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Inversion sequences avoiding 021 and another pattern of length four

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We study the enumeration of inversion sequences that avoid pattern 021 and another pattern of length four. We determine the generating trees for all possible pattern pairs and compute the corresponding generating functions. We introduce the concept of d-regular generating trees and conjecture that for any 021-avoiding pattern τ, the generating tree $\mathcal{T}(\{021,\tau\})$ is d-regular for some integer d.

Keywords: Pattern-avoiding inversion sequences, generating functions, generating trees, kernel method, Catalan numbers, Motzkin numbers

1 Introduction

An integer sequence $e = e_0e_1 \cdots e_n$ is called an *inversion sequence* of length n if $0 \leq e_i \leq i$ for each $0 \leq i \leq n$. We use $\mathbf{I}_n$ to denote the set of inversion sequences of length n. There is a bijection between $\mathbf{I}_n$ and the set of permutations of length $n+1$, denoted by S_{n+1}. Let τ be a word of length k over the alphabet $[k] := \{0, 1, \cdots, k-1\}$, we say that an inversion sequence $e \in \mathbf{I}_n$ contains the pattern τ if there is a subsequence of length k in e that has the same relative order with τ; otherwise, we say that e avoids the pattern τ. For instance, $e = 010213 \in \mathbf{I}_5$ avoids the pattern 201 because there is no subsequence $e_je_ke_l$ of length three in e with $j < k < l$ and $e_k < e_l < e_j$. On the other hand, $e = 010213$ contains the patterns 010 and 0012. For a given pattern τ, we use $\mathbf{I}_n(\tau)$ to denote the set of all τ-avoiding inversion sequences of length n. Similarly, for a given set of patterns B, we set $\mathbf{I}_n(B) = \cap_{\tau\in B}\mathbf{I}_n(\tau)$. The first results on the pattern-avoiding inversion sequences were obtained by Mansour and Shattuck (2015) and Corteel et al. (2016) for the patterns of length three. Later, Martinez and Savage (2018) generalized and extended the notion of pattern-avoidance for the inversion sequences to triples of binary relations that lead to new conjectures and open problems. Various other pattern-avoidance conditions such as vincular patterns, pairs of patterns, and longer patterns are also studied for inversion sequences; for relevant results, see Auli and Elizalde (2021); Bouvel et al. (2018); Beaton et al. (2019); Cao et al. (2019); Chern (2023); Duncan and Steingrímsson (2011); Callan et al. (2023); Lin (2018, 2020); Mansour and Shattuck (2022); Yan and Lin (2020–2021); Lin and Fu (2021); Lin and Yan (2020) and references therein. In the context of inversion

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sequences, two sets of patterns B_1 and B_2 are said to be Wilf equivalent if $|\mathbf{I}_n(B_1)| = |\mathbf{I}_n(B_2)|$ for all $n \geq 0$, that is, they have the same counting sequence. Note that there are 13 patterns of length three and 75 of length four up to order isomorphism. Hong and Li (2022) and Yan and Lin (2020–2021) studied the Wilf-equivalence classification for inversion sequences avoiding the length-four and pairs of length-three patterns, respectively. The current results show that pattern-avoiding inversion sequences have rich structures and provide a unifying framework for many well-known counting sequences. We can highlight some interesting results as follows: the large Schröder numbers enumerate $\mathbf{I}_n(021)$, the odd-indexed Fibonacci numbers enumerate $\mathbf{I}_n(012)$, the Euler up/down numbers enumerate $\mathbf{I}_n(000)$, the Bell numbers enumerate $\mathbf{I}_n(011)$, and powers of two enumerate $\mathbf{I}_n(001)$; see the above references for details. The $\mathbf{I}_n(021)$ counting sequence also satisfies a nice four-term recurrence relation, see Lin and Kim (2021). These results indicate that each case deserves further examination, and there are already some important recent results in this direction. For instance, Lin and Kim (2018) provided an intriguing bijection that preserves six set-valued statistics between $\mathbf{I}_n(021)$ and $(2413, 4213)$-avoiding permutations. Inversion sequences avoiding a pattern of length three and an additional longer pattern is a possible next step for this research program. In this paper, we study inversion sequences avoiding 021 and another pattern of length four. As we see in the next sections, the generating tree method combined with the kernel method works nicely in this case and leads to a complete enumeration of all possible cases. As a summary of the main results of this paper, we present a list of the generating functions for all cases in Table 1 and the corresponding counting sequences in Corollary 3.9. Even though the method works well, some cases are very subtle and technically demanding. The pattern class $\mathbf{I}_n(021, 0000)$ is an example of such a case. It has a long list of succession rules that require some non-trivial methods to solve the equations for the generating functions derived from the kernel method; for the details, see Section 3.1. Furthermore, we observe a special structure in the generating trees corresponding to the inversion sequences avoiding 021 and another pattern that avoids 021. We formulate this observation in Conjecture 2.1. The results of the present paper and the note of Mansour and Yıldırım (2022) provide evidence for the validity of this conjecture. We will use the algorithmic approach based on generating trees developed in Kotsireas et al. (2024) and the kernel method. Our results mainly follow the following five-step procedure: (i) an educated guess of the rules of the generating tree based on the algorithm's output, (ii) verification of the previous step, (iii) translating the rules of the generating tree into a one-parameter infinite system of equations involving related generating functions, (iv) using bivariate generating functions and obtaining a finite system of equations, (v) using kernel method to obtain an expression for the generating function of the pattern class. In most cases, obtaining a nice closed formula for the enumerating sequence of the class is possible. We will provide the details of this procedure in the following section. Inversion sequences avoiding multiple patterns look like a promising research direction. There are still no enumerating formulas for the avoidance sets $\mathbf{I}_n(100)$ and $\mathbf{I}_n(120)$, and Wilf-equivalent $\mathbf{I}_n(201)$ and $\mathbf{I}_n(210)$. The enumeration of the pattern 010 is recently solved by Testart (2022). The generating tree method also has some successful applications for pattern-avoiding permutations; for earlier and recent results, see Banderier et al. (2002); Hou and Mansour (2008); West (1996); Mansour et al. (2022).

We organized the paper as follows: In Section 2, we outline the details of our procedure that leads to the main result of this paper and present some examples. In Section 3, we study the generating functions for all possible pattern pairs of 021 and τ where τ is a pattern of length four that avoids 021. Note that we only need to consider 42 patterns of length four that avoid 021. There are some patterns τ of length four that avoids 021 but $\mathbf{I}_n(021, \tau) = \mathbf{I}_n(021)$ because all inversion sequences begin with letter 0. For instance, $\mathbf{I}_n(021, 3102) = \mathbf{I}_n(021)$.

We use C_n and M_n to denote the Catalan and Motzkin numbers, respectively. Their generating functions are $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ and $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$. For a letter or number s, we use s^m to denote the string of m s, that is, $s^m = \underbrace{ss\cdots s}_{\text{m times}}$.

2 An algorithmic approach based on generating trees

For each class of inversion sequences avoiding a fixed set of patterns, there is a corresponding generating tree that encodes the elements of the class as its vertices under some specific rules. We refer the reader to Section 2 of Kotsireas et al. (2024) for the details.

We use $\mathcal{I}_B = \cup_{n=0}^{\infty}\mathbf{I}_n(B)$ to denote the set of all inversion sequences avoiding the pattern set B. The corresponding generating tree $\mathcal{T}(B)$ is a rooted, labeled, plane tree whose vertices are the objects of $\mathcal{I}_B$ with the following properties: (i) each element of $\mathcal{I}_B$ appears exactly once in the tree; (ii) element of size n appears at level n in the tree (the root has level 0); (iii) there is a set of succession rules that determine the number of children and their labels for each vertex. The tree $\mathcal{T}(B)$ will be empty if no inversion sequence of arbitrary length avoids the set B. Otherwise, the root's label will always be 0, that is, $0 \in \mathcal{T}(B)$. From the root, whose level is zero, we construct the remainder of the tree $\mathcal{T}(B)$ in a recursive way where the n^{th} level of the tree consists exactly the elements of $\mathbf{I}_n(B)$ arranged in such a way that the parent of an inversion sequence $e_0e_1\cdots e_n \in \mathbf{I}_n(B)$ is the unique inversion sequence $e_0e_1\cdots e_{n-1} \in \mathbf{I}_{n-1}(B)$. We obtain the children of $e_0e_1\cdots e_{n-1} \in \mathbf{I}_{n-1}(B)$ from the set $\{e_0e_1\cdots e_{n-1}e_n \mid e_n = 0, 1, \ldots, n\}$ by obeying the restrictions of the patterns in B. We arrange the nodes from the left to the right so that if $e = e_0e_1\cdots e_{n-1}i$ and $e' = e_0e_1\cdots e_{n-1}j$ are children of the same parent $e_1\cdots e_{n-1}$, then e appears on the left of e' if $i < j$. Based on this initial tree $\mathcal{T}(B)$, we define an equivalence relation on the set of the nodes of this tree and obtain a second representation of the tree corresponding to the class $\mathcal{I}_B$ which is more efficient for enumerating purposes. We relabel the vertices of the tree $\mathcal{T}(B)$ as follows. Define $\mathcal{T}(B; e)$ as the subtree of the inversion sequence e as the root and its descendants in $\mathcal{T}(B)$. We say that e is *equivalent* to e', denoted by $e \sim e'$, if and only if $\mathcal{T}(B; e) \cong \mathcal{T}(B; e')$ (in the sense of plane trees). Let $\mathcal{T}'(B)$ be the same tree $\mathcal{T}(B)$ where we replace each node e by the first node $e' \in \mathcal{T}(B)$ from top to bottom and from left to right in $\mathcal{T}(B)$ such that $\mathcal{T}(B; e) \cong \mathcal{T}(B; e')$. Clearly, the generating tree $\mathcal{T}'(B)$ has a root 0, for any B such that $0 \notin B$.

Let B be any set of patterns, and let $\mathcal{T}'(B)$ be the generating tree for the class $\mathbf{I}_n(B)$. The *length* of a node $v \in \mathcal{T}'(B)$ is defined to be the number of letters in v, and it is denoted by $len(v)$. For any $k \geq 1$, let $D_k(B)$ be the multi-set of all nodes of length k at level $k-1$ in $\mathcal{T}'(B)$. For each node $v \in D_k(B)$, we denote the multi-set of all children of v at level k in $\mathcal{T}'(B)$ by $N_k(B; v)$. A generating tree $\mathcal{T}'(B)$ is said to be *d-regular* if there exists $k \geq 1$ such that

- the number of different nodes in $D_r(B)$ equals d, for all $r > k$;
- for any $v \in D_r(B)$ and $w \in N_r(B; v)$, the number of occurrences of w in $N_r(B; v)$ does not depend on r, whenever $r > k$.

$\mathcal{T}'(B)$ is 0-regular if and only if the generating tree $\mathcal{T}'(B)$ is finite. We present an example of d- regular and non-regular trees to illustrate the definition. If $B = \{021, 0123\}$, then the generating tree $\mathcal{T}'(B)$ has the following rules

$$a_m \rightsquigarrow a_{m+1}b_mb_{m-1}\cdots b_1, \quad b_m \rightsquigarrow b_{m+1}^2c^m, \quad c \rightsquigarrow c^2, \tag{1}$$

where $a_m = 0^m$, $b_m = 0^m1$, and $c = 012$. Note that $D_1(B) = \{a_1\}$, $D_2(B) = \{a_2, b_1\}$, $D_3(B) = \{a_3, b_2, b_2, b_2, c\}$, and $D_k(B) = \{a_k, \underbrace{b_{k-1}, \ldots, b_{k-1}}_{2^{k-1}-1 \text{ times}}\}$ for all $k \geq 4$. Moreover, the child a_{k+1} of $a_k \in D_k(B)$ occurs in $N_k(B; a_k)$ exactly once and the child b_k of $b_{k-1} \in D_k(B)$ occurs exactly twice. Hence, $\mathcal{T}'(B)$ is 2-regular.

Let $B = \{000, 0011\}$. The generating tree $\mathcal{T}'(B)$ has the following rules:

$$a_m \rightsquigarrow b_m^{m+1} a_{m+1}, \quad b_m \rightsquigarrow b_m^{m+2},$$

where $a_m = 012 \cdots m$ and $b_m = a_m0$. Note that $D_k(B)$ contains a_{k-1} exactly once and b_{k-2} exactly $k-1$ times, for all $k \geq 1$. Thus, $\mathcal{T}'(B)$ is not d-regular for any d.

