})$$
$$\leq v^*_{t+1}(C_U, x_{lt} + \delta_l, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, x_{lt} + \delta_l, y_{t+1}).$$

(6) If $C_U - x_{ut} - \alpha < \delta_u \leq C_U - x_{ut}$ and $C_L - x_{lt} < \delta_l$, both sides of the inequality reduce to $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \delta_u, C_L, y_{t+1})$.

(7) If $C_U - x_{ut} < \delta_u$, both sides of the inequality reduce to zero.

Hence $v^*_{t+1}(.)$ satisfies properties (i)-(iii). We now will prove that $v^*_t(.)$ satisfies properties (a)–(c).

(a) First we prove that $v^*_t(x_{ut} + \alpha, x_{lt}, y_t) - v^*_t(x_{ut}, x_{lt} + \alpha, y_t) \leq v^*_t(x_{ut} + \alpha, x_{lt} + \beta, y_t) - v^*_t(x_{ut}, x_{lt} + \alpha + \beta, y_t)$. Let $a = a^*_t(x_{ut} + \alpha, x_{lt}, y_t)$ and $c = a^*_t(x_{ut}, x_{lt}+\alpha+\beta, y_t)$. Also, let $b = b^*_t(x_{ut}+\alpha, x_{lt}, y_t)$ and $d = b^*_t(x_{ut}, x_{lt}+\alpha+\beta, y_t)$. Lemma 5.1.3 implies that $C_U \geq x_{ut} + \alpha - a$ and $C_U \geq x_{ut} - c$. Lemma 5.1.4 implies that $C_L \geq x_{lt} + a - b$ and $C_L \geq x_{lt} + \alpha + \beta + c - d$. We consider the following three scenarios to prove the statement:

(a1) Suppose that $\beta + b > d$ and $\alpha + c > a$: We show that $(c, b) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$: Since $(a, b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$, note that $b - c \leq x_{lt} + a - c < x_{lt} + \alpha$ and $b \leq x_{lt} < x_{lt} + \alpha$. Since $(c, d) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha + \beta, y_t)$, note that $x_{ut} \geq c$. Hence $(c, b) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$. We also show that $(a, d) \in \mathbb{U}(x_{ut} + \alpha, x_{lt} + \beta, y_t)$: Since $(c, d) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha + \beta, y_t)$, note that $x_{ut} + \alpha \geq c + \alpha > a$. Since $(a, b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$, note that $d - a \leq d + x_{lt} - b < x_{lt} + \beta$ and $d < b + \beta < x_{lt} + \beta$. Hence $(a, d) \in \mathbb{U}(x_{ut} + \alpha, x_{lt} + \beta, y_t)$. Thus:

$$
\begin{aligned}
&v_t^*(x_{ut} + \alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt} + \alpha, y_t)\\
&\leq R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(c, b, y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - c + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + \alpha + c - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\leq R(a, d, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + \beta + a - d + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(c, d, y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - c + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + \alpha + \beta + c - d + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\leq v_t^*(x_{ut} + \alpha, x_{lt} + \beta, y_t) - v_t^*(x_{ut}, x_{lt} + \alpha + \beta, y_t).
\end{aligned}
$$

Note that $R(a, b, y_t) - R(c, b, y_t) = (E(a) - E(c))p_t = R(a, d, y_t) - R(c, d, y_t)$. The second inequality holds since $v_{t+1}^*(.)$ satisfies property (i).

(a2) Suppose that $\beta + b > d$ and $\alpha + c \leq a$: We show that $(a - \alpha, b) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$: Since $(a, b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$, note that $a - \alpha \leq x_{ut}$, $b \leq x_{lt} < x_{lt} + \alpha$, $a - \alpha < a \leq C_R$, and $b - (a - \alpha) \leq x_{lt} + \alpha$. Since $(c, d) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha + \beta, y_t)$, note that $-C_P \leq c \leq a - \alpha$. Hence $(a - \alpha, b) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$. We also show that $(c + \alpha, d) \in \mathbb{U}(x_{ut} + \alpha, x_{lt} + \beta, y_t)$: Since $(c, d) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha + \beta, y_t)$, note that $c + \alpha \leq x_{ut} + \alpha$, $-C_P \leq c \leq c + \alpha$, and $d - (c + \alpha) \leq x_{lt} + \beta$. Since $(a, b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$, note that $c + \alpha \leq a \leq C_R$

and $d < b+\beta \leq x_{lt}+\beta$. Hence $(c+\alpha, d) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$. Thus:

$$\begin{aligned}
& v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t) \\
& \leq R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(a-\alpha, b, y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& = (E(a)-E(a-\alpha))p_t \\
& \leq (E(c+\alpha)-E(c))p_t \\
& = R(c+\alpha, d, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}-c+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+\alpha+\beta+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}-c+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+\alpha+\beta+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \leq v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha+\beta, y_t).
\end{aligned}$$

The second inequality holds since $E(.)$ is a concave function and $p_t > 0$.

(a3) Suppose that $\beta + b \leq d$: We show that $(c, d-\beta) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$: Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$, note that $d-\beta \geq b \geq 0$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha+\beta, y_t)$, note that $x_{ut} \geq c$, $d-\beta-c \leq x_{lt}+\alpha$, and $d \leq x_{lt}+\alpha+\beta$. Hence $(c, d-\beta) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$. We also show that $(a, b+\beta) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$: Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha+\beta, y_t)$, note that $b+\beta \leq d \leq C_R$. Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$, note that $x_{ut}+\alpha \geq a$, $b+\beta-a \leq x_{lt}+\beta$, and $b+\beta \leq x_{lt}+\beta$. Hence $(a, b+\beta) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$. Thus:

$$
\begin{aligned}
&v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t)\\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(a,b+\beta,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&= -\theta\beta p_t\\
&= R(c,d-\beta,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\alpha+\beta+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\alpha+\beta+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\le v_t^*(x_{ut}, x_{lt}+\alpha, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha+\beta, y_t).
\end{aligned}
$$

(b) Next we prove that $v_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t) - v_t^*(x_{ut}+\beta, x_{lt}+\alpha, y_t) \le v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t)$. Let $a = a_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t)$ and $c = a_t^*(x_{ut}, x_{lt}+\alpha, y_t)$. Also, let $b = b_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t)$ and $d = b_t^*(x_{ut}, x_{lt}+\alpha, y_t)$. Lemma 5.1.3 implies that $C_U \ge x_{ut}+\alpha+\beta-a$ and $C_U \ge x_{ut}-c$. Lemma 5.1.4 implies that $C_L \ge x_{lt}+a-b$ and $C_L \ge x_{lt}+\alpha+c-d$. We consider the following five scenarios to prove the statement:

