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When are the products of normal operators normal?

by
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To the memory of Peter Jonas, friend and collaborator

Abstract

Given two normal operators A and B on a Hilbert space it is known that, in general, AB is not normal. The question on characterizing those pairs of normal operators for which their products are normal has been solved for finite dimensional spaces by F.R. Gantmaher and M.G. Krein in 1930, and for compact normal operators by N.A. Wiegmann in 1949. Actually, in the afore mentioned cases, the normality of AB is equivalent with that of BA, and a more general result of F. Kittaneh implies that it is sufficient that AB be normal and compact to obtain that BA is the same. On the other hand, I. Kaplansky had shown that it may be possible that AB is normal while BA is not. When no compactness assumption is made, but both of AB and BA are supposed to be normal, the Gantmaher-Krein-Wiegmann Theorem can be extended by means of the spectral theory of normal operators in the von Neumann's direct integral representation.

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1 Introduction

A normal operator N is a bounded operator in a Hilbert space such that it commutes with its adjoint, $NN^* = N^*N$. It is a remarkable fact that this simple algebraic condition is strong enough to ensure that a normal operator is, when the ambient Hilbert space is transformed by an isometric isomorphism, similar to the multiplication by a function on an L^2 space (see e.g. L.A. Steen [16] for a historical perspective on this subject). Thus, when considered as a single operator, a normal operator has the best spectral theory one can dream to. However, given two normal operators A and B on the same Hilbert space

$\mathcal{H}$, it is known that, in general, the products AB and BA may not be normal, for example, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. However, as a consequence of the Fuglede's Theorem (see Theorem 2.2 below) if, in addition, A and B commute, then AB is normal. But there is a simple and striking example showing that the general picture transcends the commutativity: if both A and B are two arbitrary unitary operators (isometric isomorphisms) on the same Hilbert space, then both AB and BA are unitary operators, and hence normal.

Our attention on the normality of the products of normal operators has been drawn during the work on [17], when the normality of operators in the Pauli algebra representations became of interest in connection with some questions in polarization optics. In that case, when the operators are modeled by complex two-by-two matrices, it can be proven that, assuming that A and B are normal and do not commute, then AB is normal if and only if BA is normal if and only if both A and B are multiples of unitary operators (that corresponds to the so-called *retarders*, in polarization optics). When similar problems are to be considered in quantum optics, they have to refer to normal operators on infinite dimensional Hilbert spaces, e.g. see [2].

To our knowledge, the history of the problem on the normality of the products of normal operators starts with the following theorem of F.R. Gantmaher and M.G. Krein published in 1930.

Theorem 1.1 (F.R. Gantmaher and M.G. Krein [6])**.** *Let A and B be normal $n \times n$ complex matrices. The following assertions are equivalent:*

(i) *AB is normal.*

(ii) *BA is normal.*

(iii) *There exists an orthogonal decomposition*

$$\mathbb{C}^n = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_k$$

such that $\mathcal{R}_j$ are reducing subspaces for both A and B and, with respect to this decomposition, we have the representations

$$A = \bigoplus_{j=1}^{k} A_j, \quad B = \bigoplus_{j=1}^{k} B_j,$$

where A_j and B_j are multiples of unitary operators on $\mathcal{R}_j$, for all $j = 1, \ldots, k$.

The article [6] was most likely unknown in the Western mathematical literature and this theorem was independently discovered by N.A. Wiegmann in 1948, cf. [19]. A generalization of this result to compact operators was published by the same author one year later in [20], see Theorem 3.8.

Under compactness assumptions the normality of AB and BA are equivalent but, as I. Kaplansky showed in [9], there exist normal operators A and B with the property that AB is normal while BA is not. This question was later considered by F. Kittaneh in [10] who showed that it is sufficient to assume that A and B^* be hyponormal and that AB be compact and normal in order to conclude that BA is normal (and compact) as well. Going further in the spirit of Gantmaher-Krein and Wiegmann results and beyond compactness assumptions, one can ask the question of describing those normal operators A and B for which both of AB and BA are normal.

In this paper we prove the Gantmaher-Krein-Wiegmann Theorem, on one hand by explaining its simple linear algebra roots and, on the other hand, by using more modern tools and in a way that, combined with the powerful tool of direct integral Hilbert space representations in the spectral multiplicity theory for normal operators, allows us to obtain a generalization to the noncompact case.

The difficulty, in the case of either finite rank or compact operators, relies on two obstructions: the first one is the question of symmetry (AB normal if and only if BA normal) and the second one is a characterization of normality in terms of eigenvalues and their multiplicities only. The symmetry of the normality of AB and BA holds as long as some compactness assumptions are enforced. We explain this part carefully in Section 3 by keeping our exposition elementary through simple linear algebra ideas. As for the second obstruction, a result due to I. Kaplansky [9] that is based on Fuglede-Putnam Theorem implies that, given normal operators A and B such that AB is normal and performing "polar decompositions" $A = U|A|$ and $B = V|B|$, then BA is normal if and only if $|A|$ commutes with $|B|$, U commutes with $|B|$, and V commutes with $|A|$ (the fact that U commutes with $|A|$ and V commutes with $|B|$ is a consequence of the normality of A and B, cf. Section 2). So, roughly speaking, only the noncommutativity of U and V makes the trouble. Once isolated, this obstruction can be treated in two ways: either perform a "spectral surgery" on the various decompositions of the operators A and B, or by means of a construction that brings together the absolute values $|A|$ and $|B|$ in only one normal operator, to which the Spectral Theorem can be applied. The first variant is used in all the afore mentioned articles but, in the direct integral Hilbert space representations, difficulties related to measurable choices show up if this approach is chosen. Note that, in view of Kaplansky's counter-example, when no compactness assumptions are made, we may have AB normal while BA is not. The noncompact generalization in this paper is considering only the symmetric case, i.e. both of AB and BA are assumed to be normal. The asymmetric case remains open: a different approach and, most probably, different techniques are needed.

Since the formalism of direct integral representation for normal operators (a particular case of the reduction theory to factors of J. von Neumann [12]) is quite involved, we have included a sequence of preliminary subsections in Section 4 on direct integrals of Hilbert spaces, decomposable operators, diagonalizable operators, and a part of the Spectral Multiplicity Theorem for normal operators; detailed proofs can be found in [4], [1], and [8].

I want to express my gratitude to Tiberiu Tudor for asking questions that triggered my interest for this problem, to Petru Cojuhari who, jointly with Serguei

Naboko, provided a copy of the article of F.R. Gantmaher and M.G. Krein [6] and, last but not least, to the unknown referee for carefully reading a preliminary version of this article and making pertinent and very useful remarks.

2 Normal Operators

This section sets the background for the theory of normal operators; proofs can be seen in most of the textbooks on operator theory, e.g. see J.B. Conway [3]. Throughout this article all the Hilbert spaces are supposed to be complex.

We record, for the beginning, a few elementary facts on normal operators, e.g. see Proposition II.2.16 in [3].

