FORMAL GARCH PERFORMANCE IN A COMPUTABLE DYNAMIC GENERAL EQUILIBRIUM FRAMEWORK

A THESIS

Submitted to the Faculty of Management
and the Graduate School of Business Administration
of Bilkent University
in Partial Fulfillment of the Requirements
For the Degree of
Master of Science in Business Administration

ALİ BORA YİĞİTBAŞIOĞLU

ANKARA, AUGUST 1998

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I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science in Business Administration.

Assist. Prof. Aslihan Salih

I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science in Business Administration.

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Approved for the Graduate School of Business Administration.

Prof. Dr. Subidey Togan

ABSTRACT

FORMAL GARCH PERFORMANCE IN A COMPUTABLE DYNAMIC GENERAL EQUILIBRIUM FRAMEWORK

ALİ BORA YİĞİTBAŞIOĞLU
Master of Science in Business Administration

Supervisor: Assist. Prof. Aslıhan Salih
August, 1998

This study uses a Computable Dynamic General Equilibrium setting based on Brock's (1979, 1982) intertemporal growth and asset pricing models and applies this framework as a formal test to study the out-of-sample forecast performance of Bollerslev's (1986) GARCH (1,1) and Classical Historical Volatility forecasts. The solution to Brock's growth model reflects the utility maximizing behavior of the consumer and profit maximizing behavior of producers, and is a framework that has recorded some remarkable successes in mirroring the real economy. All existing studies have used a sample realized variance in the forecast horizon to test the out-of-sample performance of conditional variance forecasting models. The realized variance is simply an approximation to the true distribution of variance in the forecast horizon, and is often an unfair benchmark of performance. Simulation of Brock's model enables one to obtain the true distribution of asset returns and their variance at all times. The true distribution reflects all the possible states of a simulated economy, which is shown to mimic all the properties observed in empirical financial data. This framework affords the luxury of comparing the out-of-sample forecasts from various models with the true variance in the forecast horizon. The results jointly demonstrate that the GARCH (1,1) model performs significantly better than the Classical Historical Volatility when the true variance is used as the forecast comparison benchmark. It is concluded that the use of realized variance for out-of-sample performance is highly misleading, especially for short-run forecasts.

Key words: GARCH, Classical Historical Volatility Forecast, Out-of-sample forecast performance, Computable General Equilibrium Model, Benchmark, Realized Variance, True Variance.

ÖZET

HESAPLANABİLİR DİNAMİK GENEL DENGE ÇERÇEVESİNDE RESMİ GARCH PERFORMANSI

ALİ BORA YİĞİTBAŞIOĞLU
Master of Science in Business Administration

Tez Yöneticisi: Yrd. Doç. Dr. Aslıhan Salih
Ağustos, 1998

Bu çalısma, Brock'un (1979, 1982) Büyüme Modelini Hesaplanabilir Dinamik Genel Denge çerçevesinde kullanarak, Bollerslev'in (1986) GARCH ve Classical Historical Volatility modellerinin ileriye dönük tahmin performanslarını resmi bir test ortamında incelemektedir. Brock'un modelinin çözümü tüketicinin fayda fonksiyonunu maksimize edişini ve üreticilerin kârlarını maksimize edişlerini yansıtmakta, gerçek ekonomiyi çok yakından simüle etmektedir. Bütün akademik çalışmalar, tahmin penceresinde gerçekleşen varyans ölçüsünü şartlı volatilite modellerini değerlendirmekte kullanmaktadırlar. Fakat gerçekleşen varyans, gerçek varyans ölçüsünün sadece yaklaşık bir tahmini olmakla beraber, genellikle adil olmayan bir benchmark'dır. Brock'un modelini sımüle ederek gerçek varyansı bulmak mümkündür. Elde edilen gerçek varyans ekonominin mümkün olan bütün durumlarım özetlemektedir. Gerçek varyans ile karşılaştırıldıgında GARCH modelinin ileriye dönük performansının Classical Historical Volatility tahminlerinden çok daha iyi olduğu görülmektedir. Aynı anda, gerçekleşen varyansm performans benchmark'ı olarak kullanılışının yanıltıcılığıda sergilenmektedir.

Anahtar Kelimeler: GARCH, Classical Historical Volatility Tahminleri, İleriye Dönük Tahmin Performansı, Hesaplanabir Dinamik Genel Denge Modeli, Benchmark, Gerçekleşen Varyans, Gerçek Varyans.

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In acknowledging my intellectual indebtedness to the people mentioned above, I assume full responsibility for the shortcomings of this study.

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CHAPTER I

INTRODUCTION

Financial market volatility is of central importance for a variety of market participants. The literature is in agreement that financial market volatility is predictable (see Bollerslev, Chou and Kroner (1992), Engle (1993), Engle and Ng(1993)) and time-varying (Schwert (1989). This has important implications for portfolio and asset management (Merton, 1980), and the pricing of primary and derivative assets (Baillie and Myers(1991) and Engle, Hong, Kane, and Noh (1992)). Expected volatility is a key ingredient of the pricing mechanism of such diverse financial assets as bond and equity futures, options, and common stocks (Ritzman, 1991). Estimates about volatility are routinely used by participants in the derivatives markets for hedging.

Performance of different volatility forecasting models have been proposed and tested in the literature. Among all the volatility forecasting models, the GARCH family has received the greatest attention in the literature. ARCH and GARCH modeling in finance has enjoyed a prominent and varied history in the literature since its inception by Engle (1982) and generalization by Bollerslev (1986). It is an indispensable, often integral, part of many textbooks devoted to finance and econometrics. An overview of ARCH modeling in Tim, Chou, and Kroner (1992) has suggested and cited its use in the implementation and tests of competing asset pricing theories, market microstructure models, information transmission mechanisms, dynamic hedging strategies, and the pricing of derivative securities.

While most researchers agree on the predictability of stock market volatility (Engle and Ng, 1993; Engle, 1993), there is considerable variation of opinion on how this predictability should be modeled. The proliferation of research in the light of the evidence for predictability has led to a variety of approaches. The ARCH and GARCH models of Engle (1982) and Bollerslev (1986) respectively, and their many variants have been extensively used, and are among the broad class of parametric models that incorporate time dependency into the conditional variance underlying an asset return process[1].

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[1] As a prelude to further discussion, some notation on the difference between conditional and unconditional moments, central to the intuition underlying ARCH and GARCH, is necessary. The following discussion on conditional and unconditional moments is based on Engle (1993):

Let y_t be the return on an asset received in period t. Let E represent mathematical expectation. Then the mean of the return can be called μ, and

$$E y_t = \mu \qquad (1.1)$$

This is the unconditional mean, which is not a random variable. The conditional mean, m_t, uses information from the previous periods, and can generally be forecast more accurately. It is given by:

$$m_t \equiv E[y_t / F_{t-1}] \equiv E_{t-1}[y_t] \qquad (1.2)$$

This is in general a random variable of the information set F_{t-1}. Note that, although y_t-μ can be forecast, y_t-m cannot, using the information in F_{t-1} alone.

The unconditional and conditional variances can be defined in a similar way, as:

$$\sigma^2 \equiv E[y_t-\mu]^2 = E[y_t-m_t]^2 + E[m_t-\mu]^2 \qquad (1.3)$$

$$h_t \equiv E_{t-1}[y_t-m_t] \qquad (1.4)$$

The conditional variance thus potentially depends on the information set.

Models incorporating time dependence in the conditional variance, such as ARCH and GARCH, attempt to capture the well-known phenomenon of volatility clustering in financial data. Volatility clustering refers to the common occurrence in financial data whereby large shocks in magnitude to the return of a security tend to be followed by similarly high shocks of either sign.

The ARCH and GARCH type models of the conditional variance use the past values of the realized variance and shocks to forecast future variance. In so doing, the assumption of a constant variance over time is abandoned, and a structure governing the impact of past news is imposed (Engle and Ng, 1993). As such, the ARCH and GARCH type models explicitly incorporate recent news into the forecast of the future variance.

Alternative models of the volatility processes have also been proposed in the literature. Some of the most prominent are the historical variance models of Parkinson (1980), Garman and Klass (1980), Beckers (1983), and Abrahamson(1987), which will be discussed at length in Section II. These historical variance estimators use past historical price data to estimate a constant volatility parameter[2].

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[2] Canina and Figlewski (1993), Figlewski (1994), Lamoureux and Lastrapes (1993), and Pagan and Schwert (1990) use the historical variance estimator in a comparative framework with ARCH and GARCH type models, and evaluate their respective performance in producing forecasts. A very readable and intuitive account of the historical volatility estimator is provided by Ritzman (1991).

Historical volatility could be used as an ex-ante estimate of what market participants believed the volatility of an asset's return would be over some future period in time (Gordon, 1991). However, historical volatility is unlikely to reflect investors' changed expectations at a given point in time, such as in the event of a release of good or bad news (Ritzman (1991), Gordon (1991)). Consequently, a measure incorporating the markets expectation of aggregate future variance, the implied volatility, has been proposed as the market proxy for an asset's average volatility over the remaining lifetime of the option written on the asset. The performance and limitations of the implied volatility model have been studied by Day and Lewis (1992), Canina and Figlewski (1993), and Lamoureux and Lastrapes (1993). Implied volatility is calculated from the Black-Scholes option valuation formula for a European Call Option:

$$C = C(S_0, X, \sigma, r, T) \tag{1.5}$$

where C is the call option price, S_0 is the stock price today, X is the option strike price, r is the risk-free rate, T is the time remaining to maturity, and σ is the volatility. All variables but the latter are known, and thus σ can be solved for as follows:

$$\sigma_{implied} = \sigma(S_0, X, C, r, T) \tag{1.6}$$

The market price of the (call) option thus reflects the market expectation of the assets average volatility over the remaining life of the option.

Although the Classical Historical Variance and Implied Volatility models fail to accomodate for the time-varying property of volatility, there is considerable evidence in the literature that volatility is in fact changing over time. For example, Schwert (1989) shows that the variations of volatility for monthly stock returns on the period 1857-1987 range from a low of 2% in the early 1960's to a high of 20% in the early 1930's. As noted before, taking this feature into account appeared to be vital for many areas of the financial literature: continuous-time models and option pricing, CAPM, investment theory, amongst others (Engle and Ng, 1993). Another line of attack into modeling the time variation in volatility that has been developed since the early 1980's are continuous-time diffusion models with stochastic volatility (Hull and White(1980), Wiggins (1987), Chesney and Scott (1989))[3].

As mentioned above, volatility forecasting is a vital ingredient in many applications. As a result, the performance of different volatility forecasts have been studied extensively in the literature. These studies have followed two directions. In-sample tests of volatility attempt to determine the model that fits the data well[4]. However,

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[3] Hansson and Hordahl have investigated the performance of stochastic volatility models using Swedish OEX index data (1996), and Nelson (1990) has shown that a GARCH process can be interpreted as a discrete-time approximation of a diffusion model with stochastic volatility, thus connecting the two approaches.

[4] Studies doing in-sample tests of ARCH and GARCH point out the need to use a large number of observations for the model to fit the data well (Figlewski (1994), Pagan and Schwert (1991)). Figlewski (1994) reports the difficulties in implementing ARCH estimation to a group of data sets. Lamoureux and Lastrapes (1993) observe that sophisticated ARCH type models with many parameters are needed to fit a data well, and the more parameters the model has, the worst its performance is in making forecasts "out-of sample". This study encounters little difficulty in fitting a

it is a well known fact that it is an easier task to develop a model that fits one's data well than it is to construct a model that makes good out-of-sample forecasts.

For out-of-sample tests, the definition of realized volatility is very important. The common practice is to compute realized volatility from the time-series return realization of the forecast horizon. Out-of-sample tests use forecasts obtained using the estimates of the model parameters, which are found in-sample[5], and subsequently compare these forecasts to realized volatility[6]. In the case of ARCH and GARCH, the in-sample is used to obtain maximum likelihood estimates of the ARCH parameters, then these estimates are used in an ARCH forecast equation to make forecasts for the out-of-sample period. To evaluate the forecasts, forecast error criteria such as Root Mean Squared Error (RMSE) and Mean Absolute Error (MAE) are used.

This approach ignores the possibility of different return realizations for each time, which is vital for drawing conclusions about performance, as each different return realization in the forecast horizon would lead to a different conclusion about the performance of a model. In other words, a formal out-of-sample performance test of a given model would require that one should have the true distribution of returns for each

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GARCH (1,1) model to a simulated group of 1,000 sets of data, of 1,500 days each- chosen to correspond roughly to one business cycle.

[5] Using a fixed window history of price data.

[6] In the case of the classical historical volatility estimator, for example, a sample or history of returns (what is referred to as the "in-sample") is taken, the standard deviation within this sample is computed, and this becomes the average standard deviation (the historical volatility) for the forecast period ahead (what is referred to as the "out-of-sample").

time from which one can calculate the true variance. The performance of forecasts could then be judged on how closely they predict this true variance parameter in the forecast horizon. However in real life one can only observe one realization out of that distribution for each time, and this realized variance would be the best estimate of the true variance[7] over the forecast horizon[8].

For longer forecast periods, realized volatility calculations used in the literature will be increasingly closer the true variance with large number of observations that resemble the true distributions. However for short forecast horizons this approach is extremely problematic. As a forecaster, one has to think about all the possible outcomes and probabilities assigned to those before making any forecasts.

In marked contrast to any existing work in the literature, this study proposes to find and use "true" daily volatility to explore the performance of GARCH. "True" variance on a given day is calculated via simulation of a computable general equilibrium model based on Akdeniz (1998) and Akdeniz and Dechert's (1997) solution to Brock's (1979, 1982) multifirm stochastic growth model.

The wide range of findings and disagreements in the literature on GARCH performance[9] can be better understood in the light of the failure to account for the "true"

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[7] Volatility and Variance are used interchangeably in this study. For reference, Variance is Volatility *squared*.
[8] Assuming the realized return series is ergodic.
[9] See Figlewski (1994), Engle and Ng (1993), Lamoureux and Lastrapes (1993), Day and Lewis (1992), Pagan and Schwert (1990).

volatility as a performance basis when evaluating out-of-sample forecast performance of ARCH and GARCH. As observed by Bollerslev, Kroner, and Chou (1992), further developments concerning the identification and formulation of equilibrium models justifying empirical specifications for the observed heteroskedasticity remains a very important area for research. This study also aims to explore the performance of the GARCH specification for observed heteroskedasticity in contrast to naive estimators such as the classical historical volatility estimates.

In practice it is impossible to determine the true probability distributions of real economic time-series, but this study uses a stylized solution to Brock's (1979, 1982) intertemporal asset pricing model in which the true distribution of returns is calculated, and thus known. This setting is particularly chosen because of its success in simulating real life business cycle data, and reconciling some of the contentions to which the Capital Asset Pricing Model (CAPM) has been subject. This suggests that it will be a powerful tool in resolving the forecast performance debate among various volatility forecasts including the GARCH family of models.

This study uses a particular form of Akdeniz (1998) and Akdeniz and Dechert's (1997) solution to Brock's (1979, 1982) model to test the performance of out-of-sample forecasts of Bollerslev's (1986) GARCH(1,1) model. Brock's model is simulated by using Akdeniz and Dechert's (1997) solution 1,000 times to obtain 1,000 daily stock return data sets. Each simulated data set consists of the return realization for three firms over a period of 1,500 days (approximately one business cycle). The true distribution on any day can be obtained from these 1,000 simulation observations of the return process for that day.

The results demonstrate that the GARCH (1,1) model performs significantly better than historical volatility as measured by the MAE and RMSE criteria, when the true variance is used as benchmark. For forecasts 7-days ahead, GARCH (1,1) performs better than Classical Historical Volatility in 865 cases out of 1,000 simulations. It is found that Classical Historical Volatility has an MAE 4.832 times greater than GARCH (1,1), and an RMSE 4.003 times greater than GARCH across all simulations in this horizon. For a 22-day forecast period, Classical Historical Volatility is outperformed by GARCH(1,1) in 800 cases in 1,000 simulations, and has MAE 2.759 times as much, and RMSE 2.324 times as much as GARCH (1,1). Even more strikingly, it is found that if one were to use out-of-sample realized variance to evaluate forecast performance (as done in the literature) in each simulation, the RMSE and MAE for GARCH (1,1) would increase approximately three-fold, while decreasing significantly for Historical Volatility estimates. This demonstrates that the GARCH (1,1) model does a very good job of forecasting the true variance parameter in short-term horizons.

Using a single time series realization to study GARCH performance in the short-term is analogous to "mixing apples and oranges". An unfair realized volatility measure is certain to penalize GARCH forecast performance. The benefit of foreknowledge in the form of the "true" volatility as a benchmark for forecast comparisons is used to demonstrate the potential of the GARCH family of conditional volatility models. Whilst highlighting an ingredient that has been missing in the relevant literature, this study offers a means of explaining why most studies have not been sufficiently able to provide a more accurate picture of how well GARCH type models perform.

This study proceeds as follows. Chapter II. provides an overview of the literature, including more detailed discussions of the conditional variance estimators used in the literature, and goes on to discuss the computational general equilibrium model of Brock (1979, 1982) and the solution to the model by Akdeniz and Dechert (1997) that is utilized here, as well as highlighting the role of computational economics in the literature. Chapter III. discusses the model and the numerical solution. Chapter IV. details the results, and Chapter V. presents conclusions.

CHAPTER II

LITERATURE REVIEW

II.1 Introduction

Volatility is important to financial analysts for several important reasons. Estimates about volatility together with information about central tendency allow the analyst to assess the likelihood of experiencing a particular outcome. Volatility forecasts are particularly important for traders, portfolio managers and investors. Investors committed to avoiding risk, for example, may choose to reduce their exposure to assets for which high volatility is forecasted Traders may opt to buy options whose volatility they believe to be underpriced in the market, using their own subjective forecasts for volatility. The value of a derivative asset, such as options or swaps, depends very sensitively on the volatility of the underlying asset. Volatility forecasts for the underlying assets' return are therefore routinely used by participants in the derivatives markets for hedging (see Baillie and Myers (1991), and Engle, Hong, Kane, and Noh (1992)). In a market where such forces operate, equilibrium asset prices would be expected to respond to forecasts of volatility (Engle, 1993).

Scholars and practitioners have long recognized that asset returns exhibit volatility clustering (Engle and Rothschild, 1992). It is only in the last decade and a half that statistical models have been developed that are able to accommodate and account for this dependence, starting most prominently with Engle's (1982) ARCH and Bollerslev's (1986) GARCH. A natural byproduct of such models is the ability to forecast both in the short and in the long run.