(b1) Suppose that $\alpha + c > a$ and $b \le d$: Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note that $c \le x_{ut} < x_{ut}+\beta$, $d \le x_{lt}+\alpha$, and $d - c \le x_{lt}+\alpha$. Hence $(c,d) \in \mathbb{U}(x_{ut}+\beta, x_{lt}+\alpha, y_t)$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note that $a < c+\alpha \le x_{ut}+\alpha$. Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha+\beta, x_{lt}, y_t)$, note that $b-a \le x_{lt}$

and $b \leq x_{lt}$. Hence $(a, b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$. Thus:

$$\begin{aligned}
&v_t^*(x_{ut} + \alpha + \beta, x_{lt}, y_t) - v_t^*(x_{ut} + \alpha, x_{lt}, y_t) \\
&\leq R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} + \alpha + \beta - a + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - R(a, b, y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\leq R(c, d, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} + \beta - c + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + \alpha + c - d + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - R(c, d, y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - c + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + \alpha + c - d + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\leq v_t^*(x_{ut} + \beta, x_{lt} + \alpha, y_t) - v_t^*(x_{ut}, x_{lt} + \alpha, y_t).
\end{aligned}$$

The second inequality holds if $d - b \leq \alpha + c - a$: Since $v_{t+1}^*(.)$ satisfies properties (ii) and (iii),

$$\begin{aligned}
&v_{t+1}^*(\min\{x_{ut} + \alpha + \beta - a + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}) \\
&- v_{t+1}^*(\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + a - b + r_{l(t+1)}, C_L\}, y_{t+1}) \\
&\leq v_{t+1}^*(\min\{x_{ut} + \beta - b - c + d + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + \alpha + c - d + r_{l(t+1)}, C_L\}, y_{t+1}) \\
&\quad - v_{t+1}^*(\min\{x_{ut} - b - c + d + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + \alpha + c - d + r_{l(t+1)}, C_L\}, y_{t+1}) \\
&\leq v_{t+1}^*(\min\{x_{ut} + \beta - c + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + \alpha + c - d + r_{l(t+1)}, C_L\}, y_{t+1}) \\
&\quad - v_{t+1}^*(\min\{x_{ut} - c + r_{u(t+1)}, C_U\}, \\
&\quad \min\{x_{lt} + \alpha + c - d + r_{l(t+1)}, C_L\}, y_{t+1}).
\end{aligned}$$

The second inequality also holds if $\alpha + c - a < d - b$: Since $v_{t+1}^*(.)$ satisfies properties (iii) and (ii),

$$
\begin{aligned}
&v^*_{t+1}(\min\{x_{ut}+\alpha+\beta-a+r_{u(t+1)},C_U\},\\
&\quad \min\{x_{lt}+a-b+r_{l(t+1)},C_L\},y_{t+1})\\
&-v^*_{t+1}(\min\{x_{ut}+\alpha-a+r_{u(t+1)},C_U\},\\
&\quad \min\{x_{lt}+a-b+r_{l(t+1)},C_L\},y_{t+1})\\
&\leq v^*_{t+1}(\min\{x_{ut}+\alpha+\beta-a+r_{u(t+1)},C_U\},\\
&\quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)},C_L\},y_{t+1})\\
&\quad -v^*_{t+1}(\min\{x_{ut}+\alpha-a+r_{u(t+1)},C_U\},\\
&\quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)},C_L\},y_{t+1})\\
&\leq v^*_{t+1}(\min\{x_{ut}+\beta-c+r_{u(t+1)},C_U\},\\
&\quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)},C_L\},y_{t+1})\\
&\quad -v^*_{t+1}(\min\{x_{ut}-c+r_{u(t+1)},C_U\},\\
&\quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)},C_L\},y_{t+1}).
\end{aligned}
$$

(b2) Suppose that $\alpha+c>a$ and $d<b$: Since $(c,d)\in\mathbb{U}(x_{ut},x_{lt}+\alpha,y_t)$, note that $c\leq x_{ut}<x_{ut}+\beta$. Since $(a,b)\in\mathbb{U}(x_{ut}+\alpha+\beta,x_{lt},y_t)$, note that $b\leq x_{lt}<x_{lt}+\alpha$ and $b-c<b-a+\alpha\leq x_{lt}+\alpha$. Hence $(c,b)\in\mathbb{U}(x_{ut}+\beta,x_{lt}+\alpha,y_t)$. Since $(c,d)\in\mathbb{U}(x_{ut},x_{lt}+\alpha,y_t)$, note that $a<c+\alpha\leq x_{ut}+\alpha$. Since $(a,b)\in\mathbb{U}(x_{ut}+\alpha+\beta,x_{lt},y_t)$, note that $d-a<b-a\leq x_{lt}$ and $d<b\leq x_{lt}$. Hence $(a,d)\in\mathbb{U}(x_{ut}+\alpha,x_{lt},y_t)$. Thus:

$$
\begin{aligned}
&v_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t) - v_t^*(x_{ut}+\beta, x_{lt}+\alpha, y_t)\\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(c,b,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\beta-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\alpha+c-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(c,b,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\alpha+c-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\le R(a,d,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+a-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\le v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t).
\end{aligned}
$$

Note that $R(a,b,y_t)-R(a,d,y_t) = \theta(b-d)p_t = R(c,b,y_t)-R(c,d,y_t)$. The second and third inequalities hold since $v_{t+1}^*(.)$ satisfies properties (ii) and (i).

(b3) Suppose that $\alpha+c \le a$ and $d \le b$: Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha+\beta, x_{lt}, y_t)$, note that $a-\alpha \le x_{ut}+\beta$, $a-\alpha < a \le C_R$, $b \le x_{lt} < x_{lt}+\alpha$, and $b-(a-\alpha) \le x_{lt}+\alpha$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note that $-C_P \le c \le a-\alpha$. Hence $(a-\alpha, b) \in \mathbb{U}(x_{ut}+\beta, x_{lt}+\alpha, y_t)$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note that $c+\alpha \le x_{ut}+\alpha$, $-C_P \le c \le c+\alpha$, and $d-(c+\alpha) \le x_{lt}$. Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha+\beta, x_{lt}, y_t)$, note that $c+\alpha \le a \le C_R$ and $d \le b \le x_{lt}$. Hence $(c+\alpha, d) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$. Thus:

$$
\begin{aligned}
& v_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t) - v_t^*(x_{ut}+\beta, x_{lt}+\alpha, y_t) \\
& \leq R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(a-\alpha,b,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& = (E(a)-E(a-\alpha))p_t \\
& \leq (E(c+\alpha)-E(c))p_t \\
& = R(c+\alpha,d,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}-c+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}-c+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \leq v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t).
\end{aligned}
$$

The second inequality holds since $E(.)$ is a concave function.

(b4) Suppose that $\alpha + c \leq a < \alpha+\beta+c$ and $b < d$: Recall from scenario (b3) that $-C_P \leq a-\alpha \leq \min\{x_{ut}+\beta, C_R\}$ when $\alpha + c \leq a$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note that $d \leq x_{lt}+\alpha$ and $d-a+\alpha \leq d-c \leq x_{lt}+\alpha$. Hence $(a-\alpha, d) \in \mathbb{U}(x_{ut}+\beta, x_{lt}+\alpha, y_t)$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note that $c+\alpha \leq x_{ut}+\alpha$, $-C_P \leq c \leq c+\alpha$, and $b-(c+\alpha) < d-(c+\alpha) \leq x_{lt}$. Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha+\beta, x_{lt}, y_t)$, note that $b \leq x_{lt}$ and $c+\alpha \leq a \leq C_R$. Hence $(c+\alpha, b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$. Thus:

$$
\begin{aligned}
&v_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t) - v_t^*(x_{ut}+\beta, x_{lt}+\alpha, y_t) \\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\big\}, \\
&\quad \min\{x_{lt}+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - R(a-\alpha,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\big\}, \\
&\quad \min\{x_{lt}+a-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\big\}, \\
&\quad \min\{x_{lt}+\alpha+c-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - R(a-\alpha,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\big\}, \\
&\quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\le R(c+\alpha,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}-c+r_{u(t+1)}, C_U\big\}, \\
&\quad \min\{x_{lt}+\alpha+c-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}-c+r_{u(t+1)}, C_U\big\}, \\
&\quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\le v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t),
\end{aligned}
$$

Note that $R(a,b,y_t) - R(a-\alpha,d,y_t) = (E(a)+\theta b - E(a-\alpha) - \theta d)p_t \le (E(c+\alpha)+\theta b - E(c) - \theta d)p_t = R(c+\alpha,b,y_t) - R(c,d,y_t)$. The second and third inequalities hold since $v_{t+1}^*(.)$ satisfies properties (i) and (iii).

