THE GEOMETRY OF SHEAVES ON SITES

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By

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The Geometry of Sheaves on Sites
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

THE GEOMETRY OF SHEAVES ON SITES

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In this work, we study doing geometry on sheaves on sites. Categories of our sites consist of objects that are building blocks for a given geometry. Generalized spaces then will be sheaves on these sort of sites. Next we introduce the notion of varieties, and show the relationship between certain class of varieties known as diffeologies with the category of smooth manifolds. Along the way, the notion of schemes will be generalized as a variety on symmetric monoidal categories. And we show how a differential geometric construction on a site can be translated to a construction on generalized spaces.

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Keywords: Sheaves, Sites, Geometry.

ÖZET

SİTELERDEKİ ŞEAFLERİN GEOMETRİSİ

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Bu çalışmada, sitelerdeki şeafler üzerinde geometri yapmayı inceliyoruz. Sitelerimizin kategorileri belirli bir geometri için yapı taşları olan nesnelerden oluşur. Genelleştirilmiş uzaylar o zaman bu tür sitelerde sheafler olacak. Daha sonra varyete kavramını tanıtıyoruz ve diffeolojiler olarak bilinen belirli varyetelerin kategorisi ile pürüzsüz manifoldlar kategorisinin ilişkisini inceliyoruz. Yol boyunca, schemeler de simetrik monodyal kategoriler üzerinde varyeteler olarak genelleştirilecektir. Ayrıca, bazı iyi bilinen geometrik yapıların genelleştirilmiş uzaylar üzerine yapılara nasıl çevrilebileceğini göstereceğiz.

Anahtar sözcükler: Şeafler, Siteler, Geometri.

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Chapter 0

Introduction

The usual categories where we do geometry on, lack enough (co)limits. Intuitively speaking, they lack enough solutions spaces for equations, fiber products etc. The category Open_{C^∞} is an important example of such categories.

The remedy for the aforementioned problem is to use the Yoneda embedding functor, and fully faithful embed our category C into the category of presheaves on C, i.e. $\widehat{C}$. We know that the category $\widehat{C}$ has all limits and colimits.

Yet, another problem occurs. the embedding functor doesn't preserve finite colimits. An example of such situation is given in Chapter 6. This is exactly the reason why the notion of sheaf on site was invented, to deal with the colimit problem. As soon as the category C is equipped with a Grothendieck topology, we will be able to form nerves of the coverings and then, The Yoneda embedding of C into the category of sheaves on site $Sh(C, \tau)$, $C \hookrightarrow Sh(C, \tau)$, will preserve the colimits that is taken along the nerves. This makes the category $Sh(C, \tau)$ to be a very good environment for doing geometry.

In this work, we will take the first steps toward studying geometry on the category of sheaves on sites(a.k.a. topoi) in a systematic way. We have adopted and closely follow the approach taken in [Pau].

In the first chapter, We will introduce the notion of (pre)sheaf and some important notions related to it which will be used on later chapters.

Second chapter is the second step towards understanding the notion of sheaves on a site, namely the notion of sites. Sites will be discussed with some nice examples there.

In Chapter 3, we will mention basics of monoidal categories which specifically play an important role in generalization of schemes.

Chapter 4 is devoted to categorical approach to calculus which is based on the notion of tangent category, and was introduced by Daniel Quillen [Pau].

In Chapter 5, we will show how categorical calculus combined with Monoidal category, can be used in the differential geometric context.

And finally, in Chapter 6, we begin studying the notion of sheaves on sites in a geometric context. we will call an object of the category $Sh(C, \tau)$, a generalized space. Certain objects of this category which are called varieties will be introduced, and finally we will make a connection between diffeologies and the category of smooth manifolds.

Chapter 1

Sheaves

1.1 Definitions and Examples

Definition 1.1.1. A presheaf is a contravariant functor $F : C^{op} \longrightarrow M$ from an arbitrary category C to a category M.

