

# A theorem of Jon F. Carlson on filtrations of modules

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## Abstract

We give an alternative proof to a theorem of Carlson [J.F. Carlson, Cohomology and induction from elementary abelian subgroups, *Quart. J. Math.* 51 (2000) 169–181] which states that if  $G$  is a finite group and  $k$  is a field of characteristic  $p$ , then any  $kG$ -module is a direct summand of a module which has a filtration whose sections are induced from elementary abelian  $p$ -subgroups of  $G$ . We also prove two new theorems which are closely related to Carlson's theorem.

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## 1. Introduction

Let  $G$  be a finite group and let  $k$  be a field of characteristic  $p > 0$ . Let  $\mathcal{H}$  be a collection of subgroups of  $G$ . We say that a  $kG$ -module  $M$  is filtered by modules induced from  $\mathcal{H}$  if there is a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

such that for each  $i = 1, 2, \dots, n$ , there is a subgroup  $H_i \in \mathcal{H}$  and a  $kH_i$ -module  $W_i$  such that  $M_i/M_{i-1} \cong W_i \uparrow_{H_i}^G$ . We consider the following theorem of Carlson.

**Theorem 1.1** (Carlson [5]). *Any  $kG$ -module  $M$  is a direct summand of a module that is filtered by modules induced from elementary abelian  $p$ -subgroups.*

This theorem provides another way to see the role of elementary abelian  $p$ -groups in modular representation theory, and has many applications, including Chouinard's theorem for finitely generated modules (see Theorem 8.2.12 in [7]). Carlson proves Theorem 1.1 by first reducing it to  $p$ -groups and then by showing that it follows from the following statement by induction.

**Theorem 1.2** (Carlson [5]). *Suppose that  $G$  is a  $p$ -group which is not elementary abelian. Then there is a sequence  $H_1, \dots, H_n$  of maximal subgroups of  $G$  such that  $k \oplus \Omega^{1-n}(k) \oplus (\text{proj})$  has a filtration*

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = k \oplus \Omega^{1-n}(k) \oplus (\text{proj})$$

where  $L_i/L_{i-1} \cong \Omega^{1-i}(k) \uparrow_{H_i}^G$  for  $i = 1, \dots, n$ .

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The maximal subgroups  $H_1, \dots, H_n$  in the above theorem are not necessarily distinct. In fact, for  $p > 2$  we take  $n = 2k$  and  $H_{2i} = H_{2i-1}$  for all  $i = 1, \dots, k$ . Carlson proves the existence of such a filtration using Serre’s theorem on vanishing products in group cohomology (see [Theorem 3.1](#) for the statement of Serre’s theorem) and a hypercohomology calculation with coefficients in a chain complex obtained from Serre’s theorem. [Theorem 1.2](#) plays an important role in [6] where the authors use this filtration to find an upper bound for the dimensions of critical endo-trivial modules.

In this paper we give an alternative proof to [Theorem 1.2](#) using  $L_\zeta$ -modules. The  $L_\zeta$ -modules are defined as follows: Associated to a cohomology class  $\zeta \in H^n(G, k)$ , there is a module  $L_\zeta$  defined as the kernel of the representing homomorphism  $\hat{\zeta} : \Omega^n(k) \rightarrow k$ . Modules of this form are called  $L_\zeta$ -modules. They are commonly used to relate cohomology theory with modular representation theory; for example, they appear in many results about varieties of modules (see [1–3,7,8]).

In our proof for [Theorem 1.2](#), we still use Serre’s theorem, but we avoid the hypercohomology calculation. Given  $\zeta_1, \dots, \zeta_n \in H^1(G, \mathbb{F}_p)$  satisfying the conclusion of Serre’s theorem, i.e.,  $\beta(\zeta_1) \cdots \beta(\zeta_n) = 0$  where  $\beta$  is the Bockstein map, we observe that  $L_{\beta(\zeta_1) \cdots \beta(\zeta_n)} = \Omega(k) \oplus \Omega^{2n}(k)$  has a filtration whose sections are isomorphic to Heller shifts of  $L_{\beta(\zeta_i)}$ ’s. This is an easy consequence of a known exact sequence for the  $L_\zeta$  when  $\zeta$  is a product of two cohomology classes (see [Proposition 2.3](#)). Next, we show that for every  $\zeta \in H^1(G, \mathbb{F}_p)$  with kernel  $H \leq G$ , there is a 2-step filtration for  $L_{\beta(\zeta)} \oplus (proj)$  such that the sections are induced from  $H$  (see [Lemmas 2.4](#) and [2.5](#)). In fact, for  $p = 2$ , the argument is much simpler since, in this case, Serre’s theorem is true without Bocksteins, and we have  $L_\zeta \cong k \uparrow_H^G$  for every  $\zeta \in H^1(G, \mathbb{F}_2)$  with kernel  $H$ . We present our alternative proof in [Section 3](#).

In the rest of the paper, we prove two theorems which are variations of Carlson’s theorems. The first one is a generalization of [Lemmas 2.4](#) and [2.5](#), and it is strong enough to imply [Theorem 1.2](#) when it is applied to a suitable extension.

**Theorem 1.3.** *Let  $\zeta$  be the cohomology class in  $H^n(G, k)$  which is represented by the extension*

$$E : 0 \rightarrow k \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow k \rightarrow 0.$$

*Then,  $L_\zeta \oplus (proj)$  has a filtration*

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = L_\zeta \oplus (proj)$$

*with  $L_i/L_{i-1} \cong \Omega^{n-i+1}(M_{i-1})$  for  $i = 1, \dots, n$ .*

Here the notation  $(proj)$  means that the statement is true after adding a suitable projective summand. We will be using this notation throughout the paper.

In [Section 5](#), we introduce the varieties of modules, and prove

**Theorem 1.4.** *Let  $\mathcal{H}$  be a collection of subgroups of  $G$ . Then, for a finitely generated  $kG$ -module  $M$ , the following are equivalent:*

- (i)  $V_G(M) = \bigcup_{H \in \mathcal{H}} \text{res}_{G,H}^*(V_H(M \downarrow_H^G))$ .
- (ii) *There exists a finitely generated  $kG$ -module  $V$  such that  $M \oplus V$  is filtered by modules induced from  $\mathcal{H}$ .*

We conclude the paper with the following application:

**Corollary 1.5.** *Let  $G$  be an elementary abelian 2-group. If  $\zeta \in H^n(G, k)$  is represented by the extension*

$$E : 0 \rightarrow k \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow k \rightarrow 0$$

*where  $M_i$ ’s are direct sums of modules induced from proper subgroups, then  $\zeta$  is a non-zero scalar multiple of a product of one dimensional classes in  $H^1(G, \mathbb{F}_2)$ . In particular,  $E$  is equivalent to an extension coming from a topological group action on a sphere.*

There is a similar result for  $p > 2$  under stronger conditions. This result is also proved in [Section 5](#).

Throughout this paper,  $G$  always denotes a finite group,  $k$  is a field of characteristic  $p > 0$ . We assume that all  $kG$ -modules are finitely generated, and all tensor products are over  $k$  unless otherwise stated clearly.



