EXCEPTIONAL BELYI COVERINGS

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By
Cemile Kürkoğlu
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Alexander Klyachko(Advisor)

Prof. Dr. Ali Sinan Sertöz

Assist. Prof. Dr. Konstyantyn Zheltukhin

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

EXCEPTIONAL BELYI COVERINGS

Cemile Kürkoğlu
M.S. in Mathematics
Advisor: Prof. Dr. Alexander Klyachko
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Exceptional Belyi covering is a connected Belyi covering uniquely determined by its ramification scheme or the respective dessin d'enfant. Well known examples are cyclic, dihedral, and Chebyshev coverings. We add to this list a new infinite series of rational exceptional coverings together with the respective Belyi functions. We shortly discuss the field of definition of a rational exceptional covering and show that it is either $\mathbb{Q}$ or its quadratic extension. Existing theories give no upper bound on degree of the field of definition of an exceptional covering of genus 1. It is an open question whether the number of such coverings is finite or infinite. Maple search for an exceptional covering of genus $g > 1$ found none of degree 18 or less. Absence of exceptional hyperbolic coverings is a mystery we could not explain.

Keywords: exceptional Belyi covering, dessin d'enfant.

ÖZET

ÖZEL BELYİ ÖRTMELERİ

Cemile Kürkoğlu
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Özel Belyi örtmesi ya dallanma şeması ile ya da karşılık gelen "çocuk çizimi" ile tek türlü belirlenmiş bağlantılı Belyi örtmesidir. İyi bilinen örnekleri devirli örtmeler, dihedral örtmeler ve Chebyshev örtmeleridir. Bu listeye, karşılık gelen Belyi fonksiyonları ile birlikte rasyonel tek türlü belirlenmiş örtmelerin sonsuz serilerini ekledik. Rasyonel tek türlü belirlenmiş örtmenin tanım cismini kısaca tartıştık ve gösterdik ki bu cisim ya $\mathbb{Q}$ ya da onun ikinci dereceden bir genişlemesidir. Var olan teoriler cinsi 1 olan tek türlü belirlenmiş bir örtmenin tanım cisminin derecesini sınırlamıyor. Bu tür örtmelerin sayısının sonlu ya da sonsuz olduğu ucu açık bir sorudur. Maple araştırması derecesi 18 ya da daha az ve cinsi $g > 1$ olan tek türlü belirlenmiş örtmeyi bulamadı. Tek türlü belirlenmiş hiperbolik örtmelerin yokluğu, açıklayamadığımız bir gizemdir.

Anahtar sözcükler: özel Belyi örtmesi, çocuk çizimi.

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Chapter 1

Introduction

This thesis introduces the term "exceptional Belyi coverings" which are connected Belyi coverings uniquely determined by the corresponding ramification schemes.

In Chapter 2, we give some elementary information about compact Riemann surfaces and algebraic curves. Then we define what an unramified covering is. Starting from this, we give the definitions of a ramified covering and its ramification points. We then turn back to the topology of Riemann surfaces by introducing fundamental group and monodromy. In this way, we mention the ramification scheme of a covering. Lastly, we state Riemann Existence Theorem.

In Chapter 3, we begin with a definition of a Belyi covering, which is the covering of the Riemann sphere with ramification points belonging in the set $\{0, 1, \infty\}$. After giving some examples to Belyi functions, we state the famous Belyi Theorem. Then, we introduce "dessin d'enfant" which is an embedding of a bicoloured connected graph into a compact topological surface. We state the most important result of this chapter: the one-to-one correspondence between Belyi coverings and dessins d'enfants. In this chapter, we will also explain the action of absolute Galois group on dessins. We define ”rational (of genus 0), elliptic (of genus 1) and hyperbolic (of greater genus)” coverings and finish the chapter by giving some more examples.

In Chapter 4, we want to count Belyi coverings with a given ramification scheme by defining the Eisenstein number and giving the relation of this number to irreducible characters of S_n. Therefore we state basic notions from representation theory too. But before giving a generalized formula for Eisenstein number of coverings, we state the formula for a special case: the corresponding dessins for the polynomial coverings, namely bicoloured trees. Then we give a general formula and end this chapter.

In Chapter 5, we are ready to introduce ”exceptional Belyi coverings” which are uniquely determined either by the respective ramification schemes or the respective dessins d'enfants. We give Klein's coverings: cyclic, dihedral, and coverings of regular polyhedra. We also give the example of Chebyshev covering and then add other infinite series that we found. Then we define this exceptional coverings for the special case: exceptional polynomial coverings and give a classification of them. We limit ourselves to genus 0 case in this chapter. Lastly, we slightly modify the Eisenstein formula for the case of exceptional Belyi coverings. We developed a MAPLE algorithm to obtain coverings of given genus and degree. Moreover, we are able to present a table in APPENDIX A including all rational exceptional coverings up to degree 6 and some of degree 7 with respective ramification schemes, dessins and Belyi functions.

In Chapter 6, we mention elliptic exceptional Belyi coverings. We list this type of coverings up to degree 12 in a table in APPENDIX B with the help of MAPLE using the same code in Chapter 5. However our knowledge is limited in contrast to rational coverings. Thus, our table does not include dessins or the respective Belyi pairs. We also state that the hyperbolic exceptional coveringscannot be found using our MAPLE routines.

In Chapter 7, we introduce the terms "form", "principal homogeneous space", "forms of special algebras (e.g. matrix algebras)", "generalized quaternions". Then we state how they are related, in fact, how those notions are in one-to-one correspondence with each other. We then relate these information to the problem of determining the field of definition of a rational exceptional Belyi covering. Here, we do not go any further for higher genus case.

In Chapter 8, we give a conclusion to sum up our previous discussions and introduce some open questions .

To wrap up, we define exceptional Belyi coverings as Belyi coverings uniquely defined by ramification schemes or by corresponding dessins. In order to understand exceptional Belyi coverings, we introduce general theoretical information about arbitrary Belyi coverings and emphasize the correspondence of them with simple, and at the same time profound objects dessins d'enfants. Then we discuss exceptional coverings of genus 0 and genus 1 case separately. We are able to interpret rational coverings using MAPLE algorithms and to give a detailed table in APPENDIX A. But, we do not have so much to say for the elliptic case. We still introduce a table for genus 1 coverings in APPENDIX B. However, we conclude that MAPLE gives no such hyperbolic covering and this is a real mystery why it is so.

Chapter 2

Riemann Surfaces and Algebraic Curves

We give basic notions from algebraic geometry: we describe the compact Riemann surfaces, their properties and the correspondence of these surfaces with algebraic curves. We also give the definitions of unramified and ramified coverings. Then we describe fundamental group and monodromy. We lastly state the Riemann Existence Theorem. All of these results can be found in [7] and [12]. Since it can be considered as a "preliminaries" chapter, we omit the proofs of theorems and propositions.

2.1 Compact Riemann surfaces

Definition 2.1.1. ***A topological surface*** *S is a Hausdorff topological space provided with a collection $\{\varphi_i : U_i \to \varphi_i(U_i)\}$ of homeomorphisms (called* ***charts****) from open subsets $U_i \subset X$ (called* ***coordinate neighbourhoods****) to open subsets $\varphi_i(U_i) \subset \mathbb{C}$ such that:*

(i) the union $\bigcup_i U_i$ covers the whole space X; and

(ii) whenever $U_i \cap U_j \neq \emptyset$, the transition function

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is a homeomorphism (Figure 2.1).

A collection of charts fulfilling these properties is called a (topological) ***atlas****, and the inverse φ_i^{-1} of a chart is called a* ***parametrization****.*

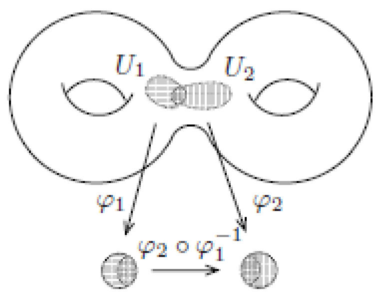


Figure 2.1: The transition function between two coordinate charts

Definition 2.1.2. ***A Riemann surface*** *is a connected topological surface such that the transition functions of the atlas are holomorphic mappings between open subsets of the complex plane $\mathbb{C}$ (rather than mere homeomorphisms).*

Example 2.1.1. *The simplest Riemann surfaces are determined by one chart. For example, every connected open subset U in the plane, the complex plane $\mathbb{C}$ itself, the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the upper halfplane $\mathbb{H} = \{z \in \mathbb{C} : Imz > 1\}$.*

Example 2.1.2. *$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is known as the* ***extended complex plane*** *or the* ***Riemann sphere****. It is a compact, connected Hausdorff topological space. In order to show the Riemann surface structure, we will use the following charts:*

$$U_1 = \mathbb{C},\ \psi_1(z) = z$$

$$U_2 = (\mathbb{C} \cup \{\infty\})\backslash\{0\},\ \psi_2(z) = \begin{cases} 1/z & z \neq \infty \\ 0 & z = \infty \end{cases}$$

So the transition function will be

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}, z \to 1/z$$

Example 2.1.3. *The **unit sphere***

$$\mathbb{S}^2 = \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 + t^2 = 1\}$$

is not homeomorphic to an open subset of the plane. Therefore, a Riemann surface structure cannot be defined on the sphere by a single chart. Considering the charts

$$U_1 = \mathbb{S}^2\backslash\{(0,0,1)\}, \quad \varphi_1(x, y, t) = \frac{x}{1-t} + i\frac{y}{1-t}$$

$$U_2 = \mathbb{S}^2\backslash\{(0,0,-1)\}, \quad \varphi_2(x, y, t) = \frac{x}{1+t} - i\frac{y}{1+t}$$

we can find the transition function which is $\varphi_2 \circ \varphi_1^{-1}(z) = 1/z$*. This function is defined over* $\varphi_1(U_1 \cap U_2) = \mathbb{C}\backslash\{0\}$*. Here* z *is the complex variable variable in* $\varphi_1(U_1)$*.*

Remark 2.1.1. *We can identify* $\mathbb{S}^2$ *with* $\widehat{\mathbb{C}}$ *through stereographic projection with* U_1 *and* U_2 *in Example* 2.1.3*. Therefore the Riemann sphere can also be thought as the unit sphere in the real space* $\mathbb{R}^3$*. From now on we will denote the Riemann sphere by* $\mathbb{P}^1$*.*

Example 2.1.4. ***(A complex torus)*** *Identify every* $w \in \mathbb{C}$ *with its images under all translations by Gaussian integers, that is complex numbers whose real and imaginary parts are both integer numbers. The classes for this equivalence relation are*

$$[w] = \{w + n + mi : n, m \in \mathbb{Z}\}.$$

The corresponding quotient set

$$\mathbb{C}/\Lambda, \ \text{where } \Lambda = \mathbb{Z} \oplus \mathbb{Z}i.$$

This set is a compact, connected Hausdorff space.

If $U \subset \mathbb{C}$ is an open set such that no pair of its points belong to the same equivalence class, that is if the canonical projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is injective when restricted to U, then we define a coordinate chart by $\varphi_U := (\pi|_U)^{-1} : \pi(U) \to U$.

Suppose that two such coordinate neighbourhoods $\pi(U)$ and $\pi(V)$ have non-empty intersection. Then, the transition functions take in each connected component the form

$$\varphi_V \circ \varphi_U^{-1}(z) = z + \lambda, \lambda \in \Lambda$$

and the parallelogram in the figure below can be chosen as the fundamental domain, i.e., a subset of $\mathbb{C}$ containing at least one representative of every equivalence class and exactly one except in the boundary. Opposite sides are identified to form the quotient space, thus $\mathbb{C}/\Lambda$ is topologically a torus. In this way, $\mathbb{C}/\Lambda$ is a compact Riemann surface (Figure 2.2).

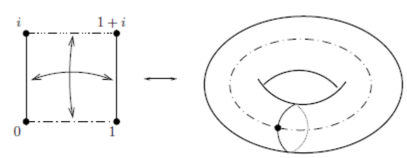


Figure 2.2: $\mathbb{C}/\Lambda$ is topologically a torus.

A compact Riemann surface can also be considered as a complex manifold of dimension 1, or equivalently, a real manifold of dimension two. Moreover, it is oriented.

Definition 2.1.3. *Let S be a compact Riemann surface. A holomorphic mapping $f : S \to \mathbb{P}^1$ is called a* ***meromorphic function*** *on S. For some point $x \in S$, if $f(x) = 0$, then x is called a* ***zero*** *of f and if $f(x) = \infty$, then x is called a* ***pole*** *of f.*

Remark 2.1.2.

(i) *There exist many nonconstant meromorphic functions on each Riemann surface.*

(ii) *When $S = \mathbb{P}^1$, every meromorphic function is a rational function of one variable.*

(iii) *A meromorphic function over $\mathbb{P}^1$ with a single pole at ∞ is a polynomial.*

Definition 2.1.4. *A biholomorphic bijection of two compact Riemann surfaces is a* ***(complex) isomorphism*** *of these surfaces. An isomorphism from a surface to itself is called* ***automorphism*** *of the surface. All automorphisms of a compact Riemann surface form a group with composition as group operation and this group is denoted by* $\mathrm{Aut}(S)$.

Example 2.1.5. *The groups of automorphisms of $\mathbb{P}^1$ is*

$$\begin{aligned}\mathrm{Aut}(\mathbb{P}^1) &= \left\{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\} \\ &= \mathbb{P}GL(2, \mathbb{C}), \textit{the group of Möbius transformations}\end{aligned}$$

Topologically, any compact Riemann surface is homeomorphic to a sphere with handles. The number of handles, g, is called **the genus**. $g = 0$ yields a sphere, $g = 1$ a torus, and $g = 2$ a double torus.

Definition 2.1.5. *Any compact Riemann surface can be triangulated (the proof is mainly based on the existence of meromorphic functions on each Riemann surface), i.e., the surface is a finite union of subsets homeomorphic to triangles such that triangles are pairwise disjoint or their intersection is either a vertex or an edge.*

The genus g of a compact Riemann surface X may also be expressed in the following way:

Proposition 2.1.1. *Let S be a compact Riemann surface of genus g.*

(i) Let v, e and f be the number of vertices, edges and faces of a given triangulation of S. Then the integer

$$\chi(S) := v - e + f$$

called the Euler–Poincaré characteristic of S, independent of the triangulation.

(ii) The genus and the Euler–Poincaré characteristic are related by

$$\chi(S) = 2 - 2g$$

From now on, we will use the notation S_g to express a compact Riemann surface S together with its genus g.

Theorem 2.1.1. ***(Uniformization Theorem)***

Every simply connected Riemann surface is isomorphic to upper half plane $\mathbb{H}$, or the complex plane $\mathbb{C}$ or the Riemann sphere $\mathbb{P}^1$.

Theorem 2.1.2. ***(Uniformization of compact Riemann surfaces)***

- $\mathbb{P}^1$ *is the only compact Riemann surface of genus 0.*
- *Every compact Riemann surface of genus 1 can be described in the form $\mathbb{C}/\Lambda$, where Λ is a lattice, that is $\Lambda = w_1\mathbb{Z} \oplus w_2\mathbb{Z}$ for two complex numbers w_1, w_2 such that $w_1/w_2 \notin \mathbb{R}$ acting on $\mathbb{C}$ as a group of translations.*
- *Every compact Riemann surface of genus greater than 1 is isomorphic to a quotient $\mathbb{H}/K$, where $K \subset \mathbb{P}SL(2,\mathbb{R})$ acts freely and properly discontinuosly.*

We end this section identifying Riemann surfaces with algebraic curves. An *algebraic curve* C can be described as

$$C = \{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0 \text{ for some polynomial } p \text{ in two variables.}\}$$

An algebraic curve is an algebraic variety of dimension 1, since an **algebraic variety** is defined as the solution set of a system of polynomial equations. An algebraic variety is **irreducible** iff it is not the union of two distinct varieties. Every algebraic variety X may be written as

$$X = X_1 \cup X_2 \cup ... \cup X_k$$

where the X_i's are irreducible and $X_i \neq X_j$ for $i \neq j$; this decomposition is unique up to a reindexing. The varieties X_i are called the **irreducible components** of X.

Theorem 2.1.3. *There is a one-to-one correspondence between irreducible algebraic curves and compact Riemann surfaces.*

2.2 Ramified coverings

We will describe both unramified and ramified coverings first considering S and S' as topological surfaces. Then we will generalize these notions to the case of compact Riemann surfaces.

Definition 2.2.1. *Let S and S' be path-connected topological surfaces and $f : S \to S'$ be a continuous map. If for all $y \in S'$, there is a neighbourhood U of y and a discrete set D such that $f^{-1}(U) \subset S$, is homeomorphic to $U \times D$, then (S, S', f) is called (unramified)* ***covering*** *of S' by S.*

Definition 2.2.2.

- *The connected components of $f^{-1}(U)$ are called the* ***sheets*** *of the covering.*

- *For an element y in S', f^{-1} is called the **fiber** over y.*
- *The cardinality of the discrete set D is the **degree** of the covering f denoted by **deg(f)**.*
- *deg(f)=n means that the covering is n**-sheeted** and when $n < \infty$, the covering is **finite-sheeted**.*

Definition 2.2.3. *Let $f_1 : S^1 \to S'$ and $f_2 : S^2 \to S'$ be two unramified coverings of S'. If the following diagram is commutative with the existence of a homeomorphism $u : S^1 \to S^2$ then we say that f_1 and f_2 are **isomorphic**.*

$$\begin{tikzcd} X_1 \arrow[rr, "u"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

The following examples illustrate unramified coverings and with the help of these examples, we will introduce the notion *ramified coverings*:

Example 2.2.1. *Let us consider the mapping from $\mathbb{S}^1$ to $\mathbb{S}^1$, where $\mathbb{S}^1$ denotes the unit circle. This mapping is defined by $f(z) = z^n$ with $z \in \mathbb{C}$ and $|z| = 1$. For any point $z \in S^1$, the preimage $f^{-1}(z)$ has n points in S^1. This is an unramified covering of the unit circle by itself with degree n. The same mapping could have been expressed as $\varphi \mapsto n\varphi \ mod\ 2\varphi$. Here phi $mod\ 2\varphi$ denotes the angle of a point in S^1.*

Example 2.2.2. *Now consider the two annuli $S = \{(r, \varphi)|0 \leq r_1 < r < r_2\}$ and $S' = \{(r, \varphi)|0 \leq r_1^n < r < r_2^n\}$. The function*

$$f : (r, \varphi) \mapsto (r^n, n\varphi \ mod\ 2\pi)$$

or equivalently,

$$f : z \mapsto z^n$$

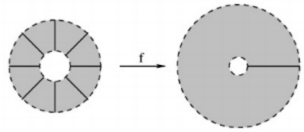


Figure 2.3: $f : z \mapsto z^8$

is an unramified covering of S' by S. Figure 2.3 above illustrates the case for $n = 8$.

Remark 2.2.1. *In Example 2.2.2, $r_1 = 0$ indicates that the annuli are open disks at $(0, 0)$. If we add this point to S and S', the resulting mapping is still continuous and all the points in S' except its center have n preimages. Here the center of S' has a single preimage, the center of S. This mapping is called* ***ramified covering*** *of one open disk by another. The single preimage, namely the center of S, is called a* ***critical point*** *and has* ***multiplicity*** *(or* ***order****) n. We call the center of S a* ***critical value****. Critical values are usually called* ***ramification points****.*

Now we want to state the ramified covering of the Riemann sphere $\mathbb{P}^1$ by a compact Riemann surface S.

Proposition 2.2.1. *A non-constant meromorphic function $f : S \to \mathbb{P}^1$ is a ramified covering of $\mathbb{P}^1$.*

Determining local coordinates around x and $y = f(x)$ such that $x \neq \infty$ and $y \neq \infty$, we define a *critical point* of f is $x \in S$ such that $f'(x) = 0$. Moreover, these local coordinates, say s and t, can also be taken in such a way that $x = 0$, $y = 0$ and so $f(s) = s^n$. Here n will be the *degree* (or *multiplicity*, or *order*) of the critical point. Similar to the definitions above, the value of f at a critical point will be a *critical value.*

The *isomorphism* of two ramified coverings of $\mathbb{P}^1$ is defined to be with a commutative diagram again and with the existence of a *biholomorphic isomorphism*

$u : S^1 \to S^2$:

$$\begin{tikzcd} X_1 \arrow[rr, "u"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & \mathbb{C}P^1 & \end{tikzcd}$$

Remark 2.2.2. *Notice that S is not only a topological space but a Riemann surface and therefore f is not only a continuous function, it is now an analytic function. So we defined the previous terms "critical point", "degree", "critical value" in an analytic point of view. Also notice that the definition of "isomorphism of two (ramified) coverings" involves not only a homeomorphism but a biholomorphic isomorphism.*

2.3 Fundamental group and monodromy

Let S and S' be two compact Riemann surfaces and let $f : S \to S'$ be an unramified covering of S'. We will characterize those unramified coverings by introducing the concepts *fundamental group* and *monodromy*. Since these notions are related to topology, we will keep in mind the fact that S and S' are basically (compact) topological surfaces by the definition of Riemann surfaces.