Based on our results (see Table 1), we have the following conjecture.

Conjecture 2.1 *For any 021-avoiding pattern τ, the generating tree $\mathcal{T}'(\{021, \tau\})$ is d-regular for some d.*

We will use the following procedure to study the generating functions for the sequences

$$\{|\mathbf{I}_n(\{021, \tau\})|\}_{n \geq 0}$$

where τ is any pattern of length four that avoids 021.

The five-step Procedure: The main results of this paper are applications of the five-step procedure described briefly in the introduction. We will now outline the details of the procedure for the pattern set $B = \{021, 1002\}$. First, we obtain the rules of the generating tree $\mathcal{T}'(B)$ by using the algorithm developed in Kotsireas et al. (2024). We then get an explicit formula for the generating function $F_B(x) = \sum_{n\geq 0} |\mathbf{I}_n(B)| x^{n+1}$ from the rules of generating tree in a systematic way that can be programmed in software.

- **Step 1- An educated guess for the rules of $\mathcal{T}'(B)$:** Based on the output of the algorithm of Kotsireas et al. (2024), we guess that the rules of the generating tree $\mathcal{T}'(B)$ can be described as follows: the generating tree $\mathcal{T}'(B)$'s root is denoted by a_1, and the succession rules are:

$$\begin{array}{ll} a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1, & b_m \rightsquigarrow c_m b_{m+1} d_m d_{m-1} \cdots d_1, \\ c_m \rightsquigarrow f c_{m+1} e_m e_{m-1} \cdots e_1 g, & d_m \rightsquigarrow e_m d_{m+1} d_m \cdots d_1, \\ e_m \rightsquigarrow h e_{m+1} e_m \cdots e_1 g, & f \rightsquigarrow f^2, \\ g \rightsquigarrow h e_1 g, & h \rightsquigarrow h, \end{array}$$

  where $m \geq 1$, $a_m = 0^m$, $b_m = a_m1$, $c_m = b_m0$, $d_m = a_m12$, $e_m = a_m102$, $f = 0100$, $g = 0103$, and $h = 01020$. In Kotsireas et al. (2024), elementary examples show how the algorithm works, so we refer the reader to it for the details. We use $\mathcal{R}$ to denote the (proposed) set of rules of the tree.

- **Step 2- Verifying the set of rules for $\mathcal{T}'(B)$:** We prove that the proposed set of rules $\mathcal{R}$ hold for any level in the tree. Note that we have to show that when we add a letter $j = 0, 1, \ldots, len(v)$ for the father v of a rule, we obtain a rule in $\mathcal{R}$. For instance, consider the rule $h \rightsquigarrow h$, we can add a letter $j = 0, 1, \ldots, 5$:

  - since the inversion sequences avoid B, we see that $\mathcal{T}'(B; h0) \sim \mathcal{T}'(B; h)$.

– $h1 = 010201$ contains 021, so $h1$ is not a child of h;

– $hj = 01020j$, $j = 2, 3, 4, 5$, contains 1002, so hj is not a child of h.

Thus the rule $h \rightsquigarrow h$ holds for $\mathcal{T}'(B)$. Similarly, we can show that the rules $\mathcal{R}$ describe all the rules of $\mathcal{T}'(B)$.

- **Step 3- From the rules of $\mathcal{T}'(B)$ to a one-parameter infinite system of equations:** For each rule of the type $v \rightsquigarrow v^{(1)}v^{(2)}\cdots v^{(\ell)}$ in $\mathcal{R}$, we define the generating function $A_v(x)$ as the generating function for the number of nodes at level n in the subtree $\mathcal{T}'(B;v)$ of $\mathcal{T}'(B)$, where the root of this subtree is the vertex v that stays at level 0. Then each rule of the type $v \rightsquigarrow v^{(1)}v^{(2)}\cdots v^{(\ell)}$ can be translated into an equation for the generating functions as

$$A_v(x) = x + x\sum_{j=1}^{\ell} A_{v^{(j)}}(x).$$

Note that if we have a finite number of rules, then there will be a finite system of equations. For the above rules, we have

$$\begin{array}{ll}
A_m(x) & = x + xA_{m+1}(x) + xB_m(x) + \cdots + xB_1(x),\\
B_m(x) & = x + xC_m(x) + xB_{m+1}(x) + xD_m(x) + \cdots + xD_1(x),\\
C_m(x) & = x + xF(x) + xC_{m+1}(x) + xE_m(x) + \cdots + xE_1(x) + xG(x),\\
D_m(x) & = x + xE_m(x) + xD_{m+1}(x) + \cdots + xD_1(x),\\
E_m(x) & = x + xH(x) + xE_{m+1}(x) + \cdots + xE_1(x) + xG(x),\\
F(x) & = x + 2xF(x),\\
G(x) & = x + xH(x) + xE_1(x) + xG(x),\\
H(x) & = x + xH(x).
\end{array}$$

where $A_m(x) = A_{a_m}(x)$, $B_m(x) = A_{b_m}(x)$, $C_m(x) = A_{c_m}(x)$, $D_m(x) = A_{d_m}(x)$, $E_m(x) = A_{e_m}(x)$, $F(x) = A_f(x)$, $G(x) = A_g(x)$, and $H(x) = A_h(x)$.

- **Step 4- A finite system of equations for bivariate generating functions:** There are many different methods to solve such system of recurrence relations. In this paper, we observe that for each set of patterns $B = \{021, \tau\}$ with a 021-avoiding four-letter pattern τ, the system of recurrence relations is linear, and it is parametrized by one parameter m.

We will use bivariate generating functions to solve the recurrence relations. Consider $A_m(x)$ parametrized by one parameter m, we define the bi-variate generating function as

$$A(x,v) = \sum_{m\geq s} A_m(x)v^{m-s},$$

where s indicates the minimal value of m such that the recurrence for the generating function

$A_m(x)$ holds. From the above recurrence relations, we obtain

$$\begin{array}{ll} A(x,v) & = \frac{x}{1-v}(1+B(x,v)) + \frac{x}{v}(A(x,v)-A(x,0)), \\ B(x,v) & = \frac{x}{1-v}(1+D(x,v)) + xC(x,v) + \frac{x}{v}(B(x,v)-B(x,0)), \\ C(x,v) & = \frac{x}{1-v}(1+F(x)+E(x,v)+G(x)) + \frac{x}{v}(C(x,v)-C(x,0)), \\ D(x,v) & = \frac{x}{1-v}(1+D(x,v)) + xE(x,v) + \frac{x}{v}(D(x,v)-D(x,0)), \\ E(x,v) & = \frac{x}{1-v}(1+H(x)+E(x,v)+G(x)) + \frac{x}{v}(E(x,v)-E(x,0)), \\ F(x) & = \frac{x}{1-2x}, \\ G(x) & = \frac{x+x(1-x)E_1(x)}{(1-x)^2}, \\ H(x) & = \frac{x}{1-x}. \end{array}$$

- **Step 5- Obtaining the generating function $F_B(x)$ with the kernel method:** In order to solve such system, we use kernel method several times. We apply the algorithm in Hou and Mansour (2008). Let $v_0 = \frac{1-\sqrt{1-4x}}{2}$, we consider the equation of $E(x,v)$ with $v = v_0$, which leads to

$$E(x,0) = \frac{1-\sqrt{1-4x}}{(1-x)(1-2x+\sqrt{1-4x})}.$$

Then by setting this into the same equation of $E(x,v)$, we obtain

$$E(x,v) = \frac{x(1-2x-\sqrt{1-4x})}{(1-x)(1-2x+\sqrt{1-4x})(v^2-v+x)}.$$

For the second step, we take the equation of $D(x,v)$ at limit $v = v_0$, which leads to

$$D(x,0) = \frac{(1-2x)\sqrt{1-4x}-(1-x)(1-4x)}{(1-4x)(1-x)}.$$

Then by setting this into the same equation of $D(x,v)$, we obtain

$$D(x,v) = \frac{(v-1)(4v^2x^2-6v^2x-8vx^2+4x^3+v^2+7vx-2x^2-v)}{2\sqrt{1-4x}(1-x)(v^2-v+x)^2} + \frac{(v-1+x)(2v^2x-v^2-4vx+2x^2+v)}{2(1-x)(v^2-v+x)^2}$$

Similarly, we solve other cases. Let $v_0 = x$, for $C(x,v)$ with $v = v_0$, we first obtain $C(x,0)$ and then obtain an explicit formula for $C(x,v)$. The case of $B(x,v)$ and $A(x,v)$ follow with the same $v = v_0$. Hence we obtain

$$A(x,0) = \frac{x}{1-x} + \frac{x}{1-x}B(x,x),$$

which leads to that the generating function $A(x,0)$ is given by

$$\frac{(1-2x)(2x^2-4x+1)}{2x^2(1-x)\sqrt{1-4x}} - \frac{(2x^2-2x+1)(2x^6-12x^5+25x^4-34x^3+24x^2-8x+1)}{2x^2(1-x)^5(1-2x)}.$$

Clearly, $A(x,0) = A_1(x)$ is the generating function for the number of inversion sequences in $\mathbf{I}_n(B)$.

An explicit formula for the enumerating sequence $\{|\mathbf{I}_n(\{021, 1002\})|\}_{n\geq 1}$ follows from the generating function:

$$\frac{1}{2}\binom{2n+4}{n+2} - \frac{5}{2}\binom{2n+2}{n+1} + \frac{5}{2}\binom{2n}{n} + \frac{1}{2}\sum_{j=0}^{n-1}\binom{2j}{j} + 2^n - \frac{1}{24}(n^4 - 2n^3 + 11n^2 + 14n + 12).$$

All other cases are based on similar techniques as described above. For some $\{021, \tau\}$-avoiding inversion sequence classes, we observe that finding the root of the kernel requires the solution of a third or fourth-order polynomial. We will provide the details of the proofs of such cases in the following sections.

3 Four letter patterns

In this section, we consider the set of inversion sequences of length n that avoid both 021 and τ, where τ is any pattern of length four that avoids 021. We summarize our results in Table 1; the results follow from the procedure discussed in the previous section (for the Maple file which includes all the computations, we refer the reader to Mansour (2022b)). We obtain exact formulas for the generating functions

$$F_{\{021,\tau\}}(x) = \sum_{n\geq 0} |\mathbf{I}_n(\{021, \tau\})| x^{n+1}.$$

Most of them are algebraic of degree at most 6 and only involve the square root of $1 - 4x$. We will give all the details of the above procedure for five cases to show how we find analytic expressions for the generating functions. In Corollary 3.9, we summarize the exact enumerating formulas for all pattern cases except the ones whose generating functions don't lead to a nice closed formula.

Tab. 1: Succession rules for the generating trees $\mathcal{T}'(B)$ and generating functions $F_B(x)$ for pattern set $B = \{021, \tau\}$, where τ is any pattern of length four that avoids 021.

