ON CONSTRUCTION OF DARBOUX INTEGRABLE DISCRETE MODELS

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The problem of discretization of Darboux integrable equations is considered. Given a Darboux integrable continuous equation, one can obtain a Darboux integrable differential-discrete equation, using the integrals of the continuous equation. In the present paper, the discretization of the differential-discrete equations is done using the corresponding characteristic algebras. New examples of integrable discrete equations are obtained.

Keywords: hyperbolic differential-discrete equations, hyperbolic discrete equations, Darboux integrability, characteristic algebras.

1. Introduction

Darboux integrability is a concept that allows to obtain a general solution of a certain type hyperbolic equations. It was developed in works of Laplace, Darboux, Goursat and other people. The continuous Darboux integrable equations have many applications and are reasonably well studied, see a review paper [21]. In recent years the problem of discretization, preserving integrability property, of such equations has generated a lot of interest. The integrable discrete models play an important role in many areas of physics, see [12]. Construction of such models is also important for general classification problem of discrete integrable systems. In the present paper we developed a new approach for discretization of Darboux integrable equations.

Let us give necessary definitions. A hyperbolic equation

$$w_{xy} = h(w, w_x, w_y) \tag{1}$$

is called Darboux integrable if it admits two nontrivial functions

$$J(w, w_y, w_{yy}, \dots) \quad \text{and} \quad G(w, w_x, w_{xx}, \dots),$$

depending on a finite number of variables, such that for all solutions of (1) we have

$$D_x J = 0 \quad \text{and} \quad D_y G = 0, \tag{2}$$

where D_x is the total x-derivative operator and D_y is the total y-derivative operator. The functions $J(w, w_y, w_{yy}, \dots)$ and $G(w, w_x, w_{xx}, \dots)$ are called x- and y-integrals, respectively. For the detailed discussion of the Darboux integrable equations see [1, 20, 21] and references therein.

First, one can look for a differential-discrete equation which is a discretization of a continuous equation (1). This differential-discrete equation should also be Darboux integrable. The notion of Darboux integrable differential-discrete equation was introduced in [11]. Let us consider an equation

$$t_{1x} = g(t, t_1, t_x), \tag{3}$$

where $t(n,x)$ is a function of a continuous variable x, a discrete variable n and $t_1 = Dt(n,x) = t(n+1,x)$ (D is the shift operator and $D^k t(n,x) = t(n+k,x) = t_k$, $k \in \mathbb{Z}$). Differential-discrete equation (3) is called Darboux integrable if it admits two functions

$$I(t, t_x, t_{xx}, \dots) \quad \text{and} \quad F(\dots, t_{-1}, t, t_1, t_2, t_{-2}, \dots),$$

depending on a finite number of variables, such that for all solutions of (3) we have

$$D_x F = 0 \quad \text{and} \quad DI = I. \tag{4}$$

Such Darboux integrable differential-discrete equations and discrete equations (the definition of Darboux integrable discrete equation is given in the next section) are actively studied nowadays, see [2–7].

It was proposed in [9] to use x- or y- integrals of a continuous equation (1) to obtain its discretization. That is, one looks for a differential-discrete equation that admits a given x- or y-integral as its n-integral. This approach allowed to construct many differential-discrete equations, see [10–19]. Moreover, the constructed equations turned out to admit also an x-integral, that is the equations are Darboux integrable. Now, one can take a constructed differential-discrete equation and consider its further discretization using the corresponding x-integral. However, if one tries to find a discrete equation corresponding to a given integral, one obtains a complicated functional equation to be solved, see [9] for some examples. In many cases the discretization is not found.

The Darboux integrability can be also defined in terms of characteristic algebras, which are Lie–Rinehart algebras, see [13–15]. We propose to use the characteristic algebras for the discretization.

The paper is organized as follows. In Section 2 we give necessary definitions and description of our approach to discretization. In Section 3 we give examples of discretization for differential-discrete Darboux integrable equations.

2. Preliminaries

In what follows we always assume that $t, t_{\pm 1}, t_{\pm 2}, \dots$ and $t_x, t_{xx}, t_{xxx} \dots$ are independent dynamical variables. Derivatives of variables $t, t_{\pm 1}, t_{\pm 2}, \dots$ and shifts of variables $t_x, t_{xx}, t_{xxx}, \ldots$ are expressed in terms of the dynamical variables using (3).