Definition 2.3.1. *Let S' be a compact topological surface. A continuous path*

$$\gamma : I = [0,1] \to S'$$

*is called **a loop with base point** p_0 if $\gamma(0) = \gamma(1) = p_0$. The image $\gamma([0,1])$ is an oriented path both starting and ending at $p_0 = 0$, thus it is also called a loop. Two loops $\alpha, \beta : I \to S'$ with same base point p_0 are said to be **homotopic**, if they can be deformed to each other through a continuous family $\{\gamma_s : I \to S'\}_{s\in I}$ of loops with base point p_0, this means that there is a continuous map*

$$\gamma : I \times I \longrightarrow S' \qquad (t,s) \longmapsto \gamma(t,s) =: \gamma_s(t)$$

such that $\gamma_s(0) = \gamma_s(1) = p_0$ *for all* $s \in I$ *and* $\gamma_0(t) = \alpha(t), \gamma_1 = \beta(t)$ *for all* $t \in I$. *The set of homotopy classes of loops can be endowed with a group structure by means of the following composition law*

$$[\alpha] \circ [\beta] = [\alpha\beta], \text{ where } \alpha\beta(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

This group is called the ***fundamental group*** *of* S', *and it is denoted by* $\pi_1(S', p_0)$ *or, very often, simply by* $\pi_1(S')$ *(different base points will lead to equivalent groups).*

Now let us again consider the covering $f : S \to S'$. Let $P = f^{-1}(p_0)$ and so P can equally identified with the discrete set D in the definition of covering. We will now define *monodromy* as the action of the fundamental group $\pi_1(S', p_0)$ on P:

Definition 2.3.2. *Take an arbitrary* γ *in* $\pi_1(S', p_0)$. *Note that* $f^{-1}(\gamma)$ *corresponds to* $|D|$*-many oriented curves in* S. *By definition,* γ *maps* p_0 *to* p_0 *and so each of these curves gives a mapping from* P *to* P. *Moreoever, one can also find the inverse of this mapping by using the invertibility of* γ *in the fundamental group. This discussion leads to the fact that* γ *gives the bijection* $P \to P$. *So we can write the following group homomorphism:*

$$\pi_1(S', p_0) \to \{\text{Bijections on } P\}$$

The ***monodromy group*** *is defined to be the image of this homomorphism.*

Corollary 2.3.1. *There is a one-to-one correspondence between the following sets:*

- *Unramified coverings of* S'.
- *The subgroups of* $\pi_1(S', p_0)$ *(up to conjugacy).*

Now let $f : S \to \mathbb{P}^1$ be a covering of the Riemann sphere ramified at k points $s_1, ..., s_k$ with degree n. The fundamental group of $\mathbb{P}^1 \backslash \{s_1, ..., s_k\}$ is a free group with $k-1$ generators. But we want to preserve the symmetry among the punctures by choosing a generator for each puncture. So the fundamental group will have k generators and a single identity. We need a loop enclosing each s_i to form the fundamental group: For each γ_i enclosing s_i, the product $\gamma_1\gamma_2...\gamma_k$ will enclose all punctures. By the discussion above the identity $\gamma_1\gamma_2...\gamma_k = id$ leads to the identity $g_1g_2...g_k = id$, where g_i is the corresponding permutation in S_n for γ_i acting on P. $\langle\gamma_1, \gamma_2, ..., \gamma_k | \gamma_1\gamma_2...\gamma_k = id\rangle$ is the fundamental group and $\langle g_1, g_2, ..., g_k | g_1g_2...g_k = id\rangle$ is the monodromy group as a subgroup of S_n. For each fiber $f^{-1}(s_i)$, let the cycle structure of g_i be denoted by λ_i. λ_i's are the **ramification indices** and the expression $[\lambda_1][\lambda_2]...[\lambda_k]$ is called the **ramification scheme** of the covering.

We end this section with the famous formula:

Theorem 2.3.1. ***(Riemann-Hurwitz formula)*** *Let S_g and $S'_{g'}$ be two compact Riemann surfaces with genera g and g' respectively. Consider the ramified covering $f : S_g \to S'_{g'}$ (ramified at k points) of degree n with ramification index $[\lambda_1][\lambda_2]...[\lambda_k]$ Then, the Riemann-Hurwitz formula is*

$$2 - 2g = n(2 - 2g') - \sum_i (\lambda_i - 1)$$

Remark 2.3.1. *This can also be stated using Euler-Poincaré characteristic:*

$$\chi(S) = n\chi(S') - \sum_i (\lambda_i - 1)$$

Remark 2.3.2. *If $S' = \mathbb{P}^1$ with $g' = 0$, the formula is*

$$2 - 2g = 2n - \sum_i (\lambda_i - 1)$$

2.4 Riemann existence theorem

Proposition 2.2.1 and Remark 2.1.2 (i) indicate the fact that every compact Riemann surface can be represented by a ramified covering of the Riemann sphere $\mathbb{P}^1$.

Definition 2.4.1. *The sequence* $[g_1, ...g_k]$ *is called a* ***k-constellation****(or simply, a* ***constellation****) if the permutations* $g_i \in S_n$ *satisfies the following:*

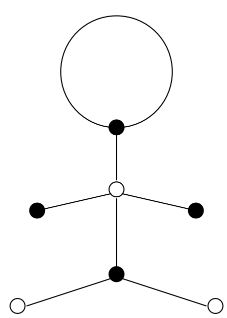
- $G = \langle g_1, g_2, ..., g_k \rangle$ *acts transitively on* $\{1, 2, ..., n\}$.
- $g_1 g_2 ... g_k = id$.

Theorem 2.4.1. ***(Riemann existence theorem)*** *Consider the set* $\{s_1, ..., s_k\}$ *with each* $s_i \in \mathbb{P}^1$ *fixed. Then for any constellation* $[g_1, ..., g_k]$*, where each* g_i *is a permutation in* S_n*, there exists a compact Riemann surface* S *and a meromorphic function* $f : S \to \mathbb{P}^1$ *(a ramified covering of* $\mathbb{P}^1$*) such that* $s_1, s_2, ..., s_k$ *are the critical values (the ramification points) of* f *and* $g_1, g_2, ..., g_k$ *are the corresponding monodromy permutations. Moreover,* f *is unique up to isomorphism.*

Chapter 3

Belyi Coverings

A Belyi covering is a ramified covering of the Riemann sphere with ramification points belonging to the set $\{0, 1, \infty\}$. We will give examples of Shabat polynomials ([20]). Then we will state Belyi's Theorem ([3]). We will give two parts of the proof from [7] and [11] respectively. Then we introduce the notion *dessin d'enfant* which is a French synonym for "child-drawing". The following example from [22] is a dessin d'enfant and gives us an idea about why this name is indeed chosen:



Although the dessin above looks like a simple bicoloured graph, it also carries a topological structure ([7] and [22]). The theory of dessins d'enfants was first introduced by Grothendieck ([8]) and triggered by Belyi's Theorem. Later we will state that dessins and Belyi coverings are in one-to-one correspondence ([24]). We will also mention the action of the absolute Galois group on dessins ([12] and [7]). More details about this action can be found in [16]. We conclude this chapter by giving examples from [21], [4] and [24].

3.1 Belyi coverings and Belyi theorem

Definition 3.1.1. *Let S_g be a compact Riemann surface with genus g. A* ***Belyi covering*** $\beta : S_g \to \mathbb{P}^1$ *is a ramified covering of* $\mathbb{P}^1$ *with ramification points belonging to the set* $\{0, 1, \infty\}$.

Remark 3.1.1. *Let ramification points of a covering of* $\mathbb{P}^1$ *correspond to at most* 3 *different critical values. If some of them are not equal to* $0, 1,$ *or* ∞*, then we can apply a suitable linear fractional transformation (the elements of* $\mathrm{Aut}(\mathbb{P}^1)$*) to make those critical values lie in* $\{0, 1, \infty\}$*. Thus, a Belyi covering is a covering of* $\mathbb{P}^1$ *ramified at most 3 points.*

Remark 3.1.2. *We will call* β *a* ***Belyi function*** *when the genus* $g = 0$ *and in other cases, we will not only express* β*, instead we will write* (S_g, β) *and call it a* ***Belyi pair****.*

Example 3.1.1. *Let* $S = \mathbb{P}^1$.

$$\beta : z \mapsto z^n$$

is a Belyi function. This is because the single root of the equation $\beta'(z) = nz^{n-1} = 0$ *is* 0 *and* $\beta(0) = 0$.

Example 3.1.2. *If* $S = \mathbb{P}^1$*, then consider* ***the Belyi polynomial***

$$\beta_{m,n} = z \mapsto \frac{1}{\mu} z^m (1-z)^n$$

where $\mu = \frac{m^m n^n}{(m+n)^{m+n}}$. *Then using logarithmic differentiation,*

$$\frac{\beta'_{m,n}}{\beta_{m,n}}(z) = (\ln \beta)'(z) = [n \ln z + m \ln(1-z)]' = \frac{n-(m+n)z}{z(1-z)}.$$

Thus

$$\beta'_{m,n}(z) = z^{m-1}(1-z)^{n-1}(n-(m+n)z) = 0.$$

The critical points are 0, 1 *and* $\frac{n}{m+n}$. *So the critical values are* $\beta_{m,n}(0) = 0, \beta_{m,n}(1) = 0$ *and* $\beta_{m,n}(\frac{n}{m+n}) = 1$. *Therefore,* $\beta_{m,n}$ *is a Belyi function.*

Example 3.1.3. ***(Chebyshev polynomials)*** $\cos n\varphi$ *can be expressed as a polynomial of degree* n *in* $\cos \varphi$:

$$\cos n\varphi = T_n(\cos \varphi),$$

where T_n *is the* n*-th Chebyshev polynomial.*

Let $z = \cos \varphi$. *Now the Chebyshev polynomial can also be expressed as*

$$T_n(z) = \cos n(\arccos z).$$

Then

$$T'_n(z) = n \sin n(\arccos z) \frac{1}{\sqrt{1-z^2}}$$

The critical points of T_n *are the zeros of* $\sin n(\arccos z)$. *So, we are looking for points* z *such that*

$$\arccos z = 0 \text{ and } \arccos z = \frac{k\pi}{n}, k \in \mathbb{Z}.$$

Now we find that $z = 1$ *and* $z = \cos(\frac{k\pi}{n})$. *Therefore, the critical values are* $T_n(1) = 1$ *and* $T_n(\cos(\frac{k\pi}{n})) = \pm 1$.

We showed that this Chebyshev polynomial is also a Belyi function.

Remark 3.1.3.

- *We did not state explicitly but the polynomials above are ramified at ∞ (Remark* 2.1.2 *(iii)).*
- *A polynomial with at most two critical values is called a* ***Shabat polynomial****, or a* ***generalized Chebyshev polynomia****l. Thus, the examples above are Shabat polynomials.*

Recall that compact Riemann surfaces and irreducible algebraic curves are in one-to-one correspondence. Now let K be some subfield of $\mathbb{C}$. A compact Riemann surface S is said to *be defined over* K if it is isomorphic to the Riemann surface of some curve defined over K, i.e., the coefficients of that curve lie in K. The problem is to decide when S is defined over a number field -a finite extension of $\mathbb{Q}$-, or equivalently, to decide when S is defined over $\overline{\mathbb{Q}}$, the field of algebraic numbers, since a number field is a subfield of $\overline{\mathbb{Q}}$.

Now we are ready to give the main theorem of this section:

Theorem 3.1.1. ***(Belyi)*** *Let S be a compact Riemann surface. The following statements are equivalent:*

(a) S is defined over $\overline{\mathbb{Q}}$.

(b) S admits a meromorphic function $f : S \to \mathbb{P}^1$ with at most three ramification points.

Proof.
(a)⇒(b): It is enough to show the existence of a meromorphic function $f : S \to \mathbb{P}^1$ ramified over a set of rational values $\{0, 1, \infty, \lambda_1, ..., \lambda_n\} \subset \mathbb{Q} \cup \{\infty\}$. To see this;

we observe that after composing with the Möbius transformations $T(x) = 1 - x$ and $M(x) = 1/x$ if necessary, we can assume that $0 < \lambda_1 < 1$. Therefore, λ_1 can be written in the desired form $\lambda_1 = \frac{m}{m+n}$. Composing now f with the rational function P_{λ_1} we would get $P_{\lambda_1} \circ f$ with strictly less branching values (ramification points), namely $\{0, 1, \infty, P_{\lambda_1}(\lambda_2), ..., P_{\lambda_1}(\lambda_n)\}$. From here the problem is solved inductively.
In order to show the existence of such a function $f : S \to \mathbb{P}^1$, write S in the form $S = S_F$ with

$$F(X,Y) = p_0(X)Y^n + p_1(X)Y^{n-1} + ... + p_n(X) \in \overline{\mathbb{Q}}[X,Y]$$

and consider

$$\begin{aligned} S_F &\xrightarrow{\text{x}} \mathbb{P}^1 \\ (x,y) &\longmapsto x \end{aligned}$$

Here S_F is a (unique) compact and connected Riemann surface that contains $S_F^X = \{(x,y) \in \mathbb{C}^2 | F(x,y) = 0, F_Y(x,y) \neq 0, p_0(x) \neq 0\}$.
Denote the set of branching values of $\mathtt{x}$ by $B_0 = \{\mu_1, ..., \mu_s\}$. Since the branching points of x (critical points) lie in the finite set $S \backslash S_F^X$, each μ_i is either a zero of $p_0(X)$ or the point $\infty \in \mathbb{P}^1$, or the first coordinate of a common zero of the polynomials $F, F_Y \in \overline{\mathbb{Q}}$. B_0 is contained in $\overline{\mathbb{Q}} \cup \{\infty\}$. Now, if $B_0 \subset \mathbb{Q} \cup \{\infty\}$, we are done. If not, the following inductive argument begins:
Let $m_1(T) \in \mathbb{Q}[T]$ be the minimal polynomial of $\{\mu_1, ..., \mu_s\}$, i.e. the monic polynomial of lowest degree that vanishes at the points $\mu_1, ..., \mu_s$ (or at $\mu_1, ..., \mu_{s-1}$ if one of them, say μ_s, equals ∞). Equivalently, $m_1(T)$ is the product of the minimal polynomials of all algebraic numbers μ_i, avoiding repetition of factors. Denote the roots of $m_1'(T)$ by $\beta_1, ..., \beta_d$ and their minimal polynomial by $p(T)$. By definition $\deg(p(T)) \leq \deg(m_1'(T))$.
Since the identity $\text{Branch}(g \circ f) = \text{Branch}(g) \cup \text{Branch}(f)$ for f and g holds, the set

of branching values of

$$S_F \xrightarrow{\mathrm{x}} \mathbb{P}^1 \xrightarrow{m_1} \mathbb{P}^1$$
$$(x,y) \longmapsto x \longmapsto m_1(x)$$

is

$$B_1 = m_1(\{\text{roots of } m_1'\}) \cup \{0,\infty\}.$$

Again, if $B_0 \subset \mathbb{Q} \cup \{\infty\}$, we are done. If not, denote the minimal polynomial of the branching value set of m_1, that is $m_1(\{\text{roots of } m_1'\}) = \{m_1(\beta_1), ..., m_1(\beta_d)\}$ by $m_2(T) \in \mathbb{Q}[T]$. Clearly, $[\mathbb{Q}(m_1(\beta_i)) : \mathbb{Q}] \geq [\mathbb{Q}(\beta_i) : \mathbb{Q}]$, which means that the degree of the minimal polynomial of $m_1(\beta_i)$ is lower or equal to the degree of the minimal polynomial of β_i. Moreover, elementary Galois theory shows that two algebraic numbers β_i, β_j have the same minimal polynomial if and only if $\sigma(\beta_i) = \beta_j$ for some field embedding $\sigma : \mathbb{Q}(\beta_i) \to \overline{\mathbb{Q}}$. But in that case $\sigma(m(\beta_i)) = m(\beta_j)$ and so $m(\beta_i)$ and $m(\beta_j)$ also have the same minimal polynomial. Therefore,

$$\deg(m_2(T)) \leq \deg(p(T)) \leq \deg(m_1'(T)) < \deg(m_1(T)). \tag{3.1}$$

Next the set of branching values of $m_2 \circ m_1 \circ \mathrm{x}$ is $B_2 = m_2(\{\text{roots of } m_2'\}) \cup m_2(B_1)$. By construction, $m_2(B_1) \subset \mathbb{Q} \cup \{infty\}$; in fact $m_2(B_1)$ consists of the points $0, \infty$ and $m_2(0)$. Now if the whole set B_2 is contained in $\mathbb{Q} \cup \{\infty\}$ we have finished. If not, we continue the process denoting the minimal polynomial of $m_2(\{\text{roots of } m_2'\})$ by $m_3(T) \in \mathbb{Q}[T]$ and looking at the set B_3 of branching values of $m_3 \circ m_2 \circ m_1 \circ x$, which is given by

$$B_3 = m_2(\{\text{roots of } m_3'\}) \cup m_3 \circ m_2(B_2)$$

etc.

This process ends when $B_k \subset \mathbb{Q} \cup \{\infty\}$, something that must happen after finitely many steps since by (3.1) we have the inequality $\deg(m_i(T)) \leq \deg(m_{i+1}(T)) - 1$. $\square$

For the other part of the proof, we consider the corresponding algebraic curve for the compact Riemann surface S and then continue in this way. Before giving the proof, we state the following facts:

Let C be an algebraically closed field of characteristic 0 and let $f : S \to \mathbb{P}^1$ be a meromorphic function defined from a curve S over C to the Riemann sphere (projective line) $\mathbb{P}^1$.

Definition 3.1.2. *The* ***moduli field of f*** *is the field $M(S,f) := C^{U(S,f)}$ fixed by the subgroup $U(S,f)$ of $U(S)$ consisting of all $\sigma \in Aut(C)$ such that there exists an isomorphism $t_\sigma : S^\sigma \to S$ of varieties over C such that the following diagram commutes:*

$$\begin{tikzcd} S^\sigma \arrow[r, "t_\sigma"] \arrow[d, "f^\sigma"'] & S \arrow[d, "f"] \\ (\mathbb{P}^1)^\sigma \arrow[r, "Proj(\sigma)"] & \mathbb{P}^1 \end{tikzcd}$$

Here, $Proj(\sigma)$ means the automorphism of the scheme $\mathbb{P}^1 = Proj(C[T_0,T_1])$ induced by the extension of the automorphism $\sigma \in Aut(C)$ to $C[T0,T_1]$ (denoted by σ again). Obviously, $M(S) \subseteq M(S,f)$.

Theorem 3.1.2. *The curve S/C and the meromorphic function f are defined over a finite extension of $M(S,f)$. If f is a Galois covering (i.e. if the corresponding extension of function fields is Galois), then S/C and f are defined over $M(S,f)$ itself.*

Proposition 3.1.1. *Let D be a finite set of (closed) points of $\mathbb{P}^1$ and let $d \geq 1$ be a natural number. Then there are at most finitely many isomorphism classes of pairs (S,f) where $S/\mathbb{C}$ is a curve and $f : S \to \mathbb{P}^1$ is a meromorphic function of varieties over $\mathbb{C}$ of degree d whose critical values lie in D.*

Corollary 3.1.1. *Let $S/\mathbb{C}$ be a curve, let $f : S \to \mathbb{P}^1_{\mathbb{C}}$ be a meromorphic function and let K be a subfield of $\mathbb{C}$ such that the critical values of f are K-rational. Then the moduli field of f is contained in a finite extension of K.*

Now we are ready to give the proof:

Proof.
(b)⇒(a): Assume that there is a meromorphic function $f : S \to \mathbb{P}^1_{\mathbb{C}}$ as above. Compose f with an appropriate fractional linear transformation, we may assume that the critical values of f lie in $D := \{0, 1, \infty\}$. Then the moduli field $M(S, t)$ is a number field by Corollary 3.1.1. Now, Theorem 3.1.2 shows that S is defined over a (maybe, bigger) number field. □

3.2 Dessins d'enfants and Belyi coverings

Definition 3.2.1. *A **dessin d'enfant**, or simply a **dessin**, is a pair $(X, \mathcal{D})$ where X is an oriented compact topological surface, and $\mathcal{D} \subset X$ is a finite graph such that:*

(i) $\mathcal{D}$ is connected.

