A Solution to the Diagonalization Problem by Constant Precompensator and Dynamic Output Feedback

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Abstract—**A solution is obtained for the problem of diagonalization (row by row decoupling) by a constant precompensator and a dynamic output feedback compensator of a $p \times m$ linear time-invariant system. The solvability condition is compact and concerns the dimension of a single subspace defined via the concepts of "essential rows" and "static kernels" associated with the transfer matrix. A characterization of the set of all solutions to the problem is also given. In solving this dynamic feedback problem, we also obtain a complete solution to its state-feedback counterpart, namely, the restricted state-feedback problem of diagonalization.**

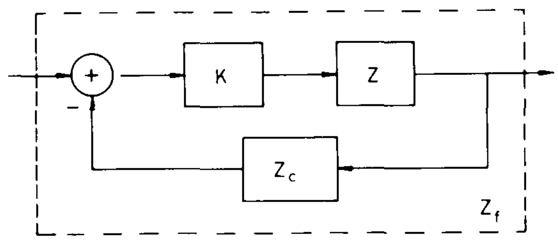

Fig. 1.

I. Introduction

NONINTERACTING control of linear multivariable systems occupied a great deal of attention since the early work of Morgan [10], Rekasius [14], Falb and Wolovich [7], Morse and Wonham [11], Basile and Marro [1], Cremer [3], and others. From a pure point of view of "design," the algebraic theory of noninteracting control is unsatisfactory and here "adaptive" or "almost noninteracting control" approaches like that of [18] (also see [17]) or [19] may be preferable. However, as the algebraic (or "geometric" in the sense of Wonham [17]) studies of systems taught us, such puristic approaches yield many clues as to the finer structure of control systems and a knowledge of this is a prerequisite for a good theory of design.

There is now a renewed interest in noninteracting control problems mainly because most of these problems remain among the unsolved in spite of the maturity attained in the algebraic and geometric techniques used in dealing with such problems. A recent solution in [9] and [5] to the diagonalization by the state feedback problem of [10] is another reason for the revival of interest.

This paper presents a solution to the diagonalization problem by dynamic output feedback and a constant precompensator of a $p \times m$ transfer matrix. In the special case $p = m$, the problem was stated and solved in [2] by an extension of a result of [16]. The aspect of internal stability has been later incorporated by [6].

(P) Problem Definition: Let Z be a strictly proper transfer matrix of size $p \times m$. Determine a $p \times p$ dynamic feedback compensator Z_c and an $m \times p$ constant precompensator K such that the closed-loop transfer matrix $Z_f = (I + ZKZ_c)^{-1}ZK$ is diagonal and nonsingular.

Fig. 1 illustrates the type of compensation used on Z. The problem (P) can be stated as an open-loop problem by a recent result of [12]. (P) *is solvable if and only if there exists a constant $m \times p$ matrix K such that ZK is* DCDD. The concept of DCDD is defined in Section II and it is easy to see that "if ZK is DCDD, then it is row proper." This suggests the idea of solving the problem in the following two stages.

(P1) Determine a constant matrix L for which ZL is row proper.

(P2) Determine a matrix K in the class of solutions to (P1) for which ZK is DCDD. One immediately observes that (P1) is closely related to another diagonalization problem which is the state feedback version of (P); see [15]. To be precise: *There exists a constant matrix L of size $m \times p$ such that ZL is row proper if and only if a reachable realization of Z can be diagonalized by restricted static state feedback.* (This result, although well known, does not seem to be explicitly stated in the literature.) Solutions to this subproblem have been given in [8] and [4]. In Theorem (3.5), we present an alternative but closely related solution to (P1) in a form most suitable to our purpose and examine the set of solutions to (P1) in detail, obtaining a characterization in Theorem (4.2). This is a new result on (P1).

An important feature of our solution in Theorem (3.11) to (P) is that the solvability condition is in closed form and can be checked by a finite algorithm. Our solution technique also leads in Section IV to a characterization of the set of all solutions to problems (P1) and (P2) above. This yields the set of all solutions (K, Z_c) to (P) in Section IV, by making use of some results in [13]. One aspect of the solvability condition given in Theorem (3.11) which is of interest but not addressed in this paper is the relation between S_q and the more familiar geometric subspaces of, say, [17]. One can, however, foresee that such a relation will be involved enough to deserve inspection on its own.

Remark 1: Given the compensation scheme above, one might consider more general versions of (P): i) Z_f may be required to be block diagonal with specified block sizes; ii) the precompensator may be external to the feedback loop; iii) the requirement of the internal stability of the closed loop may be added; iv) two groups of outputs: "controlled" and "measured" may be distinguished. Although this paper does not directly address these generalized problems, we shall comment on some in Remarks 2 and 3.

II. Essential Row Form, Row Properness, and Diagonal Causality Degree Dominance

In this section, we define and examine various concepts like "the essential rows of a constant matrix," "row properness of a

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rational matrix," and "diagonal causality degree dominance of a transfer matrix." These concepts and their properties are used in obtaining the main results of Section III.

Let K be in $R^{p\times m}$. Following [3], we call the ith row k_i of K *an essential row of* K iff it is linearly independent of the other rows of K. It is easy to see that a basis matrix J of the left kernel of $K(JK = 0)$ has the property that k_i is essential iff the ith column of J is identically zero. An immediate consequence of this is that the spaces spanned by the essential rows of K and the nonessential rows of K have zero intersection.

The *essential kernel* $N_e(K)$ *of* K is defined by

$$N_e(K) := \{l \text{ in } R^m : k_i l = 0 \text{ for all nonessential rows } k_i \text{ of } K\}.$$

In case K had full row rank, we let $N_e(K) = R^m$ for convenience. Let us say that K is in *essential row form (erf)* iff all nonzero rows of K are linearly independent, i.e., iff all nonessential rows (if any) of K are identically zero. We now list some easy but useful facts, pertaining to the notion of "essential rows," stated without proofs as they are direct consequences of the definitions.

Lemma (2.1): Let K be in $R^{p\times m}$ and L be in $R^{m\times n}$.

i) Ker $K \subseteq N_e(K)$.

ii) K is in erf iff $N_e(K) = R^m$.

iii) All the nonessential rows of K are zero in KL iff Im $L \subseteq N_e(K)$.

iv) If Im $L = N_e(K)$, then KL is in erf.

A $p \times m$ rational matrix Z can be uniquely represented as

$$Z = \Gamma Z_h + Y \tag{2.2}$$

where Γ is diagonal and of the form $\Gamma = \operatorname{diag}\{z^{\mu_i}\}$ with μ_i denoting the ith row causality-degree of Z, Z_h is in $R^{p\times m}$, and Y is a rational matrix having strictly less row degrees than Z, i.e., Y is such that $\Gamma^{-1}Y$ is strictly proper. If a row i of Z is zero, then we let $\Gamma_{ii} = 1$ to have Γ nonsingular for all Z. The representation (2.2) of Z will be referred to as *the standard representation of* Z. We call Z *row proper* iff Z_h is of full row rank. If Z is proper, then $\mu_i \le 0$ is the order (multiplicity) of the zero at infinity of the ith row of Z; see, e.g., [5]. If Z is strictly proper and has full row rank, then $\Gamma^{-1} =: D$ is a strictly polynomial matrix and it will be called *the row degree matrix of* Z. Let Z be in $R(z)^{p\times m}$ and let L be in $R^{m\times n}$. By the standard representation of ZL it readily follows that

$$\text{Ker } L \subseteq \text{Ker } (ZL)_h. \tag{2.3}$$

The relation of $(ZL)_h$ to Z_hL is usually quite intricate. One can prove, however, the following fact.