A criteria for the existence of n- and x-integrals of a differential discrete equation (3) can be formulated in terms of the so-called characteristic algebras.

Let us introduce the criteria for the existence of x-integral first. Define an operator

$$Z = t_x \frac{\partial}{\partial t} + t_{1x} \frac{\partial}{\partial t_1} + t_{-1x} \frac{\partial}{\partial t_{-1}} + \ldots, \tag{5}$$

which corresponds to the total derivative operator D_x and an operator

$$W = \frac{\partial}{\partial t_x}. \tag{6}$$

We have that $ZF = 0$ and $WF = 0$. Clearly, the function F is also annulled by all possible commutators of these operators. Thus we define the characteristic x-algebra, denoted by L_x, as a Lie–Rinehart algebra generated by the operators Z and W. The algebra L_x is considered over the ring of functions depending on a finite number of dynamical variables. In general, all algebras and linear spaces of operators introduced in this paper are considered over the ring of functions depending on finite number of dynamical variables.

Theorem 1 ([11]). *Eq.* (3) *admits a nontrivial x-integral if and only if the corresponding characteristic x-algebra L_x is finite-dimensional.*

Now we introduce the criteria for the existence of the n-integral of a differential-discrete equation (3). Following [11] we define an operator

$$Y_0 = \frac{\partial}{\partial t_1} \tag{7}$$

and operators

$$Y_k = D^{-k} \frac{\partial}{\partial t_1} D^k, \qquad k = 1, 2, 3, \ldots, \tag{8}$$

$$X_k = \frac{\partial}{\partial t_{-k}}, \qquad k = 1, 2, \ldots. \tag{9}$$

Theorem 2 ([11]). *Eq.* (3) *admits a nontrivial n-integral if and only if the following conditions are satisfied:*

1. *The linear space generated by the operators $\{Y_k\}_{k=0}^{\infty}$ has a finite dimension. Let us denote the dimension by N.*
2. *The Lie–Rinehart algebra generated by the operators $\{Y_k\}_{k=0}^{N}$ and $\{X_k\}_{k=1}^{N}$ has a finite dimension.*

The characteristic n-algebra, denoted by L_n, is a Lie–Rinehart algebra generated by the operators $\{Y_k\}_{k=0}^{N}$ and $\{X_k\}_{k=1}^{N}$ from the above theorem.

Now, we consider a discrete equation case. Assume that a function $u(n,m)$ depends on two discrete variables n and m. For the function $u(n,m)$ we have the shift operator D such that $Du(n,m) = u(n+1,m) = u_1$, the shift with respect to the first variable, and the shift operator $\bar{D}$, $\bar{D}u(n,m) = u(n,m+1) = u_{\bar{1}}$, the shift with respect to the second variable. Note that $D^k u(n,m) = u(n+k,m) = u_k$ and $\bar{D}^k u(n,m) = u(n,m+k) = u_{\bar{k}}$, $k \in \mathbb{Z}$. We study a discrete equation

$$u_{1\bar{1}} = f(u, u_1, u_{\bar{1}}). \tag{10}$$

In what follows we always assume that $u, u_{\pm 1}, u_{\pm 2}, \dots$ and $u_{\pm\bar{1}}, u_{\pm\bar{2}}, \dots$ are independent dynamical variables. Also D^k shifts the variables $u_{\bar{1}}, u_{\bar{2}}, \dots$ and $\bar{D}^k$ shifts the variables $u_1, u_2, \dots$ which are expressed in terms of the dynamical variables using (10).

A sequence of functions $\{J_k(n, u_{-j}, \dots, u_r)\}_{k=-\infty}^{\infty}$, depending on finite number of dynamical variables $u_{-j}, \dots, u_r$, is called an m-integral for a discrete equation (10) if $\bar{D}J_i = J_{i+1}$, $\quad i \in \mathbb{Z}$. Note that a shift of an m-integral $\{D^p J_k(n, u_{-j}, \dots, u_r)\}_{k=-\infty}^{\infty}$, where p is fixed, is also an m-integral. The notion of an n-integral for a discrete equation (10) is defined in a similar way. Eq. (10) is called Darboux integrable if it admits nontrivial m- and n-integrals. A criteria for existence of m- and n-integrals can be formulated in terms of the so-called characteristic algebras. We consider the existence criteria for the m-integral (for the n-integral the corresponding criteria is formulated in a similar way). Following [21] we define operators

$$\tilde{Y}_0 = \frac{\partial}{\partial u_{\bar{1}}} \tag{11}$$

and

$$\tilde{Y}_k = \bar{D}^{-k} \frac{\partial}{\partial u_{\bar{1}}} \bar{D}^k, \qquad k = 1, 2, \dots, \tag{12}$$

$$\tilde{X}_k = \frac{\partial}{\partial u_{-k}}, \qquad k = 1, 2, \dots. \tag{13}$$