(ii) $\mathcal{D}$ is bicoloured.

*(iii) $X \backslash \mathcal{D}$ is the union of finitely many topological discs, which we call **faces** of $\mathcal{D}$.*

To fully understand the definition above, we need to recall some basic terms:

- A *graph* consists of vertices and edges.
- A *connected* graph when there is a path between every pair of vertices.

- A *map* is a graph embedded in a compact oriented two-dimensional manifold such that (1) the edges do not cross each other and (2) the complement of the graph in the surface is a disjoint union of "faces" homeomorphic to open disks.

- Each vertex of a *bicoloured* has one of two colours such that each edge connects different colours. A *bipartite graph* admits such a colouring.

- A *hypermap* is a bicoloured map.

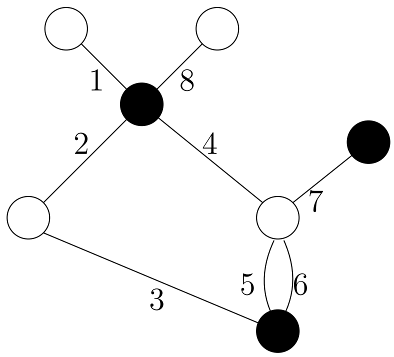
So Definition 3.2.1 shows that a dessin is a connected bipartite map or simply a connected hypermap. What is important here is the fact that dessins are not only abstract graphs but also embeddings into a topological surface. Therefore, we denote a dessin by $(X, \mathcal{D})$ alongside the corresponding surface X. When the underlying surface is clear, we simply express a dessin as $\mathcal{D}$.

Remark 3.2.1. *The genus of $(X, \mathcal{D})$ is simply the genus of the topological surface X.*

Definition 3.2.2. *We consider two dessins $(X_1, \mathcal{D}_1)$ and $(X_2, \mathcal{D}_2)$ equivalent when there exists an orientation-preserving homeomorphism from X_1 to X_2 whose restriction to $\mathcal{D}_1$ induces an isomorphism between the coloured graphs $\mathcal{D}_1$ and $\mathcal{D}_2$.*

Now suppose that the edges of the dessin are numbered from the set $\Omega = \{1, 2, 3, ...\}$. Each edge joins a black vertex to a white vertex, and incident with every black vertex, we have some of these edges. Using the anticlockwise orientation of the surface gives us a cyclic permutation of these edges. Thus if we have b black vertices, we have a permutation σ_0 that is a product of b disjoint cycles. Similarly, if we have w white vertices then we get a permutation σ_1 consisting of w disjoint cycles, again using the anticlockwise orientation. We then find that the permutation $\sigma_2 := (\sigma_0 \sigma_1)^{-1}$ describes the edges going around a face, each cycle of length u corresponding to a $2u$-gonal face.

Example 3.2.1. *Let* $\sigma_0 = (1248)(365)(7)$ *and* $\sigma_1 = (1)(23)(4567)(8)$*. So* $\sigma_2 = (18473)(25)(6)$ *and the corresponding diagram is the following:*



Definition 3.2.3. $\langle \sigma_0, \sigma_1 \rangle$ *is called the* ***permutation representation pair*** *of the dessin.*

Remark 3.2.2. *By the definition of a dessin,* ***connectedness*** *guarantees that the group* $\langle \sigma_0, \sigma_1 \rangle$ *is transitive.*

Now we will focus on the importance of *dessins d'enfants*: the relation of them to Belyi coverings.

- Let $\beta : S \to \mathbb{P}^1$ be a Belyi covering of the Riemann sphere by a compact Riemann surface S. Take the segment $[0, 1] \subset \mathbb{P}^1$, color the point 0 in black ($\bullet$), color the point 1 in white ($\circ$), so that the segment itself looks like $\bullet\!\!-\!\!\!-\!\!\!-\!\!\circ$ and take the preimage $\beta^{-1}([0, 1]) \subset S$. This is a hypermap embedded(in a very specific way) in the surface S. All black vertices are the roots of the equation $\beta(x) = 0$ and all white vertices are the roots of the equation $\beta(x) = 1$. The multiplicities of these preimages correspond to the degrees of vertices or faces. Inside each face of the hypermap there exists a (single) pole of β (a root of the equation $\beta(x) = \infty$), the multiplicity of the pole being equal to the degree of the face. These poles are called as the "centers of faces". We sometimes denote the centers of faces by $*$. There are no other critical points of β other than the set of black and white vertices and the centers of faces.

- Conversely, for any hypermap (of any genus) there exists a corresponding Belyi pair. This is a consequence of *Riemann Existence Theorem.*

More precisely, we have the following theorem:

Theorem 3.2.1.

$$\{\textit{Equivalence classes of dessins}\} \longleftrightarrow \{\textit{Equivalence classes of Belyi pairs}\}$$

Remark 3.2.3.

- *Note that the hypermap specifies not only a Belyi covering; it also specifies a Riemann surface on which this function is defined.*

- *There is one other facet of dessins which is "triangle decomposition": a collection of triangles covering the underlying surface of dessin so that the intersection of two triangles consists of a union of edges or vertices. T (This is not exactly the same thing with "triangulation of a surface" due to the fact that the intersection of triangles can be more than one edge.) Now join the center of each face to the black vertices and white vertices adjacent to this face, we obtain a triangulation of the hypermap. We call a triangle* positive *(resp.* negative*), if its vertices taken in the counter clockwise direction are labeled as* $(0, 1, \infty)$ *(resp.* $(0, \infty, 1)$*). The image under the Belyi covering of all positive triangles is the upper half-plane, and for the negative ones, the lower half-plane.*

Now we will define the monodromy group of a Belyi covering.

Proposition 3.2.1. *The permutation representation pair of a dessin d'enfant and the monodromy of the corresponding Belyi pair are determined by each other.*

Proof. Let $\beta : S \to \mathbb{P}^1$ be a Belyi covering with degree n. The fundamental group $\pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty\}, p_0)$ is a free group with 2 generators $\langle \gamma_0, \gamma_1 \rangle$, where γ_0 is the loop around 0 and γ_1 is the loop around 1.
The monodromy homomorphism

$$M_\beta : \pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty\}, p_0) \to S_n$$

is determined by $M_\beta(\gamma_0) = \sigma_{\gamma_0}^{-1}$ and $M_\beta(\gamma_1) = \sigma_{\gamma_1}^{-1}$. If x_j in the fiber $\beta^{-1}(p_0)$ which lies in the edge j of the dessin $(\mathcal{D}, S)$, then the lift of γ_0 with the initial point x_j ends at $x_{\gamma_0(j)}$ since $\beta(z) = z^n$ in a neighbourhood of a point in $\beta^{-1}(0)$ (a white vertex in $\mathcal{D}$). In this way, $\sigma_{\gamma_0} = \sigma_0$, and similarly, $\sigma_{\gamma_1} = \sigma_1$. $\square$

Remark 3.2.4. *Let the monodromy group of a Belyi covering is*

$$\langle g_0, g_1, g_\infty | g_0, g_1, g_\infty = id \rangle.$$

So the permutation representation pair for the corresponding dessin will be $\langle g_0, g_1 \rangle$ *and* $g_\infty = (g_0 g_1)^{-1}$. *If the cycle structure of* g_i *are* λ_i, *then the Belyi covering is determined by its* ***ramification scheme*** $[\lambda_\infty][\lambda_0][\lambda_1]$.

Theorem 3.2.2. *There is a one-to-one correspondence between the followings:*

- *Dessins* $(\mathcal{D}, S_g)$
- *Belyi coverings of* $\beta : S_g \to \mathbb{P}^1$ *with degree* n.
- *The solutions of the monodromy group relation* $g_0 g_1 g_\infty = id$, *where* $g_i \in S_n$.

There is one last notion the "field of definition" for Belyi coverings, or for the corresponding dessins:

Definition 3.2.4. *Let* S *be a compact Riemann surface and* $\beta : S \to \mathbb{P}^1$ *be a Belyi function. A* ***field of definition*** *of a Belyi pair* (S, β), *or a dessin denfant, is a number field* K *such that both the algebraic curve* C *(corresponding to* S*) and the Belyi function* β *can be defined with coefficients in* K.

A dessin can have many fields of definition: first of all, if some K is a field of definition, every field containing it is also a field of definition.

3.3 The action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Definition 3.3.1. *The* ***universal Galois group****, or the* ***absolute Galois group*** *is the group of auto- morphisms of algebraic numbers* $\overline{\mathbb{Q}}$ *and denoted by* $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*.* $\mathbb{Q}$ *is the fixed field of* Γ*.*

Let $k \subset \overline{\mathbb{Q}}$ be a number field(finite extensions of $\mathbb{Q}$).

Fact 1 Every automorphism k may be extended to an automorphism of $\overline{\mathbb{Q}}$.

Fact 2 **(The main theorem of Galois theory)** Subgroups of Γ of finite index are in one-to-one correspondence with finite extensions of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}$.

- Consider first the genus $g = 0$ case. Let $\mathcal{D}$ be a dessin, and let β be the corresponding Belyi function. Then β is a rational function, and according to Belyi's theorem β may be chosen in such a way that all its coefficients are algebraic numbers. Let K be the normal extension of $\mathbb{Q}$ generated by the coefficients. Now, let us act on all these numbers simultaneously by an automorphism of K; note that, according to Fact 1, this is the same thing as to act by an automorphism α of $\overline{\mathbb{Q}}$, that is, by an element $\alpha \in \Gamma$. The result f^α of such an action is once more a Belyi function (due to the Theorem 3.3.1 below). The dessin $\mathcal{D}^\alpha$ which corresponds to β^α is, by definition, the result of the action of α on $\mathcal{D}$.

- For a dessin of genus $g > 1$, the discussion above is valid with the only exception: acting on a Belyi pair not a Belyi function. Let $\mathcal{D}$ be a dessin, and let (S, β) be a corresponding Belyi pair. This curve S may be realized as an algebraic curve in $\mathbb{P}^k$ a , that is, as a solution of a system of homogeneous algebraic equations in homogeneous coordinates $(x_0 : x_1 : ... : x_k)$. According to Belyi's theorem the coefficients of these equations can be taken from $\overline{\mathbb{Q}}$. The function β is a rational function in the variables $x_0, ..., x_k$ whose coefficients, again by Belyi's theorem, can be taken from $\overline{\mathbb{Q}}$. Now we act on all the above algebraic numbers simultaneously by an automorphism $\alpha \in \Gamma$, and we get a new Belyi pair (S^α, β^α) which produces a new dessin $\mathcal{D}^\alpha$.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, dashed] \arrow[d] & \mathcal{D}^\alpha \\
(S, \beta) \arrow[r, "\alpha"] & (S^\alpha, \beta^\alpha) \arrow[u]
\end{tikzcd}$$

Remark 3.3.1. *All orbits of the action of* Γ *on dessins are finite.*

Theorem 3.3.1. *Let* $\mathcal{D}$ *be a dessin. The following properties of* $\mathcal{D}$ *remain invariant under the action of the absolute Galois group:*

(1) The number of edges.

(2) The number of white vertices, black vertices and faces.

(3) The degree of the white vertices, black vertices and faces.

(4) The genus.

(5) The monodromy group.

(6) The automorphism group.

We conclude this section with the following theorem describing another facet of the action of Γ:

Theorem 3.3.2. *The restriction of the action of Γ to dessins of genus g is faithful for every g.*

3.4 Examples

Definition 3.4.1. *Let S be a compact Riemann surface with genus g and $\beta : S_g \to \mathbb{P}^1$ be a Belyi covering.*

- *If $g = 0$, β is called a* ***rational Belyi covering****.*
- *If $g = 1$, β is called an* ***elliptic Belyi covering****.*
- *If $g > 1$, β is called a* ***hyperbolic Belyi covering****.*

First we will give examples for rational Belyi coverings. The simplest case for rational Belyi coverings is polynomial Belyi coverings. If we put a single pole to infinity, we get Shabat polynomials.

Theorem 3.4.1. *There is a bijection between the set of combinatorial bicolored plane trees and the set of equivalence classes of Shabat polynomials.*

Remark 3.4.1. *Calculation of Shabat polynomials: Let the critical points be 0 and 1 and put the single pole at ∞. Denote the ramification indices as follows:*

$\lambda_0 = \alpha_1, ..., \alpha_p$

$\lambda_1 = \beta_1, ..., \beta_q$

$\lambda_\infty = n$ *(the degree of the covering)* $= p + q - 1$ *(Riemann-Hurwitz formula)*

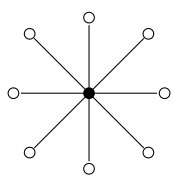
Denote the coordinates corresponding to black vertices by $a_1, ..., a_p$ and to white vertices by $b_1, ..., b_q$. So we get the following equations:

$$P(x) = C(x - a_1)^{\alpha_1} ... (x - a_p)^{\alpha_p}$$

$$P(x) - 1 = C(x - b_1)^{\beta_1} ... (x - b_q)^{\beta_q}$$

All a_i and b_j are distinct here and take $C = 1$. In this state, we have to solve the equations simultaneously by either using analytical techniques or with the help of computer programs, i.e., MAPLE.

Example 3.4.1. *In the figures below the Belyi functions and ramification schemes are given with the corresponding dessins for star-trees and for Chebyshev polynomials.*



Star-Tree

$$[1][n][\underbrace{1, 1, ..., 1}_{n \text{ times}}]$$

$$\beta : z \mapsto z^n$$

Chain-Tree

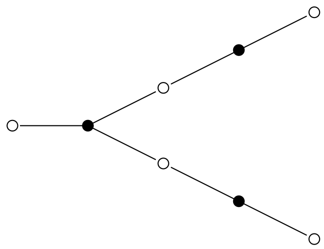
$$[2\text{m}][\underbrace{2, 2, ..., 2}_{\text{m times}}][\underbrace{2, 2, ..., 2}_{\text{m-1 times}}, 1, 1], \text{ or}$$

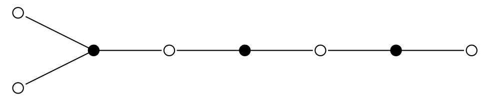
$$[2\text{m}+1][\underbrace{2, 2, ..., 2}_{\text{m times}}, 1][\underbrace{2, 2, ..., 2}_{\text{m times}}, 1]$$

$$T_n(\cos\varphi) = \cos n\varphi$$

Remark 3.4.2. *Every plane tree has a unique and canonical geometric form. So, a transformation can change the tree's size or position, but does not change its geometric form.*

Example 3.4.2. *There are "conjugate" trees with a common ramification scheme: The following trees have the ramification scheme $[7][3, 2, 2][2, 2, 1, 1, 1]$ with the corresponding polynomials $P(x) = x^3(x^2 - 2x \pm a)^2$, where $a = \frac{1}{7}(34 \pm 6\sqrt{21})$ and they are both defined over the field $\mathbb{Q}(\sqrt{21})$.*



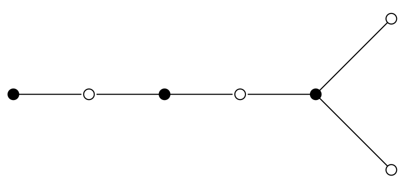


Example 3.4.3. *These three dessins are defined over cubic fields, permutable by* $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *They lie in the decomposition of the polynomial*

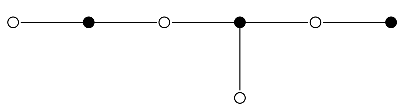
$$25x^3 - 12x^2 - 24x - 16 = 25(x-a)(x-a_+)(x-a_-).$$

We agree that $a \in \mathbb{R}$, $Im(a_+) > 0$, $Im(a_-) < 0$ *and get the following dessins:*

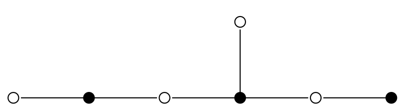
D_a :



D_{a_+} :

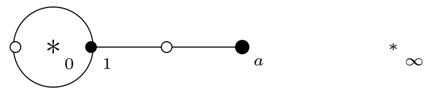


D_{a_-} :



Now we will give an example of a rational Belyi covering which is not a tree.

Example 3.4.4. *Let the dessin be as follows:*



We put one of the poles to the center of one of the faces and the other pole to ∞*. We also denoted the roots of the corresponding function with* 1 *and* a*. Now the Belyi covering will be of the form*

$$\beta(x) = K\frac{(x-1)^3(x-a)}{x}$$

We wrote β *of degree* 4 *since the number of edges of the dessin give us the degree. Moreoever, black vertices are of multiplicity 1 and 3. So, the numerator should be in the form above and denominator is of degree 1, since we have a single pole at* $x = 0$*. By subtracting the degree of the denominator from the one of the numerator, we find the multiplicity of the pole at* ∞ *as* 3*. Similarly,*

$$\beta(z) - 1 = K\frac{(x^2+bx+c)^2}{x}.$$

If we solve those equations simultaneously, we find $K = -\frac{1}{64}$ *and* $a = 9$*. The roots of* $\beta(z)-1$ *corresponding to white vertices are* $3 \pm 2\sqrt{3}$*.*

The following example will be for the case genus $g = 1$.

Example 3.4.5. *Consider the elliptic curve* $E_1 : y^2 = x(x-1)(x-(3+2\sqrt{3})$*. The meromorphic function on* E_1 *which is the projection on the first coordinate:*

$$\rho : E_1 \to \mathbb{P}^1$$
$$(x, y) \mapsto x$$

The projection ρ *is not a Belyi covering since the ramification points(the critical values) are* $0, 1, 3+2\sqrt{3},$ *and* ∞*. In the previous example* $\beta(x) = -\frac{(x-1)^3(x-9)}{64x}$ *sends these values to* $0, 1,$ *and* ∞*. Therefore*

$$(x, y) \to x \to -\frac{(x-1)^3(x-9)}{64x}$$

will give us a Belyi covering.
But E_1 *has a conjugate curve* $E_2 : y^2 = x(x-1)(x-(3-2\sqrt{3})$ *since* $3-2\sqrt{3}$ *is the conjugate of* $3+2\sqrt{3}$.
The corresponding dessins (Figure 3.1*) both have the same ramification scheme* $[6,2][6,1,1][4,2,2]$ *and they are defined over the field* $\mathbb{Q}(\sqrt{3})$.

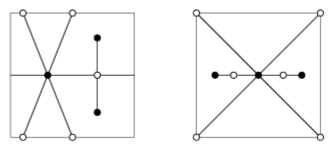


Figure 3.1: Dessins for E_1 and E_2 respectively

Chapter 4

Counting Coverings with a Given Ramification Scheme

We will define the Eisenstein number of a covering ([13]). Dessin of a polynomial covering is a bicoloured planar (plane) tree as we have seen before. They were first studied by G. Shabat ([20]). Counting planar trees is a classical combinatorial problem solved by Tutte ([23]) using the Eisenstein number. Then we will give a formula for the general case: counting coverings with given ramification schemes. This formula includes both Eisentein number and irreducible characters of symmetric groups ([10]). So, before stating this formula, we will give some preliminaries from representation theory ([6], [12], [15] and [17]).

4.1 Tutte formula for counting polynomial coverings

Definition 4.1.1. *Let β be a Belyi covering of $\mathbb{P}^1$. The centralizer in S_n of the monodromy group of β is called the **automorphism group of** β and denoted by* $\operatorname{Aut}\beta$.

Definition 4.1.2. *Let S be a compact Riemann surface. $\sum_{\beta:S\to\mathbb{P}^1}\frac{1}{\operatorname{Aut}\beta}$ is called **Eisenstein number** of coverings of $\mathbb{P}^1$.*

Remark 4.1.1.

