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AM/FM signal estimation with micro-segmentation and polynomial fit

Zeynel Deprem · A. Enis Çetin · Orhan Arıkan

Abstract Amplitude and phase estimation of AM/FM signals with parametric polynomial representation require the polynomial orders for phase and amplitude to be known. But in reality, they are not known and have to be estimated. A well-known method for estimation is the higher-order ambiguity function (HAF) or its variants. But the HAF method has several reported drawbacks such as error propagation and slowly varying or even constant amplitude assumption. Especially for the long duration time-varying signals like AM/FM signals, which require high orders for the phase and amplitude, computational load is very heavy due to nonlinear optimization involving many variables. This paper utilizes a micro-segmentation approach where the length of segment is selected such that the amplitude and instantaneous frequency (IF) is constant over the segment. With this selection first, the amplitude and phase estimates for each micro-segment are obtained optimally in the LS sense, and then, these estimates are concatenated to obtain the overall amplitude and phase estimates. The initial estimates are not optimal but sufficiently close to the optimal solution for subsequent processing. Therefore, by using the initial estimates, the overall polynomial orders for the amplitude and phase are estimated. Using estimated orders, the initial amplitude and phase functions are fitted to the polynomials to obtain the final signal. The method does not use any multivariable nonlinear optimization and is efficient in the sense that the MSE performance is close enough to the Cramer–Rao bound. Simulation examples are presented.

Keywords Non-stationary signal · AM/FM signal · Polynomial phase signals

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1 Introduction

In many practical signal applications involving amplitude and/or phase modulation, we encounter discrete-time signals which can be represented as

$$s[n] = a[n]e^{j\emptyset[n]}, \tag{1}$$

where $a[n]$ and $\emptyset[n]$ are the real amplitude and phase functions, respectively. The amplitude is assumed to be positive, and the phase is assumed to be continuous. Such signals are common in AM/FM communication, radar, sonar applications, and in many other practical problems. The problem is to estimate time-varying signal $s[n]$ from its noisy version

$$x[n] = s[n] + w[n] = a[n]e^{j\emptyset[n]} + w[n] \quad n = 0, 1, \ldots, N-1, \tag{2}$$

where $w[n]$ is a zero-mean white Gaussian complex noise process with unknown variance σ^2.

A widely used method is the parametric approach where the amplitude and phase functions are represented as a linear combination of some known basis functions p_k and g_k given by

$$a[n] = \sum_{k=0}^{P} a_k g_k[n], \tag{3}$$

$$\emptyset[n] = \sum_{k=0}^{Q} b_k \, p_k[n], \qquad (4)$$

and the coefficients a_k and b_k corresponding to the basis functions need to be estimated. The parameters a_k and b_k are real-valued amplitude and phase coefficients. In general, basis functions can be any functions which are square integrable and span the space of real and integrable functions in a given observation interval. Also, they can be selected to be different for amplitude and phase functions. In this paper, they are assumed to be polynomials both for the amplitude and the phase. Therefore, P and Q correspond to orders for amplitude and phase polynomials, respectively.

The first approach tried for the estimation of the parameters in (3) and (4) is the maximum likelihood (ML) estimation [6,13]. For the Gaussian noise case, the ML estimation of the parameters is equivalent to the LS optimization [6]. The ML estimation of the parameters requires a multivariable nonlinear optimization problem to be solved. Therefore, the solution requires iterative techniques like nonlinear conjugate gradient (NL-CG) or quasi-Newton type algorithms and is computationally intensive [6,13]. All mentioned methods do not ensure the convergence to a global minimum. Therefore, another requirement is a good initial estimate avoiding possible local minima.

Compared with the ML approach, a suboptimal but practical method is the parameter estimation with higher-order ambiguity function (HAF) [10,16], and [1]. If the amplitude $a[n]$ in (1) is constant, then $s[n]$ is a polynomial phase signal (PPS) of order Q. The basic principal of the HAF transformation is that, when $s[n] = a_0 e^{j\emptyset[n]}$ is multiplied by $e^{-j\emptyset[n-\tau]}$, τ being a constant integer, then the order of resultant PPS will be decreased by one. Therefore, a PPS of order Q, when transformed with HAF of the same order, will produce a pure-tone signal of the frequency, which is proportional to the highest coefficient b_Q. Therefore, the highest order coefficient estimate b_Q can be obtained from Qth order HAF. Then, by multiplying $s[n]$ with $e^{-jb_Q n^Q}$ will decrease the order of $s[n]$ to $Q-1$. By repeating the process iteratively, all coefficients of PPS are estimated.

An algorithm which uses both time-frequency distribution and HAF is given in [5] where the components of a multi-component signal are estimated in a sequential manner. There are several drawbacks of this successful method. Although there are various attempts to handle slowly varying amplitude case [16], the amplitude is assumed to be constant in general [14]. But in (1), this is not the case in some cases. Furthermore, there may be error propagation in successive order reduction with HAF algorithm. Therefore, the method requires a good SNR. There are approaches [7], which try to reduce the effect of this propagation, but due to higher polynomials order still there is error propagation.

There are some other methods, which do not use polynomial modeling for the estimation of amplitude and instantaneous frequency (IF). In [2], a method is given which estimates IF using empirical mode decomposition (EMD). In [4], the amplitude and instantaneous frequency changes on signal are obtained from the spectrogram.

In order to handle more general cases, signals with both time varying amplitude and long durations need to be considered. With long duration, we mean a time interval of the signal over which the amplitude and phase functions cannot be expressed with polynomials of order 3 or less. Therefore, when long duration is the case, higher polynomial orders (>3) need to be handled. One approach to deal with the problem is the segmentation. In [8], a method is proposed where the long duration signal is divided into non-sequential and overlapping segments and for each segment a parametric ML approach is used to estimate the signal corresponding to that segment. Then, the overlapping segments are aggregated with windowing. A segmentation strategy is also given. The segments are determined depending on their local energy on time-frequency plane. Each segment is represented at most with a polynomial of order 3. The proper length and the order for each segment are determined by trying orders from 1 to 3 and lengths from a predetermined range (15–60 samples). The proper order and segment length, which gives a polynomial fit error less than a predetermined threshold, are selected for each segment with trail. The main motivation behind this approach is that the optimization with orders less than 3 is easier.

The main drawback with the method is that we still need to run a nonlinear multivariable optimization algorithm for each segment and we have the same local minima and computation problems of the ML method. Therefore, simulated annealing (SA) which is a stochastic method is used to find the parameters for each segment in [8].

1.1 Contribution of the paper

In this paper, we propose a different segmentation strategy and use an overall polynomial order estimation method for the signal estimation. The proposed segmentation is called as micro-segmentation in the sense that the segment length is selected such that the amplitude function over the segment can be assumed to be constant and the phase function can be assumed to be linear. In other words, the assumption is that the rate of change in amplitude and phase function is slow compared with signal itself. Therefore, we can select a micro-segment such that the amplitude and frequency can be nearly constant over the selected segment. With this selection of segments, the total number of parameters which need to be estimated for a micro-segment is 3. Since the related ML or LS optimization is separable, two of parameters are expressed in terms of the third one. This will reduce the optimization

Fig. 1 Micro-segmentation for an AM/FM signal with orders $Q = 10$ and $P = 8$
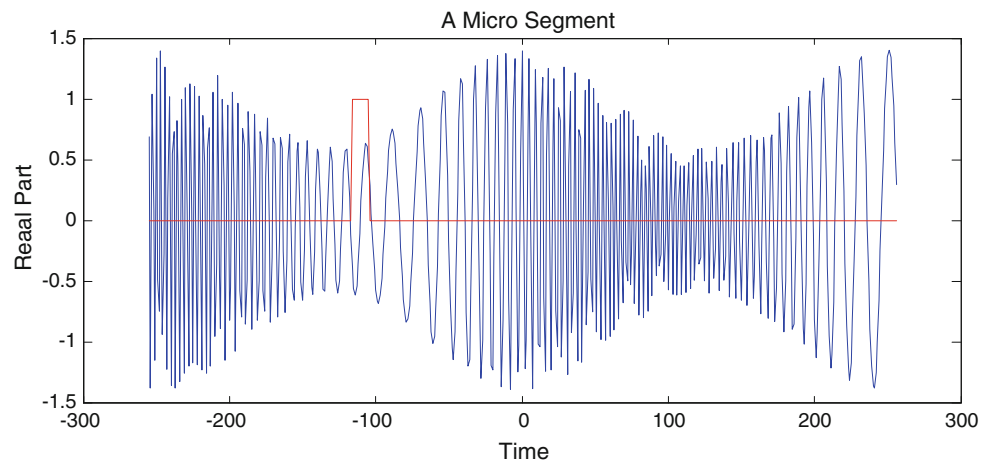
with 3 parameters to a 1-D search and subsequent substitution. It will be shown that the optimal polynomial parameters for both amplitude and phase of the micro-segment are simply obtained using Discrete-Time Fourier Transform DTFT (FFT). Segments are non-overlapping. The overall amplitude and phase estimates for the signal in (1) are obtained by concatenating microsegments. An example AM/FM signal and a micro-segment are shown in Fig. 1.