Lemma (2.4): Let Z be in $R(z)^{p\times m}$. If L in $R^{m\times n}$ is such that ZL is row proper, then Z_hL is in erf.

Proof: Let $Z = \Gamma Z_h + Y$ and $ZL = \Gamma_L(ZL)_h + Y_L$ be the standard representations of Z and ZL. Since the ith row degree of ZL is less than or equal to the ith row degree of Z, we have that $\Gamma^{-1}\Gamma_L$ is proper. Consequently, $\Gamma^{-1}Y_L$ is strictly proper. We can write

$$Z_hL = \Gamma^{-1}\Gamma_L(ZL)_h + \Gamma^{-1}Y_L - \Gamma^{-1}YL$$

where the last two terms are strictly proper. It follows that $Z_hL = (\Gamma^{-1}\Gamma_L)_0(ZL)_h$, where $(A)_0$ denotes the coefficient of z^0 in the Laurent series expansion of A. By row properness of ZL, we have all the rows of $(ZL)_h$ linearly independent. On the other hand, $(\Gamma^{-1}\Gamma_L)_0$ is a diagonal matrix of 0's and 1's at its diagonal entries. Therefore, all the nonzero rows of Z_hL are linearly independent. ∎

Consider now, a nonsingular and strictly proper Z in $R(z)^{p\times p}$; Z is called *diagonally causality degree dominant (DCDD)* iff

$$\deg(Z_{ij}) \le \deg(Z_{ii}) + \deg(Z_{jj}); \qquad i, j = 1, \cdots, p;\ i \ne j$$

where deg(·) denotes the causality degree [13] of its argument. This degree inequality implies, by strict properness of Z, that deg $(Z_{ij}) <$ deg (Z_{ii}) for all $i \ne j$. Hence, a DCDD Z is also row proper. In fact, if Z is DCDD, then Z_h is diagonal and nonsingular by the latter inequality. A more compact way of expressing the notion of DCDD is: *Z is DCDD iff $D(Z)_{\text{off}}D$ is proper*, where D is the row degree matrix of Z, and $(Z)_{\text{off}}$ denotes the $p \times p$ matrix obtained from Z on replacing its diagonal entries by zeros. (For a more detailed discussion of DCDD, the reader is referred to [12].)

Let a strictly proper transfer matrix Z in $R(z)^{p\times m}$ be of full row rank. Let $Z^{(i)}$ denote the $p \times m$ matrix obtained from Z on replacing its ith row by a zero row. Also let A^+ denote the *strictly polynomial part* of a rational matrix A. Note that any rational matrix A can be uniquely written as $A = A^- + A^+$, where A^- is the proper part and A^+ is the strictly polynomial part in its Laurent series expansion. We can now state a still different characterization for the notion of DCDD. (Contrast the case $L = I$.)

Lemma (2.5): Let L be in $R^{m\times p}$ such that ZL is nonsingular and let l_i denote the ith column of L. The $p \times p$ transfer matrix ZL is DCDD if and only if

$$[(D_L)_{ii}D_LZ^{(i)}]^+ l_i = 0; \qquad i = 1, \cdots, p \tag{2.6}$$

where D_L is the row degree matrix of ZL.

Proof: The matrix ZL is DCDD iff $D_L(ZL)_{\text{off}}D_L$ is proper. Noting that the ith column of $(ZL)_{\text{off}}$ is given by $Z^{(i)}l_i$, this holds iff $(D_L)_{ii}D_LZ^{(i)}l_i$ is proper. This in turn holds iff (2.6) is valid. ∎

The question we pose at the end of this section is the following. Given a strictly proper and full row rank Z in $R(z)^{p\times m}$, when is there a K in $B^{m\times p}$ such that ZK is DCDD and the ith row degrees of Z and ZK are the same? The answer to this question is given by Theorem (2.8) and is the key to the resolution of (P2).

Let D be the row degree matrix of Z and let $[K_i : P_i]$ be a minimal polynomial basis matrix of Ker $[D_{ii}DZ^{(i)}]^+$; $i = 1, \cdots, p$, where K_i denotes the columns of degree zero (constant columns) and P_i denotes the columns of strictly positive degrees in the basis matrix. Note that although the minimal polynomial basis of a rational vector space is nonunique, Im K_i is nevertheless unique by the invariance of the column degrees of a minimal polynomial basis. Define

$$\boldsymbol{K}(Z) := [K_1 : \cdots : K_p]$$

which is in $R^{m\times k}$ with $k := \Sigma_i \dim\{\text{Ker } [D_{ii}DZ^{(i)}]^+ \cap R^m\}$ and i ranging from 1 to p. Note that some of the blocks in $\boldsymbol{K}(Z)$ might be missing depending on whether $R^m \cap \text{Ker } [D_{ii}DZ^{(i)}]^+ = \{0\}$ or nonzero. Observe that

$$Z_h^{(i)}K_i = 0; \qquad i = 1, \cdots, p. \tag{2.7}$$

In fact, let $Z = D^{-1}Z_h + Y$ be the standard representation of Z and note that $0 = [D_{ii}DZ^{(i)}]^+K_i = [D_{ii}Z_h^{(i)} + D_{ii}DY^{(i)}]^+K_i$ implies $Z_h^{(i)}K_i = -D_{ii}^{-1}[D_{ii}DY^{(i)}]^+K_i$, where the right-hand side is strictly proper. This yields (2.7).

Theorem (2.8): There exists K_o in $R^{m\times p}$ such that ZK_o is DCDD and has the same row degrees as Z if and only if rank $[Z_h\boldsymbol{K}] = p$, where $\boldsymbol{K} := \boldsymbol{K}(Z)$. Such a K_o, when it exists, is given by $K_o = \boldsymbol{K}U$, where $U = \operatorname{diag}\{u_i\}$ is a constant matrix satisfying $Z_hK_iu_i = e_iw_i$ for $i = 1, \cdots, p$ with e_i denoting the ith column of the identity matrix and $0 \ne w_i$ is in R.

Proof:

[Only if] Let D be the row degree matrix of Z. If $D(ZK_o)_{\text{off}}D$ is proper, then the ith column k_{oi} of K_o is in Ker $[D_{ii}DZ^{(i)}]^+$ by Lemma (2.5). Thus, k_{oi} is in Im K_i, i.e., $k_{oi} = K_iu_i$ for some constant column vector u_i. It follows on comparing the standard representations of Z and ZK_o that $(ZK_o)_h = Z_hK_o$ which is nonsingular and diagonal by the fact that ZK_o is DCDD. By (2.7), we have

$$Z_hK_o = Z_h[K_1u_1 : \cdots : K_pu_p] = Z_h\boldsymbol{K}U$$

where $U := \operatorname{diag}\{u_i\}$. It follows that rank $[Z_h K] = p$ and $Z_h K_i u_i = e_i w_i$ with $w_i := (Z_h K_o)_{ii}$ by this equality.