Beginning of Table 1				
No.	τ	d	Rules of $\mathcal{T}'(\{021,\tau)$	$F_{\{021,\tau\}}(x)$
1	0000	6	Theorem 3.11	Theorem 3.11
2	0001	2	$a_0 \rightsquigarrow b_0c_1, b_0 \rightsquigarrow ea_1f, e \rightsquigarrow e, f \rightsquigarrow egf, g \rightsquigarrow eha_1f$ $h \rightsquigarrow eh, i \rightsquigarrow hi, a_m \rightsquigarrow e^3b_ma_ma_{m-1}\cdots a_1f$ $b_m \rightsquigarrow e^3ha_{m+1}a_m\cdots a_1f, c_m \rightsquigarrow a_md_mc_mc_{m-1}\cdots c_1,$ $d_m \rightsquigarrow ib_mc_{m+1}c_m\cdots c_1$ $a_m = 0^21^2\cdots(m-1)^2m, b_m = a_mm$ $c_m = 01^22^2\cdots(m-1)^2m, d_m = c_mm$ $e = 000, f = 002, g = 0022, h = 00111, i = 0111$	$\frac{9x^5-6x^4-x^3+5x^2-4x+1}{2x^2(1+x)(1-x)^3\sqrt{1-2x-3x^2}} + \frac{2x^6-6x^5+x^4+3x^3-4x^2+3x-1}{2x^2(1+x)(1-x)^3}$
3	0010	2	$a_1 \rightsquigarrow a_2b_1, a_m \rightsquigarrow a_{m+1}a_m\cdots a_2c$ $b_m \rightsquigarrow a_{m+2}b_{m+1}b_m\cdots b_1, c \rightsquigarrow a_2c$ $a_m = 0^m, b_m = 01^m, c = 002$	$\frac{x}{\sqrt{1-4x}}$
4	0011	2	$a_1 \rightsquigarrow a_2b_1, b_1 \rightsquigarrow a_3b_2b_1, a_m \rightsquigarrow a_{m+1}a_m\cdots a_2c$ $b_m \rightsquigarrow a_{m+2}b_{m+1}a_ma_{m-1}\cdots a_2c, c \rightsquigarrow a_2c$ $a_m = 0^m, b_m = 01^m, c = 002$	$\frac{2x^3+2x^2-4x+1}{2x^2(1-x)} - \frac{(1-2x)\sqrt{1-4x}}{2x^2(1-x)}$
5	0012	2	$a_1 \rightsquigarrow a_2b_1, b_1 \rightsquigarrow a3b_2b_1, a_m \rightsquigarrow a_{m+1}c^m$ $b_m \rightsquigarrow a_{m+2}b_{m+1}c^m, c \rightsquigarrow c^2$ $a_m = 0^m, b_m = 01^m, c = 001$	$\frac{x(2x^4-5x^3+8x^2-4x+1)}{(1-x)^4(1-2x)}$
6	0100	3	$a_m \rightsquigarrow a_{m+1}b_mb_{m-1}\cdots b_1, b_m \rightsquigarrow c_mb_{m+1}b_m\cdots b_1$ $c_m \rightsquigarrow c_{m+1}c_m\cdots c_1d, d \rightsquigarrow c_1d$ $a_m = 0^m, b_m = a_m1, c_m = b_m0, d = 0103$	$\frac{(1-3x)^2}{2x^2\sqrt{1-4x}} - \frac{(1-3x)(1-x)}{2x^2}$
	0110	3	$a_m \rightsquigarrow a_{m+1}b_mb_{m-1}\cdots b_1, b_m \rightsquigarrow c_mb_{m+1}b_m\cdots b_1$ $c_m \rightsquigarrow c_{m+1}c_m\cdots c_1d, d \rightsquigarrow c_1d$ $a_m = 0^m, b_m = a_m1, c_m = b_m1, d = 0113$	

Continuation of Table 1				
No.	τ	d	Rules of $\mathcal{T}'(\{021,\tau\})$	$F_{\{021,\tau\}}(x)$
7	0101 0111	2 2	$a_m \rightsquigarrow a_{m+1}b_mb_{m-1}\cdots b_1, b_m \rightsquigarrow a_{m+1}b_{m+1}b_m\cdots b_1$ $a_m = 0^m, b_m = a_m1$	Theorem 3.2
8	0102	3	$a_m \rightsquigarrow a_{m+1}b_mb_{m-1}\cdots b_1, b_m \rightsquigarrow db_{m+1}c_mc_{m-1}\cdots c_1$ $c_m \rightsquigarrow ec_{m+1}c_m\cdots c_1$ $a_m = 0^m, b_m = a_m1, c_m = b_m2, d = 010, e = 0120$	$\frac{4x^6-12x^5+24x^4-27x^3+19x^2-7x+1}{2x(1-2x)(1-x)^4}$ $-\frac{\sqrt{1-4x}}{2x(1-x)}$
9	0112	2	$a_m \rightsquigarrow a_{m+1}b_mb_{m-1}\cdots b_1, b_m \rightsquigarrow cb_{m+1}b_m\cdots b_1,$ $c \rightsquigarrow c^2$ $a_m = 0^m, b_m = a_m1, c = 011$	$\frac{1-4x+5x^2-4x^3-(1-x)^2\sqrt{1-4x}}{2x(1-2x)(1-x)}$
10	0120 0122	3 3	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, b_m \rightsquigarrow b_{m+1}^2c_1\cdots c_m$ $c_m \rightsquigarrow c_1\cdots c_{m+1}$ $a_m = 0^m, b_m = a_m1, c_m = b_m2$	$\frac{1-2x}{2(1-x)}\left(\frac{1}{\sqrt{1-4x}}-1\right)$
11	0123	2	$a_m \rightsquigarrow a_{m+1}b_mb_{m-1}\cdots b_1, b_m \rightsquigarrow b_{m+1}^2c^m, c \rightsquigarrow c^2$ $a_m = 0^m, b_m = a_m1, c = 012$	$\frac{x(1-7x+21x^2-30x^3+22x^4-8x^5)}{(1-x)^3(1-2x)^3}$
12	1000	4	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, b_m \rightsquigarrow b_1\cdots b_{m+1}c_m$ $c_m \rightsquigarrow c_1\cdots c_{m+1}d_me, d_m \rightsquigarrow d_1\cdots d_{m+1}fg$ $e \rightsquigarrow efc_1, f \rightsquigarrow d_1fg, g \rightsquigarrow fg$ $a_m = 0^m, b_m = a_m1, c_m = b_m0, d_m = c_m0$ $e = 0103, f = 01003, g = 01004$	
	1100	4	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, b_m \rightsquigarrow b_1\cdots b_{m+1}c_m$ $c_m \rightsquigarrow c_1\cdots c_{m+1}d_me, d_m \rightsquigarrow d_1\cdots d_{m+1}fg$ $e \rightsquigarrow efc_1, f \rightsquigarrow d_1fg, g \rightsquigarrow fg$ $a_m = 0^m, b_m = a_m1, c_m = b_m1, d_m = c_m0$ $e = 0113, f = 01103, g = 01104$	$\frac{(1-x)(1-2x)^2}{2x^3}$ $+\frac{2x^5-56x^4+78x^3-44x^2+11x-1}{2x^3\sqrt{1-4x}^3}$
13	1001 1011	3 3	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, b_m \rightsquigarrow c_mb_1\cdots b_{m+1}$ $c_m \rightsquigarrow a_{m+2}b_1\cdots b_{m+1}c_{m+1}$ $a_m = 0^m, b_m = a_m1, c_m = b_m0$	
	1101	3	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, b_m \rightsquigarrow c_mb_1\cdots b_{m+1}$ $c_m \rightsquigarrow a_{m+2}b_1\cdots b_{m+1}c_{m+1}$ $a_m = 0^m, b_m = a_m1, c_m = b_m1$	Theorem 3.1
14	1002	5	$a_m \rightsquigarrow a_{m+1}b_mb_{m-1}\cdots b_1,$ $b_m \rightsquigarrow c_mb_{m+1}d_md_{m-1}\cdots d_1$ $c_m \rightsquigarrow fc_{m+1}e_me_{m-1}\cdots e_1g, d_m \rightsquigarrow e_md_{m+1}d_m\cdots d_1$ $e_m \rightsquigarrow he_{m+1}e_m\cdots e_1g, f \rightsquigarrow f^2$ $g \rightsquigarrow he_1g, h \rightsquigarrow h$ $a_m = 0^m, b_m = a_m1, c_m = b_m0$ $d_m = a_m12, e_m = a_m102, f = 0100$ $g = 0103, h = 01020$	$\frac{(1-2x)(2x^2-4x+1)}{2x^2(1-x)\sqrt{1-4x}}$ $-\frac{x^2(2x^2-2x+1)(2x^2-12x+25)}{2(1-x)^5(1-2x)}$ $+\frac{(2x^2-2x+1)(34x^3-24x^2+8x-1)}{2x^2(1-x)^5(1-2x)}$
15	1010	4	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, b_m \rightsquigarrow c_mb_1\cdots b_{m+1}$ $c_m \rightsquigarrow c_{m+1}d_mb_1\cdots b_{m+1}, d_m \rightsquigarrow d_1\cdots d_{m+1}ef$ $e \rightsquigarrow d_1ef, f \rightsquigarrow ef$ $a_m = 0^m, b_m = a_m1, c_m = b_m0, d_m = c_m1$ $e = 01013, f = 01014$	
	1110	4	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, b_m \rightsquigarrow c_mb_1\cdots b_{m+1}$ $c_m \rightsquigarrow c_{m+1}d_mb_1\cdots b_{m+1}, d_m \rightsquigarrow d_1\cdots d_{m+1}ef$ $e \rightsquigarrow d_1ef, f \rightsquigarrow ef$ $a_m = 0^m, b_m = a_m1, c_m = b_m1, d_m = c_m1$ $e = 01113, f = 01114$	Theorem 3.3
20	1012	3	$a_m \rightsquigarrow a_{m+1}b_mb_{m-1}\cdots b_1, b_m \rightsquigarrow c_mb_{m+1}b_m\cdots b_1$ $c_m \rightsquigarrow dc_{m+1}b_{m+1}b_m\cdots b_1, d \rightsquigarrow d^2$ $a_m = 0^m, b_m = a_m1, c_m = b_m0, d = 0101$	Theorem 3.5
21	1020 1022	5 5	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, b_m \rightsquigarrow c_mb_{m+1}d_1\cdots d_m$ $c_m \rightsquigarrow c_{m+1}^2e_1\cdots e_{m-1}fg, d_m \rightsquigarrow e_md_1\cdots d_{m+1}$ $e_m \rightsquigarrow e_1\cdots e_{m+1}fg, f \rightsquigarrow fg, g \rightsquigarrow e_1fg$ $a_m = 0^m, b_m = a_m1, c_m = b_m0, d_m = b_m2$ $e_m = d_m0, f = 0103, g = 0102$	$\frac{2-15x+34x^2-22x^3}{2x^3(1-x)\sqrt{1-4x}}$ $+\frac{2x^4+4x^3-16x^2+11x-2}{2x^3(1-x)}$
23	1023	5	$a_m \rightsquigarrow a_{m+1}b_mb_{m-1}\cdots b_1,$ $b_m \rightsquigarrow c_mb_{m+1}d_md_{m-1}\cdots d_1$	

