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A Swan length theorem and a Fong dimension theorem for Mackey algebras

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Abstract

We first present some results about Mackey algebras of p-groups over fields of characteristic p, including their primitive idempotents and decompositions of their simple and principal indecomposable modules under restriction. We then apply these results together with a Green's indecomposability theorem for Mackey algebras to obtain Mackey algebra versions of some classical results of group algebras which are mostly related to restriction, induction and dimensions of modules. Our results about dimensions include Mackey algebra analogues of Dickson's theorem, Swan's theorem and Fong's dimension formula.

Keywords: Mackey functor; Dimension; Fong's dimension formula

1. Introduction

Swan's theorem on the composition length of a module and Fong's theorem on the dimension are two classic applications of Clifford theory of group algebras. The theory of Mackey functors is rather different from the theory of group algebras. For instance, restriction does not preserve dimension. Nevertheless, we shall be using a Clifford theory of Mackey functors to obtain analogues of Swan's theorem and Fong's dimension theorem.

Mackey functors were introduced by J.A. Green [6] and A. Dress [5] to axiomatize representation theory of finite groups, unifying several notions like representation rings, G-algebras and cohomology. Besides the definitions of Mackey functors given in [5,6], there is another one

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introduced by J. Thévenaz and P. Webb in [12] which identifies Mackey functors of a finite group G over a commutative unital ring R with modules of an R-free R-algebra $\mu_R(G)$, called the Mackey algebra of G over R, allowing one to adopt many module theoretical constructions. J. Thévenaz and P. Webb in [11] constructed the simple Mackey functors explicitly. We will use these comprehensive references [11,12] frequently.

As usual in Section 2, we have collected some crucial notions about Mackey functors. The structure of Mackey algebra $\mu_{\mathbb{F}}(G)$ of a p-group G over an algebraically closed field $\mathbb{F}$ of characteristic p is studied in Section 3. By using [12], we observe that $\sum_{H\leqslant G} t_H^H = 1$ is an orthogonal decomposition of the unity of $\mu_{\mathbb{F}}(G)$ as a sum of primitive idempotents, and the principal indecomposable $\mu_{\mathbb{F}}(G)$-module $\mu_{\mathbb{F}}(G)t_H^H$ is the projective cover of $S_{H,\mathbb{F}}^G$. We also observe that the center of $\mu_{\mathbb{F}}(G)$ is a local algebra. Then we provide explicit decompositions of simple and principal indecomposable $\mu_{\mathbb{F}}(G)$-modules under restriction.

In later sections we use a Mackey functor version of Green's indecomposability theorem to prove some results on lengths and dimensions of modules, including Mackey functor versions of some theorems of Dickson, Swan, Dade and Fong. For many of them we need some extra assumptions on group G to guarantee that $\mu_{\mathbb{F}}(G)$-modules satisfy analogues of the theorems mentioned above. These additional assumptions are needed because of some structural differences between Mackey algebras and group algebras. For a proper subgroup H of G, the subalgebra $\mu_{\mathbb{F}}(H)$ of $\mu_{\mathbb{F}}(G)$ is not unital, principal indecomposable $\mu_{\mathbb{F}}(G)$-modules may have vertices different than 1, and indecomposable $\mu_{\mathbb{F}}(G)$-modules may have vertices which are not p-groups. These are among the main differences which prevent us from finding exact analogues of the results mentioned above.

Let P be a p-subgroup of G and S be a Sylow p-subgroup of G containing P, and let V be a finitely generated P-projective $\mathbb{F}G$-module. Then it is known that $|S:P|$ divides $\dim_{\mathbb{F}} V$. In particular, if V is principal indecomposable, equivalently $P=1$, then the order of a Sylow p-subgroup of G divides $\dim_{\mathbb{F}} V$. This is a theorem of Dickson. In Section 4 we provide Mackey algebra versions of these results. Let T be a finitely generated indecomposable $\mu_{\mathbb{F}}(G)$-module. We first show that $\dim_{\mathbb{F}} \downarrow_P^G T$ is divisible by $\dim_{\mathbb{F}} \mu_{\mathbb{F}}(P)t_1^1$ if T is 1-projective, which is reminiscent of Dickson's theorem. Also it is shown that $|S:P|$ divides $\dim_{\mathbb{F}} \downarrow_P^G T$ if P is normal and T is P-projective.

Suppose that N is a normal subgroup of G with G/N is p-solvable. If U is a simple $\mathbb{F}G$-module then by a theorem of Swan the composition length of $\downarrow_N^G U$ divides $|G:N|$. We prove in Section 5 that a similar result holds for Mackey algebras. Namely, if M is a simple $\mu_{\mathbb{F}}(G)$-module with $\downarrow_N^G M \neq 0$ then the composition length of $\downarrow_N^G M$ divides $|G:N|$. In Section 6, we provide a Fong dimension theorem for Mackey algebras.

In Section 7 we prove a result for Mackey algebras of direct products of groups which has some applications for Mackey algebras of nilpotent groups. For instance, if G is nilpotent and H is a subgroup of G such that the Sylow p-subgroup of H is not normal in the Sylow p-subgroup of G, then the dimension of every simple $\mu_{\mathbb{F}}(G)$-module of the form $S_{H,V}^G$ is divisible by p.

Throughout the paper, G denotes a finite group, R denotes a commutative unital ring, $\mathbb{K}$ denotes a field, and $\mathbb{F}$ denotes an algebraically closed field of characteristic $p>0$. We write $H \leqslant G$ to indicate that H is a subgroup of G. Let $H \leqslant G \geqslant K$. The notation $H =_G K$ means that K is G-conjugate to H and $H \leqslant_G K$ means that H is G-conjugate to a subgroup of K. By the notation $gH \subseteq G$ we mean that g ranges over a complete set of representatives of left cosets of H in G, and by $HgK \subseteq G$ we mean that g ranges over a complete set of representatives of double cosets of (H,K) in G. Also we put $\overline{N}_G(H) = N_G(H)/H$, ${}^gH = gHg^{-1}$, $H^g = g^{-1}Hg$ for $g \in G$, and $|G:H|$ for the index of H in G. Furthermore, for a module U we denote by $J(U)$

and $Soc(U)$ the Jacobson radical and the socle of U, respectively. We will mainly work over a field $\mathbb{F}$ which is algebraically closed and of characteristic $p > 0$.

2. Preliminaries

In this section, we briefly summarize some crucial material on Mackey functors. For the proofs, see Thévenaz–Webb [11,12]. Recall that a Mackey functor for G over R is such that, for each subgroup H of G, there is an R-module $M(H)$; for each pair $H, K \leqslant G$ with $H \leqslant K$, there are R-module homomorphisms $r^K_H : M(K) \to M(H)$ called the restriction map and $t^K_H : M(H) \to M(K)$ called the transfer map or the trace map; for each $g \in G$, there is an R-module homomorphism $c^g_H : M(H) \to M({}^gH)$ called the conjugation map. The following axioms must be satisfied for any $g, h \in G$ and $H, K, L \leqslant G$ [1,6,11,12].

(M_1) If $H \leqslant K \leqslant L$, $r^L_H = r^K_H r^L_K$ and $t^L_H = t^L_K t^K_H$; moreover $r^H_H = t^H_H = \mathrm{id}_{M(H)}$.
(M_2) $c^{gh}_K = c^g_{{}^hK} c^h_K$.
(M_3) If $h \in H$, $c^h_H : M(H) \to M(H)$ is the identity.
(M_4) If $H \leqslant K$, $c^g_H r^K_H = r^{{}^gK}_{{}^gH} c^g_K$ and $c^g_K t^K_H = t^{{}^gK}_{{}^gH} c^g_H$.
(M_5) (Mackey Axiom) If $H \leqslant L \geqslant K$, $r^L_H t^L_K = \sum_{HgK \subseteq L} t^H_{H\cap {}^gK} r^{{}^gK}_{H\cap {}^gK} c^g_K$.