In our alternative proof for [Theorem 1.2](#), the main ingredient is the following exact sequence:

**Proposition 2.3.** *If  $\zeta_1 \in H^r(G, k)$  and  $\zeta_2 \in H^s(G, k)$ , then there is an exact sequence*

$$0 \rightarrow \Omega^r(L_{\zeta_2}) \rightarrow L_{\zeta_1 \cdot \zeta_2} \oplus (\text{proj}) \rightarrow L_{\zeta_1} \rightarrow 0.$$

**Proof.** See Lemma 5.9.3 on page 191 of [2].  $\square$

For the rest of the section, we assume  $G$  is a (finite)  $p$ -group. Recall that

$$H^1(G, \mathbb{F}_p) \cong \text{Hom}(G, \mathbb{Z}/p).$$

Given a one dimensional class  $\zeta \in H^1(G, \mathbb{F}_p)$ , the kernel of the corresponding homomorphism is usually referred as the kernel of  $\zeta$ .

**Lemma 2.4.** *Let  $G$  be a 2-group, and  $\zeta$  be a cohomology class in  $H^1(G, \mathbb{F}_2)$  considered as a class in  $H^1(G, k)$ . Then  $L_\zeta \cong \Omega(k) \uparrow_H^G$  where  $H$  is the kernel of  $\zeta$ .*

**Proof.** We have the following commutative diagram

$$\begin{array}{ccccccc} & L_\zeta & \xlongequal{\quad} & L_\zeta & & & \\ & \downarrow & & \downarrow & & & \\ 0 & \longrightarrow & \Omega(k) & \longrightarrow & P_0 & \longrightarrow & k \longrightarrow 0 \\ & & \hat{\zeta} \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & k & \longrightarrow & P_0/L_\zeta & \longrightarrow & k \longrightarrow 0 \end{array}$$

from which we obtain that the extension class of the sequence  $0 \rightarrow k \rightarrow P_0/L_\zeta \rightarrow k \rightarrow 0$  corresponds to  $\zeta$  under the isomorphism  $H^1(G, k) \cong \text{Ext}_{kG}^1(k, k)$ . There is an extension of the form

$$0 \rightarrow k \rightarrow k \uparrow_H^G \rightarrow k \rightarrow 0$$

with extension class equal to  $\zeta$ . From the equivalence of the exact sequences we get

$$k \uparrow_H^G \cong P_0/L_\zeta \cong \Omega^{-1}(L_\zeta) \oplus (\text{proj}).$$

Taking the first Heller shift of this isomorphism, we obtain  $L_\zeta \cong \Omega(k \uparrow_H^G)$ . Note that for  $p$ -groups,  $\Omega(k \uparrow_H^G) \cong \Omega(k) \uparrow_H^G$ . We conclude that  $L_\zeta \cong \Omega(k) \uparrow_H^G$ .  $\square$

In the case where  $p$  is an odd prime we have a similar result.

**Lemma 2.5.** *Let  $G$  be a finite  $p$ -group, and let  $\beta(\zeta)$  be the Bockstein of a one dimensional class  $\zeta \in H^1(G, \mathbb{F}_p)$ . Consider  $\beta(\zeta)$  as a class in  $H^2(G, k)$ . Then,  $L_{\beta(\zeta)} \oplus (\text{proj})$  has a filtration  $0 = M_0 \subseteq M_1 \subseteq M_2 = L_{\beta(\zeta)} \oplus (\text{proj})$  with the property  $M_2/M_1 \cong \Omega(k) \uparrow_H^G$  and  $M_1 \cong \Omega^2(k) \uparrow_H^G$  where  $H$  is the kernel of  $\zeta$ .*

**Proof.** By Proposition 5.7.6 in [7], there is an extension of the form

$$0 \rightarrow k \rightarrow k \uparrow_H^G \rightarrow k \uparrow_H^G \rightarrow k \rightarrow 0$$

with extension class equal to  $\beta(\zeta)$ . Thus, we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^2(k) & \longrightarrow & P_1 & \longrightarrow & P_0 \longrightarrow k \longrightarrow 0 \\ & & \widehat{\beta(\zeta)} \downarrow & & \downarrow f_1 & & \downarrow f_0 \quad \parallel \\ 0 & \longrightarrow & k & \longrightarrow & k \uparrow_H^G & \longrightarrow & k \uparrow_H^G \longrightarrow k \longrightarrow 0 \end{array}$$

where the leftmost homomorphism is the representing homomorphism for  $\beta(\zeta)$ .

Thus, we have the diagram

$$\begin{array}{ccccccccc}
 & & \ker \tilde{f}_1 & \xrightarrow{\cong} & \ker f_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 0 & \longrightarrow & k & \longrightarrow & P_1/L_{\beta(\zeta)} & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow & 0 \\
 & & \parallel & & \downarrow \tilde{f}_1 & & \downarrow f_0 & & \parallel & & \\
 0 & \longrightarrow & k & \longrightarrow & k \uparrow_H^G & \longrightarrow & k \uparrow_H^G & \longrightarrow & k & \longrightarrow & 0
 \end{array}$$

where  $\ker f_0 = \Omega(k \uparrow_H^G) \oplus (\text{proj})$ . The first vertical short exact sequence in the above diagram gives us the sequence

$$0 \rightarrow \Omega(k \uparrow_H^G) \oplus (\text{proj}) \rightarrow \Omega^{-1}(L_{\beta(\zeta)}) \oplus (\text{proj}) \rightarrow k \uparrow_H^G \rightarrow 0.$$

If we tensor this exact sequence with  $\Omega(k)$  over  $k$ , we get

$$0 \rightarrow \Omega^2(k \uparrow_H^G) \oplus (\text{proj}) \rightarrow L_{\beta(\zeta)} \oplus (\text{proj}) \rightarrow \Omega(k \uparrow_H^G) \oplus (\text{proj}) \rightarrow 0.$$

Note that we can cancel projective modules at both ends of the sequence since projective  $kG$ -modules are also injective. Thus, we get an exact sequence

$$0 \rightarrow \Omega^2(k \uparrow_H^G) \rightarrow L_{\beta(\zeta)} \oplus (\text{proj}) \rightarrow \Omega(k \uparrow_H^G) \rightarrow 0$$

which gives the desired filtration.  $\square$

Throughout the paper, we will come across the situations where we will need to cancel projective modules from both ends of exact sequences as we did in the above proof. More generally, we will need to cancel projective modules from sections of filtrations of modules. We quote the following lemma from [7]. The proof easily follows from the fact that projective  $kG$ -modules are injective.

**Lemma 2.6.** *Suppose that  $M$  is a  $kG$ -module which has a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$$

with  $M_i/M_{i-1} \cong X_i \oplus P_i$  for some projective modules  $P_i$ . Then,  $M \cong M' \oplus P$  for some projective module  $P$  such that  $M'$  has a filtration

$$0 = M'_0 \subseteq M'_1 \subseteq \dots \subseteq M'_n = M'$$

with  $M'_i/M'_{i-1} \cong X_i$  for all  $i = 1, \dots, n$ .

### 3. The alternative proof

The aim of this section is to give a proof for Theorem 1.2 using  $L_\zeta$ -modules. First, we recall Serre’s theorem on vanishing products in group cohomology.