- *The definitions above are also valid for arbitrary coverings $f : X \to Y$ where X and Y are compact Riemann surfaces.*
- *The Eisenstein number can also be expressed using the notation* $\operatorname{Aut}\mathcal{D}$ *where $\mathcal{D}$ is the corresponding dessin for the Belyi covering β.*

Tutte found the Eisenstein number of planar trees with n edges and given degrees $d_i^\bullet$ and d_j° of black and white vertices. Clearly, $\sum_i d_i^\bullet = \sum_j d_j^\circ = n$, the number of edges of T, or what is the same, the degree of the respective covering. The degrees $d^\bullet = (d_1^\bullet, d_2^\bullet, ...)$ and $d^\circ = (d_1^\circ, d_2^\circ, ...)$ give two partitions of n. In practice it is more convenient to deal with partitions $q^\bullet = (q_1^\bullet, q_2^\bullet, ...)$ and $q^\circ = (q_1^\circ, q_2^\circ, ...)$ where $q_i^\bullet$ and q_i° is the number of black and white vertices of degree i. Observe that

$$n = \sum_i iq_i^\bullet = \sum_j jq_j^\circ = \sum_i q_i^\bullet + \sum_j q_j^\circ - 1. \tag{4.1}$$

We'll often use the last two sums and introduce for them special notations

$$\sigma^\bullet = \sum_i q_i^\bullet, \qquad \sigma^\circ = \sum_j q_j^\circ. \tag{4.2}$$

In this notations the (slightly modified) Tutte result may be stated as follows:

Theorem 4.1.1. ***(Tutte formula)***

$$\sum_T \frac{1}{|\operatorname{Aut} T|} = \frac{1}{\sigma^\bullet \sigma^\circ} \binom{\sigma^\bullet}{q^\bullet} \binom{\sigma^\circ}{q^\circ}, \tag{4.3}$$

where the sum is extended over all planar trees T *with given degrees* $d^\bullet, d^\circ$ *of black and white vertices. Parentheses in right hand side* $\binom{\sigma^\bullet}{q^\bullet}$ *stand for multinomial coefficient* $\binom{q_1^\bullet+q_2^\bullet+\ldots+q_k^\bullet}{q_1^\bullet,q_2^\bullet,\ldots,q_k^\bullet}$.

Before giving the proof for Tutte formula, we will state some basic definitions:

A *tree* is a connected finite graph containing no polygon. We consider only trees with at least two vertices. Such a tree is said to be *planted* when one *monovalent*(of *degree*, or *valency* 1)vertex is specified as the *root*. If in addition one or more other monovalent vertices are specified as *secondary roots* we say the tree is *doubly planted*. Let T be any planted or doubly planted tree. A *proper vertex* of T is any vertex which is not a root or a secondary root. A *plane tree* is a tree which is embedded in the Euclidean plane. Two planted or doubly planted plane trees are *equivalent* if and only if each can be transformed into the other by an orientation-preserving homeomorphism of the plane onto itself which maps root onto root and proper vertices onto proper vertices. For doubly planted plane trees this implies that secondary roots are mapped onto secondary roots. In what follows we do not distinguish between equivalent planted or doubly planted plane trees. We determine the number of planted plane trees having a given partition. A planted or doubly planted tree is *k-colored* when to each of its proper vertices there is assigned a unique member of a given set of k colors, denoted by $C_1, C_2, \ldots C_k$ subject to the condition that no two adjacent vertices of the tree may have the same color.(Since we are focusing on dessins-bicolored trees, k will always be 2).

Now we are ready to give the proof of Tutte formula:

Proof. Let T be any bicolored (2-colored) planted or doubly planted plane tree with colors black(C_1) and white(C_2). If the color of the vertex joined to the root is black we say that T has *basic color black*.

Let $q^\bullet$ and q° represent the *partition* of T where their i^{th} coordinates correspond to the number of black and white vertices with valencies i as stated above. Now let $f(x_1, x_2)$ be a function having partial derivatives of all orders with respect to x_1 and x_2, but otherwise arbitrary and define

$$\pi(q^\bullet) = \prod_{i=1}^{\infty} \left\{ \frac{1}{(i-1)!} \left(\frac{\partial}{\partial x_1} \right)^{i-1} f(x_1) \right\}^{q_i^\bullet} \tag{4.4}$$

and

$$\pi(q^\circ) = \prod_{i=1}^{\infty} \left\{ \frac{1}{(i-1)!} \left(\frac{\partial}{\partial x_2} \right)^{i-1} f(x_2) \right\}^{q_i^\circ}. \tag{4.5}$$

Let $a_\bullet(q^\bullet, q^\circ)$ denote the number of bicolored planted plane trees of basic colour black and with two color-partitions and $a_\circ(q^\bullet, q^\circ)$ denote the number of bicolored planted plane trees of basic colour white and with two color-partitions. Similarly let $a_\bullet(s; q^\bullet, q^\circ)$(*resp.* $a_\circ(s; q^\bullet, q^\circ)$) be the number of doubly planted bicolored plane trees with s secondary roots, with basic color black (resp. white), and with the same two color-partitions. We write

$$A_{\bullet,\circ} = [a_\bullet(q^\bullet, q^\circ) + a_\circ(q^\bullet, q^\circ)]\pi(q^\bullet)\pi(q^\circ) \tag{4.6}$$

$$A_{\bullet,\circ}(s) = [a_\bullet(s; q^\bullet, q^\circ) + a_\circ(s; q^\bullet, q^\circ)]\pi(q^\bullet)\pi(q^\circ) \tag{4.7}$$

$$J = A_\bullet + A_\circ. \tag{4.8}$$

We are led to the following equations:

$$A_\bullet(s) = (J - A_\bullet)A_\bullet(s+1) + \frac{1}{s!}\left(\frac{\partial}{\partial x_1} \right)^s f(x_1),$$

$$A_\circ(s) = (J - A_\circ)A_\circ(s+1) + \frac{1}{s!}\left(\frac{\partial}{\partial x_2} \right)^s f(x_2)$$

$$A_\bullet = (J - A_\bullet)A_\bullet(1) + f(x_1),$$
$$A_\circ = (J - A_\circ)A_\circ(1) + (x_2)$$

The first pair of equations is valid for any positive integers s. Multiplying the first pair of equations by A^s, summing over s, and adding the second pair of equations, we obtain the pair of functional equations

$$A_\bullet = f(x_1 + J - A_\bullet) = f(x_1 + A_\circ), A_\circ = f(x_2 + J - A_\circ) = f(x_2 + A_\bullet). \tag{4.9}$$

Hence using Lagrange formula for 2 variables

$$\begin{aligned}
A_\bullet &= f(x_1 + f(x_2 + A_\bullet)), \\
&= \sum_{n=1}^{\infty} \frac{1}{n!}\left[\left(\frac{d}{da}\right)^{n-1} f^n(x_1 + f(x_2 + a))\right]_{a=0} \\
&= \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\partial}{\partial x_2}\right)^{n-1} f^n(x_1 + f(x_2)) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\partial}{\partial x_2}\right)^{n-1} \sum_{p=0}^{\infty} \frac{f^p(x_2)}{p!}\left(\frac{\partial}{\partial x_1}\right)^{p} f^n(x_1), \\
&= \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{1}{n!p!}\left(\frac{\partial}{\partial x_1}\right)^{p} f^n(x_1)\left(\frac{\partial}{\partial x_2}\right)^{n-1} f^p(x_2).
\end{aligned} \tag{4.10}$$

We write

$$\sigma^\bullet = \sum_i q_i^\bullet, \qquad \sigma^\circ = \sum_i q_i^\circ$$

and

$$\delta^\bullet = \sum_i^\infty i q_i^\bullet, \qquad \delta^\circ = \sum_i^\infty i q_i^\circ.$$

Let $r(q^\bullet, q^\circ)$ be the coefficient of

$$\prod_{i=1}^{\infty}\left\{\left(\frac{\partial}{\partial x_1}\right)^{i-1} f(x_1)\right\}^{q_i^\bullet}\left\{\left(\frac{\partial}{\partial x_2}\right)^{i-1} f(x_2)\right\}^{q_i^\circ}$$

in $A_\bullet$. Then by (4.4) and (4.5) we have

$$a_\bullet(q^\bullet, q^\circ) = r(q^\bullet, q^\circ) \prod_{i=1}^{\infty} \{(i-1)!\}^{q_i^\bullet + q_i^\circ}.$$

By (4.10) $r(q^\bullet, q^\circ) = 0$ unless there are integers p_1 and p_2 such that

$$\begin{aligned} p_1 &= \sum_{i=1}^{\infty} (i-1) q_i^\bullet = \delta^\bullet - \sigma^\bullet, \\ p_2 &= \sum_{i=1}^{\infty} q_i^\bullet = \sigma^\bullet, \\ p_2 - 1 &= \sum_{i=1}^{\infty} (i-1) q_i^\circ = \delta^\circ - \sigma^\circ, \\ p_1 &= \sum_{i=1}^{\infty} q_i^\circ = \sigma^\circ. \end{aligned}$$

Such integers exist if and only if

$$\delta^\bullet = \sigma^\bullet + \sigma^\circ = \delta^\circ + 1, \tag{4.11}$$

a pair of equations which expresses some elementary properties of a bicolored planted tree with basic color black.

Suppose $q^\bullet$ and q° satisfy (4.11). Then, by (4.10), $r(q^\bullet, q^\circ)$ is the coefficent of

$$\prod_{i=1}^{\infty} \left\{ \left(\frac{\partial}{\partial x_1} \right)^{i-1} f(x_1) \right\}^{q_i^\bullet} \left\{ \left(\frac{\partial}{\partial x_2} \right)^{i-1} f(x_2) \right\}^{q_i^\circ}$$

in

$$\frac{1}{\sigma^\bullet! \sigma^\circ!} \left(\frac{\partial}{\partial x_1} \right)^{\sigma^\circ} f^{\sigma^\bullet}(x_1) \left(\frac{\partial}{\partial x_2} \right)^{\sigma^\bullet - 1} f^{\sigma^\circ}(x_2).$$

Hence

$$r(q^\bullet, q^\circ) = \frac{\sigma^\circ!}{\prod \{(i-1)!\} q_i^\bullet} \times \frac{\sigma^\bullet!}{\prod (q_i^\bullet!)} \times \frac{(\sigma^\bullet! - 1)}{\prod \{(i-1)!\} q_i^\circ} \times \frac{\sigma^\circ!}{\prod (q_i^\circ!)} \times \frac{1}{\sigma^\bullet! \sigma^\circ!}.$$

We deduce that

$$a_\bullet(q^\bullet, q^\circ) = \frac{(\sigma^\bullet! - 1)\sigma^\circ!}{\prod(q_i^\bullet!)\prod(q_i^\circ!)} \tag{4.12}$$

if $q^\bullet$ and q° satisfy (4.11), and $a_\bullet(q^\bullet, q^\circ) = 0$ otherwise. $\square$

Remark 4.1.2. *This formula by Tutte is the number of ordinary (rooted) bicolored plane trees. Respectively, the number of the non-isomorphic ordinary (rooted) bicolored plane trees, each one of them counted with the factor* $\frac{1}{|\operatorname{Aut} T|}$*, is equal to our very first formula* (4.3)*.*

The most surprising consequence of the equation (4.3) is that the Eisenstein number of trees(i.e. polynomial Belyi coverings) depends only on multiplicities $q^\bullet$, q_j° of vertices' degrees, rather than on degrees $d^\bullet$, d° themselves.

4.2 Symmetric group S_n, its characters and linear representations

Basic Notions from Representation Theory

(1) A *representation* of a finite group G on a finite dimensional complex vector space V is a group homomorphism $\rho : G \to GL(V)$ of G to the group of automorphisms of V. The dimension of V is also called the *degree (dimension) of* ρ.
(2) Two representations V and V' are called *isomorphic*, denoted $V \simeq V'$, if there is a G-equivariant isomorphism from V to V', and write $V \cong V'$ if such an isomorphism has been fixed.
(3) A representation V of G is called *irreducible* if it contains no proper subspace which is invariant under the action of G.
(4) Any representation of G is a direct sum of irreducible ones.
(5) (*Schur's Lemma*) Let V and V' be two irreducible representations of G. Then

the complex vector space $Hom_G(V, V')$ is 0-dimensional if $V \not\simeq V'$ and 1-dimensional if $V \simeq V'$. The space $Hom_G(V, V)$ is canonically isomorphic to $\mathbb{C}$.
(6) If (V, ρ) is a representation of G, its *character* is defined as the function $\chi_\rho(g) = tr(\rho(g), V)$ from G to $\mathbb{C}$. For two representations (V, ρ) and (V', ρ') of G, the character corresponding to the vector space $V \otimes V'$ is $\chi_\rho . \chi_{\rho'}$ (The product of two characters is again a character, but not necessarily an irreducible one. However, the product of a character with a linear character (of dimension 1) is irreducible).
(7) (*First orthogonality relation*) Let (V, ρ) and (V', ρ') be two irreducible representations of G. Then

$$\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \overline{\chi_{\rho'}(g)} = \begin{cases} 1 & \text{if } \rho \simeq \rho', \\ 0 & \text{otherwise} \end{cases}$$

Equivalently,

$$\sum_{c \in \mathcal{C}} |C| \chi_\rho(C) \overline{\chi_{\rho'}(C)} = |G| \delta_{\rho,\rho'} \ (\rho, \rho' \in \mathcal{R}),$$

where $\mathcal{C}$ is the set of conjugacy classes in G and $\mathcal{R}$ is the set of isomorphism classes of irreducible representations.
(8) *The group algebra* $\mathbb{C}[G]$ is the set of linear combinations $\sum_{g \in G} \alpha_g [g]$ of formal symbols $[g]$ $(g \in G)$, with the obvious addition and multiplication. Let G be a finite group. Then there is a canonical (GxG)-equivariant algebra isomorphism

$$\mathbb{C}[G] \cong \bigoplus_{i \in I} End_{\mathbb{C}}(V_i)$$

sending $[g]$ to the collection of linear maps $\pi_i(g) : V_i \to V_i$.
(9) The cardinality of $\mathcal{R}$ is finite and $\sum_{\rho \in \mathcal{R}} (\dim \rho)^2 = |G|$.
(10) The sets $\mathcal{C}$ and $\mathcal{R}$ have the same cardinality: there are as many irreducible representations of G as there are conjugacy classes in G.
(11) (*Second orthogonality relation*) Let $C_1, C_2 \in \mathcal{C}$. Then

$$\sum_{\rho \in \mathcal{R}} \chi_\rho(C_1) \overline{\chi_\rho(C_2)} = \begin{cases} |G|/|C_1| & \text{if } C_1 = C_2, \\ 0 & \text{otherwise} \end{cases}$$

This formula agrees with (9) if $C_1 = C_2 = 1$, since $\chi_\rho(1) = \dim \rho$.

For small numbers n, character tables of S_n can easily be calculated using the properties of characters and orthogonality relations above. S_n has the following irreducible representations:

- 1-dimensional representation $\mathbf{1}$ (the trivial representation, $V = \mathbb{C}$ with all elements of G acting as $+1$)
- ε_n (the *sign* representation, $V = \mathbb{C}$ with odd permutations acting as -1)
- (n-1)-dimensional irreducible representation $\mathbf{St}_n$, i.e.,

  $$\{(x_1, ..., x_n) \in \mathbb{C}^n | x_1 + ...x_n = 0\}$$

  which is the subspace of $\mathbb{C}^n$.

For $n = 2$ and $n = 3$, (9) indicates that these are the only irreducible representations (and $\mathbf{St}_n \simeq \varepsilon_n$ for $n = 2$), with character tables given by

C	Id	(1,2)
$\|C\|$	1	1
$\mathbf{1}$	1	1
ε_2	1	-1

C	Id	(1,2)	(1,2,3)
$\|C\|$	1	3	2
$\mathbf{1}$	1	1	1
ε_3	1	-1	1
$\mathbf{St}_3$	2	0	1

For $n = 4$, the character table must take the form

C	Id	(1,2)	(1,2,3)	(1,2)(3,4)	(1,2,3,4)
$\|C\|$	1	6	8	3	6
$\mathbf{1}$	1	1	1	1	1
ε_4	1	-1	1	1	-1
A	2	0	-1	2	0
$\mathbf{St}_4$	3	1	0	-1	-1
$\mathbf{St}_4 \otimes \varepsilon_4$	3	-1	0	-1	1

for some 2-dimensional irreducible representation A of S_4. The set A can be explicitly given by

$$A = \Big\{(x_s)_{s\in S} \mid \sum x_s = 0\Big\},$$

with S is the 3-element set of decompositions of $\{1,2,3,4\}$ into two subsets of cardinality 2.

Remark 4.2.1. *Observe the general fact that the character values $\chi_\rho(g)$ of S_n are all integers.*

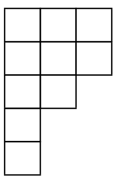
More generally, we will summarize the irreducible representations of S_n using *Young diagrams*:

Theorem 4.2.1. *In any symmetric group S_n, the conjugacy classes correspond naturally to the partitions of n, that is, expressions of n as a sum of positive integers $a_1, ..., a_k$ where the correspondence associates to such a partition the conjugacy class of a permutation consisting on disjoint cycles of length $a_1, ..., a_k$.*

The number of irreducible representations of S_n is the number of conjugacy classes (as in (10)), which is the number of partitions of n, that is, the number of ways to write $n = \lambda_1 + + \lambda_k$, with $\lambda_1 \geq ... \geq \lambda_k$.

Definition 4.2.1. *To a partition $\lambda = (\lambda_1, , \lambda_k)$ is associated a* ***Young diagram*** *with k rows lined up on the left and λ_i boxes in the i^{th} row. The* ***conjugate partition*** *$\lambda' = (\lambda'_1, , \lambda'_r)$ to the partition λ is defined by interchanging rows and columns in the Young diagram, that is, reflecting the diagram on the diagonal.*

For example, the diagram below is that of the partition (3,3,2,1,1) whose conjugate is that of the partition (5,3,2).



For a given Young diagram, number the boxes consecutively as shown below:

1	2	3
4	5	
6	7	
8		

More generally, define a tableau on a given Young diagram to be a numbering of the boxes by the integers $1, ..., n$. Given a tableau, define two subgroups of the symmetric group

$$P = P_\lambda = \{g \in S_n : g \text{ preserves each row}\}$$

$$Q = Q_\lambda = \{g \in S_n : g \text{ preserves each column}\}.$$

In the group algebra $\mathbb{C}S_n$, we introduce two elements corresponding to these subgroups: we set

$$a_\lambda = \sum_{g \in P} e_g \qquad b_\lambda = \sum_{g \in Q} sgn(g) \cdot e_g$$

where e_g is the standard basis elements of $\mathbb{C}^n$ and $sgn(g) = \pm 1$.

Now define *Young symmetrizer*

$$c_\lambda = a_\lambda \cdot b_\lambda.$$

For example, when $\lambda = (n)$, $c_{(n)} = a_{(n)} = \sum_{g \in S_n} e_g$, and the image of $c_{(n)}$ on $V^{\otimes n}$ is $Sym^n V$ (symmetric power). When $\lambda = (1, ..., 1)$, $c_{(1,...,1)} = b_{(1,...,1)} = \sum_{g \in S_n} sgn(g) \cdot e_g$, and the image of $c_{(n)}$ on $V^{\otimes n}$ is $\wedge^n V$ (exterior power).

Theorem 4.2.2. *Some scalar multiple of c_λ is idempotent, that is, $c_\lambda^2 = d_\lambda c_\lambda$ and the image of c_λ (by right multiplication on $\mathbb{C}S_n$) is an irreducible representation V_λ of S_n. Moreover, $d_\lambda = n!/dimV_\lambda$ and every irreducible representation of S_n can be obtained in this way from a unique partition.*

Example 4.2.1. *For any positive integer n, the trivial representation corresponds to the partition $n = n$ while the sign representation corresponds to the partition $n = 1 + ... + 1$. The standard representation $V = \boldsymbol{St}_n$ corresponds to the partition $n = (n-1) + 1$. Moreover, each exterior power $\wedge^k V$ is irreducible for $0 \leq k \leq n-1$, and it corresponds to the partition $n = (n-k) + 1 + ... + 1$.*

Now we want to compute the character of symmetric groups. Let λ be a partition of n. The *Murnaghan-Nakayama Rule* gives a formula for the value of the character χ^λ, on the conjugacy class C_μ(the conjugacy class corresponding to the partition μ) in terms of *rim-hook tableaux.*

Example 4.2.2. *A rim-hook tableau of shape $\lambda = (5, 4, 3, 3, 1)$ and content $\mu = (6, 3, 3, 2, 1, 1)$ is the following tableau and note that the columns and row are weakly increasing, and for each i, the set $H_i(T)$ of cells containing an i is contiguous.*

1	1	1	4	4
1	2	3	3	
1	2	3		
1	2	6		
5				

Theorem 4.2.3. ***(Murnaghan-Nakayama Rule,*** 1937***)*** *Let T be rim-hook tableau with shape λ and content μ.*

$$\chi^{\lambda}(C_{\mu}) = \sum_{T} \prod_{i}^{n} (-1)^{1+ht(H_i(T))}$$

Here ht denotes the heights of $H_i(T)$.