Theorem 3 ([21]). *Eq.* (10) *admits a nontrivial n-integral if and only if the following conditions are satisfied:*

1. *The linear space generated by the operators $\{\tilde{Y}_k\}_{k=0}^{\infty}$ has a finite dimension. Let us denote the dimension by $\tilde{N}$.*
2. *The Lie–Rinehart algebra generated by the operators $\{\tilde{Y}_k\}_{k=0}^{\tilde{N}}$ and $\{\tilde{X}_k\}_{k=1}^{\tilde{N}}$ has a finite dimension.*

The Lie–Rinehart algebra generated by the operators $\{\tilde{Y}_k\}_{k=0}^{\tilde{N}}$ and $\{\tilde{X}_k\}_{k=1}^{\tilde{N}}$ from the above theorem is called the characteristic m-algebra, denoted by $\tilde{L}_m$.

Given a Darboux integrable differential-discrete equation (3) we have an x-integral $F(t, t_1, \dots, t_j)$. We would like to find a discrete equation (10) that admits same function F as its m-integral. In general, a function F generates an m-integral for

Eq. (10) if on all solutions of Eq. (10) we have

$$\bar{D}F = h(F), \tag{14}$$

for some function h, see [21]. Let us assume that $\bar{D}F = F$, for simplicity. The equality $\bar{D}F = F$ in general gives a complicated equation for the function f in (10), namely

$$F(u, u_1, \dots, u_j) = I(u_{\bar{1}}, u_{1\bar{1}}, \dots, u_{j\bar{1}}) = I(u_{\bar{1}}, f, f(u_1, u_2, f), \dots), \tag{15}$$

that is not easy to solve.

To find the discrete equation we propose to use characteristic algebras. Operators of the characteristic algebra L_x of a given differential-discrete equation and operators of characteristic algebra $\tilde{L}_m$ of the corresponding discrete equation annul the same function F. The operator $\tilde{Y}_1 \in \tilde{L}_m$ has the form (see [21])

$$\tilde{Y}_1 = \frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial u_1} + \frac{1}{\alpha_{-1}} \frac{\partial}{\partial u_{-1}} + \cdots, \tag{16}$$

where

$$\alpha = \bar{D}^{-1} \frac{\partial}{\partial u_{\bar{1}}} f. \tag{17}$$

We assume that the operator $\tilde{Y}_1$ can be identified with the operator

$$[X, Z] = \frac{\partial}{\partial t} + \frac{\partial t_{1x}}{\partial t_x} \frac{\partial}{\partial t_1} + \frac{\partial t_{-1x}}{\partial t_x} \frac{\partial}{\partial t_{-1}} + \cdots, \tag{18}$$

$[X, Z] \in L_x$ (note that $\dfrac{\partial t_{1x}}{\partial t_x} = \left(\dfrac{\partial t_{-1x}}{\partial t_x}\right)^{-1}$). Thus, the coefficient $\alpha = \bar{D}^{-1} \dfrac{\partial}{\partial u_{\bar{1}}} f$ is identified with the coefficient $\dfrac{\partial}{\partial t_x} g$. So, if we take function $\dfrac{\partial}{\partial t_x} g(t, t_1, t_x)$ and replace the variables as follows $t = u_{\bar{1}}$, $t_1 = u_{1\bar{1}}$ and $t_x = A(u, u_1)$ (the function A to be found later) we obtain an equation for $\dfrac{\partial}{\partial u_{\bar{1}}} f$,

$$\frac{\partial f}{\partial u_{\bar{1}}} = \frac{\partial}{\partial t_x} g(t, t_1, t_x)|_{t=u_{\bar{1}},\, t_1=u_{1\bar{1}},\, t_x=A(u,u_1)}. \tag{19}$$

The above equation determines the function f up to some unknown functions of u, u_1. The unknown functions can be found using (14).