Another possible definition of Mackey functors for G over R uses the Mackey algebra $\mu_R(G)$ [1,12]: $\mu_{\mathbb{Z}}(G)$ is the algebra generated by the elements r^K_H, t^K_H, and c^g_H, where H and K are subgroups of G such that $H \leqslant K$, and $g \in G$, with the relations (M_1)–(M_7).

(M_6) $\sum_{H\leqslant G} t^H_H = \sum_{H\leqslant G} r^H_H = 1_{\mu_{\mathbb{Z}}(G)}$.
(M_7) Any other product of r^K_H, t^K_H and c^g_H is zero.

A Mackey functor M for G, defined in the first sense, gives a left module $\widetilde{M}$ of the associative R-algebra $\mu_R(G) = R \otimes_{\mathbb{Z}} \mu_{\mathbb{Z}}(G)$ defined by $\widetilde{M} = \bigoplus_{H\leqslant G} M(H)$. Conversely, if $\widetilde{M}$ is a $\mu_R(G)$-module then $\widetilde{M}$ corresponds to a Mackey functor M in the first sense, defined by $M(H) = t^H_H \widetilde{M}$, the maps t^K_H, r^K_H, and c^g_H being defined as the corresponding elements of the $\mu_R(G)$. Moreover, homomorphisms and subfunctors of Mackey functors for G are $\mu_R(G)$-module homomorphisms and $\mu_R(G)$-submodules, and conversely.

Theorem 2.1. *(See [12].) Letting H and K run over all subgroups of G, letting g run over representatives of the double cosets $HgK \subseteq G$, and letting J runs over representatives of the conjugacy classes of subgroups of $H^g \cap K$, then $t^H_{{}^gJ} c^g_J r^K_J$ comprise, without repetition, a free R-basis of $\mu_R(G)$.*

Given a simple Mackey functor M for G over R, there is a unique, up to G-conjugacy, subgroup H of G, called a minimal subgroup of M, such that $M(H)$ is nonzero. Moreover, for such an H the $R\overline{N}_G(H)$-module $M(H)$ is simple, see [11].

Theorem 2.2. *(See [11].) Given a subgroup $H \leqslant G$ and a simple $R\overline{N}_G(H)$-module V, then there exists a simple Mackey functor $S^G_{H,V}$ for G, unique up to isomorphism, such that H is a minimal subgroup of $S^G_{H,V}$ and $S^G_{H,V}(H) \cong V$. Moreover, up to isomorphism, every simple*

Mackey functor for G has the form $S^G_{H,V}$ for some $H \leqslant G$ and simple $R\overline{N}_G(H)$-module V. Two simple Mackey functors $S^G_{H,V}$ and $S^G_{H',V'}$ are isomorphic if and only if, for some element $g \in G$, we have $H' = {}^gH$ and $V' \cong c^g_H(V)$.

We now recall the definitions of restriction, induction and conjugation for Mackey functors [1,9,11,12]. Let M and T be Mackey functors for G and H, respectively, where $H \leqslant G$, then the restricted Mackey functor $\downarrow^G_H M$ is the $\mu_R(H)$-module $1_{\mu_R(H)}M$ and the induced Mackey functor $\uparrow^G_H T$ is the $\mu_R(G)$-module $\mu_R(G)1_{\mu_R(H)} \otimes_{\mu_R(H)} T$, where $1_{\mu_R(H)}$ denotes the unity of $\mu_R(H)$. For $g \in G$, the conjugate Mackey functor $|^g_H T = {}^gT$ is the $\mu_R({}^gH)$-module T with the module structure given for any $x \in \mu_R({}^gH)$ and $t \in T$ by $x.t = (\gamma_{g^{-1}} x \gamma_g)t$ where γ_g is the sum of all c^g_X with X ranging over subgroups of H. Obviously, one has $|^g_L S^L_{H,V} \cong S^{{}^gL}_{{}^gH, c^g_H(V)}$. The subgroup $\{g \in N_G(H)\colon {}^gT \simeq T\}$ of $N_G(H)$ is called the inertia group of T in $N_G(H)$.

Given $H \leqslant G \geqslant K$ and a Mackey functor M for K over R, the following is the Mackey decomposition formula for Mackey algebras [12], which will be of great use,

$$\downarrow^L_H \uparrow^L_K M \cong \bigoplus_{HgK \subseteq L} \uparrow^H_{H\cap {}^gK} \downarrow^{{}^gK}_{H\cap {}^gK} |^g_K M.$$

We let B^G_R denote Burnside functor for G, see [12].

Theorem 2.3. *(See [12].)*

(i) $\mu_R(G)t^H_H \simeq \uparrow^G_H B^H_R$.
(ii) *$B^G_{\mathbb{F}}$ is indecomposable if and only if G is a p-group.*
(iii) *Let G be a p-group. Then $S^G_{H,\mathbb{F}}(K)$ is nonzero if and only if $H =_G K$.*

As a last result in this section, we record a Mackey algebra version of Green's indecomposability theorem which will be used frequently.

Theorem 2.4. *(See [13].) Let N be a normal subgroup of G. Let S be a finitely generated indecomposable Mackey functor for N over $\mathbb{F}$, and let L be the inertia group of S. Then, $\uparrow^G_N S$ is an indecomposable Mackey functor for G over $\mathbb{F}$ if and only if L/N is a p-group.*

3. Mackey functors of p-groups

In this section, using some results from [12], we find an orthogonal primitive idempotent decomposition of the unity of Mackey algebra of a p-group over $\mathbb{F}$, an algebraically closed field of characteristic p. As in [12] we let $P^G_{H,V}$ denote the projective cover of the simple Mackey functor $S^G_{H,V}$ for G.

Proposition 3.1. *Let G be a p-group. Then for any subgroup H of G,*

(i) *The idempotent $t^H_H \in \mu_{\mathbb{F}}(G)$ is primitive.*
(ii) $\mu_{\mathbb{F}}(G)t^H_H \simeq P^G_{H,\mathbb{F}} \simeq \uparrow^G_H B^H_{\mathbb{F}}$.

Proof. (i) The idempotent t_H^H is primitive if and only if $\mu_{\mathbb{F}}(G)t_H^H$ is an indecomposable $\mu_{\mathbb{F}}(G)$-module. Indeed, $\mu_{\mathbb{F}}(G)t_H^H \simeq \uparrow_H^G B_{\mathbb{F}}^H$ is indecomposable by 2.3 and 2.4.

(ii) As $\sum_{H\leqslant G} t_H^H = 1$ is an expression of 1 as a sum of primitive orthogonal idempotents, the projective cover $P_{H,\mathbb{F}}^G$ of the simple $\mu_{\mathbb{F}}(G)$-module $S_{H,\mathbb{F}}^G$ must be isomorphic to $\mu_{\mathbb{F}}(G)t_X^X$ for some subgroup X of G. Therefore the simple modules $\mu_{\mathbb{F}}(G)t_X^X/J(\mu_{\mathbb{F}}(G))t_X^X$ and $S_{H,\mathbb{F}}^G$ must be isomorphic, implying that t_X^X does not annihilate $S_{H,\mathbb{F}}^G$, that is $S_{H,\mathbb{F}}^G(X) \neq 0$. By 2.3(iii) $X =_G H$. $\square$

The previous result can be proved without using 2.4. Indeed, it follows by 2.3 and [12, 8.6], a result which express $\uparrow_H^G B_{\mathbb{K}}^H$ explicitly as a direct sum of principal indecomposable $\mu_{\mathbb{K}}(G)$-modules, for large enough fields $\mathbb{K}$.