Continuation of Table 1

No.	τ	d	Rules of $\mathcal{T}'(\{021,\tau)$	$F_{\{021,\tau\}}(x)$
			$c_m \rightsquigarrow c_{m+1}^2 f^{m+1}, d_m \rightsquigarrow e_m d_{m+1} d_m \cdots d_1$ $e_m \rightsquigarrow e_{m+1} f^{m+2}, f \rightsquigarrow f^2$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 0, d_m = a_m 12$ $e_m = d_m 0, f = 0102$ $e = 0110, f = 0113, g = 01120$	$-\frac{(2x^2-2x+1)\sqrt{1-4x}}{2x(1-x)^2(1-2x)}$ $-\frac{16x^9-76x^8+212x^7-342x^6}{2x(1-x)^5(1-2x)^3}$ $-\frac{374x^5-286x^4+151x^3-53x^2+11x-1}{2x(1-x)^5(1-2x)^3}$
26	1102	4	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1, b_m \rightsquigarrow b_{m+1} b_m \cdots b_1 c_m$ $c_m \rightsquigarrow e c_{m+1} d_m d_{m-1} \cdots d_1 f, d_m \rightsquigarrow g d_{m+1} d_m \cdots d_1 f$ $e \rightsquigarrow e^2, f \rightsquigarrow g d_1 f, g \rightsquigarrow g$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 1, d_m = c_m 2$ $e = 0110, f = 0113, g = 01120$	$\frac{-4x^5+31x^4-46x^3+30x^2-9x+1}{2x^2(1-x)^2(1-2x)\sqrt{1-4x}}$ $-\frac{4x^6-15x^5+35x^4-40x^3+25x^2-8x+1}{2x^2(1-x)^3(1-2x)}$
28	1120	4	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1, b_m \rightsquigarrow b_{m+1} b_m \cdots b_1 c_m$ $c_m \rightsquigarrow c_{m+1}^2 d_m d_{m-1} \cdots d_1, d_m \rightsquigarrow d_{m+1} d_m \cdots d_1 e,$ $e \rightsquigarrow d_1 e$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 1, d_m = c_m 2, e = 0113$	$\frac{(3x-1)\sqrt{1-4x}+5x^2-5x+1}{x(1-4x)}$
29	1200	4	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m, b_m \rightsquigarrow b_{m+1}^2 c_1 \cdots c_m$ $c_m \rightsquigarrow d_m c_1 \cdots c_{m+1}, d_m \rightsquigarrow d_1 \cdots d_{m+1} e, e \rightsquigarrow d_1 e$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 0, e = 01204$	
	1220	4	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m, b_m \rightsquigarrow b_{m+1}^2 c_1 \cdots c_m$ $c_m \rightsquigarrow d_m c_1 \cdots c_{m+1}, d_m \rightsquigarrow d_1 \cdots d_{m+1} e, e \rightsquigarrow d_1 e$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 2, e = 01224$	$\frac{22x^3-29x^2+10x-1}{2x(1-x)\sqrt{1-4x}^3} + \frac{(1-5x)(1-2x)}{2x(1-4x)}$
30	1202	4	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1, b_m \rightsquigarrow b_{m+1}^2 c_m c_{m-1} \cdots c_1$ $c_m \rightsquigarrow d_m c_{m+1} c_m \cdots c_1, d_m \rightsquigarrow d_{m+1} d_m \cdots d_1$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 0$	Theorem 3.7
31	1203	4	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1, b_m \rightsquigarrow b_{m+1}^2 c_m c_{m-1} \cdots c_1$ $c_m \rightsquigarrow c_{m+1} d_m d_{m-1} \cdots d_1 e, d_m \rightsquigarrow d_{m+1} d_m \cdots d_1 f$ $e \rightsquigarrow e^2, f \rightsquigarrow f$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 4$ $e = 0120, f = 01230$	$\frac{(1-2x)}{2(1-x)^2\sqrt{1-4x}}$ $-\frac{16x^8-76x^7+188x^6-270x^5}{2(1-x)^5(1-2x)^3}$ $-\frac{246x^4-145x^3+53x^2-11x+1}{2(1-x)^5(1-2x)^3}$
32	1220	4	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1, b_m \rightsquigarrow b_{m+1}^2 c_m c_{m-1} \cdots c_1$ $c_m \rightsquigarrow d_m c_{m+1} c_m \cdots c_1, d_m \rightsquigarrow d_{m+1} d_m \cdots d_1 e,$ $e \rightsquigarrow d_1 e$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 2, e = 01224$	$\frac{(1-2x)(1-5x)}{2x(1-4x)} + \frac{22x^3-29x^2+10x-1}{2x(1-x)\sqrt{1-4x}^3}$
33	1230	4	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1, b_m \rightsquigarrow b_{m+1}^2 c_m c_{m-1} \cdots c_1$ $c_m \rightsquigarrow c_{m+1}^2 d_m d_{m-1} \cdots d_1, d_m \rightsquigarrow d_{m+1} d_m \cdots d_1$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 3$	$\frac{x(3-16x+26x^2-16x^3-(1-2x+2x^2)\sqrt{1-4x})}{2(1-x)^2(1-2x)(1-4x)}$

End of Table 1

Theorem 3.1 *Let $B \in \{\{021, 1001\}, \{021, 1011\}, \{021, 1101\}\}$. Then, the generating function $F_B(x)$ satisfies $F_B(x) = x(1 + F_B(x))^2(1 + F_B^2(x))$. Moreover, for all $n \geq 1$,*

$$|\mathbf{I}_n(B)| = \frac{1}{n+1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{j} \binom{2n+2}{n-2j}.$$

Proof: By the generating tree succession rules for $\mathcal{T}'(\{021, 1001\})$, $\mathcal{T}'(\{021, 1011\})$, and $\mathcal{T}'(\{021, 1101\})$ in Table 1, it follows that for all x,

$$F_{\{021,1001\}}(x) = F_{\{021,1011\}}(x) = F_{\{021,1101\}}(x).$$

Define $A_m(x)$, $B_m(x)$, and $C_m(x)$ to be the generating functions for the number of nodes at level n in $\mathcal{T}'(\{021, 1001\}; a_m)$, $\mathcal{T}'(\{021, 1001\}; b_m)$, and $\mathcal{T}'(\{021, 1001\}; c_m)$, respectively, where its root stays at level 0. Then

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + xB_1(x) + \cdots + xB_m(x),\\ B_m(x) &= x + xC_m(x) + xB_1(x) + \cdots + xB_{m+1}(x),\\ C_m(x) &= x + xA_{m+2}(x) + xB_1(x) + \cdots + xB_{m+1}(x) + xC_{m+1}(x). \end{aligned}$$

Define $G(x,v) = \sum_{n\geq 1} G_m(x)v^{m-1}$ for all $G \in \{A,B,C\}$. Then the above recurrences can be written as

$$\begin{aligned} A(x,v) &= \tfrac{x}{1-v} + \tfrac{x}{v}(A(x,v) - A_1(x)) + \tfrac{x}{1-v}B(x,v),\\ B(x,v) &= \tfrac{x}{1-v} + xC(x,v) + \tfrac{x}{1-v}B(x,v) + \tfrac{x}{v}(B(x,v) - B_1(x)),\\ C(x,v) &= \tfrac{x}{1-v} + \tfrac{x}{v^2}(A(x,v) - A_1(x) - vA_2(x)) + \tfrac{x}{1-v}B(x,v)\\ &\quad + \tfrac{x}{v}(B(x,v) + C(x,v) - B_1(x) - C_1(x)). \end{aligned}$$

By finding $A(x,v)$ from the first equation and $C(x,v)$ from the second equation, we obtain that the second equation can be written as

$$\begin{aligned} &\tfrac{v^4-2v^3x+2v^2x^2+(x-v)^3}{vx^2(1-v)(x-v)}B(x,v)\\ &= \tfrac{v}{x-v}A_1(x) + \tfrac{x-v-vx}{vx}B_1(x) - (A_2(x) + C_1(x)) + \tfrac{(vx+v-x)(v-x)+x^2}{x(1-v)(v-x)}. \end{aligned}$$

Let v_1, v_2, v_3, v_4 be the four roots of $K(v) = v^4 - 2v^3x + 2v^2x^2 + (x-v)^3 = 0$ as functions of x, where (here $i^2 = -1$)

$$\begin{aligned} v_1 &= x + x^{\frac{4}{3}} + \tfrac{2}{3}x^{\frac{5}{3}} + \tfrac{2}{3}x^2 + \tfrac{82}{81}x^{\frac{7}{3}} + \tfrac{349}{243}x^{\frac{8}{3}} + \cdots,\\ v_2 &= x - \tfrac{1+\sqrt{3}i}{2}x^{\frac{4}{3}} - \tfrac{1-\sqrt{3}i}{3}x^{\frac{5}{3}} + \tfrac{2}{3}x^2 - \tfrac{41(1+\sqrt{3}i)}{81}x^{\frac{7}{3}} - \tfrac{349(1-\sqrt{3}i)}{486}x^{\frac{8}{3}} + 2x^3 + \cdots,\\ v_3 &= x - \tfrac{1-\sqrt{3}i}{2}x^{\frac{4}{3}} - \tfrac{1+\sqrt{3}i}{3}x^{\frac{5}{3}} + \tfrac{2}{3}x^2 - \tfrac{41(1-\sqrt{3}i)}{81}x^{\frac{7}{3}} - \tfrac{349(1+\sqrt{3}i)}{486}x^{\frac{8}{3}} + 2x^3 + \cdots,\\ v_4 &= 1 - x - 2x^2 - 6x^3 - 21x^4 - 82x^5 - \cdots. \end{aligned}$$

By taking $v = v_j$, $j = 1,2,3$, we obtain

$$\frac{v_j}{x-v_j}A_1(x) + \frac{x - v_j - v_jx}{xv_j}B_1(x) - (A_2(x) + C_1(x)) + \frac{(v_jx + v_j - x)(v_j - x) + x^2}{x(1-v_j)(v_j-x)} = 0.$$

Solving this system for $A_1(x)$, $B_1(x)$, and $A_2(x) + C_1(x)$, we obtain

$$A_1 = \frac{v_1v_2v_3(1+x^2) + (v_1+v_2+v_3)(1-x)x^2 + x^4 - (v_1v_2 + v_1v_3 + v_2v_3)x}{x^2(1-v_1)(1-v_2)(1-v_3)}.$$

Note that $v_1+v_2+v_3+v_4 = 1+2x$, $v_1v_2v_3v_4 = x^3$, and $v_4(v_1+v_2+v_3)+v_3(v_1+v_2)+v_2v_1 = x(3+2x)$. Thus, $v_3(v_1+v_2) + v_2v_1 = x(3+2x) - v_4(1+2x-v_4)$. Hence, the generating function A_1 can be written as

$$A_1 = \frac{x^2(x^2+1) - x(x^2+x+2)v_4 + (x^2+x+1)v_4^2 - v_4^3}{x(v_4^3 - 2xv_4^2 + (2x^2+x)v_4 - x^3)}.$$

By using a computer programming, we can express v_4 in terms of A_1. Given that $K(v_4) = 0$, we find that the generating function A_1 satisfies $A_1 = x(1+A_1)^2(1+A_1^2)$. By the Lagrange inversion formula, we complete the proof. □

From Table 1, we see that the patterns 0101 and 0111 when paired with 021 have the same generating trees. Our next result enumerates these pattern pairs.

Theorem 3.2 *The generating functions $F_{\{021,0101\}}(x)$ and $F_{\{021,0111\}}(x)$ are given by $f(x)$, where $f(x) = \frac{x(1+f(x))}{1-x(1+f(x))^2}$. Moreover, for all $n \geq 1$,*

$$|\mathbf{I}_n(\{021,0101\})| = |\mathbf{I}_n(\{021,0111\})| = \sum_{i=1}^{n+1} \frac{1}{i} \binom{n}{i-1} \binom{2n+2-i}{i-1}.$$

Proof: By the generating tree succession rules for $\mathcal{T}'(\{021,0101\})$ and $\mathcal{T}'(\{021,0111\})$ in Table 1, we have

$$F_{\{021,0101\}}(x) = F_{\{021,0111\}}(x).$$

By translating the rules of the generating tree $\mathcal{T}'(\{021,0101\})$ to generating function, we obtain

$$\begin{aligned} A(x,v) &= \tfrac{x}{1-v} + \tfrac{x}{v}(A(x,v) - A(x,0)) + \tfrac{x}{1-v}B(x,v), \\ B(x,v) &= \tfrac{x}{1-v} + \tfrac{x}{v}(A(x,v) - A(x,0)) + \tfrac{x}{v}(B(x,v) - B(x,0)) + \tfrac{x}{1-v}B(x,v). \end{aligned}$$

Finding $A(x,v)$ from the first equation and substituting it into the second equation, we obtain

$$\begin{aligned} B(x,v) &= \tfrac{x}{1-v} + \tfrac{x}{v}\left(\tfrac{x(vA(x,0)+vB(x,v)-A(x,0)+v)}{(1-v)(v-x)} - A(x,0)\right) \\ &+ \tfrac{x}{v}(B(x,v) - B(x,0)) + \tfrac{x}{1-v}B(x,v). \end{aligned}$$

The kernel of this equation is given by $K(v) = \frac{v^3-(1+x)v^2+x(x+2)v-x^2}{(1-v)(v-x)v}$. The equation $K(v)$ has three roots, that is, $K(v_j) = 0$ for $j = 1,2,3$, where

$$v_j = \frac{2\sqrt{1-4x-2x^2}}{3} \cos\left(\frac{1}{3}\arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^{\,3}}\right) + \frac{(2j+1)\pi}{3}\right) + \frac{x+1}{3}.$$

By taking either $v = v_1$ or $v = v_2$ into the equation, we get a system of equations. Solving this system for $A(x,0)$ and $B(x,0)$, we obtain

$$A(x,0) = \frac{(x+1)v_1v_2 - x(v_1+v_2)x}{x(1-v_1)(1-v_2)}.$$

Note that $v_1 + v_2 + v_3 = 1 + x$ and $v_1v_2v_3 = x^2$, we have

$$A(x,0) = \frac{v_3^2 - (1+x)v_3 + x(1+x)}{v_3^2 - xv_3 + x^2}.$$

Since $K(v_3) = 0$, we have that $A(x,0)$ satisfies the equation

$$A(x,0) = \frac{x(1+A(x,0))}{1-x(1+A(x,0))^2}.$$

By the Lagrange inversion formula, we obtain

$$A(x,0) = \sum_{i\geq 1}\sum_{j\geq 0} \frac{1}{i}\binom{i-1+j}{i-1}\binom{i+2j}{i-1}x^{i+j},$$

which, by comparing coefficient of x^n, we complete the proof. □

Theorem 3.3 *Let* $B \in \{\{021, 1010\}, \{021, 1110\}\}$. *Then,*

$$F_B(x) = \frac{x}{1-x}\left(1 + \frac{P(x)}{Q(x)}\right)$$

with
$P(x) = x(1-x)((x-1)\sqrt{1-4x}-3x+1)+((1-x)^2\sqrt{1-4x}-x^2+4x-1)v_3-x(\sqrt{1-4x}+1)v_3^2$,
$Q(x) = x^2(x-1)(\sqrt{1-4x}-2x+1) - 2x((x-1)\sqrt{1-4x}-x^2+3x-1)v_3 - (1-4x+2x^2+(1-2x)\sqrt{1-4x})v_3^2$,
where v_3 *is defined in* (2).