(b5) Suppose that $\alpha+\beta+c < a$ and $b < d$:

Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha+\beta, x_{lt}, y_t)$, note that $c+\beta < a-\alpha < a \le C_R$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note that $-C_P \le c < c+\beta \le x_{ut}+\beta$, $d \le x_{lt}+\alpha$, and $d-(c+\beta) < d-c \le x_{lt}+\alpha$. Hence $(c+\beta, d) \in \mathbb{U}(x_{ut}+\beta, x_{lt}+\alpha, y_t)$. Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha+\beta, x_{lt}, y_t)$, note that $a-\beta < a \le C_R$, $a-\beta \le x_{ut}+\alpha$, and $b \le x_{lt}$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note that $-C_P \le c < c+\alpha < a-\beta$ and $b-a+\beta < b-c-\alpha < d-c-\alpha \le x_{lt}$. Hence $(a-\beta, b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$. Thus:

$$
\begin{aligned}
& v_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t) - v_t^*(x_{ut}+\alpha, x_{lt}, y_t) \\
& \leq R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(a-\beta,b,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+a-b-\beta+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \leq R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+a-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(a-\beta,b,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}+\alpha+\beta-a+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+a-d-\beta+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \leq R(c+\beta,d,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}-c+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+\alpha+\beta+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\big\{x_{ut}-c+r_{u(t+1)}, C_U\big\}, \\
& \quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \leq v_t^*(x_{ut}+\beta, x_{lt}+\alpha, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t).
\end{aligned}
$$

Note that $R(a,b,y_t) - R(c+\beta,d,y_t) = (E(a)+\theta b - E(c+\beta) - \theta d)p_t \leq (E(a-\beta)+\theta b - E(c) - \theta d)p_t = R(a-\beta,b,y_t) - R(c,d,y_t)$ since $E(.)$ is a concave function. The second and third inequalities hold since $v_{t+1}^*(.)$ satisfies properties (i) and (iii).

(c) Last we prove that $v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}+\beta, y_t) \leq v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}, y_t)$. Let $a = a_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t)$ and $c = a_t^*(x_{ut}, x_{lt}, y_t)$. Also, let $b = b_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t)$ and $d = b_t^*(x_{ut}, x_{lt}, y_t)$. Lemma 5.1.3 implies that $C_U \geq x_{ut}+\alpha-a$ and $C_U \geq x_{ut}-c$. Lemma 5.1.4 implies that $C_L \geq x_{lt}+\beta+a-b$ and $C_L \geq x_{lt}+c-d$. We consider the following three scenarios to prove the statement:

(c1) Suppose that $b > \beta + d$: Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$, note that $a \leq x_{ut}+\alpha$, $b-\beta \leq x_{lt}$, and $b-\beta-a \leq x_{lt}$. Since $b-\beta > d \geq 0$, we obtain $(a, b-\beta) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $c \le x_{ut}$, $d+\beta \le x_{lt}+\beta$, and $d+\beta-c \le x_{lt}+\beta$. Since $b > d+\beta$, we obtain $(c, d+\beta) \in \mathbb{U}(x_{ut}, x_{lt}+\beta, y_t)$. Thus:

$$\begin{aligned}
& v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_t^*(x_{ut}+\alpha, x_{lt}, y_t) \\
& \le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt}+\beta+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(a, b-\beta, y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt}+\beta+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& = \theta\beta p_t \\
& = R(c, d+\beta, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt}+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt}+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \le v_t^*(x_{ut}, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}, y_t).
\end{aligned}$$

(c2) Suppose that $b \le \beta + d$ and $c < a$: Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$, note that $a \le x_{ut}+\alpha$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $d \le x_{lt}$ and $d-a < d-c \le x_{lt}$. Hence $(a,d) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$. Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$, note that $b \le x_{lt}+\beta$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $c \le x_{ut}$ and $b-c \le d-c+\beta \le x_{lt}+\beta$. Hence $(c,b) \in \mathbb{U}(x_{ut}, x_{lt}+\beta, y_t)$. Thus:

$$\begin{aligned}
&v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_t^*(x_{ut}+\alpha, x_{lt}, y_t)\\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\beta+a-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(a,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+a-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\le R(c,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\beta+c-b+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+c-d+r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big]\\
&\le v_t^*(x_{ut}, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}, y_t).
\end{aligned}$$

Note that $R(a,b,y_t) - R(a,d,y_t) = \theta(b-d)p_t = R(c,b,y_t) - R(c,d,y_t)$. The second inequality holds if $\alpha \le a-c$: Since $v_{t+1}^*(.)$ satisfies properties (i) and (iii),

$$\begin{aligned}
&v_{t+1}^*(\min\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\beta+a-b+r_{l(t+1)}, C_L\}, y_{t+1})\\
&- v_{t+1}^*(\min\{x_{ut}+\alpha-a+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+a-d+r_{l(t+1)}, C_L\}, y_{t+1})\\
&\le v_{t+1}^*(\min\{x_{ut}-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\alpha+\beta+c-b+r_{l(t+1)}, C_L\}, y_{t+1})\\
&\quad - v_{t+1}^*(\min\{x_{ut}-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\alpha+c-d+r_{l(t+1)}, C_L\}, y_{t+1})\\
&\le v_{t+1}^*(\min\{x_{ut}-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+\beta+c-b+r_{l(t+1)}, C_L\}, y_{t+1})\\
&\quad - v_{t+1}^*(\min\{x_{ut}-c+r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt}+c-d+r_{l(t+1)}, C_L\}, y_{t+1}).
\end{aligned}$$

The second inequality also holds if $a-c < \alpha$: Since $v_{t+1}^*(.)$ satisfies properties (iii) and (i),

$$
\begin{aligned}
& v_{t+1}^*(\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt} + \beta + a - b + r_{l(t+1)}, C_L\}, y_{t+1}) \\
& \quad - v_{t+1}^*(\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt} + a - d + r_{l(t+1)}, C_L\}, y_{t+1}) \\
& \leq v_{t+1}^*(\min\{x_{ut} - c + r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt} + \beta + a - b + r_{l(t+1)}, C_L\}, y_{t+1}) \\
& \quad - v_{t+1}^*(\min\{x_{ut} - c + r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt} + a - d + r_{l(t+1)}, C_L\}, y_{t+1}) \\
& \leq v_{t+1}^*(\min\{x_{ut} - c + r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt} + \beta + c - b + r_{l(t+1)}, C_L\}, y_{t+1}) \\
& \quad - v_{t+1}^*(\min\{x_{ut} - c + r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1}).
\end{aligned}
$$