Given that volatility is predictable (Bollerslev (1992), Engle (1993), and Engle and Ng(1993)), it is clear that one must choose between alternative models of forecasting volatility to obtain an "optimal" forecast model. In particular, the choice must be made based on a clear set of criteria, such as long run or short run performance in a specific forecast horizon. Equally important, the criteria for performance must be carefully scrutinized for bias towards a particular model. What follows is a description of the most popular volatility forecasting models in the literature.

II.2 Forecasting Models in the Literature

II.2.1 GARCH(p,q):

The GARCH(p,q) was originally proposed by Bollerslev (1986) and is a generalization of Engle's (1982) ARCH(p). A GARCH(p,q) model specifies that the conditional variance depends only on the past values of the dependent variable and this relationship is summarized in the following equations:

$$y_t = x_t\pi + \varepsilon_t \tag{2.1}$$

$$\varepsilon_t \mid \Psi_{t-1} \sim N(0, \sigma_t^2) \tag{2.2}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \tag{2.3}$$

$$= \omega + \alpha(L)\varepsilon_t^2 + \beta(L)\sigma_t^2 \tag{2.4}$$

In the GARCH(1,1) case, (2.4) reduces to:

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{2.5}$$

Here y_t refers to the return on day t, $x_t\pi$ is the set of regressors, ε_t is the residual error of regression which is conditionally distributed as a normal random variable with mean zero and variance σ_t^2. Here σ_t^2 is the conditional variance at time t, and is a function of the intercept long run variance $\frac{\omega}{1-\alpha-\beta}$, the squared residual from yesterday ε_{t-1}^2 (the ARCH term) and yesterday's forecast variance σ_{t-1}^2 (the GARCH term).

This specification makes sense in financial settings where an agent predicts today's variance by forming a weighted average of a long term average or constant variance, the forecast of yesterday (the volatility), and what was learned yesterday (or equivalently the shock). If asset returns were large in either way, the agent increases his estimate of the variance for the next day (Engle, 1996). This specification of the conditional variance equation takes the familiar phenomenon of volatility clustering in financial data into account, which is the property that large returns are more likely to be followed by large returns of either positive or negative sign, rather than small returns.

II.2.2 Historical Volatility

The most commonly used measure of volatility in financial analysis is standard deviation (Ritzman,1991). Standard deviation is computed by measuring the difference between the value of each observation on returns and the sample's mean, squaring each difference, taking the average of the squares, and then taking the square root of this average. In mathematical terms, this is equivalent to

$$\sigma_{1,T} = \sqrt{\frac{1}{T-1}\sum_{i=1}^{T}(R_i - \overline{R})^2} \qquad (2.6)$$

Where R_i is the return on day i, $\overline{R}$ is the mean return in the sample 1, ..., T and the sum of squared differences is divided by T-1 because the data under consideration is a sample, and not the population.

The classical historical volatility estimates are computed in a similar fashion. The classical historical volatility estimate as defined by Parkinson (1980), Garman and Klass (1980), and other authors, as a forecast going N periods forward is simply the standard deviation of returns computed from the previous N periods. Equation (2.6) is thus the classical historical volatility estimate of the average volatility during days T+1, T+2, ..., 2T. An alternative classical volatility estimate, motivated by the lognormality of stock price returns property (Black and Scholes, 1973), can be obtained by computing the logarithms of one plus the returns, squaring the differences of these values from their average, and taking the squared root of the average of the squared differences. For short time frames (of the order of one or two months) it does not make much difference which version of the classical historical estimator is used (Ritzman, 1991).

Alternate estimators have been derived in an attempt to improve on the efficiency of the classical historical volatility (CHV) estimator[10], which simply uses the variance of the market close-to-close return:

$$V_t^R = (\ln(R_{t+1} / \ln R_t) - \overline{R})^2 \tag{2.7}$$

$$V^R = \frac{1}{N-1} \sum_{t=1}^{N} V_t^R \tag{2.8}$$

[10] As defined by Parkinson, Garman & Klass, and others.

Where C_t denotes closing price at time t, and N is the number of periods. The classical historical volatility estimator is an estimate of the volatility using past history of data, and is used as a forecast of volatility over some future period.

All of these approaches are similar in that they assume the stock price follows a diffusion process with zero drift and constant volatility:

$$\frac{dS}{S} = \sigma dz \tag{2.9}$$

Based on this assumption, several authors develop measures which include more than the closing prices as estimators of the variance term σdz.

Parkinson's PK estimator (1980) of the variance of returns depends only on the high and low values over each period. The statistic V_i^{PK} is the variance estimate over period i. To obtain an estimate of average variance over N periods, the high observed price H_i and the low observed price L_i, is cumulated to obtain V^N, the average variance over N days:

$$V_i^{PK} = \left[\frac{(\ln(H_i) - \ln(L_i))^2}{4 \ln 2} \right] \tag{2.10}$$

$$V^N = \frac{1}{N} \sum_{i=1}^{N} V_i^{PK} \tag{2.11}$$

The Garman and Klass (1980) estimator is an extension of the Klass estimator and develops a more accurate method of estimating the variance of the displacement (or diffusion constant σ) in a random walk. Garman and Klass (1980) assume that the logarithm of stock prices follows a Brownian motion with zero drift, regardless of whether the market is open or closed. Their basic formula is given in equation (2.12), where O_i is the assets' opening price and C_i is the closing price. The left-hand term is the squared difference between the open-to-high and open-to-close returns, while the right hand term is the squared open-to-close return:

$$V_i^{GK} = .5\left[\ln\left(\frac{H_i}{O_i}\right) - \ln\left(\frac{L_i}{O_i}\right)\right]^2 - [2\ln(2) - 1] * \left(\ln\frac{C_i}{O_i}\right)^2 \qquad (2.12)$$

$$= .5[\ln H_i - \ln L_i]^2 - .39\left(\ln\frac{C_i}{O_i}\right)^2 \qquad (2.13)$$

The formula for situations when opening prices are not available (and the previous day's close is used instead) is given by equation (2.14):

$$V_i^{GK} = .5[\ln H_i - \ln L_i]^2 - .39\left(\ln\frac{C_i}{C_{i-1}}\right)^2 \qquad (2.14)$$

In this form, it can be seen that the Garman and Klass (1980) estimator is really a weighted average of the PK estimator and the classical historical estimator.

When the market of interest does not trade 24 hours a day, the estimator to be used is given in equation (2.15). This version is simply a linear combination of the previous estimator (2.14) and a new estimator of the variance during the proportion of the day f when the market is closed:

$$V_i^{GK2} = .12\left[\frac{\left(\ln\frac{O_i}{C_{i-1}}\right)^2}{f}\right] + .88\left[\frac{V_i^{GK}}{1-f}\right] \tag{2.15}$$

$$\text{V} = \frac{1}{N}\sum_{i=1}^{N} V_i^{GK} \tag{2.16}$$

II.2.3 Implied Volatility

There can be situations where an ex-ante estimate of what market participants believe the volatility of an assets' return to be is needed. In such a case, a historical estimate, such as those described in section II.2.2, could be used. However, the historical estimate is unlikely to reflect investors' changed expectations at a given point in time (e.g., in the event of the release of bad news). Consequently, it is necessary to focus some attention on the concept of implied volatility, which is an attempt to estimate the aggregate expectations of future variance that the market has at a given point in time. If we believe that investors and speculators price options according to the Black and Scholes (1973) option pricing formula, then the price of a European Call Option is given by

$$C = C(S_0, X, \sigma, r, T) \tag{2.17}$$

This also suggests that the implied standard deviation (ISD) of the option should be given by:

$$\sigma = \sigma(S_0, X, C, r, T) \tag{2.18}$$

Equation (2.18) can be solved by numerical search procedures, such as Newton Raphson or the method of bisection (Ritzman (1991), Gordon (1991))[11].

II.2.4 Stochastic Volatility

The development here borrows heavily from the discussion of stochastic volatility to be found in Hansson and Hördahl (1996), Taylor (1994), Harveys, Ruiz and Shephard

[11] Latane and Rendleman (1976), Patell and Wolfson (1981), Scmalansee and Trippi (1978), MacBeth and Melville (1979), Manaster and Rendleman (1982) have examined the properties and investigated the difficulties with the use of ISDs in practice.

(1994), and Ruiz (1994).

From a theoretical point of view, it is useful to start with a continuous time specification for the price of an asset, while a discrete time approximation is generally used for estimation purposes. It is assumed, following Scott (1987) and Wiggins (1987), that the return of an asset dP/P follows a Geometric Brownian Motion while the logarithm of volatility follows an Ornstein-Uhlenbeck process:

$$dP / P = \alpha dt + \sigma dW_t, \tag{2.19a}$$

$$d \ln \sigma = \tau(\zeta - \ln \sigma)dt + \varphi dW_2, \tag{2.19b}$$

$$dW_1 dW_2 = \rho dt, \tag{2.19c}$$

Where $P(t)$ denotes the price of an asset at time t, α is the return drift, τ is a parameter which governs the speed of adjustment of log-volatility to its long-term mean ζ, φ determines the variance of the log-volatility process, and $\{W_1(t), W_2(t)\}$ is a two-dimensional Wiener process with correlation ρ.

In a discrete time approximation of the models above[12] the continuous return of an asset r_t, corrected for the unconditional mean μ, is a martingale difference while the

[12] See Taylor (1994), Harvey, Ruiz, and Shephard (1994), and Ruiz (1994) for details.

logarithm of squared volatility $\ln \sigma_t^2$ follows an AR(1) process:

$$r_t \equiv \ln(P_t) - \ln(P_{t-1}) - \mu = \exp(h_t/2)\varepsilon_t,\ \varepsilon_t \sim IN[0.1] \qquad (2.20)$$

$$h_{t+1} = \delta + \phi h_t + \eta_t, \qquad \eta_t \sim IN[0, \sigma_\eta^2] \qquad (2.21)$$

$$\ln \sigma_{t+1}^2 = h_{t+1} \qquad (2.22)$$

where δ, and ϕ are constants. The stochastic volatility for period t is σ_t or $\exp(h_t/2)$, and the realized value of the volatility process h_t is in general not observable. The AR(1) process of the logarithm of variance, h_t, is stationary if $|\phi| < 1$ and it follows a random walk if $\phi = 1$.

II.3 Conditional Volatility Models in the Literature and Performance Studies

Performance of different volatility forecasts have been proposed and their performance tested in the literature. Among all the volatility forecasting models, the GARCH family has received the greatest attention in the literature. ARCH and GARCH modeling in finance has enjoyed a prominent and varied history in the literature since its inception by Engle (1982) and generalization by Bollerslev (1986). An entire volume of the Journal of Econometrics was devoted to its use in research in 1992. It is an indispensable, often integral, part of many textbooks devoted to finance and econometrics. An overview of ARCH modeling in Tim, Chou, and Kroner (1992) has suggested and cited its use in the implementation and tests of competing asset pricing theories, market microstructure models, information transmission mechanisms, dynamic hedging strategies, and the pricing of derivative securities.

While it has been recognized for quite some time that the uncertainty of speculative prices, as measured by the variances and covariances, are changing over time [e.g Mandelbrot (1963) and Fama (1965)], it is only somewhat recently that applied researchers in financial and monetary economics have started explicitly modeling time variation in second or higher-order moments. The Autoregressive Conditional Heteroskedasticity (ARCH) model of Engle (1982) and its various extensions have emerged as one of the most noteworthy tools for characterizing changing variances. More than several hundred research papers applying this modeling strategy have already appeared. Volatility is a central variable which permeates many financial instruments and has a central role in many areas of finance. As one simple example, volatility is vitally important in asset pricing models as well as in the determination of option prices. From an empirical viewpoint, it is therefore of no small importance to carefully model any temporal variation in the volatility process. It is important therefore that the performance of the ARCH model and its various extensions be tested in vitro in a simulated dynamic general equilibrium model that has already corroborated the theoretical results of the

CAPM and where the true distribution of the returns is known in its ex-ante form. It is a fact worth mentioning that there has been scant attention to this type of performance analysis of the GARCH family models in the relevant literature.

Implied and historical variance estimators have also received wide attention, and their empirical performance has been analyzed, both on a stand-alone basis, and in comparison with other conditional variance estimators. Figlewski (1994) examines the empirical performance of different historical variance estimators and of the GARCH(1,1) model for forecasting volatility in important financial markets over horizons up to five years. He finds that historical volatility computed over many past periods provides the most accurate forecasts for both long and short horizons, and that root mean squared forecast errors are substantially lower for long term than for short term volatility forecasts. He also finds that, with the exception of one out of five data sets used, the GARCH model tends to be less accurate and much harder to use than the simple historical volatility estimator for his application.

In another paper, Canina and Figlewski (1993) compare implied volatility estimators to historical volatility for S&P 100 index options, and find implied volatility to be a poor forecast of subsequent realized volatility. They also find that in aggregate and across sub-samples separated by maturity and strike price, implied volatility has virtually no correlation with future volatility, and does not incorporate the information contained in recent observed volatility.

Pagan and Schwert (1990) compare several statistical models for monthly stock return volatility using US data from 1834-1925. They use a two-step conditional variance estimator [Davidian and Carroll (1987)], a GARCH (2,1) model, an EGARCH(1,2) model [Nelson (1988)], Hamilton's (1989) two-state switching-regime model, a nonparametric kernel estimator based on the Nadaraya (1964) and Watson (1964) Kernel estimator, and a nonparametric flexible Fourier form estimator [Gallant (1981)]. They find that taking the 1835-1925 period as the sample, nonparametric procedures tended to give a better explanation of the squared returns than any of the parametric models. Of the parametric models, Nelson's EGARCH comes closest to the explanatory power of the nonparametric models, because it reflects the asymmetric relationship between volatility and past returns. However, they also find that Nonparametric models fare worse in out-of-sample prediction experiments than the parametric models.

Lamoureux and Lastraspes (1993) examine the behavior of measured variances from the options market and the underlying stock market, under the joint hypothesis that markets are informationally efficient and that option prices are explained by a particular asset pricing model. They observe that under this joint hypothesis, forecasts from statistical models of the stock-return process such as GARCH should not have any predictive power above the market forecasts as embodied in option implied volatilities. Using in-sample and out-of-sample tests, they show that this hypothesis can be rejected, and find that implied volatility helps predict future volatility. Using the analytical framework of the Hull and White (1987) model, they characterize stochastic volatility in their data by using the GARCH model. Their out-of-sample tests show that GARCH does not outperform the classical historical volatility estimator significantly. GARCH, Historical Volatility, and Implied Volatility are each used to forecast the mean of the daily variance over the remaining life of the option. For each day in the forecast horizon, each forecast is compared to the actual mean of the daily realized variance. In a bid to compare their results with other studies, they refer to Akgiray (1989). Using the root mean squared error (RMSE) criterion for stock index data, Akgiray (1989) finds that, at a forecast horizon of 20 days, GARCH variance forecasts are convincingly superior to historical volatility. Lamoureux and Lastrapes (1993) replicate Akgiray's (1989) analysis with a 100 day forecast horizon and find that the relative rankings of historical volatility and GARCH are overturned. They also find that implied variance tends to underpredict realized variance, as evinced by a significantly positive mean error (ME) in their study. They also observe that, as GARCH weights the most recent data more heavily, GARCH overstates the frequency of large magnitude shocks, which leads to good in-sample fit and excellent short-term forecasts, but poor long term forecasts (see also Akgiray(1989) and Nelson (1992)).

Day and Lewis (1992) compare the information content of the implied volatilities from call options on the S&P 100 index to GARCH and Nelson's EGARCH models of conditional volatility. By using the implied volatility as an exogenous regressor in the ARCH and EGARCH models, they examine the with-in sample incremental information content of implied volatilities and the forecast from GARCH and EGARCH models.

Their out-of-sample forecast comparisons suggest that short-run market volatility is difficult to predict, and they conclude that they are unable to make strong statements concerning the relative information content of GARCH forecasts and implied volatilities.

Jorion (1995) examines the information content and forecast power of implied standard deviations (ISDs) using Chicago Mercantile Exchange options on foreign currency futures. Defining the realized volatility in the conventional sense[13], he regresses ***realized*** volatility on forecast volatility[14] and a time series volatility specification modeled as GARCH (1,1), in the following way:

$$\sigma_{t,T} = a + b_1\sigma_t^{ISD} + b_2\sigma_t^{GAR} + \varepsilon_{t,T} \tag{2.23}$$

This regression is used to test whether GARCH (1,1) forecasts have predictive power beyond that contained in σ^{ISD}. In other words, Jorion tests whether the coefficient of the GARCH forecast, b_2, is significantly different from zero. GARCH forecasts are obtained by successively solving for the expected variance for each remaining day of the options life, then averaging over all days. He finds that MA(20) and GARCH (1,1) forecasts have lower explanatory power than ISDs[15]. He concludes that his results indicate option-implied forecasts of future volatility outperform statistical time-series models such as

13 The realized volatility of returns from day t to day T is thus the standard deviation of returns in days t, t+1, ..., T (Ritzman, 1991).
14 Where the forecast volatility is taken to be the ISD forecast of volatility of returns over the remaining contract life.
15 Jorion finds that the slope coefficient b_2 of the GARCH (1,1) forecast becomes small and insignificant, implying low forecast power compared to ISD.

GARCH in the foreign exchange market[16], although the volatility forecasts of ISDs are biased- suggesting ISDs to be too volatile. It is worth extra emphasis that Jorion conforms to the existing practice in the literature of utilizing the realized variance, in this case in the foreign exchange market for three currencies, for forecast comparison regressions.

West and Cho (1995) compare the out-of-sample forecast performance of univariate homoskedastic, GARCH[17], autoregressive, and nonparametric models for conditional variances, using five bilateral (Canada, France, Germany, Japan, and the United Kingdom) weekly exchange rates for the dollar, for the period 1973-1989. They compare the out-of-sample realization of the square of the weekly change in an exchange rate with the value predicted by a given model of the conditional variance for horizons of one, twelve, and twenty four weeks. The performance measure that they use is the mean squared prediction error (MSPE), using rolling and expanding samples for the out-of-sample forecasts. For one-period horizons, they find some evidence favoring GARCH models. For twelve and twenty-four-week ahead forecasts of the squared weekly change, they find little basis for preferring one model over another. At the one-week-ahead forecast horizon, they report that GARCH (1,1) produces slightly better forecasts in the MSPE sense, but they conclude that statistical tests cannot reject the null hypothesis that the MSPE from GARCH (1,1) is equal to the MSPE from other models at all horizons considered. They observe that, based on this inability to reject the null, there are no viable

[16] This is in sharp contrast to Canina and Figlewski (1993), who find that ISDs have poor performance in the U.S. stock market.
[17] More specifically, GARCH (1,1) and IGARCH (1,1) models are used. These are selected because in-sample diagnostics are found to be better than other GARCH models.

reasons for preferring one model over another at all the horizons considered. It thus appears that GARCH performance leaves something to be desired, and although the GARCH models perform well, endorsement of these models is moderate. Although the in-sample evidence in their study clearly suggests that a homoskedastic model should be strongly dominated by the other models, they find this not to be the case. This is a surprising result, as the authors themselves acknowledge. There is one comment in their paper that is definitely worth reproducing in the context of this study: "...it might be largely a matter of chance which model produces the smallest RMSPE[18]". This element of chance is best explained and understood in terms of the use of realized out-of-sample variance, and this observation of West and Cho is one of the few veiled references in the literature to the arbitrariness of evaluating performance on the basis of a sample realization in the forecast horizon (which introduces the "chance" factor in performance referred to in the quotation), and the lack of formal tests of performance.