This rule implies the following:

Proposition 4.2.1.

$$dim\ V_{\lambda} = \frac{n!}{\prod(Hook\ lengths)},$$

*where **hook length** of a box in a Young diagram is the number of squares directly below or directly to the right of the box, including the box once.*

4.3 Burnside Theorem

Fact Topologically, coverings $\pi : X \to Y$ of degree n unramified outside k points $y_i \in Y$ are classified by conjugacy classes of homomorphisms $\pi_1 \to S_n$ of the fundamental group $\pi_1 = \pi_1(Y \backslash \{y_1, y_2, ..., y_k\})$, which is known to be defined by the unique relation

$$c_1 c_2 ... c_k [f_1, h_1][f_2, h_2]...[f_g, h_g] = 1, \qquad f_i, h_i \in S_n, c_i \in C_i$$

where g is the genus of Y and the brackets denote the commutator $[f, h] = fhf^{-1}h^{-1}$. Thus the coverings of Riemann sphere $\pi : X \to \mathbb{P}^1$ of given degree n and ramification indices are parametrized by solutions of the equation

$$c_1 c_2 ... c_k = 1, \qquad c_i \in C_i, \tag{4.13}$$

up to conjugacy, where cycle lengths of the conjugacy class $C_i \subset S_n$ are equal to ramification indices of points in fibers $\pi^{-1}(y_i)$.

The following theorem gives the number of solutions of the equation (4.13) for an arbitrary group G in terms of irreducible characters:

Theorem 4.3.1. ***(Burnside)***

$$\#\{c_1c_2...c_k = 1 | c_i \in C_i\} = \frac{|C_1||C_2|...|C_k|}{|G|} \sum_{\chi} \frac{\chi(c_1)\chi(c_2)...\chi(c_k)}{(\chi(1)^{k-2})}. \tag{4.14}$$

Proof. If C is any conjugacy class of G, then the element $e_C = \sum_{g \in C}[g]$ is central and hence, by *Schur's Lemma*, acts on any irreducible representation π of G as multiplication by a scalar $v_\pi(C)$. Since each element $g \in C$ has the same trace $\chi_\pi(g) = \chi_\pi(C)$, we find

$$|C|\chi_\pi(C) = \sum_{g \in C} \chi_\pi(g) = tr(\pi(e_C), V) = tr(v_\pi(C).Id, V) = v_\pi(C)\text{dim } \pi$$

and hence

$$v_\pi(C) = \frac{|C|}{\text{dim } \pi}\chi_\pi(C) = \frac{\chi_\pi(C)}{\chi_\pi(1)}|C|.$$

Now we compute the trace of the action by left multiplication of the product of the elements $e_{C_1}, ..., e_{C_k}$ on both sides of (8) in the previous section. On the one hand, this product is the sum of the elements $[c_1...c_k]$with $c_i \in C_i$ for all i, and since the trace of left multiplication by $[g]$ on $\mathbb{C}[G]$ is clearly $|G|$ for $g = 1$ and 0 otherwise, the trace equals $|G|\#\{c_1c_2...c_k = 1|c_i \in C_i\}$. On the other hand, the product of the e_{C_i} acts as scalar multiplication by $\prod v_\pi(C_i)$ on π and hence also on the $(\text{dim } \pi)^2$-dimensional space $End_{\mathbb{C}}(\pi)$. The formula (4.14) follows immediately. $\square$

4.4 Eisenstein number of coverings and characters of S_n

Using the previous notation in Burnside formula,

Theorem 4.4.1. *The formula for Eisenstein number of coverings $\pi : X \to \mathbb{P}^1$ with prescribed ramification indices is as follows:*

$$\sum_{\pi:X\to\mathbb{P}^1} \frac{1}{|\operatorname{Aut}\pi|} = \frac{|C_1||C_2|...|C_k|}{(n!)^2} \sum_{\chi} \frac{\chi(c_1)\chi(c_2)...\chi(c_k)}{(\chi(1)^{k-2})}. \qquad (4.15)$$

Proof. In view of the Burnside formula it is sufficient to show that

$$\#\{c_1c_2...c_k = 1 | c_i \in C_i \subset S_n\} = \sum_{\pi:X\to\mathbb{P}^1} \frac{n!}{|\operatorname{Aut}\pi|}. \qquad (4.16)$$

According to Fact 1; a solution $\{c_1, c_2, ..., c_k\}$ of the equation in the left-hand side of (4.6) corresponds to a ramified covering $\pi : X \to \mathbb{P}^1$ and

$$\operatorname{Aut}\pi \cong C(c_1, c_2, ..., c_k),$$

where $C(c_1, c_2, ...c_k)$ is the centralizer of the set $\{c_1, c_2, ..., c_k\}$ in S_n. Hence the number of solutions conjugate to $\{c_1, c_2, ..., c_k\}$ is equal to

$$[S_n : C(c_1, c_2, ..., c_k)] = \frac{n!}{|\operatorname{Aut}\pi|}$$

and (4.6) follows. □

Now if we turn to Belyi coverings β with 3 respective conjugacy classes of monodromy permutations for ramification points 0,1 and ∞, the formula above will be as follows:

$$\sum_{\beta:S\to\mathbb{P}^1} \frac{1}{|\operatorname{Aut}\beta|} = \frac{|C_0||C_1||C_\infty|}{(n!)^2} \sum_{\chi} \frac{\chi(c_1)\chi(c_2)\chi(c_3)}{\chi(1)}.$$

More specifically, this formula for polynomial coverings of degree n with ramification scheme $n, d^\bullet, d^\circ$ will be:

$$\frac{|C_n||C_{d^\bullet}||C_{d^\circ}|}{(n!)^2} \sum_{\chi} \frac{\chi(n)\chi(d^\bullet)\chi(d^\circ)}{\chi(1)},$$

or equivalently,

$$\frac{1}{z(d^\bullet)z(d^\circ)}\sum_{0\le l\le n}(-1)^l l!(n-l-1)!\chi_l(d^\bullet)\chi_l(d^\circ), \tag{4.17}$$

where $z(d^\bullet)$ is the order of centralizer of a permutation with cycle structure $d^\bullet$.

This is because the character $\chi(n)$ of n-cycle is equal to $(-1)^l$ for hook Young diagram with leg of length l and 0 otherwise. Therefore the sum in can be restricted to characters χ_l of *exterior powers* $V_l = \wedge^l V_{st}$, $0 \le l \le n-1$ of the standard irreducible representation V_{st} of dimension $n-1$. Recall that the representation $\wedge^l V_{st}$ is irreducible and its Young diagram is a hook with leg of length l. Clearly $\chi_l(1) = \dim V_l = \binom{n-1}{l}$ and $|C_n| = (n-1)!$.

Chapter 5

Exceptional Belyi Coverings

A Belyi covering is uniquely defined by either ramification scheme or by respective dessin d'enfant. We want to characterize such *unique* coverings completely defined by respective ramification schemes; i.e. the ramification indices in fibers 0, 1, and ∞. Well-known examples are cyclic, dihedral and Chebyshev coverings. Cyclic coverings, dihedral coverings and the coverings of regular polyhedra ([5], [14]) are known as Klein's coverings. We found a number of rational exceptional coverings which include Klein's coverings and Chebyshev polynomials. But we also found new infinite series. In Section 3.2, we give a classification of exceptional polynomial coverings by Adrianov ([1]). Lastly, we state the formulae for Eisenstein number of coverings in Chapter 4 for exceptional coverings. We created an algorithm in MAPLE which finds all coverings with a given genus and degree. We determined all rational exceptional Belyi coverings up to degree 15. We managed to calculate all respective Belyi functions up to degree 6 and also for some of degree 7. Thus, we gave the table of all rational exceptional Belyi coverings up to degree 6 with ramification schemes, Belyi functions and dessins d'enfants. We also added some rational exceptional coverings of degree 7 to this table. Some rational coverings come from modular curves.

A table to illustrate rational coverings with ramification schemes, dessins and Belyi functions is given in APPENDIX A.

5.1 Definition and examples

Definition 5.1.1. *Let S_g be a compact Riemann surface with genus g and*

$$\beta : S_g \to \mathbb{P}^1$$

be a Belyi covering. β is said to be an ***exceptional Belyi covering*** *iff it is uniquely determined by its ramification scheme.*

Remark 5.1.1. *There exists unique dessin with given degrees of vertices which corresponds to exceptional Belyi covering.*

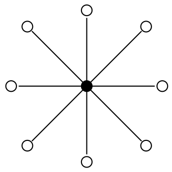
There are some examples illustrating these exceptional coverings: Klein studied the first three examples by classifying $G \subset \mathbb{P}GL(2, \mathbb{C})$ as cyclic, dihedral, cubic etc. where $G = Aut(\mathbb{P}^1, \beta)$ in the natural projection $\mathbb{P}^1 \to \mathbb{P}^1/G$.

Example 5.1.1. *(Cyclic Covering)(Infinite Series) This is a rational covering*

$$\begin{aligned} \beta : \mathbb{P}^1 &\to \mathbb{P}^1 \\ z &\mapsto z^n \end{aligned}$$

The covering is called as "cyclic" due to the fact that the group $G = \mathrm{Aut}(\mathbb{P}^1, \beta)$ in the natural projection $\mathbb{P}^1 \to \mathbb{P}^1/G$ is cyclic.

The ramification scheme is of the form $[n][n]\underbrace{[1, 1, ..., 1]}_{n\ times}$.



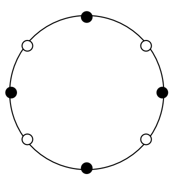
Star-Tree

Example 5.1.2. *(Dihedral Covering)(Infinite Series)*

$$\beta : z \mapsto z^n + \frac{1}{z^n}$$

The covering is called as "dihedral" due to the fact that the group $G = \mathrm{Aut}(\mathbb{P}^1, \beta)$ *in the natural projection* $\mathbb{P}^1 \to \mathbb{P}^1/G$ *is dihedral.*

The ramification scheme is $[p, p]\underbrace{[2, 2, ..., 2]}_{n/2\ times}\underbrace{[2, 2, ..., 2]}_{n/2\ times}$, *where* $2p = n$.



Example 5.1.3. *In three-dimensional space, a **Platonic solid** is a regular, convex polyhedron. It is constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex. Five solids meet those criteria, and each is named after its number of faces: tetrahedron (4 faces), cube-hexahedron- (6 faces), octahedron (8 faces), icosahedron (12 faces), dodecahedron (24 faces).*

*Every polyhedron has a **dual** (or **polar**) polyhedron with faces and vertices interchanged:*

- *The tetrahedron is self-dual (i.e. its dual is another tetrahedron).*
- *The cube and the octahedron form a dual pair.*

- *The dodecahedron and the icosahedron form a dual pair.*

Every polyhedron has an associated symmetry group, which is the set of all transformations (Euclidean isometries) which leave the polyhedron invariant. There are 3 *symmetry groups of finite orders (not* 5 *since this group of a polyhedron coincides with that of its dual):*

- *The symmetry group of tetrahedron is*

$$A_4 = \langle r, s | s^2 = r^3 = (sr)^3 = 1 \rangle$$

- *The symmetry group of octahedron(cube) is*

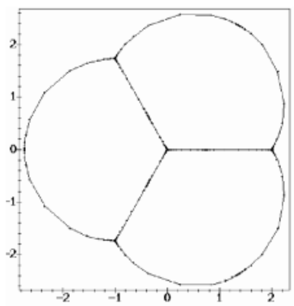
$$S_4 = \langle r, s | s^2 = r^3 = (sr)^4 = 1 \rangle$$

- *The symmetry group of icosahedron(dodecahedron) is*

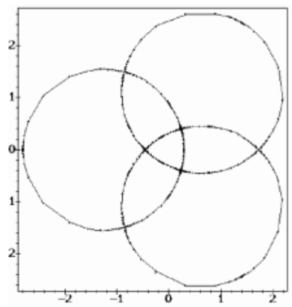
$$A_5 = \langle r, s | s^2 = r^3 = (sr)^5 = 1 \rangle$$

The Belyi functions for platonic solids were computed by Felix Klein as stated in [9].

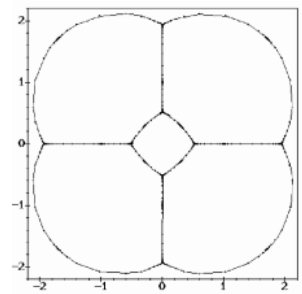
It is obvious that if f is a Belyi function for a map, then $1/f$ is a Belyi function for the dual map.



Tetrahedron

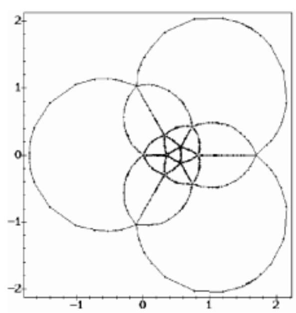


Octahedron

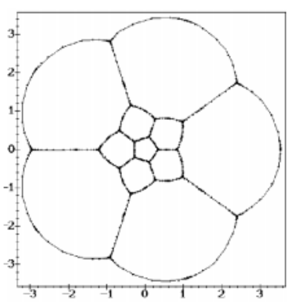


Cube

$f(z) = -64\frac{(z^3+1)}{(z^3-8)^3 z^3}$ $\quad f(z) = -108\frac{(z^8-14z^4+1)^3}{(z^4+1)^4 z^4}$ $\quad f(z) = -108\frac{(z^4+1)^4 z^4}{(z^8-14z^4+1)^3}$



Icosahedron



Dodecahedron

$f(z) = 1728\frac{(z^{10}-11z^5-1)^5 z^5}{(z^{20}+228z^{15}+494z^{10}-228z^5+1)^3}$ $\quad f(z) = 1728\frac{(z^{20}+228z^{15}+494z^{10}-228z^5+1)^3}{(z^{10}-11z^5-1)^5 z^5}$

Example 5.1.4. *(Chebyshev Covering)(Infinite Series)*
As we have seen in Chapter 3*, the Chebyshev polynomial is* $T_n(\cos x) = \cos nx$*. The corresponding dessin with the ramification scheme*

$$\begin{cases} [n][1,2,2...,2][1,2,2...,2] & ;\ n \text{ is odd.} \\ [n][1,1,2,2...,2][2,2,...,2] & ;\ n \text{ is even.} \end{cases}$$

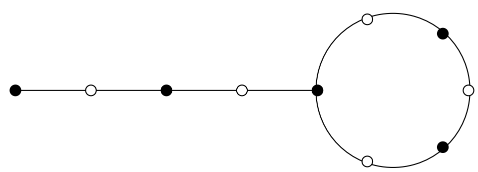
is

Chain-Tree

Now we are going to introduce new infinite series we found.

There is an "interpolating series between Chebyshev and dihedral covering". The ramification scheme for this series is

$$\begin{cases} [p,q][2,2...,2,3][1,2,2,...,2] & ;\ \text{n=p+q is odd.} \\ [p,q][1,2,2...,2,3][2,2...,2] & ;\ \text{n=p+q is even.} \end{cases}$$



The respective Belyi function can be expressed as

$$f(t) = e^{qti}\left[\frac{\alpha - e^{ti}}{1-\alpha e^{ti}}\right]^p + e^{-qti}\left[\frac{\alpha - e^{-ti}}{1-\alpha e^{-ti}}\right]^d,$$

where $\alpha = \frac{q+p}{q-p}$.

If $p = q$, this turns out to be a dihedral covering and if $q = 0$, then it will be a Chebyshev covering.

We used MAPLE to calculate the respective Belyi function and also we could have seen the real graphs and dessins simultaneously again using our MAPLE code. We will give the MAPLE routines for the rational exceptional Belyi covering of degree 9 of this type (One can obtain other series by simply changing the degree in the code):

```
>restart:
digits:=20:
deg:=9;
d:=3;
n:=deg-d; alpha:=(n+d)/(n-d);
```

```
'pole'=(n^2+d^2)/(n^2-d^2);
if d=0 then pole:=1 else pole:=(alpha+1/alpha)/2 fi:
#FlexPoint=1;
print(`In general n=deg-d`, 'alpha=(n+d)/(n-d)', `and the function has a pole at`, 'a=(n^2+d^2)/(n^2-d^2)');
## a=pole
'f0=exp(n*I*t)*((alpha-exp(I*t))/(1-alpha*exp(I*t)))^d+exp(-n*I*t)*((alpha-exp(-I*t))/(1-alpha*exp(-I*t)))^d';
f0:=exp(n*I*t)*((alpha-exp(I*t))/(1-alpha*exp(I*t)))^d+exp(-n*I*t)*((alpha-exp(-I*t))/(1-alpha*exp(-I*t)))^d;
convert(%,trig):
expand(%):
print(`Converting into a rational functiion of`, x=cos(t));
f0:=simplify(%):
f0:=sort(subs(cos(t)=x,f0)/2):
#f0:=factor(f0);
#convert(f_0,parfrac);
#[fsolve(numer(f_0),x,complex)];
#map(x->abs(x),%);
##factor(%);convert(%,parfrac);
'f=(f0+1)/2';
f:=factor((f0+1)/2); ## To get ramification points 0,1;
print(`The basic equation`);
`f-1`=sort(factor(f-1));
rt0:=[fsolve(numer(f),x,complex)]:
rt1:=[fsolve(numer(f)=denom(f),x,complex)]:
rt:=[op(rt0),op(rt1)]:
Imrt:=map(z->Im(z),rt):
bnd:=max(op(Imrt)):
#expand(subs(x=x+pole,f));
#expand(f);
#g:=sort(simplify(%));
#f:=convert(f, parfrac,x);
print(`Exceptional Belyi covering with ramification scheme`);
if type(deg,even) and d>0 then [d,deg-d],[1,seq(2, i=1..deg/2-2),3],[seq(2, i=1..deg/2)];
elif type(deg,even) and d=0 then [deg],[1,1,seq(2, i=1..deg/2-1)],[seq(2, i=1..deg/2)] fi;
if type(deg,odd) and d>0 then [d,deg-d],[seq(2, i=1..(deg-3)/2),3],[1,seq(2,i=1..(deg-1)/2)];
elif type(deg,odd) and d=0 then [deg],[1,seq(2, i=1..(deg-1)/2)],[1,seq(2, i=1..(deg-1)/2)] fi;
cr0:=map(x->[x,-1],[op({fsolve(numer(f))})]):
cr_0:=plot(cr0, style=point,symbol=circle,color=BLACK):
cr1:=map(x->[x,1],[op({fsolve(numer(f-1))})]):
cr_1:=plot(cr1, style=point,symbol=cross, color=BLACK):
pl2:=plot([2*f-1,1,0,-1,[pole,t,t=-10..10]], x=-1.5..2*pole-1,
y=-bnd-3/2..bnd+3/2, color=[red,grey,grey,grey,grey],discont=true,numpoints=200,axes=boxed):
#plot(exp(n*I*t)*((alpha-exp(I*t))/(1-alpha*exp(I*t)))^d+exp(-n*I*t)*((alpha-exp(-I*t))/(1-alpha*exp(-I*t)))^d,
#t=-Pi..0,axes=boxed); with(plots,rootlocus,complexplot,display):
pl0:=rootlocus( f/factor(normal(f-1)), x, 0..1, scaling=constrained,symbol=cross,
symbolsize=10,thickness=2,numpoints=300):
```

```
pl1:=rootlocus( factor(normal(f-1))/f, x, 0..1, scaling=constrained,symbol=circle, symbolsize=10, thickness=2):
display({pl0, pl1,pl2,cr_0,cr_1}, axes=boxed, scaling=constrained,labels=[``,``],
title=`Real plot and Dessin d'enfant`,titlefont=[COURIER,10]); #l := [1+2*I, 3+4*I, 5-I, 7-8*I];
#complexplot(rt0, x=-2..10, style=point,color=red,symbol=circle,axes=boxed,scaling=constrained);
#complexplot(rt1, x=-2..10, style=point,symbol=circle,color=blue,axes=boxed);
```

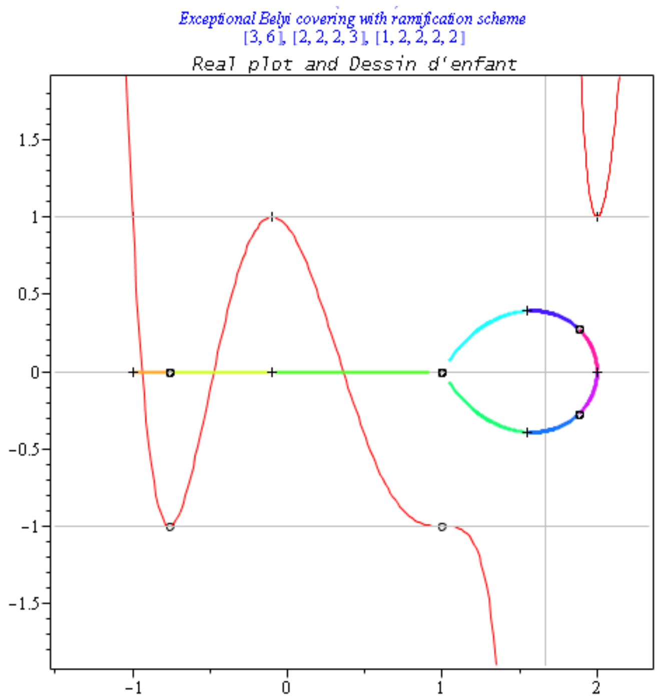


Figure 5.1: Interpolating series of degree 9

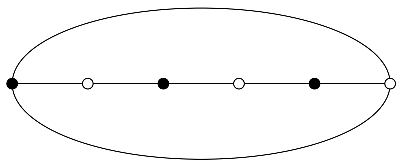
As we can see from the MAPLE code above, $d = p$ and $n = q$. So, the respective Belyi function for the ramification scheme $[3, 6][2, 2, 2, 3][1, 2, 2, 2, 2]$ is

$$f(t) = e^{6ti}\left[\frac{3 - e^{ti}}{1 - 3e^{ti}}\right]^3 + e^{-6ti}\left[\frac{3 - e^{-ti}}{1 - 3e^{-ti}}\right]^3$$

This function has a pole at $\frac{5}{3}$ (In general, the pole is at $\frac{n^2+d^2}{n^2-d^2}$).

