HIGHER DIMENSIONAL SPHERICALLY SYMMETRIC GRAVITATIONAL THEORIES

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS AND THE INSTITUTE OF ENGINEERING AND SCIENCES OF BILKENT UNIVERSITY IN PARCIAL FULFELINENT OF THE REQUIREMENTS FOR THE DEOREE OF

MASTER OF SCIENCE

By Emre Sermutlu September 1994



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## ABSTRACT

#### HIGHER DIMENSIONAL SPHERICALLY SYMMETRIC GRAVITATIONAL THEORIES

Emre Sermutlu M.S. in Mathematics Supervisor: Prof. Dr. Metin Gürses September 1994

We consider all possible theories in spherically symmetric Riemannian geometry in D-dimensions. We find solutions to such theories, in particular black hole solutions of the low energy limit of the string theory in Ddimensions.

Keywords : Gravitation, higher dimensions, black holes, low energy limit, spherical symmetry.

# ÖZET

# YÜKSEK BOYUTLARDA KÜRESEL SİMETRİK GRAVİTASYON TEORİLERİ

Emre Sermutlu Matematik Yüksek Lisans Tez Yöneticisi: Prof. Dr. Metin Gürses Eylül 1994

D boyutlu küresel simetrik Riemann geometrisinde mümkün olan bütün teorileri ele alarak çözümleri inceledik. Bir özel hal olarak sicim teorisinin düşük enerji limitinde kara delik çözümleri bulduk.

Anahtar Kelimeler : Gravitasyon, yüksek boyutlar, kara delikler, düşük enerji limiti, küresel simetri.

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## Chapter 1

#### Introduction

In the classical relativity, the lagrangian contains only the Ricci scalar. On the other hand we learned from the low energy limit of string theory that the classical lagrangian contains all possible invariants constructed from the curvature tensor and the matter fields. Depending upon the order of the string tension parameter, this lagrangian is an infinite series expansion in these invariants, i.e.

$$L = \sqrt{-g}R + \sum_{n=1}^{\infty} \alpha^n L_n.$$
(1.1)

Here  $\alpha$  is the inverse of the string tension and  $L_n$  are functions containing the invariants up to  $n^{th}$  order. There are several examples where the sum is terminated at some value n. For instance when n = 2 we have Gauss-Bonnet theory

$$L = \sqrt{-g}R + \alpha (R^{ijkl}R_{ijkl} - 4R^{ij}R_{ij} + R^2).$$
 (1.2)

For different n we have the Lovelock theorem: In d-dimensions divergencefree second order symmetric tensors constructed from the metric and its first two derivatives are given by

$$A_{j}^{i} = \sum_{p=1}^{m-1} a_{p} \delta_{jj_{1} \dots j_{2p}}^{ih_{1} \dots h_{2p}} R_{h_{1}h_{2}}^{j_{1}j_{2}} R_{h_{3}h_{4}}^{j_{3}j_{4}} \cdots R_{h_{2p-1}h_{2p}}^{j_{2p-1}j_{2p}} + a\delta_{j}^{i}, \qquad (1.3)$$

where  $m = \frac{1}{2}n$  if n is even,  $m = \frac{1}{2}(n+1)$  if n is odd.

Boulware and Deser [2] found two solutions for the Einstein plus Gauss-Bonnet lagrangian, one of them asymptotically flat, the other asymptotically anti-de Sitter.

Wiltshire [4] slightly generalized the previous results by including a Maxwell field. Wheeler [5] has considered the most general second-order gravity theory in arbitrary dimensions and analyzed asymptotically flat spherically symmetric static solutions, and cosmological solutions. Whitt [6] extended Wheeler's work to non-static space-times.

The Schwarzschild solution which describes the uncharged black holes in general relativity also describes (to a good approximation) uncharged black holes in string theory. But, this is not the case for charged black holes. The dilaton has a coupling to  $F^2$ , so the Reissner-Nordström solution is not even an approximate solution of string theory.

Charged black holes were first analyzed by Gibbons-Maeda [7] and GHS (Garfinkle, Horowitz, Strominger) [8] independently.

Black hole solutions depend on dimension and number of physical parameters. In this work we find D-dimensional solutions with three independent parameters, mass M, electric charge Q, and the dilaton charge  $\Sigma$ . We also extend the Levi-Civita Bertotti-Robinson metrics to D-dimensions and prove that they solve all field equation arising from a variational principle.

#### Chapter 2

# Spherically Symmetric Riemannian Geometry

The metric of a static and spherically symmetric D-dimensional spacetime is given by:

$$ds^{2} = -A^{2}dt^{2} + B^{2}dr^{2} + C^{2}d\Omega_{D-2}^{2}, \qquad (2.1)$$

where A,B,C depend only on r.  $d\Omega_{D-2}^2$  is the metric on  $S_{D-2}$ . The metric can be rewritten as  $g_{ij} = -A^2 t_i t_j + B^2 k_i k_j + C^2 h_{ij}$ , where  $t_i = \delta_i^t$ ,  $k_i = \delta_i^r$ ,  $h_{ij}$  =metric on D-2 sphere for  $i, j \ge 2$ ,  $h_{0i} = h_{1i} = 0$ .

Christoffel symbols are given by

$$\Gamma^{i}_{jm} = \frac{1}{2}g^{in}(g_{jn,m} + g_{nm,j} - g_{jm,n})$$
(2.2)

$$\Gamma_{jm}^{i} = -AA't^{i}(t_{j}k_{m} + t_{m}k_{j}) + AA'k^{i}t_{j}t_{m} + BB'k^{i}k_{j}k_{m} - CC'k^{i}h_{jm} + CC'(h_{j}^{i}k_{m} + h_{m}^{i}k_{j}) + \Gamma_{(s)jm}^{i}, \qquad (2.3)$$

where  $\Gamma_{(s)}$  is the christoffel symbol on D-2 sphere.

The Riemann tensor is given by

$$R^i_{jml} = \Gamma^i_{jl,m} - \Gamma^i_{jm,l} + \Gamma^i_{nm}\Gamma^n_{jl} - \Gamma^i_{nl}\Gamma^n_{jm}$$
(2.4)

$$R_{ijml} = \left(AA'' - \frac{AA'B'}{B}\right) t_{[i}k_{j]}t_{[m}k_{l]} + \left(CC'' - \frac{CC'B'}{B}\right) k_{[i}h_{j][m}k_{l]} + \frac{AA'CC'}{B^2} t_{[m}h_{l][j}t_{i]} - \frac{C^2C'^2}{B^2} h_{i[m}h_{l]j} + C^2R_{sijml}, \qquad (2.5)$$

where  $R_{sijml} = h_{i[m}h_{l]j}$ . Riemann tensor can be rewritten as

$$R_{ijml} = g_{jl}S_{im} - g_{jm}S_{il} + g_{im}S_{jl} - g_{il}S_{mj} + \eta_2 H_{ijk\cdots n}H_{ml}^{k\cdots n}$$
(2.6)

where

$$S_{ij} = \eta_0 M_{ij} + \eta_1 k_i k_j + \frac{1}{2} \eta_3 g_{ij}, \qquad (2.7)$$

$$M_{ij} = H_{im\cdots n} H_j^{m\cdots n} - \frac{1}{2(D-2)} H^2 g_{ij}$$
(2.8)

which turns out to be

$$M_{ij} = \frac{(D-3)!}{C^{2(D-3)}} (h_{ij} - \frac{1}{2C^2} g_{ij}), \qquad (2.9)$$

the tensor  $H_{ij...k}$  is the volume form of  $S_{D-2}$ , i.e.

$$H_{ij\cdots k} = -\sqrt{h}\epsilon_{ij\cdots k}.$$
(2.10)