Proposition 2.1. *Given a bounded linear operator $N \in \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, the following assertions are equivalent:*

(i) *N is normal, that is, $N^*N = NN^*$.*

(ii) *For all $x \in \mathcal{H}$ it holds $\|Nx\| = \|N^*x\|$.*

(iii) *$N = U|N|$ with $|N| = (N^*N)^{1/2}$, U is unitary and commutes with $|N|$.*

(iv) *The real part $\operatorname{Re}(N) = (N + N^*)/2$ and the imaginary part $\operatorname{Im}(N) = (N - N^*)/2\mathrm{i}$ commute.*

Consequently, if $N \in \mathcal{B}(\mathcal{H})$ is a normal operator then $\ker(N) = \ker(N^*)$ and $\overline{\operatorname{Ran}(N)} = \overline{\operatorname{Ran}(N^*)}$.

There are a few different theorems that are known under the name of *The Spectral Theorem for Normal Operators*. The one that we consider in this article uses the notion of spectral measure, that is, a mapping $E\colon \mathfrak{B} \to \mathcal{B}(\mathcal{H})$, where $\mathfrak{B}$ denotes the σ-algebra of Borel sets in the complex plane, subject to the following properties:

(a) For all $\Delta \in \mathfrak{B}$, $E(\Delta) = E(\Delta)^*$;

(b) $E(\emptyset) = 0$ and $E(\mathbb{C}) = I$;

(c) For all $\Delta_1, \Delta_2 \in \mathfrak{B}$ we have $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$;

(d) For any sequence $(\Delta_n)_{n\in\mathbb{N}}$ of mutually disjoint Borel subsets of $\mathbb{C}$ we have

$$E(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} E(\Delta_n),$$

where the convergence is in the strong operator topology.

As is well-known, the range of a spectral measure consists only of orthogonal projections and the notion of *support* of a spectral measure can be defined. Once we have a spectral measure E as above, for any $h, k \in \mathcal{H}$ the mapping

$E_{h,k}(\Delta) = \langle E(\Delta)h, k\rangle$ turns out to be a scalar (in general, complex valued) countably additive measure on $\mathbb{C}$, with total variation $\leq \|h\|\|k\|$. Then for any bounded Borel complex function f one can define the following integral

$$\rho(f) = \int f(z)\mathrm{d}E(z) \in \mathcal{B}(\mathcal{H}),$$

in the sense that, for all $h, k \in \mathcal{H}$

$$\langle \rho(f)h, k\rangle = \int f(z)\mathrm{d}E_{h,k}(z),$$

and it turns out that $\rho\colon B(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$ is a $*$-representation of the C^*-algebra $B(\mathbb{C})$, of all bounded Borel complex functions, in $\mathcal{B}(\mathcal{H})$. According to The Spectral Theorem for Normal Operators, if N is a normal operator then there exists a unique spectral measure E, subject to the following properties:

(i) $N = \int z\mathrm{d}E(z)$.

(ii) For any open set G such that $G \cap \sigma(N) \neq \emptyset$, we have $E(G) \neq 0$;

(iii) For any $A \in \mathcal{B}(\mathcal{H})$, we have $AN = NA$ if and only if A commutes with the spectral measure E, that is, $AE(\Delta) = E(\Delta)A$ for all $\Delta \in \mathfrak{B}$.

Condition (ii) says that the support of the spectral measure E is exactly $\sigma(N)$, the spectrum of the normal operator N.

In this respect, we recall also the celebrated B. Fuglede's Theorem [5] that sheds some light on condition (iii) in the Spectral Theorem; actually, the original theorem of Fuglede is more general, since it refers to unbounded normal operators, while the finite dimensional case was proved earlier by J. von Neumann [11].

Theorem 2.2 (B. Fuglede [5])**.** *If $N \in \mathcal{B}(\mathcal{H})$ is a normal operator and $A \in \mathcal{B}(\mathcal{H})$ arbitrary is such that $AN = NA$ then $AN^* = N^*A$ as well.*

An immediate consequence of Fuglede's Theorem is that if $A, B \in \mathcal{B}(\mathcal{H})$ are two normal operators that commute, then each one commutes with the adjoint of the other and hence their product AB is normal as well. This is, probably, the most popular sufficient condition of normality for the product of two normal operators.

There is a generalization (in the bounded case they are actually equivalent) of Fuglede's Theorem:

Theorem 2.3 (C.R. Putnam [14])**.** *If $M, N \in \mathcal{B}(\mathcal{H})$ are normal operators and $A \in \mathcal{B}(\mathcal{H})$ such that $MA = AN$, then $M^*A = AN^*$ as well.*

We end this section with a consequence of Putnam's Theorem.

Lemma 2.4 (I. Kaplansky [9])**.** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that A and AB are normal. Then BA is normal if and only if B commutes with $|A|$.*

Proof: Since A and AB are normal, by Proposition 2.1 let $A = U|A|$, where $U \in \mathcal{B}(\mathcal{H})$ is unitary and commutes with $|A| = (A^*A)^{1/2}$. If, in addition, B commutes with $|A|$, then

$$U^*ABU = U^*U|A|BU = B|A|U = BU|A| = BA,$$

and hence BA is normal as well (as unitary equivalent with the normal operator AB).

Conversely, if BA is normal let $M = AB$ and $N = BA$. Since $MA = ABA = AN$, by Theorem 2.3 it follows that $M^*A = AN^*$, that is, $B^*A^*A = AA^*B^*$ and, taking into account that $A^*A = AA^*$, this means that B^* commutes with A^*A, and so does B. □

3 Compact Normal Operators

An operator A on a Hilbert space $\mathcal{H}$ is *compact* if $\{Ah \mid \|h\| \leq 1\}$ has compact closure in $\mathcal{H}$, equivalently, if $\{Ah \mid \|h\| \leq 1\}$ is compact in $\mathcal{H}$, equivalently, A can be uniformly approximated by finite rank operators (that is, operators whose ranges are finite dimensional subspaces). The collection of all compact operators in a Hilbert space makes a uniformly closed ideal in $\mathcal{B}(\mathcal{H})$, stable under taking adjoints. In this section we present what is known on the normality of the products AB and/or BA, when A and B are normal operators and in the presence of various compactness assumptions. The first subsection gathers classical results on compact operators, for details see [3], and then focuses on a characterization of normality of a compact operator in terms of eigenvalues and their (algebraic) multiplicities only.

3.1 Eigenvalues and Singular Numbers

According to the spectral theory of compact operators on Hilbert spaces, the spectrum of A is discrete with 0 the only possible accumulation point. Moreover, any point $\lambda \in \sigma(A) \setminus \{0\}$ is an eigenvalue of finite algebraic multiplicity, more precisely, λ is isolated in $\sigma(A)$ and the Riesz projection corresponding to the spectral set $\{\lambda\}$ has finite rank, the algebraic multiplicity of λ. We consider $\lambda_1(A), \ldots, \lambda_n(A), \ldots$ an enumeration of the non-zero eigenvalues of A, in the decreasing order of their moduli (absolute values), and counted with their algebraic multiplicities. For our purposes it is not important the order we use for counting eigenvalues with the same absolute values. In general, we get either a finite or an infinite sequence $(\lambda_n(A))_{n=1}^N$, where N is either a natural number or the symbol $+\infty$, depending whether A has finite rank or not.