Proof: By the generating tree succession rules for $\mathcal{T}'(\{021, 1010\})$ and $\mathcal{T}'(\{021, 1110\})$ in Table 1, we have

$$F_{\{021,1010\}}(x) = F_{\{021,1110\}}(x).$$

Define $A_m(x)$, $B_m(x)$, $C_m(x)$, $E(x)$, and $F(x)$ to be the generating functions for the number of nodes at level $n \geq 0$ in $\mathcal{T}'(\{021, 1010\}; a_m)$, $\mathcal{T}'(\{021, 1010\}; b_m)$, $\mathcal{T}'(\{021, 1010\}; c_m)$, $\mathcal{T}'(\{021, 1010\}; d_m)$, $\mathcal{T}'(\{021, 1010\}; e)$, and $\mathcal{T}'(\{021, 1010\}; f)$, respectively, where its root stays at level 0. Then

$$\begin{aligned}
A_m(x) &= x + xA_{m+1}(x) + xB_1(x) + \cdots + xB_m(x),\\
B_m(x) &= x + xC_m(x) + xB_1(x) + \cdots + xB_{m+1}(x),\\
C_m(x) &= x + xC_{m+1}(x) + xD_m(x) + xB_1(x) + \cdots + xB_{m+1}(x),\\
D_m(x) &= x + xD_1(x) + \cdots + xD_{m+1}(x) + xE(x) + xF(x),\\
E(x) &= x + xD_1(x) + xE(x) + xF(x),\\
F(x) &= x + xE(x) + xF(x).
\end{aligned}$$

Define $G(x, v) = \sum_{n\geq 1} G_m(x)v^{m-1}$ for all $G \in \{A, B, C, D\}$. Then the above recurrence can be written as

$$\begin{aligned}
A(x, v) &= \tfrac{x}{1-v} + \tfrac{x}{v}(A(x, v) - A_1(x)) + \tfrac{x}{1-v}B(x, v),\\
B(x, v) &= \tfrac{x}{1-v} + xC(x, v) + \tfrac{x}{1-v}B(x, v) + \tfrac{x}{v}(B(x, v) - B_1(x)),\\
C(x, v) &= \tfrac{x}{1-v} + \tfrac{x}{v}(C(x, v) - C_1(x)) + xD(x, v) + \tfrac{x}{1-v}B(x, v) + \tfrac{x}{v}(B(x, v) - B_1(x)),\\
D(x, v) &= \tfrac{x}{1-v} + \tfrac{x}{v}(D(x, v) - D_1(x)) + \tfrac{x}{1-v}D(x, v) + \tfrac{x}{1-v}(E(x) + F(x)),\\
E(x) &= x + xD_1(x) + xE(x) + xF(x),\\
F(x) &= x + xE(x) + xF(x).
\end{aligned}$$

By taking $v = \frac{1-\sqrt{1-4x}}{2}$, the last three equations lead to

$$\begin{aligned}
D_1(x) &= \tfrac{(2x-1)\sqrt{1-4x}+2x^2-4x+1}{2x^3},\\
E(x) &= \tfrac{(x-1)\sqrt{1-4x}-3x+1}{2x^2},\\
F(x) &= \tfrac{1-2x-\sqrt{1-4x}}{2x}.
\end{aligned}$$

By finding $C(x, v)$ from the third equation, then substituting it into the second equation, we obtain

$$\begin{aligned}
&\tfrac{v^3-(vx+v-x)(v-x)}{v(1-v)(x-v)}B(x, v)\\
&= \tfrac{x(vx+v-x)}{v(v-x)}B_1(x) + \tfrac{x^2}{x-v}C_1(x) - \tfrac{(vx-v-2x+1)v\sqrt{1-4x}}{2(v-x)(v^2-v+x)}\\
&+ \tfrac{(2v^3x^2-v^3x-6v^2x^2+2vx^3+v^3+5v^2x+6vx^2-2x^3-2v^2-4vx+v)}{2(v^2-v+x)(1-v)(v-x)}.
\end{aligned}$$

Let v_1, v_2, v_3 be the four roots of $v^3 - (vx + v - x)(v - x) = 0$ as functions of x,

$$v_j = \frac{2\sqrt{1-4x-2x^2}}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^{\,3}}\right) + \frac{(2j+1)\pi}{3}\right) + \frac{x+1}{3}. \qquad (2)$$

By taking $v = v_j$, $j = 1, 2$, we obtain

$$\begin{aligned}&\tfrac{x(v_jx+v_j-x)}{v_j(v_j-x)}B_1(x) + \tfrac{x^2}{x-v_j}C_1(x) - \tfrac{(v_jx-v_j-2x+1)v_j\sqrt{1-4x}}{2(v_j-x)(v_j^2-v_j+x)} \\ &+\tfrac{(2v_j^3x^2-v_j^3x-6v_j^2x^2+2v_jx^3+v_j^3+5v_j^2x+6v_jx^2-2x^3-2v_j^2-4v_jx+v_j)}{2(v_j^2-v_j+x)(1-v_j)(v_j-x)} = 0.\end{aligned}$$

Solving this system for $B_1(x)$ and $C_1(x)$, we have explicit formulas for $B_1(x)$ and $C_1(x)$:

$$B_1(x) = \frac{\begin{array}{r}x(2x^2(x^2-2x+2)+x(x-2)(\sqrt{1-4x}-2x+3)v_3 \\ -(x-1)(\sqrt{1-4x}-2x+3)v_3^2)(2x-3+\sqrt{1-4x})\end{array}}{\begin{array}{r}2(x^2-2x+2)(x^2(x-1)(\sqrt{1-4x}-2x+1) \\ -2x((x-1)\sqrt{1-4x}-x^2+3x-1)v_3+(4x-2x^2-1+(2x-1)\sqrt{1-4x})v_3^2)\end{array}},$$

$$C_1(x) = \frac{x((1+x)\sqrt{1-4x}+x-1-x(\sqrt{1-4x}+1)v_3+(\sqrt{1-4x}+1)v_3^2)}{\begin{array}{r}x^2(1-x)(\sqrt{1-4x}-2x+1)+2x((x-1)\sqrt{1-4x}-x^2+3x-1)v_3 \\ +(1-4x+2x^2+(1-2x)\sqrt{1-4x})v_3^2\end{array}}.$$

Hence, by the equation of $B(x, v)$, we obtain an explicit formula for $B(x, v)$. In particular, we have $B(x, x) = R(x)$. Therefore, by equation of $A(x, v)$ with $v = x$, we have that $A_1(x) = \frac{x}{1-x} + \frac{x}{1-x}B(x, x)$, which completes the proof. $\square$

Remark 3.4 *By the proof of Theorem 3.3 and using that $v = v_3$ is a root of $v^3 - (vx+v-x)(v-x) = 0$, we get that $f = F_{\{021,1010\}}(x)$ satisfies*

$$\begin{aligned}&x(x^6-6x^5+19x^4-32x^3+27x^2-9x+1) \\ &+(x-1)(6x^6-28x^5+58x^4-57x^3+35x^2-10x+1)f \\ &+x(15x^6-76x^5+159x^4-170x^3+94x^2-23x+2)f^2 \\ &+x(20x^6-84x^5+139x^4-117x^3+50x^2-11x+1)f^3+x^3(15x^4-46x^3+50x^2-21x+3)f^4 \\ &+x^5(6x^2-10x+3)f^5+x^7f^6=0.\end{aligned}$$

Theorem 3.5 *The generating function $F_{\{021,1012\}}(x)$ is given by*

$$-\frac{((3x^2-3x+1)v_3^2+(-2x^3+2x-1)v_3+2x^4-2x^2+x)}{(1-x)(1-2x)(v_3^2-v_3x+x^2)},$$

where $v_3 = \frac{2\sqrt{1-4x-2x^2}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^{\,3}}\right)+\frac{\pi}{3}\right)+\frac{x+1}{3} = 1-x-2x^2-\cdots$ is a root of $v^3-(1+x)v^2+x(2+x)v-x^2=0$.

Proof: By translating the rules of the generating tree $\mathcal{T}'(\{021, 1012\})$ in Table 1 to generating functions, we have

$$\begin{aligned}A(x,v) &= \tfrac{x}{1-v}+\tfrac{x}{v}(A(x,v)-A(x,0))+\tfrac{x}{1-v}B(x,v), \\ B(x,v) &= \tfrac{x}{1-v}+xC(x,v)+\tfrac{x}{v}(B(x,v)-B(x,0))+\tfrac{x}{1-v}B(x,v), \\ C(x,v) &= \tfrac{x}{1-v}+\tfrac{x}{v}(C(x,v)-C(x,0))+\tfrac{x}{1-v}D(x)+\tfrac{x}{v}(B(x,v)-B(x,0))+\tfrac{x}{1-v}B(x,v), \\ D(x) &= \tfrac{x}{1-2x}.\end{aligned}$$

Finding $C(x,v)$ from the third equation and substituting its expression into the second equation, we obtain

$$\begin{array}{ll} B(x,v) & = \frac{x}{1-v} + \frac{x^2}{(1-\frac{x}{v})(1-v)}\Big(1 + \frac{x}{1-2x} + \frac{1-v}{v}(B(x,v) - B(x,0) - C(x,0)) + B(x,v)\Big) \\ & + \frac{x}{v}(B(x,v) - B(x,0)) + \frac{x}{1-v}B(x,v), \end{array}$$

The kernel of this equation is given by $K(v) = \frac{v^3-(1+x)v^2+x(2+x)v-x^2}{(1-v)(v-x)v}$. The equation $K(v) = 0$ has three roots, that is, $K(v_j) = 0$ for $j = 1, 2, 3$, where

$$v_j = \frac{2\sqrt{1-4x-2x^2}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^{\,3}}\right) + \frac{(2j+1)\pi}{3}\right) + \frac{x+1}{3}.$$

By taking either $v = v_1$ or $v = v_2$ into the equation, we get a system of equations. Solving this system for $B(x,0)$ and $C(x,0)$, we obtain

$$\begin{array}{ll} B(x,0) & = \frac{(x-1)^2 v_1 v_2}{x(1-2x)(1-v_1)(1-v_2)}, \\ C(x,0) & = \frac{(2x^2-1)v_1v_2 + x(1-x-x^2)(v_1+v_2) + 2x^3 - x^2}{x^2(1-2x)(1-v_1)(1-v_2)}. \end{array}$$

Note that $v_1 + v_2 + v_3 = 1 + x$ and $v_1v_2v_3 = x^2$, we have

$$A(x,0) = -\frac{((3x^2-3x+1)v_3^2 + (-2x^3+2x-1)v_3 + 2x^4 - 2x^2 + x)}{(1-x)(1-2x)(v_3^2 - v_3x + x^2)},$$

as claimed. □

Remark 3.6 *By the proof of Theorem 3.5 and using that* $v = v_3$ *is a root of* $v^3 - v^2x + vx^2 - v^2 + 2vx - x^2 = 0$*, we get that* $f = F_{\{021,1012\}}(x)$ *satisfies*

$$\begin{array}{l} x(2x^2-2x+1)(x^6-2x^5+13x^4-24x^3+19x^2-7x+1) \\ +(1-2x)(1-x)(x^7-x^6+24x^5-51x^4+50x^3-27x^2+8x-1)f \\ +x(x^3+6x^2-6x+2)(1-2x)^2(1-x)^2f^2 + x(1-2x)^3(1-x)^3f^3 = 0. \end{array}$$

Theorem 3.7 *The generating function* $F_{\{021,1202\}}(x)$ *is given by*

$$\frac{x(2(1-3x)v_3^2 - (1-6x^2)v_3 + x(1+x)(1-3x))}{(1-x)(1-6x)(v_3^2 - xv_3 + x^2)},$$

where $v_3 = \frac{2\sqrt{1-4x-2x^2}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^{\,3}}\right) + \frac{\pi}{3}\right) + \frac{x+1}{3} = 1 - x - 2x^2 - \cdots$ *is a root of* $v^3 - (1+x)v^2 + x(x+2)v - x^2 = 0$.