There is also another series we found which is of odd degree:

The ramification scheme is $[1,p,p][\underbrace{2,2,2,...,2}_{p-1 \text{ times}},3][2,2,...,2,3]$.



The MAPLE code for this series is the following (We give the code for degree 13, again one can change the input degree):

```
>restart:
n:=6:deg:=2*n+1:
## unassign(seq('a||i',i=1..n-1)):
a_1:=n->if type(n,odd) then (2*n^2-2) else -2*n^2-1 fi:
a1:=a_1(n):
a_n1:= n-> if type(n,odd) then (-1)^((n-1)/2 mod 2)*(2*n-1)^n else (-1)^(n/2 mod 2)*(2*n-1)^n fi:
a||(n-1):=a_n1(n):
p:=1+add(a||i*z^i,i=1..n-1):
q:=subs(z=-z,p):
eq_1:=2*((4*n^2-1)*z^2+1)^n+q^2*(z-1)^3=p^2*(z+1)^3:
f:=p^2*(z+1)^3/2/((4*n^2-1)*z^2+1)^n:
expand(rhs(eq_1)):collect(%,z):
eq:=lhs(eq_1)-rhs(eq_1):eq:=expand(%):
eq:=collect(eq,z):
cff:=[coeffs(eq,z)]:
st:=time():sol:=solve(cff,[seq(a||i,i=2..n-2)]):
print(Time=time()-st):
                                 Time = 0.015
with(plots,rootlocus,complexplot,display):
id:=subs(op(sol),eq_1):f:=subs(op(sol),f):
cr0:=map(x->[x,-1],[op({fsolve(numer(f))})]):
cr_0:=plot(cr0, style=point,symbol=circle,color=BLACK):
cr1:=map(x->[x,1],[op({fsolve(numer(f-1))})]):
cr_1:=plot(cr1, style=point,symbol=cross, color=BLACK):
## f:=normal(subs(z=1/z,f));
pl2:=plot([2*f-1,1,0,-1], z=-1.2..1.2,
y=-1.6..1.6, color=[red,grey,grey,grey#,grey,grey,grey
],discont=true,numpoints=200,axes=boxed):
```

```
pl0:=rootlocus( f/(f-1), z, 0..1, scaling=constrained,symbol=cross,symbolsize=10,thickness=2,numpoints=300):
pl1:=rootlocus( factor(normal(f-1))/f, z, 0..1, scaling=constrained,symbol=circle, symbolsize=10, thickness=2):
print(`Fundamental identity of the exceptional covering with ramification scheme`,
[1,n,n],[seq(2,i=1..n-1),3],[seq(2,i=1..n-1),3]);
id;
print(`and the respective Belyi function`);
'f'=f,`slope at infinity`='(1/2)*((2*n-1)/(2*n+1))^n',(1/2)*((2*n-1)/(2*n+1))^n,evalf(1/2*((2*n-1)/(2*n+1))^n);
display({pl0, pl1,pl2,cr_0,cr_1}, axes=boxed, scaling=unconstrained,labels=[``,``],
 title=`Plot of 2f-1 and Dessin d'enfant`,titlefont=#[HELVETICA,10]);#
[COURIER,10]);
```

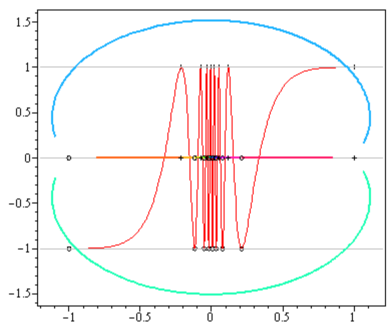


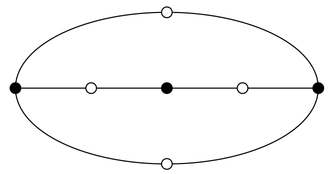
Figure 5.2: Series of odd degree

As we can see from the figure above, the respective Belyi function (calculated by using the MAPLE code above) for the ramification scheme $[1,6,6][2,2,2,2,2,3][2,2,2,2,2,3]$ is very complicated:

$$f(z) = \frac{1}{2}\frac{(1-129z-6915z^2+236675z^3+4142595z^4-63321987z^5-356457217z^6+2562890625z^7)^2(z+1)^3}{(255z^2+1)^8}.$$

There are more exceptional series for which we do not know the respective Belyi covering. The following series of even degree is one of them:

The ramification scheme is $[2,p,p][3,3,2,,2...,2][2,2,...,2]$.



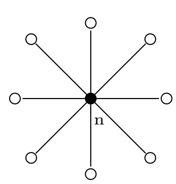
There are $2p$ edges on the middle line segment.

A complete classification of of rational exceptional Belyi coverings is still an open problem. In particular we do not know whether or not the number of exceptional series is finite or infinite.

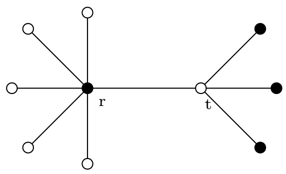
5.2 Classification of exceptional polynomial coverings

Remember that dessin d'enfants corresponding to polynomial coverings are trees.For a given ramification scheme, the number of trees in its family gives the upper bound for the degree of the corresponding field (or fields, if there are several orbits). For example, if the tree is unique, it is defined over $\mathbb{Q}$. This remark leads to the following problem: *For a given k, enumerate all ramification scheme $[n][\alpha][\beta]$ such that there exist exactly k trees with this passport.* We call this problem *"the inverse enumeration problem"* referring to bicolored plane trees. It goes without saying that the similar question may be asked about plane maps, plane hypermaps, maps of genus 1, etc.

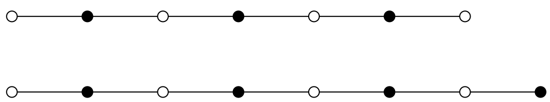
For trees and for $k = 1$ this problem was solved by N. Adrianov ([1], [12]):

(1) 

$[n][n][\underbrace{1, 1, ..., 1}_{\text{n times}}]$(the "star-trees")

(2) 

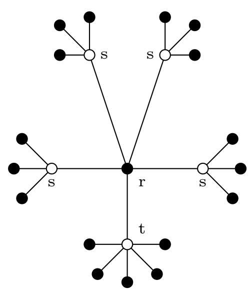
$$[n][r,\underbrace{1,1,...,1}_{\text{t-1 times}}][t,\underbrace{1,1,...,1}_{\text{r-1 times}}](n=r+t-1)$$

(3) 

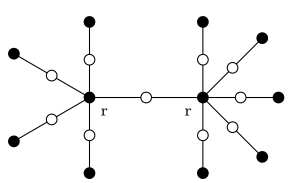
$$[2\text{m}][\underbrace{2,2,...,2}_{\text{m times}}][\underbrace{2,2,...,2}_{\text{m-1 times}},1,1]$$

$$[2\text{m+1}][\underbrace{2,2,...,2}_{\text{m times}},1][\underbrace{2,2,...,2}_{\text{m times}},1]$$

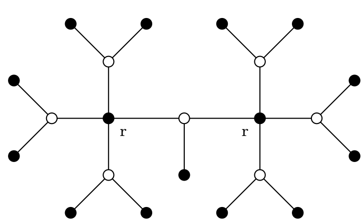
(the "chain-trees")

(4) 

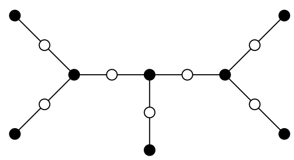
$$[n][r,\underbrace{1,1,...,1}_{\text{p times}}][\underbrace{s,s,...,s}_{\text{q times}},t](n=r+p=qs+t)$$

(5) 

$$[n][r,s,\underbrace{1,1,...,1}_{\text{p times}}][\underbrace{2,2,...,2}_{\text{q times}},t](n=r+s+p=2q)$$

(6) 

$$[n][r,r,\underbrace{1,1,...,1}_{\text{p times}}][\underbrace{3,3,...,3}_{\text{q times}},t](n=2r+p=3q)$$

(7)  [14][3, 3, 3, 1, 1, 1, 1, 1][2, 2, 2, 2, 2, 2, 2]

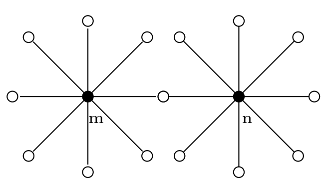
We see that the six first classes are,in fact, infinite series of passports and trees, while the seventh one is a single tree which does not enter into any series; one may call it a *sporadic tree*. We may also note that the second class is in fact a particular case of the fourth one, with $s = 1$.

Example 5.2.1. *The corresponding Belyi function for the class* (2) *is*

$$\beta(x) = \frac{(1-x)^{r+1}}{r!} \frac{d^r}{dx^r} \Big(\frac{x^{r+t+1} - 1}{x - 1} \Big).$$

Example 5.2.2. *The respective Belyi function for gluing of two star trees (note that it also belongs to one of the classes above) is the* Belyi polynomial

$$\beta(x) = x^m (1-x)^n \frac{(m+n)^{m+n}}{m^m n^n}.$$



5.3 Counting exceptional Belyi coverings

We stated Burnside formula in Chapter 4 to find the Eisenstein number of coverings with given ramification schemes and gave also a special case for Belyi coverings β

ramified over three points:

$$\sum_{\beta:S\to\mathbb{P}^1}\frac{1}{|\operatorname{Aut}\beta|}=\frac{|C_0||C_1||C_\infty|}{(n!)^2}\sum_{\chi:irreducible}\frac{\chi(c_1)\chi(c_2)\chi(c_3)}{\chi(1)}, \tag{5.1}$$

where $\chi(c_i)$ are the irreducible characters of permutations c_i in the ramification scheme and $|C_i|$ are the size of conjugacy classes with representatives c_i. Also, we stated Tutte formula for bicolored trees(corresponding to polynomial coverings):

$$\sum_T\frac{1}{|\operatorname{Aut}T|}=\frac{1}{\sigma^\bullet\sigma^\circ}\binom{\sigma^\bullet}{q^\bullet}\binom{\sigma^\circ}{q^\circ}, \tag{5.2}$$

where $q_i^\bullet$, q_i° is the number of black and white vertices of degree i respectively and $\sigma^\bullet=\sum_i q_i^\bullet$, $\sigma^\circ=\sum_j q_j^\circ$.

The sums on the left-hand side in two formulas above simply reduces to the inverse of an integer when the case is exceptional Belyi coverings: By definition, these Belyi coverings are unique, so the Eisenstein number will be $\frac{1}{|\operatorname{Aut}\beta|}$ and $\frac{1}{|\operatorname{Aut}T|}$.

Remark 5.3.1. *However, the formulae above do not distinguish connected dessin d'enfants and therefore this should be checked; i.e. the monodromy action should be transitive.*

We developed a MAPLE algorithm finding all exceptional Belyi coverings with a given genus and degree:

```
with(combinat);
with(group);
with(combstruct);
#######################
n:=N: # degree of a covering with three singular fibers
gen:=g: #genus of the covering
print(`Genus`=gen,`Degree`=n);
tot:=time():
ABC:=iterstructs(Partition(n+2-2*gen),size=3):##
while not finished(ABC) do
```

```
st:=time():
abc:=nextstruct(ABC): # the number of points in singular fibers
a:=abc[1]:b:=abc[2]:c:=abc[3]:
print('a'=a,'b'=b,'c'=c,`degree`=n);
#n:=a+b+c:       # degree of the covering to ensure that genus=1,
                 # *provided* the covering consists of one component
sol:=0: ## count combinatorail types of covering print('a'=a,'b'=b,'c'=c,degree=n);
unit:=[seq(1,i=1..n)]: ## class of identical permutation
S||n:=permgroup(n, {[[1,2]],[[seq(i,i=1..n)]]}): #symmetric group S_n
P_a := iterstructs(Partition(n),size=a): #ramification indices at fiber a
P_b := iterstructs(Partition(n),size=b): #ramification indices at fiber b
P_c := iterstructs(Partition(n),size=c): #ramification indices at fiber c
while not finished(P_a) do p_a:=nextstruct(P_a):
red_a:=map(x->if x=1 then NULL else x fi,p_a): ## removing cycles of length one
C_a:=SnConjugates(S||n,red_a): # order of the conjugacy class of p_a
while not finished(P_b) do p_b:=nextstruct(P_b):
red_b:=map(x->if x=1 then NULL else x fi,p_b):
C_b:=SnConjugates(S||n,red_b): # order of the conjugacy class of p_b
while not finished(P_c) do p_c:=nextstruct(P_c):
red_c:=map(x->if x=1 then NULL else x fi,p_c):
C_c:=SnConjugates(S||n,red_c): # order of the conjugacy class of p_c
     Rep := iterstructs(Partition(n)): ## Young diagrams = irreps of S_n
     Cov:=0: ## Counting coverings
     while not finished(Rep) do
        rep:=nextstruct(Rep):
Cov:=Cov+Chi(rep,p_a)*Chi(rep,p_b)*Chi(rep,p_c)/Chi(rep,unit):
     od;
        Cov:=Cov*C_a*C_b*C_c/(n!)^2: ## Eisenstein number of coverings

     g:=1/2*(add(x-1, x in p_a)+add(x-1, x in p_b)+add(x-1, x in p_c))-n+1; ## genus, should be = 1.
     if Cov>0 and type(1/Cov,integer)
     then sol:=sol+1:
       print(`Ramification indices `= p_a, p_b, p_c);
        print(genus=g,`Eisenstein number of coverings`=Cov):
     fi:
od:
P_c := iterstructs(Partition(n),size=c): #ramification indices at fiber c
od:
P_b := iterstructs(Partition(n),size=b): #ramification indices at fiber b
od:
if sol>0 then print(`The number of combinatorial types`=sol, `time`=time()-st)
else print(`No solution`,`time`=time()-st) fi:
od:
print(`Total time`=time()-tot);
```

Note that N and g are inputs for degree and genus respectively.

This code allows us to find all rational exceptional Belyi coverings (of genus 0) up to degree 15. Then, we had a very extensive MAPLE calculation enabling us to determine all Belyi functions up to degree 6 in the genus 0 case. Besides, we could have managed to calculate some Belyi functions of degree 7 with respective ramification schemes. Some rational coverings come from modular curves. In order to fully present our results, we gave a table in APPENDIX A for all rational coverings up to degree 6 and some of degree 7 with respective ramification schemes, Belyi functions and dessins d'enfants.

Our MAPLE routines for computation of Belyi functions are based on "the fundamental identities". The fundamental identities are formed according to the fact that the ramification schemes illustrate the degree of the faces and the degree of the vertices in the dessin where black vertices correspond to the roots of the equation $f(x) = 0$ and white vertices correspond to $f(x) = 1$. Let us illustrate this with two examples we choose from the table in APPENDIX A.

Example 5.3.1. *Consider the line* $R5.3$ *in our table with ramification scheme* $[5][2,3][1,1,1,2]$. *Notice that this is a polynomial (Shabat) covering and call it* f. *We know that the roots of* $f(x) = 0$ *correspond to black vertices in the respective dessin with degrees* 2 *and* 3. *Similarly, the roots of* $f(x) = 1$ *correspond to white vertices in the respective dessin with degrees* $1, 1, 1$ *and* 2. *Therefore,* f *has to be of the form* $f(x) = (x - a_1)^2(x - a_2)^3$. *On the other hand,* $f - 1$ *can be written as* $(x^3 + b_1x^2 + b_2x + b_3)(x^2)$. *If we write the equation*

$$(x - a_1)^2(x - a_2)^3 = 1 + (x^3 + b_1x^2 + b_2x + b_3)(x^2),$$

then this will be our "fundamental identity". We then find all coefficients with the help of MAPLE and take the Belyi function simply as f.

What happens in a rational covering which is not a polynomial?

Example 5.3.2. *Consider the line* $R6.8$ *in our table with ramification scheme* $[3, 3][2, 2, 2][2, 2, 2]$ *which is the dihedral covering. Notice that* 3 *is the degree of both of the faces. So, again we can form the fundamental identity as follows:*

$$(x^3 + a1x^2 + a2x + a3)^2 = b1x^3 + (x^3 + c1x^2 + c2x + c3)^2$$

After solving the coefficients, we cannot simply take the left-hand side of the identity this time. This is because, this function has a pole of degree 3 *and we have to express it in the denominator by writing the function as a fraction. Therefore, our corresponding Belyi function should be of the form*

$$\frac{(x^3 + a1x^2 + a2x + a3)^2}{b1x^3}$$

The Belyi functions for infinite series were given in Section 5.3.

Chapter 6

Exceptional Belyi Coverings with Genus $g > 0$

Recall that an exceptional Belyi covering with genus 1 is usually called as *elliptic* exceptional covering. We have not so much to say for elliptic coverings in contrast to rational coverings (of genus 0) as we mentioned in Chapter 5. However, we still have a table for elliptic coverings up to degree 12 in APPENDIX B. We are going to illustrate with an example why we could not give all the dessins for elliptic coverings. We will also give another example referring the term "double-covering" as it is often used in our table. On the other hand, MAPLE did not find any exceptional Belyi covering with genus ≥ 2, which are called as *hyperbolic* coverings.

6.1 Exceptional Belyi coverings with genus 1

An*elliptic* exceptional Belyi covering is an exceptional Belyi covering $\beta : S \to \mathbb{P}^1$ when the compact Riemann surface S has genus $g = 1$.

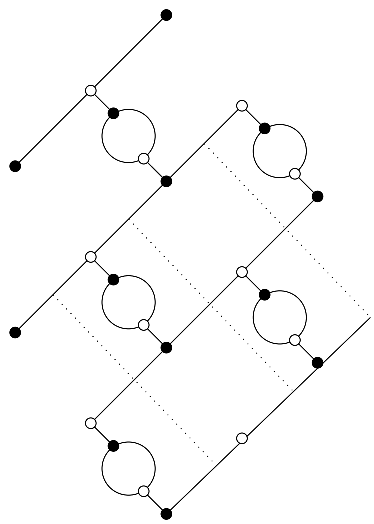
The MAPLE code in section 5.3 also allows us to determine elliptic coverings. However, our knowledge for elliptic coverings is limited in contrast to rational coverings. Our results are limited up to degree 12 and also bounded by ramification schemes, monodromy permutations and Eisenstein numbers. We are not able to list all Belyi pairs for elliptic coverings. Similarly, the respective dessins are not easy to be drawn. Still we gave a table in APPENDIX B for elliptic coverings up to degree 12 and we added some descriptions regarding whether they come from modular curves or they are referred as $double/triple - coverings$. The code shows that there is no elliptic covering of degree 7.

Recall that we are working on torus when $g = 1$ and Example 6.1.1 shows why we did not give respective dessins in APPENDIX B.

Example 6.1.1. *The following dessin d'enfant is drawn to be like a "wall-paper". Remember the fact that a compact Riemann surface of genus* 1 *is equivalently a torus. However, trying to draw the respective dessin over this surface is neither an easy nor an illustrative thing to do. Therefore, we consider torus as* $\mathbb{C}/\lambda$*, where* λ *is a lattice.*

The respective ramification scheme for the following dessin is $[1,5][3,3][3,3]$*. This corresponds to the line* $E6.4$ *in our table. We also have the information of monodromy permutation of this dessin:*

$$(1)(2,3,4,5,6) \mid (4,2,1)(3,5,6) \mid (1,6,4)(2,3,5).$$



There are some descriptions in our table such as double covering, triple covering, or k-covering for some integer k. Let us elaborate these notions with an example from the table.