An issue which naturally comes to mind is the selection of micro-segment length over which the amplitude and the IF are constant. A method is also proposed for proper selection of the micro-segment length. The method is robust enough to select flexibly the segment length in an allowed range.

The overall amplitude and phase estimates obtained by concatenation of the corresponding micro-segmentation estimates are not optimal in LS sense. But they are good enough to be used for overall polynomial order estimation for both amplitude and phase function. Therefore, another contribution of the paper is the estimation of the polynomial orders for overall amplitude and phase functions. The micro-segmentation-based estimate for both amplitude and phase is fitted to a polynomial where the degree of polynomial is searched in range. If we observe the differential fit error evolution in a trial range, it will be seen that there is high contrast around actual polynomial order. This is a result of sufficient SNR improvement at micro-segmentation step, which clarifies the main shape of the amplitude and phase functions. Therefore, by detecting this order, a proper order can be estimated for both amplitude and phase function. It will be shown that with this polynomial order, the estimates obtained with micro-segmentation are significantly improved approaching the Cramer–Rao bound.

The rest of the paper is organized as follows. In Sect. 2, the micro-segmentation-based estimates are derived. In Sect. 3, overall polynomial order estimation is explained together with several methods which can be applied to the micro-segmentation estimate for making the task of polynomial

order estimation easier. In Sect. 4, the performance analysis for parametric ML approach is determined to be used as a reference for comparison purposes. In Sect. 5, simulation results are presented.

2 Micro-segment based estimation

Given an amplitude and frequency modulated signal of form (2), we define a segment of length $N_\mu < K$

$$x_\mu[m] = x[n_0 + m], \quad 0 \leq m \leq N_\mu \tag{5}$$

where n_0 is the starting instant of the segment. The amplitude and phase functions are similarly defined as

$$A_\mu[m] = a[n_0 + m] \tag{6}$$

and

$$\emptyset_\mu[m] = \emptyset[n_0 + m], \tag{7}$$

The segment length is selected such that the amplitude and frequency functions are constant over the segment. In other words, the assumption is that the rate of change in signal $s[n]$ is higher, at least K times, than the rate of change in amplitude and IF function. Then, we can express the micro-segment signal as

$$x_\mu[m] = a_\mu e^{j(2\pi f_\mu m + \varphi_\mu)} + w_\mu[m] \tag{8}$$

where a_μ, f_μ and φ_μ are constants and $w_\mu[m]$ is the noise corresponding to the segment. With this selection of the segment, we have the same parameter estimation problem with set of unknown parameters reduced to 3. Therefore, we can define the least square optimization problem as,

$$J(a_\mu, f_\mu, \varphi_\mu) = \sum_{m=0}^{N_\mu-1} \left| x_\mu[m] - a_\mu e^{j(2\pi f_\mu m + \varphi_\mu)} \right|^2 \tag{9}$$

Fig. 2 Noisy signal with SNR = 10 dB and micro-segmentation-based initial amplitude, frequency (IF), and phase estimates with $N = 512$, $N_\mu = 12$
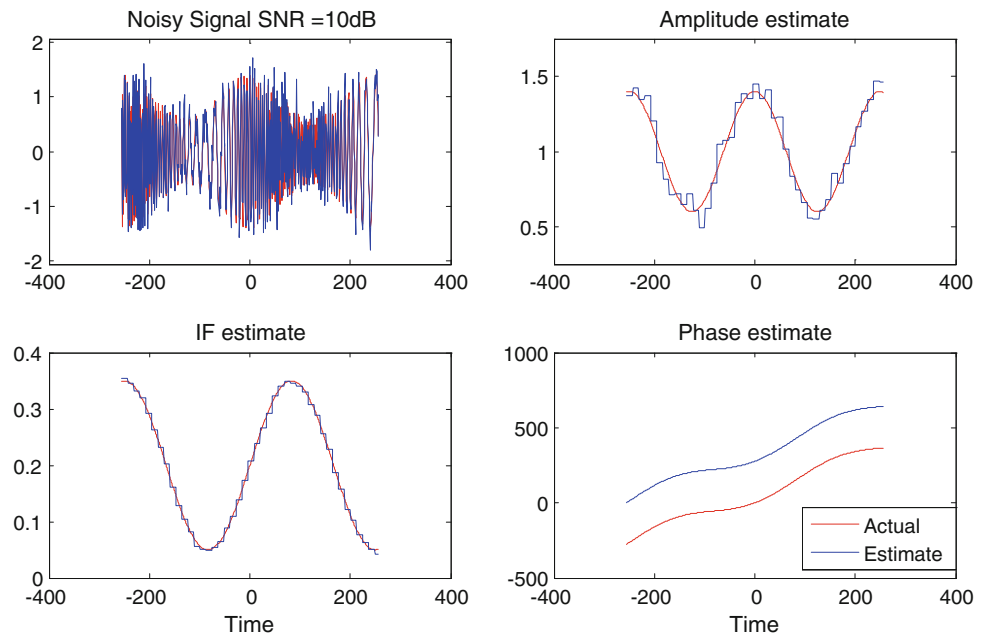
where the parameter set is minimized as follows:

$$\left\{ \hat{a}_\mu, \hat{f}_\mu, \hat{\varphi}_\mu \right\} = \operatorname{argmin}_{a_\mu, f_\mu, \varphi_\mu} J(a_\mu, f_\mu, \varphi_\mu) \qquad (10)$$

Although we have 3 parameters to be optimized, the objective function is separable; therefore, we can first solve $\{\hat{a}_\mu, \hat{\varphi}_\mu\}$ in terms of f_μ, then substitute these values in (9) to solve for $\hat{f}_\mu$ [6]. With this method, the frequency estimate $\hat{f}_\mu$ can be solved as

$$\hat{f}_\mu = \operatorname*{argmax}_{f_\mu} \left| X_\mu \left(e^{j2\pi f_\mu} \right) \right|^2 \qquad (11)$$

and using this value $\{\hat{a}_\mu, \hat{\varphi}_\mu\}$ are estimated as

$$\hat{a}_\mu e^{j\hat{\varphi}_\mu} = \frac{1}{N_\mu} X_\mu \left(e^{j2\pi f_\mu} \right) = \frac{1}{N_\mu} \mathrm{DTFT}\{x_\mu[n]\}_{f=f_\mu}. \quad (12)$$

In other words, for micro-segment parameter optimization, first we need to compute DTFT of the sequence $x_\mu[m]$ and then find the frequency $\hat{f}_\mu$ at which the square magnitude of DTFT is maximum and using this frequency compute $\{\hat{a}_\mu, \hat{\varphi}_\mu\}$ from (12).

The overall estimates corresponding to amplitude and phase function are obtained by concatenating micro segment estimates corresponding to them. The amplitude is obtained by

$$\hat{a}_{ms}[n_0 + m] = \hat{A}_\mu[m] = \hat{a}_\mu, \qquad 0 \le m \le N_\mu \qquad (13)$$

and similarly phase is obtained by

$$\hat{\emptyset}_{ms}[n_0 + m] = \hat{\varphi}_\mu[m] = 2\pi \hat{f}_\mu m + \hat{\varphi}_\mu, 0 \le m \le N_\mu \quad (14)$$

and the signal estimate is obtained as

$$\hat{s}_{ms}[n] = \hat{a}_{ms}[n] e^{j\hat{\emptyset}_{ms}[n]} \qquad (15)$$

where n_0 is the starting instant of a segment, $\hat{a}_{ms}[n]$, $\hat{\emptyset}_{ms}[m]$, and $\hat{s}_{ms}[n]$ are the overall amplitude, phase and signal estimates.

There are three considerations with this approach. Although we assumed that the amplitude is positive, if $\{\hat{a}_\mu, \hat{f}_\mu, \hat{\varphi}_\mu\}$ is a solution to the optimization problem in (10), then $\{-\hat{a}_\mu, \hat{f}_\mu, \hat{\varphi}_\mu + (2k+1)\pi, k \ integer\}$ is also a solution. That is, there is an ambiguity in solution. Therefore, whenever $\hat{a}_\mu$ is negative, it should be replaced with absolute value and π should be added to $\hat{\varphi}_\mu$ to remove ambiguity.