We record an immediate consequence of 3.1.

Corollary 3.2. *Let G be a p-group. Then for any subgroups H and K of G,*

(i) $\mu_{\mathbb{F}}(G)t_H^H \simeq \mu_{\mathbb{F}}(G)t_K^K$ *if and only if* $H =_G K$.
(ii) *If* $H \leqslant K$ *then* $\uparrow_K^G P_{H,\mathbb{F}}^K \simeq P_{H,\mathbb{F}}^G$.

Proof. (i) Obvious by 3.1(ii).

(ii) Using 3.1(ii), $\uparrow_L^G P_{H,\mathbb{F}}^L \simeq \uparrow_L^G \uparrow_H^L B_{\mathbb{F}}^H \simeq \uparrow_H^G B_{\mathbb{F}}^H \simeq P_{H,\mathbb{F}}^G$. $\square$

The group algebra $\mathbb{F}G$ is local if G is a p-group. For Mackey algebras we have

Proposition 3.3.

(i) *If G is a p-group, then the center of $\mu_{\mathbb{F}}(G)$ is a local algebra, so* 1 *is a block idempotent of $\mu_{\mathbb{F}}(G)$.*
(ii) *If the center of $\mu_{\mathbb{K}}(G)$ is a local algebra then $\mathbb{K}$ is of characteristic $p > 0$ and G is a p-group.*

Proof. (i) For any subgroups H and K of G, the primitive idempotents t_H^H and t_K^K of $\mu_{\mathbb{F}}(G)$ lie in the same block because $t_H^H\mu_{\mathbb{F}}(G)t_K^K$ is nonzero (it contains $t_1^H r_1^K$). As $\sum_{H\leqslant G} t_H^H = 1$ is an orthogonal decomposition, 1 must be a block idempotent.

(ii) The Burnside algebra $B_{\mathbb{K}}(G)$ embeds into the center of $\mu_{\mathbb{K}}(G)$, see [12]. Hence $1 \in B_{\mathbb{K}}(G)$ is a primitive idempotent and the result follows by [4]. $\square$

For a p-group G, the radical $J(\mathbb{F}G)$ of $\mathbb{F}G$ is the kernel of the augmentation map. We see now that a similar result holds for Mackey algebras. Consider a map $\mu_R(G) \to R$ whose image at the basis element t_G^G is 1 and at the other basis elements $t_{{}^gJ}^H c_J^g r_J^K$ are 0, see 2.1 for the basis elements. It is a routine checking that the above map is an R-algebra epimorphism. We let ψ denote its restriction to the center $Z(\mu_R(G))$ of $\mu_R(G)$. If G is a p-group then as $Z(\mu_{\mathbb{F}}(G))$ is local it is clear that the radical of $Z(\mu_{\mathbb{F}}(G))$ is equal to the kernel of ψ, which is also equal to the set of all elements of $Z(\mu_{\mathbb{F}}(G))$ annihilating the simple $\mu_{\mathbb{F}}(G)$-module $S_{G,\mathbb{F}}^G$.

For future use we next state a Mackey algebra version of Nakayama's reciprocities for group algebras whose first three parts depend on the adjointness of restriction and induction functors and whose last part depend on the symmetricity of group algebras. For Mackey algebras, we have the same adjointness properties of restriction and induction functors, see [11]. So a slight modification of the proof of the result for group algebras implies

Proposition 3.4. *Let $\mathbb{K}$ be algebraically closed and H be a subgroup of G. Let S be a simple $\mu_{\mathbb{K}}(G)$-module and T a simple $\mu_{\mathbb{K}}(H)$-module, and let $P(S)$ and $P(T)$ denote their projective covers. Then,*

- (i) *The multiplicity of S as a simple constituent of $\uparrow_H^G T/J(\uparrow_H^G T)$ is equal to the multiplicity of T as a simple constituent of $Soc(\downarrow_H^G S)$.*
- (ii) *The multiplicity of $P(T)$ as a direct summand of $\downarrow_H^G P(S)$ is equal to the multiplicity of S as a simple constituent of $\uparrow_H^G T$.*
- (iii) *The multiplicity of $P(S)$ as a direct summand of $\uparrow_H^G P(T)$ is equal to the multiplicity of T as a simple constituent of $\downarrow_H^G S$.*
- (iv) *If $\mu_{\mathbb{K}}(H)$ is a symmetric algebra, then the multiplicity of T as a simple constituent of $\downarrow_H^G P(S)$ is equal to the multiplicity of S as a simple constituent of $\uparrow_H^G P(T)$.*

By Clifford theory for Mackey algebras the restriction of a simple $\mu_R(G)$-module to a subnormal subgroup is semisimple [13]. In the next result we give an explicit description of the restriction of simple Mackey functors of p-groups over characteristic p.

Corollary 3.5. *Suppose that G is a p-group. Then for any $K \leqslant G \geqslant L \geqslant H$,*

(i) $$\downarrow_L^G P_{K,\mathbb{F}}^G \simeq \bigoplus_{LgK\subseteq G} P_{L\cap^g K,\mathbb{F}}^L.$$

(ii) $$\downarrow_L^G S_{H,\mathbb{F}}^G \simeq \bigoplus_{LgN_G(H)\subseteq G:\ {}^gH\leqslant L} S_{{}^gH,\mathbb{F}}^L.$$

Proof. (i) By 3.1(ii) $P_{K,\mathbb{F}}^G \simeq \uparrow_K^G B_{\mathbb{F}}^K$, and using the Mackey decomposition formula we get

$$\downarrow_L^G P_{K,\mathbb{F}}^G \simeq \downarrow_L^G \uparrow_K^G B_{\mathbb{F}}^K \simeq \bigoplus_{LgK\subseteq G} \uparrow_{L\cap^g K}^L B_{\mathbb{F}}^{L\cap^g K}.$$

And by 3.1(ii) each $\uparrow_{L\cap^g K}^L B_{\mathbb{F}}^{L\cap^g K}$ is isomorphic to $P_{L\cap^g K,\mathbb{F}}^L$, implying the result.

(ii) For each $X \leqslant_L L$ let m_X be the multiplicity of $S_{X,\mathbb{F}}^L$ as a simple constituent of $\downarrow_L^G S_{H,\mathbb{F}}^G$. Then by 3.4(iii) m_X is the multiplicity of $P_{H,\mathbb{F}}^G$ as a direct summand of $\uparrow_L^G P_{X,\mathbb{F}}^L$. Moreover by 3.2(ii) we have $\uparrow_L^G P_{X,\mathbb{F}}^L \simeq P_{X,\mathbb{F}}^G$. Therefore $m_X = 1$ if $H =_G X$, and $m_X = 0$ otherwise. Since $\downarrow_L^G S_{H,\mathbb{F}}^G$ is semisimple we must have

$$\downarrow_L^G S_{H,\mathbb{F}}^G \simeq \bigoplus_{X\leqslant_L L:\ X=_G H} S_{X,\mathbb{F}}^L.$$

Finally it is clear that the index set $\{X \leqslant_L L:\ X =_G H\}$ is the desired index set. $\square$

4. Relative projectivity

The object of this section is to provide some applications of Green's indecomposability theorem for Mackey algebras 2.4 and the results of the previous section. Specifically, we obtain Mackey algebra versions of some classical results about dimensions of modules of group algebras, and we derive some results related to restriction and induction of Mackey functors.