(c3) Suppose that $b \leq \beta + d$ and $a \leq c$: Since $(a, b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt} + \beta, y_t)$, note that $b \leq x_{lt} + \beta$ and $b - a \leq x_{lt} + \beta$. Since $(c, d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $a \leq c \leq x_{ut}$. Hence $(a, b) \in \mathbb{U}(x_{ut}, x_{lt} + \beta, y_t)$. Since $(c, d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $c \leq x_{ut} < x_{ut} + \alpha$, $d \leq x_{lt}$, and $d - c \leq x_{lt}$. Hence $(c, d) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$. Thus:

$$
\begin{aligned}
& v_t^*(x_{ut} + \alpha, x_{lt} + \beta, y_t) - v_t^*(x_{ut}, x_{lt} + \beta, y_t) \\
& \leq R(a, b, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt} + \beta + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(a, b, y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt} + \beta + a - b + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \leq R(c, d, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} + \alpha - c + r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \quad - R(c, d, y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - c + r_{u(t+1)}, C_U\}, \\
& \quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
& \leq v_t^*(x_{ut} + \alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}, y_t).
\end{aligned}
$$

The second inequality holds if $\beta + d - b \leq c - a$: Since $v^*_{t+1}(.)$ satisfies properties (ii) and (iii),

$$\begin{aligned}
&v^*_{t+1}(\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + \beta + a - b + r_{l(t+1)}, C_L, y_{t+1})\\
&\quad - v^*_{t+1}(\min\{x_{ut} - a + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + \beta + a - b + r_{l(t+1)}, C_L\}, y_{t+1})\\
&\leq v^*_{t+1}(\min\{x_{ut} + \alpha + \beta - b - c + d + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1})\\
&\quad - v^*_{t+1}(\min\{x_{ut} + \beta - b - c + d + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1})\\
&\leq v^*_{t+1}(\min\{x_{ut} + \alpha - c + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1})\\
&\quad - v^*_{t+1}(\min\{x_{ut} - c + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1}).
\end{aligned}$$

The second inequality also holds if $c - a < \beta + d - b$: Since $v^*_{t+1}(.)$ satisfies properties (iii) and (ii),

$$\begin{aligned}
&v^*_{t+1}(\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + \beta + a - b + r_{l(t+1)}, C_L\}, y_{t+1})\\
&- v^*_{t+1}(\min\{x_{ut} - a + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + \beta + a - b + r_{l(t+1)}, C_L\}, y_{t+1})\\
&\leq v^*_{t+1}(\min\{x_{ut} + \alpha - a + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1})\\
&\quad - v^*_{t+1}(\min\{x_{ut} - a + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1})\\
&\leq v^*_{t+1}(\min\{x_{ut} + \alpha - c + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1})\\
&\quad - v^*_{t+1}(\min\{x_{ut} - c + r_{u(t+1)}, C_U\},\\
&\quad \min\{x_{lt} + c - d + r_{l(t+1)}, C_L\}, y_{t+1}).
\end{aligned}$$

Hence $v_t^*(.)$ satisfies properties (a)–(c). $\square$

Proof of Lemma 5.1.6. Suppose that $\alpha > 0$ and $\beta > 0$. Fix z_t and y_t. Without loss of generality, we assume that $\alpha \geq \beta$. We consider the following sixteen cases to show that $V_t(z_t, z_{ut}, y_t)$ is concave in z_{ut}, that is,

$$\begin{aligned}
&V_t(z_t, z_{ut}, y_t) - V_t(z_t, z_{ut} + \alpha, y_t) \\
&= \mathbb{E}\Big[v_{t+1}^*\Big(\min\{z_{ut} + r_{u(t+1)}, C_U\}, \\
&\quad \min\{z_t - z_{ut} + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{z_{ut} + \alpha + r_{u(t+1)}, C_U\}, \\
&\quad \min\{z_t - z_{ut} - \alpha + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\leq \mathbb{E}\Big[v_{t+1}^*\Big(\min\{z_{ut} + \beta + r_{u(t+1)}, C_U\}, \\
&\quad \min\{z_t - z_{ut} - \beta + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{z_{ut} + \alpha + \beta + r_{u(t+1)}, C_U\}, \\
&\quad \min\{z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}, C_L\}, y_{t+1}\Big)\Big] \\
&= V_t(z_t, z_{ut} + \beta, y_t) - V_t(z_t, z_{ut} + \alpha + \beta, y_t).
\end{aligned}$$

(1) If $r_{u(t+1)} \leq C_U - z_{ut} - \alpha - \beta$ and $z_t - z_{ut} + r_{l(t+1)} \leq C_L$, by properties (a) and (b) of Lemma 5.1.5, $v_{t+1}^*(z_{ut} + r_{u(t+1)}, z_t - z_{ut} + r_{l(t+1)}, y_{t+1}) - v_{t+1}^*(z_{ut} + \alpha + r_{u(t+1)}, z_t - z_{ut} - \alpha + r_{l(t+1)}, y_{t+1}) \leq v_{t+1}^*(z_{ut} + \beta + r_{u(t+1)}, z_t - z_{ut} - \beta + r_{l(t+1)}, y_{t+1}) - v_{t+1}^*(z_{ut} + \alpha + \beta + r_{u(t+1)}, z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}, y_{t+1})$.

(2) If $r_{u(t+1)} \leq C_U - z_{ut} - \alpha - \beta$ and $z_t - z_{ut} - \beta + r_{l(t+1)} \leq C_L < z_t - z_{ut} + r_{l(t+1)}$, by Lemma 5.1.2 and by properties (b) and (a) of Lemma 5.1.5, $v_{t+1}^*(z_{ut} + r_{u(t+1)}, C_L, y_{t+1}) - v_{t+1}^*(z_{ut} + \alpha + r_{u(t+1)}, \min\{z_t - z_{ut} - \alpha + r_{l(t+1)}, C_L\}, y_{t+1}) \leq v_{t+1}^*(z_{ut} + r_{u(t+1)}, C_L, y_{t+1}) - v_{t+1}^*(z_{ut} + \alpha + r_{u(t+1)}, C_L - \alpha, y_{t+1}) \leq v_{t+1}^*(z_{ut} + \beta + r_{u(t+1)}, C_L, y_{t+1}) - v_{t+1}^*(z_{ut} + \alpha + \beta + r_{u(t+1)}, C_L - \alpha, y_{t+1}) \leq v_{t+1}^*(z_{ut} + \beta + r_{u(t+1)}, z_t - z_{ut} - \beta + r_{l(t+1)}, y_{t+1}) - v_{t+1}^*(z_{ut} + \alpha + \beta + r_{u(t+1)}, z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}, y_{t+1})$.