3. Examples

In this section we consider the discretization of several differential-discrete Darboux integrable equations. To our knowledge the obtained Darboux integrable discrete equations (25), (36) and (45) are new.

EXAMPLE 1. Consider a differential-discrete equation

$$t_{1x} = \frac{t_1 + 1}{t + 1} t_1 t_x \tag{20}$$

with an x-integral $F = \dfrac{(t_1 + 1)(t + 1)}{t_1}$ (see [16]). We are looking for a discrete equation (10) with a corresponding m-integral

$$\tilde{F} = \frac{(u_1 + 1)(u + 1)}{u_1}. \tag{21}$$

We assume that $u_{1\bar{1}} = f(u, u_1, u_{\bar{1}})$ satisfies Eq. (19), so we have

$$\frac{\partial u_{1\bar{1}}}{\partial u_{\bar{1}}} = \frac{u_{1\bar{1}} + 1}{u_{\bar{1}} + 1} u_{1\bar{1}}. \tag{22}$$

The solution of the above equation is

$$\frac{(u_{1\bar{1}} + 1)(u_{\bar{1}} + 1)}{u_{1\bar{1}}} = A(u, u_1). \tag{23}$$

Now we find $A(u, u_1)$ using the integral (21). Assuming that the equality $\bar{D}\tilde{F} = \tilde{F}$ holds we find $A = \dfrac{(u_1 + 1)(u + 1)}{u_1}$. Thus we obtain a discrete equation

$$\frac{(u_{1\bar{1}} + 1)(u_{\bar{1}} + 1)}{u_{1\bar{1}}} = \frac{(u_1 + 1)(u + 1)}{u_1}, \tag{24}$$

or

$$u_{1\bar{1}} = \frac{u_1(u_{\bar{1}} + 1)}{1 + u + uu_1 - u_1 u_{\bar{1}}} \tag{25}$$

with the m-integral (21). One can also find an n-integral for the given equation

$$\tilde{G} = \left(\frac{1 + (u_{\bar{1}})^{(-1)^n}}{1 + u^{(-1)^n}} \right)^{(-1)^n}. \tag{26}$$

Hence, Eq. (25) is Darboux integrable.

REMARK 1. In general, for equations of the form

$$t_{1x} = K(t, t_1) t_x \tag{27}$$

one has an x-integral of the form $F(t, t_1)$, where the function F satisfies the equation

$$F_t + K(t, t_1) F_{t_1} = 0, \tag{28}$$

see [9]. For such equations our approach leads to an obvious general discrete equation

$$F(u, u_1) = F(u_{\bar{1}}, u_{1\bar{1}}) \tag{29}$$

that admits an m-integral $F(u, u_1)$. The existence of n-integrals for such equations requires further investigation. Some results on a similar classification problem can be found in [6].

Example 2. Consider a differential-discrete equation

$$t_{1x} = \frac{t_1}{t} t_x + t_1^2 + t t_1 \tag{30}$$

with an x-integral $F = \left(1 + \frac{t_1}{t_2}\right)\left(1 + \frac{t_1}{t}\right)$ (see [16]). We are looking for a discrete equation (10) with a corresponding m-integral

$$\tilde{F} = \left(1 + \frac{u_1}{u_2}\right)\left(1 + \frac{u_1}{u}\right). \tag{31}$$

We assume that $u_{1\bar{1}} = f(u, u_1, u_{\bar{1}})$ satisfies Eq. (19), so

$$\frac{\partial u_{1\bar{1}}}{\partial u_{\bar{1}}} = \frac{u_{1\bar{1}}}{u_{\bar{1}}}. \tag{32}$$

The solution of the above equation is

$$u_{1\bar{1}} = A(u, u_1) u_{\bar{1}}. \tag{33}$$

Now we find $A(u, u_1)$ using the integral (31). Assuming that the equality $\bar{D}\tilde{F} = \tilde{F}$ holds, we find $A = \frac{u_1}{u}$. Thus, we obtain a discrete equation

$$u_{1\bar{1}} = \frac{u_1}{u} u_{\bar{1}}. \tag{34}$$

Eq. (34) has the m-integral (31) and an n-integral $\tilde{G} = \frac{u_{\bar{1}}}{u}$. Hence, Eq. (34) is Darboux integrable.