**Theorem 3.1** (Serre [9]). *Suppose that  $G$  is a  $p$ -group which is not elementary abelian. Then there is a sequence  $\zeta_1, \dots, \zeta_n \in H^1(G, \mathbb{F}_p)$  of nonzero elements such that*

$$\begin{aligned}
 \zeta_1 \zeta_2 \cdots \zeta_n &= 0 \quad \text{if } p = 2, \\
 \beta(\zeta_1) \beta(\zeta_2) \cdots \beta(\zeta_n) &= 0 \quad \text{if } p > 2.
 \end{aligned}$$

Now, we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** First let’s assume  $p = 2$ . Let  $\zeta_1, \dots, \zeta_n$  be classes in  $H^1(G, \mathbb{F}_2)$  satisfying the conclusion of Serre’s theorem. Then, we have

$$L_{\zeta_1 \dots \zeta_n} \cong \Omega(k) \oplus \Omega^n(k).$$

By Proposition 2.3, for each  $i = 1, \dots, n - 1$ , there is an exact sequence of the form

$$0 \rightarrow \Omega^{n-i}(L_{\zeta_i}) \rightarrow L_{\zeta_i \dots \zeta_n} \oplus P_i \rightarrow L_{\zeta_{i+1} \dots \zeta_n} \rightarrow 0$$

where  $P_1, \dots, P_{n-1}$  are projective modules. By adding projective summands to the last two terms of the exact sequences above, we find exact sequences of the form

$$0 \rightarrow \Omega^{n-i}(L_{\zeta_i}) \rightarrow L_{\zeta_i \dots \zeta_n} \oplus Q_i \rightarrow L_{\zeta_{i+1} \dots \zeta_n} \oplus Q_{i+1} \rightarrow 0$$

for  $i = 1, \dots, n - 1$  where

$$Q_i = \bigoplus_{k=i}^{n-1} P_k.$$

Using these exact sequences, we obtain a filtration for

$$L_{\zeta_1 \dots \zeta_n} \oplus Q_1 \cong \Omega(k) \oplus \Omega^n(k) \oplus Q_1$$

as follows: let  $M_n = L_{\zeta_1 \dots \zeta_n} \oplus Q_1$  and  $M_1 = \Omega^{n-1}(L_{\zeta_1})$ . Then,  $M_n/M_1 \cong L_{\zeta_2 \dots \zeta_n} \oplus Q_2$ . Choose  $M_2$  such that  $M_2/M_1 \cong \Omega^{n-2}(L_{\zeta_2})$ . Then the exact sequence

$$0 \rightarrow \Omega^{n-2}(L_{\zeta_2}) \rightarrow L_{\zeta_2 \dots \zeta_n} \oplus Q_2 \rightarrow L_{\zeta_3 \dots \zeta_n} \oplus Q_3 \rightarrow 0$$

gives that  $M_n/M_2 \cong L_{\zeta_3 \dots \zeta_n} \oplus Q_3$  which will be the middle term of the next exact sequence. Continuing this way we obtain a filtration

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = \Omega(k) \oplus \Omega^n(k) \oplus Q_1$$

with  $M_n/M_i \cong L_{\zeta_{i+1} \dots \zeta_n} \oplus Q_{i+1}$  and  $M_i/M_{i-1} \cong \Omega^{n-i}(L_{\zeta_i})$  for  $i = 1, \dots, n$ . Tensoring the entire filtration by  $\Omega^{-n}(k)$ , and cancelling the projective summands from sections as in Lemma 2.6, we obtain a filtration

$$0 = L_0 \subseteq \dots \subseteq L_n = \Omega^{1-n}(k) \oplus k \oplus (proj)$$

with  $L_i/L_{i-1} \cong \Omega^{-i}(L_{\zeta_i})$ . Note that by Lemma 2.4, we have  $L_{\zeta_i} \cong \Omega^1(k \uparrow_{H_i}^G)$ . Thus,

$$L_i/L_{i-1} \cong \Omega^{-i}(\Omega(k \uparrow_{H_i}^G)) \cong \Omega^{1-i}(k \uparrow_{H_i}^G) \cong \Omega^{1-i}(k) \uparrow_{H_i}^G$$

where the last isomorphism is true because  $G$  is a  $p$ -group. So, the proof for  $p = 2$  is complete.

Now, assume  $p > 2$ . As above, we can obtain a filtration

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = \Omega(k) \oplus \Omega^{2n}(k) \oplus (proj)$$

such that  $M_i/M_{i-1} \cong \Omega^{2n-2i}(L_{\beta(\zeta_i)})$  for  $i = 1, \dots, n$ . By Lemma 2.5,  $L_{\beta(\zeta_i)} \oplus (proj)$  has a filtration with sections isomorphic to  $\Omega^2(k) \uparrow_H^G$  and  $\Omega^1(k) \uparrow_H^G$ . After adding projective modules to each  $M_i$ , we can assume  $M_i/M_{i-1}$  has a filtration with sections isomorphic to  $\Omega^{2n-2i+2}(k) \uparrow_H^G$  and  $\Omega^{2n-2i+1}(k) \uparrow_H^G$ . For each  $i = 1, \dots, n$ , let  $N_i$  be the  $kG$ -module satisfying  $M_{i-1} \subseteq N_i \subseteq M_i$  with  $N_i/M_{i-1} \cong \Omega^{2n-2i+2}(k) \uparrow_{H_i}^G$  and  $M_i/N_i \cong \Omega^{2n-2i+1}(k) \uparrow_{H_i}^G$ . By taking  $L_{2i} = M_i$  and  $L_{2i-1} = N_i$ , tensoring everything with  $\Omega^{-2n}(k)$ , and cancelling the projective summands on the sections, we obtain a filtration

$$0 = L_0 \subseteq \dots \subseteq L_{2n} = k \oplus \Omega^{1-2n}(k) \oplus (proj)$$

where

$$L_j/L_{j-1} \cong \Omega^{1-j}(k) \uparrow_{H_i}^G$$

when  $j = 2i$  or  $j = 2i - 1$ . This completes the proof.  $\square$

Now, we explain briefly how Theorem 1.1 follows from Theorem 1.2. The details of this argument can be found on page 166 of [7]. First note that it is enough to prove Theorem 1.1 for  $M = k$ . The general case follows by tensoring everything with  $M$ . Also note that if  $P$  is a Sylow  $p$ -subgroup of  $G$ , then  $k$  is a summand of  $k \uparrow_P^G$ . So, it is enough to prove Theorem 1.1 for  $p$ -groups. To see this, suppose that there is a  $kP$ -module  $V$  such that  $k \uparrow_P^G \oplus V$  has a filtration whose sections are induced from elementary abelian  $p$ -subgroups. Inducing the entire filtration to  $G$ , we get a filtration for  $k \uparrow_P^G \oplus V \uparrow_P^G$ , and hence conclude that  $k$  is a direct summand of a finitely generated module which has a filtration with the desired properties.