Here the scalars are given by

$$\eta_{0} = \frac{C^{2(D-2)}}{(D-3)!} \left( \frac{A''}{AB^{2}} - \frac{A'B'}{AB^{3}} - \frac{C''}{CB^{2}} + \frac{B'C'}{B^{3}C} \right),$$

$$\eta_{1} = \frac{A'C'}{AC} - \frac{C''}{C} + \frac{B'C'}{BC},$$

$$\eta_{2} = \frac{C^{2(D-3)}}{(D-4)!} \left( 1 - \frac{C'^{2}}{B^{2}} + \frac{A'C'C}{AB^{2}} - \frac{A''C^{2}}{AB^{2}} + \frac{A'B'C^{2}}{AB^{3}} + \frac{CC''}{B^{2}} - \frac{B'C'C}{B^{3}} \right),$$
(2.11)

$$\eta_3 = -\frac{A'C'}{AB^2C}.$$
 (2.12)

The Ricci tensor, Ricci scalar and Einstein tensor can be computed from Riemann as follows:

$$R_{ij} = \left[\frac{(D-3)!}{2C^{2(D-2)}}(\eta_0(D-4)+\eta_2) + \frac{1}{B^2}\eta_1 + \eta_3(D-1)\right]g_{ij} + \eta_1(D-2)k_ik_j + [\eta_0(D-2)+\eta_2]M_{ij},$$
(2.13)

$$R = \frac{(D-3)!}{C^{2(D-2)}} [\eta_0(D-4)(D-1) + \eta_2(D-2)] + \frac{2\eta_1(D-1)}{B^2} + \eta_3 D(D-1),$$
(2.14)

$$G_{ij} = -\left[\frac{(D-3)!}{2C^{2(D-2)}}(\eta_0(D-4)(D-2) + \eta_2(D-3)) + \frac{1}{B^2}\eta_1(D-2) + \frac{1}{2}\eta_3(D-1)(D-2)\right]g_{ij} + \eta_1(D-2)k_ik_j + [\eta_0(D-2) + \eta_2]M_{ij}.$$
(2.15)

The covariant derivatives of  $H_{ij\ldots k}$  and  $k_i$  are given as

$$\nabla_l H_{ij...m} = -\rho[(D-2)H_{ij...m}k_l + k_i H_{lj...m} + k_j H_{il...m} + \dots + k_m H_{ij...l}], \quad (2.16)$$

$$\nabla_{i}k_{j} = \rho_{1}g_{ij} + \rho_{2}M_{ij} + \rho_{3}k_{i}k_{j} \tag{2.17}$$

where

$$\rho = \frac{C'}{C} \qquad \rho_1 = \frac{A'C + AC'}{2AB^2C}$$

$$\rho_2 = \frac{C^{2(D-2)}}{(D-3)!} \frac{AC' - A'C}{AB^2C} \qquad \rho_3 = -\frac{(AB)'}{AB}.$$
(2.18)

The covariant derivatives of  $H_{ij\dots k}$  and  $k_i$  are expressed only in terms of themselves and the metric tensor. Riemann tensor is given in terms of  $H_{ij\dots k}$ ,  $g_{ij}$  and  $k_i$ . Hence we have the following theorem:

**Theorem 1** Covariant derivatives of the Riemann tensor  $R_{ijkl}$ , the tensor  $H_{ij}$  and the vector  $k_i$  at any order are expressible only in terms of  $H_{ij\cdots k}$ ,  $g_{ij}$ ,  $k_i$ .

Since contraction of  $k^i$  with  $H_{ij\cdots k}$  vanishes, the only symmetric tensors constructable out of  $H_{ij\cdots k}$ ,  $g_{ij}$  and  $k_i$  are  $M_{ij}$ , the metric tensor  $g_{ij}$  and  $k_i k_j$ . Then the following theorems hold:

**Theorem 2** Any second rank symmetric tensor constructed out of the Riemann tensor, anti-symmetric tensor  $H_{ij\dots k}$ , dilaton field  $\phi = \phi(r)$  and their covariant derivatives is a linear combination of  $M_{ij}$ ,  $g_{ij}$  and  $k_i k_j$ .

Let this symmetric tensor be  $E'_{ij}$ . Then we have

$$E'_{ii} = \sigma_1 M_{ij} + \sigma_2 g_{ij} + \sigma_3 k_i k_j \tag{2.19}$$

where  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  are scalars which are functions of the metric functions, invariants constructed out of the curvature tensor  $R_{ijkl}$ ,  $H_{ij}$  and the dilaton field.

**Theorem 3** Any vector constructed out of the Riemannian tensor  $R_{ijkl}$ ,  $H_{ij...k}$  the dilaton field  $\phi = \phi(r)$  and their covariant derivatives is proportional to  $k_i$ .

Let this vector be  $E'_i$ . Hence

$$E_i' = \sigma k_i \tag{2.20}$$

where  $\sigma$  is a scalar like  $\sigma_1$ ,  $\sigma_2$ ,  $\sigma_3$ .

Theorem 2 has an application:

**Theorem 4** In a spherically symmetric, D-dimensional spacetime, the coefficients  $a_i$  in the identity

$$a_1 R_{imnl} R_j^{mnl} + a_2 R_{im} R_j^m + a_3 R_{jmin} R^{mn} + a_4 R_{ij} + a_5 g_{ij} = 0$$
 (2.21)

can be found in terms of the  $\eta_i$ .

This can be easily seen if we write each of the symmetric tensors as a combination of  $M_{ij}$ ,  $g_{ij}$ ,  $k_i k_j$ . Let us denote these tensors as follows:

$$R_i^a R_{ja} = \beta_1 g_{ij} + \beta_2 M_{ij} + \beta_3 k_i k_j$$

$$R_{jaib} R^{ab} = \beta_4 g_{ij} + \beta_5 M_{ij} + \beta_6 k_i k_j$$

$$R_i^{abc} R_{jabc} = \beta_7 g_{ij} + \beta_8 M_{ij} + \beta_9 k_i k_j$$

$$R_{ij} = f_1 g_{ij} + f_2 M_{ij} + f_3 k_i k_j.$$
(2.22)

Thus, we obtain

$$\beta_{7}a_{1} + \beta_{1}a_{2} + \beta_{4}a_{3} + f_{1}a_{4} + a_{5} = 0,$$
  

$$\beta_{8}a_{1} + \beta_{2}a_{2} + \beta_{5}a_{3} + f_{2}a_{4} = 0,$$
  

$$\beta_{9}a_{1} + \beta_{3}a_{2} + \beta_{6}a_{3} + f_{3}a_{4} = 0.$$
(2.23)

Provided we know  $a_4$  and  $a_5$ , we can compute  $a_1$ ,  $a_2$ ,  $a_3$ .

Let's denote

$$\begin{vmatrix} \beta_7 & \beta_1 & \beta_4 \\ \beta_8 & \beta_2 & \beta_5 \\ \beta_9 & \beta_3 & \beta_6 \end{vmatrix} = \frac{1}{\Psi},$$
(2.24)

then,

$$a_{1} = -\Psi[(\beta_{2}\beta_{6} - \beta_{3}\beta_{5})(f_{1}a_{4} + a_{5}) + (\beta_{3}\beta_{4} - \beta_{1}\beta_{6})(f_{2}a_{4}) + (\beta_{1}\beta_{5} - \beta_{2}\beta_{4})(f_{3}a_{4})],$$
(2.25)

$$a_{2} = -\Psi[(\beta_{5}\beta_{9} - \beta_{6}\beta_{8})(f_{1}a_{4} + a_{5}) + (\beta_{6}\beta_{7} - \beta_{4}\beta_{8})(f_{2}a_{4}) + (\beta_{4}\beta_{8} - \beta_{5}\beta_{7})(f_{3}a_{4})],$$
(2.26)

$$a_{3} = -\Psi[(\beta_{3}\beta_{8} - \beta_{2}\beta_{9})(f_{1}a_{4} + a_{5}) + (\beta_{1}\beta_{9} - \beta_{3}\beta_{7})(f_{2}a_{4}) + (\beta_{2}\beta_{7} - \beta_{1}\beta_{8})(f_{3}a_{4})].$$
(2.27)

We will give  $\beta$ 's explicitly in appendix.