If we assume that the equality

$$\bar{D}\tilde{F} = \tilde{F}^{-1} \tag{35}$$

holds then we find

$$A = -\left(\frac{|-1 + (-1)^n|\, u + |1 + (-1)^n|\, u_1}{2(u + u_1)}\right)^{(-1)^n}.$$

Thus we obtain a discrete equation

$$u_{1\bar{1}} = -\left(\frac{|-1 + (-1)^n|\, u + |1 + (-1)^n|\, u_1}{2(u + u_1)}\right)^{(-1)^n} u_{\bar{1}}. \tag{36}$$

Eq. (36) has the m-integral (31) and an n-integral $\tilde{G} = \frac{u_{\bar{2}}}{u}$. Hence, Eq. (36) is Darboux integrable.

Example 3. Consider a differential-discrete equation

$$t_{1x} = t_x + t_1^2 - t^2 \tag{37}$$

with an x-integral

$$F = \frac{(t - t_2)(t_1 - t_3)}{(t - t_3)(t_1 - t_2)} \tag{38}$$

(see [2]). We are looking for a discrete equation (10) with m-integral

$$\tilde{F} = \frac{(u - u_2)(u_1 - u_3)}{(u - u_3)(u_1 - u_2)}. \tag{39}$$

We assume that $u_{1\bar{1}} = f(u, u_1, u_{\bar{1}})$ satisfies Eq. (19), so

$$\frac{\partial u_{1\bar{1}}}{\partial u_{\bar{1}}} = 1. \tag{40}$$

The solution of the above equation is

$$u_{1\bar{1}} = u_{\bar{1}} + A(u, u_1). \tag{41}$$

We find $A(u, u_1)$ using the integral (39). Assuming that the equality $\bar{D}\tilde{F} = \tilde{F}$ holds we find $A = \frac{1}{u} - \frac{1}{u_1}$. Thus we obtain a discrete equation

$$u_{1\bar{1}} = u_{\bar{1}} + \frac{1}{u} - \frac{1}{u_1} \tag{42}$$

with m-integral (39). One can also find an n-integral for the given equation $\tilde{G} = u_{\bar{1}} + \frac{1}{u}$. Hence, Eq. (42) is Darboux integrable.

Example 4. Consider a differential-discrete equation

$$t_{1x} = (1 + Re^{t+t_1})t_x + \sqrt{R^2 e^{2(t+t_1)} + 2Re^{t+t_1}}\sqrt{t_x^2 - 4} \tag{43}$$

with an x-integral

$$F_1 = \sqrt{Re^{2t_1} + 2e^{t_1 - t}} + \sqrt{Re^{2t_1} + 2e^{t_1 - t_2}} \tag{44}$$

(see [9]). The corresponding discrete equation is given by

$$e^{-u_{1\bar{1}}} = e^{-u_{\bar{1}}}\left(e^u\sqrt{e^{-u-u_1} + \frac{R}{2}} + e^u\sqrt{\frac{R}{2}}\right)^2 + \sqrt{2R}\left(e^u\sqrt{e^{-u-u_1} + \frac{R}{2}} + e^u\sqrt{\frac{R}{2}}\right) \tag{45}$$

Eq. (45) admits the m-integral

$$\tilde{F}_1 = \sqrt{Re^{2u_1} + 2e^{u_1 - u}} + \sqrt{Re^{2u_1} + 2e^{u_1 - u_2}}\,. \tag{46}$$

One can also find an n-integral for the given equation

$$\tilde{G}_1 = \frac{e^{-u} + e^{-u_{\bar{1}}}}{e^{-u_{\bar{1}}} + e^{-u_{\bar{2}}}}. \tag{47}$$

Hence, Eq. (45) is Darboux integrable. The derivation of this example is given below.

Now, let us derive the discrete equation given in Example 4. We are looking for a discrete equation (10) which is a discretization of (43). Since

$$\frac{\partial}{\partial t_x}\left((1+Re^{t+t_1})t_x+\sqrt{R^2e^{2(t+t_1)}+2Re^{t+t_1}}\sqrt{t_x^2-4}\right)$$
$$=(1+Re^{t+t_1})+\sqrt{R^2e^{2(t+t_1)}+2Re^{t+t_1}}\,\frac{t_x}{\sqrt{t_x^2-4}}, \quad (48)$$

we assume that

$$\frac{\partial f}{\partial u_{\bar 1}}=1+Re^{u_{\bar 1}+f}+T(u,u_1)\sqrt{R^2e^{2(u_{\bar 1}+f)}+2Re^{u_{\bar 1}+f}}, \quad (49)$$

where T is a function of u, u_1. By solving (49) we get

$$\sqrt{e^{-f-u_{\bar 1}}+\frac{R}{2}}=e^{-u_{\bar 1}}E(u,u_1)+C(u,u_1). \quad (50)$$