Example 6.1.2. *Consider the line* $E6.1$ *in APPENDIX B with the ramification scheme* $[6][3,3][2,2,2]$. *Recall from Chapter* 3, *Section* 3.4 *how we form elliptic coverings: Consider the rational covering corresponding to* $[3][3][1,1,1]$ *which is* $z \mapsto x^3$. *It is a star-tree with one black vertex and 3 white vertices. This shows that one black vertex corresponds to 0 with multiplicity 3, the solution of* $x^3 = 0$ *and each white vertex corresponds to the solutions of* $x^3 = 1$, *with multiplicity* 1. *The roots are* 1, ξ *and* ξ^2, *where* $\xi = e^{\frac{2\pi}{3}}$. *Now consider the elliptic curve including the square roots of the roots of* $x^3 - 1$. *Call this curve* $E : y^2 = (x^3 - 1)$ *and consider the projection* $E \to \mathbb{P}^1$ *by* $(x, y) \mapsto x$.

$$E \to \mathbb{P}^1 \to \mathbb{P}^1$$

The composition in this way yields to a Belyi covering. We call this new covering as a double covering of the rational covering.

6.2 Exceptional Belyi coverings with genus $g \geq 2$

An exceptional *hyperbolic* Belyi covering is an exceptional Belyi covering $\beta : S \to \mathbb{P}^1$ when the compact Riemann surface S has genus $g \geq 2$.

The calculations via MAPLE (the code in Section 5.3) could not lead to an exceptional Belyi covering with genus $g \geq 2$ of degree 18 or less. The absence of exceptional hyperbolic Belyi coverings is a mystery we could not explain.

Chapter 7

Arithmetics of Rational Exceptional Belyi Coverings

We are trying to find the (minimal) field of definition of a rational exceptional Belyi covering (an exceptional Belyi covering of genus 0); i.e. the smallest field in which both the Riemann surface, or equivalently the corresponding algebraic curve, and the coefficients of the covering are defined. We will define "principal homogeneous space", "form", "quadratic form", "generalized Quaternion algebra", "matrix algebra" and then state the correspondences between them ([18] and [19]). Using these correspondences, we will be able to state the field of definition of a rational exceptional Belyi covering. We will not discuss the field of definition of exceptional coverings of higher genus. The case for elliptic coverings (coverings of genus 1) is more complicate and the degree of the field of definition may be arbitrarily big, in contrast to rational coverings.

7.1 Basic notions

Definition 7.1.1. *A **principal homogeneous space** for a group G is a non-empty set X on which G acts freely and transitively, i.e. for any $x, y \in X$ there exists a unique g in G such that $xg = y$ where denotes the (right) action of G on X.*

Definition 7.1.2. *Let k be a field, K a field of extension of k and X an "object" defined over k. An object "Y" will be a K/k**-form (k-form)** if Y becomes isomorphic to X when the ground field is extended to K.*

Definition 7.1.3. *An algebra over a field $\mathbb{F}$ is said to be a **form of matrix algebra** $Mat(2,\mathbb{F})$ iff $A \otimes \overline{\mathbb{F}} \simeq Mat(2,\overline{\mathbb{F}})$. Here $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$.*

Example 7.1.1. *Consider "the Hamilton quaternions"*

$$\boldsymbol{H} = \langle 1, i, j, k | i^2 = j^2 = k^2 = -1, ijk = -1 \rangle.$$

The operations are as follows:

$$\begin{aligned} ij &= k & ji &= -k \\ jk &= i & kj &= -i \\ ki &= j & ik &= -j \end{aligned}$$

Any element $q \in \boldsymbol{H}$ is of the form $q = x + yi + zj + tk$ where $x, y, z, t \in \mathbb{R}$. Let q_1 and q_2 be two elements in $\boldsymbol{H}$ such that $q_1 = x_1 + y_1 i + z_1 j + t_1 k$ and $q_2 = x_2 + y_2 i + z_2 j + t_2 k$. Then addition and multiplication are defined as follows:

- $q_1 + q_2 = (x_1 + x_2) + (y_1 + y_2)i + (z_1 + z_2)j + (t_1 + t_2)$
- $q_1 q_2 = (x_1 x_2 - y_1 y_2 - z_1 z_2 - t_1 t_2) + (x_1 y_2 - y_1 x_2 - z_1 t_2 - t_1 z_2)i$
  $+ (x_1 z_2 - y_1 t_2 + z_1 x_2 - t_1 y_2)j + (x_1 t_2 - y_1 z_2 - z_1 y_2 - t_1 a_2)k$
  (using the product formulae of quaternionic units i, j and k)

$\boldsymbol{H}$ is an algebra over $\mathbb{R}$. We can also consider it as a vector space over $\mathbb{R}$ of dimension 4.

The "conjugate" of an element q is defined as $\overline{q} = x - yi - zj - tk$. $q\overline{q} = x^2 + y^2 + z^2 + t^2$ and $\|q\| := \sqrt{q\overline{q}}$ is called as the "norm" of q.

$\mathbb{H}$ can be represented as 2×2 complex matrices: If we denote

$$\boldsymbol{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \boldsymbol{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then, $i^2 = j^2 = -1$ and $k = ij = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

More generally any element of $\boldsymbol{H}$ $a+bi+cj+dk$ can be represented with $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$.

$\mathbb{H} \otimes \mathbb{C} \cong Mat(2, \mathbb{C})$ and so $\mathbb{H}$ is a real form of matrix algebra $Mat(2, \mathbb{C})$.

Remark 7.1.1. *The notion of a quaternion algebra can be seen as a generalization of the Hamilton quaternions to an arbitrary base field. "Generalized quaternion algebra", denoted by $\boldsymbol{H}_{\alpha,\beta}$, is a central simple algebra over a field $\mathbb{F}$ that has dimension 4 over $\mathbb{F}$.*

$$\boldsymbol{H}_{\alpha,\beta} = \langle 1, i, j, k \mid i^2 = \alpha, j^2 = \beta, ij = k = -ji \rangle$$

and

$$jk = \beta i = -kj$$
$$ki = \alpha j = -ik$$

Any element q is of the form $q = x + yi + zj + k$ and the conjugate of q is of the form $\overline{q} = x - yi - zj - tk$. So, $q\overline{q} = x^2 - \alpha y^2 - \beta z^2 + t^2\alpha\beta$ (due to the operations above; e.g., $ijij = i(-ij)j = -i^2j^2 = -\alpha\beta$). $\|q\| = \sqrt{q\overline{q}}$ is the norm of q.

Definition 7.1.4. *Two quaternion algebras are **isomorphic** if and only if the respective norm forms are isomorphic over $\mathbb{F}$.*

Theorem 7.1.1. *Isomorphism classes of quaternion algebras $\boldsymbol{H}_{\alpha,\beta}$ over $\mathbb{F}$ is equal to equivalence classes of quadratic forms $\alpha y^2 + \beta z^2 - \alpha\beta t^2$.*

Remark 7.1.2. *Why we ignore the x^2 in the norm is the fact that we are interested in the space of purely imaginary quaternions, i.e., $\overline{q} = -q$.*

Theorem 7.1.2.

- $\boldsymbol{H}_{\alpha,\beta} \simeq Mat(2, \mathbb{F})$ *iff the respective conic $C : \alpha x^2 + \beta y^2 - \alpha\beta z^2$ has a rational point; i.e. a point belonging to C is over $\mathbb{F}$.*
- *If the isomorphism does not occur, then, $\boldsymbol{H}_{\alpha,\beta}$ is a division algebra, i.e., all nonzero elements are invertible.*

Definition 7.1.5. *An algebraic curve C is a **form of** $\mathbb{P}^1$ **over** $\mathbb{F}$ iff $C \otimes \overline{\mathbb{F}} \simeq \mathbb{P}^1(\overline{\mathbb{F}})$.*

Theorem 7.1.3. *There is a one-to-one correspondence between the followings:*

- *Forms of $\mathbb{P}^1$ over $\mathbb{F}$.*
- *Forms of $Mat(2, \mathbb{F})$.*
- *Generalized quaternion algebras.*
- *Principal homogeneous space of $\mathbb{P}GL(2, \mathbb{F})$.*

7.2 Field of definition of a rational exceptional Belyi covering

Recall that a rational exceptional Belyi covering $\beta : \mathbb{P}^1 \to \mathbb{P}^1$ is *uniquely* determined by the respective ramification scheme or by the respective dessin d'enfant. The important thing is that the term *uniqueness* here refers to *uniqueness up to a fractional linear transformation*. This is because

$$\begin{aligned}\mathrm{Aut}(\mathbb{P}^1) &= \left\{z \mapsto \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad - bc \neq 0\right\} \\ &= \mathbb{P}GL(2, \mathbb{C}), \text{ the group of Möbius transformations} \\ &= \mathrm{Aut}(Mat(2, \mathbb{C})).\end{aligned}$$

So,

(i) β may or may not be defined over $\mathbb{Q}$.

(ii) However the collection $\mathcal{B}$ of all β with given dessins is indeed defined over $\mathbb{Q}$.

(iii) $PGL(2, \mathbb{C})$ acts on $\mathcal{B}$ in such a way that for any $\beta_1, \beta_2 \in \mathcal{B}$, there exists (unique) $a, b, c, d \in \mathbb{C}$ such that $\beta_2(z) = \beta_1\left(\frac{az+b}{cz+d}\right)$.

Thus $\mathcal{B}$ is a principal homogeneous space for the group $\mathbb{P}GL(2, \mathbb{C})$.

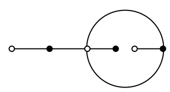
Now we will give a result due to the statements in the previous section: Let $\beta \in \mathcal{B}$ be an exceptional Belyi rational covering. Then the following statements are equal:

- β is defined over $\mathbb{Q}$.
- Principal homogeneous space of β is trivial; i.e. it is isomorphic to $\mathbb{P}GL(2, \mathbb{C})$.
- The curve $C \simeq \mathbb{P}^1$.

- There is a rational point in C.

According to our calculations via MAPLE, there is only one rational exceptional curve which is not defined over $\mathbb{Q}$, instead over a quadratic extension of $\mathbb{Q}$:

Example 7.2.1. *We begin with the ramification scheme* $[3,3][1,2,3][1,1,4]$ *and the respective dessin is*



We calculated the respective Belyi function f *using MAPLE: First we defined the "fundamental identity" based on the fact that the ramification scheme* $[3,3][1,2,3][1,4,4]$ *illustrates the degree of the faces and the degree of the vertices in the dessin where black vertices correspond to the roots of the equation* $f(x) = 0$ *and white vertices correspond to* $f(x) = 1$. *For instance, the ramification structure* $[1,2,3]$ *tells us that there are* 3 *black vertices (*3 *roots of* $f(x) = 0$*) with degrees* $1, 2,$ *and* 3. *Similarly,* $[3,3]$ *is for the degrees of the faces and* $[1,1,4]$ *is for the degree of the white vertices. So, we wrote the left-hand side of the fundamental identity accordingly. After expressing the fundamental identity in terms of coefficients*

$a_1, a_2, a_3, b_1, c_1, c_2, c_3$ and then calculated these coefficients all dependent on the number α, which we found at last that $\alpha = \frac{13}{3} + \frac{4\sqrt{10}}{3}$. More precisely,

$$
\begin{aligned}
a_1 &= -\frac{\alpha}{2} + \frac{9}{2} \\
a_2 &= \alpha \\
a_3 &= \frac{\alpha}{6} - \frac{35}{18} \\
b_1 &= -\frac{6400(3\alpha - 8)}{729} \\
c_1 &= \frac{2(3\alpha - 8)}{3} \\
c_2 &= -\frac{2\alpha}{9} - \frac{11}{27} \\
c_3 &= 1
\end{aligned}
$$

The resulting Belyi function is obviously closely related to the fundamental identity in such a way that it both contains the left-hand side of the fundamental identity and the pole at infinity of degree 3. Thus, the Belyi function can be thought as a fraction with numerator as the left-hand side of the fundamental identity and denominator with the expression z^3. So, this Belyi function is defined over $\mathbb{Q}(\sqrt{10})$. Then we made MAPLE draw both the real graph and the respective dessin d'enfant simultaneously. Both the MAPLE routines and the graphs (Figure 7.1) can be found below.

```
>restart:
eq_1:=(z+a1)*(z+a2)^2*(z+a3)^3=b1*z^3+(z^2+c1*z+c2)*(z+c3)^4;
eq:=lhs(eq_1)-rhs(eq_1):eq:=expand(%):
eq:=collect(eq,z):
cff:=[coeffs(eq,z)];
sol:=solve(cff,[a1,a2,a3,b1,c1,c2,c3]);sol[4]
## Take the label from the above equation
alias(alpha=RootOf(3-26*_Z+3*_Z^2,label = _L1)):
alias(beta=RootOf(16*_Z^16-47258883-29250396*_Z+530449560*_Z^2+1222754292*_Z^3-1838570778*_Z^4-2452109436*
_Z^5+6605347452*_Z^6-6265790868*_Z^7+3474444525*_Z^8-1271083016*_Z^9+320873572*_Z^10-56778912*_Z^11+7029272*
_Z^12-596896*_Z^13+33168*_Z^14-1088*_Z^15,label = _L2)):
sol[4];sol[-2]:
#sol4:=subs(c3=1,sol[4]):
```

```
sol4:=map(x->evala(x),sol[4]):
subs(op(sol4),eq_1):
id:=subs(c3=1,%):
#factor(lhs(%))=1+factor(rhs(%)-1):
#subs(z=10*z,%);id:=factor(lhs(%))=1+factor(rhs(%)-1):
#evala(9/2*(2057*alpha-1065)/(17200*alpha-3939)):
#evala(1/6*(2762531376687229*alpha-322627907986089)/(2045012873516581*alpha-239172834390216)):
id := (z-1/2*alpha+9/2)*(z+alpha(id))^2*(z+1/6*alpha-35/18)^3 = -6400/729*(3*alpha-8)*z^3+(z^2+2/3*(3*alpha-8)*
z-2/9*alpha-11/27)*(z+1)^4;
Digits:=15:
with(plots,rootlocus,complexplot,display):
#alias(alpha=13/3+4/3*10^(1/2));
alias(alpha=RootOf(3-26*_Z+3*_Z^2,label = _L1)):
alpha=(13+4*10^(1/2))/3;
id := (z-1/2*alpha+9/2)*(z+alpha)^2*(z+1/6*alpha-35/18)^3 = -6400/729*(3*alpha-8)*z^3+(z^2+2/3*(3*alpha-8)*
z-2/9*alpha-11/27)*(z+1)^4:
f:=lhs(id)/(-6400/729*(3*alpha-8)*z^3);
###############################################
## This is to test that involution z->1/z transforms one covering into another
#f:=subs(alpha=1/alpha,f);
f:=subs(z=1/z,f);#f:=subs(z=1/z,f);
##############################################
cr0:=map(x->[x,-1],[op({fsolve(numer(f))})]):
cr_0:=plot(cr0, style=point,symbol=circle,color=BLACK):
cr1:=map(x->[x,1],[op({fsolve(expand(numer(f-1)))})]):
cr_1:=plot(cr1, style=point,symbol=cross, color=BLACK):
pl2:=plot([2*f-1,1,0,-1,[0,t,t=-5..5]], z=-15..6,#-13..2,
y=-3/2..3/2, color=[red,grey,grey,grey,grey#,grey,grey
],discont=true,numpoints=200,axes=boxed):
pl0:=rootlocus( f/(f-1), z, 0..1, scaling=constrained,symbol=cross,symbolsize=10,thickness=2,numpoints=300):
pl1:=rootlocus( (f-1)/f, z, 0..1, scaling=constrained,symbol=circle, symbolsize=10, thickness=2,numpoints=300):
print(`The fundamental identity for exceptional covering with ramification scheme [3,3][1,2,3][1,1,4]`);
(z-1/2*alpha+9/2)*(z+alpha)^2*(z+1/6*alpha-35/18)^3 = -6400/729*(3*alpha-8)*z^3+(z^2+2/3*(3*alpha-8)*z
-2/9*alpha-11/27)*(z+1)^4, 3*alpha^2-26*alpha+3=0;
display({pl0,pl1,pl2,cr_0,cr_1}, axes=boxed, scaling=unconstrained,labels=[``,``],
title=`Real plot and Dessin d'enfant`,titlefont=[COURIER,10]);
```

$$\alpha = \frac{13}{3} + \frac{4}{3}\sqrt{10}$$

$$f := -\frac{729}{6400}\,\frac{\left(z - \frac{1}{2}\alpha + \frac{9}{2}\right)(z+\alpha)^2\left(z + \frac{1}{6}\alpha - \frac{35}{18}\right)^3}{(3\alpha - 8)\,z^3}$$

$$f := -\frac{729}{6400}\,\frac{\left(\frac{1}{z} - \frac{1}{2}\alpha + \frac{9}{2}\right)\left(\frac{1}{z}+\alpha\right)^2\left(\frac{1}{z} + \frac{1}{6}\alpha - \frac{35}{18}\right)^3 z^3}{3\alpha - 8}$$

The fundamental identity for exceptional covering with ramification scheme [3,3][1,2,3][1,1,4]

$$\left(z - \frac{1}{2}\alpha + \frac{9}{2}\right)(z+\alpha)^2\left(z + \frac{1}{6}\alpha - \frac{35}{18}\right)^3 = -\frac{6400}{729}(3\alpha-8)\,z^3 + \left(z^2 + \frac{2}{3}(3\alpha-8)\,z - \frac{2}{9}\alpha - \frac{11}{27}\right)(z+1)^4,\ 3\alpha^2 - 26\alpha + 3 = 0$$

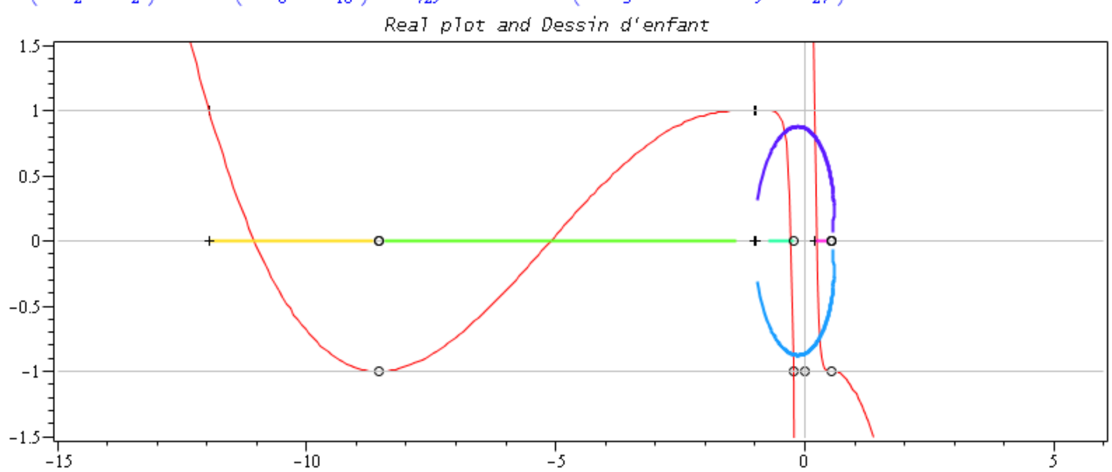


Figure 7.1: The Belyi function defined over $\mathbb{Q}(\sqrt{10})$

Chapter 8

Open Questions and Conclusion

In the thesis we study exceptional Belyi coverings, that is finite coverings of $S \to \mathbb{P}^1$ ramified over three points $0, 1, \infty$ that are uniquely determined up to automorphism of S by by ramification scheme or the respective dessin.

(a) First of all we develop an algorithm that allows to find all such coverings for a given genus and degree. We identified all such coverings of genus 0 up to degree 15 and calculate their Belyi functions For degree ≤ 6 and for some covering of degree 7. Besides we found a number of infinite series of exceptional rational coverings. Not all rational exceptional Belyi coverings defined over $\mathbb{Q}$.

(b) We described in a similar way the situation with elliptic exceptional Belyi coverings. However our knowledge about this type of coverings is limited.

(c) To our surprise we have found no hyperbolic exceptional Belyi covering and it is a mystery why there are no exceptional coverings of this type.

There are few questions that we cannot answer:

(1) Find Belyi functions for all polynomial exceptional Belyi coverings.

(2) Find all exceptional dessins of genus zero.

(3) Can every rational exceptional Belyi covering be defined over a "real" quadratic extension of $\mathbb{Q}$?

(4) Are there infinitely many elliptic exceptional Belyi coverings?

(5) Find fields of definition for elliptic exceptional Belyi coverings.

(6) Is there aa hyperbolic exceptional Belyi covering?