The other consideration is finding the frequency at which square magnitude DTFT of $x_\mu[m]$ is the maximum. Usually DTFT is computed via FFT at discrete frequencies. To find the maximum point, we need to sample the DTFT in a fine grid. Therefore, the frequency which maximizes $|X_\mu(e^{j2\pi \hat{f}_\mu})|^2$ is found by searching over the FFT frequency bins followed by an interpolation step to refine the frequency estimate. The interpolation is obtained by fitting the maximum frequency bin of FFT and several other points around it to a second-order polynomial. The analytic maximum of this fitted polynomial gives the refined maximum of $|X_\mu(e^{j2\pi f_\mu})|^2$.

The third and crucial consideration is the concatenation of the micro-segments. While the overall amplitude estimate can be easily obtained by concatenation of micro-segment amplitudes, for phase concatenation, we need to remove phase ambiguity of $k2\pi$ between subsequent segments. This is achieved by checking the phase difference between phase at first time index of current micro-segment and phase at last time index of previous micro-segment. With this arrangement, an unwrapped phase sequence is obtained.

Fig. 3 Real parts of the noisy signal (*top blue*) with initial SNR = 10 dB and micro-segmentation estimate (*bottom*) with $N = 512$, $N_\mu = 12$ where SNR is improved to 18.1 dB
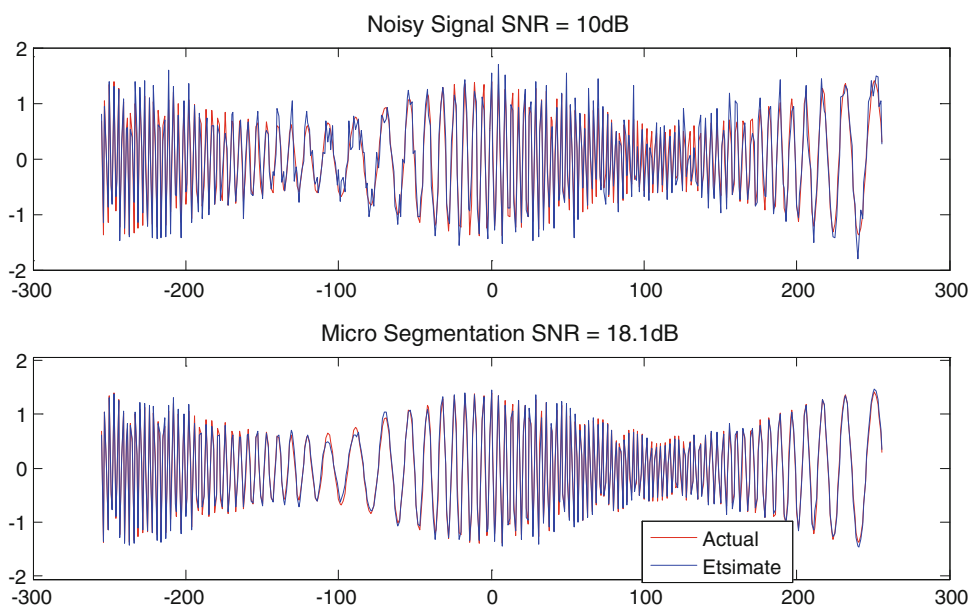
In Fig. 2, we show the initial amplitude, phase, and frequency estimates obtained from a noisy long duration signal of $N = 512$ with micro-segmentation using $N_\mu = 12$. The amplitude polynomial order is 10, and the phase polynomial order is 8. In Fig. 3, real parts of the noisy signal and micro-segmentation-based estimates are shown on top of the noiseless signal where the SNR is the time average SNR defined as

$$\text{SNR} = 10 \log \frac{\sum_{n=0}^{N-1} |s[n]|^2}{\sum_{n=0}^{N-1} |\hat{s}[n] - s[n]|^2}. \tag{16}$$

A similar definition is also used in [6]. But some other definitions which define the SNR based on the domain where the signal is represented can be found in [15]. In Fig. 2 for the phase estimate, we see a constant phase offset of $k2\pi$. Therefore, the true and estimated phase functions are parallel to each other.

From Fig. 2, we observe that the IF and phase estimates are better compared with amplitude estimate. Since the phase is unwrapped, it is monotonically increasing. Therefore, the noise on it is not seen from the figure as a result of scale resolution.

Now, the question is how to select the micro-segment length. Before determining a suitable length, it should be noted that we have two constraints that need to be satisfied. First, in order to have a good estimate, the number of data points in a micro-segment should be sufficiently large compared with number of parameters to be estimated. In our case, it is 3. Since we have a complex signal, the data size are twice the size of segment N_μ. Therefore, we should have $2N_\mu \gg 3$. On the other hand, as we made an assumption at the beginning that the rate of change in signal $s[n]$ is at least K times greater than that of both amplitude and IF functions,

we need to select a segment length in the range defined by

$$2 \ll N_\mu < K \tag{17}$$

where, K is defined as follows,

$$K = \frac{\min(f_{\text{signal}})}{\max\left[\max\left(f_{\text{amp}}\right), \max\left(f_{\text{IF}}\right)\right]}. \tag{18}$$

Though we can search for an optimum length in this range, it is not our intention to get a further optimized estimate at this stage. As it will be seen in Sect. 3, initial results obtained at this stage will be sufficient for subsequent processing.

3 Polynomial fit-based estimation

3.1 Polynomial order estimation

In mathematical analysis, according to Weierstrass approximation theorem [12], every continuous function defined on a closed interval $[a, b]$ can be uniformly approximated as closely as desired by a polynomial function. In fact, the polynomial functions are among the simplest functions for continuous function representation. Increasing the order for representation will achieve better approximation. Therefore, as the order is increased, more detailed parts of the function will be fitted or approximated.

In most of the denoising applications, since the noise spectrum is white and actual signal spectrum is low-pass compared to noise, the majority of high-frequency energy of the noisy signal is due to the noise. Therefore, removing the high frequencies beyond a properly selected cut-off, frequency will remove most of the noise. In our case, we can
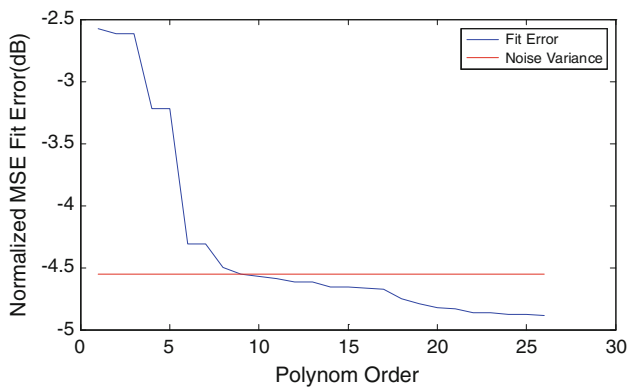
Fig. 4 Normalized fit error (MSE) versus polynomial order

Fig. 5 Fit error and differential fit error versus polynomial order (phase function) at SNR = 5 dB

Fig. 6 Normalized MSE fit error and differential fit error versus polynomial (amplitude function) at SNR = 5 dB

also assume that both the amplitude and phase functions are low-pass functions compared to noise.

A similar approach can be used when approximating a noisy signal with polynomial functions. While the actual noiseless signal can be approximated with a polynomial of order P which gives an approximation error e_{pf}, a higher-order $P_{noisy} > P$ will be needed to approximate the noisy function with the same error. Because the noise part contains more high-frequency fluctuations than the actual signal and require higher polynomial orders to approximate it. But what we want is to approximate the actual function rather than noise. Therefore, when approximating the noisy function if take a cut-off order $\hat{P}$ which is smaller than P_{noisy} and as close as possible to the actual order P, then the approximation error will be larger than e_{pf}, but we will eliminate most of the noise and get an estimate of the function. Then, the problem is how to find the order P or an estimate for it.

One method is to use the noise variance. If we know the noise variance, then we can search for a range of orders and select the order for which the approximation error to noisy function is less than or equal the noise variance. Because if we are getting a lower approximation error than the noise variance, that means we are fitting to the noise rather than the noiseless function. Therefore, we should not go beyond that order. In Fig. 4, we can see the mean square fit error (MSE), normalized to function energy, for an initial phase estimate. The straight line in figure is the noise variance. But the problem with this method is that we do not know the noise variance.