A Mackey functor M for G over $\mathbb{K}$ is said to be H-projective for some subgroup H of G if M is a direct summand of $\uparrow_H^G \downarrow_H^G M$, equivalently M is a direct summand of $\uparrow_H^G T$ for some Mackey functor T for H. For an indecomposable Mackey functor M, up to conjugacy there is a unique minimal subgroup H of G, called the vertex of M, so that M is H-projective. Although these notions are very similar to the corresponding notions in group algebras, there are some crucial differences. For example, projective Mackey functors may not be 1-projective. Indeed, over any field the projective indecomposable Mackey functor $P_{H,V}^G$ has vertex H, see [9,12]. These differences are major obstacles to the obtaining Mackey algebra versions of the classical results about group algebras. That is why we usually need some extra assumptions for Mackey algebras to satisfy the similar results.

Let V be a principal indecomposable $\mathbb{F}G$-module. Then $\dim_{\mathbb{F}} V$ is divisible by the order of a Sylow p-subgroup of G. This is one of the earliest result in modular representation theory, known as Dickson's theorem, see [7, Corollary 7.16, p. 91]. We begin by obtaining a Mackey algebra version of Dickson's theorem.

Proposition 4.1. *Let P be a p-subgroup of G. Then, given any 1-projective indecomposable $\mu_{\mathbb{F}}(G)$-module M, there is a positive integer n such that $\downarrow_P^G M \simeq n\mu_{\mathbb{F}}(P)t_1^1$ as $\mu_{\mathbb{F}}(P)$-modules. In particular $\dim_{\mathbb{F}} \mu_{\mathbb{F}}(P)t_1^1$ divides $\dim_{\mathbb{F}} \downarrow_P^G M$.*

Proof. We may write $\downarrow_P^G M \simeq T_1 \oplus \cdots \oplus T_n$ as a direct sum of indecomposable $\mu_{\mathbb{F}}(P)$-modules T_i. Then each T_i is a direct summand of $\downarrow_P^G M$, and as M is 1-projective by applying the Mackey decomposition formula we see that each T_i is 1-projective. Therefore T_i is a direct summand of $\uparrow_1^P T$ for some indecomposable $\mu_{\mathbb{F}}(1)$-module T. As $\mu_{\mathbb{F}}(1) \simeq \mathbb{F}$, $T \simeq \mathbb{F}$ and T_i is a direct summand of $\uparrow_1^P T \simeq \mu_{\mathbb{F}}(P)t_1^1 \otimes_{\mathbb{F}} \mathbb{F} \simeq \mu_{\mathbb{F}}(P)t_1^1$. Moreover by 3.1 the $\mu_{\mathbb{F}}(P)$-module $\mu_{\mathbb{F}}(P)t_1^1$ is indecomposable, implying that each T_i is isomorphic to $\mu_{\mathbb{F}}(P)t_1^1$. $\square$

Let V be a finitely generated P-projective $\mathbb{F}G$-module where P is a p-subgroup of G. Then $|S : P|$ divides $\dim_{\mathbb{F}} V$ where S is a Sylow p-subgroup of G containing P, see [8, Theorem 7.5, p. 293]. We next give a Mackey algebra version of this result.

Proposition 4.2. *Suppose M is a finitely generated P-projective $\mu_{\mathbb{F}}(G)$-module where P is a normal p-subgroup of G. If S is a Sylow p-subgroup of G, then $|S : P|$ divides $\dim_{\mathbb{F}} \downarrow_P^G M$.*

Proof. Evidently $P \subseteq S$. As M is P-projective it is a direct summand of $\uparrow_P^G \downarrow_P^G M$. Using the Mackey decomposition formula we see that $\downarrow_S^G M$ is a direct summand of

$$\downarrow_S^G \uparrow_P^G \downarrow_P^G M \simeq |G : S| \uparrow_P^S \downarrow_P^G M.$$

We write $\downarrow_P^G M \simeq T_1 \oplus \cdots \oplus T_r$ as a direct sum of indecomposable $\mu_{\mathbb{F}}(P)$-modules T_i. By 2.4 each $\uparrow_P^S T_i$ is an indecomposable $\mu_{\mathbb{F}}(S)$-module. Therefore

$$\downarrow_S^G M \simeq n_1 \uparrow_P^S T_1 \oplus \cdots \oplus n_r \uparrow_P^S T_r$$

for some integers $n_i \geqslant 0$. Consequently,

$$\downarrow_P^G M \simeq \downarrow_P^S \downarrow_S^G M \simeq n_1 \downarrow_P^S \uparrow_P^S T_1 \oplus \cdots \oplus n_r \downarrow_P^S \uparrow_P^S T_r.$$

Finally, using the Mackey decomposition formula we see that

$$\downarrow_P^S \uparrow_P^S T_i \simeq \bigoplus_{gP \subseteq S} {}^g T_i$$

for each i. In particular, for any i, $|S : P|$ divides $\dim_{\mathbb{F}} \downarrow_P^S \uparrow_P^S T_i$, implying the result. $\square$

For group algebras the dimension of $\downarrow_H^G \uparrow_H^G W$ is divisible by $|G : H|$ which may not be the case for Mackey algebras unless H is normal. This is one of the reasons why we assumed the normality of P in the previous result. In fact, let G be a p-group having a nonnormal subgroup H of order p. Then for any $g \in G$ it is clear that $H \cap {}^g H = H$ if and only if $g \in N_G(H)$, and $H \cap {}^g H = 1$ otherwise. By using the Mackey decomposition formula we see that $\dim_{\mathbb{F}} \downarrow_H^G \uparrow_H^G S_{H,\mathbb{F}}^H = |N_G(H) : H|$ is not divisible by $|G : H|$. In particular, the normality of P cannot be removed from the statement of 4.2 so as to leave a correct statement.

Corollary 4.3. *Let G have a normal Sylow p-subgroup S, and let M be a finitely generated indecomposable $\mu_{\mathbb{F}}(G)$-module with $\dim_{\mathbb{F}} \downarrow_S^G M = 1$. Then S is a vertex of M.*

Proof. As S is a Sylow p-subgroup, $\mathbb{F}$ is of characteristic p and $\downarrow_S^G M \neq 0$ it follows that M is S-projective, see [12]. Let Q be a vertex of M and Q' be a vertex of $\downarrow_S^G M$. Then $Q \subseteq S$, and if $Q' \neq S$ then we choose a maximal subgroup P of S that contains Q'. Then $\downarrow_S^G M$ is a finitely generated P-projective $\mu_{\mathbb{F}}(S)$-module and P is normal in S. Therefore 4.2 implies that $|S : P| = p$ divides $\dim_{\mathbb{F}} \downarrow_P^S \downarrow_S^G M = \dim_{\mathbb{F}} \downarrow_P^G M$ which is impossible because $\downarrow_P^G M \subseteq \downarrow_S^G M$. Therefore S is a vertex of $\downarrow_S^G M$. As Q is a vertex of M it is a direct summand of $\uparrow_Q^G \downarrow_Q^G M$. This implies that $\downarrow_S^G M$ is a direct summand of $\downarrow_S^G \uparrow_Q^G \downarrow_Q^G M$ which is, by the Mackey decomposition formula, isomorphic to $\bigoplus_{gS \subseteq G} \uparrow_{{}^g Q}^S \downarrow_{{}^g Q}^G M$. Moreover as $\dim_{\mathbb{F}} \downarrow_S^G M = 1$, the module $\downarrow_S^G M$ and the modules $\downarrow_{{}^g Q}^G M$ must be indecomposable. Finally by 2.4 we see that $\downarrow_S^G M \simeq \uparrow_{{}^g Q}^S \downarrow_{{}^g Q}^G M$ for some $g \in G$, in particular $\downarrow_S^G M$ is ${}^g Q$-projective. As S is a vertex of $\downarrow_S^G M$ we conclude that $S = Q$. $\square$

Remark 4.4. Let G have a p'-subgroup H. Then, any finitely generated indecomposable H-projective $\mu_{\mathbb{F}}(G)$-module is projective.