We may also write the Riemann tensor in the form

$$R_{ijml} = g_{jl}S_{im} - g_{jm}S_{il} + g_{im}S_{jl} - g_{il}S_{mj} + e_2F_{ij}F_{ml}, \qquad (2.28)$$

where

$$F_{ij} = \frac{AB}{C^2} (t_i k_j - t_j k_i).$$
 (2.29)

Other tensors are defined similar to the previous case, i.e.

$$S_{ij} = e_0 \,\widetilde{M_{ij}} + e_1 \,k_i k_j + \frac{1}{2} e_3 \,g_{ij}, \qquad (2.30)$$

$$\widetilde{M_{ij}} = F_{mj}F_i^m - \frac{1}{4}F^2 g_{ij} = \frac{1}{C^2}(h_{ij} - \frac{1}{2C^2}g_{ij}).$$
(2.31)

The scalars are given as

$$e_{0} = C^{2} - \frac{C'^{2}C^{2}}{B^{2}} + \frac{C^{3}C'A'}{AB^{2}},$$

$$e_{1} = \eta_{1},$$

$$e_{2} = -C^{2} \left( 1 - \frac{C'^{2}}{B^{2}} + \frac{A'C'C}{AB^{2}} - \frac{A''C^{2}}{AB^{2}} + \frac{A'B'C^{2}}{AB^{3}} + \frac{CC''}{B^{2}} - \frac{B'C'C}{B^{3}} \right),$$

$$e_{3} = \eta_{3}.$$

$$(2.32)$$

Notice that when D = 4,  $\widetilde{M_{ij}} = M_{ij}$ , and  $e_2 = -\eta_2$ .

The covariant derivatives of  $F_{ij}$  ,  $k_i$  and  $t_i$  are given as

$$\nabla_l F_{ij} = -\frac{3C'}{C} F_{ij} k_l - \frac{AC'}{BC^3} (t_j g_{il} - t_i g_{jl}), \qquad (2.33)$$

$$\nabla_i k_j = \rho_1 g_{ij} + \tilde{\rho_2} \widetilde{M_{ij}} + \rho_3 k_i k_j, \qquad (2.34)$$

$$\nabla_i t_j = \frac{A'}{A} (t_i k_j + t_j k_i). \tag{2.35}$$

where  $\tilde{\rho}_2 = \frac{(D-3)!}{C^{2(D-4)}} \rho_2$ . The covariant derivatives of  $F_{ij}$  and  $k_i$  are expressed in terms of themselves, metric tensor and  $t_i$ . So we have a similar theorem for an extended set:

**Theorem 5** Covariant derivatives of the Riemann tensor  $R_{ijkl}$ , the antisymmetric tensor  $F_{ij}$ , and the vectors  $k_i$  and  $t_i$  at any order are expressible only in terms of  $F_{ij}$ ,  $k_i$ ,  $t_i$  and  $g_{ij}$ .

### Chapter 3

# Schwarzschild and Reissner-Nordström Solutions

These are the earliest spherically symmetric solutions, obtained in 1916, so they deserve a short rewiew here.

Let's take the line element as

$$ds^{2} = -A^{2}dt^{2} + B^{2}dr^{2} + r^{2}d\Omega^{2}.$$
(3.1)

In other words, take C(r) = r. Our equation is

$$G_{ij} = 0. (3.2)$$

We have already computed  $G_{ij}$  in terms of  $g_{ij}$ ,  $k_i k_j$  and  $M_{ij}$  (2.15), so equating the coefficients of these terms to zero, we obtain three equations:

$$\frac{(D-3)!}{2C^{2(D-2)}} [\eta_0(D-4)(D-2) + \eta_2(D-3)] + \frac{\eta_1}{B^2}(D-2) + \frac{\eta_3}{2}(D-1)(D-2) = 0,$$
(3.3)

$$\eta_1(D-2) = 0, \tag{3.4}$$

$$\eta_0(D-2) + \eta_2 = 0, \tag{3.5}$$

(3.4) gives

$$\frac{A'}{A} + \frac{B'}{B} = 0 \tag{3.6}$$

so  $AB = c_1$  but the boundary condition for the metric to be Lorentzian at infinity requires that  $c_1 = 1$ .

Thus

$$AB = 1. \tag{3.7}$$

Using (3.5), we can find  $\eta_2$  in terms of  $\eta_0$ . Inserting  $\eta_2$  and substituting  $B = \frac{1}{A}$  in (3.3) gives

$$AA'' + A'^{2} + \frac{(D-2)}{r}AA' = 0, \qquad (3.8)$$

which is actually

$$u' + (D-2)\frac{u}{r} = 0 \tag{3.9}$$

where u = A A'.

Thus  $A^2 = c_2 + c_3 r^{-D+3}$ . Inserting this in (3.5), we find that  $c_2 = 1$ ,

so

$$A^{2} = 1 - \frac{2m}{r^{D-3}},$$
  

$$B^{2} = \left(1 - \frac{2m}{r^{D-3}}\right)^{-1},$$
(3.10)

where 2m is the integration constant  $c_3$ .

So we obtain the Schwarzschild solution. For Reissner-Nordström solution we have to take electrostatic charge Q into account.

Our equation is

$$G_{ij} = T_{ij} \tag{3.11}$$

where

$$T_{ij} = -2\left[F_{iN}F_{j}^{N} - \frac{1}{4}F_{MN}F^{MN}g_{ij}\right],$$
(3.12)

$$F_{tr} = Q \, \frac{AB}{r^{D-2}}.\tag{3.13}$$

 $T_{ij}$  turns out to be

$$T_{ij} = -2 Q^2 \frac{r^{2(D-4)}}{(D-3)!} M_{ij}.$$
(3.14)

Thus we obtain the same set of equations except for (3.5).

Proceeding in a similar manner, instead of (3.8) we obtain

$$AA'' + A'^{2} + \frac{(D-2)}{r}AA' = -2\frac{(D-3)}{(D-2)}\frac{Q^{2}}{r^{2(D-2)}},$$
(3.15)

which is actually

$$\frac{(r^{D-2}AA')'}{r^{D-2}} = -2\frac{(D-3)}{(D-2)}\frac{Q^2}{r^{2(D-2)}},$$
(3.16)

so we find

$$A^{2} = \frac{-2Q^{2}}{(D-2)(D-3)} r^{2(D-3)}.$$
(3.17)

But this is the non-homogeneous solution, we have to add the solution of (3.8) to this.

Thus

$$A^{2} = 1 - \frac{2m}{r^{D-3}} - \frac{2Q^{2}}{(D-2)(D-3)r^{2(D-3)}},$$
  

$$B^{2} = \left(1 - \frac{2m}{r^{D-3}} - \frac{2Q^{2}}{(D-2)(D-3)r^{2(D-3)}}\right)^{-1}$$
(3.18)

Here integration constants are found by the asymptotic behaviour of the solution:

$$\lim_{r \to \infty} r^{D-3} (A^2 - 1) = -2M,$$
  
$$\lim_{r \to \infty} r^{D-2} F_{tr} = Q.$$
 (3.19)

### Chapter 4

# Levi-Civita Bertotti-Robinson Metric

#### 4.1 Einstein's equation

An example is the Levi-Civita Bertotti-Robinson (LCBR) metric in Ddimensions

$$g_{ij} = \frac{q^2}{r^2} \left( -t_i t_j + c_0^2 k_i k_j + r^2 h_{ij} \right)$$
(4.1)

where  $h_{ij}$  is the metric on  $S_{D-2}$ ,  $t_i = \delta_i^t$ ,  $k_i = \delta_i^r$ , q and  $c_0$  are constants.