[If] By (2.7), we have rank $[Z_h K_i] \le 1$ for $i = 1, \cdots, p$. If $p =$ rank $[Z_h K]$, then it must be that rank $[Z_h K_i] = 1$. Let u_i be such that $Z_h K_i u_i = e_i$ for $i = 1, \cdots, p$ and let $U := \operatorname{diag}\{u_i\}$. Note that such u_i's exist again by (2.7). Defining $K_o := KU$ we have that $Z_h K_o = I$ and that $[D_{ii} D Z^{(i)}]^+ K_i u_i = 0$, for all $i = 1, \cdots, p$, where $K_i u_i$ is actually the ith column of K_o. Since $Z_h K_o = I$, the ith row degree of ZK_o is $-\deg(D_{ii})$ which is the ith row degree of Z. Therefore, by Lemma (2.5), ZK_o is DCDD. ■

We close this section by the following result which will be used in the next section.

Lemma (2.9): Let $K := K(Z)$ and $K_0 := K(ZKL)$ for a given strictly proper transfer matrix Z and a constant nonsingular L. Then, K_0 is of full row rank if and only if Z and ZKL have the same row degrees.

Proof: The "only if" part readily follows since multiplication on the right by a constant full row rank matrix leaves the row degrees unchanged. To see the "if" part, let $K = [K_1 : \cdots : K_p]$ and $K_0 = [K_{01} : \cdots : K_{0p}]$. Let D be the row degree matrix of Z, and hence also of ZKL by hypothesis. By definition of K_{0i}, for $i = 1, \cdots, p$, we have

$$\begin{aligned}\operatorname{Im}(K_{0i}) &= \operatorname{Ker}[D_{ii} D(ZKL)^{(i)}]^+ \cap R^q \\ &= \operatorname{Ker}\{[D_{ii} D(Z)^{(i)}]^+ KL\} \cap R^q\end{aligned}$$

where q is the number of columns of K. This implies, as Ker $(KL) \subseteq$ Ker $\{[D_{ii} D(Z)^{(i)}]^+ KL\} \cap R^q$, that Ker $(KL) \subseteq$ Im $K_{0i} \subseteq$ Im K_0. Let N_i be any constant matrix such that $KLN_i = K_i$. By the definition of K_i, we have $(D_{ii} D Z^{(i)})^+ K_i = 0$ implying that $[D_{ii} D(ZKL)^{(i)}]^+ N_i = 0$. Hence, Im $N_i \subseteq$ Im K_{0i} and Im $K_i \subseteq$ Im (KLK_{0i}). Consequently, Im $(KL) \subseteq$ Im $(KLK_0) \subseteq$ Im (KL) and we have Im $(KL) =$ Im (KLK_0). Therefore,

$$\begin{aligned}\dim[\operatorname{Im}(KL)] &= \dim[\operatorname{Im} K_0] - \dim[\operatorname{Ker}(KL) \cap \operatorname{Im} K_0] \\ &= \dim(\operatorname{Im} K_0) - \dim(\operatorname{Ker} KL)\end{aligned}$$

yielding that $\dim[\operatorname{Im} K_0] = R^q$ or that K_0 is of full row rank. ■

III. Bringing a Transfer Matrix to Row Proper and DCDD Forms by a Constant Precompensator

Throughout this section, Z in $R(z)^{p \times m}$ is a strictly proper transfer matrix of full row rank. In this section, we resolve the following two problems introduced in Section I.

(P1) Determine L in $R^{m \times m}$ such that ZL is row proper.

(P2) Determine K in $R^{m \times p}$ that ZK is DCDD.

A solution L^* to (P1) is called *maximal* iff for any solution L to (P1) one has deg (ith row of ZL) $\le$ deg (ith row of ZL^*). A maximal solution K^* to (P2) is similarly defined. The existence of maximal solutions to both problems will also be established in this section. As mentioned in Section I, the problem (P1) is equivalent to finding a restricted state feedback such that the closed-loop transfer matrix is row decoupled (i.e., it is nonsingular and diagonal). The problem (P2), on the other hand, is equivalent to the diagonalization problem by dynamic output feedback and constant precompensator (P) by a main result of [12]. We now restate this result in our terminology.

Lemma (3.1): The problem (P) is solvable if and only if (P2) is solvable. ii) If K is a solution to (P2), then the pair

$$(K, Zc) := (K, -[(ZK)^{-1}]_{\text{off}})$$

is a solution to (P).

Proof: By [12, Lemma (1)], (P) has a solution iff there exists on $m \times p$ constant matrix K such that ZK is nonsingular and DCDD. The expression for the solution to (P) follows by the proof of the same lemma. ■

In the subsequent paragraphs, we first obtain an alternative solution to those of [8] and [4] for (P1) by vigorously exploiting the notion of "essential row form." The main reason for our interest in this alternative solution is that it is most suitable for obtaining a solution to (P2).

Let us define a sequence $\{N_i : i \ge 1\}$ of matrices in $R^{m \times m}$ and a sequence $\{V_i : i \ge 1\}$ of subspaces of R^m as follows.

$$\operatorname{Im}(N_1) = N_e(Z_h) \text{ and } \operatorname{Im} N_{i+1} = N_e(ZN_1 \cdots N_i) \quad \text{for } i \ge 1. \tag{3.2a}$$

$$V_1 = \operatorname{Im} N_1 \text{ and } V_i = \operatorname{Im}(N_1 \cdots N_i) \quad \text{for } i \ge 1. \tag{3.2b}$$

We first observe that the subspace V_i is independent of the choice of matrices $\{N_j : j \le i\}$ for all i. We prove this fact by the following lemma.

Lemma (3.3): Let $\{N_i : i \ge 1\}$ and $\{M_i : i \ge 1\}$ be two sequences of matrices in $R^{m \times m}$ satisfying (3.2a). Then

$$\operatorname{Im}(N_1 \cdots N_i) = \operatorname{Im}(M_1 \cdots M_i) \quad \text{for all } i \ge 1.$$

Proof: The required equality will be established by induction on i. For $i = 1$, the equality clearly holds. If, for some $j \ge 1$, one has Im $(N_1 \cdots N_j) =$ Im $(M_1 \cdots M_j)$, then $M_1 \cdots M_j = N_1 \cdots N_j K$ for some nonsingular K in $R^{m \times m}$. It follows on comparing the standard representations of $ZN_1 \cdots N_j$ and $ZM_1 \cdots M_j$ that $(ZM_1 \cdots M_j)_h = (ZN_1 \cdots N_j)_h K$. Since Im $N_{j+1} = N_e(ZN_1 \cdots N_j)_h$ and Im $M_{j+1} = N_e(ZM_1 \cdots M_j)_h$ by definition, we further have that $(ZN_1 \cdots N_j)_h K M_{j+1}$ is in erf, by Lemma (2.1-iv). Hence, by Lemma (2.1-iii), Im $(KM_{j+1}) \subseteq N_e(ZN_1 \cdots N_j)_h =$ Im N_{j+1}. Using nonsingularity of K, we similarly obtain Im $K^{-1}N_{j+1} \subseteq$ Im M_{j+1}. Therefore, Im $(KM_{j+1}) =$ Im N_{j+1}, i.e., $M_{j+1} = K^{-1}N_{j+1}L$ for some nonsingular L in $R^{m \times m}$; yielding $M_1 \cdots M_{j+1} = N_1 \cdots N_{j+1}L$ or Im $(M_1 \cdots M_{j+1}) =$ Im $(N_1 \cdots N_{j+1})$. ■

As $\{V_i : i \ge 1\}$ is a decreasing sequence, i.e., $V_i \supseteq V_{i+1}$ for all $i \ge 1$, there is an integer $r \le \dim V_1 \le m$ at which $V_r = V_{r+1}$.