When  $G$  is a  $p$ -group, and  $M$  is the trivial module  $k$ , Theorem 1.1 follows from Theorem 1.2 by an induction. Note that by Theorem 1.2, there is a filtration

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_n \cong k \oplus \Omega^{1-n}(k) \oplus (\text{proj})$$

where  $L_i/L_{i-1} \cong \Omega^{1-i}(k) \uparrow_{H_i}^G$  for  $i = 1, \dots, n$ . If any of the subgroups  $H_i$  is not an elementary abelian  $p$ -group, then we can apply Theorem 1.2 to  $H_i$  and refine the above sequence further until we reach the stage that all the subgroups involved are elementary abelian  $p$ -subgroups.

We note that the following version of Theorem 1.1 is also true.

**Theorem 3.2** (Carlson [5]). *There exists a finitely generated  $kG$ -module  $V$  such that  $k \oplus V$  has a filtration*

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_n = k \oplus V$$

where for every  $i = 1, \dots, n$ , the sections  $L_i/L_{i-1}$  are isomorphic to  $\Omega^{n_i}(k) \uparrow_{E_i}^G$  for some integer  $n_i$  and some maximal elementary abelian  $p$ -subgroup  $E_i$  of  $G$ .

We will use this version later in Section 5.

#### 4. A filtration theorem for $L_\zeta$ -modules

The main purpose of this section is to prove Theorem 1.3 stated in the introduction. We also prove an important corollary which will be useful later in Section 5.

**Definition 4.1.** Let  $E$  be an  $n$ -fold extension of  $kG$ -modules with extension class  $\alpha$  in  $\text{Ext}_{kG}^n(A, B)$ . Suppose that  $\tilde{E} : 0 \rightarrow \Omega^{-n+1}(B) \rightarrow M \rightarrow A \rightarrow 0$  is an extension whose class is associated to  $\alpha$  under the isomorphism

$$\text{Ext}_{kG}^n(A, B) \cong \text{Ext}_{kG}^1(A, \Omega^{-n+1}(B)).$$

Then, we say  $\tilde{E}$  is a contraction of  $E$ .

**Lemma 4.2.** *Let  $E$  be an  $n$ -fold extension of  $kG$ -modules*

$$E : 0 \rightarrow B \rightarrow M_{n-1} \rightarrow M_{n-2} \rightarrow \dots \rightarrow M_0 \rightarrow A \rightarrow 0$$

and let

$$\tilde{E} : 0 \rightarrow \Omega^{-n+1}(B) \rightarrow M \rightarrow A \rightarrow 0$$

be a contraction of  $E$ . Then,  $M \oplus (\text{proj})$  has a filtration

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_n = M \oplus (\text{proj})$$

with  $L_i/L_{i-1} \cong \Omega^{1-i}(M_{i-1})$  for  $i = 1, \dots, n$ .

**Proof.** Consider the following commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & B & \longrightarrow & M_{n-1} & \longrightarrow & M_{n-2} & \longrightarrow & M_{n-3} & \longrightarrow & \dots \\ & & \parallel & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & B & \longrightarrow & I(M_{n-1}) & \longrightarrow & K_{n-2} & \longrightarrow & M_{n-3} & \longrightarrow & \dots \\ & & & & \downarrow & & \downarrow & & & & \\ & & & & \Omega^{-1}(M_{n-1}) & \xlongequal{\quad} & \Omega^{-1}(M_{n-1}) & & & & \end{array}$$

where  $K_{n-2}$  is the push out and  $I(M_{n-1})$  is the injective hull of  $M_{n-1}$ . From the second horizontal exact sequence, we obtain an extension

$$E' : 0 \rightarrow \Omega^{-1}(B) \oplus (\text{proj}) \rightarrow K_{n-2} \rightarrow M_{n-3} \rightarrow \dots \rightarrow M_0 \rightarrow A \rightarrow 0$$

whose extension class corresponds to  $\alpha$  under the isomorphism

$$\text{Ext}_{kG}^{n-1}(A, \Omega^{-1}(B)) \cong \text{Ext}_{kG}^n(A, B).$$

Let

$$\tilde{E} : 0 \rightarrow \Omega^{-n+1}(B) \rightarrow M \rightarrow A \rightarrow 0$$

be a contraction of  $E$ . Note that  $\tilde{E}$  is also a contraction of  $E'$ . So, by induction,  $M \oplus (\text{proj})$  has a filtration

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_{n-1} = M \oplus (\text{proj})$$

with  $L_i/L_{i-1} \cong \Omega^{1-i}(M_{i-1})$  for  $i = 1, \dots, n-2$  and  $L_{n-1}/L_{n-2} \cong \Omega^{2-n}(K_{n-2})$ .

To finish the proof, we need to refine the above filtration at the  $L_{n-1}/L_{n-2}$  section. For this, consider the exact sequence  $0 \rightarrow M_{n-2} \rightarrow K_{n-2} \rightarrow \Omega^{-1}(M_{n-1}) \rightarrow 0$ . After tensoring this exact sequence with  $\Omega^{2-n}(k)$ , and cancelling projective summands from both ends of the sequence, we get a filtration for  $\Omega^{2-n}(K_{n-2}) \oplus (\text{proj})$  with sections isomorphic to  $\Omega^{2-n}(M_{n-2})$  and  $\Omega^{1-n}(M_{n-1})$ . By adding projective summands to  $L_n$  if necessary, we can assume  $L_{n-1}/L_{n-2}$  also have a similar filtration. So, there exists a  $kG$ -module  $\tilde{L}_{n-1}$  such that

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_{n-2} \subseteq \tilde{L}_{n-1} \subseteq L_{n-1} = M \oplus (\text{proj})$$

is a filtration having the desired properties.  $\square$

Now, we are ready to prove [Theorem 1.3](#).

**Proof of Theorem 1.3.** Let  $\zeta$  be the cohomology class in  $H^n(G, k)$  which is represented by the extension

$$E : 0 \rightarrow k \rightarrow M_{n-1} \rightarrow M_{n-2} \rightarrow \dots \rightarrow M_0 \rightarrow k \rightarrow 0.$$

Tensoring the exact sequence in [Lemma 2.2](#) with  $\Omega^{1-n}(k)$ , and cancelling the projective summands from both ends, we obtain a short exact sequence

$$\tilde{E} : 0 \rightarrow \Omega^{1-n}(k) \rightarrow \Omega^{-n}(L_\zeta) \oplus (\text{proj}) \rightarrow k \rightarrow 0$$

with extension class corresponding to  $\zeta$  under the isomorphism

$$\text{Ext}_{kG}^n(k, k) \cong \text{Ext}_{kG}^1(k, \Omega^{1-n}(k)).$$

So,  $\tilde{E}$  is a contraction of  $E$ . Applying [Lemma 4.2](#), we obtain a filtration

$$0 = T_0 \subseteq T_1 \subseteq \dots \subseteq T_n = \Omega^{-n}(L_\zeta) \oplus (\text{proj})$$

with  $T_i/T_{i-1} = \Omega^{1-i}(M_{i-1})$  for  $i = 1, \dots, n$ . Tensoring the entire system with  $\Omega^n(k)$ , and eliminating the projective summands if necessary, gives the desired filtration.  $\square$

Note that [Theorem 1.2](#) follows from [Theorem 1.3](#) as a corollary. To see this, first observe that for a  $p$ -group which is not elementary abelian, there is a sequence of maximal subgroups  $H_1, \dots, H_n$  and an exact sequence

$$E : 0 \rightarrow k \rightarrow C_{2n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow k \rightarrow 0$$

such that  $C_{2i-2} \cong C_{2i-1} \cong k \uparrow_{H_i}^G$  for  $i = 1, \dots, n$  and the class of  $E$  in  $\text{Ext}_{kG}^{2n}(k, k)$  is zero. This is just a consequence of Serre's theorem (see Corollary 3.4 in Carlson [5]). We now apply [Theorem 1.3](#) to this sequence and conclude [Theorem 1.2](#).