(3) If $r_{u(t+1)} \leq C_U - z_{ut} - \alpha - \beta$ and $z_t - z_{ut} - \alpha - \beta + r_{l(t+1)} \leq C_L < z_t - z_{ut} - \beta + r_{l(t+1)}$, by properties (b) and (c) of Lemma 5.1.5 and by

Lemma 5.1.2, $v^*_{t+1}(z_{ut}+r_{u(t+1)}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut}+\beta+r_{u(t+1)}, C_L, y_{t+1}) \leq v^*_{t+1}(z_{ut}+\alpha+r_{u(t+1)}, C_L-\alpha, y_{t+1}) - v^*_{t+1}(z_{ut}+\alpha+\beta+r_{u(t+1)}, C_L-\alpha, y_{t+1}) \leq v^*_{t+1}(z_{ut}+\alpha+r_{u(t+1)}, z_t-z_{ut}-\alpha-\beta+r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(z_{ut}+\alpha+\beta+r_{u(t+1)}, z_t-z_{ut}-\alpha-\beta+r_{l(t+1)}, y_{t+1}) \leq v^*_{t+1}(z_{ut}+\alpha+r_{u(t+1)}, \min\{z_t-z_{ut}-\alpha+r_{l(t+1)}, C_L\}, y_{t+1}) - v^*_{t+1}(z_{ut}+\alpha+\beta+r_{u(t+1)}, z_t-z_{ut}-\alpha-\beta+r_{l(t+1)}, y_{t+1})$.

(4) If $r_{u(t+1)} \leq C_U - z_{ut} - \alpha - \beta$ and $C_L < z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}$, by properties (b) and (c) of Lemma 5.1.5 (which together imply the concavity of $v^*_{t+1}(\cdot, C_L, y_{t+1})$), $v^*_{t+1}(z_{ut}+r_{u(t+1)}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut}+\beta+r_{u(t+1)}, C_L, y_{t+1}) \leq v^*_{t+1}(z_{ut}+\alpha+r_{u(t+1)}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut}+\alpha+\beta+r_{u(t+1)}, C_L, y_{t+1})$.

(5) If $C_U - z_{ut} - \alpha - \beta < r_{u(t+1)} \leq C_U - z_{ut} - \beta$ and $z_t - z_{ut} + r_{l(t+1)} \leq C_L$, by properties (b) and (a) of Lemma 5.1.5 and by Lemma 5.1.2, $v^*_{t+1}(z_{ut}+r_{u(t+1)}, z_t-z_{ut}+r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(z_{ut}+\beta+r_{u(t+1)}, z_t-z_{ut}-\beta+r_{l(t+1)}, y_{t+1}) \leq v^*_{t+1}(C_U-\beta, z_t-z_{ut}+r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t-z_{ut}-\beta+r_{l(t+1)}, y_{t+1}) \leq v^*_{t+1}(C_U-\beta, z_t-z_{ut}-\alpha+r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t-z_{ut}-\alpha-\beta+r_{l(t+1)}, y_{t+1}) \leq v^*_{t+1}(\min\{z_{ut}+\alpha+r_{u(t+1)}, C_U\}, z_t-z_{ut}-\alpha+r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t-z_{ut}-\alpha-\beta+r_{l(t+1)}, y_{t+1})$.

(6) If $C_U - z_{ut} - \alpha - \beta < r_{u(t+1)} \leq C_U - z_{ut} - \beta$ and $z_t - z_{ut} - \alpha + r_{l(t+1)} \leq C_L < z_t - z_{ut} + r_{l(t+1)}$, by Lemma 5.1.2 and by properties (b) and (a) of Lemma 5.1.5, $v^*_{t+1}(z_{ut}+r_{u(t+1)}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut}+\beta+r_{u(t+1)}, \min\{z_t-z_{ut}-\beta+r_{l(t+1)}, C_L\}, y_{t+1}) \leq v^*_{t+1}(z_{ut}+r_{u(t+1)}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut}+\beta+r_{u(t+1)}, C_L-\beta, y_{t+1}) \leq v^*_{t+1}(C_U-\beta, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L-\beta, y_{t+1}) \leq v^*_{t+1}(C_U-\beta, z_t-z_{ut}-\alpha+r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t-z_{ut}-\alpha-\beta+r_{l(t+1)}, y_{t+1}) \leq v^*_{t+1}(\min\{z_{ut}+\alpha+r_{u(t+1)}, C_U\}, z_t-z_{ut}-\alpha+r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t-z_{ut}-\alpha-\beta+r_{l(t+1)}, y_{t+1})$.

(7) If $C_U - z_{ut} - \alpha - \beta < r_{u(t+1)} \leq C_U - z_{ut} - \beta$ and $z_t - z_{ut} - \alpha - \beta + r_{l(t+1)} \leq C_L < z_t - z_{ut} - \alpha + r_{l(t+1)}$, since $\alpha \geq \beta$, by properties (b) and (c) of Lemma 5.1.5 (which together imply the concavity of $v^*_{t+1}(\cdot, C_L, y_{t+1})$) and by Lemma 5.1.2, $v^*_{t+1}(z_{ut}+r_{u(t+1)}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut}+\beta+r_{u(t+1)}, C_L, y_{t+1}) \leq v^*_{t+1}(C_U-\beta, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \leq v^*_{t+1}(\min\{z_{ut}+\alpha+r_{u(t+1)}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(C_U, z_t-z_{ut}-\alpha-\beta+$

$r_{l(t+1)}, y_{t+1})$.

(8) If $C_U - z_{ut} - \alpha - \beta < r_{u(t+1)} \leq C_U - z_{ut} - \beta$ and $C_L < z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}$, by properties (b) and (c) of Lemma 5.1.5 (which together imply the concavity of $v^*_{t+1}(\cdot, C_L, y_{t+1})$) and by Lemma 5.1.2, $v^*_{t+1}(z_{ut} + r_{u(t+1)}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \beta + r_{u(t+1)}, C_L, y_{t+1}) \leq v^*_{t+1}(C_U - \beta, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \leq v^*_{t+1}(\min\{z_{ut} + \alpha + r_{u(t+1)}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

(9) If $C_U - z_{ut} - \beta < r_{u(t+1)} \leq C_U - z_{ut}$ and $z_t - z_{ut} + r_{l(t+1)} \leq C_L$, since $\alpha \geq \beta$, by Lemma 5.1.2 and by properties (a) and (c) of Lemma 5.1.5 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), $v^*_{t+1}(z_{ut} + r_{t+1}, z_t - z_{ut} + r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t - z_{ut} - \alpha + r_{l(t+1)}, y_{t+1}) \leq v^*_{t+1}(C_U, z_t - z_{ut} + r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t - z_{ut} - \alpha + r_{l(t+1)}, y_{t+1}) \leq v^*_{t+1}(C_U, z_t - z_{ut} - \beta + r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}, y_{t+1})$.

(10) If $C_U - z_{ut} - \beta < r_{u(t+1)} \leq C_U - z_{ut}$ and $z_t - z_{ut} - \beta + r_{l(t+1)} \leq C_L < z_t - z_{ut} + r_{l(t+1)}$, since $\alpha \geq \beta$, by Lemma 5.1.2 and by properties (a) and (c) of Lemma 5.1.5 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), $v^*_{t+1}(z_{ut} + r_{u(t+1)}, C_L, y_{t+1}) - v^*_{t+1}(C_U, z_t - z_{ut} - \alpha + r_{l(t+1)}, y_{t+1}) \leq v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L - \alpha, y_{t+1}) \leq v^*_{t+1}(C_U, z_t - z_{ut} - \beta + r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}, y_{t+1})$.