Lemma (3.4): The following statements are equivalent:

i) $V_r = V_{r+1}$;

ii) $(ZN_1 \cdots N_r)_h$ is in erf;

iii) $ZN_1 \cdots N_r$ and $ZN_1 \cdots N_r N_{r+1}$ have the same row degrees.

Proof: [ii) implies i)] By ii) and (2.1-ii), N_{r+1} is nonsingular yielding i). [i) implies iii)] By i), $N_1 \cdots N_r N_{r+1} = N_1 \cdots N_r M$ for some nonsingular constant M implying $ZN_1 \cdots N_{r+1} = ZN_1 \cdots N_r M$. This last equality clearly yields the row degree equality as M is nonsingular. [iii) implies ii)] By iii), the row degrees of $ZN_1 \cdots N_r$ remain unchanged when multiplied by N_{r+1} on the right. It must be that $(ZN_1 \cdots N_r)_h$ is in erf since otherwise, by the definition of N_{r+1} and by (2.1-iii), there would have been degree reductions in those rows of $ZN_1 \cdots N_{r+1}$ that correspond to the nonessential rows in $(ZN_1 \cdots N_r)_h$. ■

This result implies that $V_j = V_{j+1}$ for all $j \ge r$, since by Lemma (2.1-ii) N_{r+1}, as well as N_j for $j \ge r + 1$, have to be nonsingular. We have thus shown that the decreasing sequence of subspaces $\{V_i : i \ge 1\}$ converge to V_r in at most m steps. We can now give a solution to (P1).

Theorem (3.5): The problem (P1) has a solution if and only if

$$\dim[V_r/(V_r \cap \operatorname{Ker} Z)] = p.$$

In case (PI) is solvable, a maximal solution L^* is given by $L^* = N_1 \cdots N_r$ which is related to any other solution L of (P1) by $L = L^*M$, for some M in $R^{m \times m}$.

Proof: Let L be a solution to (P1) for Z, i.e., L in $R^{m \times m}$ is such that $(ZL)_h$ has full row rank. By Lemma (2.2), $Z_h L$ is in erf and by Lemma (2.1-iii) $L = N_1 M_1$ for some M_1 in $R^{m \times m}$. Now, M_1 is a solution to (P1) for the transfer matrix ZN_1 implying as above that $M_1 = N_2 M_2$ for some M_2 in $R^{m \times m}$. Thus, by induction, we have $L = N_1 \cdots N_i M_i$, for some M_i in $R^{m \times m}$ and for any $i \geq 1$. In particular, $L = N_1 \cdots N_r M_r$. On the other hand, as $ZL = ZN_1 \cdots N_r M_r$, and hence, $ZN_1 \cdots N_r$ has full row rank, we further have

$$p = \dim [\mathrm{Im}\,(ZN_1 \cdots N_r)] = \dim [\mathrm{Im}\,(N_1 \cdots N_r)] - \dim [\mathrm{Ker}\, Z \cap \mathrm{Im}\,(N_1 \cdots N_r)]$$

or equivalently, $\dim [V_r/(V_r \cap \mathrm{Ker}\, Z)] = p$. Conversely, suppose $p = \dim [V_r/(V_r \cap \mathrm{Ker}\, Z)]$ so that $\dim [\mathrm{Im}\,(ZN_1 \cdots N_r)] = p$. Consider $L^* = N_1 \cdots N_r$ for which ZL^* has full row rank. By Lemma (3.4), we know further that $(ZL^*)_h$ is in erf. Consequently, $(ZL^*)_h$ must have all its rows essential, i.e., $(ZL^*)_h$ must have full row rank. It follows that L^* is a solution to (P1). The constructed solution L^* is related, by above, to any other solution L by $L = L^* M_r$, for some M_r in $R^{m \times m}$. Now, L^* is a "maximal" solution to (P1) in the sense that the ith row degree of ZL^* is greater or equal to the ith row degree of ZL for a solution L to (P1). ∎

Given any strictly proper transfer matrix Y of size $p \times t$, let $L(Y)$ denote a basis matrix for the limit $V_r(Y)$ as i increases of the sequence $\{V_i(Y); i \geq 1\}$, where $V_i(Y)$ is the subspace given by (3.2) in which Z is replaced by Y. Theorem (3.5) shows that a maximal solution to (P1) for Y is given by the completion $[L(Y):0]$ of $L(Y)$ to a $t \times t$ square matrix provided $\dim \{V_r(Y)/[V_r(Y) \cap \mathrm{Ker}\, Y]\}$ is equal to p.

We now consider (P2) for Z. Let us define two sequences of constant matrices $\{L_i : i \geq 1\}$ and $\{K_i : i \geq 1\}$ by

$$L_1 = L(Z),\ K_1 = K(ZL_1),$$

$$L_{i+1} = L(ZL_1 \cdots L_i K_i),\ K_{i+1} = K(ZL_1 \cdots L_i K_i L_{i+1}); \qquad i \geq 1 \tag{3.6}$$

and a sequence of strictly proper transfer matrices $\{Y_i : i \geq 0\}$ by

$$Y_0 := Z,\ Y_i := ZL_1 \cdots L_i K_i; \qquad i \geq 1$$

where $K(Y)$ for a transfer matrix Y has been defined prior to Theorem (2.8) in Section II. The decreasing sequence of subspaces

$$\{S_i := \mathrm{Im}\,(L_1 K_1 \cdots L_i K_i) : \ i \geq 1\}$$

will play a crucial role in the resolution of (P2).

We first show that S_i is independent of the choice of matrices $\{L_j : 1 \leq j \leq i\}$ and $\{K_j : 1 \leq j \leq i\}$.

Lemma (3.7): If $\{\underline{L}_i : i \geq 1\}$ and $\{\underline{K}_i : i \geq 1\}$ are two sequences of matrices also satisfying (3.6), then

$$\mathrm{Im}\,(\underline{L}_1 \underline{K}_1 \cdots \underline{L}_i \underline{K}_i) = \mathrm{Im}\,(L_1 K_1 \cdots L_i K_i); \qquad i \geq 1.$$