Proof: By translating the rules of the generating tree $\mathcal{T}'(\{021,1202\})$ in Table 1 to generating functions, we have

$$\begin{array}{ll} A(x,v) & = \frac{x}{1-v} + \frac{x}{v}(A(x,v) - A(x,0)) + \frac{x}{1-v}B(x,v), \\ B(x,v) & = \frac{x}{1-v} + \frac{x}{1-v}C(x,v) + \frac{2x}{v}(B(x,v) - B(x,0)), \\ C(x,v) & = \frac{x}{1-v} + \frac{x}{v}(C(x,v) - C(x,0)) + \frac{x}{1-v}C(x,v) + xD(x,v), \\ D(x,v) & = \frac{x}{1-v} + \frac{x}{1-v}C(x,v) + \frac{x}{v}(C(x,v) + D(x,v) - C(x,0) - D(x,0)). \end{array}$$

Finding $D(x,v)$ from the fourth equation and substituting its expression into the third equation, we obtain

$$\begin{aligned} C(x,v) &= \tfrac{x}{1-v}(1+C(x,v)) + \tfrac{x^2}{(1-v)(v-x)}((v-1)(C(x,0)+D(x,0)) + C(x,v) + v) \\ &+ \tfrac{x}{v}(C(x,v) - C(x,0)). \end{aligned}$$

The kernel of this equation is given by $K(v) = \frac{v^3-(1+x)v^2x+x(2+x)v-x^2}{(1-v)(v-x)v}$. The equation $K(v) = 0$ has three roots, namely, $K(v_j) = 0$ for $j = 1, 2, 3$, where

$$v_j = \frac{2\sqrt{1-4x-2x^2}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^{\,3}}\right) + \frac{(2j+1)\pi}{3}\right) + \frac{x+1}{3}.$$

By taking either $v = v_1$ or $v = v_2$ into the equation, we get a system of equations. Solving this system for $B(x,0)$ and $C(x,0)$, we obtain

$$\begin{aligned} C(x,0) &= \tfrac{v_1v_2}{x(1-v_1)(1-v_2)}, \\ D(x,0) &= -\tfrac{((1+x)v_1-x)((1+x)v_2-x)}{x^2(1-v_1)(1-v_2)}. \end{aligned}$$

Thus, by expressions of $C(x,0)$ and $D(x,0)$, we have explicit formulas for $C(x,v)$ and $D(x,v)$.

Hence, by considering the equation of $B(x,v)$ with $v = 2x$, we obtain

$$B(x,0) = \frac{(1+3x)v_1v_2 - 3x(v_1+v_2) + x}{(1-6x)(1-v_1)(1-v_2)},$$

which, by using this expression and $C(x,v)$, we obtain an explicit formula for $B(x,v)$.

By considering the equation of $A(x,v)$ with $v = x$, we obtain

$$\begin{aligned} A(x,0) &= \tfrac{x}{1-x} + \tfrac{x}{1-x}B(x,x) \\ &= \tfrac{(1-2x-6x^2)v_1v_2 - 2x(1-3x)(v_1+v_2) + x(1-4x)}{(1-x)(1-6x)(1-v_1)(1-v_2)}. \end{aligned}$$

Note that $v_1 + v_2 + v_3 = 1 + x$ and $v_1v_2v_3 = x^2$, we have

$$A(x,0) = \frac{x(2(1-3x)v_3^2 - (1-6x^2)v_3 - x(1+x)(1-3x))}{(1-x)(1-6x)(v_3^2 - v_3x + x^2)},$$

as claimed. $\square$

Remark 3.8 *By the proof of Theorem 3.7 and using that v_3 is a root of*

$$v^3 - (vx + v - x)(v - x) = 0,$$

we get that $f = F_{\{021,1202\}}(x)$ satisfies

$$x^2(2x^2 - 2x + 1) - x(1-x)(8x^2 - 3x + 1)f + x(12x-1)(1-x)^2f^2 + (1-6x)(1-x)^3f^3 = 0.$$

Before we study the pair of 021 and 0000 in Section3.1, we present the following corollary that states explicit formulas for $|\mathbf{I}_n(\{021,\tau)|$ where τ is any 021-avoiding four-letter pattern except three cases, $1110, 1012$ and 1202. For these three patterns, we could not succeed to find explicit formulas. The formulas follow from determining the coefficient of the term x^n in the generating functions $F_{\{021,\tau\}}(x)$, where we omit the details.

Corollary 3.9 *For all* $n \geq 0$,

$$
\begin{aligned}
|\mathbf{I}_n(\{021,0001\})| &= \tfrac{(4n-25)(-1)^n}{32} - \tfrac{n(n+1)-1}{4} + \tfrac{1}{32}3^{n+4} \\
&\quad + \textstyle\sum_{j=0}^{n+1}\left(\tfrac{(4j-39)(-1)^j}{32} + \tfrac{1}{4}j^2 - j + \tfrac{1}{2} - \tfrac{1}{32}3^{j+2}\right) M_{n+1-j},\\
|\mathbf{I}_n(\{021,0010\})| &= \tbinom{2n}{n},\\
|\mathbf{I}_n(\{021,0011\})| &= C_{n+2} + 1 - \textstyle\sum_{j=0}^{n+1} C_j,\\
|\mathbf{I}_n(\{021,0012\})| &= 2^{n+3} - \tfrac{(n+1)(2n^2+7n+24)}{6} - 3,\\
|\mathbf{I}_n(\{021,0100\})| &= |\mathbf{I}_n(\{021,0110\})| = \tfrac{n^2+n+6}{8(2n+3)(2n+5)}\tbinom{2n+6}{n+3},\\
|\mathbf{I}_n(\{021,0101\})| &= |\mathbf{I}_n(\{021,0111\})| = \textstyle\sum_{j=1}^{n+1}\tfrac{1}{j}\tbinom{n}{j-1}\tbinom{2n+2-j}{j-1},\\
|\mathbf{I}_n(\{021,0102\})| &= 2^{n+1} - \tfrac{(n+1)(n^2+2n+12)}{6} - 1 + \textstyle\sum_{j=0}^{n+1} C_j,\\
|\mathbf{I}_n(\{021,0112\})| &= C_{n+1} - 2^{n+1} + 1 + \textstyle\sum_{j=0}^{n} 2^{n-j} C_j,\\
|\mathbf{I}_n(\{021,0120\})| &= |\mathbf{I}_n(\{021,0122\})| = \tfrac{1}{2}\tbinom{2n+2}{n+1} - \tfrac{1}{2}\textstyle\sum_{j=1}^{n}\tbinom{2j}{j},\\
|\mathbf{I}_n(\{021,0123\})| &= 2^{n-1}(n^2-3n+4) + \tfrac{n(n+1)}{2} - 1,\\
|\mathbf{I}_n(\{021,1000\})| &= |\mathbf{I}_n(\{021,1100\})|\\
&= \tfrac{n^5+2n^4+23n^3+46n^2+120n+48}{2(n+1)(n+2)(n+3)(n+4)}\tbinom{2n}{n},\\
|\mathbf{I}_n(\{021,1001\})| &= |\mathbf{I}_n(\{021,1011\})| = |\mathbf{I}_n(\{021,1101\})|\\
&= \tfrac{1}{n+1}\textstyle\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\tbinom{n+1}{j}\tbinom{2n+2}{n-2j},\\
|\mathbf{I}_n(\{021,1002\})| &= \tfrac{1}{2}\tbinom{2n+6}{n+3} - \tfrac{5}{2}\tbinom{2n+4}{n+2} + \tfrac{5}{2}\tbinom{2n+2}{n+1} + \tfrac{1}{2}\textstyle\sum_{j=0}^{n}\tbinom{2j}{j}\\
&\quad + 2^{n+1} - \tfrac{1}{24}(n^4+2n^3+11n^2+34n+36),\\
|\mathbf{I}_n(\{021,1020\})| &= |\mathbf{I}_n(\{021,1022\})|\\
&= \tbinom{2n+8}{n+4} - \tfrac{13}{2}\tbinom{2n+6}{n+3} + \tfrac{21}{2}\tbinom{2n+4}{n+2} - \tfrac{1}{2}\textstyle\sum_{j=0}^{n+1}\tbinom{2j}{j} - \tfrac{1}{2},\\
|\mathbf{I}_n(\{021,1023\})| &= \textstyle\sum_{j=0}^{n+1}(2^{j+1}-j-1)C_{n+1-j}\\
&\quad + \tfrac{n(3n^3+22n^2+129n+398)}{24} + 2^{n-1}(n^2-3n-52) + 24,\\
|\mathbf{I}_n(\{021,1102\})| &= \tfrac{1}{2}\tbinom{2n+6}{n+3} - \tfrac{21}{4}\tbinom{2n+4}{n+2} + \tbinom{2n+2}{n+1} + \tfrac{(n+1)^2}{2} - 2^n + 1\\
&\quad + \tfrac{1}{2}\textstyle\sum_{j=1}^{n+3}(2^{j-2}-3j+8)\tbinom{2n+6-2j}{n+3-j}\\
|\mathbf{I}_n(\{021,1120\})| &= 4^n - \tfrac{n}{2(2n+3)}\tbinom{2n+4}{n+2},\\
|\mathbf{I}_n(\{021,1200\})| &= |\mathbf{I}_n(\{021,1220\})|\\
&= \tfrac{n+4}{2(n+2)}\tbinom{2n+2}{n+1} + \textstyle\sum_{j=0}^{n-1}(2j+1)\tbinom{2j}{j} - 4^n,\\
|\mathbf{I}_n(\{021,1203\})| &= \tfrac{n+1}{24}(n^3+n^2-2n-108) + 2^{n-1}(n^2-11n+28) - \tfrac{19}{2}\\
&\quad + \tfrac{1}{2}\tbinom{2n+2}{n+1} - \tfrac{1}{2}\textstyle\sum_{j=2}^{n+1}(j-1)\tbinom{2n+2-2j}{n+1-j},\\
|\mathbf{I}_n(\{021,1220\})| &= \tfrac{n+4}{2(n+2)}\tbinom{2n+2}{n+1} - 4^n + \textstyle\sum_{j=0}^{n-1}(2j+1)\tbinom{2j}{j},\\
|\mathbf{I}_n(\{021,1230\})| &= \tfrac{1}{3}(2\cdot 4^n+1) + \textstyle\sum_{j=1}^{n}(\tfrac{j}{2} - 2^{j-1})\tbinom{2n+2-2j}{n+1-j}.
\end{aligned}
$$

Remark 3.10 *The first ten terms of the counting sequences of* $\mathbf{I}_n(\{021,0010\})$, $\mathbf{I}_n(\{021,0011\})$,

$\mathbf{I}_n(\{021,0101\})$, $\mathbf{I}_n(\{021,0120\})$, $\mathbf{I}_n(\{021,0123\})$, $\mathbf{I}_n(\{021,1120\})$ *match with A000984, A279557, A106228, A279561, A116757, A319028 in OEIS, respectively.*