Amin and Ng (1990) study the asymmetric/leverage effect in volatility in option pricing. The asymmetric/leverage effect is a property of stock returns which has been the subject of much recent study (Black (1976), Christie (1982), French, Schwert and Stambaugh (1987), Nelson (1990), Schwert (1990)) and refers to the phenomenon whereby whereby stock return volatility is higher following bad news than good news. They find that a comparison of the mean absolute option pricing error under the GARCH, EGARCH, and Glosten, Jagganathan, and Runkle (GJR, 1992) models relative to the

[18] root mean squared performance error.

mean absolute option pricing error under a constant volatility model favors EGARCH and GJR models.

Hansson and Hördahl (1996) estimate the conditional variance of daily Swedish OMX-index returns for the period January 1984-February 1996 with stochastic volatility (SV) and GARCH models. They find that the best in-sample fit is provided by the asymmetric unrestricted SV model with a seasonal effect. An evaluation of the forecasting power of the model is shown to provide better out-of-sample forecasts than GARCH models. They conclude in their study that the SV model specification is the preferred model for forecasting purposes.

Long memory processes and modeling aspects is another area that is exciting considerable interest in the GARCH literature. Bollerslev and Mikkelsen (1996) introduce a new class of fractionally integrated GARCH and EGARCH models to characterize U.S. financial market volatility. Recent empirical evidence indicates that apparent long-run dependence in U.S. stock market volatility is best accounted for by a mean-reverting fractionally integrated process[19]. After in-sample estimation of various[20] models on the Standard and Poors index, the authors simulate the price paths of options of different maturities with three alternative pricing schemes, and make option price forecasts for the three different EGARCH and an AR(3) data-generating mechanisms. At

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19 In such a process, a shock to the optimal forecast (grounded in the model) of the future conditional variance decays a slow hyperbolic rate.
20 AR(3), AR-EGARCH, AR-IEGARCH, AR-FIEGARCH.

70-days to maturity, the IEGARCH model results in the highest forecast prices, whereas the homoskedastic AR model uniformly produces the lowest valuations. The prices for EGARCH and FIEGARCH are very close. As the maturity increases to 270 days, the prices generated by FIGARCH are between EGARCH and IEGARCH valuations. The AR(3) model consistently underprices long and short horizon options, and the FIGARCH model appears to be the best candidate for characterizing the long-run dependencies in the volatility process of the underlying asset.

II.4 The Computable Dynamic General Equilibrium Model

II.4.1 Introduction

This study uses the method employed by Akdeniz and Dechert (1997) to solve the multifirm stochastic growth model of Brock (1979). Brock's model reflects the utility maximizing behavior of the consumer and the profit maximizing behavior of the producers. The computational solution to Brock's model produces results that resemble the "real" economy very closely and hence it is a very powerful tool with a rich variety of possible future applications. Many empirical anomalies, in particular the predictions of the Capital Asset Pricing Models (CAPM) and the contentions surrounding whether its predictions hold or not, have been resolved in the computational economy framework via the approach by Akdeniz (1998) and Akdeniz and Dechert (1997)[21]. The power of this computational framework as a general tool is thus particularly chosen to study the performance of the GARCH model, and the economy that it simulates can be made to mirror stock market data very closely, having all the properties that empirical data possess such as excess kurtosis, conditional moment dependence on time, volatility clustering, etc. This is just one of the many possible applications that Brock's model, via the numerical solution of Akdeniz and Dechert (1997), can be put in to use.

The next section will discuss the mulifirm stochastic growth model and asset pricing model of Brock (1979, 1982), and prepare the ground for discussions on the particular numerical solution method adopted for this study, which will be presented in Section III.

II.4.2 The Growth Model

The model used as the basis in this study is the optimal growth model with uncertainty. The long-run behavior of the deterministic one-sector optimal growth model has been studied by Cass (1965) and Koopmans (1965), who showed the existence of a steady state solution. The extension of this model to include uncertainty can be found in Brock and Mirman (1973) and Brock (1979, 1982). This study utilizes the case of

[21] For details on more specific successes of this approach, the reader is directed to their papers.

discrete Markov process driven technology shocks. There is an infinitely lived representative consumer who has preferences over sequences of consumption given by the following expression:

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t u(c_t)\right\}, \qquad 0<\beta<1, \tag{2.24}$$

where c_t denotes consumption of the single good, and E_0 denotes expectation conditional on information at date zero. There are N production processes in this economy that can be used to produce one type of good that can be consumed or added to the capital stock. The following discusses the growth model in some depth, and borrows heavily from Akdeniz and Dechert (1997). The key elements of the model are recapitulated as follows[22].

$$\max_{c_t, x_{it}} \quad E\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right] \tag{2.25}$$

subject to:

$$x_t = \sum_{i=1}^{N} x_{it} \tag{2.26}$$

$$y_{t+1} = \sum_{i=1}^{N} f_i(x_{it}, \xi) \tag{2.27}$$

$$c_t + x_t = y_t \tag{2.28}$$

$$c_t x_{it} \geq 0 \tag{2.29}$$

and y_0 historically given

22 For a more detailed exposition, the reader is referred to Brock (1979), and Akdeniz and Dechert (1997).

Where:

E	mathematical expectation
β	discount factor (constant) on future utility
u	utility function of consumption[23]
c_t	consumption at date *t*
x_t	capital stock at date *t*
y_t	output at date *t*
f_i	production function of process *i* plus undepreciated capital
x_{it}	capital allocated to process *i* at date *t*
δ_i	depreciation rate for capital installed in process *i*
ξ_t	random shock

Observe here that $f_i(x_{it},\xi_t) = g_i(x_{it},\xi_t) + (1-\delta_i)x_{it}$, where $g_i(x_{it},\xi_t)$ is the production function of process *i*, and $(1-\delta_i)x_{it}$ is remaining (undepreciated) capital after production in process i.

The optimal policy functions for consumption and investment for the representative consumer are obtained via maximization of the expected value of the discounted sum of utilities over all possible consumption paths and capital allocations[24]. The following explanation is offered by Brock to elucidate the working of the model:

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23 The utility function is characterized by $u : \Re_+ \to \Re$ is strictly concave, strictly increasing, and continuously differentiable with $u(0)=0$, $U'(0)=\infty$ and $U'(\infty)=0$.

24 The x's at date t must be measurable with respect to the x_j's through date t-1.

"There are N different processes. At date t it is decided how much to consume and how much to hold in the form of capital. It is assumed that capital goods can be costlessly transformed into consumption goods on a one-for-one basis. After it is decided how much to hold in the form of capital , then it is decided how to allocate capital across the N processes. After the allocation is decided nature reveals the value of r_t, and $g_i(x_{it}, r_t)$ units of new production are available from process i at the end of period t. But $\delta_i x_{it}$ units of capital have evaporated at the end of period t. Thus, the net new produce is $g_i(x_{it}, r_t)$- $\delta_i x_{it}$ from process i. The total produce available to be divided into consumption and capital stock at date t+1 is given by

$$\sum_{i=1}^{N}[g_i(x_{it}, r_t) - \delta_i x_{it}] + x_t = \sum_{i=1}^{N}[g_i(x_{it}, r_t) + (1-\delta_i)x_{it}]$$

$$\equiv \sum_{i=1}^{N} f_i(x_{it}, r_t) \equiv y_{t+1},$$

where

$$f_i(x_{it}, r_t) \equiv g_i(x_{it}, r_t) + (1-\delta_i)x_{it}$$

denotes the total amount of produce emerging from process i at the end of period t. The produce y_{t+1} is divided into consumption and capital stock at the beginning of date $t+1$, and so on it goes".

This study introduces a notational difference. Observe that Brock uses "r_t" for his shock parameter, whereas in this study the shock parameter is denoted "ξ_t". The main assumptions for the model in this study are[25]:

(A1) the functions u and f_i are concave, increasing, twice continuously differentiable, and satisfy the Inada conditions;

(A2) the stochastic process is a discrete Markov process with eight states of uncertainty for each production process;

(A3) the maximization problem has a unique optimal solution.

Then the first order conditions for the intertemporal maximization can be written as:

$$u'(c_{t-1}) = \beta E_{t-1}\left[u'(c_t) f_i'(x_{it}, \xi_t)\right] \tag{2.30}$$

$$\lim_{t \to \infty} \beta^t E_{t-1}\left[u'(c_t) x_{it}\right] = 0 \tag{2.31}$$

25 For a detailed explanation of the model see Brock (1982).

It is worth mentioning that Equation (2.30) will be the one used in the following sections to obtain a numerical solution to the growth model. Observing that the problem given by equations (2.25) to (2.29) is time stationary, the optimal levels of c_t, x_t, x_{it} are functions of the output level y_t, and can be written as:

$$c_t = g(y_t) \qquad x_t = h(y_t) \qquad x_{it} = h_i(y_t) \tag{2.32}$$

Now the aim is to solve the growth model for the optimal investment functions, h_i, and then to analyze the underlying implications of the asset pricing model. As a further simplification, note that the first two functions in equation (2.32) can be expressed in terms of these investment functions:

$$h(y) = \sum_{i=1}^{N} h_i(y) \tag{2.33}$$

$$c(y) = y - h(y) \tag{2,34}$$

II.4.3 The Asset Pricing Model

Brock's (1982) asset pricing model has many characteristics in common with the Lucas (1978) model. The primary feature seperating the two models is that Brock's model includes production. The inclusion of shocks to the production processes in the model directly links the uncertainty in asset prices to economic fluctuations in output levels and profits.

The asset pricing model closely resembles the growth model discussed in Section II.4.2. There is an infinitely lived representative consumer whose preferences are given by equation (2.24 & 2.25). The production side consists of N different firms. Firms maximize profits by renting capital from the consumer side:

$$\pi_{i,t+1} = f_i(x_{it}, \xi_t) - r_{it} x_{it} \tag{2.34}$$

Each firm takes the decision of hiring capital after the shock ξ_t is revealed. The interest rate r_{it} in industry i at time t is determined within the model. Following the convention in the literature, asset shares are normalized so that there is one (perfectly divisible) equity share for each firm in the economy. Thus, ownership of a proportion of the share of firm i at time t grants the consumer the right to that proportion of firm i's profits at time t+1. Following Lucas (1978), it is assumed that the optimal levels of output, consumption, asset prices, and capital form a rational expectations equilibrium.

The representative consumer then solves the following problem:

$$\max \quad E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \tag{2.35}$$

subject to: $c_t + x_t + P_t \cdot Z_t \leq \pi_t \cdot Z_{t-1} + P_t \cdot Z_{t-1} + \sum_{i=1}^{N} r_{i,t-1} x_{i,t-1}$ (2.36)

$$c_t,\ Z_t,\ x_{it} \geq 0 \tag{2.37}$$

$$r_{it} = f_i'(x_{it}, \xi_t) \tag{2.38}$$

$$\pi_{it} = f_i(x_{i,t-1}, \xi_{t-1}) - f_i'(x_{i,t-1}, \xi_{t-1})x_{i,t-1} \tag{2.39}$$

Where:

P_{it} price of one share of firm *i* at date *t*.

Z_{it} number of shares of firm *i* owned by the consumer at date *t*.

π_{it} profits of firm *i* at date *t*.

Again, for details of the model, refer to Brock (1982). First order conditions derived from the maximization problem are:

$$P_{it}u'(c_t) = \beta E_t\left[u'(c_{t+1})(\pi_{i,t+1} + P_{i,t+1})\right] \tag{2.40}$$

and

$$u'(c_t) = \beta E_t\left[u'(c_{t+1})f_i'(x_{i,t+1}, \xi_{t+1})\right] \tag{2.41}$$

The first order conditions yield the prices for the assets. However, the transversality conditions

$$\lim_{t\to\infty} \beta^t E_0\left[u'(c_t)\sum_i P_{it}Z_{it}\right] = 0 \tag{2.42}$$

$$\lim_{t\to\infty} \beta^t E_0\left[u'(c_t)x_t\right] = 0 \tag{2.43}$$

are required to fully characterize the optimum[26]. This condition, as noted by Judd (1992), suggests that we are looking for a bounded solution[27] to the growth model. Brock (1979) has shown that there is a duality between the growth model (2.25-2.29) and the asset pricing model (2.35-2.39). Thus the solution to the growth model also solves the asset pricing model. Once the solution to the former is found, equation (2.40) can be used to solve for the asset prices, and asset returns. To be more specific, once the optimal policy functions, $h_i(y)$, for the growth problem are obtained asset prices and asset returns can easily be calculated for given values of shocks.

II.5 The Model and Computational Economics in the Literature

In general no closed form solutions for stochastic growth models exists, except for the specific cases of logarithmic utility and Cobb-Douglas production functions with 100 percent discounting, and carefully paired CES production and utility functions[28]. The recent advances in computer hardware and computational methods have enabled economists to study these models. As a result, more and more economists have been using computational methods to solve dynamic economic models over the past two decades.

26 Brock (1982) shows that the transversality condition holds in this model.
27 The optimal solution remains in a bounded interval: $0<a<y_t<b<\infty$ for all t.
28 See Benhabib and Rustichini (1994).

Sims (1990) solves the stochastic growth model by backward-solving, a class of methods that generates simulated values for exogenous forcing processes in a stochastic equilibrium model from a specified assumed distribution for Euler-equation disturbances, with a particular nonlinear form for the decision rule. The backward-solved simulations, whose solution paths were shown to be stationary (Sims, 1989), are applied to a one-sector neoclassical growh model with decision rule generated from a linear-quadratic approximation. Baxter, Crucini, and Rouwenworst (1990) solve the stochastic growth model by a discrete-state-space Euler equation approach. They focus on the Euler equations that characterize equilibrium behavior, and then compute approximations to equilibrium decision rules. Their approach is "exact" in the sense that their approximate decision rules converge to the true decision rule as the grid over which the decision rules are computed becomes arbitrariliy fine. Their approach to simulute the economy's respononse to shocks relies on Tauschen's (1986) method to generate a sequence for the technology shocks, and using their approximate policy function together with the resource constraint to determine associated equilibrium values of output, consumption, and investment. Benhabib and Rustichini (1994) provide exact solutions for a class of stochastic programming problems in growth theory, specifically those involving pairs of constant relative risk aversion utility functions and CES technologies. They incorporate depreciation schemes into their model, and generalize the solutions for the well-known case of logarithmic utility coupled with Cobb-Douglas production functions[29].

29 A more thorough description and comparison of some of the various methods can be found in Taylor and Uhlig (1990), and Danthine and Donaldson (1995).

A series of important observations made by Judd (1995) emphasize the relevance of computational methods for science in general and economics in particular. Judd (1995), in surveying the interaction between economic theory and computational economics, discusses the methodological issues raised by the idea of "computational theory", and the problems associated with its dissemination and exposition. It is important to evidence what computational methods *is not*. At the most obvious, computational economics is not a theoretical proof of a proposition, in the sense that a proof is a logical-deductive process in which a proposition is formulated, and auxiliary assumptions are added (such as linear demand, constant costs, etc.) to the basic ones in order to make a proof of the proposition possible. Computational methods have complementary strengths and weaknesses relative to theoretical methods. They can approximately solve an arbitrary example of the basic theory, and determine the nature of any individual point in the solution space being explored. The advantage is that computational methods do not need as many, if any, auxiliary assumptions to compute the equilibrium. The supposed weakness is that they can do this only one example, one point, at a time, and in the end can examine only a finite number of points. Some argue that deductive theory has a superiority in that its results are error free. This has to be balanced with the self-evident limitations of an approach that is exclusively limited to a small subset of examples that obey the conditions imposed to make the analysis tractable. Although computational methods often involve error, their superior range offers an advantage.

Thus the vital conclusion drawn by Judd is that computational methods are a complement to theory, and can provide powerful insights for the investigation of those systems that are not analytically tractable. An exhaustive list of episodes are recounted, notably the work of Kydland and Prescott (1982), who showed that fairly simple dynamic general equilibrium models could display the type of economic fluctuations that we observe in the macroeconomy.

Brock's model has been used for the last twenty years, and has been extensively cited in the literature. Nonetheless, some researchers have used a linear investment function specification, while others starting with Kydland and Prescott (1982) have used a quadratic approximation to the value function which similarly results in a linear policy function. Consequently, these studies have not succeeded in exploiting the numerous implications that exist in Brock's model.

In this study the starting point is the use of numerical methods to obtain solutions for Brock's (1979) growth model for any type of utility and production functions. The projection method of Judd (1992) is used to solve the stochastic growth model for the optimal investment functions, profits, and returns. The method proposed by Akdeniz and Dechert (1997) and Akdeniz (1998) is generalized to include discrete Markov-process driven shocks to both the output levels and elasticities. This refinement induces serial correlation in the squared returns of the process, a manifestation which closely mirrors empirical asset return phenomena. The likelihood of subsequent high shocks in return of either sign, after a high shock is initially recorded in the system is increased, thus forcing volatility clustering and fat tailedness (excess kurtosis) in the return data, which agrees with empirical stock market return data.

CHAPTER III

THE MODEL

III.1 Introduction

This study uses a particular form of Akdeniz (1998) and Akdeniz and Dechert's (1997) solution to Brock's (1979, 1982) model to test the performance of out-of-sample forecasts of Bollerslev's (1986) GARCH(1,1) model. Akdeniz (1998) and Akdeniz and Dechert's (1997) model is simulated 1,000 times to obtain 1,000 daily stock return data sets. Each simulated data set consists of the return realization for a chosen firm over a period of 1,500 days (one business cycle).