In the rest of the section, we study some consequences of [Theorem 1.3](#). We first introduce some more terminology: Given a  $kG$ -module  $M$ , let  $J_G(M)$  denote the kernel of the homomorphism

$$M \otimes_k - : \text{Ext}_{kG}^*(k, k) \rightarrow \text{Ext}_{kG}^*(M, M)$$

defined by tensoring an extension with  $M$  (over  $k$ ). Note that  $J_G(M)$  can also be considered as the annihilating ideal of  $\text{Ext}_{kG}^*(M, M)$  as a  $\text{Ext}_{kG}^*(k, k)$ -module. We now recall the following well known theorem.

**Theorem 4.3.** Let  $\zeta \in H^n(G, k)$  and let  $M$  be a  $kG$ -module. Then,  $\zeta \in J_G(M)$  if and only if

$$L_\zeta \otimes M \cong \Omega^n(M) \oplus \Omega(M) \oplus (\text{proj}). \quad \square$$

**Proof.** See Proposition 9.7.5 in [7].  $\square$

Combining this theorem with Theorem 1.3, we obtain

**Corollary 4.4.** *Let  $M$  be a finitely generated  $kG$ -module such that  $J_G(M)$  includes a product of Bocksteins of one dimensional classes. Then, for each integer  $s$ , there exists a finitely generated  $kG$ -module  $V$  such that  $\Omega^s(M) \oplus V$  has a filtration*

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_m = \Omega^s(M) \oplus V$$

where for every  $i = 1, \dots, m$ , we have  $L_i/L_{i-1} \cong \Omega^{s+i-1}(M \downarrow_{H_i}^G) \uparrow_{H_i}^G$  for some maximal subgroup  $H_i$  of  $G$ .

**Proof.** Suppose that  $\zeta = \beta(u_1) \cdots \beta(u_n) \in J_G(M)$  where  $u_i$ 's are one dimensional classes in  $H^1(G, \mathbb{F}_p)$ . Note that  $\zeta$  is represented by an extension which is the Yoneda splice of extensions of the form

$$0 \rightarrow k \rightarrow k \uparrow_{H_i}^G \rightarrow k \uparrow_{H_i}^G \rightarrow k \rightarrow 0$$

where  $H_i$  is the kernel of  $u_i$ . So, by Theorem 1.3, there is a filtration for  $L_\zeta \oplus (proj)$

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_{2n} = L_\zeta \oplus (proj)$$

where the  $j$ th section  $L_j/L_{j-1}$  is isomorphic to  $\Omega^{n+1-j}(k) \uparrow_{H_i}^G$  when  $j = 2i$  or  $j = 2i - 1$ . Tensoring  $L_i$ 's with  $M$ , we obtain a filtration

$$0 = (L_0 \otimes M) \subseteq (L_1 \otimes M) \subseteq \dots \subseteq (L_{2n} \otimes M) = (L_\zeta \otimes M) \oplus (proj)$$

such that

$$\begin{aligned} (L_j \otimes M)/(L_{j-1} \otimes M) &\cong (\Omega^{n+1-j}(k) \uparrow_{H_i}^G) \otimes M \\ &\cong \Omega^{n+1-j}(M \downarrow_{H_i}^G) \uparrow_{H_i}^G \oplus (proj) \end{aligned}$$

where  $j = 2i$  or  $j = 2i - 1$ . By Theorem 4.3, we have  $L_\zeta \otimes M \cong \Omega^n(M) \oplus \Omega(M) \oplus (proj)$ . So, tensoring the entire system with  $\Omega^{s-n}(k)$ , and eliminating the projective summands, we obtain the desired filtration.  $\square$

Note that the filtration length  $m$  in Corollary 4.4 depends on the number of one dimensional classes whose product of Bocksteins is in  $J_G(M)$ . This suggests the following definition:

**Definition 4.5.** The cohomology length of a  $kG$ -module  $M$ , denoted by  $chl_G(M)$ , is defined as the smallest positive integer  $n$  such that there exist non-zero elements  $u_1, u_2, \dots, u_n \in H^1(G, \mathbb{F}_p)$  such that

$$\begin{aligned} u_1 u_2 \cdots u_n &\in J_G(M) \quad \text{if } p = 2, \\ \beta(u_1) \beta(u_2) \cdots \beta(u_n) &\in J_G(M) \quad \text{if } p > 2. \end{aligned}$$

If no such integer exists, then we set  $chl_G(M) = \infty$ .

It is easy to see that if  $chl_G(M) = n$ , then there is a filtration as in Corollary 4.4 of length  $2n$ . Note that for  $p = 2$ , we have a filtration of length  $n$ . So, the cohomology length of a module is an interesting invariant to consider if one is interested in finding filtrations like in Corollary 4.4 of shortest length.

Note that there is a notion of cohomology length for  $p$ -groups (which are not elementary abelian) as a consequence of Serre's theorem. The cohomology length of a  $p$ -group  $G$ , denoted by  $chl(G)$ , is defined as the minimal  $m$  such that the product in Serre's theorem vanish. We can declare  $chl(G) = \infty$  for groups where Serre's theorem does not hold. Then, it is clear that  $chl(G) = chl_G(k)$ . Also note that for any  $kG$ -module  $M$ , we have  $chl_G(M) \leq chl(G)$ . In general, the cohomology length of a  $kG$ -module  $M$  can be much smaller than the cohomology length of  $G$ . For example, if  $G$  is an elementary abelian  $p$ -group and  $H$  a maximal subgroup of  $G$ , then  $chl(G) = \infty$  whereas  $chl_G(k \uparrow_H^G) = 1$ .

### 5. Varieties of modules

In this section we introduce the varieties of modules, and prove [Theorem 1.4](#) and [Corollary 1.5](#) stated in the introduction. As in the previous sections,  $k$  denotes a field of characteristic  $p > 0$ . We do not assume that  $k$  is algebraically closed and denote the algebraic closure of  $k$  by  $K$ . Let  $V_G(k)$  denote the maximal ideal spectrum of  $H^\bullet(G, k)$  where  $H^\bullet(G, k) = H^*(G, k)$  for  $p = 2$ , and  $H^\bullet(G, k) = H^{ev}(G, k)$ , the ring of even dimensional classes, for  $p > 2$ . Since  $H^\bullet(G, k)$  is a finitely generated commutative  $k$ -algebra,  $V_G(k)$  is a finite dimensional homogeneous affine variety, where a point in  $V_G(k)$  can be viewed as a  $k$ -linear ring homomorphism  $H^\bullet(G, k) \rightarrow K$ . Two such ring homomorphisms correspond to the same point in  $V_G(k)$  if and only if they are in the same orbit under the Galois action of  $\text{Aut}_k(K)$ .