Proof: Let us show that given a $p \times t$ transfer matrix Y and a nonsingular constant matrix M, one has

$$L(YM) = M^{-1} L(Y) H, \tag{3.8}$$

$$K(YM) = M^{-1} K(Y) J \tag{3.9}$$

for some constant nonsingular matrices H and J. To see the first equality, note by (3.2) that $V_r(YM) = M^{-1} V_r(Y)$ for any constant nonsingular M. Thus, any basis matrices for $V_r(Y)$ and $V_r(YM)$ are related by (3.8) as claimed. To see (3.9), let $K(Y) = [K_1 : \cdots : K_p]$, where K_i is the submatrix consisting of the constant columns of a minimal polynomial basis matrix $[K_i : P_i]$ for the kernel of $[D_{ii} D Y^{(i)}]^+$ with D being the row degree matrix of Y. Since D is also the row degree matrix of YM for any nonsingular constant matrix M and since $(YM)^{(i)} = Y^{(i)} M$, it follows that $M^{-1}[K_i : P_i]$ is a minimal polynomial basis of $\mathrm{Ker}\, [D_{ii} D (YM)^{(i)}]^+$. Note that $M^{-1} K_i$ is the submatrix consisting of the constant columns of $M^{-1}[K_i : P_i]$. Consequently, $M^{-1} K(Y)$ is a possible choice for $K(YM)$. By the invariance of $\mathrm{Im}\, K_i$ under different choices of minimal polynomial bases for $\mathrm{Ker}\, [D_{ii} D Y^{(i)}]^+$, it follows that $K(Y)$ and $K(YM)$ are related as in (3.9) for a nonsingular "block diagonal" constant matrix J. We can now prove the lemma by induction on i. Let $i = 1$ and note that $\underline{L}_1 = L_1 M_1$ for some constant nonsingular M_1 as $\underline{L}_1$ and L_1 are basis matrices for $V_r(Z)$. By (3.9) this implies that $\underline{K}_1 = M_1^{-1} K_1 J_1$ for some constant nonsingular J_1 as $\underline{K}_1 = K(ZL_1 M_1)$ and $K_1 = K(ZL_1)$. Thus, $\underline{L}_1 \underline{K}_1 = L_1 K_1 M_1$ and the lemma holds for $i = 1$. If $\underline{L}_1 \underline{K}_1 \cdots \underline{L}_j \underline{K}_j = L_1 K_1 \cdots L_j K_j M_j$ for some $j \geq 1$, then by definition $\underline{L}_{j+1} = L(\underline{Y}_j)$, $L_{j+1} = L(Y_j M_j)$, $\underline{K}_{j+1} = K(\underline{Y}_j \underline{L}_{j+1})$, $K_{j+1} = K(Y_j L_{j+1})$. By (3.8) and (3.9), it follows that $\underline{L}_{j+1} = M_j^{-1} L_{j+1} H_j$ and $\underline{K}_{j+1} = H_j^{-1} K_{j+1} J_j$ for some constant nonsingular H_j and J_j. This yields $\underline{L}_1 \underline{K}_1 \cdots \underline{L}_{j+1} \underline{K}_{j+1} = L_1 K_1 \cdots L_{j+1} K_{j+1} J_j$ proving the lemma for $i = j + 1$. ∎

Lemma (3.10): The following statements hold: i) If Y_q and Y_{q+1} have the same row degrees, then $(Y_q)_h$ and $(Y_{q+1})_h$ are both in erf. ii) If $S_q = S_{q+1}$, then Y_q and Y_{q+1} have the same row degrees. iii) If $S_q = S_{q+1}$, then $S_i = S_j$ for all i, j such that $q \leq i \leq j$.

Proof: i) Since Y_q and Y_{q+1} have the same row degree matrix, it follows by their standard representations that

$$(Y_{q+1})_h = (Y_q)_h L_{q+1} K_{q+1}.$$

By the definition of L_{q+1}, we have that $\mathrm{Im}\,(L_{q+1} K_{q+1}) \subseteq N_e(Y_q)$ which implies by Lemma (2.1-iii) that all nonessential rows of $(Y_q)_h$ are zero in $(Y_{q+1})_h$. By the definition of the highest coefficient matrix and by the equality of the row degrees again, there cannot be any nonzero and nonessential rows of $(Y_q)_h$, i.e., $(Y_q)_h$ is in erf. It also follows that $(Y_{q+1})_h$ is in erf since the existence of a nonzero nonessential row in $(Y_{q+1})_h$ implies that the corresponding row in $(Y_q)_h$ is nonzero and nonessential. [ii), iii)] If $S_q = S_{q+1}$, then $L_1 K_1 \cdots L_q K_q = L_1 K_1 \cdots L_{q+1} K_{q-1} M$ for some full row rank constant matrix M. Thus, $Y_q = Y_{q+1} M$ implying that the ith row degree of Y_{q+1} is less than or equal to the ith row degree of Y_q. On the other hand, $Y_{q+1} = Y_q L_{q+1} K_{q+1}$, which yields the equality of row degrees. By part i) of this lemma, $(Y_{q+1})_h$ is in erf which implies by Lemma (2.1-ii) and the definition of L_{q+2} that L_{q+2} is nonsingular. Now by Lemma (2.9), K_{q+2} is of full row rank. Consequently, $\mathrm{Im}\,(L_1 K_1 \cdots L_{q+1} K_{q+1}) = \mathrm{Im}\,(L_1 K_1 \cdots L_{q+2} K_{q+2})$, or $S_{q+1} = S_{q+2}$. By induction, the statement ii) follows. ∎

As a consequence of this lemma, it follows that the decreasing sequence of subspaces $\{S_i : i \geq 1\}$ converges to S_q in at most $q \leq \dim\,(\mathrm{Im}\, L_1) \leq m$ steps.

We can now state the first main result of the paper.

Theorem (3.11): The problem (P2) has a solution if and only if

$$\dim [S_q/(S_q \cap \mathrm{Ker}\, Z)] = p.$$

In case (P2) is solvable, there exists a maximal solution K^* which is given by $K^* = L_1 K_1 \cdots L_{q+1} K_{q+1} U$ for some constant matrix U satisfying $(Y_q L_{q+1})_h K_{q+1} U = I$.

Proof: Let K be a solution to (P2) so that ZK is DCDD (and row proper) and the completion $[K:0]$ of K to an $m \times m$ matrix is a solution to (P1) for Z. By Theorem (3.5), there exists J_1 in $R^{m \times p}$ such that $K = L_1 J_1$. Note that J_1 is a solution to (P2) for ZL_1 implying by Lemma (2.5) that $[D_{ii} D (ZL_1)^{(i)}]^+ j_i = 0$ for each column j_i of j_1, where D is the row degree matrix of $ZK = ZL_1 J_1$. Let E be the row degree matrix of ZL_1 and note that $E^{-1} D$ is polynomial and ED^{-1} is proper. Hence,

$$[E_{ii} E (ZL_1)^{(i)}]^+ = \{E_{ii} ED_{ii}^{-1} D^{-1} [D_{ii} D (ZL_1)^{(i)}]^+\}^+$$

where $E_{ii} ED_{ii}^{-1} D^{-1}$ is proper. This implies that $[E_{ii} E (ZL_1)^{(i)}]^+ j_i = 0$ for each column j_i of J_1. Therefore, $\mathrm{Im}\, J_1$ is in $\mathrm{Im}\, K_1$, i.e., $J_1 = K_1 M_1$ for some M_1. We now observe that M_1 is a solution to (P1) for the transfer matrix $ZL_1 K_1$. By induction, it follows that

$$K = L_1 K_1 \cdots L_i K_i M_i; \qquad i \geq 1$$

for constant matrices M_i. Since Im $(L_1 \cdots L_iK_i) \supseteq$ Im $(L_1 \cdots L_{i+1}K_{i+1})$ at some integer q, we must have the equality for subsequent images. Since ZK is nonsingular, S_q = Im $(L_1 \cdots L_qK_q)$ is nonzero and in point of fact, dim $[S_q/(S_q \cap \text{Ker } Z)] = p$ as rank $(ZL_1 \cdots L_qK_q) = p$. Conversely, if this equality holds, then rank $(Y_q) = p$ and since $S_q = S_{q+1}$ we have, by Lemma (3.10), that Y_q and