Bollerslev's (1986) GARCH(1,1) model for conditional variance is then fitted to the first 1,300 days of ***each*** data set. In-sample diagnostic tests are used to evaluate the model for these data sets. These diagnostic in-sample tests are elaborated in Section IV. Forecasts are then made for 7 and 22 days ahead (i.e. for the out-of-sample period [1301, ..., 1307] and [1301, ..., 1322]), and average forecast volatilities over the horizon are computed for each data set. These forecasts are compared to the "true" variance on a given day, which is computed from the daily true distribution of the simulated return data sets, and to the realized variance[30] in the forecast horizon. The GARCH (1,1) mean absolute forecast error is then calculated over all 1,000 simulations using both benchmarks. This is simply the average of the absolute value of all the forecast errors across each data set, which is computed by subtracting the GARCH(1,1) forecast of average *variance* over the forecast horizon from the "true" variance[31] at the end points of the forecast horizon (in this case days 1307 and 1322 for 7 and 22-day ahead horizons)[32]. A similar procedure is then carried out to compute the RMSE of the GARCH forecasts using both benchmarks at 7 and 22-day horizons. Finally, the whole procedure is repeated for forecasts obtained from the CHV estimator. The equations below define the MAE and RMSE specifically for the GARCH model forecasts 7-days ahead using true variance as benchmark. The other cases follow in exactly the same way:

$$RMSE_7^{GARCH} = \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}(\bar{v}_i^{GAR7} - v^{TRUE7})^2} \tag{3.1}$$

The right hand side of equation (3.1) refers to RMSE computed using the GARCH (1,1) model at the 7-day horizon. Here v^{TRUE7} refers to the 1307th day true variance, and is a constant that is subtracted from each GARCH forecast, denoted by $\bar{v}_i^{GAR7}$, where the i refers to the forecast in a particular simulation, and is summed over all simulations. The equation below defines MAE for GARCH at the 7-day horizon, and is self-explanatory.

30 Realized variance is obtained for each simulated return set via programming in EVIEWS. Refer to Appendix C.
31 The procedure is similar when computing MAE using realized variance as benchmark. Instead of subtracting true variance on the 1307th and 1322nd days, the realized variance for the horizons are calculated for each return data set and subtracted from the GARCH forecasts of average variance in the two horizons.
32 This procedure is explained at greater length in Chapter IV.

$$MAE_7^{GARCH} = \frac{1}{1000}\left\{\sum_{i=1}^{1000}\left|\overline{v}_i^{GAR7} - v^{TRUE7}\right|\right\} \quad (3.2)$$

GARCH(1,1) estimation and forecasting is done through programming in EVIEWS. In EVIEWS, heteroskedasticity-consistent covariance estimator and the Berndt-Hall-Hall-Hausman (BHHH) algorithm is used to estimate the GARCH(1,1) equation. The BHHH algorithm for maximizing likelihood is preferred over the alternative Marquardt algorithm because of its speed, and the heteroskedasticity consistent covariance estimator option is chosen to accommodate for the fact that what is actually being done is quasi maximum likelihood estimation (maximum likelihood with an invalid assumption) -as the residuals of estimation are not conditionally normal- and hence need to be corrected for by robust standard errors (Bollerslev and Woodridge, 1991).

The next section discusses the GARCH(1,1) model and forecasting in EVIEWS, and the particular form adopted in this application of the numerical solution used by Akdeniz (1998) and Akdeniz and Dechert (1997) to solve Brock's (1979, 1982) growth model.

III.2 GARCH (1,1)

A full discussion of the analytical issues concerning the GARCH model can be found in the seminal paper by Bollerslev (1986), and in Hamilton (1994). This section focuses, instead, on the implementation of the GARCH (1,1) model in EVIEWS, and particularly on forecasting in EVIEWS after the GARCH(1,1) equation has been estimated for each return data set in the "in-sample" (days [1,..., 1300]).

Recapitulating section II.2.1, the GARCH (1,1) model is described by the following equations:

$$y_t = x_t \pi + \varepsilon_t \tag{3.3}$$

$$\varepsilon_t | \Psi_{t-1} \sim N(0, \sigma_t^2) \tag{3.4}$$

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{3.5}$$

A useful insight can be obtained by lagging equation (3.5) by one period and by substituting for the lagged variance on the right hand side. Then an expression with two lagged squared returns, and a two period lagged variance is obtained. By successively substituting for the lagged conditional variance, an illuminating expression is found:

$$\sigma_t^2 = \frac{\omega}{1-\beta} + \alpha \sum_{j=1}^{\infty} \beta^{j-1} \varepsilon_{t-j}^2 \tag{3.6}$$

An ordinary sample variance (c.f. classical historical variance estimator) would give each of the past squares an equal weight rather that declining weights (as in GARCH). Thus the GARCH variance is like a sample variance but it emphasizes the most recent observations. Since σ_t^2 is the one day ahead forecast variance based on past information, it is termed the conditional variance.

EVIEWS uses the return history of days [1, ...,1300], for each data set in turn, to estimate the parameters $\{\pi\}$ of the conditional mean (equation 3.3) and the parameters $\{\alpha, \beta, \omega\}$ of the conditional variance equation (equation 3.5).

The estimated α and β parameters convey a measure of volatility persistence in the system. As noted by Bollerslev (1986), $\alpha + \beta < 1$ is necessary for wide-sense stationarity; $\alpha + \beta = 1$ implies a unit root, and $\alpha + \beta > 1$ signifies an explosive process (see Bollerslev (1986), Hendry (1996)). The closer $\alpha + \beta$ is to unity, the more persistent are the volatility shocks, and the less rapidly their effect dies out.

Forecasts in EVIEWS in the out-of-sample horizon are computed according to the following expression:

$$\sigma_t^2 = \omega + (\alpha + \beta)\sigma_{t-1}^2 \tag{3.7}$$

Thus,

$$\sigma_{1301}^2 = \omega + (\alpha + \beta)\sigma_{1300}^2 \tag{3.8}$$

$$\sigma_{1302}^2 = \omega + (\alpha + \beta)\sigma_{1301}^2 \tag{3.9}$$

$$= \omega + (\alpha + \beta)\left[\omega + (\alpha + \beta)\sigma_{1300}^2\right]$$

and so on.

σ_{1300}^2 is computed using the information available in days [1, ..., 1299]. However, no new information of the (unknown) future variance is allowed to enter when making forecasts of the conditional variance out-of-sample. Since no new information is incorporated into the forecast, the GARCH forecast of conditional variance quickly converges to its long-run variance as the forecast horizon increases (Bollerslev(1986), Figlewski (1994)):

$$\sigma_{long-run}^2 = \frac{\omega}{1 - \alpha - \beta} \tag{3.10}$$

at a rate that depends on the value of $(\alpha + \beta)$.

III.3 The Numerical Solution

With the exception of restricted special cases of the utility and production functions, there generally are no closed form solutions for the optimal investment functions. Thus, numerical techniques are necessary for analysis of the properties of the asset pricing model. The numerical procedure adopted in this study is the projection method of Judd (1992). Alternative methods have been proposed in the literature[33], but Judd's method is chosen because it is fast, extremely accurate[34], and because the entire investment function can be estimated. The following discussion on the numerical method used in this study borrows heavily from Akdeniz and Dechert (1997) and their notation and equations are reproduced with their permission.

Judd's method takes advantage of the time stationarity of the solutions instead of solving for a specific solution to the Euler equation (2.30), and solves for the optimal investment functions $h_i(y)$. To be able to achieve this, optimal policy functions are characterized as Chebyshev sums:[35]

$$h(y, a_i) = \sum_{j=1}^{n} a_{ij} \psi_j(y) \tag{3.11}$$

[33] These methods are discussed in a special volume of the Journal of Business & Economic Statistics (January 1990).
[34] This method yields results that are accurate to within \$1 in \$1,000,000.
[35] See Rivlin (1990) for a description of approximation methods using Chebyshev polynomials.

Here the term ψ_j refers to the j-1 Chebyshev polynomial which has been shifted to the estimation range $[y_{min}, y_{max}]$[36]

$$\psi_j(y) = T_{j-1}\left(2\frac{y - y_{min}}{y_{max} - y_{min}} - 1\right) \tag{3.12}$$

and n is the number of coefficients used.

Observing that along an optimal solution the following relationship will hold:

$$x_{i,t+1} = h_i(y_t)$$

and

$$\begin{aligned} x_{i,t+2} &= h_i(y_{t+1}) \\ &= h_i\left(\sum_{i=1}^{N} f_i(h_i(y_t),\xi_t)\right) \end{aligned}$$

then define the residual functions from the Euler equation as:

$$\begin{aligned} \Re_i(y,a) = u'\left(y - \sum_j h(y,a_j)\right) - \\ \beta E\left[u'\left(\sum_j \left[f_j(h(y,a_j),\xi) - h(\sum_k f_k(h(y,a_k),\xi),a_j)\right]\right) f_i'(h(y,a_i),\xi)\right] \end{aligned} \tag{3.13}$$

36 This estimation interval is chosen such that this interval contains the observed ergodic distribution of y_t.

In numerical simulations Akdeniz and Dechert (1997) use a discrete probability space for the random shocks. In this study, a simple Markov process governs the arrival of the shocks. In what follows, the expectation notation will be used instead of writing out sums explicitly. The residual functions defined in equation (3.11) will all be equal to zero when evaluated at the optimal policy functions, for all values of y. In order to solve for the optimal values of the coefficients of the Chebyshev sums, a discrete set of y's corresponding to the zeros of a Chebyshev polynomial of order m[37] is taken. Then the projection[38] of the residual functions so defined is:

$$B_{ij}(a) = \sum_{k=1}^{m} \Re_i(y_k, a)\psi_j(y_k) \qquad (3.14)$$

and the coefficients are sought for which these projections are simultaneously equal to zero. This is accomplished via a Newton-Raphson[39] numerical routine.

Once the growth model is solved for the optimal investment functions, equation (3.6) is solved for the asset pricing functions. It is noted that this can be reduced to solving a set of linear equations. Then define:

$$G_i(y) = P_i(y)u'(c(y))$$

and

---

37 Here m is taken to be greater than n, the number of coefficients.
38 See Fletcher (1984) for the use of the Galerkin projection method in numerical analysis.
39 A C language routine is utilized, based on Akdeniz and Dechert (1997) and Press et al (1992).

$$b_i(y) = \sum_s u'(c(Y(y,\xi_s)))\pi_i(y,\xi_s)q_s$$

where a discrete (Markov) random variable is used with $P\{\xi = \xi_s\} = q_s$.[40]

Then, a solution to

$$\beta^{-1}G_i(y) = \sum_s G_i(Y(y,\xi_s))q_s + b_i(y) \tag{3.15}$$

is looked for the functions G_i. A way of achieving this is through the use of Chebyshev approximations to G_i. The coefficients of the approximating polynomials are then solved for. Letting

$$G_i(y) = \sum_{j=1}^{n} c_{ij}\psi_j(y) \tag{3.16}$$

with ψ_j is defined as before. Then let $y_1,\ldots,y_m$ be the zeros of ψ_m where m>n+1. Then one needs to find $c_{i1},\ldots,c_{in}$ such that, for k=1.....,m

$$\beta^{-1}\sum_{j=1}^{n} c_{ij}\psi_j(y_k) = \sum_s \sum_{j=0}^{n} c_{ij}\psi_j((Y(y_k,\xi_j))q_s + b_i(y_k)$$

[40] For programming purposes, a discrete random variable must be used. This discrete random variable is assigned from an 8-state Markov process.

$$= \sum_{j=1}^{n}\left(\sum_{s}\psi_j(Y(y_k,\xi_s))q_s\right)c_{ij} + b_i(y_k) \tag{3.17}$$

Note that this is linear in the coefficients. However, as there are m>n equations here, in general they cannot all be expected to hold. The solution will be taken to be those coefficients that minimize the sum of the least square errors between the left and right hand sides of the equations. Define **T** to be the *m x n* matrix whose *k, j* element is $\psi_j(y_k)$, and define **M** to be the *m x n* matrix whose *k, j* element is $\sum_s \psi_i(Y(y_k,\xi_s))q_s$. Also let $\mathbf{b}_i$ be the vector of elements of $b_i(y_k)$. Then in matrix form $\mathbf{a}_i$ is sought that minimizes the following expression below

$$\left[(\beta^{-1}T - M)a_i - b_i\right]'\left[(\beta^{-1}T - M)a_i - b_i\right] \tag{3.18}$$

The solution that minimizes (3.18) is $A = \beta^{-1}T - M$. Then, using the least squares principles the solution can be written as:

$$a_i = (A'A)^{-1}A'b_i.$$

Akdeniz and Dechert (1997) prove that equation (3.15) defines a contraction mapping of modulus β on the space of bounded continuous functions so that the procedure outlined above is well defined and has a unique solution[41]. The next section discusses the discrete

[41] For details, refer to their paper (1997).

Markov process that governs the arrival of shocks to the output and elasticities. The explicit form of the Markov chain is also provided. The parameters corresponding to the (Markov) states are reported in the appendix.

III.3.3 A Discrete Markov Process for the Shocks

In this application the Akdeniz (1998) and Akdeniz and Dechert (1997) model with 8 possible states of uncertainty is used[42]. The states vary from state 1 (very bad shock) to state 8 (very favorable shock). The states in between, namely states 2 to 7, have shocks that are going from bad to good in a stepwise increasing manner. The occurrence of state 1 results in an inordinately bad shock, compared to any other state; likewise occurrence of state 8 results in an disproportionally good shock, compared to neighboring states. For states 3 to 6 (intermediate shocks), the economy is equally likely to move to any other state. For states 1,2 7, and 8, the economy is constrained to move to one of 3 contiguous states. This calibration of the economy aims to increase the long run time spent by the economy in extreme states and induces excess kurtosis and volatility clustering in the return data. The Markov transition matrix between states is given in the Table 1 in the next page:

42 see Appendix F.

Table 1: Markov Transition Matrix Governing the Shocks

	state 1	*state 2*	*state 3*	*state 4*	*state 5*	*state 6*	*state 7*	*state 8*
state 1	*1/3*	*1/3*	*0*	*0*	*0*	*0*	*0*	*1/3*
state 2	*1/3*	*1/3*	*1/3*	*0*	*0*	*0*	*0*	*0*
state 3	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*
state 4	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*
state 5	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*
state 6	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*	*1/8*
state 7	*0*	*0*	*0*	*0*	*0*	*1/3*	*1/3*	*1/3*
state 8	*1/3*	*0*	*0*	*0*	*0*	*0*	*1/3*	*1/3*

III.3.4 The Numerical Solution

This section presents the solution to the model that is used to simulate the economy. This study uses the constant relative risk aversion utility function (CRRA),

$$u(c)=\frac{c^{\gamma}}{\gamma}$$

where γ, the utility curvature is set equal to -1.37 to match postwar U.S. data[43]. In

[43] Campbell and Cochrane (1994) estimate that a CRRA utility function with a utility curvature parameter of -1.37 matches postwar US data. This value is used to match the data to selected U.S. indices (see Chapter IV.)

keeping with the common practice in the literature, the value of the daily discount parameter β is taken to equal 0.99987413. The production side of the economy is characterized by firms with the Cobb-Douglas production functions:

$$f(x,\xi) = \theta(\xi)x^{\alpha(\xi)} + (1-\delta(\xi))x$$

where ξ is the shock parameter in the production function. The values of α and β are chosen at random, and a value of δ is picked such that it corresponds to values that agree with aggregate data. A glance at Appendix F reports the wide variability that is used for the shock parameters in different states of the economy.

Despite the fact that the estimation problem appears to be a relatively straightforward computational exercise, there are a variety of difficulties that appear in the implementation of a numerical procedure that converges to the desired optimal policy functions. These problems are discussed in Akdeniz and Dechert (1997). To produce an economy that mirrors real financial data in its many properties requires a considerable variation in the parameters used in the numerical model. The Markov model for the shocks produces the desired autocorrelation in squared returns that is very widespread in financial data. Very bad shocks at the extreme states drives fat-tailedness in the return data. The relatively larger negative extreme shocks forces negative skewness, in keeping with what is observed in the equity markets. Forcing the numerical model to converge to the desired policy functions while mimicking all the desired properties of empirical financial data is not easy in practice.

CHAPTER IV

RESULTS

Using Akdeniz (1998) and Akdeniz and Dechert's (1997) numerical solution to Brock's (1979, 1982) stochastic growth and asset pricing models, 1,000 simulations are done to obtain 1,500 daily returns (per simulation). Akdeniz's computational dynamic general equilibrium model has enjoyed considerable success in simulating the real economy, most specifically in its ability to produce levels of output and consumption data that closely mirror real data, and in standing up to the predictions of the CAPM. This study has sought to fit the numerical model to reflect real financial data by a change in the parameters of the model, and by introduction of a Markov process driving the shocks to output levels and elasticities. Real financial data, for example returns of indices such as that of S&P500, exhibit volatility clustering, excess kurtosis, negative skewness, strong autocorrelation in the squared return time series, and first order autocorrelation in the returns time series. The numerical model in this study induces returns time series that have all these properties. Each simulation produces daily returns for three firms. For ease of exposition, only the first firm is considered. The following table presents a set of descriptive statistics for all 1,000 simulations:

Table 2: Average, Minimum, and Maximum Kurtosis and Skewness, 1,000 Simulations

	average	minimum	maximum
kurtosis	3.944	3.021	4.848
skewness	-0.7452	-1.043	-0.431

The average kurtosis and skewness are computed through programming in EVIEWS[1]. Table 2 shows that the data in the simulation model is pronouncedly leptokurtic and that there is an asymmetric response of volatility to news- as seen by the significantly negative average skewness. Thus, an increase in volatility is likely to be greater following a large downward move in return, referred to as the leverage effect[2].. These two findings -leptokurtosis and significantly negative skewness- closely mirror empirical financial data of many types, including equities[3], and the simulated economy captures many of the empirical regularities of asset returns. A comparative base for kurtosis and skewness in the data in relation to daily return distributions in United States indices is provided[4] in Table 3:

1 see appendix
2 first noted by Black (1973), subsequently studied by Nelson (1990); Glosten, Jagannathan and Runkle (1993); Engle (1993) and others.
3 see Bollerslev, Engle and Kroner, 1998.
4 The distributional characteristics of all indices (except S&P500) are reported from Salih and Kurtas (1998). The kurtosis and skewnesses refer to the distribution of returns in *pre and post-index futures* introduction periods.