From the above description, it is easy to see that any ring homomorphism  $f : H^\bullet(G, k) \rightarrow H^\bullet(H, k)$  induces a continuous map  $f^* : V_H(k) \rightarrow V_G(k)$  of corresponding varieties. In particular, for every  $H \leq G$ , the restriction homomorphism  $\text{res}_{G,H} : H^\bullet(G, k) \rightarrow H^\bullet(H, k)$  induces a map  $\text{res}_{G,H}^* : V_H(k) \rightarrow V_G(k)$  on varieties.

For a finitely generated  $kG$ -module  $M$ , the support variety of  $M$ , denote by  $V_G(M)$ , is defined as the variety of the ideal  $J_G(M)$  where  $J_G(M)$  is the annihilator of  $\text{Ext}_{kG}^*(M, M)$  in  $H^\bullet(G, k)$ . Observe that we have  $\text{res}_{G,H}^*(V_H(M)) \subseteq V_G(M)$  as a consequence of the obvious inclusion for ideals  $\text{res}_{G,H}(J_G(M)) \subseteq J_H(M)$ . So,  $\text{res}_{G,H}^*$  induces a map  $\text{res}_{G,H}^* : V_H(M) \rightarrow V_G(M)$  on the varieties of the module  $M$ .

The following is a list of properties of varieties which we will need in this section.

**Lemma 5.1.** *Suppose that  $k$  is a field of characteristic  $p > 0$ ,  $G$  is a finite group, and  $H$  is a subgroup of  $G$ . Let  $M, M_1, M_2, M_3$  be  $kG$ -modules, and  $N$  a  $kH$ -module.*

- (i)  $V_G(M) = 0$  if and only if  $M$  is projective.
- (ii)  $V_G(M) = V_G(\Omega^n(M))$  for all integers  $n$ .
- (iii) If  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  is exact, then  $V_G(M_2) \subseteq V_G(M_1) \cup V_G(M_3)$ . In particular  $V_G(M_1 \oplus M_2) = V_G(M_1) \cup V_G(M_2)$ .
- (iv)  $V_G(N \uparrow_H^G) = \text{res}_{G,H}^*(V_H(N))$ .
- (v)  $(\text{res}_{G,H}^*)^{-1}(V_G(M)) = V_H(M \downarrow_H^G)$ .
- (vi)  $V_G(M_1 \otimes M_2) = V_G(M_1) \cap V_G(M_2)$ .
- (vii) Let  $\zeta \in \text{Ext}_{kG}^n(k, k)$ . Then  $V_G(L_\zeta) = V_G(\zeta)$  where  $V_G(\zeta)$  is the variety of the ideal generated by  $\zeta$ .

Proofs of these statements can be found in [2,4,7,8]. Note that the above properties hold for an arbitrary field if and only if they hold for an algebraically closed field. To see this, observe that the ring homomorphism  $\phi : H^\bullet(G, k) \rightarrow H^\bullet(G, K) \cong H^\bullet(G, k) \otimes_k K$  defined by  $\phi(\zeta) = 1 \otimes \zeta$  induces a map  $\phi^* : V_G(K) \rightarrow V_G(k)$  on varieties which is finite to one. For a  $kG$ -module  $M$ , we have  $J_G(K \otimes M) \cong K \otimes J_G(M)$ , which gives  $V_G(K \otimes M) = (\phi^*)^{-1}(V_G(M))$ . Thus proving these results for  $V_G(M \otimes K)$  will give corresponding results for  $V_G(M)$ . For details of this argument, we refer the reader to Theorem 10.4.2 and Remark 10.4.3 in [7].

**Lemma 5.2.** *Let  $E$  be an elementary abelian  $p$ -group, and  $\mathcal{M}$  be a collection of the maximal subgroups of  $E$ . Suppose  $M$  is a finitely generated  $kG$ -module such that*

$$V_E(M) = \bigcup_{D \in \mathcal{M}} \text{res}_{E,D}^*(V_D(M \downarrow_D^E)).$$

*Then,  $J_E(M)$  includes a product of Bocksteins of one dimensional classes in  $H^1(E, \mathbb{F}_p)$  whose kernels are in  $\mathcal{M}$ .*

**Proof.** Applying  $(\phi^*)^{-1}$  to the given equality, we obtain

$$V_E(K \otimes M) \subseteq \bigcup_{D \in \mathcal{M}} \text{res}_{E,D}^*(V_D(K \otimes M \downarrow_D^E)) \subseteq \bigcup_{D \in \mathcal{M}} \text{res}_{E,D}^*(V_D(K)).$$

For each  $D \in \mathcal{M}$ , choose a one dimensional class  $x_D \in H^1(E, \mathbb{F}_p)$  such that the kernel of  $x_D$  is  $D$ . Then, it is clear that  $\beta(x_D)$  is in the kernel of the restriction map  $\text{res}_{E,D}$ , so we have

$$\text{res}_{E,D}^*(V_D(K)) = V_E(\ker \text{res}_{E,D}) \subseteq V_E(\beta(x_D))$$

for each  $D$ . This gives

$$V_E(K \otimes M) \subseteq V_E \left( \prod_{D \in \mathcal{M}} \beta(x_D) \right).$$

By Hilbert’s Nullstellensatz,

$$u = \left( \prod_{D \in \mathcal{M}} \beta(x_D) \right)^r \in J_E(K \otimes M)$$

for some  $r > 0$ . Since  $J_E(K \otimes M) = K \otimes J_E(M)$  and  $u \in H^\bullet(E, \mathbb{F}_p)$ ,  $u$  lies in  $J_E(M)$ .  $\square$

Now, we are ready to prove [Theorem 1.4](#), which is our main result in this section.

**Proof of Theorem 1.4.** (ii)  $\Rightarrow$  (i) Suppose that there exists a  $kG$ -module  $V$  such that  $M \oplus V$  has a filtration

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_n = M \oplus V$$

where for each  $i = 1, \dots, n$ , the  $i$ th section  $L_i/L_{i-1}$  is isomorphic to  $W_i \uparrow_{H_i}^G$  for some subgroup  $H_i \in \mathcal{H}$  and some  $kH_i$ -module  $W_i$ . Applying the properties (ii)–(iv) listed in [Lemma 5.1](#), we obtain

$$V_G(M) \subseteq \bigcup_i V_G(W_i \uparrow_{H_i}^G) = \bigcup_i \text{res}_{G, H_i}^*(V_{H_i}(W_i))$$

which gives

$$V_G(M) \subseteq \bigcup_{H \in \mathcal{H}} \text{res}_{G, H}^* V_H(k).$$

Note that by property (v), we have

$$V_G(M) \cap \text{res}_{G, H}^*(V_H(k)) = \text{res}_{G, H}^*(V_H(M \downarrow_H^G))$$

for all  $H \in \mathcal{H}$ , so we obtain

$$V_G(M) = \bigcup_{H \in \mathcal{H}} \text{res}_{G, H}^*(V_H(M \downarrow_H^G))$$

as desired.