Notation: In what follows $Open_X$ stands for the category of open subsets of topological space X, and inclusions i.e.

$$Hom_{Open_X}(U,V) = \begin{cases} U \hookrightarrow V & \text{if } U \subset V \\ \emptyset & \text{otherwise} \end{cases}$$

In this chapter, we will restrict ourselves to the following specific presheaves:

Definition 1.1.2. Suppose that X is a topological space and C is a category. A presheaf F of C on X is a contravariant functor $Open_X^{op} \longrightarrow C$.

The above definition means that F is an assignment such that

1. For every open set $U \subseteq X$, $F(U) \in Ob(C)$.

2. For every inclusion $i : U \hookrightarrow V$ in X, a morphism $res_U^V := F(i) : F(V) \longrightarrow F(U)$ is called a restriction map exists and satisfies two conditions:

   (a) $res_U^U = id_{F(U)}$ for every open set $U \subseteq V$.

   (b) For inclusions of open sets $U \hookrightarrow V \hookrightarrow W$ in X, the composition of restriction maps exists i.e. $res_U^W = res_U^V \; o \; res_V^W$.

Elements of $F(U)$ are called sections over U. When $U = X$, the elements of $F(X)$ are called global sections. As a convention, for opens $U \subseteq V$ in X, and
$s \in F(V), \; s_{|U} := res_U^V(s)$.

Example 1.1.3. 1. Presheaf of functions: Let E be a set. For every open subset $U \subset X$ of topological space X, we define $Map_E(U)$ to be the set of all functions from U to E. Then $Map_E(-) : Open_X^{op} \to E$ with restriction maps $res_U^V : Map_E(U) \to Map_E(V)$ being the usual restriction of maps from U to V $(V \subset U)$ is a presheaf.

2. Presheaf of smooth functions on topological space X: This presheaf is defined as

$$F : Open_X^{op} \longrightarrow k - \text{algebras}$$

$$U \mapsto C^\infty(U)$$

where $k = \mathbb{R}$ or $\mathbb{C}$, and restriction maps being the natural restriction functions i.e. for opens $V \subset U \subset X$,

$$F(U) \longrightarrow F(V)$$

$$f \mapsto f_{|V}$$

3. $E-$valued constant presheaf: Let E be a set. A constant presheaf with value E, is defined as $F(U) = E, \; \forall \; U \subseteq X$ and restriction maps $res_V^U = id_E \; \forall \; V \subseteq U \subseteq X$.

4. Presheaf of bounded functions: This sheaf is defined as $F : Open_X^{op} \longrightarrow \text{Set}$ such that for every open $U \subseteq X, \; F(U)$ is the set of bounded $\mathbb{R}$ (or $\mathbb{C}$) valued functions on U, and restrictions maps are (like previous examples) natural restriction functions.

For any two presheaves F and G, $Hom(F, G)$ is not empty. This fact turns presheaves into a category. A morphism $f \in Hom(F, G)$ is actually a family of morphisms:

Definition 1.1.4. For any two presheaves, F, G on X, their morphisms $Hom(F, G)$ is the set of natural transformations i.e. every $f \in Hom(F, G)$, $f : F \longrightarrow G$ is a family of maps $f_U : F(U) \to G(U)$ (for all opens $U \subseteq V$) such that $V \subseteq U \subseteq X$, the following diagram commutes:

$$\begin{tikzcd} F(U) \arrow[r, "res^U_V"] \arrow[d, "f_U"'] & F(V) \arrow[d, "f_V"] \\ G(U) \arrow[r, "res^U_V"'] & G(V) \end{tikzcd}$$

Definition 1.1.5. A presheaf F on topological space X is a called a sheaf iff it satisfies the following two axioms:

1. Uniqueness axiom: For open set $U \subset X$ and an open covering $U = \bigcup\limits_i U_i$ of U, if $s, s' \in F(U)$ such that $s_{|U_i} = s'_{|U_i}$ $\forall i \in I$, then $s = s'$.
2. Gluing axiom: For every open set $U \subset X$ and every open covering $U = \bigcup\limits_i U_i$, if $s_i \in F(U_i)$ $(\forall i \in I)$ such that $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ $(\forall i, j \in I)$, then there exist $s \in F(U)$ such that $s_{|U_i} = s$ for all $i \in I$.