Table 3: Distributional Characteristics of Simulated Data versus a Selection of Indices

INDEX	SKEWNESS	KURTOSIS
SIMULATION	-0.7452	3.944
NYSE	-0.380	3.919
RUSSEL 2000	-0.914	3.739
S&P500[5]	-0.691	3.873

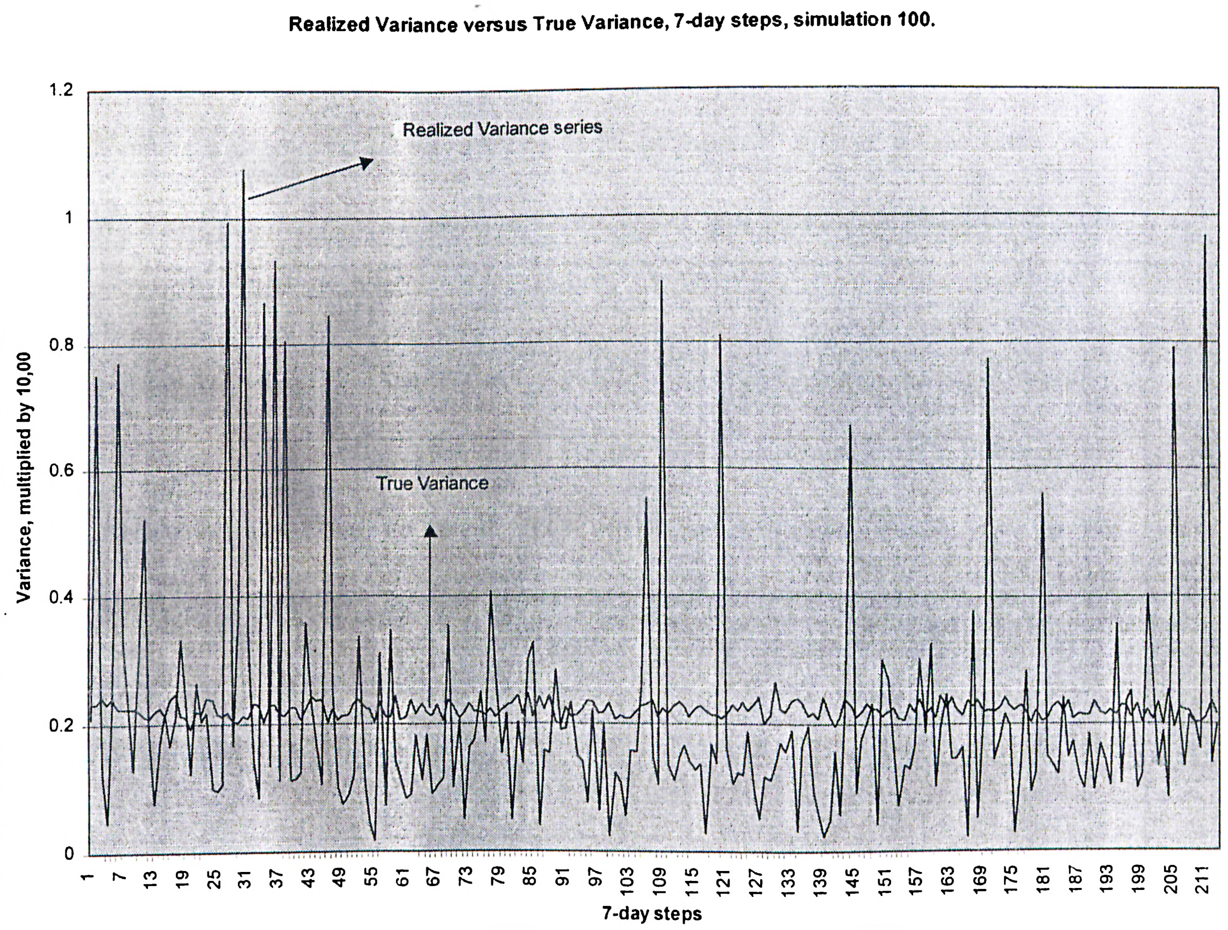
Table 3 compares skewness and kurtosis across three U.S. indices and the simulated data that this study uses. U.S. indices are chosen for comparison because the numerical model is calibrated to U.S postwar utility curvature and discount parameters. It is immediately apparent that the simulated data has distributional characteristics that closely mimic real return data

As noted before, because volatility forecasting is an integral part of many applications in the finance and econometrics literature, and because ARCH type models for the conditional variance have received such wide attention, the out of sample forecast performance of many models have been tested. All existing studies use realized volatility to judge the performance of volatility forecasting models. The following two figures demonstrate how realized volatility can be an unfair benchmark for performance. For notational convenience, all the variance parameters are multiplied by 10,000 in the

[5] The S&P500 composite index returns are computed for the period 1st January 1985 to 1st January 1987.

remainder of this study.

Figure 1: Realized Variance versus True Variance, 7-day steps



The darker series in Figure 1 is a time-series plot of realized variance, from only one sample time series chosen randomly from the simulation. The realized variance is computed as the variance per 7-day period in the simulation sample return time series. In 1,500 days, there are 214 such 7-day periods, hence the abscissa in the graph runs until the 214th 7-day period. The smoother series is the true variance time series which is calculated from the true probability of multiples of the 7th day[6]. Thus realized variance in this sample computed over days 1,..., 7 would be expected to correspond closely to the true variance on day 7, and so on, for realized volatility to be a fair measure as a benchmark. A look at Figure 2 shows this is not the case. In fact, the sample realized variance seems to spend most of its time below the time series for true variance, and frequently peaks dramatically (to an order of 450% of the true variance on the same day). This shows that using realized variance as a benchmark can be quite misleading. This is especially true for out of sample performance studies, as the error in forecast using a realized sample variance as a benchmark could be potentially huge, whereas the error computed relative to the true variance might be much less, especially when the model is actually forecasting the true variance quite well. The following figure repeats the procedure described above for multiples of 22-days:

6 That is to say days 7, 14, 21, 28,..., 1498.

Figure 2: Realized Variance versus True Variance, 22-day steps

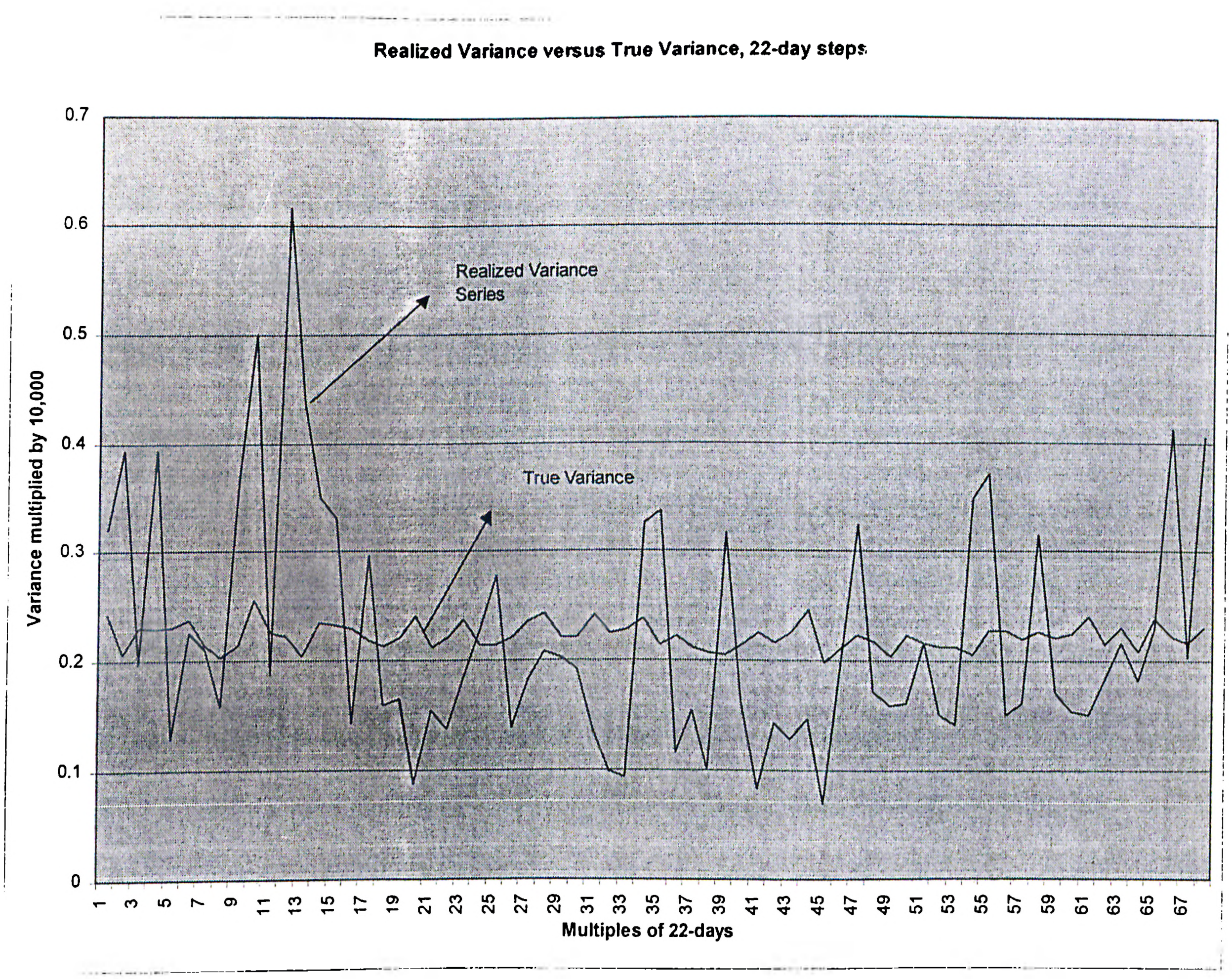


Figure 2 conveys a similar message, albeit in a less pronounced way. This is to be expected as the peakness of the realized variance is smoothed by incorporating more data. The total number of 22-day periods in this case is 68, and these multiples of 22-days are plotted in the abscissa. Realized variance is the variance in each 22-day period, and corresponds to the peaked series in the figure. The less peaked series is a plot of multiples of 22nd day true variances (days 22,44,66,...,1498). Although the relative difference of the two series is less pronounced than in Figure 1, the same pattern of high bias is observed in the (simulation) sample realization. These two figures jointly provide evidence that realized volatility is not a reliable benchmark to compare forecast variances from conditional variance models. It is evident that this has serious implications for performance studies, not least because it is precisely the case that realized variance is biased in favor of historical variance estimators, which have been proposed as alternatives to ARCH models. Using it as a benchmark is sure to favor historical estimates and penalize ARCH models.

This study uses the GARCH (1,1) model[7]. With each simulated return set, GARCH is estimated on the first 1300 days (the sub-sample) and out-of-sample forecasts are computed for the next 7 and 22 days, according to the procedure described in equations (3.5) to (3.7). It is known that GARCH forecasts perform best in the short-run, and that long run forecasts simply converge to the long-run variance (equation (3.8)). Therefore short-run forecast performance is concentrated upon.

GARCH estimation and forecasts are carried out in EVIEWS. The following discussion illustrates the procedure with an example. One simulation is chosen at random[8], and the distributional characteristics, diagnostic tests, and forecasts are studied. General guidelines for dynamic misspecification, remaining ARCH effect tests, the

7 abbreviated to GARCH throughout the remainder of this paper.
8 to be specific, simulation 260 is used. The results are, on the whole, generalizable to other data sets.

Ljung-Box (1977) diagnostic test, and a particular instance where the GARCH model rapidly adjusts to a series of bad shocks to produce very accurate forecasts (in contrast to Classical Historical Volatility which performs very poorly) are addressed here. Attention is then turned to overall performance in out-of-sample of GARCH and Classical Historical Volatility[9]. Mean Absolute (forecast) Error[10] and Root Mean Square (forecast) Error[11] are computed, as described in Section III.1, and final results are given.

IV.1 An Example[12]

The example considered is one of 1,000 simulations. The return series is named "ret260", and the table below is the standard EVIEWS descriptive statistics window:

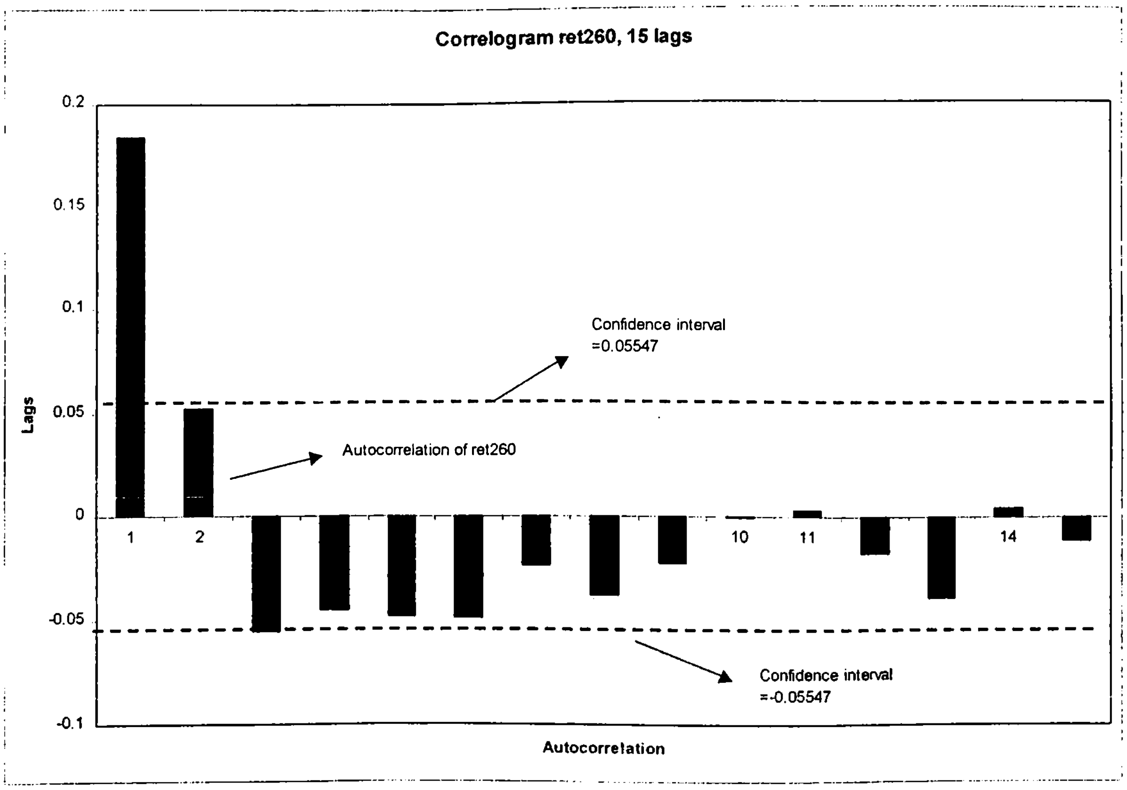
Table 4: Distributional Characteristics for ret260

Mean	**0.002494**
Median	0.003280
Maximum	0.016368
Minimum	-0.016778
Std. Dev.	0.004689
Skewness	-0.773897
Kurtosis	4.196743

9 abbreviated to CHV throughout the remainder of this paper.
10 MAE.
11 RMSE.
12 It is emphasized that the example reflects all the other simulated series in that exactly the same estimation procedures are carried out on these.

Descriptive statistics for series ret260 conform to the pattern found earlier in the average simulated data characteristics. There is excess kurtosis and leverage effect in the sample return series. Next the correlogram of the return process is looked at in Figure 3:

Figure 3: Correlogram of ret260 for 15 Lags



The return series has significant positive autocorrelation in the first lag. The Ljung-Box (1978) Q-statistic on the first fifteen lags follows a χ^2 distribution with degrees of freedom equal to the number of lags. It is common practice in the literature to compute the LB test statistic at 15 degrees of freedom. This statistic has a critical value of

24.99 at the 5% level. Figure 3 confirms that the original return260 series has high serial correlation in the levels, and is found to have Ljung-Box Q-statistic of 79.595 at 15 lags.

The squared return, taken as a proxy for the variance, is called var260. It exhibits a much greater degree of autocorrelation and a slow decay, confirming strong time dependence in volatility. The correlogram of var260 up to 15 lags is given in Figure 4.

***Figure 4: Correlogram of var260 for 15 Lags*:**

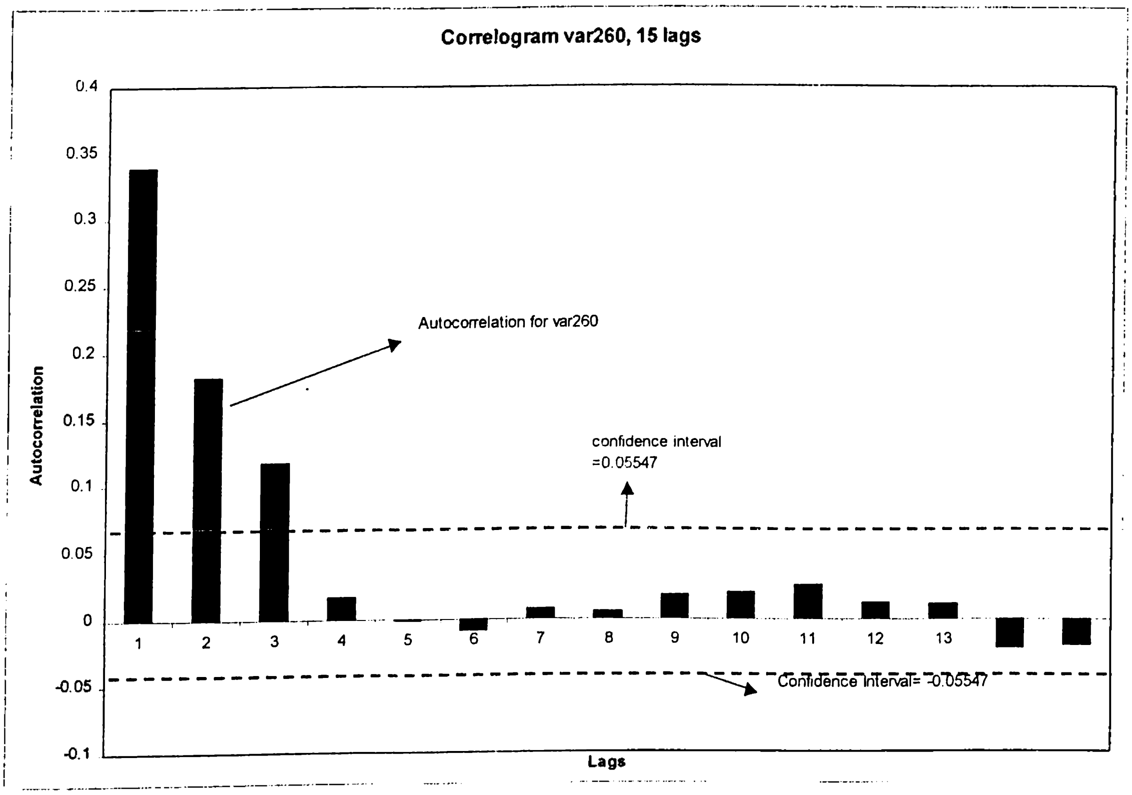


Figure 4 reports very strong autocorrelation for the squared return series, decaying gradually over 3 lags. The LB test statistic at 15 lags has a value of 241.77, and a probability of 0.0000 at all lags. The descriptive statistics and correlogram of ret260 and var260 are very good examples of what numerous financial data have been found to exhibit, in particular to those which ARCH type tests are applied. Thus the raw data provide extremely convincing evidence of conditional heteroskedasticity.