(i)  $\Rightarrow$  (ii) Note that by [Theorem 3.2](#), there exists a finitely generated  $kG$ -module  $V$  such that  $k \oplus V$  has a filtration

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_n = k \oplus V$$

where for every  $i = 1, \dots, n$ , the sections  $L_i/L_{i-1}$  are isomorphic to  $\Omega^{n_i}(k) \uparrow_{E_i}^G$  for some integer  $n_i$  and some maximal elementary abelian  $p$ -subgroup  $E_i$  of  $G$ . Tensoring this system with  $M$ , we obtain a filtration

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M \oplus (M \otimes V)$$

where  $M_i = M \otimes L_i$ , and hence

$$M_i/M_{i-1} \cong (L_i/L_{i-1}) \otimes M \cong \Omega^{n_i}(k) \uparrow_{E_i}^G \otimes M \cong (\Omega^{n_i}(M \downarrow_{E_i}^G) \oplus P_i) \uparrow_{E_i}^G$$

where  $P_i$  is a projective  $kE_i$ -module. Since  $E_i$  is a  $p$ -group,  $P_i$  is a free  $kE_i$ -module. Thus,

$$M_i/M_{i-1} \cong \Omega^{n_i}(M \downarrow_{E_i}^G) \uparrow_{E_i}^G \oplus F_i$$

for some free  $kG$ -module  $F_i$ .

If for some  $i \in \{1, \dots, n\}$ , the ideal  $J_{E_i}(M \downarrow_{E_i}^G)$  includes a product of Bocksteins of one dimensional classes, then by [Corollary 4.4](#) there is a finitely generated  $kE_i$ -module  $W$  such that  $\Omega^{n_i}(M \downarrow_{E_i}^G) \oplus W$  has a filtration

$$0 = T_0 \subseteq T_1 \subseteq \dots \subseteq T_m = \Omega^{n_i}(M \downarrow_{E_i}^G) \oplus W$$

with

$$T_j/T_{j-1} \cong \Omega^{m_j}(M \downarrow_{D_j}^G) \uparrow_{D_j}^{E_i}$$

where for  $j = 1, \dots, m$ , we have  $m_j = n_i + j - 1$  and  $D_j$  is a maximal subgroup of  $E_i$ . Inducing the entire filtration to  $G$ , we obtain a filtration

$$0 = S_0 \subseteq S_1 \subseteq \dots \subseteq S_m = \Omega^{m_i}(M \downarrow_{E_i}^G) \uparrow_{E_i}^G \oplus W \uparrow_{E_i}^G$$

where  $S_j = T_j \uparrow_{E_i}^G$  and

$$S_j/S_{j-1} \cong \Omega^{m_j}(M \downarrow_{D_j}^G) \uparrow_{D_j}^G.$$

Finally, adding the free summand  $F_i$  to the last section, we obtain a filtration

$$0 = U_0 \subseteq U_1 \subseteq \dots \subseteq U_m = (M_i/M_{i-1}) \oplus W \uparrow_{E_i}^G$$

where

$$U_j/U_{j-1} \cong \Omega^{m_j}(M \downarrow_{D_j}^G) \uparrow_{D_j}^G \oplus F_j$$

for some free summand  $F_j$  (only the last section has a nontrivial free summand).

We can use this filtration to refine the original filtration for  $M \oplus (M \otimes V)$  as follows: let  $\hat{U}_j$  denote the pre-image of  $U_j$  under the quotient map  $M_i \oplus W' \rightarrow (M_i \oplus W')/M_{i-1}$  where  $W' = W \uparrow_{E_i}^G$ . We have

$$M_{i-1} = \hat{U}_0 \subseteq \hat{U}_1 \subseteq \dots \subseteq \hat{U}_m = M_i \oplus W'$$

with

$$\hat{U}_j/\hat{U}_{j-1} \cong U_j/U_{j-1} \cong \Omega^{m_j}(M \downarrow_{D_j}^G) \uparrow_{D_j}^G \oplus F_j.$$

Splicing this into the filtration

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_{i-1} \subseteq M_i \oplus W' \subseteq M_{i+1} \oplus W' \subseteq \dots \subseteq M_n \oplus W' = M \oplus V'$$

where  $V' = (M \otimes V) \oplus W'$ , we obtain a new filtration for  $M \oplus V'$  where the  $i$ th section is replaced by a sequence of modules with quotients induced from maximal subgroups of  $E_i$ . Applying this process to every section, we can assume that there is a  $kG$ -module  $V$  such that  $M \oplus V$  has a filtration

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M \oplus V$$

with

$$M_i/M_{i-1} \cong \Omega^{n_i}(M \downarrow_{E_i}^G) \uparrow_{E_i}^G \oplus F_i$$

where  $F_i$  is a free  $kG$ -module and  $E_i$  is an elementary abelian  $p$ -group such that the ideal  $J_{E_i}(M \downarrow_{E_i}^G)$  does not have any element which is a product of Bocksteins of one dimensional classes. Note that by [Lemma 5.2](#), this implies that for any collection of maximal subgroups  $\mathcal{M}_i$  of  $E_i$

$$V_{E_i}(M \downarrow_{E_i}^G) \neq \bigcup_{D \in \mathcal{M}_i} \text{res}_{E_i, D}^*(V_D(M \downarrow_D^{E_i})).$$

We claim that this forces  $E_i$  to be conjugate to a subgroup  $H$  in  $\mathcal{H}$ . To see this, consider the following calculation. We have

$$V_{E_i}(M \downarrow_{E_i}^G) = (\text{res}_{G, E_i}^*)^{-1}(V_G(M)) \subseteq \bigcup_{H \in \mathcal{H}} (\text{res}_{G, E_i}^*)^{-1}(\text{res}_{G, H}^*(V_H(k)))$$

which gives

$$V_{E_i}(M \downarrow_{E_i}^G) \subseteq \bigcup_{H \in \mathcal{H}} V_{E_i}(k \uparrow_H^G \downarrow_{E_i}^G) = \bigcup_{H \in \mathcal{H}} \left( \bigcup_{H \setminus G/E_i} V_{E_i}(k \uparrow_{sH \cap E_i}^{E_i}) \right).$$

From this we obtain

$$V_{E_i}(M \downarrow_{E_i}^G) = \bigcup_L \text{res}_{E_i, L}^*(V_L(M \downarrow_L^{E_i}))$$

where  $L$  runs over the subgroups of the form  ${}^g H \cap E_i$  over the set of double cosets  $H \setminus G/E_i$  and over  $H \in \mathcal{H}$ . But, no such equality exists for a collection of proper subgroups, so we must have  $E_i = {}^g H \cap E_i \leq {}^g H$  for every  $g \in H \setminus G/E_i$  and  $H \in \mathcal{H}$ . Thus  $E_i$  is conjugate to a subgroup of some  $H \in \mathcal{H}$  for all  $i = 1, \dots, n$ .

To complete the proof, we observe that  $\Omega^{n_i}(M \downarrow_{E_i}^G) \uparrow_{E_i}^G \cong \Omega^{n_i}(M \downarrow_{E_i^g}^G) \uparrow_{E_i^g}^G$  for all  $g \in G$ , so by replacing  $E_i$ 's with their conjugates, we can assume  $E_i$ 's are subgroups of  $H$ . Thus  $M \oplus V$  has a filtration with sections induced from subgroups in  $\mathcal{H}$ .  $\square$

From the above proof it is clear that the following is also true.