Remark 1.1.6. [Wed] The category of presheaves of C on topological space X will be denoted by ${}^{Psh}/_{X}$. The category of sheaves of C on X is going to be denoted by ${}^{Sh}/_{X}$. ${}^{Sh}/_{X}$ is a full subcategory of ${}^{Psh}/_{X}$. Both ${}^{Psh}/_{X}$ and ${}^{Sh}/_{X}$ are (co)complete i.e. limits and colimits always exist.

Definition 1.1.7. For a pair of morphisms $A \underset{g}{\overset{f}{\rightrightarrows}} B$ (where $A, B \in Ob(C)$), an equalizer of (f, g) is a pair $(Eq(f, g), h)$, where $Eq(f, g)$ is an object, h is a monomorphism $Eq(f, g) \xrightarrow{h} A$ such that $foh = goh$, and $(Eq(f, g), h)$ is subject to the universality condition. This means that for every map $h' : X \to A$, there exist a unique map $\lambda : X \to Eq(f, g)$ such that $h' = \lambda$ o h, i.e. the following diagram is commutative:

$$\begin{tikzcd} Eq(f,g) \arrow[r, "h"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B. \\ X \arrow[u, dashed, "\lambda"] \arrow[ur, "h'"'] & & \end{tikzcd}$$

.

Definition 1.1.8. A diagram $K \xrightarrow{h} A \underset{g}{\overset{f}{\rightrightarrows}} B$ is called exact iff $K \cong Eq(f, g)$.

There is a relationship between presheaf F being a sheaf, and exactness of a certain diagram:

Theorem 1.1.9. *[Kat] Let F be a presheaf on X, $U \subset X$ an open subset of X, and $X = \bigcup_i U_i$ an open covering of X such that $U_i \subseteq X$ are open subsets; F is a sheaf on X iff the following diagram is exact: $F(U) \xrightarrow{\varepsilon} \prod_{i \in I} F(U_i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$. Here, $\varepsilon(s) = \left(res^U_{U_i}(s)\right)_i$, $\alpha((s_i)_i) = \left(res^{U_i}_{U_i \cap U_j}(s_i)\right)_{(i,j) \in I \times I}$, $\beta((s_i)_{i \in I}) = \left(res^{U_j}_{U_i \cap U_j}(s_j)\right)_{(i,j) \in I \times I}$.*

1.2 Stalks, Sheafification and Étalé Spaces

Let $Open_x := \{U_x \subset X : x \in X$ and U_x open neighbourhood of x in $X\}$, ordered by inclusion. $Open_x$ is full subcategory of $Open_X$. For $F \in P$, $F \in Ob({}^{Psh}/_X)$, its restriction to $Open_x$, $F : Open_x^{op} \to \text{Set}$ is a presheaf. Also $Open_x^{op}$ is a filtered category.

Now we will focus on local strusture of sheaves:

Definition 1.2.1. The stalk of F at x, denoted by F_x is defined as $F_x := \varinjlim_{U_x \in Open_x} F(U_x)$.

Remark 1.2.2. F_x has the following explicit construction: $F_x = \{(U_x, s) \mid U_x \in Open_x,\ s \in F(U_x)\}/\sim$ such that $(U_x, s) \sim (V_x, t)$ iff $\exists\ W_x \subset U_x \cap V_x$ such that $s_{|W_x} = t_{|W_x}$.

Remark 1.2.3. For every open $U_x \in Open_x$, there exist a canonical map

$$F(U_x) \longrightarrow F_x$$

$$s \mapsto s_x$$

where $s_x := [(U_x, s)] \in F_x$ (the class of (U_x, s) in F_x). s_x is called the germ of s at x.