3.1 Case B={021,0000}

In this section, we study the pair of patterns 021 and 0000. The generating tree $\mathcal{T}'(\{021,0000\})$ has a long list of succession rules and the solution to the equations for the generating functions requires some technical steps. By applying our algorithm, we found that the generating tree $\mathcal{T}'(\{021,0000\})$ has the following set of rules. The root has the label 0 and the succession rules are the following:

$$\begin{array}{ll}
0 \rightsquigarrow 0^2, 01, & 0^2 \rightsquigarrow 0^3, 0^21, 0^22,\\
01 \rightsquigarrow 0^21, 01^2, 01, & 0^3 \rightsquigarrow 0^31, 0^32, 0^33,\\
0^21 \rightsquigarrow 0^31, 0^21^2, 0^21, 0^22, & 0^22 \rightsquigarrow 0^32, 0^22^2, 0^22,\\
01^2 \rightsquigarrow 0^21^2, 01^3, a_{121}(2), 01, & 0^31 \rightsquigarrow 0^31^2, 0^31, 0^32, 0^33,\\
0^32 \rightsquigarrow 0^32^2, 0^32, 0^33, & 0^33 \rightsquigarrow 0^33^2, 0^33,\\
0^21^2 \rightsquigarrow 0^31^2, 0^21^3, a_{221}(2), 0^21, 0^22, & 0^22^2 \rightsquigarrow 0^32^2, 0^22^3, 0^21, 0^22,\\
01^3 \rightsquigarrow 0^21^3, a_{131}(2), a_{121}(2), 01, & 0^31^2 \rightsquigarrow 0^31^3, a_{321}(2), 0^31, 0^32, 0^33,\\
0^32^2 \rightsquigarrow 0^32^3, 0^31, 0^32, 0^33, & 0^33^2 \rightsquigarrow 0^3, 0^32, 0^33,\\
0^21^3 \rightsquigarrow 0^31^3, a_{231}(2), a_{221}(2), 0^21, 0^22, & 0^22^3 \rightsquigarrow 0^32^3, a_{221}(2), 0^21, 0^22,\\
0^31^3 \rightsquigarrow a_{331}(2), a_{321}(2), 0^31, 0^32, 0^33, & 0^32^3 \rightsquigarrow a_{321}(2), 0^31, 0^32, 0^33
\end{array}$$

such that for all $m \geq 2$,

$$\begin{array}{l}
a_{121}(m) \rightsquigarrow a_{221}(m), a_{122}(m), a_{121}(m), \cdots, a_{121}(2), a_{131}(m-1), \cdots, a_{131}(2), 01,\\
a_{122}(m) \rightsquigarrow a_{222}(m), a_{123}(m), a_{131}(m), \cdots, a_{131}(2), a_{121}(m), \cdots, a_{121}(2), 01,\\
a_{123}(m) \rightsquigarrow a_{223}(m), a_{121}(m+1), \cdots, a_{121}(2), a_{131}(m), \cdots, a_{131}(2), 01,\\
a_{131}(m) \rightsquigarrow a_{231}(m), a_{132}(m), a_{121}(m), \cdots, a_{121}(2), a_{131}(m), \cdots, a_{131}(2), 01,\\
a_{132}(m) \rightsquigarrow a_{232}(m), a_{133}(m), a_{121}(m+1), \cdots, a_{121}(2), a_{131}(m), \cdots, a_{131}(2), 01,\\
a_{133}(m) \rightsquigarrow a_{233}(m), a_{121}(m+1), \cdots, a_{121}(2), a_{131}(m+1), \cdots, a_{131}(2), 01,\\
a_{221}(m) \rightsquigarrow a_{321}(m), a_{222}(m), a_{221}(m), \cdots, a_{221}(2), a_{231}(m-1), \cdots, a_{231}(2), 0^21, 0^22,\\
a_{222}(m) \rightsquigarrow a_{322}(m), a_{223}(m), a_{221}(m), \cdots, a_{221}(2), a_{231}(m), \cdots, a_{231}(2), 0^21, 0^22,\\
a_{223}(m) \rightsquigarrow a_{323}(m), a_{221}(m+1), \cdots, a_{221}(2), a_{231}(m), \cdots, a_{231}(2), 0^21, 0^22,\\
a_{231}(m) \rightsquigarrow a_{331}(m), a_{232}(m), a_{221}(m), \cdots, a_{221}(2), a_{231}(m), \cdots, a_{231}(2), 0^21, 0^22,\\
a_{232}(m) \rightsquigarrow a_{332}(m), a_{233}(m), a_{221}(m+1), \cdots, a_{221}(2), a_{231}(m), \cdots, a_{231}(2), 0^21, 0^22,\\
a_{233}(m) \rightsquigarrow a_{333}(m), a_{221}(m+1), \cdots, a_{221}(2), a_{231}(m+1), \cdots, a_{231}(2), 0^21, 0^22,\\
a_{321}(m) \rightsquigarrow a_{322}(m), a_{321}(m), \cdots, a_{321}(2), a_{331}(m-1), \cdots, a_{331}(2), 0^31, 0^32, 0^33,\\
a_{322}(m) \rightsquigarrow a_{323}(m), a_{321}(m), \cdots, a_{321}(2), a_{331}(m), \cdots, a_{331}(2), 0^31, 0^32, 0^33,\\
a_{323}(m) \rightsquigarrow a_{321}(m+1), \cdots, a_{321}(2), a_{331}(m), \cdots, a_{331}(2), 0^31, 0^32, 0^33,\\
a_{331}(m) \rightsquigarrow a_{332}(m), a_{321}(m), \cdots, a_{321}(2), a_{331}(m), \cdots, a_{331}(2), 0^31, 0^32, 0^33,\\
a_{332}(m) \rightsquigarrow a_{333}(m), a_{321}(m+1), \cdots, a_{321}(2), a_{331}(m), \cdots, a_{331}(2), 0^31, 0^32, 0^33,\\
a_{333}(m) \rightsquigarrow a_{331}(m+1), a_{321}(m+1), \cdots, a_{321}(2), a_{331}(m), \cdots, a_{331}(2), 0^31, 0^32, 0^33.
\end{array}$$

where $a_{ijk}(m) = 0^i1^3\cdots(m-2)^3(m-1)^jm^k$ for all $m \geq 2$, $1 \leq i, k \leq 3$ and $j = 2, 3$.

Define

$$\begin{array}{ll}
A_r & = A_r(x) = \sum_{n\geq 0}(\text{number of nodes at level } n \text{ in } \mathcal{T}'(B;r))x^{n+1},\\
A_{ijk}(v) & = A_{ijk}(x;v) = \sum_{n\geq 0}(\text{number of nodes at level } n \text{ in } \mathcal{T}'(B;a_{ijk}(m)))x^{n+1}v^{m-2}.
\end{array}$$

Then, by translating each rule of the generating tree $\mathcal{T}'(B)$, we obtain the following sets of equations (we group them into three sets of equations):
System S1:

$$\begin{aligned}
A_0 &= x + xA_{00} + xA_{01}, & A_{00} &= x + xA_{000} + xA_{001} + xA_{002},\\
A_{01} &= x + xA_{001} + xA_{011} + xA_{01}, & A_{000} &= x + xA_{0001} + xA_{0002} + xA_{0003},\\
A_{001} &= x + xA_{0001} + xA_{0011} + xA_{001} + xA_{002}, & A_{002} &= x + xA_{0002} + xA_{0022} + xA_{002},\\
A_{011} &= x + xA_{0011} + xA_{0111} + xA_{121}(0) + xA_{01}, & A_{0001} &= x + xA_{00011} + xA_{0001} + xA_{0002} + xA_{0003},\\
A_{0002} &= x + xA_{00022} + xA_{0002} + xA_{0003}, & A_{0003} &= x + xA_{00033} + xA_{0003},\\
A_{0011} &= x + xA_{00011} + xA_{00111} + xA_{221}(0) & A_{0022} &= x + xA_{00022} + xA_{00222} + xA_{001} + xA_{002},\\
&\quad + xA_{001} + xA_{002}, & &\\
A_{0111} &= x + xA_{00111} + x(A_{131}(0) + A_{121}(0)) + xA_{01}, & A_{00011} &= x + xA_{000111} + xA_{321}(0) + xA_{0001}\\
& & &\quad + xA_{0002} + xA_{0003},\\
A_{00022} &= x + xA_{000222} + xA_{0001} + xA_{0002} + xA_{0003}, & A_{00033} &= x + xA_{000} + xA_{0002} + xA_{0003},\\
A_{00111} &= x + xA_{000111} + x(A_{231}(0) + A_{221}(0)) & A_{00222} &= x + xA_{000222} + xA_{221}(0) + xA_{001}\\
&\quad + xA_{001} + xA_{002}, & &\quad + xA_{002},\\
A_{000111} &= x + x(A_{331}(0) + A_{321}(0)) + xA_{0001} & A_{000222} &= x + xA_{321}(0) + xA_{0001} + xA_{0002}\\
&\quad + xA_{0002} + xA_{0003}, & &\quad + xA_{0003},
\end{aligned}$$

System S2:

$$\begin{aligned}
A_{121}(v) &= \tfrac{x}{1-v} + xA_{221}(v) + xA_{122}(v) + \tfrac{x}{1-v}(A_{121}(v) + vA_{131}(v) + A_{01}),\\
A_{122}(v) &= \tfrac{x}{1-v} + xA_{222}(v) + xA_{123}(v) + \tfrac{x}{1-v}(A_{131}(v) + A_{121}(v) + A_{01}),\\
A_{131}(v) &= \tfrac{x}{1-v} + xA_{231}(v) + xA_{132}(v) + \tfrac{x}{1-v}(A_{131}(v) + A_{121}(v) + A_{01}),\\
A_{133}(v) &= \tfrac{x}{1-v} + xA_{233}(v) + \tfrac{x}{v}(A_{131}(v) + A_{121}(v) - A_{131}(0) - A_{121}(0))\\
&\quad + \tfrac{x}{1-v}(A_{131}(v) + A_{121}(v) + A_{01}),\\
A_{221}(v) &= \tfrac{x}{1-v} + xA_{321}(v) + xA_{222}(v) + \tfrac{x}{1-v}(A_{221}(v) + vA_{231}(v) + A_{001} + A_{002}),\\
A_{222}(v) &= \tfrac{x}{1-v} + xA_{322}(v) + xA_{223}(v) + \tfrac{x}{1-v}(A_{231}(v) + A_{221}(v) + A_{001} + A_{002}),\\
A_{231}(v) &= \tfrac{x}{1-v} + xA_{331}(v) + xA_{232}(v) + \tfrac{x}{1-v}(A_{231}(v) + A_{221}(v) + A_{001} + A_{002}),\\
A_{233}(v) &= \tfrac{x}{1-v} + xA_{333}(v) + \tfrac{x}{v}(A_{231}(v) + A_{221}(v) - A_{231}(0) - A_{221}(0))\\
&\quad + \tfrac{x}{1-v}(A_{231}(v) + A_{221}(v) + A_{001} + A_{002}),\\
A_{321}(v) &= \tfrac{x}{1-v} + xA_{322}(v) + \tfrac{x}{1-v}(A_{321}(v) + vA_{331}(v) + A_{0001} + A_{0002} + A_{0003}),\\
A_{322}(v) &= \tfrac{x}{1-v} + xA_{323}(v) + \tfrac{x}{1-v}(A_{331}(v) + A_{321}(v) + A_{0001} + A_{0002} + A_{0003}),\\
A_{331}(v) &= \tfrac{x}{1-v} + xA_{332}(v) + \tfrac{x}{1-v}(A_{331}(v) + A_{321}(v) + A_{0001} + A_{0002} + A_{0003}),\\
A_{333}(v) &= \tfrac{x}{1-v} + \tfrac{x}{v}(A_{331}(v) + A_{321}(v) - A_{331}(0) - A_{321}(0))\\
&\quad + \tfrac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{0001} + A_{0002} + A_{0003}),
\end{aligned}$$

and

System S3:

$$\begin{aligned}
A_{123}(v) &= \frac{x}{1-v} + xA_{223}(v) + \frac{x}{v}(A_{121}(v) - A_{121}(0)) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{01}), && (3)\\
A_{132}(v) &= \frac{x}{1-v} + xA_{232}(v) + xA_{133}(v) + \frac{x}{1-v}A_{121}(v) + \frac{x}{v}(A_{121}(v) - A_{121}(0))\\
&\quad + \frac{x}{1-v}(A_{131}(v) + A_{01}), && (4)\\
A_{223}(v) &= \frac{x}{1-v} + xA_{323}(v) + \frac{x}{v}(A_{221}(v) - A_{221}(0)) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{001} + A_{002}), && (5)\\
A_{232}(v) &= \frac{x}{1-v} + xA_{332}(v) + xA_{233}(v) + \frac{x}{v}(A_{221}(v) - A_{221}(0))\\
&\quad + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{001} + A_{002}), && (6)\\
A_{323}(v) &= \frac{x}{1-v} + \frac{x}{v}(A_{321}(v) - A_{321}(0)) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{0001} + A_{0002} + A_{0003}), && (7)\\
A_{332}(v) &= \frac{x}{1-v} + xA_{333}(v) + \frac{x}{v}(A_{321}(v) - A_{321}(0))\\
&\quad + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{0001} + A_{0002} + A_{0003}). && (8)
\end{aligned}$$

In order to find the generating function $A_0 = F_{\{021,0000\}}(x)$, we have to solve a system that is obtained from S1-S2-S3. Since the expressions are too long to present here, we only describe the algorithm that leads to an explicit formula for A_0. For a Maple worksheet file for finding A_0, we refer the reader to Mansour (2022a).