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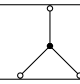
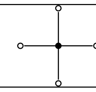
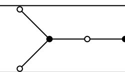
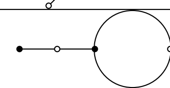
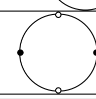
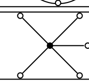
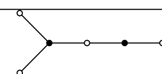
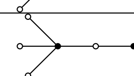
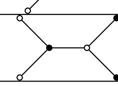
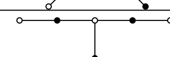
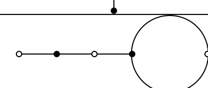
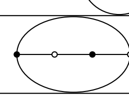
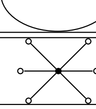
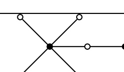
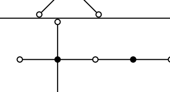
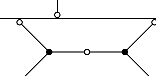
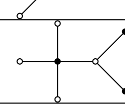
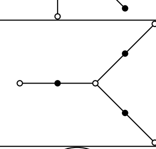
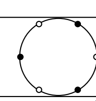
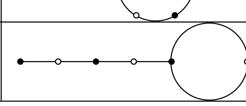
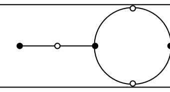
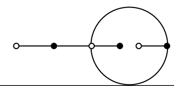
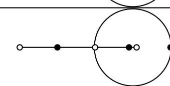
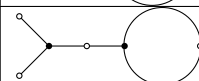
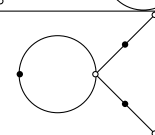
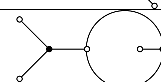
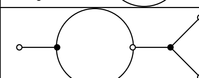
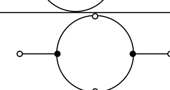
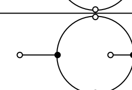
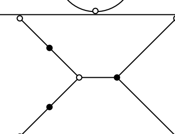
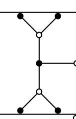
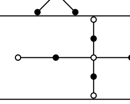
Appendix A

We gave a table of rational exceptional Belyi coverings up to degree 6 alongside with some rational coverings of degree 7.

Rational Exceptional Belyi Coverings

Label	Ramification Scheme	Belyi Function	Dessins D'enfant
R3.1	$\mid 3 \mid 3 \mid 1,1,1 \mid$	$f(z) = z^3$	
R3.2	$\mid 3 \mid 1,2 \mid 1,2 \mid$	$f(z) = 4z^3 - 3z$	
R4.1	$\mid 4 \mid 4 \mid 1,1,1,1 \mid$	$f(z) = z^4$	
R4.2	$\mid 4 \mid 2,2 \mid 1,1,2 \mid$	$f(z) = 8z^4 - 8z^2 + 1$	
R4.3	$\mid 4 \mid 1,3 \mid 1,1,2 \mid$	$f(z) = -(3z-1)(z+1)^3$	
R4.4	$\mid 1,3 \mid 1,3 \mid 2,2 \mid$	$f(z) = e^{3zi}\left[\frac{2-e^{zi}}{1-2e^{zi}}\right] + e^{-3zi}\left[\frac{2-e^{-zi}}{1-2e^{-zi}}\right]$	
R4.5	$\mid 2,2 \mid 2,2 \mid 2,2 \mid$	$f(z) = z^4 + \frac{1}{z^4}$	
R5.1	$\mid 5 \mid 5 \mid 1,1,1,1,1 \mid$	$f(z) = z^5$	
R5.2	$\mid 5 \mid 1,2,2 \mid 1,2,2 \mid$	$f(z) = 16z^5 - 20z^3 + 5z$	
R5.3	$\mid 5 \mid 2,3 \mid 1,1,1,2 \mid$	$f(z) = (3z-1)^2(2z+1)^3$	
R5.4	$\mid 5 \mid 1,4 \mid 1,1,1,2 \mid$	$f(z) = (1-12z)(3z+1)^4$	
R5.5	$\mid 5 \mid 1,1,3 \mid 1,1,3 \mid$	$f(z) = -(54z^2+9z+1)(3z-1)^3$	
R5.6	$\mid 5 \mid 1,2,2 \mid 1,1,3 \mid$	$f(z) = (1-4z)(6z^2+2z+1)^2$	
R5.7	$\mid 1,4 \mid 2,3 \mid 1,2,2 \mid$	$f(z) = e^{4zi}\left[\frac{\frac{5}{3}-e^{zi}}{1-\frac{5}{3}e^{zi}}\right] + e^{-4zi}\left[\frac{\frac{5}{3}-e^{-zi}}{1-\frac{5}{3}e^{-zi}}\right]$	
R5.8	$\mid 1,2,2 \mid 2,3 \mid 2,3 \mid$	$f(z) = \frac{1}{2}\frac{(1-9z)^2(z+1)^3}{(15z^2+1)^2}$	
R6.1	$\mid 6 \mid 6 \mid 1,1,1,1,1,1 \mid$	$f(z) = z^6$	
R6.2	$\mid 6 \mid 2,2,2 \mid 1,1,2,2 \mid$	$f(z) = 32z^6 - 48z^4 + 18z^2 - 1$	
R6.3	$\mid 6 \mid 1,5 \mid 1,1,1,1,2 \mid$	$f(z) = -(5z-1)(z+1)^5$	
R6.4	$\mid 6 \mid 2,4 \mid 1,1,1,1,2 \mid$	$f(z) = (2z-1)^2(z+1)^4$	
R6.5	$\mid 6 \mid 3,3 \mid 1,1,1,1,2 \mid$	$f(z) = (z^2+1)^3$	
R6.6	$\mid 6 \mid 1,1,4 \mid 1,1,1,3 \mid$	$f(z) = (40z^2-8z+1)(2z+1)^4$	
R6.7	$\mid 6 \mid 2,2,2 \mid 1,1,1,3 \mid$	$f(z) = (z^3+1)^2$	
R6.8	$\mid 3,3 \mid 2,2,2 \mid 2,2,2 \mid$	$f(z) = \frac{(z^3+1)^2}{4z^3}$	
R6.9	$\mid 1,5 \mid 1,2,3 \mid 2,2,2 \mid$	$f(z) = e^{5zi}\left[\frac{\frac{3}{2}-e^{zi}}{1-\frac{3}{2}e^{zi}}\right] + e^{-5zi}\left[\frac{\frac{3}{2}-e^{-zi}}{1-\frac{3}{2}e^{-zi}}\right]$	

Label	Ramification scheme	Equation	Dessins D'enfants
R6.10	$\mid 2,4 \mid 1,2,3 \mid 2,2,2 \mid$	$f(z) = e^{4zi}\left[\frac{\frac{5}{3}-e^{zi}}{1-\frac{5}{3}e^{zi}}\right]^2 + e^{-4zi}\left[\frac{\frac{5}{3}-e^{-zi}}{1-\frac{5}{3}e^{-zi}}\right]^2$	
R6.11	$\mid 3,3 \mid 1,2,3 \mid 1,1,4 \mid$	$f(z) = -\frac{729\left(z-\frac{\alpha}{2}+\frac{9}{2}\right)(z+\alpha)^2\left(z+\frac{\alpha}{6}-\frac{35}{18}\right)^3}{6400(3\alpha-8)z^3},\ \alpha = \frac{13}{3} + \frac{4\sqrt{10}}{3}$	
R6.12	$\mid 3,3 \mid 2,2,2 \mid 1,1,4 \mid$	$f(z) = -\frac{(z^3+9z^2-9z-1)^2}{256z^3}$	
R6.13	$\mid 1,5 \mid 3,3 \mid 1,1,2,2 \mid$	$f(z) = \frac{(z^2+10z+5)^3}{1728z}$	
R6.14	$\mid 1,5 \mid 2,2,2 \mid 1,1,4 \mid$	$f(z) = \frac{(z^3-5z^2+15z+5)^2}{(256z)}$	
R6.15	$\mid 2,4 \mid 3,3 \mid 1,1,1,3 \mid$	$f(z) = -\frac{(z^2+8z-2)^3}{729z^2}$	
R6.16	$\mid 1,5 \mid 3,3 \mid 1,1,1,3 \mid$	f(z)=$-\frac{(z^2-5z-5)^3}{729z}$ MODULAR	
R6.17	$\mid 2,4 \mid 3,3 \mid 1,1,2,2 \mid$	$f(z) = \frac{(z^2-2)^3}{27z^2}$ MODULAR	
R6.18	$\mid 3,3 \mid 3,3 \mid 1,1,2,2 \mid$	$f(z) = -\frac{(z^2+2z-3)^3}{64z^3}$ MODULAR	
R7.1	$\mid 7 \mid 2,2,3 \mid 1,1,1,1,3 \mid$	$f(z) = -(30z^2+6z+1)^2(4z-1)^3$	
R7.2	$\mid 7 \mid 1,1,1,1,3 \mid 1,3,3 \mid$	$f(z) = (1-3z)(2z^2+z+1)^3$	
R7.3	$\mid 7 \mid 1,2,2,2 \mid 1,1,1,4 \mid$	$f(z) = (1-4z)(20z^3+6z^2+2z+1)^2$	

Appendix B

We gave a table of elliptic exceptional Belyi coverings up to degree 12.

<table>
<tr><th colspan="5">Elliptic Exceptional Belyi Coverings</th></tr>
<tr><th>Label</th><th>Ramification scheme</th><th>Monodromy permutations</th><th>|Aut|</th><th>Description</th></tr>
<tr><td>E3.1</td><td>| 3 | 3 | 3 |</td><td>(1, 2, 3) | (1, 2, 3) | (1, 2, 3)</td><td>3</td><td>$E : y^3 = x(x-1);\ (x, y) \mapsto x.$</td></tr>
<tr><td>E4.1</td><td>| 4 | 4 | 2, 2 |</td><td>(4, 1, 2, 3) | (4, 1, 2, 3) | (1, 3)(2, 4)</td><td>4</td><td>Double covering of [②][②][①, ①]</td></tr>
<tr><td>E4.2</td><td>| 4 | 4 | 1, 3 |</td><td>(1, 2, 3, 4) | (4, 2, 1, 3) | (1)(2, 3, 4)</td><td>1</td><td>E:</td></tr>
<tr><td>E5.1</td><td>| 5 | 5 | 1, 2, 2 |</td><td>(5, 1, 2, 3, 4) | (5, 2, 1, 3, 4) | (1)(2, 4)(3, 5)</td><td>1</td><td></td></tr>
<tr><td>E5.2</td><td>| 5 | 2, 3 | 2, 3 |</td><td>(1, 2, 3, 4, 5) | (3, 1)(2, 4, 5) | (3, 5)(1, 2, 4)</td><td>1</td><td></td></tr>
<tr><td>E6.1</td><td>| 6 | 3, 3 | 2, 2, 2 |</td><td>(1, 2, 3, 4, 5, 6) | (3, 5, 1)(2, 4, 6) | (1, 4)(2, 5)(3, 6)</td><td>6</td><td>Modular; Double covering of [③][3][①, ①, ①]</td></tr>
<tr><td>E6.2</td><td>| 6 | 2, 4 | 2, 2, 2 |</td><td>(1, 2, 3, 4, 5, 6) | (1, 4)(2, 3, 5, 6) | (1, 3)(2, 5)(4, 6)</td><td>2</td><td>Double covering of [③][①, ②][①, 2]</td></tr>
<tr><td>E6.3</td><td>| 6 | 1, 5 | 2, 2, 2 |</td><td>(1, 2, 3, 4, 5, 6) | (2)(3, 1, 4, 5, 6) | (1, 2)(3, 5)(4, 6)</td><td>1</td><td></td></tr>
<tr><td>E6.4</td><td>| 1, 5 | 3, 3 | 3, 3 |</td><td>(1)(2, 3, 4, 5, 6) | (4, 2, 1)(3, 5, 6) | (1, 6, 4)(2, 3, 5)</td><td>1</td><td></td></tr>
<tr><td>E8.1</td><td>| 8 | 2, 3, 3 | 2, 2, 2, 2 |</td><td>(1, 2, 3, 4, 5, 6, 7, 8) | (1, 5)(2, 4, 7)(3, 6, 8) | (1, 4)(2, 6)(3, 7)(5, 8)</td><td>2</td><td>Double covering of [④][①, 3][①, ①, 2]</td></tr>
<tr><td>E8.2</td><td>| 2, 6 | 4, 4 | 2, 2, 2, 2 |</td><td>(1, 2)(3, 4, 5, 6, 7, 8) | (2, 3, 5, 7)(4, 1, 6, 8) | (1, 3)(2, 6)(4, 7)(5, 8)</td><td>2</td><td>Double covering of [4][①, ③][①, ①, 2]</td></tr>
<tr><td>E8.3</td><td>| 4, 4 | 2, 2, 4 | 2, 3, 3 |</td><td>(1, 2, 3, 4)(5, 6, 7, 8) | (5, 3)(4, 6)(1, 2, 7, 8) | (1, 7)(2, 4, 5)(3, 8, 6)</td><td>2</td><td>Double covering of [4][①, 3][①, ①, ②]</td></tr>
<tr><td>E8.4</td><td>| 2, 6 | 3, 5 | 2, 2, 2, 2 |</td><td>(1, 2)(3, 4, 5, 6, 7, 8) | (2, 3, 6)(4, 1, 5, 7, 8) | (1, 3)(2, 5)(4, 7)(6, 8)</td><td>1</td><td></td></tr>
<tr><td>E8.5</td><td>| 3, 5 | 3, 5 | 2, 2, 2, 2 |</td><td>(1, 2, 3)(4, 5, 6, 7, 8) | (1, 4, 6)(2, 5, 3, 7, 8) | (1, 5)(2, 7)(3, 4)(6, 8)</td><td>1</td><td></td></tr>
<tr><td>E8.6</td><td>| 1, 7 | 4, 4 | 2, 2, 2, 2 |</td><td>(1)(2, 3, 4, 5, 6, 7, 8) | (3, 1, 2, 6)(4, 5, 7, 8) | (1, 2)(3, 5)(4, 7)(6, 8)</td><td>1</td><td></td></tr>
<tr><td>E9.1</td><td>| 9 | 3, 3, 3 | 1, 2, 2, 2, 2 |</td><td>(1, 2, 3, 4, 5, 6, 7, 8, 9) | (4, 8, 2)(6, 5, 1)(3, 7, 9) | (5)(1, 4)(2, 7)(3, 8)(6, 9)</td><td>1</td><td></td></tr>
<tr><td>E9.2</td><td>| 4, 5 | 3, 3, 3 | 2, 2, 2, 3 |</td><td>(1, 2, 3, 4)(5, 6, 7, 8, 9) | (5, 7, 2)(3, 6, 1)(4, 8, 9) | (1, 5)(2, 6)(7, 9)(3, 4, 8)</td><td>1</td><td></td></tr>
<tr><td>E9.3</td><td>| 3, 3, 3 | 3, 3, 3 | 2, 2, 5 |</td><td>(1, 2, 3)(4, 5, 6)(7, 8, 9) | (4, 2, 7)(1, 5, 8)(3, 6, 9) | (1, 7)(2, 6)(3, 8, 4, 9, 5)</td><td>1</td><td></td></tr>
<tr><td>E10.1</td><td>| 5, 5 | 3, 3, 4 | 2, 2, 2, 2, 2 |</td><td>(1, 2, 3, 4, 5)(6, 7, 8, 9, 10) | (3, 7, 1)(2, 6, 8)(4, 5, 9, 10) | (1, 6)(2, 7)(3, 5)(4, 9)(8, 10)</td><td>2</td><td>Double covering of [5][②, 3][①, ①, ①, 2]</td></tr>
<tr><td>E10.2</td><td>| 5, 5 | 1, 4, 5 | 2, 2, 2, 2, 2 |</td><td>(1, 2, 3, 4, 5)(6, 7, 8, 9, 10) | (6, 9, 5, 2)(3, 4, 7, 1, 8)(10) | (1, 6)(2, 4)(3, 7)(5, 8)(9, 10)</td><td>1</td><td></td></tr>
<tr><td>E10.3</td><td>| 3, 7 | 2, 4, 4 | 2, 2, 2, 2, 2 |</td><td>(1, 2, 3)(4, 5, 6, 7, 8, 9, 10) | (2, 5)(3, 4, 7, 9)(6, 1, 8, 10) | (1, 5)(2, 4)(3, 8)(6, 9)(7, 10)</td><td>1</td><td></td></tr>
<tr><td>E10.4</td><td>| 3, 7 | 3, 3, 4 | 2, 2, 2, 2, 2 |</td><td>(1, 2, 3)(4, 5, 6, 7, 8, 9, 10) | (6, 9, 3)(4, 7, 2)(5, 1, 8, 10) | (1, 4)(2, 6)(3, 8)(5, 9)(7, 10)</td><td>1</td><td></td></tr>
<tr><td>E10.5</td><td>| 10 | 1, 3, 3, 3 | 2, 2, 2, 2, 2 |</td><td>(1, 2, 3, 4, 5, 6, 7, 8, 9, 10) | (4, 7, 2)(3, 6, 8)(5, 1, 9)(10) | (1, 4)(2, 6)(3, 7)(5, 8)(9, 10)</td><td>1</td><td></td></tr>
<tr><td>E12.1</td><td>| 1, 11 | 3, 3, 3, 3 | 2, 2, 2, 2, 2, 2 |</td><td>(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)(12) | (4, 7, 2)(3, 6, 8)(5, 1, 9)(10, 11, 12) | (1, 4)(2, 6)(3, 7)(5, 8)(9, 11)(10, 12)</td><td>1</td><td>Modular; $y^2 + y = x^3 - x^2 - 10x - 20$</td></tr>
<tr><td>E12.2</td><td>| 2, 10 | 3, 3, 3, 3 | 2, 2, 2, 2, 2, 2 |</td><td>(1, 2)(3, 4, 5, 6, 7, 8, 9, 10, 11, 12) | (2, 3, 9)(5, 7, 11)(8, 4, 1)(6, 10, 12) | (1, 3)(2, 8)(4, 7)(5, 10)(6, 11)(9, 12)</td><td>2</td><td>Modular; 2-covering of [①, ⑤][3, 3][①, ①, 2, 2]</td></tr>
<tr><td>E12.3</td><td>| 3, 9 | 3, 3, 3, 3 | 2, 2, 2, 2, 2, 2 |</td><td>(1, 2, 3)(4, 5, 6, 7, 8, 9, 10, 11, 12) | (4, 8, 2)(5, 1, 10)(3, 7, 11)(6, 9, 12) | (1, 4)(2, 7)(3, 10)(5, 9)(6, 11)(8, 12)</td><td>3</td><td>Modular; 3-covering of [①, ③][①, 3][2, 2], see E3.1</td></tr>
<tr><td>E12.4</td><td>| 4, 8 | 3, 3, 3, 3 | 2, 2, 2, 2, 2, 2 |</td><td>(1, 2, 3, 4)(5, 6, 7, 8, 9, 10, 11, 12) | (5, 8, 2)(3, 7, 10)(6, 1, 11)(4, 9, 12) | (1, 5)(2, 7)(3, 9)(4, 11)(6, 10)(8, 12)</td><td>4</td><td>Modular; 4-covering of [①, ②][3][①, 2], see E4.1</td></tr>
<tr><td>E12.5</td><td>| 5, 7 | 2, 2, 4, 4 | 2, 2, 2, 2, 2, 2 |</td><td>(1, 2, 3, 4, 5)(6, 7, 8, 9, 10, 11, 12) | (8, 2)(6, 10)(1, 3, 7, 9)(4, 5, 11, 12) | (1, 8)(2, 7)(3, 5)(4, 11)(6, 9)(10, 12)</td><td>1</td><td></td></tr>
<tr><td>E12.6</td><td>| 6, 6 | 1, 3, 4, 4 | 2, 2, 2, 2, 2, 2 |</td><td>(1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12) | (2)(5, 7, 9)(11, 6, 8, 4)(3, 1, 10, 12) | (1, 2)(3, 11)(4, 7)(5, 8)(6, 10)(9, 12)</td><td>1</td><td></td></tr>
<tr><td>E12.7</td><td>| 2, 3, 7 | 4, 4, 4 | 2, 2, 2, 2, 2, 2 |</td><td>(1, 2)(3, 4, 5)(6, 7, 8, 9, 10, 11, 12) | (5, 2, 8, 11)(4, 6, 9, 1)(7, 3, 10, 12) | (1, 8)(2, 4)(3, 6)(5, 10)(7, 11)(9, 12)</td><td>1</td><td></td></tr>
<tr><td>E12.8</td><td>| 3, 3, 6 | 3, 4, 5 | 2, 2, 2, 2, 2, 2 |</td><td>(1, 2, 3)(4, 5, 6)(7, 8, 9, 10, 11, 12) | (3, 4, 11)(7, 9, 5, 2)(6, 8, 1, 10, 12) | (1, 7)(2, 4)(3, 10)(5, 8)(6, 11)(9, 12)</td><td>1</td><td></td></tr>
</table>