This is the previous metric with the choice  $A = \frac{q}{r}$ ,  $B = \frac{qc_0}{r}$ , C = q, then

$$F_{ij} = \frac{c_0}{r^2} (t_i k_j - t_j k_i), \qquad (4.2)$$

$$\widetilde{M_{ij}} = F_{mj}F_i^m - \frac{1}{4}F^2 g_{ij}, \qquad (4.3)$$

 $e_1 = e_3 = 0 \Rightarrow S_{ij} = e_0 \widetilde{M_{ij}}, e_0 = q^2,$ 

$$R_{ijkl} = q^2 [g_{jl}\widetilde{M_{ik}} - g_{jk}\widetilde{M_{il}} + g_{ik}\widetilde{M_{jl}} - g_{il}\widetilde{M_{kj}}] + \frac{q^2(1-c_0^2)}{c_0^2}F_{ij}F_{ml}, \quad (4.4)$$

$$R_{ij} = q^2 [(D-3) + \frac{1}{c_0^2}] \widetilde{M_{ij}} + \frac{1}{2q^2} [(D-3) - \frac{1}{c_0^2}] g_{ij}, \qquad (4.5)$$

$$G_{ij} = q^2 [(D-3) + \frac{1}{c_0^2}] \widetilde{M}_{ij} + \frac{1}{2q^2} [\frac{1}{c_0^2} - (D-3)^2] g_{ij}.$$
 (4.6)

It is easily seen that

$$\nabla_l F_{ij} = 0 \quad \nabla_m R_{ijkl} = 0. \tag{4.7}$$

If we consider the Einstein's equations

$$G_{ij} = T_{ij} \tag{4.8}$$

where

$$T_{ij} = \widetilde{F_{mj}}\widetilde{F_i^m} - \frac{1}{4}\widetilde{F^2}g_{ij}$$
(4.9)

and  $\widetilde{F_{ij}}(E.M) = eF_{ij}$ , we obtain

$$q^{2}\left[(D-3) + \frac{1}{c_{0}^{2}}\right] = e^{2}, \qquad (4.10)$$
$$\frac{1}{2q^{2}}\left[\frac{1}{c_{0}^{2}} - (D-3)^{2}\right] = \lambda$$

where  $\lambda$  is the cosmological constant. To eliminate the cosmological constant, we let  $\frac{1}{c_0} = |D-3|$ . It is interesting that spacetime is conformally flat only for D = 4.

We can obtain the same equations using the expression (2.15). This time we have to choose

$$T_{ij} = \widetilde{H}_{im\cdots n} \widetilde{H}_j^{m\cdots n} - \frac{1}{2(D-2)} \widetilde{H}^2 g_{ij}$$
(4.11)

and  $\widetilde{H}_{ij\cdots k} = eH_{ij\cdots k}$ . When the cosmological constant is set equal to zero, we obtain

$$c_0^2 = \frac{1}{(D-3)^2}.$$
(4.12)

Hence the higher dimensional (D > 4) Levi-Civita Bertotti-Robinson spacetimes without cosmological constants can not be conformally flat.

#### 4.2 The Solutions of Lovelock Theory

In the following  $\delta^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}$  is the generalized Kronecker delta defined by

$$\delta_{j_{1}j_{2}...j_{N}}^{i_{1}i_{2}...i_{N}} = det \begin{vmatrix} \delta_{j_{1}}^{i_{1}} & \delta_{j_{N}}^{i_{1}} \\ \vdots & \vdots \\ \delta_{j_{1}}^{i_{N}} & \cdots & \delta_{j_{N}}^{i_{N}} \end{vmatrix}$$
(4.13)

According to a theorem by Lovelock, the only symmetric tensor  $A^{ij} = A^{ij}(g_{rs}; g_{rs,t} : g_{rs,tu})$  for which  $A^{ij}_{,j} = 0$  is

$$A_{j}^{i} = \sum_{p=1}^{m-1} a_{p} \delta_{jj_{1}\dots j_{2p}}^{ih_{1}\dots h_{2p}} R_{h_{1}h_{2}}^{j_{1}j_{2}} R_{h_{3}h_{4}}^{j_{3}j_{4}} \cdots R_{h_{2p-1}h_{2p}}^{j_{2p-1}j_{2p}} + a\delta_{j}^{i}$$
(4.14)

where a and  $a_p$  are arbitrary constants. We have found the equation for n = 5, 6

$$a_1 \delta_{jm_1m_2}^{in_1n_2} R_{n_1n_2}^{m_1m_2} + a_2 \delta_{jm_1m_2m_3m_4}^{in_1n_2n_3n_4} R_{n_1n_2}^{m_1m_2} R_{n_3n_4}^{m_3m_4} + a \delta_j^i = 0$$
(4.15)

which is

$$G_{ij} + \alpha_0 [(R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd})g_{ij} + 4(2R^a_iR_{ja} + 2R_{jaib}R^{ab} - RR_{ij} - R^{abc}_iR_{jabc})] = \lambda g_{ij}$$

$$(4.16)$$

where  $\alpha_0 = -\frac{a_2}{a_1}$ 

We know that every symmetric tensor of rank two can be written as a linear combination of  $g_{ij}$ ,  $M_{ij}$  and  $k_i k_j$ . To calculate these we need the following:

For the LCBR metric

$$ds^{2} = \frac{q^{2}}{r^{2}} \left( -dt^{2} + c_{0}^{2} dr^{2} + r^{2} d\Omega^{2} \right)$$
(4.17)

where q and  $c_0$  are constants, we have

$$\eta_{0} = \frac{1}{\alpha c_{0}^{2}},$$

$$\eta_{1} = 0,$$

$$\eta_{2} = \frac{(D-3)(c_{0}^{2}-1)}{\alpha c_{0}^{2}},$$

$$\eta_{3} = 0.$$
(4.18)

If we insert these values in  $t_1$ ,  $t_2$ ,  $t_3$  we find

$$t_{1} = \frac{1}{2q^{2}c_{0}^{2}} - \frac{(D-3)^{2}}{2q^{2}} + \frac{\alpha_{0}}{q^{4}}(D-3)(D-4)^{2}(D-5) - \frac{2\alpha_{0}}{q^{4}c_{0}^{2}}(D-3)(D-4), \quad (4.19)$$

$$t_{2} = \frac{1}{\alpha} \left( D - 3 + \frac{1}{c_{0}^{2}} \right) - \frac{4\alpha_{0}(D-3)(D-4)}{\alpha q^{2}c_{0}^{2}} \left[ 1 + c_{0}^{2}(D-5) \right], \quad (4.20)$$

$$t_{3} = 0. \quad (4.21)$$

Field equations reduce to  $t_1 = \lambda$ ,  $t_2 = t_3 = 0$ . This gives relations between constants of the theory and the constants of the metric.