Tables 5 to 8 detail the output of GARCH (1,1) applied to the ret260 series on the first 1300 days. Due to the significant first lag autocorrelation of the return, the conditional mean is regressed on a constant and its own first lag. A correct conditional mean specification is vital to the success of GARCH, and the following conditional mean specification is adopted in GARCH estimation in EVIEWS on all the simulated return series:

$$r_t = \pi_1 r_{t-1} + \pi_2 + \varepsilon_t \tag{4.1}$$

where r_t is the return in period t, π_i are coefficients to be estimated (π_2 is a constant), and ε_t is the residual.

Table 5 summarizes some preliminary estimation details, such as the sample size (in this case days 1 to 1300), the number of iterations used to maximize the likelihood, and the options used in estimation (the BHHH algorithm and Heteroskedasticity Consistent Covariance Estimator to reflect the fact that robust standard errors are used).

Table5: GARCH (1,1) Equation Label in EVIEWS

ARCH // Dependent Variable is RET260
Sample(adjusted): 2 1300
Included observations: 1299 after adjusting endpoints
Failure to improve Likelihood after 19 iterations
Bollerslev-Woodrige robust standard errors & covariance

Table 6 reports the estimates for the conditional mean in the GARCH estimation. RET260(-1) refers to the first lag on which the return time series is regressed. The use of the lagged dependent variable explains why the sample size is adjusted to 1299.

Table 6: GARCH Output for the Conditional Mean

Conditional Mean Equation				
Variable	**Coefficient**	**Std. Error**	**t-Statistic**	**Prob.**
RET260(-1)	0.135892	0.026949	5.042643	0.0000
C	0.002253	0.000115	19.63080	0.0000

It can be seen that both the lagged dependent variable (RET260(-1)) and the constant in regression are highly significant. The autocorrelation in the first lag of the return process is in agreement with empirical data[13].

Table 7 gives output summaries for the conditional variance equation. Here C refers to the ω term, ARCH(1) to the α term, and GARCH(1) to the β term in equation (3.3).

Table 7: GARCH Output for the Conditional Variance

Variance Equation				
Variable	**Coefficient**	**Std. Error**	**t-Statistic**	**Prob.**
C	9.27E-06	1.08E-06	8.591400	0.0000
ARCH(1)	0.450936	0.053984	8.353159	0.0000
GARCH(1)	0.119120	0.053104	2.243129	0.0251

This table shows that there is long-run mean reversion in the conditional variance (significant C term), and that the data exhibits significant ARCH and GARCH effects, the former being highly significant (at order 0.0000), and the latter significant at 5 percent. The maximized log-likelihood, which is not reported in the tables, is 5227.32 for this equation.

There are various ways for testing for the adequacy of the GARCH (1,1) model.

13 Sheedy (1997) studies the relationship between volatility clustering and return correlation. Also see Bertero and Mayer(1989), Boucrelle and Le Fur (1996).

The simplest way is to re-estimate with a more complex specification, and to see whether the new parameters are significantly different from zero. For this example, adding lags to both the return process and adding extra ARCH and GARCH terms are found to provide no real improvement, and additional parameter coefficients are not found to be significant. The EVIEWS diagnostic menu has a set of tests which can be applied to investigate the adequacy of a particular ARCH equation. GARCH (1,1) is examined in terms of these diagnostic tests, which are formulated in terms of standardized residuals and squared standardized residuals.

Standardized residuals are the conventional residuals divided by their one step ahead standard deviation. If the model is correctly specified, the standardized residuals would be expected to be distributed as a mean zero and variance one series, although they need not be necessarily normal. After GARCH estimation, the standardized residuals for ret260 are found to have variance 0.999960 and mean -0.029332, suggesting good fit.

Another diagnostic test is the correlogram of the standardized residuals. This is presented in Figure 5:

Figure 5: Correlogram of Standardized Residuals for 15 Lags

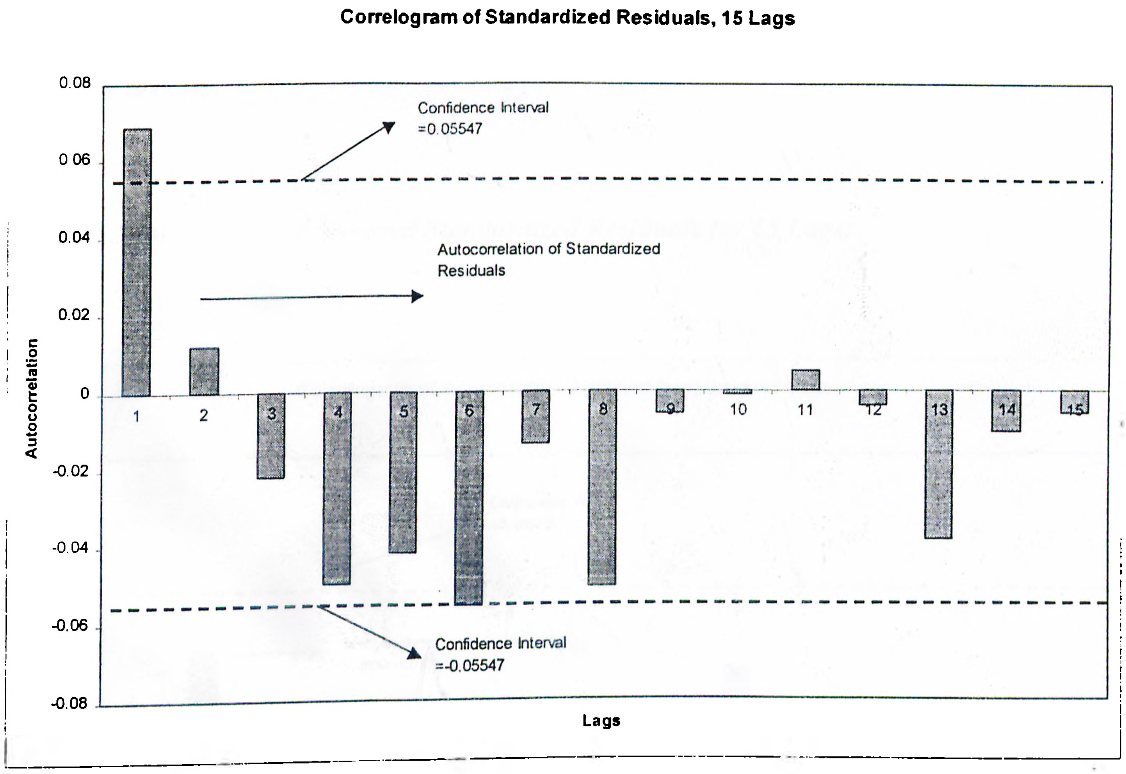


Figure 5 demonstrates that the conditional mean equation is correctly specified. The LB test statistic at 15 lags is 22.821, which is not significant at the 5% critical value of 24.99. After GARCH estimation, the null hypothesis of no autocorrelation in the standardized residuals cannot be rejected. Compared to the raw ret260 series, which had a Q-Statistic of 79.819 at 15 lags, a dramatic improvement is obtained by modeling the conditional mean in the form of equation (4.1). To test for the presence of additional ARCH effects, the squared standardized residuals and their serial correlations are looked at. The ARCH-LM test is a formal test for heteroskedasticity, and consists in regressing squared residuals on their own lagged values. Thus, failure of the ARCH-LM test to detect additional heteroskedasticity would clear the way for concluding that the GARCH (1,1) is the correctly specified model, and that it captures all the relevant ARCH effects in the data. Figure 6 reports the correlogram of the squared standardized residuals after GARCH is done.

Figure 6: Correlogram of Squared Standardized Residuals for 15 Lags:

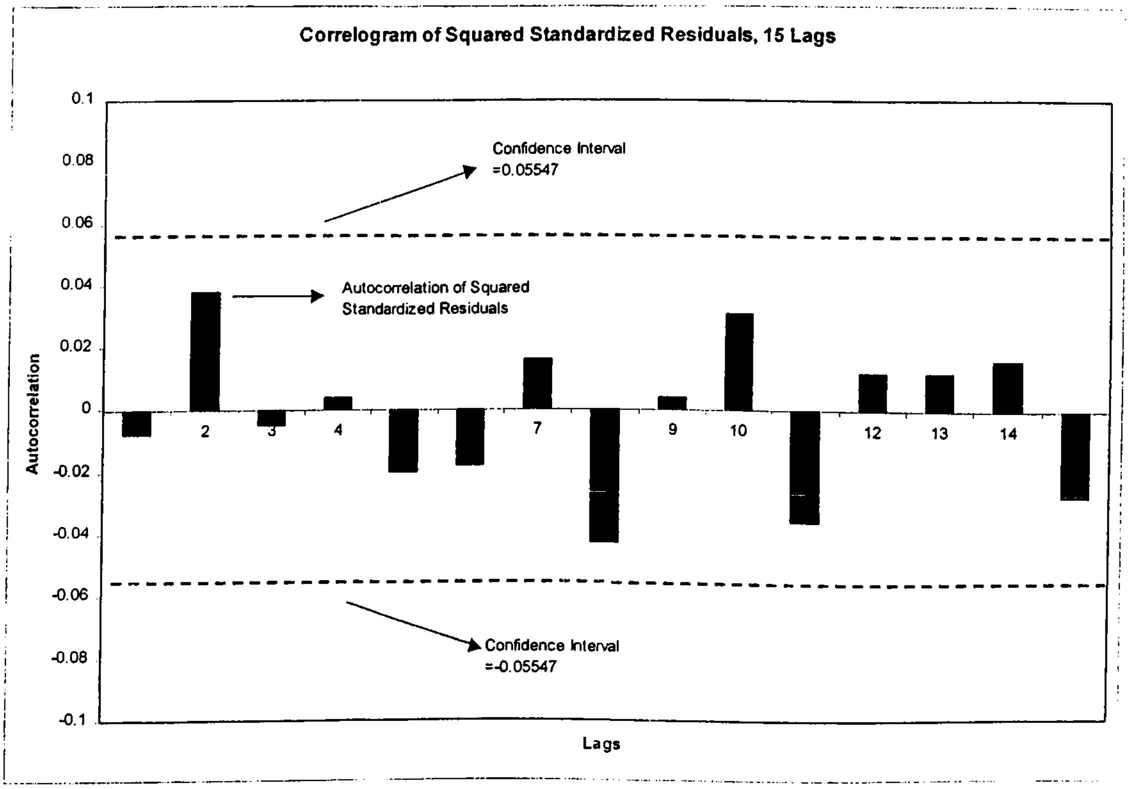


Figure 6 presents some dramatic results. The squared standardized residuals have a p-value of 0.793 at 15 lags, and the LB statistic is 10.417. Comparison with the series var260 reveals that the LB statistic has fallen from a highly significant 248.30 to 10.417, indicating that the conditional variance process is being modeled correctly with GARCH

(1,1). This result is confirmed with the ARCH-LM test at 15 lags. This test uses the following regression to test for the significance of past squared residuals:

$$u_t^2 = \beta_0 + \beta_1 u_{t-1}^2 + \beta_2 u_{t-2}^2 + \ldots + \beta_{15} u_{t-15}^2 \qquad (4.2)$$

The ARCH-LM tests for additional ARCH effects. The output is an F-statistic and a TR^2 statistic, distributed as χ^2, which is the outcome of a Lagrange Multiplier **(LM)** test with degrees of freedom equal to the number of lagged, squared residuals. The outcome of the ARCH-LM test is presented in the next page in Table 8:

Table 8: ARCH-LM Test After GARCH Estimation is done, 15 Lags.

ARCH LM Test:

F-statistic	***0.681958***		**Probability**	***0.804348***
Obs*R-squared	***10.27555***		**Probability**	***0.802058***

Sample(adjusted): 17 1300

Included observations: 1284 after adjusting endpoints

White Heteroskedasticity-Consistent Standard Errors & Covariance

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	1.020075	0.116663	8.743781	0.0000
STD_RESID^2(-1)	-0.003590	0.030159	-0.119037	0.9053
STD_RESID^2(-2)	0.040156	0.039897	1.006489	0.3144
STD_RESID^2(-3)	-0.010904	0.026851	-0.406086	0.6847
STD_RESID^2(-4)	0.005521	0.025711	0.214733	0.8300
STD_RESID^2(-5)	-0.018278	0.026386	-0.692700	0.4886
STD_RESID^2(-6)	-0.019869	0.027163	-0.731496	0.4646
STD_RESID^2(-7)	0.011925	0.026210	0.454963	0.6492
STD_RESID^2(-8)	-0.039774	0.024798	-1.603927	0.1090
STD_RESID^2(-9)	0.005970	0.026845	0.222385	0.8240
STD_RESID^2(-10)	0.031438	0.032147	0.977939	0.3283
STD_RESID^2(-11)	-0.037047	0.026433	-1.401501	0.1613
STD_RESID^2(-12)	0.010593	0.033188	0.319180	0.7496
STD_RESID^2(-13)	0.016913	0.028543	0.592536	0.5536
STD_RESID^2(-14)	0.013981	0.026639	0.524848	0.5998
STD_RESID^2(-15)	-0.028586	0.026098	-1.095322	0.2736

The ARCH-LM test confirms there are no remaining ARCH effects after GARCH(1,1) is done. Testing at 15 lags for the squared standardized residuals, the F-statistic probability is 0.804380 while the χ^2 statistic probability is 0.802058. This agrees well with the results depicted in Figure 7, where the Q-statistic at 15 lags had a probability of 0.793. Hence, the set of results reported in Tables 3 to 8 and Figures 3 to 6 provide convincing statistical evidence that the GARCH (1,1) model is adequate for the series in this example.

Before presenting overall results of GARCH performance out-of-sample, it is sensible to do a preliminary assessment of the performance of GARCH in the return series discussed in the example. To illustrate, the following is carried out: GARCH forecasts of variance in the forecast horizon (days [1301...1307]) is compared to true variance on day 1307. For reference, realized variance over the same period is also given, which is a sample variance in the out-of-sample realization.

Table 9 reports forecasts of average volatility for days [1301,..1307] for CHV and GARCH, and compares the forecasts to true volatility on day 1307. Figure 7 shows the GARCH 7-day ahead variance forecasts for days 1301,...,1307.

Table 9: Forecasts Out-of-Sample Using GARCH and CHV.

TRUE VARIANCE[14]	0.2180
FORECAST AVE. VARIANCE	
REALIZED VARIANCE	0.1730
GARCH	0.2070
CHV	0.0741

This example highlights how badly amiss the CHV model forecasts go, compared to GARCH forecasts. Recent extreme news in the raw return series adversely affects the performance of CHV, while GARCH forecasts quickly adjust and come quite close to the true 1307th day variance.

14 True variance on day 1307.

Figure 7: Out-of-Sample GARCH Forecasts compared to True and Realized Variance

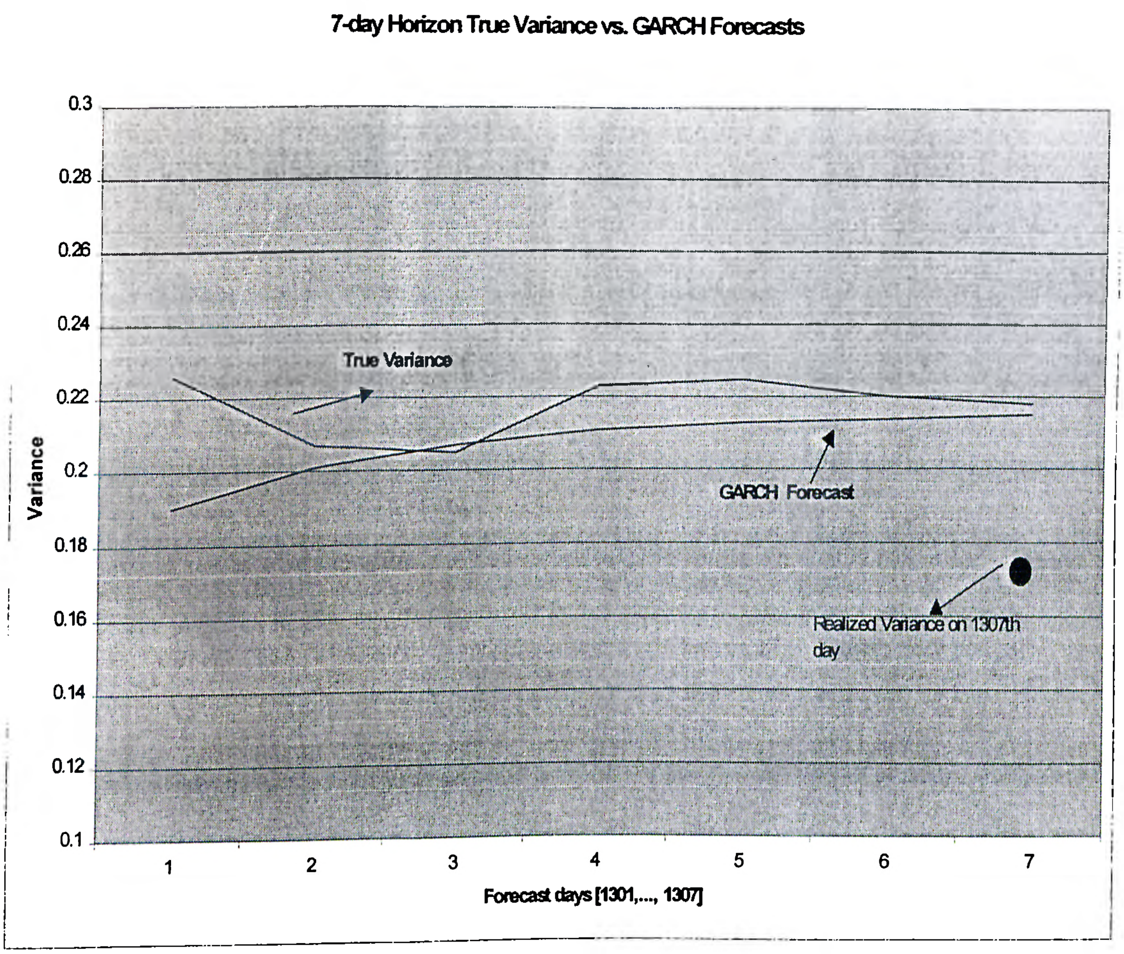


Table 9 and Figure 7 provide two key insights: that it would be concluded that GARCH performs less well if realized variance instead of true variance is used as the benchmark, and that CHV performs badly regardless of which benchmark is used, but less badly compared to realized variance. This is due to the fact that realized variance is biased towards the CHV model. Although this is merely an example, it provides the motivation to look at how GARCH performs in the aggregate across all the simulations. It also highlights an essential problematic across all studies on GARCH forecast performance, the absence of a true variance measure as benchmark. Basing GARCH forecast performance in the example on realized variance would lead to the false conclusion that GARCH performs less well than it actually does.