An alternative method for order estimation can be obtained from the change in approximation or fit error as a function of polynomial order. If we observe the change in approximation error we will see that up to an order $\hat{P}$, the rate of change will be high but beyond that it will follow a comparably flat pattern. For example looking at Fig. 4, we will see that the approximation error is decreasing sharply between 0 and 10 beyond that, the rate of change is much slower. That is, around 8–10, there is a break point. The region from 0 to 8
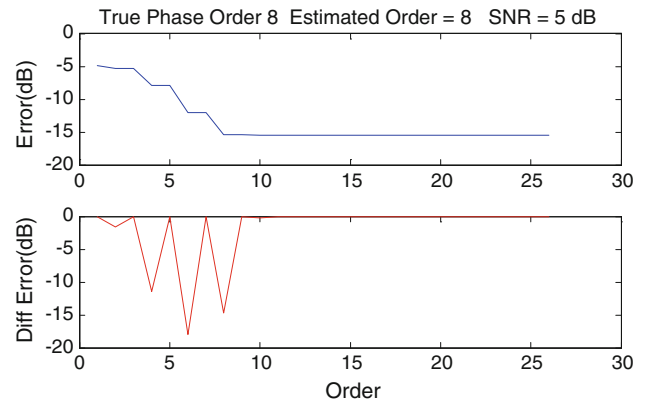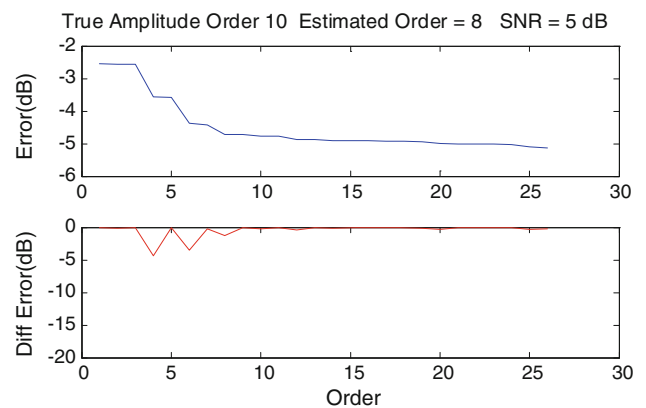
or from 0 to 10 is the area where the actual function shape is approximated but the region beyond that is the region of noise fitting. Therefore, even though we do not know the noise variance, from this rate of change pattern we can estimate an order with a proper threshold. But in order to use this method, we need to have a good SNR. Otherwise, the contrast in change pattern may not be so clear. Therefore, micro-segment-based amplitude and phase estimates will be used for this purpose. This change pattern is clearly observable from Figs. 5 and 6.

In Figs. 5 and 6, both the normalized MSE fit error and the differential fit error are plotted for the micro-segmentation phase and amplitude estimates obtained from a noisy signal with SNR = 5 dB.
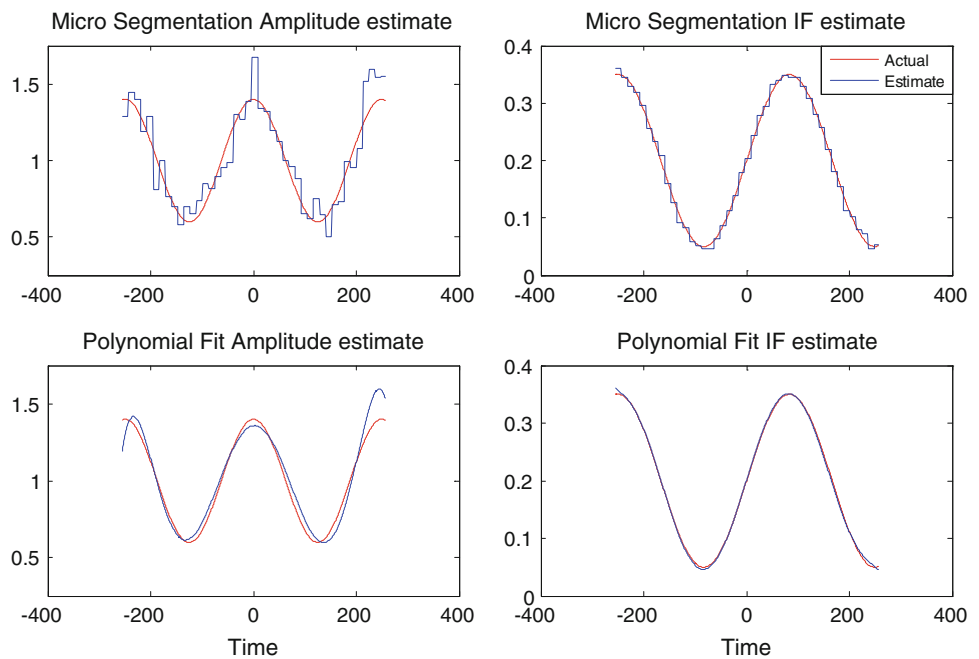
The normalized MSE fit error and differential fit error are defined as,

$$e_{pf}(p) = 10 \log \frac{\sum_{n=0}^{N-1} \left| \hat{a}_{ms}[n] - \hat{a}_{pms}[n] \right|^2}{\sum_{n=0}^{N-1} \left| \hat{a}_{ms}[n] \right|^2}$$

$$p = 1, 2, \ldots, \text{MaxPol} \quad (19)$$

$$\text{de}_{pf}(p) = e_{pf}(p) - e_{pf}(p-1) \quad p = 2, 3 \ldots, \text{MaxPol} \quad (20)$$

Fig. 7 Comparison between micro-segmentation and polynomial fit estimates

where $\hat{a}_{ms}[n]$ is overall micro-segmentation-based amplitude estimate and $\hat{a}_{pms}[n]$ is the polynomial fit of $\hat{a}_{ms}[n]$ at pth order where MaxPol is the predefined maximum search range and $\mathrm{de}_{pf}(1) = 0$. The order p around which there is a high contrast is detected by the following moving average based search:

$$\hat{p} = \min_{m_{de}(p) < \gamma_{\mathrm{ord}}} p, \qquad p = 1, 2 \ldots, \mathrm{MaxPol}, \tag{21}$$

where

$$m_{de}(p) = \frac{1}{M} \sum_{k=1}^{M} \left| \mathrm{de}_{pf}(p+k) \right| \tag{22}$$

with M being the length of moving average filter. The minimization basically detects a region during which the mean of absolute differential fit error drops below a threshold γ_{ord}.

Having the assumption that the rate of change in signal $s[n]$ is higher, at least K times, than the rate of change in amplitude and IF function, we can conclude that compared with $s[n]$ the amplitude function and unwrapped phase functions are higher resolution functions and can be down sampled by K. In other words, we can represent them with N/K samples. Therefore, this gives us an intuition that without any compression or dimensionality reduction and just by down sampling we can express the functions with N/K samples. Therefore, the basis expansions given in (3) and (4) should give us better compressed representations. With this intuition, we can select MaxPol as the smallest integer greater or equal to N/K.

But on the other hand, since the search in (22) can be implemented with an online algorithm, the search will terminate around true polynomial order plus M. In other words, if the true polynomial order is P, then the search will termi-

nate at $P + M + \varepsilon_P$ where ε_P is the integer estimation error on P and is expected to be small when SNR is good. After that the search will terminate and the corresponding $\hat{a}_{pms}[n]$ will be selected as the polynomial fit estimate for amplitude function $a[n]$.

In Fig. 7, the polynomial fit estimates corresponding to micro-segmentation-based estimates given in Fig. 2 are shown. Where true/estimated orders, obtained with this search method for amplitude and phase are 10/8 and 8/8, respectively. The function estimates are plotted on top of their noiseless versions to see the improvement. From differences between two results, it is clear that there is a good improvement. In Fig. 8, the improvement in SNR corresponding to overall signal is shown. From this figure also, we see that there is a good improvement.

3.2 Micro-segmentation-based estimate improvement

As we explained in Sect. 2, the main objective of micro-segmentation-based estimation is to get an intermediate estimate to be used for polynomial-based estimation. The goal of the micro-segmentation method is not to obtain a final estimate. But, with some effort, an even improved estimate can be obtained which will ease the task of polynomial order estimation.

Regarding the N_μ of micro-segment, apart from constraints given in (17), in general, a large window is preferable at low SNR for better noise removal and a small window is preferable for better time resolution.