If A is a compact operator then the operator $|A| = (A^*A)^{1/2}$ is compact and nonnegative. Thus, by the spectral theory for these operators, $\sigma(|A|)$ is a discrete set with 0 the only possible accumulation point and its non-zero elements denoted by $s_n(A)$, the *singular numbers* of A, arranged in decreasing order and counted with their multiplicities. An important difference with respect to the sequence of its eigenvalues is that, for $A \neq 0$, there exists at least one nontrivial singular value. A has finite rank if and only if it has a finite number of singular values. The singular numbers provide a very useful representation, the so-called *Schmidt Representation*: for any compact operator A, letting $(s_n(A)_{n=1}^N$ (where N is either a positive integer of infinite), there exist two orthonormal sequences $(x_n)_{n=1}^N$ and $(y_n)_{n=1}^N$ such that

$$A = \sum_{n=1}^{N} s_n(A) \langle \cdot, x_n \rangle y_n, \tag{3.1}$$

where, in case $N = \infty$, the series converges uniformly (that is, in the operator norm). Note that formula (3.1) substantiates the assertion that any compact operator is uniformly approximated by finite rank operators, because the operator $h \mapsto \langle h, x \rangle y$ is the prototype of the operators of rank one.

Singular numbers are related to eigenvalues of compact operators in a subtle way. Among the many relations that have been obtained so far, see e.g. the monographs of I.C. Gohberg and M.G. Krein [7] or that of B. Simon [15], we recall the celebrated Weyl Inequalities: if A is a compact operator with eigenvalues $(\lambda_n(A))_{n=1}^N$ and singular numbers $(s_n(A))_{n=1}^N$, ordered as mentioned before, then for all admissible $n = 1, 2, \ldots$

$$\prod_{k=1}^{n} |\lambda_k(A)| \leq \prod_{k=1}^{n} s_k(A), \tag{3.2}$$

$$\sum_{k=1}^{n} |\lambda_k(A)| \leq \sum_{k=1}^{n} s_k(A). \tag{3.3}$$

If A is a compact and normal operator on a Hilbert space $\mathcal{H}$, then the Spectral Theorem for Normal Operators implies that there exists an at most countable family of finite rank orthogonal and mutually orthogonal projections $\{P_n\}_{n=1}^N$ such that

$$A = \sum_{n=1}^{N} \lambda_n(A) P_n, \tag{3.4}$$

where, for $N = \infty$, the series converges uniformly in $\mathcal{B}(\mathcal{H})$. In particular, the number of its non-zero eigenvalues $(\lambda_n(A))_{n=1}^N$ is the same with the number of its singular values and, in addition, for all $n = 1, \ldots, N$ we have $|\lambda_n(A)| = s_n(A)$. The next theorem, which can be found in the monograph [7] and is a generalization of a result of F.R. Gantmaher and M.G. Krein [6], shows that, for compact operators, this spectral condition is also sufficient for the normality of A. Let us note that, in this case, the Spectral Theorem implies even more than that, namely that for all $n = 1, \ldots, N$, the Riesz projection corresponding to A and the spectral set $\{\lambda \mid |\lambda| = s_n(A)\}$ coincides with the spectral projection corresponding to $|A|$ and the spectral set $\{s_n(A)\}$, but this comes for free from the proof of the following

Theorem 3.1. *Let A be a compact operator on the Hilbert space $\mathcal{H}$ and, following the conventional notation as before, let us assume that it has exactly the same number of non-zero eigenvalues $(\lambda_n(A))_{n=1}^N$ as its singular values $(s_n(A))_{n=1}^N$ and, in addition, that for all $n = 1, \ldots, N$ we have $|\lambda_n(A)| = s_n(A)$. Then A is normal.*

The proof of this theorem is based on linear algebra ideas, as seen in the following technical result that can be tracked back to F. Klein (cf. [6]). Recall that an eigenvector z corresponding to an operator $A \in \mathcal{B}(\mathcal{H})$ and eigenvalue λ is called *simple* if there exists no vector $h \in \mathcal{H} \setminus \{0\}$ such that $Ah = \lambda(z + h)$. An eigenvalue λ of A is called *simple* if all the eigenvectors corresponding to A and λ are simple.

Lemma 3.2. *Let $\{z_i\}_{i=1}^n$ be a set of linearly independent eigenvectors corresponding respectively to the eigenvalues $\{\lambda_i\}_{i=1}^n$ of a linear operator $A \in \mathcal{B}(\mathcal{H})$, and let z be an eigenvector corresponding to the eigenvalue λ of A, such that for all $i = 1, \ldots, n$ we have $\lambda \neq \lambda_i$. If there is no vector y in $\mathcal{H}$, orthogonal to $\{z_i\}_{i\in I}$ and such that*

$$\|Ay\| > |\lambda|\|y\|, \tag{3.5}$$

then z is a simple eigenvector of A and orthogonal on $\{z_i\}_{i=1}^n$.

Proof: Let $\{x_i\}_{i=1}^n$ be the Gram-Schmidt orthonormalization of the sequence $\{z_i\}_{i=1}^n$. For all $j = 1, \ldots, n$ we have $\mathcal{L}_j = \operatorname{Lin}\{x_1, \ldots, x_j\} = \operatorname{Lin}\{z_1, \ldots, z_j\}$ is invariant under A. Then $\mathcal{L} = \bigvee_{j=1}^n \mathcal{L}_j$ is invariant under A and let $[a_{ij}]$ be the matrix representation of $A|\mathcal{L} \in \mathcal{B}(\mathcal{L})$ with respect to the orthonormal basis $\{x_j\}_{j=1}^n$, that is,

$$a_{ij} = \langle Ax_i, x_j\rangle, \quad i, j = 1, \ldots, n.$$

We first prove that $z \perp \{z_i\}_{i=1}^n$, equivalently, $z \perp \mathcal{L}$. Indeed, if this is not true, then the vector

$$y = z - \sum_{i=1}^n \langle z, x_i\rangle x_i$$

is not zero and

$$\begin{aligned} Ay = Az - \sum_{j=1}^n \langle z, x_j\rangle Ax_j &= \lambda z - \sum_{i=1}^n \sum_{j=1}^n \langle z, x_j\rangle a_{ij} x_j \\ &= \lambda z - \sum_{j=1}^n \left(\sum_{i=1}^n \langle z, x_j\rangle a_{ij} \right) x_j = \lambda(z - \sum_{j=1}^n c_j x_j) \end{aligned}$$

where we note that $\lambda \neq 0$ (due to the strict inequality in (3.5)) and we denote

$$c_j = \frac{1}{\lambda} \sum_{i=1}^n \langle z, x_i\rangle a_{ij}. \tag{3.6}$$

We now observe that λ is not an eigenvalue of the matrix $[a_{ij}]$. Indeed, this matrix has its eigenvalues exactly $\lambda_1, \ldots, \lambda_n$ and λ is different of any of these and hence, by the previous calculation, it follows that there exists $k \in \{1, \ldots, n\}$ such that $c_k \neq \langle z, x_k \rangle$. Therefore, by the fact that the Fourier coefficients $\langle z, x_i \rangle$ are the only choice for minimizing the norm $\|z - \sum_{i=1}^n \alpha_i x_i\|$ it follows that

$$\|Ay\| = |\lambda| \|z - \sum_{i=1}^n c_i x_i\| > |\lambda| \|y\|$$

which contradicts the assumption (3.5). Thus, z is orthogonal to all $z_1, \ldots, z_n$.