IV.2 Overall Forecast Performance Across All Simulations

The extended example was offered as an illustration of what is done per each simulation. To obtain overall performance, two things are done. First, GARCH and CHV forecasts are calculated for each simulated return series, the average over 7-day and 22-day forecast horizons are computed, and compared to the true variance on the 1307th or 1322nd day respectively. The criteria used to evaluate performance are the RMSE and MAE.. Next, the realized variance is calculated for each simulation over the forecast horizons[15]. This is the benchmark that has been used in the literature to test performance of GARCH. It is in this respect that the results are startling and compelling. Tables 10(a) and 10(b) report the results. Using *realized variance as benchmark*, GARCH performance is only marginally better (and only in the MAE sense) than a naive estimator such as CHV for the 7-period ahead forecast, and is much worst than CHV at the 22-day forecast horizon. The picture changes drastically when true *variance as benchmark* is used. Here GARCH has a much lower RMSE and MAE at both horizons than it had when

15 That is, the variance of return for days 1301,..., 1307 and days 1301. ...,1322 are computed for each of the 1,000 simulations. Programming for this is done in EVIEWS.

realized variance was used as benchmark. It also outperforms CHV by a large margin. This demonstrates that the GARCH model has considerably lower forecast error (both in RMSE and MAE sense) than previously found in the empirical literature when the true-distribution variance measure that accounts for all the different possible states of the economy in the forecast horizon is used as benchmark. This "true" variance is the second moment of the true return distribution that is obtained from 1,000 simulations for each day of the computable dynamic general equilibrium model. The realized variance per each simulation forecast horizon is only one possible realization of the second moment of the true return distribution in that horizon, and hence is just an estimate of the true variance. Here one can do better and use the true parameter and not just the best possible estimate based on available information in a particular time series realization. This enables one to use full information about the economy, and is the benchmark that *should* be used if one had the luxury to be able to discover it. The computable general dynamic equilibrium framework affords that luxury. The striking implications for GARCH performance are reported in Table 10.

Table 10a: Forecast Evaluation of GARCH in 1,000 Simulations[16]

CRITERION	RMSE 7-DAY	MAE 7-DAY	RMSE 22-DAY	MAE 22-DAY
GARCH~TRUE	0.0622	0.0327	0.0512	0.0303
GARCH~REALIZED	0.1890	0.0948	0.1280	0.1250
CHV~TRUE	0.2490	0.1580	0.1190	0.0896
CHV~REALIZED	0.1880	0.1250	0.0122	0.0108

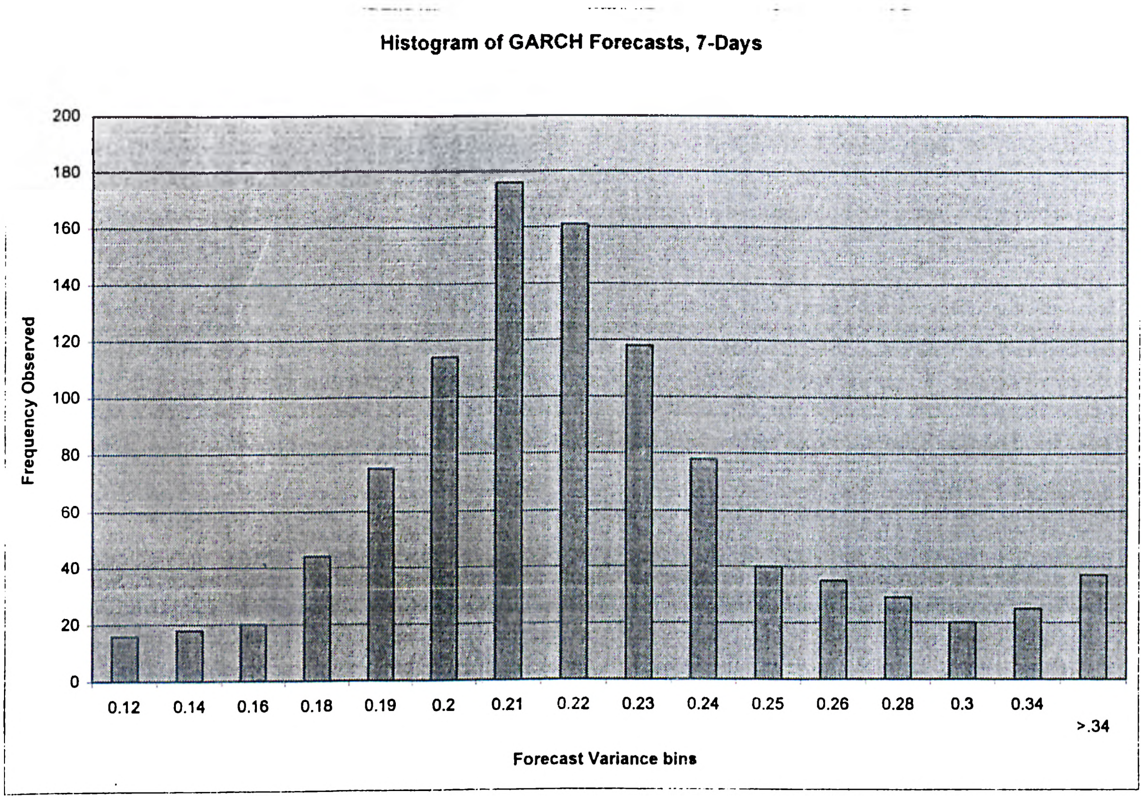
Table 11b: Ratios

MODEL	RMSE	MAE
RATIO CHV7D~TRUE / GARCH7D~TRUE	4.003	4.832
RATIO CHV22D~TRUE / GARCH22D~TRUE	2.324	2.759

In addition to documenting the unmistakable improvement in GARCH forecast performance using true variance as benchmark, Table 10 also confirms that, as the forecast horizon lengthens, the ratio of CHV forecast error decreases relative to GARCH forecast error. This is to be expected as realized variance is closer to true variance as more observations are included.

16 G7D, for example, refers to GARCH 7-day forecast and H22D pertains to classical historical variance forecast for 22-days.

Figure 8: GARCH Histogram of 7-day Ahead Forecasts Across All Simulations[17]:



This figure reports the overall GARCH forecasts across all 1,000 simulated data sets. Note that GARCH forecasts are distributed predominantly about the true variance (0.218). This demonstrates how good a job GARCH is doing in out-of-sample tests in the whole. Now comparing this with CHV forecasts, the results are all the more striking:

17 Note that the **true variance** on day 1307 is 0.0000218. As all variances are multiplied by a factor 10,000 the true variance in the histograms corresponds to ***0.218***. Similarly, for 22-day ahead forecasts, the true variance is ***0.225*** on day 1322.

Figure 9: CHV Histogram of 7-day Ahead Forecasts Across All Simulations:

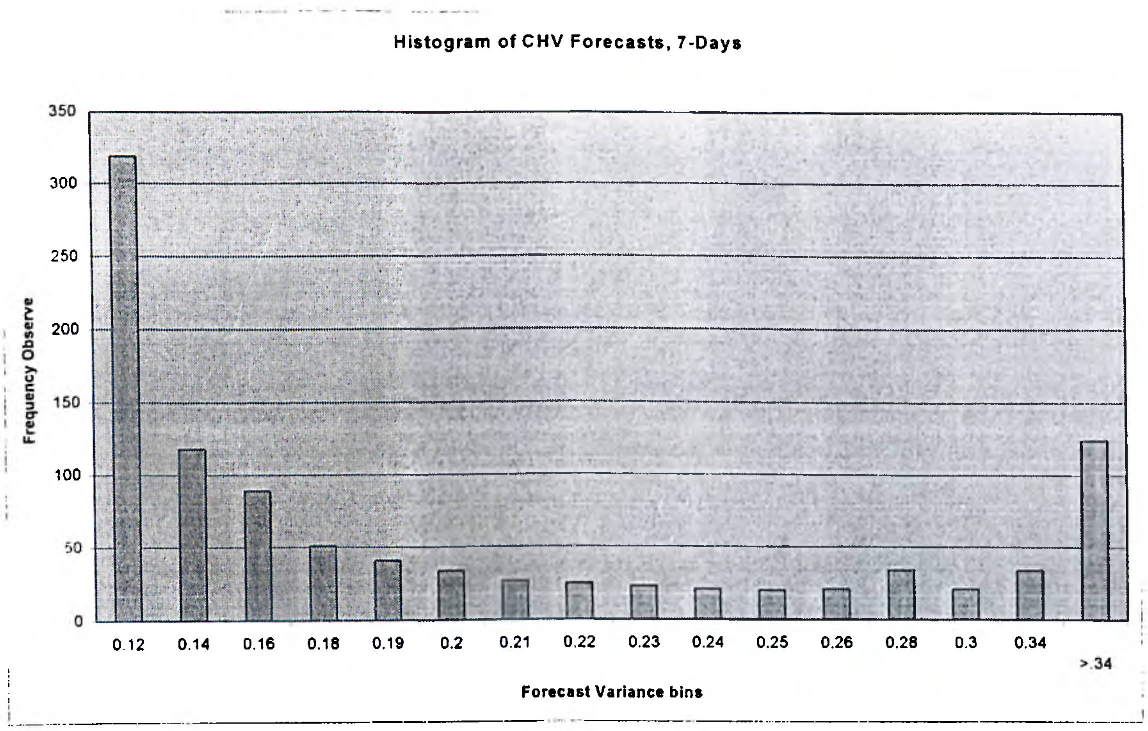
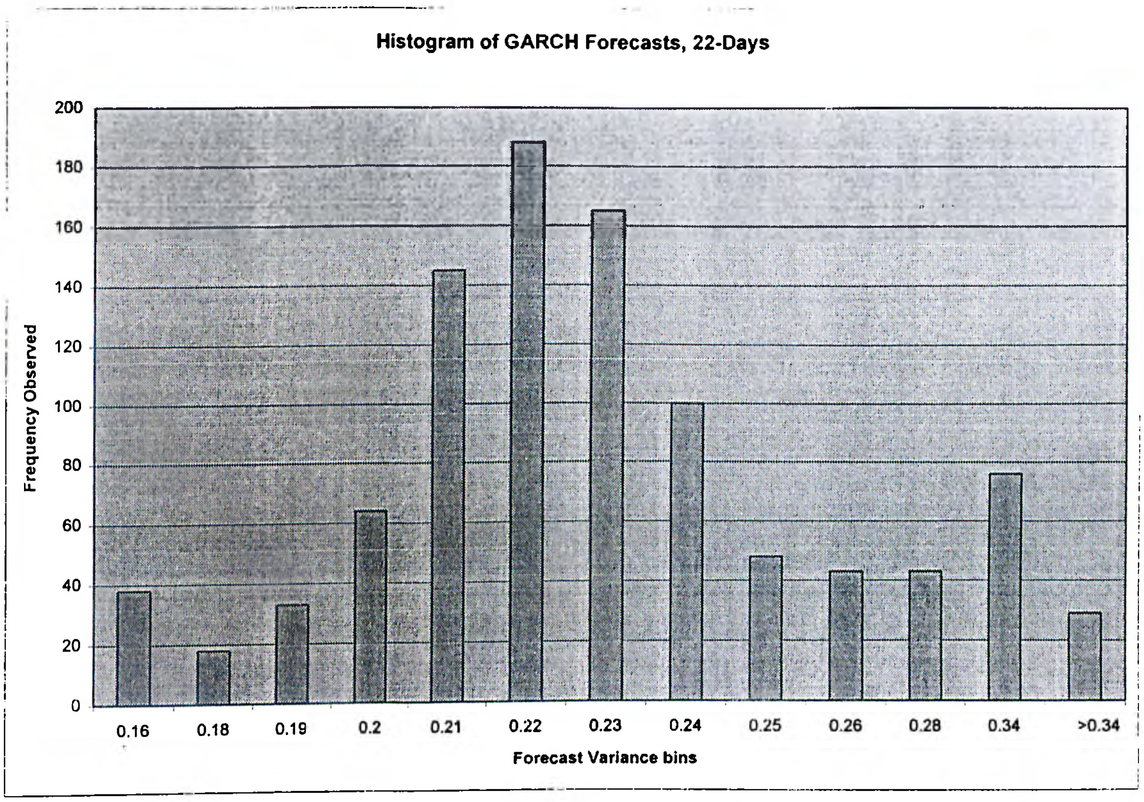


Figure 9 demonstrates that, at the 7-day ahead forecast horizon, CHV forecasts are going badly off. Observe also the low frequency with which CHV forecasts come within proximity of the true 1307th day variance. The dominating majority of CHV forecasts appear to be concentrated in very high or very low variance levels. In fact, one observes the exact opposite for CHV performance in comparison to GARCH performance[18]. Thus the CHV model's forecasts leave a lot to be desired. Figure 11 reports the histogram of GARCH forecasts at the 22-day horizon:

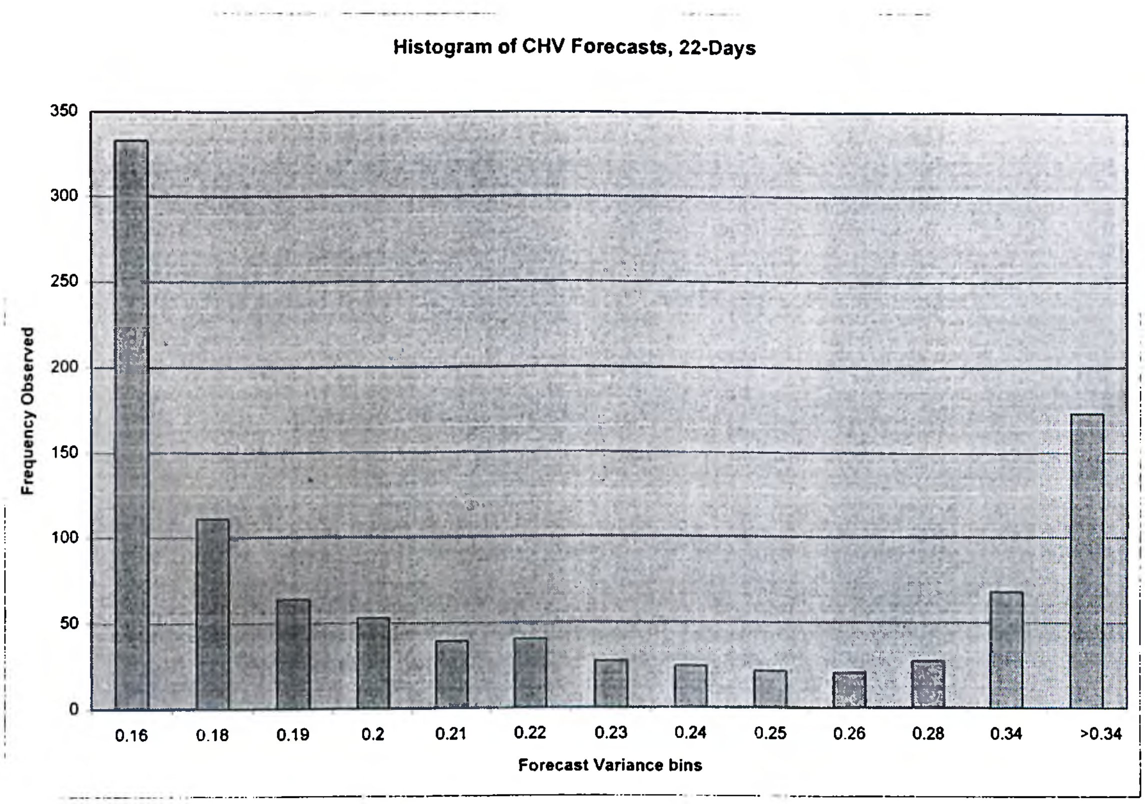
[18] See Figure 9.

Figure 10: GARCH Histogram of 22-day Ahead Forecasts Across All Simulations:



Here again, the GARCH (1,1) model forecasts very accurately in the aggregate. Note that the great majority of forecasts exhibit a high degree of proximity to the **true 1322nd day variance (0.225).** This finding too is contrasted with CHV performance at the same horizon in Figure 11:

Figure 11: CHV Histogram of 22-day Ahead Forecasts Across All Simulations:



As more information is incorporated in the Classical Historical Volatility estimate, less variation is observed in the aggregate forecasts. However, using more past information simply makes the CHV model approximate the long run variance parameter more closely. This is not necessarily more efficient for pointwise variance estimates[19] as more recent information might be more revealing for forecasts into the immediate future. At the 22-day horizon, the histogram again gives little room to uncertainty about the poorness of CHV forecasts.

[19] Pointwise in the sense that a "point", i.e. a day (such as day 1307, or 1322) is forecasted.

CHAPTER V

CONCLUSIONS

This study has applied a computable dynamic general equilibrium framework to study the performance of variance forecasting models. All previous studies have used a sample realization in the forecast horizon to test the out-of-sample performance of such models. This study has differed markedly from the literature in proposing to find and implement the real variance as the benchmark for out-of-sample forecast performance tests. The true variance for each time is obtained from the true distribution of 1,000 simulations of the computable dynamic general equilibrium model. This true distribution reflects all the possible states of the economy. The real variance thus reflects the true variability of returns in the economy in the forecast horizon. For a conditional variance model to perform well, it must be able to forecast the true variance of returns in the economy, and not just one of many possible sample realizations of the return variance, as all studies have done so far. It has moreover been shown that the computational dynamic general equilibrium framework is quite successful in mirroring reality in the equity financial markets.