Step 1: We use System **S2** to find formulas for $A_{121}(v)$, $A_{122}(v)$, $A_{131}(v)$, $A_{133}(v)$, $A_{221}(v)$, $A_{222}(v)$, $A_{231}(v)$, $A_{233}(v)$, $A_{321}(v)$, $A_{322}(v)$, $A_{331}(v)$, and $A_{333}(v)$ in terms of $A_{123}(v)$, $A_{123}(0)$, $A_{132}(v)$, $A_{132}(0)$, $A_{223}(v)$, $A_{223}(0)$, $A_{232}(v)$, $A_{232}(0)$, $A_{323}(v)$, $A_{323}(0)$, $A_{332}(v)$, and $A_{332}(0)$. We denote these set of expressions by **E1**.

Step 2: By substituting expressions of **E1** into all equations of System **S3**, and then using Gauss elimination method with respect to the six variables $A_{123}(v)$, $A_{132}(v)$, $A_{223}(v)$, $A_{232}(v)$, $A_{323}(v)$, and $A_{332}(v)$, we obtain System **NS3**.

Step 3: Define

$$K(v) = x^6 + x^3(x-2)v + (1-2x-x^2)v^2 - v^3.$$

The equation $A_{332}(v)$ in System **NS3** is given by

$$\begin{aligned}
K(v)A_{332}(v) &= \tfrac{(v+2x-1)vx^3}{1-2x}A_{323}(0) + \tfrac{(-2x^4+x^3+v^2+2vx-v)x^3}{1-2x}A_{332}(0)\\
&\quad + \tfrac{xv^2}{1-2x}(1 + A_{0001} + A_{0002} + A_{0003}). && (9)
\end{aligned}$$

We use kernel method to solve this equation. Note that the roots of $K(v) = 0$ are given by

$$\begin{aligned}
v_j &= \tfrac{2\sqrt{4x^4-2x^3+2x^2-4x+1}}{3}\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{16x^6-12x^5+27x^4-10x^3+18x^2-12x+2}{2\sqrt{4x^4-2x^3+2x^2-4x+1}^{\,3}}\right) + \tfrac{2\pi j}{3}\right)\\
&\quad + \tfrac{1-2x+x^2}{3},
\end{aligned}$$

where $j = 0, 1, 2$. Note that the first terms in the power series (in the variable $\sqrt{x}$) of v_j are given by

$$\begin{aligned}
v_0 &= 1 - 2x - x^2 - 2x^3 - 3x^4 - 8x^5 - 22x^6 - 62x^7 - 182x^8 - 548x^9 + \cdots,\\
v_1 &= x^3 - x^{\frac{7}{2}} + \tfrac{3}{2}x^4 - \tfrac{21}{8}x^{\frac{9}{2}} + 4x^5 - \tfrac{839}{128}x^{\frac{11}{2}} + 11x^6 - \tfrac{18733}{1024}x^{\frac{13}{2}} + 31x^7 - \cdots,\\
v_2 &= x^3 + x^{\frac{7}{2}} + \tfrac{3}{2}x^4 + \tfrac{21}{8}x^{\frac{9}{2}} + 4x^5 + \tfrac{839}{128}x^{\frac{11}{2}} + 11x^6 + \tfrac{18733}{1024}x^{\frac{13}{2}} + 31x^7 + \cdots.
\end{aligned}$$

Note that there are two relations for these three roots

$$v_0 + v_1 + v_2 = 1 - 2x - x^2 \text{ and } v_0 v_1 v_2 = x^6. \tag{10}$$

By taking either $v = v_1$ or $v = v_2$ into (9), we obtain a system of two equations with variables $A_{323}(0)$ and $A_{332}(0)$. Solving the system and using (10), we obtain

$$A_{323}(0) = \frac{(1 + A_{0001} + A_{0002} + A_{0003})(v_0(1 - 2x - x^2) - v_0^2 - x^3)}{x^2 v_0 (x^2 + v_0)}$$

and

$$A_{332}(0) = \frac{x(1 + A_{0001} + A_{0002} + A_{0003})}{x^2 v_0 (x^2 + v_0)}.$$

By substituting expressions of $A_{323}(0)$ and $A_{332}(0)$ into equations of $A_{323}(v)$ and $A_{332}(v)$ of **NS3** and solving for $A_{323}(v)$ and $A_{332}(v)$, we obtain

$$\begin{aligned}
A_{323}(v) &= \tfrac{(x^6-2x^5-x^3-(2x^3-2x^2+3x-1)v-(x^2-x+1)v^2)(1+A_{0001}+A_{0002}+A_{0003})}{xK(v)} \\
&\quad - \tfrac{(2x^6-2x^5+3x^4-x^3+(x^4+6x^2-5x+1)v-(x^2-3x+1)v^2)(1+A_{0001}+A_{0002}+A_{0003})v_0}{x^4K(v)} \\
&\quad - \tfrac{(x^5-x^4+x^3-(x^2-3x+1)v+(1-x)v^2)(1+A_{0001}+A_{0002}+A_{0003})v_0^2}{x^4K(v)} \\
A_{332}(v) &= \tfrac{(x^6-2x^4+2x^3-(2x^3-2x^2-2x+1)v+(1-x^2)v^2)(1+A_{0001}+A_{0002}+A_{0003})}{xK(v)} \\
&\quad - \tfrac{(2x^5-2x^4-2x^3+x^2-(4x^2-4x+1)v+(1-2x)v^2)(1+A_{0001}+A_{0002}+A_{0003})v_0}{x^3K(v)} \\
&\quad - \tfrac{(x^4-x^2+(1-2x)v-v^2)(1+A_{0001}+A_{0002}+A_{0003})v_0^2}{x^3K(v)}.
\end{aligned}$$

Step 4: We will not present the exact expressions because most of them are quite long. For the next step, we consider the equation of $A_{232}(v)$ in System **NS3**. After using the expressions of $A_{323}(v)$, $A_{323}(0)$, $A_{332}(v)$, and $A_{332}(0)$, we obtain an equation of the form

$$K(v)^2 A_{232}(v) = \alpha(v) + \beta(v) A_{232}(0) + \gamma(v) A_{223}(0),$$

which leads to

$$2K(v)K'(v)A_{232}(v) + K^2(v)A'_{232}(v) = \alpha'(v) + \beta'(v)A_{232}(0) + \gamma'(v)A_{223}(0),$$

where $f'(v)$ denotes the derivative of f with respect to v. By substituting either $v = v_1$ or $v = v_2$, we obtain a system of two equations with the variables $A_{232}(0)$ and $A_{223}(0)$. Solving this system and using (10), we obtain explicit formulas for $A_{232}(0)$ and $A_{223}(0)$ in terms of the root v_0 and x. Then, by substituting expressions of $A_{323}(0)$, $A_{332}(0)$, $A_{223}(0)$, $A_{232}(0)$, $A_{323}(v)$, and $A_{332}(v)$ into the equations of $A_{223}(v)$ and $A_{232}(v)$ of **NS3** and solving for $A_{223}(v)$ and $A_{232}(v)$, we derive explicit formulas for $A_{223}(v)$ and $A_{232}(v)$ in terms of the root v_0 and v, x.
Step 5: Now, we consider the equation of $A_{132}(v)$ in System **NS3**. After using the expressions of $A_{323}(v)$, $A_{323}(0)$, $A_{332}(v)$, $A_{332}(0)$, $A_{223}(v)$, $A_{223}(0)$, $A_{232}(v)$, and $A_{232}(0)$, we obtain an equation of the form

$$K(v)^3 A_{132}(v) = \alpha(v) + \beta(v) A_{132}(0) + \gamma(v) A_{123}(0),$$

which leads to

$$(K^3(v)A_{132}(v))'' = \alpha''(v) + \beta''(v)A_{132}(0) + \gamma''(v)A_{123}(0).$$

By substituting either $v = v_1$ or $v = v_2$, we obtain a system of two equations with the variables $A_{132}(0)$ and $A_{123}(0)$. Solving this system and using (10), we obtain explicit formulas for $A_{132}(0)$ and $A_{123}(0)$ in terms of the root v_0 and x. Then, by substituting expressions of $A_{323}(0)$, $A_{332}(0)$, $A_{223}(0)$, $A_{232}(0)$, $A_{132}(0)$, $A_{123}(0)$, $A_{323}(v)$, $A_{332}(v)$, $A_{223}(v)$, and $A_{232}(v)$ into the equations of $A_{123}(v)$ and $A_{132}(v)$ of **NS3** and solving for $A_{123}(v)$ and $A_{132}(v)$, we derive explicit formulas for $A_{123}(v)$ and $A_{132}(v)$ in terms of the root v_0 and v, x.

All the explicit expressions of these generating functions can be found in the following Maple worksheet Mansour (2022a).

Step 6: Up to now, we have explicit formulas for the generating functions $A_r(v)$ (also for $A_r(0)$), for all $r \in \{123, 132, 223, 232, 323, 332\}$. By the expressions of **E1**, we get explicit formulas for all $A_r(v)$ which leads to explicit expressions for $A_r(0)$, where $r = ijk$ and $1 \leq i, k \leq 3$ and $j = 1, 2$.

Step 7: At the last step, we solve System **S1** by using expressions $A_r(0)$ from the previous step. In particular, we obtain an explicit formula for A_0.

Theorem 3.11 *The generating function $F_{\{021,0000\}}(x)$ is given by*

$$\begin{aligned}
&-\frac{256x^9+560x^7+132x^6+792x^5+168x^4+235x^3-115x^2-31x+10}{x^2(16x^3+8x^2+11x-4)^2}\\
&\quad+\frac{(256x^9-576x^8-440x^7-1306x^6-104x^5-85x^4+335x^3+18x^2-62x+10)v_0}{x^5(16x^3+8x^2+11x-4)^2}\\
&\quad+\frac{(256x^8+112x^6-452x^5-198x^4-191x^3+56x^2+42x-10)v_0^2}{x^5(16x^3+8x^2+11x-4)^2}\\
&= x + 2x^2 + 6x^3 + 21x^4 + 78x^5 + 296x^6 + 1126x^7 + 4285x^8 + 16281x^9 + 61690x^{10}\\
&+233078x^{11} + 878164x^{12} + 3299936x^{13} + 12370320x^{14} + \cdots .
\end{aligned}$$

The computations used in Steps 1-7 are programmed in Mansour (2022a).

4 Concluding remarks

Table 1 presents the generating trees for all the cases $\mathbf{I}_n(\{021, \tau\})$ whenever τ is a four-letter pattern that avoids 021. Moreover, the table includes the explicit formulas for the corresponding generating functions $F_{\{021,\tau\}}(x)$. We see that Conjecture 2.1 holds for length-four patterns. Our method successfully solves the case 0000, but we are curious whether there is a more straightforward solution for it or not.

There are 106 patterns of length five that avoid 021. We applied our algorithm to each class and found that the Conjecture 2.1 holds for 021-avoiding five-letter patterns. More precisely, we determined the generating trees and obtained explicit formulas for the generating functions $F_{\{021,\tau\}}(x)$ whenever τ is a five-letter pattern that avoids 021. Since the computations are too long, especially for the following four cases $\tau = 00000, 00001, 00011, 00012$, we decided not to present them here. But the explicit formulas of the generating functions for five-letter pattern cases are available in Mansour and Yıldırım (2022).

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