It remains to prove that z is a simple eigenvector for A. To see this, let us assume that z is not simple, that is, there exists a vector $h \neq 0$ in $\mathcal{H}$ such that

$$Ah = \lambda(z + h).$$

Consider the vector y defined

$$y = h - \langle y, z \rangle - \sum_{i=1}^n \langle z, x_i \rangle x_i,$$

hence $y \perp \{z, z_1, \ldots, z_n\}$. Then, with the notation as in (3.6) we have

$$\begin{aligned}
Ay &= Ah - \langle y, z \rangle Az - \sum_{i=1}^n \langle z, x_i \rangle A x_i \\
&= \lambda(z + h) - \langle h, z \rangle \lambda z - \lambda \sum_{i=1}^n c_i x_i \\
&= \lambda\big(h + (1 - \langle h, z \rangle) z - \sum_{i=1}^n c_i x_i\big).
\end{aligned}$$

Since $1 - \langle h, z \rangle \neq -\langle h, z \rangle$ and using again the minimization property of the Fourier coefficients, it follows that

$$\|Ay\| > |\lambda| \|y\|,$$

which contradicts the assumption on (3.5). Thus, z is a simple eigenvector of A. $\square$

Proof of Theorem 3.1. Excluding the trivial case $A = 0$, without loss of generality we can assume that $\sigma(|A|) \setminus \{0\}$ is nontrivial. Then there exists $k_1 \in \mathbb{N}$ such that the sequence (finite or infinite) of the singular values of A is

$$s_1(A) = s_2(A) = \ldots = s_{k_1}(A) > s_{k_1+1}(A) = \ldots$$

Since $\|A\| = \|A^*A\|^{1/2} = \||A|\| = s_1(A)$, for all $y \in \mathcal{H}$ we have $\|Ay\| \leq s_1(A)\|y\|$. By assumption we have the sequence (finite or, respectively, infinite) of eigenvalues of A, counted with their multiplicities

$$\lambda_1(A), \lambda_2(A), \ldots, \lambda_{k_1}(A), \lambda_{k_1+1}(A), \ldots$$

and such that

$$|\lambda_1(A)| = |\lambda_2(A)| = \cdots = |\lambda_{k_1}(A)| = s_1(A) > s_{k_1+1}(A) = |\lambda_{k_1+1}(A)| \cdots$$

Recall that eigenvectors corresponding to different eigenvalues are linearly independent. By Lemma 3.2 it follows that any eigenvector corresponding to either of the eigenvalues $\lambda_1, \ldots, \lambda_{k_1}$ are simple and orthogonal to any eigenvector corresponding to eigenvalues λ different than any of $\lambda_1, \ldots, \lambda_{k_1}$. In particular, it follows that there exists an orthonormal family $\{z_1, \ldots, z_{k_1}\}$ of simple eigenvectors of A corresponding respectively to the eigenvalues $\lambda_1, \ldots, \lambda_{k_1}$. Then letting $\mathcal{S}_1 = \mathrm{Lin}\{z_1, \ldots, z_{k_1}\}$ we get a k_1 dimensional vector space that reduces A and such that $A|\mathcal{S}_1$ is normal.

We consider the compact operator $A_1 = A|\mathcal{H} \ominus \mathcal{S}_1$ and apply the same procedure as before, taking into account that the eigenvalues of A_1 are

$$\lambda_{k_1+1}(A), \lambda_{k_1+2}(A), \ldots$$

while its singular numbers are $s_{k_1+1}(A), s_{k_1+2}(A), \ldots$. The proof can be finished by induction. □

3.2 When is the Normality of AB Equivalent with the Normality of BA?

In this subsection we will prove that if A and B are normal operators such that AB is normal and compact, then BA is normal as well. This is a particular case of a theorem of F. Kittaneh [10] (the result of Kittaneh says that if A and B^* are hyponormal, that is, $AA^* \leq A^*A$ and $B^*B \leq BB^*$, and AB is compact and normal, then BA is compact and normal as well). The proof follows the lines in [10]. Then we present an example, due to I. Kaplansky, showing that there exist two normal operators A and B such that AB is normal, but BA is not.

We first recall some classical but useful facts.

Lemma 3.3. *If A and B are bounded operators such that both AB and BA are compact then, modulo a reordering of the eigenvalues with the same absolute value, we have $\lambda_k(AB) = \lambda_k(BA)$ for all $k = 1, 2, \ldots$.*

Proof: To see this we use the well-known fact in operator theory that for any bounded operators A and B we have $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ and then take into account the spectral theory of compact operators described before. More precisely, from the factorization

$$\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

it follows that the compact operators AB and BA have the same nonzero eigenvalues, counted with multiplicities. $\square$

Lemma 3.4. *Let A and B be normal operators such that AB is compact. Then BA is compact as well and $s_k(AB) = s_k(BA)$ for all $k = 1, 2, \ldots$.*

Proof: We have $(A^*B)^*(A^*B) = B^*AA^*B = B^*A^*AB = (AB)^*(AB)$ and compact. Hence $|A^*B|$ is compact and then, by polar decomposition, it follows that A^*B is compact. Therefore, its adjoint B^*A is compact as well. From here, reasoning in a similar way, we obtain that BA is compact.

Further on, for arbitrary $k = 1, 2, \ldots$ we have, by Lemma 3.3 and the normality of A and B,

$$\begin{aligned} s_k^2(BA) &= \lambda_k(A^*B^*BA) = \lambda_k(A^*BB^*A) \\ &= \lambda_k(B^*AA^*B) = \lambda_k(B^*AA^*B) = s_k^2(AB). \end{aligned}$$

$\square$

Theorem 3.5 (F. Kittaneh [10])**.** *If A and B are normal operators such that AB is normal and compact, then BA is normal and compact as well, and $s_k(AB) = s_k(BA)$ for all $k = 1, 2, \ldots$.*

Proof: In view of Lemma 3.4, we have that BA is compact and that $s_k(AB) = s_k(BA)$ for all $k = 1, 2, \ldots$. Then, for any $k = 1, 2, \ldots$, by Lemma 3.3 we have $s_k(BA) = s_k(AB) = |\lambda_k(AB)| = |\lambda_k(BA)|$ and hence, by Theorem 3.1 it follows that BA is a normal operator. $\square$

We conclude this subsection with an example due to I. Kaplansky [9] that shows that, in the absence of compactness assumptions there exist normal operators A and B such that AB is normal but BA is not.