For D = 5 we obtain

$$q^{2} - 4q^{2}c_{0}^{2} - 8\alpha_{0} = 2q^{4}c_{0}^{2}\lambda, \qquad (4.22)$$
$$q^{2} + 2q^{2}c_{0}^{2} - 8\alpha_{0} = 0$$

which gives  $c_0^2 = 0$  or  $q^2 = -\frac{3}{\lambda}$ 

For D = 6 we obtain

$$q^{2}(1 - 9c_{0}^{2}) + 24\alpha_{0}(c_{0}^{2} - 1) = 2q^{4}c_{0}^{2}\lambda, \qquad (4.23)$$
  

$$3q^{2}c_{0}^{2} + q^{2} - 24\alpha_{0}(1 + c_{0}^{2}) = 0.$$

which gives  $q^2 = 4\alpha_0, c_0^2 = -\frac{5}{3}$  for  $\lambda = 0$  and

$$q^{2} = \frac{-3 + \sqrt{9 + 24\alpha_{0}\lambda}}{\lambda}, \qquad (4.24)$$

$$c_{0}^{2} = \frac{q^{2} - 24\alpha_{0}}{q^{2}(3 + \lambda)},$$

for  $\lambda \neq 0$ .

So there is no solution for 5 or 6 dimensional spacetimes without a cosmological constant.

#### 4.3 Solution of the most general theory

The Lagrangian of the most general theory will be a scalar containing the Riemann tensor, metric tensor, and their derivatives, contractions and multiple products of all orders. But, according to theorem 2, all second rank symmetric tensors constructed out of these will be expressible in terms of  $g_{ij}$ ,  $M_{ij}$ , and  $k_i k_j$ .

So, whatever the theory is, we'll obtain two equations for the LCBR metric, because the coefficient of the  $k_i k_j$  term will automatically vanish. This will give us two algebraic equations for two unknown constants in the metric, namely q and  $c_0$ . These equations may or may not have a solution according to theory. For example, in the preceding case, in D = 5, 6 we have no solution.

#### Chapter 5

# Solutions of the Low Energy Limit of the String Theory

The gravitational field equations obtained from the low energy limit of the string theory can be obtained from the following lagrangian

$$L = \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{4}{(D-2)\kappa^2} (\nabla\phi)^2 - \frac{1}{4} e^{-\alpha_e \phi} F^2 \right].$$
 (5.1)

The field equations are

$$G_{ij} = \frac{8}{(D-2)} \left[ \partial_i \phi \partial_j \phi - \frac{1}{2} (\nabla \phi)^2 g_{ij} \right] - \kappa^2 e^{-\alpha_e \phi} \left[ F_i^m F_{jm} - \frac{1}{4} F^2 g_{ij} \right], \quad (5.2)$$

$$\nabla_i (e^{-\alpha_e \phi} F^{ij}) = 0, \qquad (5.3)$$

$$\partial_i(\sqrt{-g}\,g^{ij}\,\partial_j\phi) + \frac{(D-2)\kappa^2\alpha_e\sqrt{-g}}{32}e^{-\alpha_e\phi}F^2 = 0,\qquad(5.4)$$

where  $F_{ij}$  is the Maxwell and  $\phi$  is the dilaton field. Here  $i, j = 1, 2, ..., D \ge 4$ . In static spherically symmetric spacetimes, gravitational field equations first lead to

$$\frac{\eta_0(D-4)(D-2)!}{2C^{2(D-2)}} + \frac{\eta_2(D-3)(D-3)!}{2C^{2(D-2)}} + \frac{\eta_1(D-2)}{B^2} + \frac{\eta_3(D-1)(D-2)}{2} - \frac{4{\phi'}^2}{(D-2)B^2} = 0,$$
(5.5)

$$\eta_1(D-2) - \frac{8{\phi'}^2}{(D-2)} = 0, \tag{5.6}$$

$$\eta_0(D-2)! + \eta_2(D-3)! - \frac{\kappa^2 Q^2 e^{\alpha_z \phi}}{A_{D-2}^2} = 0.$$
 (5.7)

Dilaton equation is:

$$\frac{8}{D-2} \left[ \frac{A}{B} C^{D-2} \phi' \right]' - \frac{\alpha_e A B \kappa^2 Q^2 e^{\alpha_e \phi}}{2 C^{D-2} A_{D-2}^2} = 0.$$
(5.8)

From (5.5) and (5.6) we obtain

$$\left[\frac{\left(A\,C^{d}\right)'\,C}{B}\right]' = d^{2}\,A\,B\,C^{d-1}$$
(5.9)

where d = D - 3. Using the freedom in choosing the r coordinate we can let

$$A B C^{d-1} = r^{d-1}, (5.10)$$

by using (5.9) and (5.10) we obtain

$$A^2 C^{2d} = r^{2d} - 2b_1 r^d + b_2 (5.11)$$

where  $b_1$  and  $b_2$  are integration constants.

A combination of the dilaton (5.3) and gravitational field equations (5.2) gives

$$dT^{2} - \frac{d-1}{r}T + T' + \frac{8}{(d+1)^{2}}\phi'^{2} = 0, \qquad (5.12)$$

where T is defined as

$$T = \frac{(r^d + c_1) r^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)} - \frac{16\phi'}{(d+1)^2 \alpha_e}.$$
 (5.13)

Defining now

$$\alpha_e \,\phi' = \frac{k_1 \,d\,r^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)} \,\psi(\rho),\tag{5.14}$$

the equation (5.12) becomes

$$\frac{r^{2d} - 2b_1 r^d + b_2}{d r^{d-1}} \frac{d\psi}{d\rho} \frac{d\rho}{dr} = (\psi + \mu)^2 + \nu^2.$$
(5.15)

The constants are given by

$$a = \frac{(d+1)^2 \alpha_e^2}{32 d},$$
  

$$\mu = -(c_1 + b_1),$$
  

$$\nu^2 = a\mu^2 - \Delta(a+1).$$
(5.16)

Now, if we solve the auxiliary equation

$$\frac{r^{2d} - 2b_1 r^d + b_2}{d r^{d-1}} \frac{d\rho}{dr} = \rho \tag{5.17}$$

and insert  $\rho$  above, we obtain

$$\rho \frac{d\psi}{d\rho} = (\psi + \mu)^2 + \nu^2.$$
 (5.18)

Note that  $\rho$  depends on the sign of  $\Delta = b_1^2 - b_2$ .  $\phi$  can be found from  $\psi$  by

$$\phi = \frac{k_1}{\alpha_e} \int \frac{\psi(\rho)}{\rho} d\rho + \phi_0, \qquad (5.19)$$

where  $k_1 = \frac{2 a}{a+1}$ . The metric function C is connected to  $\phi$  as

$$\frac{C'}{C} = \frac{(r^d + c_1)r^{d-1}}{(r^{2d} - 2b_1r^d + b_2)} - \frac{16\phi'}{(d+1)^2\alpha_e}$$
(5.20)

this gives us

$$\ln\left(\frac{C}{c_0}\right)^d = \left(\int \frac{u+b_1+c_1}{u^2-\Delta} du\right) - \frac{\alpha_e}{2a}\phi$$
(5.21)

where  $u = r^d - b_1$ .

Metric functions A and B can be found from C through the equations (5.10) and (5.11).