(11) If $C_U - z_{ut} - \beta < r_{u(t+1)} \leq C_U - z_{ut}$ and $z_t - z_{ut} - \alpha - \beta + r_{l(t+1)} \leq C_L < z_t - z_{ut} - \beta + r_{l(t+1)}$, since $\alpha \geq \beta$, by Lemma 5.1.2, $v^*_{t+1}(z_{ut} + r_{u(t+1)}, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \leq 0 \leq v^*_{t+1}(C_U, \min\{z_t - z_{ut} - \alpha + r_{l(t+1)}, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}, y_{t+1})$.

(12) If $C_U - z_{ut} - \beta < r_{u(t+1)} \leq C_U - z_{ut}$ and $C_L < z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}$, since $\alpha \geq \beta$, by Lemma 5.1.2, $v^*_{t+1}(z_{ut} + r_{u(t+1)}, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \leq v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

(13) If $C_U - z_{ut} < r_{u(t+1)}$ and $z_t - z_{ut} + r_{l(t+1)} \leq C_L$, by properties (a) and (c) of Lemma 5.1.5 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), $v^*_{t+1}(C_U, z_t - z_{ut} + r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t - z_{ut} - \alpha + r_{l(t+1)}, y_{t+1}) \leq v^*_{t+1}(C_U, z_t - z_{ut} - \beta + r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}, y_{t+1})$.

(14) If $C_U - z_{ut} < r_{u(t+1)}$ and $z_t - z_{ut} - \beta + r_{l(t+1)} \le C_L < z_t - z_{ut} + r_{l(t+1)}$, by Lemma 5.1.2 and by properties (a) and (c) of Lemma 5.1.5 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, \min\{z_t - z_{ut} - \alpha + r_{l(t+1)}, C_L\}, y_{t+1}) \le v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L - \alpha, y_{t+1}) \le v^*_{t+1}(C_U, z_t - z_{ut} - \beta + r_{l(t+1)}, y_{t+1}) - v^*_{t+1}(C_U, z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}, y_{t+1})$.

(15) If $C_U - z_{ut} < r_{u(t+1)}$ and $z_t - z_{ut} - \alpha - \beta + r_{l(t+1)} \le C_L < z_t - z_{ut} - \beta + r_{l(t+1)}$, by Lemma 5.1.2, $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, \min\{z_t - z_{ut} - \alpha + r_{l(t+1)}, C_L\}, y_{t+1}) \le v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}, y_{t+1})$.

(16) If $C_U - z_{ut} < r_{u(t+1)}$ and $C_L < z_t - z_{ut} - \alpha - \beta + r_{l(t+1)}$, both sides become $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

Hence $V_t(z_t, z_{ut}, y_t)$ is concave in z_{ut}. $\square$

Proof of Theorem 5.1.1. First, we need to prove that $v^*_t(x_{ut}, x_{lt} + \alpha, y_t) = v^*_t(x_{ut}, x_{lt}, y_t)$ for $x_{ut} + x_{lt} \ge C_U$. Since Lemma 5.1.2 states that $v^*_t(x_{ut}, x_{lt} + \alpha, y_t) \ge v^*_t(x_{ut}, x_{lt}, y_t)$, it is sufficient to show that $v^*_t(x_{ut}, x_{lt} + \alpha, y_t) \le v^*_t(x_{ut}, x_{lt}, y_t)$. Assuming $v^*_{t+1}(x_{u(t+1)}, x_{l(t+1)} + \alpha, y_{t+1}) \le v^*_{t+1}(x_{u(t+1)}, x_{l(t+1)}, y_{t+1})$ for $x_{u(t+1)} + x_{l(t+1)} \ge C_U$, we show $v^*_t(x_{ut}, x_{lt} + \alpha, y_t) \le v^*_t(x_{ut}, x_{lt}, y_t)$ for $x_{ut} + x_{lt} \ge C_U$. Let $a = a^*_t(x_{ut}, x_{lt} + \alpha, y_t)$. We consider the following two scenarios to prove the statement:

- Suppose that $a > -x_{lt}$: Since $a \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$ and $a > -x_{lt}$, note that $a \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus, $v^*_t(x_{ut}, x_{lt} + \alpha, y_t) = R(a, 0, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\}, \min\{x_{lt} + \alpha + a, C_L\}, y_{t+1}\Big)\Big] \le R(a, 0, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - a + r_{u(t+1)}, C_U\}, \min\{x_{lt} + a, C_L\}, y_{t+1}\Big)\Big] \le v^*_t(x_{ut}, x_{lt}, y_t)$. The first inequality holds in each of the following two cases: (1) If $C_L \ge x_{lt} + a$, since $x_{ut} + x_{lt} \ge C_U$, $v^*_{t+1}(\min\{x_{ut} - a + r_{u(t+1)}, C_U\}, \min\{x_{lt} + \alpha + a, C_L\}, y_{t+1}) \le v^*_{t+1}(\min\{x_{ut} - a + r_{u(t+1)}, C_U\}, x_{lt} + a, y_{t+1})$ from the induction assumption. (2) If $x_{lt} + a > C_L$, both sides become $v^*_{t+1}(\min\{x_{ut} - a + r_{u(t+1)}, C_U\}, C_L, y_{t+1})$.

- Suppose that $a \le -x_{lt}$: Let $\hat{a} = -x_{lt}$. Since $a \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note that $\hat{a} \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus, $v_t^*(x_{ut}, x_{lt}+\alpha, y_t) = R(a, 0, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(C_U, x_{lt} + \alpha + a, y_{t+1}\Big)\Big] \le R(\hat{a}, 0, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(C_U, 0, y_{t+1}\Big)\Big] \le v_t^*(x_{ut}, x_{lt}, y_t)$ from the non-decreasing property of the payoff function as well as the induction assumption.

Using the above result, we are now ready to prove Theorem 5.1.1. Let $a = a_t^*(x_{ut}, x_{lt}, y_t)$. Fix x_{ut}, x_{lt}, and y_t. Suppose that $(x_{ut}, x_{lt}) \in \Omega$. In order to characterize the optimal water flow policy, we consider the following cases:

- Suppose that $a \ge 0$. We consider the following problem: $\max_{z_{ut}\in[0,x_{ut}]}\Big\{V_t(x_{ut}+x_{lt}, z_{ut}, y_t) + R_t^{(\mathsf{RS})}(x_{ut}-z_{ut}, y_t)\Big\}$. By Lemma 5.1.6, $\min\{S_t^{(\mathsf{RS})}, x_{ut}\}$ yields the maximum value in this problem. Taking into account the capacity constraint $a \le \min\{x_{ut}, C_R\}$, we obtain $a = \min\{x_{ut} - S_t^{(\mathsf{RS})}, C_R\}$ if $x_{ut} > S_t^{(\mathsf{RS})}$ and $a = 0$ if $x_{ut} \le S_t^{(\mathsf{RS})}$.
- Suppose that $a \le 0$. We consider the following problem: $\max_{z_{ut}\in[x_{ut},C_U]}\Big\{V_t(x_{ut}+x_{lt}, z_{ut}, y_t) + R_t^{(\mathsf{PP})}(x_{ut}-z_{ut}, y_t)\Big\}$. By Lemma 5.1.6, $\max\{S_t^{(\mathsf{PP})}, x_{ut}\}$ yields the maximum value in this problem. Taking into account the capacity constraint $-\min\{x_{lt}, C_P\} \le a$, we obtain $a = -\min\{x_{lt}, C_P, S_t^{(\mathsf{PP})} - x_{ut}\}$ if $x_{ut} < S_t^{(\mathsf{PP})}$ and $a = 0$ if $x_{ut} \ge S_t^{(\mathsf{PP})}$.