Example 3.6. We consider ℓ^2 the Hilbert space of square summable sequences of complex numbers: $x = (x_j)_{j\in\mathbb{N}}$ such that $\sum_{j=1}^\infty |x_j|^2 < \infty$ with inner product $\langle x, y\rangle = \sum_{j=1}^\infty x_j \overline{y}_j$, where $\overline{z}$ denotes the complex conjugate of z. Then let the Hilbert space $\mathcal{H} = \mathbb{C}^3 \otimes \ell^2$. Basically, this tensor notation is equivalent with saying that $\mathcal{H} = \ell^2 \oplus \ell^2 \oplus \ell^2$. On the Hilbert space $\mathcal{H}$ we consider the operator $P = (2 \oplus 1 \oplus \frac{1}{2}) \otimes I_{\ell^2}$, more precisely, P is just the operator $2I_{\ell^2} \oplus I_{\ell^2} \oplus \frac{1}{2} I_{\ell^2}$, or if infinite matrices are preferred, P is the diagonal matrix with the triple $2, 1, \frac{1}{2}$ repeated infinitely often down the diagonal. With respect to these notations, we consider the Hilbert space $\mathcal{K} = \mathcal{H} \oplus \ell^2$ and the operators $Q, R \in \mathcal{B}(\mathcal{K})$, $Q = P \oplus 0$ and $R = 0 \oplus I_{\ell^2}$, and note that $QR = RQ = 0$.

We claim that the operator $4Q^2 + 4R^2 = \big((16 \oplus 4 \oplus 1) \otimes I_{\ell^2}\big) \oplus 4I_{\ell^2}$ is unitarily equivalent to the operator $4Q^2 + R^2 = \big((16 \oplus 4 \oplus 1) \otimes I_{\ell^2}\big) \oplus I$. This follows from a canonical unitary identification of $\ell^2 \oplus \ell^2$ with ℓ^2, by matching odd and even indices on the canonical orthonormal basis. Thus, there exists U a unitary operator in $\mathcal{K}$ such that

$$U(4Q^2 + 4R^2)U^* = 4Q^2 + R^2. \tag{3.7}$$

Similarly, the operator $Q^2 + R^2$ is unitarily equivalent with the operator $Q^2 + 4R^2$, and hence there exists a unitary operator V in $\mathcal{K}$ such that

$$V(Q^2 + R^2)V^* = Q^2 + 4R^2. \tag{3.8}$$

Then let

$$A = \begin{bmatrix} 2U & 0 \\ 0 & V \end{bmatrix}, \quad B = \begin{bmatrix} Q & R \\ R & Q \end{bmatrix},$$

and note that A is normal, as the direct sum of two normal operators, and B is selfadjoint, hence normal as well.

Next we claim that the operator AB is normal. Indeed,

$$AB = \begin{bmatrix} 2UQ & 2UR \\ VR & VQ \end{bmatrix}$$

and then, taking into account of (3.7) and (3.8) we get that

$$(AB)^*(AB) = \begin{bmatrix} 4Q^2 + R^2 & 0 \\ 0 & Q^2 + 4R^2 \end{bmatrix} = (AB)(AB)^*.$$

On the other hand,

$$A^*A = \begin{bmatrix} 4I_{\mathcal{H}} & 0 \\ 0 & I_{\ell^2} \end{bmatrix}$$

and hence B does not commute with A^*A. By Lemma 2.4, it follows that BA is not normal.

3.3 The Gantmaher-Krein-Wiegmann Theorem

In this section we present the form of those normal and compact operators A and B such that AB (and hence, BA as well, cf. Theorem 3.5) is normal. In order to shed some light on how the geometry of these operators looks like, we first prove a particular case in a more or less straightforward way.

A bounded linear operator A on the Hilbert space $\mathcal{H}$ is *homogeneously normal* if there exists $\rho \geq 0$ such that $A^*A = AA^* = \rho^2 I$, equivalently, for some unitary operator $V \in \mathcal{B}(\mathcal{H})$ we have $A = \rho V$.

Proposition 3.7. *Let A be a normal compact operator and U be a unitary operator on the same Hilbert space $\mathcal{H}$. The following assertions are equivalent:*

(i) AU is normal.

(ii) UA is normal.

(iii) There exists an at most countable family of mutually orthogonal subspaces $\{\mathcal{H}_i\}_{i\in\mathcal{J}}$ of $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_{i\in\mathcal{J}}\mathcal{H}_i$, for all $i \in \mathcal{J}$ the subspace $\mathcal{H}_i$ reduces both A and U and, in addition, $A|\mathcal{H}_i$ is a homogeneously normal operator.

Proof: Let us assume that $C = AU$ is normal. Then $C^*C = CC^*$ and taking into account that U is unitary, it follows that $U^*A^*AU = AUU^*A^* = AA^*$, equivalently, $A^*AU = UAA^* = UA^*A$ (since A is normal as well). Let $\{\mathcal{H}_n\}_{n=0}^N$ (where N is either a nonnegative integer number or the symbol $+\infty$) be the spectral subspaces corresponding to A^*A and the singular numbers $(s_n(A))_{n=0}^N$. Each $\mathcal{H}_n$ reduces both A and U (the latter because U commutes with A^*A). For all admissible $n = 0, 1, \ldots$ the operator $A_n = A|\mathcal{H}_n$ in $\mathcal{B}(\mathcal{H}_n)$ has the property $\sigma(A_n) \subseteq \{\lambda \mid |\lambda| = s_n(A)\}$, equivalently, $A_n^*A_n = s_n(A)^2 I_n$ (here I_n denotes the identity operator on $\mathcal{H}_n$) and hence A_n is homogeneously normal. Since $\ker(A) = \ker(A^*A)$ is reducing both A and U, the conclusion follows.

If UA is normal then we apply the above procedure to A^*. □

The reason we prove Proposition 3.7 is because it indicates one way of approaching the proof of the general theorem from below: we look at the four operators $|A|$, $|B|$, U, and V and try to apply a "spectral surgery" by decomposing and then synthesizing back the operators A and B. This idea is used in all articles [6], [19], and [20]. Our approach uses instead a shortcut taken by "glueing" together $|A|$ and $|B|$ in a single normal operator, and then using the power of the spectral theory for normal operators, in order to obtain the required representations of A and B.

Theorem 3.8 (N.A. Wiegmann [20])**.** *Let A and B be normal compact operators on a Hilbert space $\mathcal{H}$. The following assertions are equivalent:*

(i) *AB is normal.*

(ii) *BA is normal.*

(iii) *There exists an at most countable family of mutually orthogonal subspaces $\{\mathcal{H}_i\}_{i\in\mathcal{J}}$ of $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_{i\in\mathcal{J}}\mathcal{H}_i$, the subspace $\mathcal{H}_i$ reduces both A and B and, in addition, $A|\mathcal{H}_i$ and $B|\mathcal{H}_i$ are homogeneously normal operators, for all $i \in \mathcal{J}$.*

Proof: (i)⇔(ii) This is a consequence of Theorem 3.5.