We have three different cases according to the sign of  $\Delta$ ,

Case 1 
$$\Delta > 0$$
  $\ln \rho(r) = \frac{1}{r_1^d - r_2^d} \ln \left( \frac{r^d - r_1^d}{r^d - r_2^d} \right),$   
Case 2  $\Delta = 0$   $\ln \rho(r) = -\frac{1}{r^d - r_3^d},$   
Case 3  $\Delta < 0$   $\ln \rho(r) = \frac{1}{\sqrt{-\Delta}} \left[ \arctan \left( \frac{r^d - b_1}{\sqrt{-\Delta}} \right) - \frac{\pi}{2} \right]$ 
(5.22)

where  $r_1^d = b_1 + \sqrt{\Delta}, \ r_2^d = b_1 - \sqrt{\Delta}, \ r_3^d = b_1.$ 

The integration constants  $c_0$  and  $\phi_0$  are determined through the asymptotic behaviour of the functions C(r) and  $\phi(r)$ .

$$\lim_{r \to \infty} \phi = \phi_0 = 0, \tag{5.23}$$

$$\lim_{r \to \infty} \frac{C(r)}{r} = c_0 = 1.$$
 (5.24)

To determine the remaining integration constants, we use the asymptotic behaviour of the metric, scalar field and the tensor field  $F_{ij}$  as follows:

$$\lim_{r \to \infty} r^{d} (A^{2} - 1) = -\frac{2\kappa^{2} M}{A_{d+1} (d+1)},$$
$$\lim_{r \to \infty} r^{d+1} \phi' = -\frac{\kappa \sqrt{d+1}}{2A_{d+1}} \Sigma,$$
(5.25)
$$\lim_{r \to \infty} r^{d+1} F_{tr} = \frac{Q}{A_{d+1}}.$$

CASE 1

$$\Delta > 0 \tag{5.26}$$

Which means there are two roots to the equation (5.11). According to the sign of  $\nu^2$  we have three distinct solutions

Type 1

$$\nu^{2} < 0 \qquad \lambda^{2} = -\nu^{2},$$
  

$$\psi = \frac{\lambda - \mu + c_{2}(\lambda + \mu)\rho^{2\lambda}}{1 - c_{2}\rho^{2\lambda}}.$$
(5.27)

The metric functions become

$$A^{2} = \frac{(r^{d} - r_{1}^{d})(r^{d} - r_{2}^{d})}{C^{2d}}, \qquad B^{2} = \frac{r^{2d-2} C^{2}}{(r^{d} - r_{1}^{d})(r^{d} - r_{2}^{d})},$$
$$C^{d} = (r^{d} - r_{2}^{d}) \left(\frac{1 - c_{2} \rho^{2\lambda}}{1 - c_{2}}\right)^{k_{2}} \rho^{k_{3}}. \qquad (5.28)$$

Dilaton field is given as

$$e^{\alpha_{e}\phi} = \left[\frac{(1-c_{2})\,\rho^{\lambda-\mu}}{1-c_{2}\,\rho^{2\lambda}}\right]^{k_{1}}$$
(5.29)

The constants are given by

$$k_2 = \frac{1}{(a+1)},$$
  $k_3 = \frac{(r_1^d - r_2^d)}{2} - \frac{(\nu + a\mu)}{(a+1)}.$  (5.30)

Undetermined integration constants are  $r_1$ ,  $r_2$ ,  $c_1$  and  $c_2$ . From the boundary condition (5.25) we find that

$$2M = \left[\frac{1+c_2}{1-c_2}\lambda + a\,\mu\right]e_1,$$
  

$$\Sigma = \left[\frac{1+c_2}{1-c_2}\lambda - \mu\right]e_2,$$

$$Q = \lambda \frac{\sqrt{c_2}}{1-c_2}e_3.$$
(5.31)

The constants  $e_i$  are given by

$$e_{1} = \frac{2A_{d+1}(d+1)}{(a+1)\kappa^{2}},$$

$$e_{2} = \frac{A_{d+1}\alpha_{e}(d+1)^{\frac{3}{2}}}{8(a+1)\kappa},$$

$$e_{3} = \frac{2A_{d+1}}{\kappa}\sqrt{\frac{(d+1)d}{(a+1)}}.$$
(5.32)

Note that, we have four integration constants  $(c_1, c_2, r_1^d \text{ and } r_2^d)$  but there exists only three equations to determine them. Also note that  $c_1$  does not appear in the solution directly, so we have a freedom in  $c_1$ .  $(c_2 \neq 1)$  In order to complete the solution, we need to determine the integration constants in terms of the physical parameters M,  $\Sigma$  and Q. Let us define some auxiliary variables to solve the set of algebraic equations (5.31)

$$T_{1} = \frac{1}{a+1} \left(\frac{a\Sigma}{e_{2}} + \frac{2M}{e_{1}}\right), \qquad T_{2} = \frac{1}{a+1} \left(\frac{2M}{e_{1}} - \frac{\Sigma}{e_{2}}\right),$$
  
$$T_{3} = \sqrt{1 - \frac{4Q^{2}}{e_{3}^{2}T_{1}^{2}}}.$$
 (5.33)

Then the integration constants are

$$\lambda = \pm T_3 T_1,$$
  

$$\mu = T_2,$$
  

$$c_2 = -1 + \frac{2}{1 \pm T_3}.$$
(5.34)

and

$$\Delta = \frac{a\mu^2 + \lambda^2}{a+1} \tag{5.35}$$

The reality of  $T_3$  imposes

$$M + g \Sigma \ge s |Q| \tag{5.36}$$

where g and s are given by

$$g = \frac{\alpha_e (d+1)^{3/2}}{4d \kappa} , \quad s = \frac{1}{\kappa} \sqrt{\frac{(d+1)(a+1)}{d}}$$
(5.37)

Such an inequality has been found by Gibbons and Wells for D=4.

When  $\Delta > 0$  we have two roots. In general these roots are the singular points of the space-time. If the integration constants satisfy some additional constraints one of these roots becomes regular. In this case we have a black hole solution carrying mass M, electric charge Q and scalar charge  $\Sigma$ . An invariant of the space-time is the scalar curvature is given by

$$R = A_1 \frac{\rho^{z_1} \left[ -\mu + \lambda + \frac{2c_2 \lambda \rho^{2\lambda}}{1 - c_2 \rho^{2\lambda}} \right]^2}{\left[ (r^d - r_1^d) (r^d - r_2^d) \right]^{1 + \frac{1}{d}} (1 - c_2 \rho^{2\lambda})^{\frac{2k_2}{d}}} + A_2 \frac{\rho^{z_1 + 2\lambda} \left( \frac{1 - c_2}{1 - c_2 \rho^{2\lambda}} \right)^{\frac{2k_2}{d} + 2}}{\left[ (r^d - r_1^d) (r^d - r_2^d) \right]^{1 + \frac{1}{d}}}$$
(5.38)

where

$$z_{1} = \frac{2}{d(a+1)}(a\mu + \lambda),$$

$$A_{1} = \frac{8}{d+1}\frac{k_{1}^{2}d^{2}}{\alpha_{e}^{2}},$$

$$A_{2} = \frac{\kappa^{2}}{2}\frac{(D-4)}{(D-2)}.$$
(5.39)

As  $r \to r_1$  we have a singularity unless we choose  $\mu = \lambda$ . By this choice,  $\mu = \frac{r_1^d - r_2^d}{2}$  and hence  $\rho^{z_1} \sim (r^d - r_1^d)^{\frac{2}{d}}, \rho^{2\lambda} \sim (r^d - r_1^d)^2$  around the horizon, so  $R \to 0$ .

If we insert these values for  $\mu$  and  $\lambda$  in the solution, we obtain

$$C^{d} = (r^{d} - r_{2}^{d}) \left( 1 - \frac{c_{2}}{1 - c_{2}} \frac{(r_{2}^{d} - r_{1}^{d})}{r^{d} - r_{2}^{d}} \right)^{k_{2}}$$
(5.40)

At this point, the choice of  $r_2 = 0$  gives Gibbons-Maeda solution, whereas the choice of  $c_2 = \frac{r_2^d}{r_1^d}$  gives the GHS solution. It is easy to show that Gibbons-Maeda metric is the same as the GHS metric with  $r > r_2$ .