Proof. Let M be such a module. Then M is a direct summand of $\uparrow_H^G T$ for some finitely generated indecomposable $\mu_{\mathbb{F}}(H)$-module T. As H is a p'-group, $\mu_{\mathbb{F}}(H)$ is semisimple [11], implying that T is projective. Lastly, as restriction and induction are exact functors which are two sided adjoints of each other, they send projectives to projectives [11,12]. Hence M must be projective. $\square$

5. A Swan length theorem

Suppose N is a normal subgroup of G and G/N is a p-solvable group. Let U be a simple $\mathbb{F}G$-module. Then, the composition length of $\downarrow^G_N U$ divides $|G:N|$. This is a theorem of Swan, see [10] or [7, Theorem 9.20, p. 143]. Our aim is to show that a similar result holds for Mackey algebras. We first need the following, see [13].

Theorem 5.1 *(Clifford Theorem for Mackey algebras). Let N be a normal subgroup of G and $S^G_{H,V}$ be a simple $\mu_{\mathbb{K}}(G)$-module with $H \leqslant N$. Then, there is a simple $\mu_{\mathbb{K}}(N)$-submodule $S^N_{H,W}$ of $\downarrow^G_N S^G_{H,V}$ such that*

$$\downarrow^G_N S^G_{H,V} \simeq d \bigoplus_{gL \subseteq G} |^g_N S^N_{H,W} \quad \textit{and} \quad \downarrow^{\overline{N}_G(H)}_{\overline{N}_N(H)} V \simeq d \bigoplus_{gT \subseteq N_G(H)} c^g_H(W)$$

for some positive integer d, called the ramification index of $S^G_{H,V}$ relative to N, where L and $\overline{T}$ are the respective inertia groups of $S^N_{H,W}$ and W in G and $\overline{N}_G(H)$. Furthermore, $L = NT$ and $N_L(H) = T$.

Theorem 5.2 *(Swan Length Theorem for Mackey algebras). Let N be a normal subgroup of G such that G/N is p-solvable, and let M be a simple $\mu_{\mathbb{F}}(G)$-module with $\downarrow^G_N M \neq 0$. Then the composition length of $\downarrow^G_N M$ divides $|G:N|$.*

Proof. Let $M = S^G_{H,V}$ where $H \leqslant N$. Then by 5.1

$$\downarrow^G_N S^G_{H,V} \simeq d \bigoplus_{gL \subseteq G} |^g_N S^N_{H,W} \quad \text{and} \quad \downarrow^{\overline{N}_G(H)}_{\overline{N}_N(H)} V \simeq d \bigoplus_{gT \subseteq N_G(H)} c^g_H(W)$$

where W is a simple $\overline{N}_N(H)$-module which is a direct summand of $\downarrow^{\overline{N}_G(H)}_{\overline{N}_N(H)} V$, and L and $\overline{T}$ are the respective inertia groups of $S^N_{H,W}$ and W. Moreover $L = NT$ and $N_L(H) = T$. Now

$$N_G(H)/N_N(H) = N_G(H)/\big(N_G(H) \cap N\big) \simeq \big(N_G(H)N\big)/N \leqslant G/N$$

implies that $\overline{N}_G(H)/\overline{N}_N(H)$ is p-solvable. Thus by [7, Theorem 9.20, p. 143] the composition length of $\downarrow^{\overline{N}_G(H)}_{\overline{N}_N(H)} V$, which is $d|N_G(H):T|$, divides $|N_G(H):N_N(H)|$. So there is a positive integer s such that $sd|N_G(H):T| = |N_G(H):N_N(H)|$ implying that

$$sd = \big|T : N_N(H)\big| = \big|T : N \cap N_L(H)\big| = |T : N \cap T| = |NT : N| = |L : N|.$$

Hence $sd|G:L| = |G:N|$, and so the composition length of $\downarrow^G_N M$, which is $d|G:L|$, divides $|G:N|$. $\square$

We record an immediate consequence of 5.1 and 5.2.

Corollary 5.3. *Let N be a normal subgroup of G such that G/N is p-solvable, and let M be a simple $\mu_{\mathbb{F}}(G)$-module with $\downarrow^G_N M \neq 0$. If S is a simple $\mu_{\mathbb{F}}(N)$-module which is a direct summand of $\downarrow^G_N M$, then* $\dim_{\mathbb{F}} \downarrow^G_N M$ *divides* $|G:N| \dim_{\mathbb{F}} S$.

Suppose A is a normal abelian subgroup of G so that G/A is p-solvable. Then $\dim_{\mathbb{F}} U$ divides $|G:A|$ for any simple $\mathbb{F}G$-module U, see [3,10]. As a consequence of the previous result we have the following similar result for Mackey algebras.

Corollary 5.4. *Let N be a normal abelian p-subgroup of G such that G/N is p-solvable, and let M be a simple $\mu_{\mathbb{F}}(G)$-module with $\downarrow^G_N M \neq 0$. Then* $\dim_{\mathbb{F}} \downarrow^G_N M$ *divides* $|G:N|$.

Proof. As N is abelian, 2.3(iii) implies that every simple $\mu_{\mathbb{F}}(N)$-module is one dimensional. Then the result follows. $\square$

The proof of the above result depend on the fact that for an abelian p-group G, all simple $\mu_{\mathbb{F}}(G)$-modules are one dimensional. It can be seen easily that all simple $\mu_{\mathbb{K}}(G)$-modules are one dimensional if and only if $\mathbb{K}$ is of characteristic $p > 0$ and G is a p-group such that all subgroups of G are normal.

6. A Fong dimension theorem

For any natural number n we let n_p and $n_{p'}$ denote its p- and p'-part, respectively. We denote by $P(\,)$ the projective cover of its argument.

If G is a p-solvable group and V is a simple $\mathbb{F}G$-module then $\dim_{\mathbb{F}} P(V) = |G|_p (\dim_{\mathbb{F}} V)_{p'}$. This is known as Fong's dimension theorem, see [7, Theorem 16.9, p. 230]. In this section we obtain a result which looks like Fong dimension theorem for Mackey algebras.

We now give some results related to ramification indices and restriction of principal indecomposable Mackey functors. The next two results will be the main ingredients of the proof of 6.3, a result which we suggest as a Fong dimension theorem for Mackey algebras.

Proposition 6.1. *Let N be a normal subgroup of G such that G/N is a p-group and M be a simple $\mu_{\mathbb{F}}(G)$-module with $\downarrow^G_N M \neq 0$. Then*

(i) *$P(M) \simeq \uparrow^G_N P(S)$ for any simple $\mu_{\mathbb{F}}(N)$-module S which is a direct summand of $\downarrow^G_N M$.*
(ii) *The ramification index of M relative to N is* 1.
(iii) *For any simple $\mu_{\mathbb{F}}(N)$-module S which is a direct summand of $\downarrow^G_N M$,*

$$\downarrow^G_N P(M) \simeq \bigoplus_{gN \subseteq G} |^g_N P(S) \simeq |L:N| \bigoplus_{gL \subseteq G} |^g_N P(S)$$

where L is the inertia group of S. In particular, $\downarrow^G_N P(M) \simeq |L:N| P(\downarrow^G_N M)$.

Proof. Take any simple $\mu_{\mathbb{F}}(N)$-module S which is a direct summand of $\downarrow^G_N M$. Then by 5.1 $\downarrow^G_N M \simeq d \bigoplus_{gL \subseteq G} |^g_N S$ where d is the ramification index. Using 3.4(iii) we see that d is the multiplicity of $P(M)$ as a direct summand of $\uparrow^G_N P(S)$, where by 2.4 $\uparrow^G_N P(S)$ is indecomposable.