$$Y_{q+1} = ZL_1K_1 \cdots L_{q+1}K_{q+1} = Y_qL_{q+1}K_{q+1}$$

have the same row degrees and that $(Y_{q+1})_h$ is in erf. Hence, rank (Y_q) = rank $(Y_{q+1}) = p$ and both transfer matrices are row proper with the same row degrees. By row properness of Y_{q+1}, it holds that rank $(Y_{q+1})_h$ = rank $[(Y_qL_{q+1})_hK_{q+1}] = p$. By Theorem (2.8), there exists a constant matrix U such that $(Y_qL_{q+1})_hK_{q+1}U = I$ and $K_{q+1}U$ is a solution to (P2) for Y_qL_{q+1} which has the same row degrees as $Y_{q+1}U$. Clearly, $K^* := L_1K_1 \cdots L_{q+1}K_{q+1}U$ is a solution to (P2) for Z. This solution is maximal since by the "only if" part of this proof, any solution K to (P2) for Z is given by $K = L_1K_1 \cdots L_{q+1}K_{q+1}M_{q+1}$ for some M_{q+1}. The ith row degree of ZK is less than or equal to the ith row degree of Y_qL_{q+1} by this expression for K which in turn is equal to the ith row degree of ZK^*. ■

An Example: Consider the 3×5 transfer matrix

$$Z = \begin{bmatrix} (z+1)/z^2 & 1/z & (z^3+1)/z^6 & 1/z^2 & 1/z^5 \\ 1/z^2 & 1/z^2 & (z^2+1)/z^5 & 1/z^3 & 1/z^4 \\ 0 & 1/z^2 & 0 & 1/z^3 & 1/z^5 \end{bmatrix}.$$

A basis matrix for Ker Z is given by the columns of

$$W = \begin{bmatrix} 0 & \alpha(z) \\ \beta(z)/z & \beta(z)/z^3 \\ 0 & -(z+1)/z^5 \\ -\beta(z) & 0 \\ 0 & -\beta(z) \end{bmatrix}$$

where $\alpha(z) := [(z+1)(2z^3+z+1)]/z^{10}$ and $\beta(z) := (1 - z - z^2)/z^7$. We first check the solvability of (P1). Note that

$$Z_h = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and V_1 = Im (N_1). It is easy to see that dim (Ker $Z \cap V_1$) = dim V_1 + dim (Ker Z) − dim (Im$[W{:}N_1]$) = 4 + 2 − 5 = 1 yielding dim $[V_1/(\text{Ker } z \cap V_1)] = 4 - 1 = 3 = p$. Therefore, by Theorem (3.5), (P1) is solvable for Z and, by the sufficiency part of the same theorem, a maximal solution is $L^* = N_1$ yielding

$$ZL_* = \begin{bmatrix} 1/z^2 & (z^3+1)/z^6 & 1/z^2 & 1/z^5 & 0 \\ 0 & (z^2+1)/z^5 & 1/z^3 & 1/z^4 & 0 \\ -1/z^2 & 0 & 1/z^3 & 1/z^5 & 0 \end{bmatrix}.$$

We now check the solvability of (P2) for Z. The row degree matrix of $Y_1 = ZL^* = ZL_1$ is D = diag $\{z^2, z^3, z^2\}$. We thus have

$$K_1 = [K_1 : K_2 : K_3] = \begin{bmatrix} 0 : 0\,0 : 0 \\ 0 : 0\,0 : 0 \\ 0 : 0\,0 : 0 \\ 0 : 1\,0 : 0 \\ 1 : 0\,1 : 1 \end{bmatrix}$$

where K_i is the matrix consisting of the constant columns of a minimal polynomial basis of Ker $[ZD_{ii}D(ZL^*)^{(i)}]^+$ for i = 1, 2, 3. It thus turns out that

$$S_1 = \text{Im } (L_1K_1) = \text{Im} \begin{bmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 1\,0\,1\,1 \end{bmatrix} = \text{Im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and dim $(S_1 \cap \text{Ker } Z) = 0$ yielding dim $[S_1/(S_1 \cap \text{Ker } Z)] = 2 < p$. Therefore, by Theorem (3.11), there is no solution to (P2) for Z.

Remark 2: Our technique of solution to (P2) implies the following directions concerning generalizations i), ii), and iv) mentioned in Remark 1. i) If Z_f is required to be block diagonal with specified block sizes, then one has to generalize the notions of "essential rows" and "DCDD" to, respectively, "essential block rows" and "block-DCDD"; both of which can actually be done. Thus, in principle, the generalization i) is possible although at the time of this writing these results are notationally too complex to state here. ii) This problem has been investigated in [12, Theorem 2] and in the case Z is row proper, a similar degree-dominance condition to DCDD reduces the problem again to an open loop, constant precompensator problem. The solvability of the problem (P1) is not necessary, however, for the solvability of this problem. Consequently, a nontrivial but plausible extension of the above method is called for in order to solve the general problem ii). Similar remarks apply to problem iv), the measured output case.

IV. Set of All Solutions

Let (P1) and (P2) be solvable for Z throughout this section. In Theorems (3.5) and (3.11), we have determined maximal solutions to (P1) and (P2). Since there are actually an infinite number of solutions to both problems, it would be of interest to examine the solution sets

$$L(Z) := \{L \text{ in } R^{m \times m} : ZL \text{ is row proper}\}$$

$$K(Z) := \{K \text{ in } R^{m \times p} : ZK \text{ is DCDD}\}$$

more closely. We now show that it is possible to determine all the elements of sets $L(X)$ and $K(Z)$ by an essentially finite process.

Consider the quotient sets $L_Q(Z) := \{[L]\}$ and $K_Q(Z) := \{[K]\}$ with respect to the equivalence relation "$L_1 \equiv L_2$ (respectively, $K_1 \equiv K_2$) iff ZL_1 and ZL_2 (respectively, ZK_1 and ZK_2) have the same row degrees." Obviously, each equivalence class $[L]$ and $[K]$ has infinitely many elements since if ZL is row proper, then ZLN is row proper with the same row degrees for any nonsingular N in $R^{m \times m}$ and if ZK is DCDD, then ZKM_d is also DCDD with the same row degrees for any diagonal nonsingular M_d in $R^{p \times p}$. On the other hand, the quotient sets $L_Q(Z)$ and $K_Q(Z)$ are finite as we show below.

We first review a few facts established in [13]. Given a strictly proper $p \times m$ transfer matrix Z of full row rank, an open-loop diagonalizer of Z is an $m \times p$ proper rational matrix T satisfying $ZT = \Gamma$, where Γ is a nonsingular (strictly proper) diagonal matrix. Such a Z always admits an open-loop diagonalizer and, in particular, there are some which yield the maximum causality degrees for the nonzero entries (Γ_{ii}) of Γ. These are called the *maximal open-loop diagonalizers of* Z. Thus, if Z_M is a maximal open-loop diagonalizer of Z, then $ZZ_M = \Gamma_M$ for a diagonal strictly proper matrix Γ_M which has deg $(\Gamma_M)_{ii}$ equal to the negative of the ith essential order [5] e_i; $i = 1, \cdots, p$. It can easily be shown [13] that there exists a maximal open-loop diagonalizer T_m of Z such that $\Gamma_m = ZT_m$ is in the form

$$\Gamma_m = \text{diag } \{z^{-e_1}, \cdots, z^{-e_p}\}$$

where the nonnegative integers $\{e_1, \cdots, e_p\}$ are the essential orders of Z. By [13, Lemma 1], Γ_m^{-1} is proper for any open-loop diagonalizer T of Z such that $ZT = \Gamma$.