**Corollary 5.3.** *Let  $M$  be a  $kG$ -module, and  $\mathcal{H}$  a collection of subgroups in  $G$ . Suppose that there is a  $kG$ -module  $V$  such that  $M \oplus V$  has a filtration*

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_{m-1} \subseteq L_m = M \oplus V$$

such that for each  $i = 1, 2, \dots, m$ , the quotient module  $L_i/L_{i-1}$  is induced from subgroups in  $\mathcal{H}$ . Then there exists a  $kG$ -module  $U$  such that  $M \oplus U$  has a filtration

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M \oplus U$$

such that for  $i = 1, \dots, n$ ,

$$M_i/M_{i-1} \cong \Omega^{n_i}(M \downarrow_{E_i}^G) \uparrow_{E_i}^G \oplus F_i$$

where  $F_i$  is a free  $kG$ -module, and  $E_i$  is an elementary abelian  $p$ -group included in a subgroup  $H$  in  $\mathcal{H}$ .

In particular this tells us that if we have a filtration for a  $kG$ -module  $M$  where the sections are induced modules, then there is a  $kG$ -module  $U$  such that  $M \oplus U$  has a filtration whose sections are given in terms of  $M$  up to a free module. This could be useful for proving theorems by induction for modules with complexity strictly smaller than the  $p$ -rank of the group.

We conclude the paper with the proof of Corollary 1.5 stated in the introduction.

**Proof (Proof of Corollary 1.5).** It is enough to prove the result under the assumption that  $k$  is algebraically closed. By Theorem 1.3, we have a filtration for  $L_\zeta$  with sections isomorphic to Heller shifts of  $M_i$ 's. Thus

$$V_G(L_\zeta) \subseteq \bigcup_{i=0}^{n-1} V_G(M_i).$$

Since the modules  $M_0, \dots, M_{n-1}$  are direct sums of modules induced from proper subgroups, there is a collection  $\mathcal{M} = \{H_1, \dots, H_m\}$  of maximal subgroups of  $G$  such that

$$V_G(L_\zeta) \subseteq \bigcup_j \text{res}_{G, H_j}^*(V_{H_j}(k)).$$

As in the proof of Lemma 5.2, we can choose one dimensional classes  $x_1, \dots, x_m \in H^1(G, \mathbb{F}_2)$  such that the kernel of  $x_j$  is  $H_j$  for each  $j = 1, \dots, m$ , and replace each  $\text{res}_{G, H_j}^*(V_{H_j}(k))$  with  $V_G(x_j)$ . Note that by property (vii), we have  $V_G(\zeta) = V_G(L_\zeta)$ , so we obtain

$$V_G(\zeta) \subseteq V_G(x_1 x_2 \cdots x_m).$$

Since  $H^\bullet(G, k) = H^*(G, k)$  is a polynomial algebra, we can apply Hilbert's Nullstellensatz, and conclude that there exists an integer  $t > 0$ , such that  $(x_1 x_2 \cdots x_m)^t \in (\zeta)$ . This means  $(x_1 x_2 \cdots x_m)^t = a \cdot \zeta$  for some  $a \in H^*(G, k)$ . Thus,  $\zeta = \lambda x_1^{t_1} \cdots x_m^{t_m}$  for some scalar  $0 \neq \lambda \in k$  and some integers  $t_i \geq 0$ .

Let  $X_i$  be a two point  $G$ -set with isotropy  $H_i$ . Now, consider the  $G$ -sphere  $X$  defined as the join of  $G$ -spheres  $X_i$  where we take  $t_i$  copies of  $X_i$  for each  $i$ . The cellular homology of  $X$  with coefficients in  $k$  gives an extension

$$0 \rightarrow H_m(X, k) \rightarrow C_m(X, k) \rightarrow \dots \rightarrow C_1(X, k) \rightarrow C_0(X, k) \rightarrow H_0(X, k) \rightarrow 0$$

where each  $C_m(X, k)$  is a  $kG$ -module and  $m = \sum_i t_i$ . Note that both  $H_0(X, k)$  and  $H_m(X, k)$  are isomorphic to  $k$ . We say a  $k$  by  $k$  extension  $E$  comes from a geometric group action on a sphere if after choosing appropriate isomorphisms

for  $H_0(X, k) \cong k$  and  $H_m(X, k) \cong k$ , the resulting extension is equivalent to  $E$ . Since the Euler class of join of spheres is the product of Euler classes, we can choose these isomorphisms to make the extension class equal to  $\zeta$ .  $\square$

For  $p > 2$ , the situation is more complicated. In this case  $H^\bullet(G, k)$  is not a polynomial algebra, so we do not have unique factorization. For example, if  $x_1$  and  $x_2$  are two one dimensional classes then we have  $(\beta(x_1) + x_1x_2)^p = \beta(x_1)^p$ . On the other hand, if we assume that  $\zeta$  is a class lying in the subalgebra generated by Bocksteins of one dimensional classes, then there is a similar conclusion for  $\zeta$ . Namely,  $\zeta$  is a non-zero scalar multiple of a product of Bocksteins of one dimensional classes in  $H^1(G, \mathbb{F}_p)$ . A linear action on a sphere such that the  $k$ -invariant is equal to this product can easily be constructed using a direct sum of one dimensional complex representations. Note that since for every  $\zeta \in H^\bullet(G, k)$  the  $p$ th power  $\zeta^p$  lies in the polynomial subalgebra generated by Bocksteins of one dimensional classes, we can also conclude that for every  $\zeta$  satisfying the conditions of [Corollary 1.5](#), the class  $\zeta^p$  is a non-zero scalar multiple of a product of Bocksteins of one dimensional classes in  $H^1(G, \mathbb{F}_p)$ .

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## References

- [1] D.J. Benson, *Modular Representation Theory: New Trends and Methods*, in: *Lecture Notes in Math.*, Springer-Verlag, 1984.
- [2] D.J. Benson, *Representations and Cohomology II: Cohomology of Groups and Modules*, Cambridge University Press, Cambridge, 1991.
- [3] J.F. Carlson, The variety of an indecomposable module is connected, *Invent. Math.* 77 (1984) 291–299.
- [4] J.F. Carlson, *Modules and Group Algebras*, Birkhäuser, Basel, 1996.
- [5] J.F. Carlson, Cohomology and induction from elementary abelian subgroups, *Quart. J. Math.* 51 (2000) 169–181.
- [6] J.F. Carlson, J. Thévenaz, The classification of torsion endo-trivial modules, *Ann. Math.* (in press).
- [7] J.F. Carlson, L. Townsley, L. Valero-Elizondo, M. Zhang, *Cohomology Rings of Finite Groups*, Kluwer Academic Publishers, 2003.
- [8] L. Evens, *The Cohomology of groups*, in: *Oxford Mathematical Monographs*, Clarendon Press, Oxford, New York, Tokyo, 1991.
- [9] J.-P. Serre, Sur la dimension cohomologique des groupes profinis, *Topology* 3 (1965) 413–420.