Hence $d = 1$ and $P(M) \simeq \uparrow_N^G P(S)$. Finally, observing that L is also the inertia group of $P(S)$, the last part follows by the Mackey decomposition formula applied to the first part. $\square$

Under the assumptions of 6.1, if S is a simple $\mu_{\mathbb{F}}(N)$-module then $P(\uparrow_N^G S) \simeq \uparrow_N^G P(S)$. Indeed, as induction is an exact functor sending projectives to projectives it follows that $P(\uparrow_N^G S)$ is a direct summand of $\uparrow_N^G P(S)$ from which the isomorphism is concluded by 2.4. Moreover, any primitive idempotent of $\mu_{\mathbb{F}}(N)$ stays primitive in $\mu_{\mathbb{F}}(G)$, because given a primitive idempotent e of $\mu_{\mathbb{F}}(N)$ we have $\mu_{\mathbb{F}}(G)e \simeq \uparrow_N^G \mu_{\mathbb{F}}(N)e$ which is indecomposable by 2.4. More to the point, any finitely generated N-projective indecomposable $\mu_{\mathbb{F}}(G)$-module M is of the form $\uparrow_N^G T$ for some indecomposable $\mu_{\mathbb{F}}(N)$-module T.

Proposition 6.2. *Let N be a normal subgroup of G such that G/N is a p'-group, and M be a simple $\mu_{\mathbb{F}}(G)$-module with $\downarrow_N^G M \neq 0$. Then*

(i) *For any simple $\mu_{\mathbb{F}}(N)$-module S which is a direct summand of $\downarrow_N^G M$,*

$$\downarrow_N^G P(M) \simeq d \bigoplus_{gL \subseteq G} |_N^g P(S)$$

where d is the ramification index of M relative to N and L is the inertia group of S in G. In particular, $\downarrow_N^G P(M) \simeq P(\downarrow_N^G M)$.

(ii) *p does not divide d.*

Proof. (i) Let S be any simple $\mu_{\mathbb{F}}(N)$-module which is a direct summand of $\downarrow_N^G M$. It follows by 5.1 that $\downarrow_N^G M \simeq d \bigoplus_{gL \subseteq G} |_N^g S$ where d is the ramification index and L is the inertia group of S. Let $S_1, \ldots, S_r$ be a complete set of representatives of simple $\mu_{\mathbb{F}}(N)$-modules. By the functorial properties of restriction it sends projective modules to projective modules (if the resulting module is nonzero), see [11,12]. So we may write

$$\downarrow_N^G P(M) \simeq \bigoplus_{i=1}^{r} n_i P(S_i)$$

for some integers $n_i \geqslant 0$. Note that $P(S_i) \simeq P(S_j)$ if and only if $i = j$. Then by 3.4 n_i is the multiplicity of M as a simple constituent of $\uparrow_N^G S_i$. Therefore n_i is equal to the dimension of $\mathrm{Hom}_{\mu_{\mathbb{F}}(G)}(P(M), \uparrow_N^G S_i)$. Moreover, as G/N is a p'-group, [13, Corollary 3.8] implies that $\uparrow_N^G S_i$ is semisimple. Now using the adjointness of restriction and induction we have as $\mathbb{F}$-modules

$$\begin{aligned}\mathrm{Hom}_{\mu_{\mathbb{F}}(G)}\big(P(M), \uparrow_N^G S_i\big) &\simeq \mathrm{Hom}_{\mu_{\mathbb{F}}(G)}\big(M, \uparrow_N^G S_i\big) \simeq \mathrm{Hom}_{\mu_{\mathbb{F}}(N)}\big(\downarrow_N^G M, S_i\big) \\ &\simeq d \bigoplus_{gL \subseteq G} \mathrm{Hom}_{\mu_{\mathbb{F}}(N)}\big(|_N^g S, S_i\big).\end{aligned}$$

Hence $n_i = d$ if $S_i \simeq |_N^g S$ for some $g \in G$, and $n_i = 0$ otherwise, implying the result.

(ii) Let $M = S_{H,V}^G$ where $H \leqslant N$. Then by 5.1 we can choose S as $S = S_{H,W}^N$ where W is a simple $\mathbb{F}\overline{N}_N(H)$-submodule of V. By 5.1 we know that the ramification index d of M relative to N is the same as the ramification index of V relative to $\overline{N}_N(H)$. Also note that, being isomorphic to a subgroup of G/N, the group $\overline{N}_G(H)/\overline{N}_N(H)$ is a p'-group. So by the corresponding result in the context of group algebras [8, p. 389], p does not divide d. $\square$

Theorem 6.3 *(Fong Dimension Theorem for Mackey algebras). Let G be a p-solvable group, N be a normal subgroup of G, and let M be a simple $\mu_{\mathbb{F}}(G)$-module with $\downarrow_N^G M \neq 0$.*

(i) *If N is a p'-group then*

$$\dim_{\mathbb{F}} \downarrow_N^G P(M) = n \dim_{\mathbb{F}} \downarrow_N^G M$$

for some natural number n which is a power of p.

(ii) *If N is a p-group then*

$$\dim_{\mathbb{F}} \downarrow_N^G P(M) = |G/N|_p \big(\dim_{\mathbb{F}} \downarrow_N^G M\big)_{p'} \dim_{\mathbb{F}} P(S)$$

where S is a simple $\mu_{\mathbb{F}}(N)$-module which is a direct summand of $\downarrow_N^G M$. And if X is a minimal subgroup of M then $\dim_{\mathbb{F}} P(S) = \dim_{\mathbb{F}}(\mu_{\mathbb{F}}(N)t_X^X)$.

Proof. As G/N is p-solvable we may find a chain $N = N_r \subset N_{r-1} \subset \cdots \subset N_1 = G$ where each N_i is a normal subgroup of G, and each quotient N_i/N_{i+1} has order n_i which is a p'-number or a power of p (consider the $p'p$-series of G/N).

Let $S_1 = M$ and for $i = 2, \dots, r$ choose a simple $\mu_{\mathbb{F}}(N_i)$-module S_i which is a direct summand of $\downarrow_{N_i}^{N_{i-1}} S_{i-1}$, and let d_{i-1} be the ramification index of S_{i-1} relative to N_i, and let L_i be the inertia group of S_i in N_{i-1}. So in particular $N_i \subseteq L_i \subseteq N_{i-1}$.

Then, for $i = 2, \dots, r$ it follows by 6.1 and 6.2 that

$$\downarrow_{N_i}^{N_{i-1}} S_{i-1} \simeq d_{i-1} \bigoplus_{g_i L_i \subseteq N_{i-1}} |_{N_i}^{g_i} S_i \quad \text{and} \quad \downarrow_{N_i}^{N_{i-1}} P(S_{i-1}) \simeq k_{i-1} \bigoplus_{g_i L_i \subseteq N_{i-1}} |_{N_i}^{g_i} P(S_i)$$

where $k_{i-1} = |L_i : N_i| = d_{i-1}|L_i : N_i|$ if $n_{i-1} = |N_{i-1} : N_i|$ is a power of p; and $k_{i-1} = d_{i-1}$, which is a p'-number, if $n_{i-1} = |N_{i-1} : N_i|$ is a p'-number.