Type 2

$$\nu^2 > 0$$
  
 $\psi = \nu \tan(c_2 + \nu \ln \rho) - \mu$ 
(5.41)

$$\int \frac{\psi(\rho)}{\rho} d\rho = -\ln \left[ \cos(c_2 + \nu \ln \rho) \right] - \mu \ln \rho + c_3$$
(5.42)

After similar steps as the previous type, we arrive at the solution

$$A^{2} = \frac{(r^{d} - r_{1}^{d})(r^{d} - r_{2}^{d})}{C^{2d}}, \qquad B^{2} = \frac{r^{2d-2}C^{2}}{(r^{d} - r_{1}^{d})(r^{d} - r_{2}^{d})}, \quad (5.43)$$

$$C^{d} = (r^{d} - r_{2}^{d}) \rho^{\frac{r_{1}^{d} - r_{2}^{d}}{2} - \frac{\mu k_{1}}{2}} \left[ \frac{\cos(c_{2} + \nu \ln \rho)}{\cos c_{2}} \right]^{k_{2}}$$
(5.44)

Scalar field is given as

$$e^{\alpha_e \phi} = \left[\frac{\cos c_2}{\rho^\mu \, \cos(c_2 + \nu \ln \rho)}\right]^{k_1} \tag{5.45}$$

Physical parameters are found using (5.25)

$$2M = (a\mu + \nu \tan c_2) e_1,$$
  

$$\Sigma = (-\mu + \nu \tan c_2) e_2,$$

$$Q = \frac{\nu}{\cos c_2} e_3,$$
(5.46)

$$\mu = T_2, 
\sin c_2 = \frac{e_3 T_1}{Q}, (5.47)$$

$$\nu \tan c_2 = T_1.$$

The condition  $|\sin c_2| < 1$  imposes

$$M + g \Sigma < s |Q| \tag{5.48}$$

where g and s are defined in (5.37). We also have to check the sign of  $\Delta$ .

$$(a+1)\Delta = \frac{1}{a+1}\left(\frac{a\Sigma^2}{e_2^2} + \frac{4M^2}{e_1^2}\right) - \frac{Q^2}{e_3^2}.$$
 (5.49)

Here the sign of  $\Delta$  puts a constraint on the physical variables.

Type 3

$$\nu^{2} = 0,$$
  

$$\psi = -\mu - \frac{1}{\ln \rho + c_{2}},$$
(5.50)

$$\int \frac{\psi(\rho)}{\rho} d\rho = -\ln(\ln\rho + c_2) - \mu \ln\rho + c_3, \qquad (5.51)$$

$$e^{\alpha_e \phi} = \left[\frac{c_2}{\rho^{\mu}(c_2 + \ln \rho)}\right]^{k_1},$$
 (5.52)

$$C^{d} = \left(\frac{c_{2} + \ln \rho}{c_{2}}\right)^{k_{2}} \left[ (r^{d} - r_{1}^{d})(r^{d} - r_{2}^{d}) \right]^{\frac{1}{2}} \rho^{\frac{-\mu k_{1}}{2}}.$$
 (5.53)

Physical parameters are found using (5.25)

$$2M = (a\mu - \frac{1}{c_2}) e_1,$$
  

$$\Sigma = (-\mu - \frac{1}{c_2}) e_2,$$

$$Q = \frac{e_3}{2c_2}.$$
(5.54)

The solution is

$$\begin{array}{rcl}
\mu &=& -T_2, \\
-\frac{1}{c_2} &=& T_1, \\
\end{array} (5.55)$$

(5.56)

which gives

CASE 2

$$\frac{2Q}{e_3} = T_1. (5.57)$$

This is the equality case of the inequality (5.36).

$$\Delta = 0 \tag{5.58}$$

Then there is one root to the equation (5.11). Denote it by  $r_1^d$ .

$$A C^{d} = (r^{d} - r_{1}^{d})$$
(5.59)

$$A^{2} = \frac{(r^{d} - r_{1}^{d})^{2}}{C^{2d}}, \qquad B^{2} = \frac{r^{2d-2}C^{2}}{(r^{d} - r_{1}^{d})^{2}}, \qquad (5.60)$$

$$C^{d} = (r^{d} - r_{1}^{d}) \rho^{-\frac{\mu k_{1}}{2}} \left[ \frac{\cos(c_{2} + \nu \ln \rho)}{\cos(c_{2})} \right]^{k_{2}}$$
(5.61)

Scalar field is given as

$$e^{\alpha_{e}\phi} = \left[\frac{\cos c_{2}}{\rho^{\mu} \cos(c_{2} + \nu \ln \rho)}\right]^{k_{1}}$$
(5.62)

CASE 3

$$\Delta < 0 \tag{5.63}$$

This case is similar to the previous one.

$$e^{\alpha_{e}\phi} = \left[\frac{\cos c_{2}}{\rho^{\mu} \cos(c_{2} + \nu \ln \rho)}\right]^{k_{1}},$$
 (5.64)

$$A^{2} = \frac{(r^{2d} - 2b_{1}r^{d} + b_{2})}{C^{2d}}, \qquad B^{2} = \frac{r^{2d-2}C^{2}}{(r^{2d} - 2b_{1}r^{d} + b_{2})}, \quad (5.65)$$

$$C^{d} = (r^{2d} - 2b_1r^{d} + b_2)^{\frac{1}{2}} \rho^{-\frac{\mu k_1}{2}} \left[\frac{\cos(c_2 + \nu \ln \rho)}{\cos(c_2)}\right]^{k_2}$$
(5.66)

In cases 2 and 3, the relation of physical parameters to integration constants are exactly the same as case 1 type 2, the only difference being the sign of  $\Delta$ , hence equation(5.49).

#### Chapter 6

## Conclusion

In this study, we considered D-dimensional spherically symmetric space times and obtained the Riemann tensor in compact form. Using this expression, we established certain theorems concerning any spherically symmetric theory of gravitation. As a special case, we considered the Lovelock theory in 5 and 6 dimensions with the LCBR metric, which simplifies most of the expressions. Then we considered the low energy limit of the string theory and obtained black hole solutions carrying mass, electrostatic charge, and dilaton charge. We also generalized an inequality concerning physical parameters of the black hole to D-dimensions.

# Chapter 7

# Appendix

$$M_n^n = \frac{\alpha(D-4)}{2C^2},$$
  

$$M_{in}M_j^n = \frac{\alpha^2}{4C^4}g_{ij},$$
  

$$M_{in}k^n = -\frac{\alpha}{2C^2}k_i,$$
(7.1)

where

$$M_{ij} = \alpha (h_{ij} - \frac{1}{2C^2} g_{ij}), \quad \alpha = \frac{(D-3)!}{C^{2(D-3)}}.$$
(7.2)

Using these, we can compute several contractions of  $S_{ij}$ , which in turn will be used to compute the contractions of the Riemann tensor. Now we'll define new variables that will simplify the coefficients.

$$\xi_0 = \frac{\eta_0 \alpha}{C^2}, \ \xi_1 = \frac{\eta_1}{B^2}, \ \xi_2 = \frac{\eta_2 \alpha}{C^2}, \ \xi_3 = \eta_3$$
(7.3)

$$\begin{split} S_{ij} &= \frac{C^2}{\alpha} \xi_0 M_{ij} + B^2 \xi_1 k_i k_j + \frac{\xi_3}{2} g_{ij}, \\ S_n^n &= \xi_0 \frac{(D-4)}{2} + \xi_1 + \frac{1}{2} \xi_3 D, \\ S_{in} S_j^n &= \frac{1}{4} (\xi_0^2 + \xi_3^2) g_{ij} + \xi_0 \xi_3 \frac{C^2}{\alpha} M_{ij} + (\xi_1^2 + \xi_1 \xi_3 - \xi_0 \xi_1) B^2 k_i k_j, \\ S_{in} k^n &= (-\frac{1}{2} \xi_0 + \xi_1 + \frac{1}{2} \xi_3) k_i, \end{split}$$

$$S_{in}M_{j}^{n} = \frac{\alpha}{C^{2}}(\frac{1}{4}\xi_{0}g_{ij} + \frac{C^{2}}{2\alpha}\xi_{3}M_{ij} - \frac{1}{2}\xi_{1}B^{2}k_{i}k_{j}),$$

$$S_{i}^{n}h_{jn} = \frac{1}{2}(\xi_{0} + \xi_{3})h_{ij},$$

$$S^{mn}h_{jn}h_{mi} = (\xi_{0} + \xi_{3})\frac{h_{ij}}{2C^{2}}.$$
(7.4)