Let E_m be a diagonal polynomial matrix such that

$$P_m := E_m Z$$

is a polynomial matrix and deg $(E_m)_{ii}$ is minimum for each $i = 1, \cdots, p$. Thus, $f_i :=$ deg $(E_m)_{ii}$ is the *McMillan degree* of the ith row of Z.

Theorem (4.1): If L is in $L(Z)$, then

$$e_i \le -\deg (ZL)_{ii} \le f_i; \qquad i = 1, \cdots, p.$$

Proof: Let D be the row degree matrix of ZL so that $DZL =: B$ is biproper. Thus, LB^{-1} is an open-loop diagonalizer of Z and, by the previous paragraph, Γ_m^{-1} is proper, or equivalently, $D\Gamma_m$ is polynomial, i.e.,

$$\deg (D)_{ii} = -\deg (ZL)_{ii} \ge -\deg (\Gamma_m)_{ii} = e_i; \qquad i = 1, \cdots, p.$$

Further, $P_m L = BE_m D^{-1}$ implying that deg $(D^{-1}E_m)_{ii}$ = deg $(P_m L)_{ii} \ge 0$ for $i = 1, \cdots, p$ as B is biproper. Therefore,

$$\deg (D)_{ii} = -\deg (ZL)_{ii} \le \deg (E_m)_{ii} = f_i$$

yielding the result. ∎

A consequence of this theorem is that $L_Q(Z)$ is a finite set. Moreover, since the completion $[K{:}0]$ of any K in $K(Z)$ to a square matrix is in $L(Z)$, finiteness of $L_Q(Z)$ implies that $K_Q(Z)$ is also finite. Also note that Theorem (4.1) yields upper and lower bounds on row degrees of ZL for any solution L to (P1). This is also a bound on the row degrees of ZK for any solution K to (P2).

We now turn to a characterization of the equivalence class $[L]$ and show that $[L]$ has a representative element L_g ("g" stands for "generator") which can be determined by a finite algorithm.

Theorem (4.2): Every equivalence class $[L]$ in $L_Q(Z)$ has an element L_g with the property "L is in $[L]$ if and only if $L = L_g T$ for some constant T such that rank $[(ZL_g)_h T] = p$." This element is of the form

$$L_g = L^* M_{i(1)} L^{*i(i)} \cdots M_{i(k)} L^{*i(k)}$$

where the index $i(j)$ is in $\{1, \cdots, p\}$ for $j = 1, \cdots, k$ with $k \le m - p$, $m_i(j)$ is a basis matrix for the kernel of the $i(j)$th row of

$$[ZL^* M_{i(1)} L^{*i(1)} \cdots M_{i(j-1)} L^{*i(j-1)}]_h$$

and $L^{*i(j)}$ is a maximal solution to (P1) for the transfer matrix

$$ZL^* M_{i(1)} L^{*i(1)} \cdots M_{i(j-1)} L^{*i(j-1)} M_{i(j)}$$

with L^* being a maximal solution to (P1) for Z.

Proof: Let L be in $[L]$ with ZL having the row degree matrix D. By Theorem (3.5), $L = L^*T$ for some constant T. If $\deg_i (ZL) = \deg_i (ZL^*)$ for all $i = 1, \cdots, p$, then both L and L^* are in $[L]$. By the fact that ZL is row proper with the same row degrees as ZL^*, rank $[(ZL^*)_h T] = p$. Thus, in this case $L_g := L^*$. If $\deg_{i(1)} (ZL) < \deg_{i(1)} (ZL^*)$ for some $i(1)$ in $\{1, \cdots, p\}$, then $L = L^* M_{i(1)} T_1$ for some constant matrix T_1 since $(ZL^*)_h T$ has its $i(1)$th row zero. Now, T_1 is a solution to (P1) for $ZL^* M_{i(1)}$ and, by Theorem (3.5), $T_1 = L^{*i(1)} S_1$ for some S_1 in $R^{m \times m}$. If ZL and $ZL^* M_{i(1)} L^{*i(1)}$ have the same row degree matrix, then $L^* M_{i(1)} L^{*i(1)}$ is in $[L]$. Otherwise, $\deg_{i(2)} [ZL^* M_{i(1)} L^{*i(1)}] < \deg_{i(2)} (Zl)$ for some index $i(2)$ in $\{1, \cdots, p\}$. Hence, $S_1 = M_{i(2)} T_2$ for some constant T_2 as, by the last degree inequality, $[ZL^* M_{i(1)} L^{*i(1)}]_h S_1$ has its $i(2)$th row zero. Moreover, $T_2 = L^{*i(2)} S_2$ for some S_2 since T_2 is a solution to (P1) for $ZL^* M_{i(1)} L^{*i(1)} M_{i(2)}$. We now claim that

$$\text{rank } [L^* M_{i(1)} L^{*i(1)}] > \text{rank } [L^* M_{i(1)} L^{*i(1)} M_{i(2)}]. \tag{4.3}$$

To see this, note that $[ZL^* M_{i(1)} L^{*i(1)}]_h = A_1 L^* M_{i(1)} L^{*i(1)}$ for some constant A_1, by (2.3); which has rank p by definition of $L^{*i(1)}$. Thus, p = rank $[A_1 L^* M_{i(1)} L^{*i(1)}]_h$ = dim $\{\text{Im } [L^* M_{i(1)}]\}$ $-$ dim $\{\text{Ker } A_1 \cap \text{Im } [L^* M_{i(1)} L^{*i(1)}]\}$. Since $M_{i(2)}$ is a basis matrix for the kernel of the $i(2)$th row of $[ZL^* M_{i(1)} L^{*i(1)}]_h$, we have

$$\begin{aligned} p &> \text{rank } [A_1 L^* M_{i(1)} L^{*i(1)} M_{i(2)}] \\ &= \dim \{\text{Im } [L^* M_{i(1)} L^{*i(1)} M_{i(2)}]\} \\ &\quad - \dim \{\text{Ker } A_1 \cap \text{Im } [L^* M_{i(1)} L^{*i(1)} M_{i(2)}]\} \\ &\ge \dim \{\text{Im } [L^* M_{i(1)} L^{*i(1)} M_{i(2)}]\} \\ &\quad - \dim \{\text{Ker } A_1 \cap \text{Im } [L^* M_{i(1)} L^{*i(1)}]\} \\ &= \text{rank } [L^* M_{i(1)} L^{*i(1)} M_{i(2)}] \\ &\quad + p - \text{rank } [L^* M_{i(1)} L^{*i(1)}] \end{aligned}$$

proving (4.3). Consequently, by successive application of the process above, in at most $m - p$ steps (i.e., $k \le m - p$) we arrive at

$$L = L^* M_{i(1)} L^{*i(1)} \cdots M_{i(k)} L^{*i(k)} S_k$$

for some constant S_k, where ZL and ZL_g have the same row degree matrix with $L_g := L^* M_{i(1)} L^{*i(1)} \cdots M_{i(k)} L^{*i(k)}$. The matrix L_g satisfies rank $[(ZL_g)_h S_k] = p$ as both ZL and ZL_g are row proper with the same row degrees. Conversely, given any constant matrix T such that rank $[(ZL_g)_h T] = p$, then $L := L_g T_0$ is in $[L]$ provided T_0 is chosen as either a submatrix or a completion $[T{:}0]$ of T having m columns such that rank $[(ZL_g)_h T_0] = p$. ∎