Then, as each N_i is normal in G, we have

$$\downarrow_N^G M = \downarrow_{N_r}^{N_{r-1}} \cdots \downarrow_{N_3}^{N_2} \downarrow_{N_2}^{N_1} S_1 \simeq (d_1 d_2 \cdots d_{r-1}) \bigoplus_{g_2 L_2 \subseteq N_1} \bigoplus_{g_3 L_3 \subseteq N_2} \cdots \bigoplus_{g_r L_r \subseteq N_{r-1}} |_{N_r}^{g_2 g_3 \cdots g_r} S_r,$$

$$\downarrow_N^G P(M) = \downarrow_{N_r}^{N_{r-1}} \cdots \downarrow_{N_3}^{N_2} \downarrow_{N_2}^{N_1} S_1 \simeq (k_1 k_2 \cdots k_{r-1}) \bigoplus_{g_2 L_2 \subseteq N_1} \bigoplus_{g_3 L_3 \subseteq N_2} \cdots \bigoplus_{g_r L_r \subseteq N_{r-1}} |_{N_r}^{g_2 g_3 \cdots g_r} P(S_r).$$

Therefore

$$\dim_{\mathbb{F}} \downarrow_N^G M = \left(\prod_{i=1}^{r-1} d_i\right)\left(\prod_{i=1}^{r-1} |N_i : L_{i+1}|\right) \dim_{\mathbb{F}} S_r,$$

$$\dim_{\mathbb{F}} \downarrow_N^G P(M) = \left(\prod_{i=1}^{r-1} k_i\right)\left(\prod_{i=1}^{r-1} |N_i : L_{i+1}|\right) \dim_{\mathbb{F}} P(S_r).$$

(i) Suppose N is a p'-group, then $\mu_{\mathbb{F}}(N)$ is semisimple [11]. So $P(S_r) = S_r$. Letting n be the number $\prod_{i=1}^{r-1}(k_i/d_i)$ we see that n is a power of p and $\dim_{\mathbb{F}} \downarrow_N^G P(M) = n \dim_{\mathbb{F}} \downarrow_N^G M$.

(ii) Suppose N is a p-group. Let X be a minimal subgroup of M. By 5.1 we may take S_r to be $S_{X,\mathbb{F}}^N$, whose dimension is equal to $|N : N_N(X)|$ by 2.3. Then 3.1 implies that $P(S_r) \simeq \mu_{\mathbb{F}}(N)t_X^X$. Let $\{1, 2, \ldots, r-1\} = I \uplus J$ where $i \in I$ if and only if $|N_i : N_{i+1}| = n_i$ is a power of p. So, $d_i = 1$ and $k_i = |L_{i+1} : N_{i+1}|$ for $i \in I$; and $k_i = d_i$ and p does not divide d_i for $i \in J$. Now note that

$$\prod_{i=1}^{r-1} |N_i : L_{i+1}| = \left(\prod_{i \in I} |N_i : L_{i+1}|\right)\left(\prod_{i \in J} |N_i : L_{i+1}|\right)$$

as product of its p- and p'-part, respectively. Note also that $\prod_{i=1}^{r-1} d_i$ is a p'-number and

$$\prod_{i=1}^{r-1} k_i = \left(\prod_{i=1}^{r-1} d_i\right)\left(\prod_{i \in I} |L_{i+1} : N_{i+1}|\right).$$

Therefore

$$\begin{aligned}
\left(\dim_{\mathbb{F}} \downarrow_N^G M\right)_{p'} &= \left(\prod_{i=1}^{r-1} d_i\right)\left(\prod_{i \in J} |N_i : L_{i+1}|\right), \\
\dim_{\mathbb{F}} \downarrow_N^G P(M) &= \left(\prod_{i=1}^{r-1} d_i\right)\left(\prod_{i \in I} |L_{i+1} : N_{i+1}|\right)\left(\prod_{i \in I} |N_i : L_{i+1}|\right) \\
&\quad \times \left(\prod_{i \in J} |N_i : L_{i+1}|\right) \dim_{\mathbb{F}}\left(\mu_{\mathbb{F}}(N)t_X^X\right).
\end{aligned}$$

As $(\prod_{i \in I} |L_{i+1} : N_{i+1}|)(\prod_{i \in I} |N_i : L_{i+1}|) = \prod_{i \in I} |N_i : N_{i+1}| = |G/N|_p$, the result follows. □

Let us explain how to derive, using the previous result, Fong's dimension theorem for group algebras. Given a simple $\mathbb{F}G$-module V, we use 6.3(ii) with $M = S_{1,V}^G$ and $N = 1$ to get $\dim_{\mathbb{F}} \downarrow_1^G P(M) = |G|_p (\dim_{\mathbb{F}} \downarrow_1^G M)_{p'} \dim_{\mathbb{F}}(\mu_{\mathbb{F}}(1)t_1^1)$, where, of course, $\dim_{\mathbb{F}}(\mu_{\mathbb{F}}(1)t_1^1) = 1$ and $(\dim_{\mathbb{F}} \downarrow_1^G M)_{p'} = (\dim_{\mathbb{F}} V)_{p'}$. Moreover $\dim_{\mathbb{F}} \downarrow_1^G P(M) = \dim_{\mathbb{F}} P_{1,V}^G(1) = \dim_{\mathbb{F}} P(V)$, because by [12, 12.6] we have $P_{1,V}^G(1) \simeq P(V)$. Hence $\dim_{\mathbb{F}} P(V) = |G|_p (\dim_{\mathbb{F}} V)_{p'}$.

7. Mackey functors of direct products

We give some results about Mackey functors for direct products of groups. If G_1 and G_2 are groups then $\mathbb{K}(G_1 \times G_2) \simeq \mathbb{K}G_1 \otimes_{\mathbb{K}} \mathbb{K}G_2$, and if we assume that G_1 and G_2 have coprime orders, then it follows by 2.1 that $\mu_{\mathbb{K}}(G_1 \times G_2) \simeq \mu_{\mathbb{K}}(G_1) \otimes_{\mathbb{K}} \mu_{\mathbb{K}}(G_2)$, because subgroups of $G_1 \times G_2$ are of the form $H_1 \times H_2$ for some subgroups H_i of G_i. We first recall some basic notions about tensor products of algebras, see [2, pp. 249–254]. Let A and B be finite dimensional $\mathbb{K}$-algebras and $\mathbb{K}$ be algebraically closed. If X and Y are A- and B-modules, respectively, then $X \otimes_{\mathbb{K}} Y$ becomes an $A \otimes_{\mathbb{K}} B$-module by means of the action $(a \otimes b)(x \otimes y) = ax \otimes by$. The simple modules of $A \otimes_{\mathbb{K}} B$ are, up to isomorphism, precisely the modules $X \otimes_{\mathbb{K}} Y$ where X and Y are simple modules of A and B, respectively. Moreover, given indecomposable A-modules

X_1, X_2 and indecomposable B-modules Y_1, Y_2 then $X_i \otimes_{\mathbb{K}} Y_i$ is an indecomposable $A \otimes_{\mathbb{K}} B$-module for $i = 1, 2$, and $X_1 \otimes_{\mathbb{K}} Y_1 \simeq X_2 \otimes_{\mathbb{K}} Y_2$ if and only if $X_1 \simeq X_2$ and $Y_1 \simeq Y_2$. Let X be an A-module and Y be a B-module. Identifying A and B with their images $A \otimes 1$ and $1 \otimes B$ in $A \otimes_{\mathbb{K}} B$ we see that $X \otimes_{\mathbb{K}} Y$ is isomorphic to $(\dim_{\mathbb{K}} Y)X$ and $(\dim_{\mathbb{K}} X)Y$ as A- and B-modules, respectively. In particular, if $X \otimes_{\mathbb{K}} Y$ is a projective $A \otimes_{\mathbb{K}} B$-module then X and Y are projective A- and B-modules.