We next show that $S_t^{(\mathsf{PP})}(x_{ut}, x_{lt}, y_t) \le S_t^{(\mathsf{RS})}(x_{ut}, x_{lt}, y_t)$: Fix x_{ut}, x_{lt}, and y_t. For each $\nu \in \{\mathsf{PP}, \mathsf{RS}\}$, let $S^{(\nu)} = S_t^{(\nu)}(x_{ut}, x_{lt}, y_t)$. By definition of $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t)$, the following inequalities hold.

$$V_t(x_{ut}+x_{lt}, S^{(\mathsf{PP})}, y_t) - p_t S^{(\mathsf{PP})}/\theta \ge V_t(x_{ut}+x_{lt}, S^{(\mathsf{RS})}, y_t) - p_t S^{(\mathsf{RS})}/\theta,$$
$$V_t(x_{ut}+x_{lt}, S^{(\mathsf{RS})}, y_t) - p_t \theta S^{(\mathsf{RS})} \ge V_t(x_{ut}+x_{lt}, S^{(\mathsf{PP})}, y_t) - p_t S^{(\mathsf{PP})}\theta.$$

The summation of the above inequalities implies that $p_t(\theta - 1/\theta)S^{(\mathsf{PP})} \ge p_t(\theta - 1/\theta)S^{(\mathsf{RS})}$. Since $0 < \theta$, $S_t^{(\mathsf{PP})}(x_{ut}, x_{lt}, y_t) \le S_t^{(\mathsf{RS})}(x_{ut}, x_{lt}, y_t)$.

We now show that $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t) = S_t^{(\nu)}(x_{ut}, x_{lt} + \alpha, y_t)$ if $x_{ut} + x_{lt} \geq C_U$ for each $\nu \in \{\mathsf{PP}, \mathsf{RS}\}$ and $\alpha > 0$: Since $v_t^*(x_{ut}, x_{lt} + \alpha, y_t) = v_t^*(x_{ut}, x_{lt}, y_t)$ if $x_{ut} + x_{lt} \geq C_U$, for any $z_{ut} \in [0, C_U]$ note that

$$\begin{aligned}
&V_t(x_{ut} + x_{lt}, z_{ut}, y_t) \\
&= \mathbb{E}\Big[v_{t+1}^*\Big(\min\{z_{ut} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - z_{ut}, C_L\}, y_{t+1}\Big)\Big] \\
&= \mathbb{E}\Big[v_{t+1}^*\Big(\min\{z_{ut} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} + \alpha - z_{ut}, C_L\}, y_{t+1}\Big)\Big] \\
&= V_t(x_{ut} + x_{lt} + \alpha, z_{ut}, y_t).
\end{aligned}$$

Thus, for each $\nu \in \{\mathsf{PP}, \mathsf{RS}\}$,

$$\begin{aligned}
S_t^{(\nu)}(x_{ut}, x_{lt}, y_t) &= \operatorname*{arg\,max}_{z_{ut} \in [0, C_U]} \{V_t(x_{ut} + x_{lt}, z_{ut}, y_t) + R_t^{(\nu)}(x_{ut} - z_{ut}, y_t)\} \\
&= \operatorname*{arg\,max}_{z_{ut} \in [0, C_U]} \{V_t(x_{ut} + x_{lt} + \alpha, z_{ut}, y_t) + R_t^{(\nu)}(x_{ut} - z_{ut}, y_t)\} \\
&= S_t^{(\nu)}(x_{ut}, x_{lt} + \alpha, y_t).
\end{aligned}$$

We also show that $S_t^{(\nu)}(x_{ut}, x_{lt} + \alpha, y_t) = S_t^{(\nu)}(x_{ut} + \alpha, x_{lt}, y_t)$ for each $\nu \in \{\mathsf{PP}, \mathsf{RS}\}$ and $\alpha > 0$:

$$\begin{aligned}
&S_t^{(\nu)}(x_{ut}, x_{lt} + \alpha, y_t) \\
&= \operatorname*{arg\,max}_{z_{ut} \in [0, C_U]} \{V_t(x_{ut} + x_{lt} + \alpha, z_{ut}, y_t) + R_t^{(\nu)}(x_{ut} - z_{ut}, y_t)\} \\
&= S_t^{(\nu)}(x_{ut} + \alpha, x_{lt}, y_t).
\end{aligned}$$

Lastly, we show that $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t) \leq S_t^{(\nu)}(x_{ut}, x_{lt} + \alpha, y_t)$ for each $\nu \in \{\mathsf{PP}, \mathsf{RS}\}$ and $\alpha > 0$: For each $\nu \in \{\mathsf{PP}, \mathsf{RS}\}$, let $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t) = S_1^{(\nu)}$ and $S_t^{(\nu)}(x_{ut}, x_{lt} + \alpha, y_t) = S_2^{(\nu)}$. Assume to the contrary that $S_1^{(\nu)} > S_2^{(\nu)}$. We consider the following nine cases to show that

$$\begin{aligned}
&\mathbb{E}\Big[v_{t+1}^*\Big(\min\{S_1^{(\nu)} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - S_1^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&- \mathbb{E}\Big[v_{t+1}^*\Big(\min\{S_2^{(\nu)} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - S_2^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&\leq \mathbb{E}\Big[v_{t+1}^*\Big(\min\{S_1^{(\nu)} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{S_2^{(\nu)} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} + \alpha - S_2^{(\nu)}, C_L\}, y_{t+1}\Big)\Big].
\end{aligned}$$

(1) If $r_{t+1} \le C_U - S_1^{(\nu)} < C_U - S_2^{(\nu)}$ and $x_{ut} + x_{lt} + \alpha - S_2^{(\nu)} \le C_L$, by property (a) of Lemma 5.1.5, $v^*_{t+1}(S_1^{(\nu)} + r_{t+1}, x_{ut} + x_{lt} - S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, x_{ut} + x_{lt} - S_2^{(\nu)}, y_{t+1}) \le v^*_{t+1}(S_1^{(\nu)} + r_{t+1}, x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, x_{ut} + x_{lt} + \alpha - S_2^{(\nu)}, y_{t+1})$.

(2) If $r_{t+1} \le C_U - S_1^{(\nu)} < C_U - S_2^{(\nu)}$ and $x_{ut} + x_{lt} - S_2^{(\nu)} \le C_L < x_{ut} + x_{lt} + \alpha - S_2^{(\nu)}$, by property (a) of Lemma 5.1.5 and by Lemma 5.1.2, $v^*_{t+1}(S_1^{(\nu)} + r_{t+1}, x_{ut} + x_{lt} - S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, x_{ut} + x_{lt} - S_2^{(\nu)}, y_{t+1}) \le v^*_{t+1}(S_1^{(\nu)} + r_{t+1}, C_L + S_2^{(\nu)} - S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(S_1^{(\nu)} + r_{t+1}, \min\{x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, C_L, y_{t+1})$.