Now we can compute the second rank tensors in terms of  $g_{ij},\ M_{ij}$  and  $k_i k_j$ 

$$R_{i}^{a}R_{ja} = \beta_{1}g_{ij} + \beta_{2}M_{ij} + \beta_{3}k_{i}k_{j},$$

$$R_{jaib}R^{ab} = \beta_{4}g_{ij} + \beta_{5}M_{ij} + \beta_{6}k_{i}k_{j},$$

$$R_{i}^{abc}R_{jabc} = \beta_{7}g_{ij} + \beta_{8}M_{ij} + \beta_{9}k_{i}k_{j},$$

$$R_{ij} = f_{1}g_{ij} + f_{2}M_{ij} + f_{3}k_{i}k_{j},$$
(7.5)

where

$$\beta_{1} = \frac{[(D-3)^{2}+1]}{2}\xi_{0}^{2} + \xi_{1}^{2} + (D-3)\xi_{0}\xi_{2} + \xi_{1}\xi_{2} + \frac{1}{2}\xi_{2}^{2} + (D-1)(D-4)\xi_{0}\xi_{3} + (D-4)\xi_{0}\xi_{3} + (D-4)\xi_{0}\xi_{3} + (D-1)\xi_{2}\xi_{3} + (D-1)^{2}\xi_{3}^{2},$$

$$\beta_2 = \frac{C^2}{\alpha} [(D-2)(D-4)\xi_0^2 + 2(D-2)\xi_0\xi_1 + 2(D-3)\xi_0\xi_2 + 2\xi_1\xi_2 + \xi_2^2 + 2(D-1)(D-2)\xi_0\xi_3 + 2(D-1)\xi_2\xi_3,$$

$$\beta_3 = B^2 [-2(D-2)\xi_0\xi_1 + D(D-2)\xi_1^2 + 2(D-1)(D-2)\xi_1\xi_3],$$

$$\beta_{4} = \frac{[(D-3)^{2}+1]}{2}\xi_{0}^{2} - 2\xi_{0}\xi_{1} + (D-1)\xi_{1}^{2} + (D-3)\xi_{0}\xi_{2} + \frac{1}{2}\xi_{1}\xi_{2} + \frac{1}{2}\xi_{2}^{2} + (\frac{3}{2}D-2)(D-4)\xi_{0}\xi_{3} + (3D-4)\xi_{1}\xi_{3} + \frac{3}{2}(D-2)\xi_{2}\xi_{3} + (D-1)^{2}\xi_{3}^{2},$$

$$\beta_5 = \frac{C^2}{\alpha} [(D-2)(D-4)\xi_0^2 + 2(D-2)\xi_0\xi_1 + 2(D-3)\xi_0\xi_2 + \xi_1\xi_2 + \xi_2^2 + (D-2)^2\xi_0\xi_3 + (D-2)\xi_2\xi_3],$$

$$\beta_6 = (D-2)B^2[(D-2)\xi_0\xi_1 + \xi_1\xi_2 + (D-2)\xi_1\xi_3],$$

$$\beta_{7} = (D-2)\xi_{0}^{2} - 2\xi_{0}\xi_{1} + 2\xi_{1}^{2} + 2\xi_{0}\xi_{2} + \frac{1}{(D-3)}\xi_{2}^{2} + 2(D-4)\xi_{0}\xi_{3} + 4\xi_{1}\xi_{3} + 2\xi_{2}\xi_{3} + 2(D-1)\xi_{3}^{2},$$
  
$$\beta_{8} = \frac{C^{2}}{\alpha} \left[ 2(D-4)\xi_{0}^{2} + 4\xi_{0}\xi_{1} + 4\xi_{0}\xi_{2} + \frac{2\xi_{2}^{2}}{(D-3)} + 4(D-2)\xi_{0}\xi_{3} + 4\xi_{2}\xi_{3} \right],$$
  
$$\beta_{9} = 2(D-2)B^{2}\xi_{1}(\xi_{1} + 2\xi_{3})$$
(7.6)

and

$$f_{1} = \frac{1}{2}(\xi_{0}(D-4) + \xi_{2}) + \xi_{1} + \xi_{3}(D-1),$$
  

$$f_{2} = \frac{C^{2}}{\alpha}(\xi_{0}(D-2) + \xi_{2}),$$
  

$$f_{3} = B^{2}(D-2)\xi_{1}.$$
(7.7)

If we insert these in (4.16), we obtain

$$\left[ f_1 - \frac{1}{2}R + \alpha_0 (R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} + 8\beta_1 + 8\beta_4 - 4\beta_7 - 4Rf_1) \right] g_{ij} + \left[ f_2 + \alpha_0 (8\beta_2 + 8\beta_5 - 4\beta_8 - 4Rf_2) \right] M_{ij} + \left[ f_3 + \alpha_0 (8\beta_3 + 8\beta_6 - 4\beta_9 - 4Rf_3) \right] k_i k_j = \lambda g_{ij}$$

$$(7.8)$$

We can rewrite it as

$$t_1 g_{ij} + t_2 M_{ij} + t_3 k_i k_j = 0 (7.9)$$

So, the field equations will reduce to  $t_1 = \lambda$ ,  $t_2 = t_3 = 0$ , where these coefficients are given as

$$t_{1} = -\frac{1}{2}(D-4)(D-2)\xi_{0} - (D-2)\xi_{1} - \frac{1}{2}(D-3)\xi_{2} - \frac{1}{2}(D-1)(D-2)\xi_{3} +\alpha_{0}(D-3)(D-4)[(D-3)(D-6)\xi_{0}^{2} + 4(D-3)\xi_{0}\xi_{1} + 2(D-6)\xi_{0}\xi_{2} +4\xi_{1}\xi_{2} + \frac{(D-4)(D-5)}{(D-3)^{2}}\xi_{2}^{2} + 2(D-4)(D-2)\xi_{0}\xi_{3} + 4(D-2)\xi_{1}\xi_{3} +2(D-3)\xi_{2}\xi_{3} + (D-1)(D-2)\xi_{3}^{2}],$$
(7.10)

$$t_{2} = \frac{C^{2}}{\alpha} [(D-2)\xi_{0} + \xi_{2}] - 4\frac{C^{2}}{\alpha}\alpha_{0}(D-3)(D-4) \left[ (D-4)\xi_{0}^{2} + \frac{(2D-9)}{(D-3)}\xi_{0}\xi_{2} + 2\xi_{0}\xi_{1} + \frac{2}{(D-3)}\xi_{1}\xi_{2} + \frac{(D-5)}{(D-3)^{2}}\xi_{2}^{2} + (D-2)\xi_{0}\xi_{3} + \xi_{2}\xi_{3} \right], \quad (7.11)$$

$$t_{3} = -4\alpha_{0}B^{2}(D-2)(D-4)\xi_{1}[\xi_{0}(D-3) + \xi_{2} + (D-3)\xi_{3}] + B^{2}(D-2)\xi_{1}.$$
(7.12)

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