Since $u_{1\bar 1}=f$, we have

$$\sqrt{e^{-u_{1\bar 1}-u_{\bar 1}}+\frac{R}{2}}=e^{-u_{\bar 1}}E(u,u_1)+C(u,u_1), \quad (51)$$

or

$$e^{-u_{1\bar 1}}=e^{-u_{\bar 1}}E^2(u,u_1)+e^{u_{\bar 1}}\left(C^2(u,u_1)-\frac{R}{2}\right)+2E(u,u_1)C(u,u_1). \quad (52)$$

To find the functions E, C we use the equality $\bar D F_1=F_1$. We have

$$e^{u_{1\bar 1}}\sqrt{e^{-u_{1\bar 1}-u_{\bar 1}}+\frac{R}{2}}+e^{u_{1\bar 1}}\sqrt{e^{-u_{1\bar 1}-u_{2\bar 1}}+\frac{R}{2}}$$
$$=e^{u_1}\sqrt{e^{-u_1-u}+\frac{R}{2}}+e^{u_1}\sqrt{e^{-u_1-u_2}+\frac{R}{2}}. \quad (53)$$

Using (51) we can write

$$\sqrt{e^{-u_{2\bar 1}-u_{1\bar 1}}+\frac{R}{2}}=e^{-u_{1\bar 1}}E(u_1,u_2)+C(u_1,u_2) \quad (54)$$

and rewrite (53) as

$$e^{u_{1\bar 1}}(e^{-u_{\bar 1}}E(u,u_1)+C(u,u_1))+e^{u_{1\bar 1}}(e^{-u_{1\bar 1}}E(u_1,u_2)+C(u_1,u_2))$$
$$=e^{u_1}\sqrt{e^{-u_1-u}+\frac{R}{2}}+e^{u_1}\sqrt{e^{-u_1-u_2}+\frac{R}{2}}. \quad (55)$$

By differentiating (55) with respect to $u_{\bar{1}}$ we get

$$\frac{\partial u_{1\bar{1}}}{\partial u_{\bar{1}}} = \frac{e^{-u_{\bar{1}}}E(u,u_1)}{e^{-u_{\bar{1}}}E(u,u_1) + C(u,u_1) + C(u_1,u_2)}, \tag{56}$$

which implies that

$$u_{1\bar{1}} = -\ln(e^{-u_{\bar{1}}}E(u,u_1) + C(u,u_1) + C(u_1,u_2)) + \hat{K}(u,u_1) \tag{57}$$

or

$$e^{-u_{1\bar{1}}} = K(u,u_1)(e^{-u_{\bar{1}}}E(u,u_1) + C(u,u_1) + C(u_1,u_2)). \tag{58}$$

Now comparing (58) with (52) we get $C(u,u_1) = \pm\sqrt{\frac{R}{2}}$ and $K(u,u_1) = E(u,u_1)$. Thus,

$$e^{-u_{1\bar{1}}} = E^2(u,u_1)e^{-u_{\bar{1}}} \pm \sqrt{2R}E(u,u_1). \tag{59}$$

By substituting the above expression for $e^{-u_{1\bar{1}}}$ into (55) and differentiating (55) with respect to u_2 we get

$$\frac{\partial E(u_1,u_2)}{\partial u_2} = \frac{\partial}{\partial u_2}\left(e^{u_1}\sqrt{e^{-u_1-u_2} + \frac{R}{2}}\right), \tag{60}$$

that is

$$E(u_1,u_2) = e^{u_1}\sqrt{e^{-u_1-u_2} + \frac{R}{2}} + M(u_1). \tag{61}$$

So, we rewrite (59) as

$$e^{-u_{1\bar{1}}} = e^{-u_{\bar{1}}}(e^{u}\sqrt{e^{-u-u_1} + \frac{R}{2}} + M(u))^2 \pm \sqrt{2R}e^{u}\sqrt{e^{-u-u_1} + \frac{R}{2}} + M(u). \tag{62}$$

We use (53) and have $M(u) = e^{u}\sqrt{\frac{R}{2}}$. Thus, we obtain (45).

Declaration

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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