In this work, we preferred to use the following segmentation strategy. First, we select a segment size N_μ which is

Fig. 8 Improvement in SNR with micro-segmentation and polynomial fit estimates where, "Noisy" is the noisy signal with SNR = 5 dB, "Micro" is the micro-segmentation estimate and "Pol" is the polynomial fit estimate
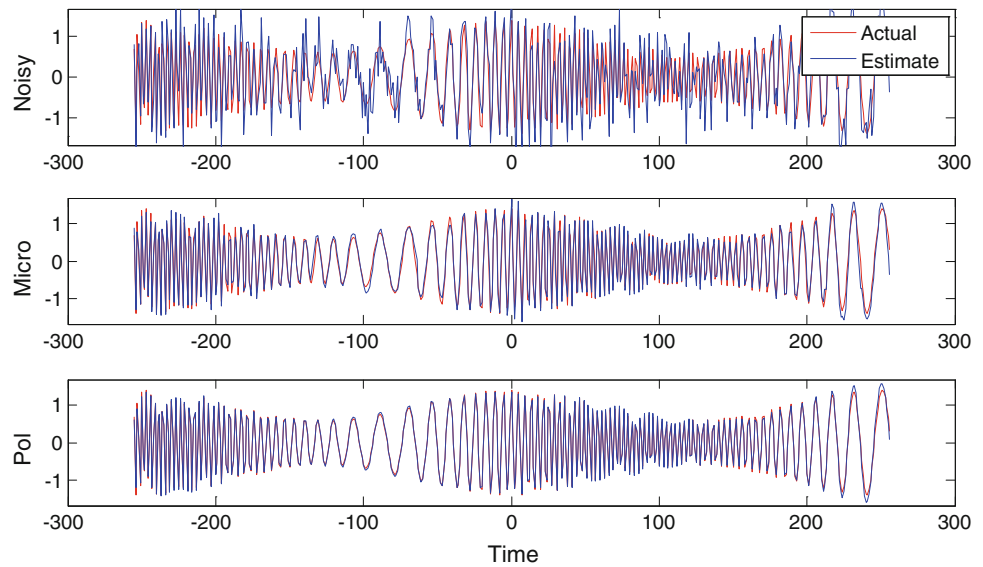
Fig. 9 Three segmentation indexes shifted by N_μ/L with $L = 3$

sufficiently large compared to the number of parameters, then we construct L segmentation indices on time axis which are shifted, as shown in Fig. 9, compared with each other by N_μ/L. Indexes are plotted for $L = 3$, and the points on lines represent the segment boundaries.

Using these L segment indexes, we obtained L overall estimates for amplitude and phase. Then, by averaging these estimates, we obtain the final micro-segment estimates as

$$\hat{a}_{ms}[n] = \frac{1}{L}\sum_{l=1}^{L}\hat{a}_{ms,l}[n] \tag{23}$$

and

$$\hat{\emptyset}_{ms}[n] = \frac{1}{L}\sum_{l=1}^{L}\hat{\emptyset}_{ms,l}[n] \tag{24}$$

where $\hat{a}_{ms,l}[n]$ and $\hat{\emptyset}_{ms,l}[n]$, $l = 1, 2, 3$ are overall estimates for each shift and $\hat{\emptyset}_{ms,l}[n]$ is the unwrapped phase. In this way, we obtained better estimates compared with single micro-segmentation estimate. Actually, this approach in effect is the sliding window with hope size N_μ/L. But we preferred this approach for the ease of concatenation. The reason for using average of shifted multiple micro-segmentation estimates is to have both sufficient number of data points for the parameter estimation and also to have a better time resolution. After determining sufficient number of data points for estimation of a_μ, f_μ and φ_μ, we selected it to be $L = 3$. In Fig. 10, the micro-segmentation estimates obtained with single shift and $L = 3$ shifts are given at SNR = 10 dB with $N_\mu = 12$. From this figure, we can clearly see that the average of multiple shifts improves the estimate.

As can be seen from Figs. 2 and 7, micro-segmentation-based IF and phase estimates are better by far compared with amplitude estimates. But, at low SNR values (SNR < 5 dB) for some micro-segments where noise is dominant, the phase or IF estimates will be too noisy. In fact, the estimates at those segments will be discontinuous and different from the neighboring segments. Such an example is shown in Fig. 11. Where a jump is observed in micro-segmentation-based IF estimate at SNR < 5 dB.

While for micro-segmentation estimate the effect of this jump is local, if we use this estimate for the subsequent polynomial fit, the effect will spread out and the polynomial fit estimate will be worse than micro-segmentation estimate. This effect can be seen also in Fig. 11. These types of jumps are indications of high noise. When these jumps are observed, we need to increase micro-segment length so as to smooth out the noise. Therefore, we used the following method. During computations whenever there is a phase or frequency jump between two micro-segments of any three shifted overall estimates $\hat{a}_{ms,l}[n]$, $\hat{\emptyset}_{ms,l}[n]$, $l = 1, 2, \ldots, L$, we restarted the micro-segment estimation with a longer segment length N_{μ_2} compared with first one. Here, the jumps are detected by comparing the difference between the IF estimate in a segment and its neighbors. If difference exceeds a given threshold γ_f, then the decision is given. Experimentally, this threshold for normalized frequency, set to $\gamma_f = 0.08$. This threshold corresponds to a phase shift of 0.16π radians.

In second round with increased segment length N_{μ_2}, we again monitored the discontinuity. But this time upon discontinuity detection, we used selective averaging of L shifted estimates rather than restarting the micro-segmentation

Fig. 10 Micro-segment estimates with single and $L = 3$ shifts at SNR $= 10$ dB with $N_\mu = 12$
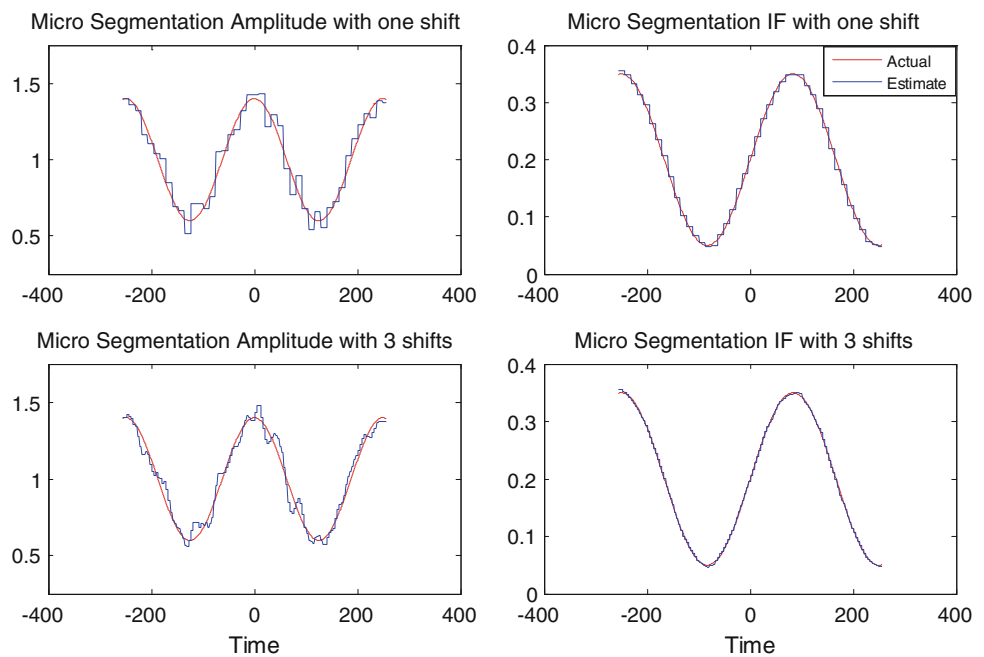
estimate with a further increased segment length. With selective averaging, we mean selecting only those estimates $\hat{a}_{ms,l}[n]$ or $\hat{\emptyset}_{ms,l}[n]$ which do not include any discontinuity. If it happens that in the second round with increased segment length all of the L shifted estimates $\hat{a}_{ms,l}[n]$ or $\hat{\emptyset}_{ms,l}[n]$, $l = 1, 2, \ldots, L$ contain such a jump, then just one of them should be selected and the polynomial fit step should be omitted. The final signal estimate should be obtained with micro-segmentation-based estimates only. This approach prevents at low SNR the performance of overall signal estimation to be not worse than micro-estimate.

To sum up, with two simple methods, that is, averaging of multiple micro-segmentation estimates and use of a longer segment length on detection of IF jumps, we tried to increase the quality of micro-segment estimate. Together with these improvements, the overall algorithm for AM/FM signal estimation is listed in Table 1.

4 Performance comparison

4.1 Cramer–Rao bound

In this section, we provide measures to be used for performance evaluation of the proposed method. The measure is the Cramer–Rao bound for parametric signal estimation. The CRB on the ML estimation of parameters of (3) and (4) is obtained in [6]. Here, the basic notation and results will be summarized. Also with a slight modification, the total bound on the signal MSE will be obtained.