(i)⇒(iii) Assume that A and B are normal and compact operators such that AB is normal, hence BA is the same. As recalled at the beginning of Section 2, we have $A = U|A|$ where $U \in \mathcal{B}(\mathcal{H})$ is unitary and commutes with $|A| = (A^*A)^{1/2}$ and, similarly, $B = V|B|$ where $V \in \mathcal{B}(\mathcal{H})$ is unitary and commutes with $|B|$. If AB and BA are normal then, by Lemma 2.4, it follows that $|A|$ commutes with $|B|$, hence the operator

$$N = |A| + \mathrm{i}|B|$$

is normal and compact. By the Spectral Theorem for compact and normal operators, see (3.4), we have

$$N = \sum_{n=1}^{\infty} \lambda_n(N) P_n,$$

where $\{P_n\}_{n=1}^{\infty}$ is a mutually orthogonal family of orthogonal projections in $\mathcal{H}$. Let $\mathcal{R}_n$ be the range of P_n and $\mathcal{R}_0 = \mathcal{H} \ominus \bigoplus_{j=1}^{N} \mathcal{R}_j$. Then

$$|A| = \mathrm{Re}(N) = \sum_{n=1}^{\infty} s_n(A) P_n, \quad |B| = \mathrm{Im}(N) = \sum_{n=1}^{\infty} s_n(B) P_n. \tag{3.9}$$

On the other hand, since both AB and BA are normal, by Lemma 2.4 it follows that A commutes with $|B|$, hence $|A|$ commutes with $|B|$, and therefore the unitary operator U commutes with $|B|$. Thus, U commutes with N, hence it commutes with all the orthogonal projections P_n, equivalently, U leaves invariant all the subspaces $\mathcal{R}_j$ for $j = 0, 1, 2, \ldots$. Equivalently,

$$U = \bigoplus_{n=0}^{\infty} U_n, \tag{3.10}$$

where U_n is a unitary operator in the space $\mathcal{R}_n$, for all $n = 0, 1, 2, \ldots$. Similarly, V commutes with N and hence

$$V = \bigoplus_{n=0}^{\infty} V_n, \tag{3.11}$$

where V_n is a unitary operator in the space $\mathcal{R}_n$, for all $n = 0, 1, 2, \ldots$.

Finally, from (3.9), (3.10), and (3.11), we get

$$A = \bigoplus_{n=1}^{\infty} s_n(A) U_n, \quad B = \bigoplus_{n=1}^{\infty} s_n(A) V_n,$$

which clearly is equivalent with the assertion (iii).

(iii)⇒(i) This is clear. ∎

4 Non-Compact Normal Operators

In this section we present a generalization of the Gantmaher-Krein-Wiegmann Theorem for the case when at least one, or both, of the normal operators A and B are not compact. As a consequence of the example of I. Kaplansky, see Example 3.6, in this general case we assume that both AB and BA are normal. On the other hand, in this general case, the Spectral Theorem for normal operators alone is not enough and we had to use the theory of decomposable operators with respect to direct integral Hilbert spaces. This theory is actually a by-product of the monumental work of von Neumann on the reduction of von Neumann algebras to factors [12]. In order to get the generalization of Gantmaher-Krein-Wiegmann Theorem, we need to have available the basic terminology and facts in the von Neumann's direct integral representation of the spectral multiplicity theory for normal operators. Details can be found in [4], [1], and [8]. To simplify things, the direct integrals are considered only on compact sets in the complex plane, even though the general theory works for Polish spaces (complete, separable, metric spaces) as well.

4.1 Direct Integrals of Hilbert Spaces

Throughout this section X will be a compact set in the complex plane and all the Hilbert spaces will be separable. A *field of Hilbert spaces* on X is a family of (separable) Hilbert spaces indexed by X, more precisely, for all $x \in X$, $(\mathcal{H}_x, \langle \cdot, \cdot \rangle_x)$ is a separable Hilbert space. Let $\mathcal{F}$ be the set of all *fields of vectors*, that is, functions $f\colon X \to \bigcup_{x \in X} \mathcal{H}_x$ subject to the property that for all $x \in X$ we have $f(x) \in \mathcal{H}_x$. $\mathcal{F}$ has a natural structure of vector space. Let μ be a probability (Borel) measure on X and let

$$\mathcal{N} = \{f \in \mathcal{F} \mid f = 0 \ \mu\text{-a.e.}\}. \tag{4.12}$$

$\mathcal{N}$ is a subspace of $\mathcal{F}$ and in the following we consider the quotient space $\mathcal{F}/\mathcal{N}$. The elements in $\mathcal{F}/\mathcal{N}$ can also be considered as elements in $\mathcal{F}$, defined μ-a.e.

The field of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}$ is called *measurable* if there exists a distinguished subspace $\mathcal{S} \subseteq \mathcal{F}$ such that

(i) For every $f \in \mathcal{S}$ the function $X \ni x \mapsto \|f(x)\|$ is μ-measurable.

(ii) If $g \in \mathcal{F}$ has the property that for all $f \in \mathcal{S}$ the function $X \ni x \mapsto \langle f(x), g(x) \rangle$ is μ-measurable, then $g \in \mathcal{S}$.

(iii) There exists a countable set $\mathcal{P} \subset \mathcal{S}$ such that for all $x \in X$, the set $\{f(x) \mid f \in \mathcal{P}\}$ spans $\mathcal{H}(x)$.

The elements in $\mathcal{S}$ are called *measurable fields of vectors*.

A field $f \in \mathcal{S}$ is called *square integrable* if $\int_X \|f(x)\|^2 \mathrm{d}\mu < \infty$. The set of all μ-equivalence classes (in $\mathcal{F}/\mathcal{N}$) of square integrable measurable fields of vectors, endowed with the usual algebraic operations of addition and multiplication with scalars, and the inner product

$$\langle f, g\rangle = \int_X \langle f(x), g(x)\rangle \mathrm{d}\mu(x),$$

forms a Hilbert space $\mathcal{H}$ denoted by $\int_X^{\oplus} \mathcal{H}_x \mathrm{d}\mu$ and called a *direct integral* of the field $\{\mathcal{H}_x\}_{x\in X}$. The construction depends to a certain extent upon the subspace $\mathcal{S}$ of measurable fields of vectors, see e.g. [4] for a detailed discussion of this fact, but, traditionally, the notation does not reflect it.

We also recall that, as a consequence of these definitions, we have:

(iv) For any $f, g \in \mathcal{H}$, the scalar function $X \ni x \mapsto \langle f(x), g(x)\rangle$ is in $L^1(\mu)$.

(v) The space $\mathcal{H}$ is an $L^\infty(\mu)$-module, that is, for all $\psi \in L^\infty(\mu)$ and all $f \in \mathcal{H}$ the function ψf is in $\mathcal{H}$.