(3) If $r_{t+1} \le C_U - S_1^{(\nu)} < C_U - S_2^{(\nu)}$ and $C_L < x_{ut} + x_{lt} - S_2^{(\nu)}$, by Lemma 5.1.2, $v^*_{t+1}(S_1^{(\nu)} + r_{t+1}, \min\{x_{ut} + x_{lt} - S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(S_1^{(\nu)} + r_{t+1}, \min\{x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, C_L, y_{t+1})$.

(4) If $C_U - S_1^{(\nu)} < r_{t+1} \le C_U - S_2^{(\nu)}$ and $x_{ut} + x_{lt} + \alpha - S_2^{(\nu)} \le C_L$, by properties (a) and (c) of Lemma 5.1.5, $v^*_{t+1}(C_U, x_{ut} + x_{lt} - S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, y_{t+1}) \le v^*_{t+1}(C_U + S_2^{(\nu)} - S_1^{(\nu)}, x_{ut} + x_{lt} - S_2^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U + S_2^{(\nu)} - S_1^{(\nu)}, x_{ut} + x_{lt} + \alpha - S_2^{(\nu)}, y_{t+1}) \le v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, x_{ut} + x_{lt} - S_2^{(\nu)}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, x_{ut} + x_{lt} + \alpha - S_2^{(\nu)}, y_{t+1})$.

(5) If $C_U - S_1^{(\nu)} < r_{t+1} \le C_U - S_2^{(\nu)}$ and $x_{ut} + x_{lt} - S_2^{(\nu)} \le C_L < x_{ut} + x_{lt} + \alpha - S_2^{(\nu)}$, by Lemma 5.1.2 and by properties (a) and (c) of Lemma 5.1.5, $v^*_{t+1}(C_U, x_{ut} + x_{lt} - S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, \min\{x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, C_L\}, y_{t+1}) \le v^*_{t+1}(C_U, x_{ut} + x_{lt} - S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, C_L + S_2^{(\nu)} - S_1^{(\nu)}, y_{t+1}) \le v^*_{t+1}(C_U + S_2^{(\nu)} - S_1^{(\nu)}, x_{ut} + x_{lt} - S_2^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U + S_2^{(\nu)} - S_1^{(\nu)}, C_L, y_{t+1}) \le v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, x_{ut} + x_{lt} - S_2^{(\nu)}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, C_L, y_{t+1})$.

(6) If $C_U - S_1^{(\nu)} < r_{t+1} \le C_U - S_2^{(\nu)}$ and $C_L < x_{ut} + x_{lt} - S_2^{(\nu)}$, by Lemma 5.1.2, $v^*_{t+1}(C_U, \min\{x_{ut} + x_{lt} - S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(C_U, \min\{x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)} + r_{t+1}, C_L, y_{t+1})$.

(7) If $C_U - S_1^{(\nu)} < C_U - S_2^{(\nu)} < r_{t+1}$ and $x_{ut} + x_{lt} + \alpha - S_2^{(\nu)} \le C_L$, by properties (a) and (c) of Lemma 5.1.5 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), $v^*_{t+1}(C_U, x_{ut} + x_{lt} - S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - S_2^{(\nu)}, y_{t+1}) \le v^*_{t+1}(C_U, x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} + \alpha - S_2^{(\nu)}, y_{t+1})$.

(8) If $C_U - S_1^{(\nu)} < C_U - S_2^{(\nu)} < r_{t+1}$ and $x_{ut} + x_{lt} - S_2^{(\nu)} \leq C_L < x_{ut} + x_{lt} + \alpha - S_2^{(\nu)}$, by properties (a) and (c) of Lemma 5.1.5 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$) and by Lemma 5.1.2, $v^*_{t+1}(C_U, x_{ut}+x_{lt}-S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut}+x_{lt}-S_2^{(\nu)}, y_{t+1}) \leq v^*_{t+1}(C_U, C_L + S_2^{(\nu)} - S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \leq v^*_{t+1}(C_U, \min\{x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

(9) If $C_U - S_1^{(\nu)} < C_U - S_2^{(\nu)} < r_{t+1}$ and $C_L < x_{ut} + x_{lt} - S_2^{(\nu)}$, by Lemma 5.1.2, $v^*_{t+1}(C_U, \min\{x_{ut}+x_{lt}-S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \leq v^*_{t+1}(C_U, \min\{x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

Hence:

$$\begin{aligned}
&\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_1^{(\nu)} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - S_1^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&- \mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_2^{(\nu)} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - S_2^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&= V_t(x_{ut} + x_{lt}, S_1^{(\nu)}, y_t) - V_t(x_{ut} + x_{lt}, S_2^{(\nu)}, y_t) \\
&\leq V_t(x_{ut} + x_{lt} + \alpha, S_1^{(\nu)}, y_t) - V_t(x_{ut} + x_{lt} + \alpha, S_2^{(\nu)}, y_t) \qquad \text{(C.1)} \\
&= \mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_1^{(\nu)} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} + \alpha - S_1^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_2^{(\nu)} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} + \alpha - S_2^{(\nu)}, C_L\}, y_{t+1}\Big)\Big].
\end{aligned}$$

By definitions of $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t)$ and $S_t^{(\nu)}(x_{ut}, x_{lt}+\alpha, y_t)$, the following inequalities hold.

$$\begin{aligned}
&V_t(x_{ut} + x_{lt}, S_1^{(\nu)}, y_t) + R_t^{(\nu)}(x_{ut} - S_1^{(\nu)}, y_t) \\
&\geq V_t(x_{ut} + x_{lt}, S_2^{(\nu)}, y_t) + R_t^{(\nu)}(x_{ut} - S_2^{(\nu)}, y_t), \\
&V_t(x_{ut} + x_{lt} + \alpha, S_2^{(\nu)}, y_t) + R_t^{(\nu)}(x_{ut} - S_2^{(\nu)}, y_t) \\
&\geq V_t(x_{ut} + x_{lt} + \alpha, S_1^{(\nu)}, y_t) + R_t^{(\nu)}(x_{ut} - S_1^{(\nu)}, y_t).
\end{aligned}$$

The summation of the above inequalities implies that

$$
\begin{aligned}
&\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_2^{(\nu)}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}-S_2^{(\nu)},C_L\},y_{t+1}\Big)\Big]\\
&-\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_1^{(\nu)}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}-S_1^{(\nu)},C_L\},y_{t+1}\Big)\Big]\\
&=V_t(x_{ut}+x_{lt},S_2^{(\nu)},y_t)-V_t(x_{ut}+x_{lt},S_1^{(\nu)},y_t)\\
&\le V_t(x_{ut}+x_{lt}+\alpha,S_2^{(\nu)},y_t)-V_t(x_{ut}+x_{lt}+\alpha,S_1^{(\nu)},y_t)\\
&=\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_2^{(\nu)}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}+\alpha-S_2^{(\nu)},C_L\},y_{t+1}\Big)\Big]\\
&\quad-\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_1^{(\nu)}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}+\alpha-S_1^{(\nu)},C_L\},y_{t+1}\Big)\Big].
\end{aligned}
$$

This leads to a contradiction with the inequality in (C.1). Thus $S_1^{(\nu)}\le S_2^{(\nu)}$. $\square$