Defining the coefficients of the amplitude and phase functions in (3) and (4) by the vectors as

$$a = [a_1 \, a_2 \ldots \ldots a_P]^T \tag{25}$$

and

$$b = [b_1 \, b_2 \ldots \ldots b_Q]^T \tag{26}$$

and using the following notation

$$n = [0, 1, 2, \ldots \ldots, N-1]^T, \tag{27}$$

$$x = x[n] = [x[0] \, x[1] \, x[2] \ldots \ldots x[N-1]]^T, \tag{28}$$

$$w = w[n] = [w[0] \, w[1] \, w[2] \ldots \ldots w[N-1]]^T, \tag{29}$$

$$e^{j\emptyset[n]} = \left[e^{j\emptyset[0]} \, e^{j\emptyset[1]} \, e^{j\emptyset[2]} \ldots \ldots e^{j\emptyset[N-1]} \right]^T \tag{30}$$

We can have,

$$x = \Phi a + w. \tag{31}$$

where Φ is a $Nx(P+1)$ matrix defined as follows

$$\Phi = \left[g_0[n] \bullet e^{j\emptyset[n]} \, g_1[n] \bullet e^{j\emptyset[n]} \, g_2[n] \bullet e^{j\emptyset[n]} \ldots \ldots g_P[n] \bullet e^{j\emptyset[n]} \right]. \tag{32}$$

where, " $\bullet$ " in (32) denotes component-by-component multiplication of vectors. With this notation and white Gaussian noise assumption, the log-likelihood function is obtained as follows. [6]

$$\Lambda = -N \left(ln\pi + 2ln\sigma \right) - \frac{1}{\sigma^2} \parallel \bar{x} - \Psi a \parallel^2 \tag{33}$$

where σ^2 is the noise variance, $\bar{x} = \left[Re\{x\}^T \, Im\{x\}^T \right]^T$ and $\Psi = \left[Re\{\Phi\}^T \, Im\{\Phi\}^T \right]^T$.

Fig. 11 The jumps in IF micro-segmentation estimate at low SNR (< 5 dB)
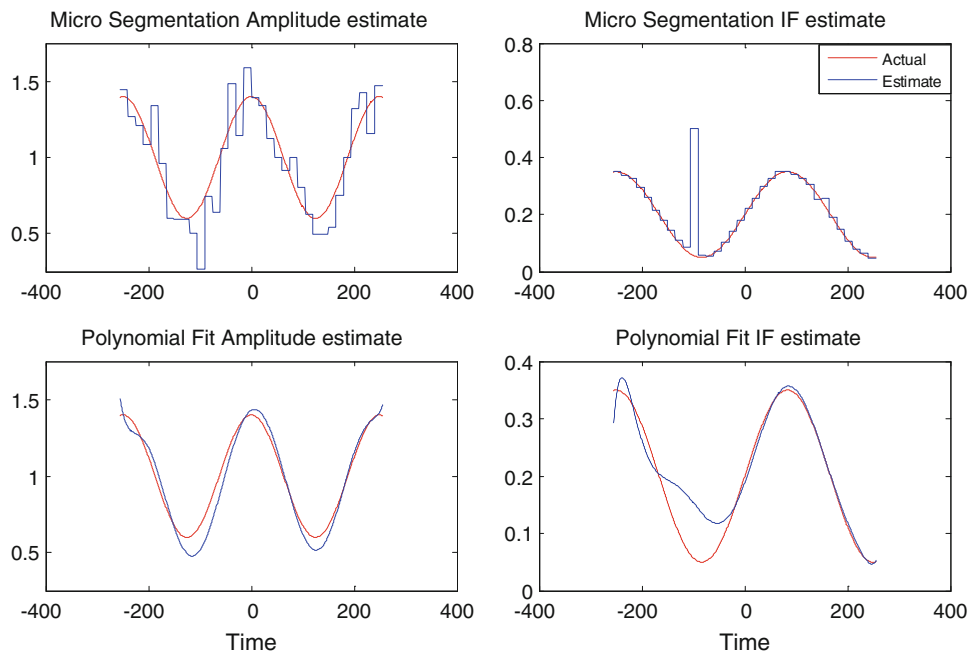
Given the likelihood function Λ in (33, the Fisher Information Matrix (FIM) for the parameter set $\boldsymbol{\theta} = [\boldsymbol{b}^T \boldsymbol{a}^T]^T$ is defined by

$$F_{ij} = -E\left\{\frac{\partial^2 \Lambda}{\partial \theta_i \partial \theta_j}\right\}. \tag{34}$$

The matrix is obtained [6] as

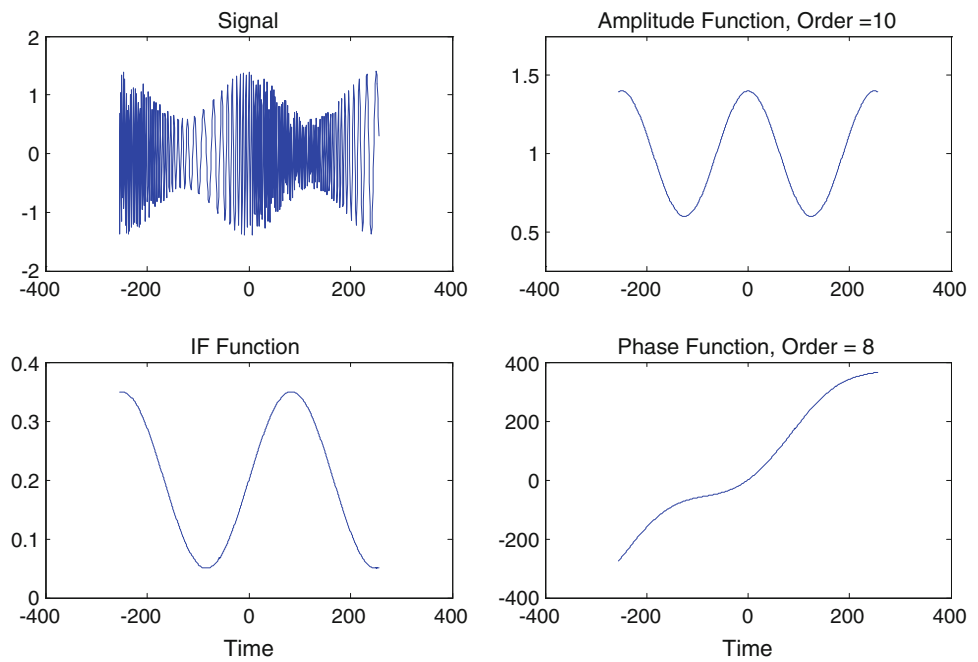$$\boldsymbol{F} = \frac{2}{\sigma^2} Re\left\{[A\Phi]^H [A\Phi]\right\}. \tag{35}$$

where

$$A = j\left[p_0[\boldsymbol{n}] \bullet s[\boldsymbol{n}] \ p_1[\boldsymbol{n}] \bullet s[\boldsymbol{n}] \ p_2[\boldsymbol{n}] \bullet s[\boldsymbol{n}] \ldots \ldots p_Q[\boldsymbol{n}] \bullet s[\boldsymbol{n}]\right]. \tag{36}$$

Table 1 AM/FM signal estimation algorithm

1	Compute shifted overall micro-segment estimates $\hat{a}_{ms,l}[n] \ \hat{\emptyset}_{ms,l}[n], l = 1, 2, \ldots, L$ with N_μ.
2	*If* all IF or $\hat{\emptyset}_{ms,l}[n], l = 1, 2, \ldots, L$ does not contain discontinuity
	Compute $\hat{a}_{ms}[n] = \frac{1}{L}\sum_{l=1}^{L}\hat{a}_{ms,l}[n], \ \hat{\emptyset}_{ms}[n] = \frac{1}{L}\sum_{l=1}^{L}\hat{\emptyset}_{ms,l}[n],$
	Set $flag = 0$ % discontinuity flag
	else
	Re compute $\hat{a}_{ms,l}[n]$ and $\hat{\emptyset}_{ms,l}[n], l = 1, 2, \ldots, L$ with N_{μ_2}
	If all $\hat{\emptyset}_{ms,l}[n], l = 1, 2, \ldots, L$ with N_{μ_2} are discontinuous
	Set $flag = 1$
	$\hat{a}_{ms}[n] = \hat{a}_{ms,1}[n], \quad \hat{\emptyset}_{ms}[n] = \hat{\emptyset}_{ms,1}[n]$
	else
	Compute $\hat{a}_{ms}[n]$ and $\hat{\emptyset}_{ms}[n]$ with selective average
	Set $flag = 0$
3	*If* $flag = 1$
	$\hat{a}[n] = \hat{a}_{ms}[n], \quad \hat{\emptyset}[n] = \hat{\emptyset}_{ms}[n]$
	else
	Find the orders $\hat{P}$ and $\hat{Q}$ for $\hat{a}_{ms}[n]$ and $\hat{\emptyset}_{ms}[n]$ with search
	$\hat{a}[n] = polynomial\_fit(\hat{a}_{ms}[n], \hat{P}) \ \hat{\emptyset}[n] = polynomial\_fit(\hat{\emptyset}_{ms}[n], \hat{Q})$
4	$\hat{s}[n] = \hat{a}[n]e^{j\hat{\emptyset}[n]}$