4.2 Decomposable Operators

Let X be a compact set in the complex plane, μ a probability measure on X, $\{\mathcal{H}_x\}_{x\in X}$ and $\{\widetilde{\mathcal{H}}_x\}_{x\in X}$ two μ-measurable fields of Hilbert spaces with respect to the subspaces $\mathcal{S}$ and, respectively, $\widetilde{\mathcal{S}}$ of μ-measurable vectors. A field $\{T_x\}_{x\in X}$ of operators, such that for all $x \in X$ we have $T_x \in \mathcal{B}(\mathcal{H}_x, \widetilde{\mathcal{H}}_x)$, is called *measurable* if for any $f = (f_x)_{x\in X} \in \mathcal{S}$ we have $(T_x f_x)_{x\in X} \in \widetilde{\mathcal{S}}$.

Let us consider the associated direct integral Hilbert spaces

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x \mathrm{d}\mu(x), \quad \widetilde{\mathcal{H}} = \int_X^{\oplus} \widetilde{\mathcal{H}}_x \mathrm{d}\mu(x),$$

with respect to a subspaces of measurable vectors $\mathcal{S}$ and, respectively, $\widetilde{\mathcal{S}}$. The fields of spaces $\{\mathcal{H}_x\}_{x\in X}$ and $\{\widetilde{\mathcal{H}}_x\}_{x\in X}$ are *isomorphic* if there exists a measurable field of unitary operators $U_x \in \mathcal{B}(\mathcal{H}_x, \widetilde{\mathcal{H}}_x)$, $x \in X$, such that $\widetilde{\mathcal{S}} = \{(U_x f(x))_{x\in X} \mid f \in \mathcal{S}\}$.

A bounded linear operator $A\colon \mathcal{H} \to \widetilde{\mathcal{H}}$ is called *decomposable* if there exists a field of operators $\{A_x\}_{x\in X}$ subject to the following conditions:

(i) For all $x \in X$, $A_x\colon \mathcal{H}_x \to \widetilde{\mathcal{H}}_x$ is a bounded linear operator.

(ii) For any measurable field $f = \{f_x\}_{x\in X} \in \mathcal{S}$ we have $\{A_x f_x\}_{x\in X} \in \widetilde{\mathcal{S}}$.

(iii) $\mu\text{-ess}\sup_{x\in X} \|A_x\| < +\infty$.

(iv) For all $f = \in \{f_x\}_{x\in X}\mathcal{H}$, we have $Af = [\{A_x f_x\}_{x\in X}]_\mu \in \widetilde{\mathcal{H}}$, that is, Af coincides with $\{A_x f_x\}_{x\in X}$ μ-a.e.

In this case, the notation

$$A = \int_X^{\oplus} A_x \mathrm{d}\mu(x),$$

is used and we also have $\|A\| = \mu\text{-ess}\sup_{x\in X} \|A_x\|$.

Note that decomposable operators are defined up to μ-negligible sets, that is, given two decomposable operators $A, A' \in \mathcal{B}(\mathcal{H}, \widetilde{\mathcal{H}})$

$$A = \int_X^{\oplus} A_x \mathrm{d}\mu(x), \quad A' = \int_X^{\oplus} A'_x \mathrm{d}\mu(x),$$

then $A = A'$ if and only the corresponding fields of operators $\{A_x\}_{x\in X}$ and $\{A'_x\}_{x\in X}$ coincide μ-a.e.

Decomposable operators have good algebraic properties. If $A, B \in \mathcal{B}(\mathcal{H}, \widetilde{\mathcal{H}})$ are decomposable

$$A = \int_X^{\oplus} A_x \mathrm{d}\mu(x), \quad B = \int_X^{\oplus} B_x \mathrm{d}\mu(x),$$

then $A + B$, λA, and A^* are also decomposable and

$$A + B = \int_X^{\oplus} (A_x + B_x) \mathrm{d}\mu(x),$$

$$\lambda A = \int_X^{\oplus} \lambda A_x \mathrm{d}\mu(x), \quad A^* = \int_X^{\oplus} A_x^* \mathrm{d}\mu(x),$$

where λ is an arbitrary complex number. Moreover, if another direct integral Hilbert space $\mathcal{K} = \int_X^{\oplus} \mathcal{K}_x \mathrm{d}\mu(x)$ is defined and $C = \int_X^{\oplus} C_x \mathrm{d}\mu(x) \in \mathcal{B}(\widetilde{\mathcal{H}}, \mathcal{K})$ is decomposable, then $CA \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is decomposable and

$$CA = \int_X^{\oplus} C_x A_x \mathrm{d}\mu(x).$$

In particular, the collection of all decomposable operators $A \in \mathcal{B}(\mathcal{H})$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

4.3 Diagonalizable Operators

Assume that X is a compact subset in the complex plane and let μ be a probability measure on X. We consider the W^*-algebra $L^\infty(X;\mu)$ of μ-essentially bounded measurable complex valued functions f defined on X, and the C^*-algebra $C(X)$ of continuous complex valued functions f defined on X. As before, we consider a direct integral Hilbert space $\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x \mathrm{d}\mu(x)$, with respect to a space $\mathcal{S}$ of μ-measurable fields of vectors.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called *(continuously) diagonalizable* if there exists $f \in L^\infty(X;\mu)$ (respectively, $f \in C(X)$) such that

$$T = \int_X^{\oplus} f(x) I_x \mathrm{d}\mu(x),$$

where I_x denotes the identity operator on $\mathcal{H}_x$ for all $x \in X$. It is clear that any diagonalizable operator is decomposable. The collection of all diagonalizable operators is an Abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, while the collection of all continuously diagonalizable operators is a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H}))$ which is weakly dense in the von Neumann algebra of all diagonalizable operators. These operators provide an algebraic characterization of decomposable operators.

Theorem 4.1. *Let us consider two direct integral Hilbert spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$*

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x \mathrm{d}\mu(x), \quad \widetilde{\mathcal{H}} = \int_X^{\oplus} \widetilde{\mathcal{H}}_x \mathrm{d}\mu(x),$$

over the space X with respect to a probability measure μ on X and let $A \in \mathcal{B}(\mathcal{H}, \widetilde{\mathcal{H}})$. The following assertions are equivalent:

(i) A is decomposable.

(ii) For any $f \in C(X)$, we have

$$A \int_X^{\oplus} f(x) I_x \mathrm{d}\mu(x) = \int_X^{\oplus} f(x) \widetilde{I}_x \mathrm{d}\mu(x)\, A \tag{4.13}$$

where $\widetilde{I}_x$ denotes the identity operator on $\widetilde{\mathcal{H}}_x$ for all $x \in X$.

(iii) The formula (4.13) *holds for all $f \in L^\infty(X;\mu)$.*

4.4 The Spectral Multiplicity Theorem in Direct Integral Representation

Let $\mathcal{H}$ be a separable Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal. Then, by the Spectral Theorem for Normal Operators there exists the spectral measure E with compact support $\sigma(N)$, as in Spectral Theorem for Normal Operators. Let μ be a *scalar spectral measure* of N, that is, a probability measure on the compact space $\sigma(N)$ such that for arbitrary Borel set B in $\sigma(N)$ we have $E(B) = 0$ if and only if $\mu(B) = 0$. The existence of a scalar spectral measure for any normal operator N is guaranteed by the existence of separating vectors e for the von Neumann algebra $W^*(N) = \{\phi(N) \mid \phi\colon \sigma(N) \to \mathbb{C}$ bounded and Borel$\}$; the measure $\mu_e(\cdot) = \langle E(\cdot)e, e\rangle$ is a scalar spectral measure for N.