Proposition 7.1. *Let G_1 and G_2 be groups with coprime orders and let H_2 be a subgroup of G_2. Suppose that $\mathbb{K}$ is algebraically closed. Then, any finitely generated $1 \times H_2$-projective indecomposable $\mu_{\mathbb{K}}(G_1 \times G_2)$-module is of the form $S \otimes_{\mathbb{K}} T$, as $\mu_{\mathbb{K}}(G_1) \otimes_{\mathbb{K}} \mu_{\mathbb{K}}(G_2)$-module, where S is a principal indecomposable $\mu_{\mathbb{K}}(G_1)$-module and T is a finitely generated H_2-projective indecomposable $\mu_{\mathbb{K}}(G_2)$-module. Conversely, given such modules S and T the $\mu_{\mathbb{K}}(G_1 \times G_2)$-module $S \otimes_{\mathbb{K}} T$ is finitely generated, $1 \times H_2$-projective and indecomposable.*

Proof. Let M be such a $\mu_{\mathbb{K}}(G_1 \times G_2)$-module. Then there is a finitely generated $1 \times H_2$-projective indecomposable $\mu_{\mathbb{K}}(1 \times G_2)$-module T such that M is a direct summand of $\uparrow_{1\times G_2}^{G_1\times G_2} T$, and there is a finitely generated indecomposable $\mu_{\mathbb{K}}(1 \times H_2)$-module U such that T is a direct summand of $\uparrow_{1\times H_2}^{1\times G_2} U$. As

$$\uparrow_{1\times H_2}^{1\times G_2} U \simeq \mu_{\mathbb{K}}(1 \times G_2) \otimes_{\mu_{\mathbb{K}}(1\times H_2)} U \simeq \mu_{\mathbb{K}}(G_2) \otimes_{\mu_{\mathbb{K}}(H_2)} U,$$

T may be regarded as a finitely generated H_2-projective indecomposable $\mu_{\mathbb{K}}(G_2)$-module. Writing $\mu_{\mathbb{K}}(G_1) \simeq S_1 \oplus \cdots \oplus S_n$ as a direct sum of principal indecomposable $\mu_{\mathbb{K}}(G_1)$-modules we note that

$$\begin{aligned}\uparrow_{1\times G_2}^{G_1\times G_2} T &\simeq \mu_{\mathbb{K}}(G_1 \times G_2) \otimes_{\mu_{\mathbb{K}}(1\times G_2)} T \simeq \big(\mu_{\mathbb{K}}(G_1) \otimes_{\mathbb{K}} \mu_{\mathbb{K}}(G_2)\big) \otimes_{\mu_{\mathbb{K}}(G_2)} T \\ &\simeq \mu_{\mathbb{K}}(G_1) \otimes_{\mathbb{K}} \big(\mu_{\mathbb{K}}(G_2) \otimes_{\mu_{\mathbb{K}}(G_2)} T\big) \simeq \mu_{\mathbb{K}}(G_1) \otimes_{\mathbb{K}} T \simeq \bigoplus_{i=1}^{n} S_i \otimes_{\mathbb{K}} T.\end{aligned}$$

Since each $S_i \otimes_{\mathbb{K}} T$ is indecomposable, M must be of the desired form.

Conversely, let S and T be given. As T is H_2-projective we may regard T as a $1 \times H_2$-projective $\mu_{\mathbb{K}}(1 \times G_2)$-module. Evidently, $S \otimes_{\mathbb{K}} T$ is a direct summand of $\mu_{\mathbb{K}}(G_1) \otimes_{\mathbb{K}} T$ which is equivalent, by what we have done above, to $\uparrow_{1\times G_2}^{G_1\times G_2} T$. Since T is $1 \times H_2$-projective the result follows. □

We next provide some applications of 7.1. For a prime number p and a group G, we denote by $O_p(G)$ and $O_{p'}(G)$ the respective largest normal p and normal p'-subgroups of G.

Corollary 7.2. *Let G be a nilpotent group and H be a p-subgroup of G. Then the dimension of the $\mu_{\mathbb{F}}(G)$-module $P_{H,V}^G$ is divisible by the dimension of $\mu_{\mathbb{F}}(O_p(G))t_H^H$.*

Proof. Applying 7.1 with $G_1 = O_{p'}(G)$ and $G_2 = O_p(G)$ we see that $P_{H,V}^G \simeq S \otimes_{\mathbb{F}} T$ for some projective indecomposable $\mu_{\mathbb{F}}(O_p(G))$-module T with vertex H. Then T is isomorphic to $\mu_{\mathbb{F}}(O_p(G))t_H^H$ by 3.1, implying the result. □

Let M be a $\mu_{\mathbb{K}}(G)$-module with vertex H. Then there is an indecomposable $\mu_{\mathbb{K}}(H)$-module U such that M is a direct summand of $\uparrow_H^G U$. Any such U is called an H-source of M.

Corollary 7.3. *Let G be a nilpotent group and H be a p-subgroup of G. If M is a finitely generated indecomposable $\mu_{\mathbb{F}}(G)$-module with vertex H and H-source U, then* $\dim_{\mathbb{F}} \uparrow_H^{O_p(G)} U$ *divides* $\dim_{\mathbb{F}} M$.

Proof. By the proof of 7.1 there is a finitely generated indecomposable $\mu_{\mathbb{F}}(O_p(G))$-module T with vertex H and H-source U such that $M \simeq S \otimes_{\mathbb{F}} T$ for some principal indecomposable $\mu_{\mathbb{F}}(O_{p'}(G))$-module S. In particular, $\dim_{\mathbb{F}} T$ divides $\dim_{\mathbb{F}} M$. As T is a direct summand of $\uparrow_H^{O_p(G)} U$ it follows by 2.4 that $T \simeq \uparrow_H^{O_p(G)} U$, finishing the proof. □

Finally we give a result on the simple $\mu_{\mathbb{F}}(G)$-modules for a nilpotent group G.

Corollary 7.4. *Let G be a nilpotent group. Then for a simple $\mu_{\mathbb{F}}(G)$-module $S_{H,V}^G$ we have* $\dim_{\mathbb{F}} S_{H,V}^G = |G : N_G(O_p(H))| \dim_{\mathbb{F}} S_{O_{p'}(H),V}^{O_{p'}(G)}$.

Proof. Considering $S_{H,V}^G$ as a simple $\mu_{\mathbb{F}}(O_{p'}(G)) \otimes_{\mathbb{F}} \mu_{\mathbb{F}}(O_p(G))$-module, there are simple modules $M_1 = S_{X,W}^{O_{p'}(G)}$ and $M_2 = S_{Y,\mathbb{F}}^{O_p(G)}$ for some respective subgroups X and Y of $O_{p'}(G)$ and $O_p(G)$ such that $S_{H,V}^G \simeq M_1 \otimes_{\mathbb{F}} M_2$. As H is isomorphic to $O_{p'}(H) \times O_p(H)$ we see that $M_1(O_{p'}(H))$ and $M_2(O_p(H))$ are nonzero. On the other hand, since $M_1(X)$ and $M_2(Y)$ are nonzero it follows that $S_{H,V}^G(XY)$ is nonzero. Hence we may take $X = O_{p'}(H)$ and $Y = O_p(H)$. Moreover as $S_{H,V}^G(H) = V$ we have $V \simeq M_1(O_{p'}(H)) \otimes_{\mathbb{F}} M_2(O_p(H)) \simeq W \otimes_{\mathbb{F}} \mathbb{F}$ as $\mathbb{F}O_{p'}(H) \otimes_{\mathbb{F}} \mathbb{F}O_p(H)$-modules. So $W \simeq V$. Finally by 2.3 $\dim_{\mathbb{F}} M_2 = |G : N_G(O_p(H))|$. □

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