It is found that the realized volatility, as used in the literature, is biased against the GARCH (1,1) conditional variance model, and using realized variance results in a relatively poor performance at 7-day and 22-day horizons. The GARCH model performs dramatically better when the true variance is used, and improvements in forecasts from 232% to 483% percent are recorded in 7-day and 22-day horizons, using the MAE and RMSE criteria[1]. The historical variance models are found to record very poor performance compared to true variance in both horizons. The computable general equilibrium framework hence provides a setting for a formal test of alternative variance forecasting models. The results have also shown that realized variance is a very precarious benchmark when evaluating the performance of different conditional variance forecast models. This is offered as an explanation of why many studies of GARCH type models have typically documented such a large disparity in forecast performance of such models. It is believed that the use of out-of-sample realized variance exemplifies a misconception at the center of the existing literature on GARCH performance.

There are a wide range of future developments which are possible in this field. No attention has been focused on using a simulated dynamic equilibrium setting to formally test the many variance forecasting models in the literature. One possibility is to extend the approach in this study to test the forecast performance of different GARCH type models, such as the EGARCH (Nelson, 1990) or the TARCH (Zakoian, 1993). A re-parametrization of Akdeniz (19960 and Akdeniz and Dechert's (1995) model could also

[1] see Table 10.

be carried out to induce further characteristics in the data that make it suitable to an alternative type of the GARCH family of models. The results of this paper would also suggest that such models would perform better than they have been claimed to do in the existing literature.

An alternative suggestion would be to compare out-of-sample performance of a battery of GARCH tests (including GARCH (1,1) and EGARCH) to Classical Historical Volatility and to Non-Parametric and Stochastic Volatility models. It is predicted that a semi-parametric GARCH model (where the return process is modeled non-parametrically, and the variance process is modeled as GARCH) would perform best in such a formal test in the dynamic general equilibrium setting. A formal dynamic general equilibrium test of the EGARCH specification forms the subject of current investigation, and it is thought that many developments are possible in the light of these findings.

It is believed that in a world where so much rides on making the right forecast of volatility, a formal setting to verify the relative performance of different models, such as the one used in this study, is an important contribution to the forecast debate. Finding the models that perform best in a setting the mirrors the real economy the most closely possible is likely to have big implications for practitioners and scholars alike.

APPENDIX A: GARCH ESTIMATION IN EVIEWS

//This program dynamically imports files into the EVIEWS workfile, does GARCH(1,1) estimation, and forecasts 7 days into the future uing the past return history from day 1 to 1300. The results are stored in the vector M.//

```
vector(1000) M
```

//the following allocates string variables to handle the dynamic data import problem//

```
%1=".dat"
for !i=1 to 1000
%2="!i"
%3=%2+%1
read(o) c:\afiles\%3 ret%2
```

//the following looks at the sample of days 1 to 1300 of each original return file, and calculates and stores GARCH(1,1) coefficients in the active workfile in EVIEWS. GARCH is done using the BHHH algorithm, and the heteroskedasticity consistent covariance estimator is chosen//

```
smpl 1 1300
equation eq%2.arch(1,1,B,H) ret%2 ret%2(-1) c
```

//now the out of sample range of days 1301 to 1307 are chosen to compute forecasts from the estimated GARCH coeffients found previously//

```
smpl 1301 1307
forecast eq%2for
forecast retf sef varf%2
smpl 1301 1307
```

//the following computes average forecasted variance in the forecast sample of days 1301 to 1307//

```
genr avarf%2=@mean(varf%2)
smpl 1301 1301
avarf%2
```

//the following stores average forecasted variance in the sample for each of the 1000 files in the (1 ... 1000) vector M, and quits the program//

```
M(!i)=avarf%2(1301)
write(e) c:\forecas\varf%2.xls varf%2
smpl 1301 1307
write(e) c:\forecas\M.xls avarf%2

next
save
```

APPENDIX B: KURTOSIS PROGRAM IN EVIEWS

The following EVIEWS program computes kurtosis for each simulation return file and stores them in the vector K.

```
vector(1000) K

%1=".dat"

for !i=1 to 1000

%2="!i"
%3=%2+%1

smpl 1 1500

read(o) c:\afiles\%3 ret%2

genr quartic%2=(ret%2-@mean(ret%2))^4

K(!i)=(1/1501)*(1/(@var(ret%2))^2)*@sum(quartic%2)

next

save
```

APPENDIX C: REALIZED VARIANCE PROGRAM

The following EVIEWS program computes the realized variance for each simulation return file for the days 1301,..., 1322 and stores them in the vector R. The case for realized variance in the horizon 1301,..., 1307 is done in the same way.

```
vector(1000) R

%1=".dat"

for !i=1 to 1000

%2="!i"
%3=%2+%1

smpl 1 1500
read(o) c:\afiles\%3 ret%2

smpl 1301 1322


genr real%2=@var(ret%2)
smpl 1301 1301
real%2
R(!i)=real%2(1301)
```

APPENDIX D: SKEWNESS PROGRAM IN EVIEWS

The following EVIEWS program calculates the skewness for each simulated return file and stores tehm in the vector S.

```
vector(1000) S

%1=".dat"

for !i=1 to 1000

%2="!i"
%3=%2+%1

smpl 1 1500
read(o) c:\afiles\%3 ret%2

genr cubed%2=(ret%2-@mean(ret%2))^3

S(!i)=(1/1500.5)*(1/(@var(ret%2))^1.5)*@sum(cubed%2)

next
```

APPENDIX E: TRUE DISTRIBUTION PROGRAM IN DELPHI

//The following program computes the true distribution for days [1,..., 1500] using all the simulated return data files. The true variance is then computed from the true distirbution for each day//

```
unit Unit1;

interface

uses
  Windows, Messages, SysUtils, Classes, Graphics, Controls, Forms, Dialogs,
  StdCtrls;

type
  TForm1 = class(TForm)
    Button1: TButton;
    Label1: TLabel;
    procedure Button1Click(Sender: TObject);
  private
    { Private declarations }
  public
    { Public declarations }
  end;

var
  Form1: TForm1;

implementation

{$R *.DFM}
var
 tab:array [1..1500,1..1000] of double;
 sum,mean :array [1..1500] of double;

procedure variance;
var i,j:integer;
temp :double;
begin

  for i:=1 to 1500 do
  begin
```

```
  sum[i]:=0;
  mean[i]:=0;
 end;

 for i:=1 to 1500 do
 begin
  for j:=1 to 1000 do
    sum[i]:=sum[i]+tab[i][j];
  mean[i]:=sum[i]/1000;
  end;

 for i:=1 to 1500 do
 begin
  for j:=1 to 1000 do
  begin
    temp:=tab[i][j]-mean[i];
    tab[i][j]:=temp*temp;
  end;
 end;

 for i:=1 to 1500 do
 begin
  sum[i]:=0;
  mean[i]:=0;
 end;

 for i:=1 to 1500 do
 begin
  for j:=1 to 1000 do
    sum[i]:=sum[i]+tab[i][j];
 mean[i]:=sum[i]/999;
 end;

end;
procedure TForm1.Button1Click(Sender: TObject);
var i,j:integer;
st,st2,filename:string;
f,output:textFILE;
g1,g2:double;
begin
  for i:=1 to 1000 do
  begin
    st:=inttostr(i);
    if length(st)=1 then
      st:='00'+st
```

```
  else if length(st)=2 then
    st:='0'+st;
  assignfile(f,'a:\'+st+'.dat');
  label1.caption:=st;
  label1.refresh;

  reset(f);
  for j:=1 to 1500 do
  begin
    readln(f,g1,g2);
    tab[j][i]:=g1;
  end;
 end;
 variance;
 assignfile(output,'var.dat');
 rewrite(output);
 for i:=1 to 1500 do
  writeln(output,mean[i]);
 closefile(output);
 closefile(f);



end;

end.
```

APPENDIX F: PRODUCTION FUNCTION PARAMETERS

Firm 1			
STATE	α	θ	δ
1	0.07	0.06	0.11402441
2	0.28	0.08	0.11402441
3	0.36	0.09	0.11402441
4	0.44	0.10	0.11402441
5	0.52	0.11	0.11402441
6	0.60	0.12	0.11402441
7	0.68	0.13	0.11402441
8	0.89	0.145	0.11402441
Firm 2			
STATE	α	θ	δ
1	0.76	0.34	0.21246614
2	0.68	0.32	0.21246614
3	0.60	0.30	0.21246614
4	0.52	0.28	0.21246614
5	0.44	0.26	0.21246614
6	0.36	0.24	0.21246614
7	0.28	0.22	0.21246614
8	0.20	0.20	0.21246614
Firm 3			
STATE	α	θ	δ
1	0.20	0.10	0.15246614
2	0.28	0.11	0.15246614
3	0.36	0.12	0.15246614
4	0.44	0.13	0.15246614
5	0.52	0.14	0.15246614
6	0.60	0.15	0.15246614
7	0.68	0.16	0.15246614
8	0.76	0.17	0.15246614

BIBLIOGRAPHY

Abrahamson, A. (1987) "Most Efficient Estimators for Stock Index Price Volatility." Working Paper, University of Denver.

Akdeniz, L. and W.D Dechert (1997) "Do CAPM Results Hold in a Dynamic Economy? A Numerical Analysis." Journal of Economic Dynamics and Control, 58: 981-1004.

Akdeniz, L. (1998) "Risk and Return in a Dynamic General Equilibrium Model." Working Paper.

Akgiray, V. (1989) "Conditional Heteroskedasticity in Time Series of Stock Returns: Evidence and Forecasts," Journal of Business, 62: 55-80.

Altay-Salih, A., R. Engle and J. Mezrich (1996) "GARCH Models in Volatility Estimation for Stock Market Indices." Working Paper.

Altay-Salih, A. and V. Kurtas, (1998) "The Impact of Stock Index Futures Introduction on the Distributional Characteristics of the Underlying Index: An International Perspective." Submitted Paper.

Amin K. and V. Ng (1994) "A comparison of Predictable Volatility Models Using Option Data." IMF Working Paper.

Baillie R.T. and R.J Myers (1991) "Modeling commodity price distributions and estimating the optimal futures hedge." Journal of Applied Econometrics 6: 109-124.

Baxter, M., M. Crucini and K. Rouwenworst (1990) "Backsolving the Stochastic Growth Model With a Particular Nonlinear Form for the Decision Rule." Journal of Business and Economic Statistics, 8: 45-47.

Beckers, S. (1981) "Standard Deviations Implied in Option Prices as Predictors of Future Stock Price Variability." Journal of Banking and Finance, 5: 362-381.

Benhabib, J. and A. Rustichini (1994) "A note on a new class of solutions to dynamic programming problems arising in economic growth."
Journal of Economic Dynamics and Control, 18: 807-813.

Black, F. (1976) "Studies in stock price volatility changes,"
Proceedings of the 1976 Business Meeting of the Business and Economic Statistics Section, American Statistics Association, 177-181.

Black F. and M. Scholes (1973) "The pricing of options and corporate liabilities." Journal of Political Economy, 81:637-659.

Bollerslev, T. (1986) "Generalized Autoregressive Conditional Heteroskedasticity." Journal of Econometrics, 31: 307-327.

Bollerslev, T. and H.O. Mikkelsen (1996) "Modeling and Pricing long memory in stock market volatility." Journal of Econometrics, 73: 151-184.

Bollerslev, T., R.Y Chou and K.F. Kroner (1992) "ARCH modeling in Finance." Journal of Econometrics, 52: 5-59.

Brock, W.A. (1979) An Integration of Stochastic Growth Theory and the Theory of Finance- Part I: The Growth Model. Pages 165-192, The Academic Press, New York.

Brock W.A. (1982) Asset Prices in a Production Economy.
Pages 1-46, The University of Chicago Press, Chicago, 1982.
Brock, W.A. and L. Mirman (1972) "Optimal economic growth and uncertainty: The discounted case." Journal of Economic Theory, 4: 479-513.

Canina, L. and S. Figlewski (1993) "The Informational Content of Implied Volatility." Working Paper.

Cass, D. (1965) "Optimum Growth in an Aggregative Model of Capital Accumulation." Review of Economic Studies, 32: 233-240.

Chesney, M. and L.O. Scott (1989) "Pricing European Currency Options: A comparison of the Modified Black-Scholes Model and the Random Variance Model." Journal of Financial and Quantitative Analysis, 24: 267-284.

Christie, A.A (1982) "The stochastic behavior of common stock variances: Value, leverage, and interest rate effects." Journal of Financial Economics, 10: 407-432.

Danthine, J.-P., and J.B. Donaldson (1995) Computing Equilibria of Nonoptimal Economies. Princeton University Press, Princeton, New Jersey.

Day, T.E., and C.M. Lewis (1992) "Stock market volatility and the information content of stock index options." Journal of Econometrics, 52: 267-288.

Engle R.F.(1993) "Statistical Models for Financial Volatility." Financial Analysts Journal, 1: 72-78.

Engle R.F. (1982) "Autoregressive Conditional Heteroscedasticity with Estimation of the Variance of United Kingdom Inflation" Econometrica, 50:4: 987-1007.

Engle, R. F., T. Hong, A. Kane and J. Noh (1992) "Arbitration Valuation of Variance Forecasts Using Simulated Options Markets." Advances in Futures and Options Research, 5:50-61.

Engle, R.F. and V.K. Ng (1993) "Measuring and Testing the Impact of News on Volatility." Journal of Finance, 48:5: 1749-1778.

Figlewski, S. (1994) "Forecasting Volatility Using Historical Data."

Working Paper.

French, K.R., G.W. Schwert and R.F. Stambaugh (1987) "Expected Stock Return and Volatility" Journal of Financial Economics, 19, 3-29.

Gallant, A.R. (1981) "On the bias in flexible functional forms and an essentially unbiased form: The Fourier flexible form." Journal of Econometrics, 15, 211-244.

Garman. M. and M. Klass (1980) "On the Estimation of Security Price Volatility from Historical Data." Journal of Business, 53: 67-78.

Glosten, J.R., R. Jagannathan and D. Runkle (1993) "On the relation between the Expected Value and the Volatility of the Nominal Excess return on Stocks."
Journal of Finance, 48:5:1779-1802.

Hamilton, J.D and G. Lin (1996) "Stock Market Volatility and the Business Cycle" Journal of Applied Econometrics, 11: 573-596.

Hamilton, J.D. (1994) Time Series Analysis. Princeton University Press, Princeton, New Jersey.

Hamilton, J.D. (1989) "A new approach to the economic analysis of nonstationary time series and the business cycle." Econometrica, 57: 357-384.

Hansson B. and Peter Hördahl (1996) "Forecasting Variance Using Stochastic Volatility and GARCH" Working Paper.

Harvey, A.C., E. Ruiz, and N. Shephard (1994) "Multivariate Stochastic Variance Models." Review of Economic Studies, 61: 247-261.

Hull, J. and A. White (1987) "The Pricing of Options on Assets with Stochastic Volatility." Journal of Finance, 442: 281-300.

Jorion, P. (1995) "Predicting Volatility in the Foreign Exchange Market." Journal of Finance, 50:2:507-528.

Judd, K.L. (1991) "Numerical Methods in Economics." Unpublished Manuscript, Hoover Institution.

Judd, K.L. (1995) "Computational Economics and Economic Theory: Substitutes or Complements?" Working Paper.

Judd, K.L. (1992) "Projection methods for solving aggregate growth models" Journal of Economic Theory, 58:410-452.

Koopmans, T. (1965) "On the concept of optimal economic growth." Pontificau Academiæ Scientiarum Scropta Varia, 28, Vatican.

Kydland, F.E. and E.C. Prescott (1982) "Time to build and aggregate fluctuations" Econometrica, 50:1345-1370.

Lamoureux, C.G. and W.D Lastrapes (1993) "Forecasting Stock-Return Variance: Toward an Understanding of Stochastic Implied Volatilities" Review of Financial Studies 6:2:293-326.

Lucas, R.E. (1978) "Asset prices in an exchange economy." Econometrica 46:1429-1445.

Mandelbrot,B. (1963) "The variation of certain speculative prices." Journal of Business 36:394-419.

Merton, R.C (1980) "On estimating the expected return on the market." Journal of Financial Economics 8:323-361.

Nadaraya, E.A. (1964) "On estimating regression."
Theory of Probability and Its Applications 9:141-142.

Nelson, D. (1990) "Conditional Heteroskedasticity in Asset Returns."
Econometrica 59:347-370.

Nelson, D. (1988) "Conditional Heteroskedasticity in Asset Returns: A new Approach" Unpublished Manuscript, University of Chicago.

Pagan, A.R. and G.W. Shwert (1990) "Alternative Models for Conditional Stock Volatility" Journal of Econometrics 45:267-290.

Parkinson, M. (1977) "Option Pricing: The American Put."
Journal of Business 50:1:21-36.

Ritzman, M. (1991) "What practitioners Need to Know About Estimating Volatility: Part I" Financial Analysts Journal, July-August, 22-25.

Ritzman, M. (1991) "What practitioners Need to Know About Estimating Volatility: Part II" Financial Analysts Journal, September- October, 10-11.

Rivlin, T.J. (1990) Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory Wiley Interscience, New York.

Ruiz, E. (1994) "Quasi- Maximum Likelihood Estimation of Stochastic Variance Models." Journal of Econometrics 63: 284-306.

Schwert, G.W. (1990) "Stock Volatility and the Crash of 87"
Review of Financial Studies 3:77-102.

Sheedy, E. (1998) "Is Correlation Constant After All?"
Working Paper, University of New South Wales, Australia.

Sims, C.A. (1990)"Backsolving the Stochastic Growth Model With a particular Nonlinear form for the Decision Rule" Journal of Business and Economic Statistics 8:45-47.

Tauchen, G. (1990) "Solving the Stochastic Growth Model by Using Quadrature Methods and Value-Function Iterations." Journal of Business and Economic Statistics 8:15-17.

Taylor, S. (1994) "Modeling Stochastic Volatility: A Review and Comparative Study." Mathematical Finance 4:183-204

Taylor, J.B. and H. Uhlig (1990) "Solving Nonlinear Stochastic Growth Models: A Comparison of Alternative Solution Methods"
Journal of Business and Economic Statistics 8:1-17.

Watson, G.S. (1964) "Smooth Regression Analysis" Sanhkya A 26:359-372.

West, K.D. and D. Cho (1995) "The predictive ability of several models of exchange rate volatility" Journal of Econometrics 69: 367-391.

Wiggins, J.B. (1987) "Option Values under Stochastic Volatility: Theory and Empirical Estimates." Journal of Financial Economics 19: 351-372.