Theorem 4.2 (First Part of The Spectral Multiplicity Theorem)**.** *Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator on the separable Hilbert space $\mathcal{H}$ and μ a scalar spectral measure of N. Then, there exists a μ-measurable field of Hilbert spaces $\{\mathcal{H}_\lambda\}_{\lambda\in\sigma(N)}$ such that, modulo a unitary identification of $\mathcal{H}$ with $\int_{\sigma(N)}^{\oplus} \mathcal{H}_\lambda \mathrm{d}\mu_\lambda$, we have*

$$(Nf)(\lambda) = \lambda f(\lambda), \text{ for all } f \in \int_{\sigma(N)}^{\oplus} \mathcal{H}_\lambda \mathrm{d}\mu(\lambda), \text{ and all } \lambda \in \sigma(N),$$

or, equivalently, N coincides with the diagonalizable operator

$$\int_{\sigma(N)}^{\oplus} \lambda I_\lambda \mathrm{d}\mu(\lambda).$$

Recall also that the function $m\colon \sigma(N) \to \mathbb{N}\cup\{\infty\}$ defined by $m(\lambda) = \dim \mathcal{H}_\lambda$ is called the *multiplicity* of the normal operator N. A second part of The Spectral Multiplicity Theorem for Normal Operators says that the triple $(\sigma(N); [\mu]; [m]_\mu)$ is a *complete set of unitary invariants for normal operators*, where $[\mu]$ denotes the class of absolute continuity of μ and $[m]_\mu$ denotes the class of functions that coincide μ-a.e. with the function m. We will not use this second part.

4.5 A Generalization of the Gantmaher-Krein-Wiegmann Theorem

Recall that a bounded linear operator A on the Hilbert space $\mathcal{H}$ is called *homogeneously normal* if there exists $\rho \geq 0$ such that $A^*A = AA^* = \rho^2 I$, equivalently, for some unitary operator $V \in \mathcal{B}(\mathcal{H})$ we have $A = \rho V$.

Theorem 4.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two normal operators on a separable Hilbert space $\mathcal{H}$. The following assertions are equivalent:*

(i) The operators AB and BA are normal.

(ii) There exists a probability measure μ on a compact subset X in the complex plane such that $\mathcal{H}$ has a direct integral representation

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}_\lambda \mathrm{d}\mu(\lambda), \tag{4.14}$$

with respect to which both A and B are direct integrals of μ-measurable fields of homogeneously normal operators, more precisely, A and B have the following representations

$$A = \int_X^{\oplus} f(\lambda) U_\lambda \mathrm{d}\mu(\lambda), \quad B = \int_X^{\oplus} g(\lambda) V_\lambda \mathrm{d}\mu(\lambda), \tag{4.15}$$

where $f, g \in L^\infty(X;\mu)$ and the operator fields U_λ and V_λ are unitary μ-a.e.

The proof follows the same trick with the normal operator N as in (4.12), but then the machinary of the spectral multiplicity theory of normal operators, described in previous subsections, is used.

Proof of Theorem 4.3. (ii)$\Rightarrow$(i). This implication is a consequence of the multiplication property for decomposable operators as explained in Subsection 4.2.

(i)$\Rightarrow$(ii). Let A and B be normal operators. As recalled in Proposition 2.1, we have $A = U|A|$ where $U \in \mathcal{B}(\mathcal{H})$ is unitary and commutes with $|A| = (A^*A)^{1/2}$ and, similarly, $B = V|B|$ where $V \in \mathcal{B}(\mathcal{H})$ is unitary and commutes with $|B|$. If AB and BA are normal then, by Lemma 2.4, it follows that $|A|$ commutes with $|B|$, hence the operator

$$N = |A| + \mathrm{i}|B|$$

is normal. Thus, by Theorem 4.2, there exist a Borel measure μ on the compact set $X = \sigma(N)$ in the complex plane, a μ-measurable field $\{\mathcal{H}_\lambda\}$ of Hilbert spaces such that $\mathcal{H} = \int_X^\oplus \mathcal{H}_\lambda \mathrm{d}\lambda$, with respect to which $N = \int_X^\oplus \lambda I_\lambda \mathrm{d}\lambda$. Since $|A|$ and $|B|$ belong to the von Neumann algebra generated by N, both of them are diagonalizable, more precisely,

$$|A| = \int_X^\oplus f(\lambda) I_\lambda \mathrm{d}\lambda, \quad |B| = \int_X^\oplus g(\lambda) I_\lambda \mathrm{d}\lambda, \tag{4.16}$$

for some functions $f, g \in L^\infty(X; \mu)$.

On the other hand, since both AB and BA are normal, by Lemma 2.4 it follows that A commutes with $|B|$, hence $|A|$ commutes with $|B|$, and therefore the unitary operator U commutes with $|B|$. Thus, U commutes with N and hence it commutes with the whole von Neumann algebra generated by N. Consequently, by Theorem 4.1, U is decomposable and hence, there exists a μ-measurable field of unitary operators $\{U_\lambda\}$ such that

$$U = \int_X^\oplus U_\lambda \mathrm{d}\lambda. \tag{4.17}$$

Similarly, there exists a μ-measurable field of unitary operators $\{V_\lambda\}$ such that

$$V = \int_X^\oplus V_\lambda \mathrm{d}\lambda. \tag{4.18}$$

The representations (4.15) follow now from (4.16), (4.17), (4.18), and the multiplication property of decomposable operators as explained in Subsection 4.2. $\square$

Remarks: (a) In Theorem 4.3.(ii) one can choose $X = [0, 1]$ without loss of generality, as follows from [13].

(b) It follows from the proof of the implication (i)⇒(ii) of Theorem 4.3 that $|A|$ and $|B|$ are continuously diagonalizable and hence, in the representation formulas (4.15), one can always choose $f, g \in C(X)$.

(c) There are some particular cases, other than compactness assumptions, when the normality of AB is equivalent with that of BA, e.g. when at least one of the operators A and B are unitary. For example, assuming that B is unitary, this follows from Lemma 2.4, but a simple argument can be used as well: since B is unitary we have $BA = B(AB)B^*$, that is, AB and BA are unitary equivalent, hence they are normal in the same time.

(d) The symmetry of the normality of AB and BA, as seen in Theorem 3.5, holds in the more general case when only AB (equivalently, BA) is supposed to be compact, but the explicit description of the operators A and B as in Theorem 4.3 still requires the direct integral Hilbert space representation, if at least one of A and B is not compact. Otherwise, this description is as in Theorem 3.8, because when both A and B are compact then the probability measure μ is discrete, in this case.

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