Theorem (4.2) yields, in effect, a finite algorithm to determine all elements of $L(Z)$ up to right multiplication by nonsingular matrices as follows. Let L^* be a maximal solution to (P1) achieving the row degree matrix D^* for ZL^*. Clearly, L^* is a generator for the equivalence class $[L^*]$. Determine all possible values for $i(1)$; these are the indexes in $\{1, \cdots, p\}$ such that (P1) has a solution for $ZL^* M_{i(1)}$. Note that if (P1) is unsolvable for any of $ZL^* M_i (i = 1, \cdots, p)$, then $[L^*]$ is the only element of $L_Q(Z)$ by the result of the theorem. Otherwise, replace Z by $ZL^* M_{i(1)}$ and repeat the above process. Since the number of choices for the ordered set $\langle i(1), i(2), \cdots, i(k)\rangle$ with $i(j)$ in $\{1, \cdots, p\}$ and $k \le m - p$ is finite, the above procedure yields all generators L_g in a finite number of steps.

Given an equivalence class $[K]$ in $K_Q(Z)$, let L_g be a generator for $[[K{:}0]]$ considered as an element of $L_Q(Z)$. Let

$$\boldsymbol{K} := \boldsymbol{K}(ZL_g) = [K_1 : \cdots : K_p\}$$

and let $U :=$ diag $\{u_i\}$ be such that $(ZL_g)_h \boldsymbol{K} U = I$. Such a constant matrix U exists by the fact that ZK is DCDD and by Theorem (2.8).

Theorem (4.4): All elements of $[K]$ are of the form

$$K = L_g \boldsymbol{K}(U + V) W_d$$

for some constant matrices V and W_d such that

$$\text{Im } (\boldsymbol{K} V_j) \subseteq \text{Ker } (ZL_g)_h \cap \text{Im } K_j; \qquad j = 1, \cdots, p$$

where V_j is the jth column of V and W_d is diagonal, nonsingular.

Proof: Let ZK be DCDD and have the same row degrees as ZL_g so that $K = L_g T$ for some constant matrix T by the fact that $[K{:}0]$ is in $[L_g]$ and by Theorem (4.2). By Theorem (2.8), $T = \boldsymbol{K}H$ for some constant matrix H = diag $\{h_1, \cdots, h_p\}$ such that $(ZL_g)_h K_i h_i = e_i w_i$, where w_i is a nonzero real number. By the definition of U, we also have that $(ZL_g)_h K_i u_i = e_i$ so that $(ZL_g)_h K_i v_i = 0$ with $v_i := h_i w_i^{-1} - u_i$ for $i = 1, \cdots, p$. Hence, $(ZL_g)_h \boldsymbol{K} V_j = 0$ for $j = 1, \cdots, p$ where V_j is the jth column of $V :=$ diag $\{v_1, \cdots, v_p\}$. It follows that Im $(\boldsymbol{K} V_j) \subseteq$ Ker $(ZL_g)_h$

and, by the special form of V_j, $\mathrm{Im}\,(KV_j) \subseteq \mathrm{Im}\,(K_j)$. This proves the claimed inclusion and with $W_d := \mathrm{diag}\,\{w_1, \cdots, w_p\}$, we also have $K = L_g KHW_d = L_g K(U + V)W_d$. Conversely, given a K in this form, by the definition of K and by Lemma (2.5), it easily follows that ZK is DCDD with the same row degrees as ZL_g. ∎

We conclude this section giving an explicit description of the set of all solutions to Problem (P). For this, we need a characterization of the set of all solutions to (P) for the case $K = I$ and Z nonsingular.

Lemma (4.5): Let Z be square and nonsingular and fix $K := I$. If Z is DCDD, then the set of all solutions to (P) is given by: $\{Z_c\} = \{Y_d(Z^{-1})_d - (Z^{-1})_{\mathrm{off}} : Y_d$ is diagonal and strictly proper rational$\}$.

Proof: If Z is DCDD, then it follows that $(Z^{-1})_{\mathrm{off}}$ is proper and $(Z^{-1})_d$ is nonsingular and biproper. Given any Z_c in the set, we have $Z(I + Z_cZ)^{-1} = Z[I + Y_d(Z^{-1})_dZ - (Z^{-1})_{\mathrm{off}}Z]^{-1} = [Z^{-1} + Y_d(Z^{-1})_d - (Z^{-1})_{\mathrm{off}}]^{-1} = [(Z^{-1})_d + Y_d(Z^{-1})_d]^{-1} = [(Z^{-1})_d]^{-1}(I + Y_d)^{-1}$, where the last expression is diagonal, nonsingular (and proper by strict properness of Y_d). Thus, every element of the set is a solution to (P). On the other hand, if $Z(I + Z_cZ)^{-1} = X_d$ for some nonsingular diagonal X_d, then $(Z_c)_d = (X_d)^{-1} - (Z^{-1})_d$, and hence $(Z^{-1})_dX_d = I - (Z_c)_dX_d$ with $(Z_c)_dX_d$ strictly proper. Hence, $Y_d := (X_d)^{-1}[Z^{-1})_d]^{-1} - I$ is strictly proper and it satisfies $Z_c = Y_d(Z^{-1})_d - (Z^{-1})_{\mathrm{off}}$. Therefore, any solution is in the above set. ∎

We can now describe the set of all solutions to (P).

Corollary (4.6): The set of all solutions of Problem (P) is given by: $\{(K, Z_c) : K$ is in $K(Z)$ and $Z_c = Y_d[(ZK)^{-1}]_d - [(ZK)^{-1}]_{\mathrm{off}}$, where Y_d is a diagonal strictly proper $p \times p$ rational matrix$\}$.

Proof: For any K in $K(Z)$, ZK is nonsingular and DCDD and it remains to describe the set of admissible Z_c such that $Z_f = (I + ZKZ_c)^{-1}ZK$ is diagonal and nonsingular. But, this is by Lemma (4.5) above. ∎

Remark 3: A procedure may be described for searching for a solution to the problem (P) with internal stability, as the problem iii) of Remark 3. Recall that the result of [6] yields a solution to this problem for a nonsingular Z. Corollary (4.6) yields an exhaustive characterization of all solutions to (P2). Thus, one can check whether a solution for iii) exists for various values of V of Theorem (4.4) and construct one by the synthesis method of [6] whenever it exists. However, note that this does not, yet, yield a solution to iii) since the set of all such V is not finite.

Remark 4: The construction of a solution to (P2) of Theorem (3.11) and the set of solutions to (P) of Corollary (4.6) can be presented in the format of an algorithm. This has been the main theme of [20] where an alternative exposition of the solvability condition to (P2) is also given.

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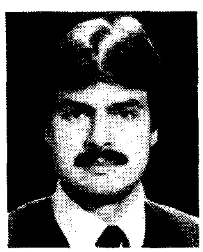

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