Fig. 12 Test signal Example 1

CRB for set $\theta = \begin{bmatrix} b^T a^T \end{bmatrix}^T$ is defined as the inverse of F, that is,

$$\mathrm{CRB}(\theta) = F^{-1} \tag{37}$$

In an actual application rather than a and b parameters, we will be interested in signal $s[n]$ itself. Therefore, we will drive the bounds on the variance of the estimate for the signal at time instant n. The signal $s[n]$ is a function of the parameter set $\theta = [b^T a^T]^T$. Having $\mathrm{CRB}(\theta)$, $\mathrm{CRB}(s[n])$ can be obtained as [11]

$$\mathrm{CRB}(s[n]) = \left(s'_n\right)^{\mathrm{H}} \mathrm{CRB}\,(\theta)\, s'_n \tag{38}$$

where

$$s'_n = \frac{\partial s[n]}{\partial \theta}. \tag{39}$$

Using (3), (4) and (37) s'_n will be obtained as

$$s'_n = [A\,[n]\ \ \Phi[n]]^T \tag{40}$$

where s'_n is simply the transpose of the row of $[A\,\Phi]$ corresponding to time instant n. Since in our application we have N time instants, we need to compute (38) for all of them. But, in order to get an overall performance indication, we will sum them up and obtain the following bound as a reference for the MSE error performance.

$$\mathrm{CRB}(s) = \sum_{n=0}^{N-1} \mathrm{CRB}(s[n]) \tag{41}$$

This is the total variance bound for the estimate of the signal values at all time instants between 0 and $N-1$.
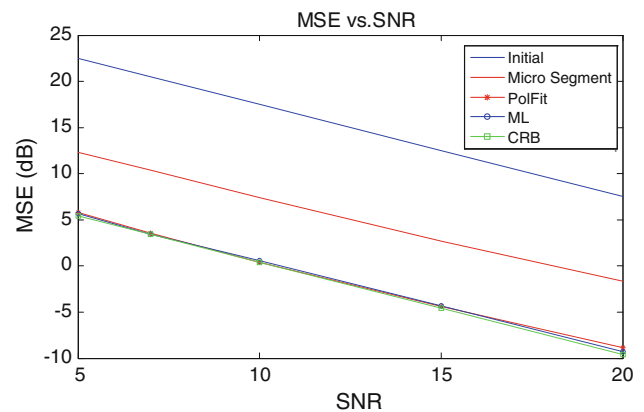

Fig. 13 MSE versus SNR for Example 1

4.2 Comparison with existing methods

Though Cramer–Rao bound is the destination or target for the performance evaluation of the proposed method. A comparison with some other method will also be useful both in terms of performance and complexity.

A known method for the estimation of signal in (1) is the parametric maximum likelihood (ML) estimation [6,13] where the amplitude and phase parameters are given by (25) and (26). As explained before with this method, the amplitude and phase orders need to be known in advance and this is one of the difficulties. Also since the estimation of parameters is found by optimization of (33) using quasi-Newton type iterative algorithms [9], good initial conditions are required to avoid convergence to local minima. Therefore, for the comparison purposes, when finding the parameters with ML
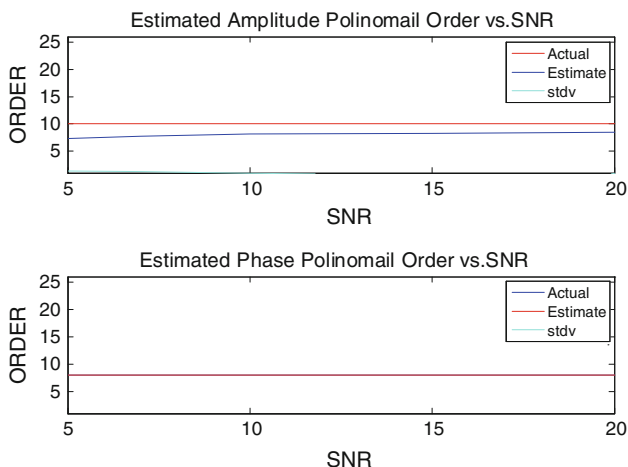
Fig. 14 True and estimated orders versus SNR for Example 1

Fig. 16 MSE versus SNR for Example 2

method we will assume that the orders, both for the amplitude and the phase are known or estimated by some other method. Also, we will use the polynomial coefficients corresponding to micro-segment amplitude and phase estimates as the initial parameters. In fact, in this type of parametric time-varying signal estimation, the time-frequency distribution based estimates are usually used as initial parameters [6,13].

The performance comparison between the proposed method and parametric ML method is given in simulation results section. Since we use the same initial parameter set both for the proposed method and ML method with BFGS, any complexity comparison should be focused on rest of the algorithms. Due to iterative nature of BFGS type algorithms, an exact comparison may not be possible. But the follow-
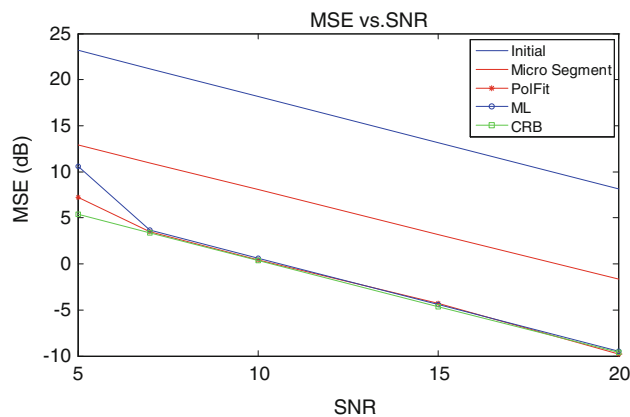
ing discussion is believed to be useful for understanding the computation complexity of proposed and ML method.

With proposed method, the final estimates are found by polynomial approximation of micro-segment estimates. Polynomial orders are also estimated from micro-segment estimates. The order estimation is based on a search between 1 and MaxPol. As explained in polynomial fit-based estimation section, a function of order P will roughly require $P + M$ steps and M was experimentally set to 5 in algorithm. Therefore, the order estimation requires $P + M$ least squares (LS) to be solved. The parameters to be found in each LS varies between 1 and $P + M$. In this regard considering a signal with amplitude order P and a phase order Q, the total complexity is equivalent to cost of solving $P + M$ LS for amplitude and the cost of $Q + M$ LS for phase. The LS can be implemented either directly with pseudo-inverse or with conjugate gradient (CG) which is expected to save computations.
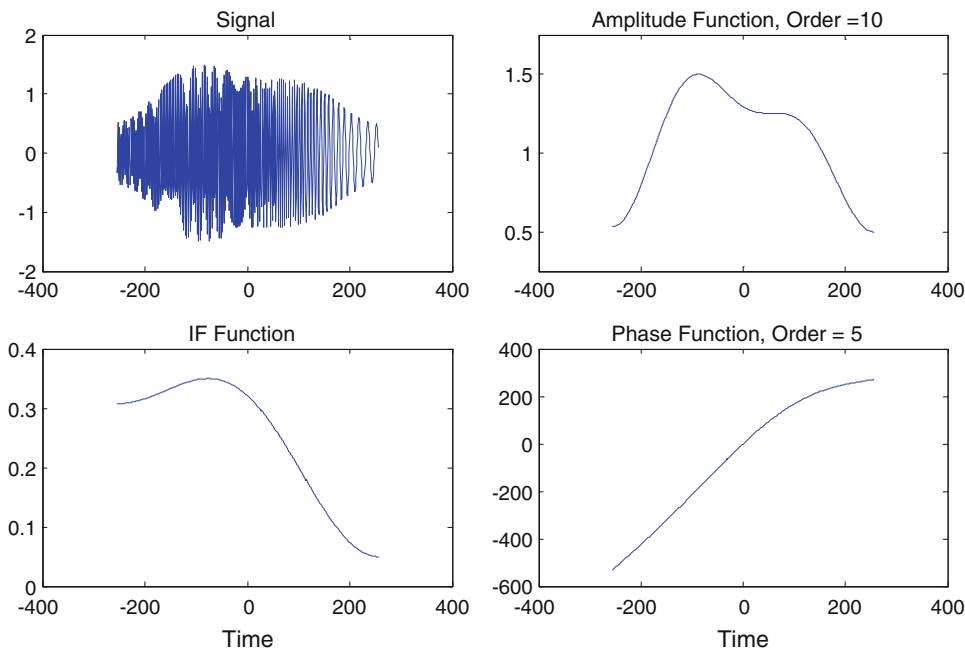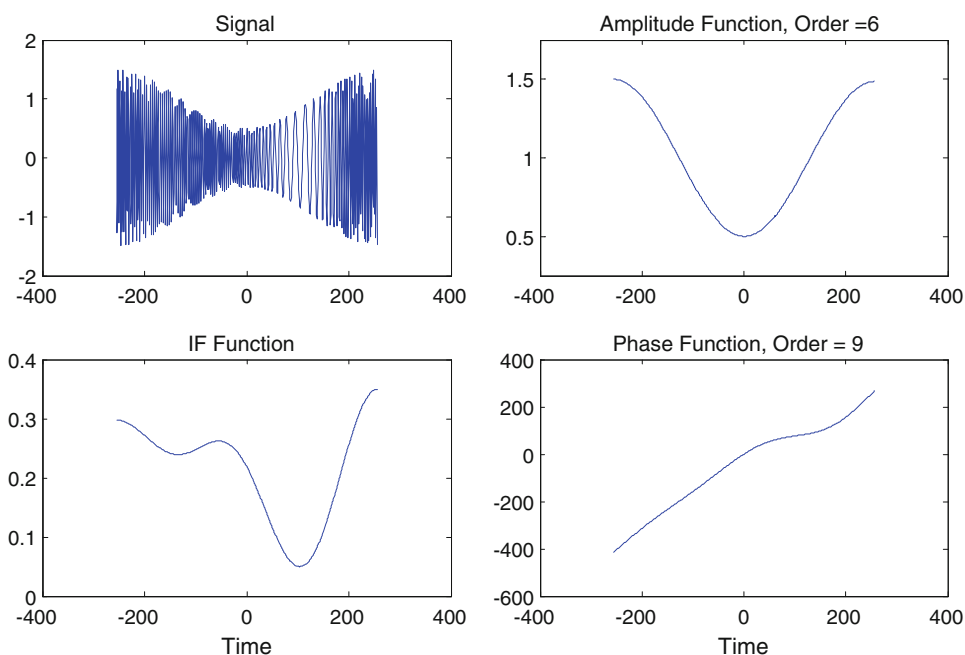
Fig. 15 Test signal Example 2

Fig. 17 Test signal Example 3

With parametric ML method, if orders are known, then we can run a quasi-Newton algorithm like BFGS [9]. These algorithms require the optimized cost function and its gradient to be computed at each iteration step. In fact, the function and gradient evaluation is done several times in line search section. The cost function and its gradient is highly nonlinear [6], and their computations complexity are very large compared to solving a LS problem with the same parameter size [3]. Considering that the convergence of the BFGS requires many steps, the computation burden of ML is apparent.

With the above reasoning, it is apparent that the proposed algorithm requires less computation than the ML method. Also, it is not affected by any convergence issues and in this respect it is robust.

5 Simulation results

Three test signals of 512 samples were used with various SNR values between 5 and 20 dB in simulation examples. The SNR definition is the one given in (16). For all examples, the length of micro-segments was selected as $N_\mu = 12$, and the second length in case of discontinuity was set to $N_{\mu_2} = 15$. The number of micro-segmentation estimates for averaging was taken as $L = 3$. The threshold for deciding on IF or phase discontinuity was set to $\gamma_f = 0.08$. The maximum polynomial order for search was set MaxPol = 26, and moving search length was set to $M = 5$. The thresholds for estimating the orders were set to $\gamma_{\text{ord}} = 0.4$ for phase and $\gamma_{\text{ord}} = 0.3$ for the amplitude. As stated before, the ML method (BFGS) was used with true or known orders and initialized with polynomial coefficients of micro-segment esti-

mates. Simulation was carried out for 400 runs for each SNR value. The MSE defined by

$$\text{MSE} = \sum_{n=0}^{N-1} \left| \hat{s}[n] - s[n] \right|^2 \tag{42}$$

was obtained for each run, and the average of each SNR value was compared with CRB in (41).

In first example (Example 1), the signal was selected with true amplitude order $P = 10$ and true phase order as $Q = 8$. The noiseless signal, its amplitude, phase, and IF functions are shown in Fig. 12. In Fig. 13, MSE versus SNR is plotted in logarithmic scale. The top line corresponds to the MSE for the noisy signal, the second line under that corresponds to MSE for micro-segmentation estimate, and the last three lines on top of each other correspond to the MSE for the polynomial estimate, ML estimate and the CRB. In Fig. 14, the true and estimated orders versus SNR are plotted. The standard deviation for 400 runs is also plotted. Based on these figures, we observe that the proposed method is very effective to estimate the AM/FM signal parameters. While the micro-segmentation-based estimate is far from the CRB, both the polynomial fit estimate and ML estimate are close to it. From Fig. 14, we observe that the standard deviation on polynomial order estimate with the proposed search is very small. This is an indication of the robustness of the proposed method for order estimation. On the other hand, while the mean estimated order for phase is nearly equal to the true value, it is under estimated for amplitude order, but still MSE is close to the CRB. In fact, an under estimate for low SNR is a desirable side effect for noise removal.
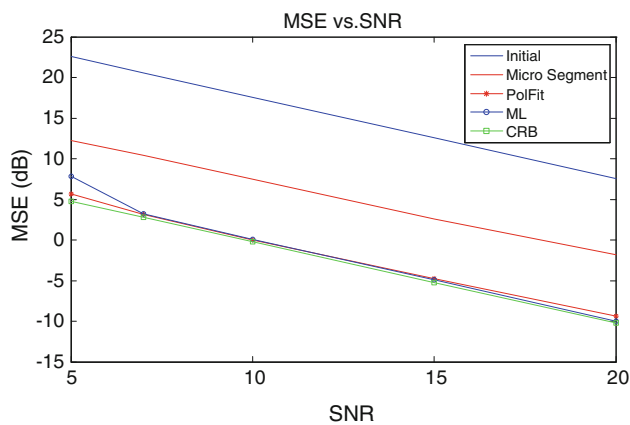
Fig. 18 MSE versus SNR for Example 3

In Example 2, given in Figs. 15 and 16, the signal was selected with true amplitude order $P = 10$ and true phase order as $Q = 5$. For this example, we can see that both the performance of proposed algorithm and the ML method are close to the CRB. While for both methods, there is a small degradation at low SNR compared with CRB, the performance of proposed method is better than ML. The order estimation performance was observed to similar to Example 1.

In Example 3, given in Fig. 17 and 18, the signal was selected with true amplitude order $P = 6$ and true phase order as $Q = 9$. For this example, also we can see that both the performance of proposed algorithm and the ML method are close to the CRB. Again, the performance of proposed method is better than ML at low SNR. The order estimation performance was observed to be similar to Examples 1 and 2.

From three examples, a clear observation is that the micro-segmentation-based IF or phase estimates are by far better than the amplitude estimate. Also, the subsequent order estimation is better by far than that of amplitude. In all examples, the mean of the phase order is nearly equal to the true order, and the standard deviation is negligible. On the other hand, while amplitude order estimate also has a negligible standard deviation, it is usually lower estimated. But still the estimated amplitude order together with phase order, improve the micro-segmentation estimate significantly.

From the three examples and Figures 12, 13, 14, 15, 16, 17 and 18 it is observed that the proposed method efficiently estimates the AM/FM signal without using any multivariable nonlinear optimization. The performance is comparable to that of the Cramer–Rao bound and is better than ML at low SNR. The method proposed for order estimation is robust enough for improving the SNR for micro-segmentation estimates.

6 Conclusion

A flexible method which does not require multivariable nonlinear optimization is proposed to estimate long duration and non-stationary AM/FM signals in white Gaussian additive noise. In the first step, amplitude and phase functions are separated optimally (the LS sense) at micro-segment level. This provides a sufficient SNR improvement compared with the initial noisy signal. In the second step, using the segmentation-based estimates of phase and amplitude functions, the corresponding orders were estimated. The MSE error performance, though not optimal, is comparable to the Cramer–Rao bound and is better than ML at low SNR. For long signals with highly nonlinear amplitude and phase functions, the method is efficient and flexible. Since the IF or phase estimates are very good compared to the amplitude, the method can be successfully applied to polynomial phase signals (PPS) with constant amplitude.

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