Proposition 1.2.4. *For a sheaf F on a topological space X, and any open $U \subseteq X$, if $s, s' \in F(U)$, then $s = s'$ iff $s_x = s'_x\ \forall x \in U$.*

Proof. If $s = s'$ then clearly $s_x = s'_x$. Conversely, if $s_x = s'_x\ \forall x \in U$, then there exist an open $x \in Open_x$ such that $res^U_{U_x}(s) = res^U_{U_x}(s')\ \ \forall x \in U$. Applying the uniqueness axiom, we get $s = s'$. □

Remark 1.2.5. For every natural transformation $\eta : F \longrightarrow F'$ we get an induced map of stalks $\eta_x := \varinjlim_{U_x} \eta_{U_x}$,

$$\eta_{U_x} : F_x \longrightarrow F'_x$$

$$s_x \mapsto s'_x$$

where $s_x := [(U_x, s)]$ and $s'_x := [(U_x, \eta_{U_x}(s))]$, $s' := \eta_{U_x}(s)$.

Remark 1.2.6. Combining the results of Note 1.4 and Note 1.5, for every fixed $x \in X$, we get a functor

$$\varphi : {}^{Psh}/_{X} \longrightarrow \text{Set}$$

$$F \mapsto F_x$$

such that the following diagram is commutative:

$$\begin{tikzcd} F(U) \arrow[r, "S \mapsto S_x"] \arrow[d, "\eta_u"'] & F_x \arrow[d, "\eta_x"] \\ F'(U) \arrow[r, "S' \mapsto S'_x"'] & F'_x \end{tikzcd}$$

Every presheaf F can be attached to a sheaf $\widetilde{F}$:

Proposition 1.2.7. *[Wed] For $F \in Ob({}^{Psh}/_{X})$, $\exists$ a pair $(\widetilde{F}, l_F)$ where $\widetilde{F} \in Ob({}^{Sh}/_{X})$ and $l_F \in Hom(F, \widetilde{F})$ such that $(\widetilde{F}, l_f)$ is universal i.e. the following diagram is commutative for every $G \in Ob({}^{Sh}/_{X})$ and any $\varphi \in Hom(F, G)$:*

$$\begin{tikzcd} F \arrow[r, "l_F"] \arrow[d, "\phi"'] & \tilde{F} \arrow[dl, dashed, "\exists!\psi"] \\ G & \end{tikzcd}$$

$\widetilde{F}$ is called the sheafification of F.

Definition 1.2.8. For every $E, X \in Ob(Top)$ and a continuous $P : E \to X$, there exist a sheaf of sections that is defined as follows: $\forall\ U \in Open_X$, $\Gamma(U, E) := F(U) = \{s : U \to E \mid s \text{ is continuous and } P \circ s = id_U\}$. For opens $V \subset U \subset X$, the restriction map is defined as

$$res^U_V : F(U) \to F(V)$$

$$s \mapsto s_{|V}$$

Remark 1.2.9. [Ten] $\Gamma(-, E)$ satisfies both axioms of sheaves.

Definition 1.2.10. Let $X \in Ob(Top)$. A pair (E, P) consists of a $E \in Ob(Top)$ and $P \in Hom_{Top}(E, X)$ is called an étalé space over X (or sheaf space over X) iff P is a local homeomorphism.

Definition 1.2.11. A morphism of étalé spaces (E, P) and (E', P') is a continuous map $f : E \to E'$ such that $P = P' \circ f$ i.e. diagram
$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "P"'] & & E' \arrow[dl, "P'"] \\ & X & \end{tikzcd}$$
is commutative. So, étalé spaces over X form a category is denoted by $\acute{E}t/_X$.

Remark 1.2.12. $\forall\ (E, p) \in \acute{E}t/_X$ the functor $\Gamma E := \Gamma(-, E)$,

$$\Gamma(-, E) : Open_X^{op} \to \text{Set}$$

$$U \mapsto \Gamma(U, E)$$

is a sheaf and for any morphism of étalé spaces $f : E \to E'$, the map $\Gamma f : \Gamma(-, E) \to \Gamma(-, E')$, pointwise defined as

$$\Gamma f_U : \Gamma(U, E) \to \Gamma(U, E')$$

$$s \mapsto f \circ s$$

(for every open U $\subset X$) is a morphism of sheaves $\Gamma f : \Gamma W \to \Gamma E'$.

Remark 1.2.13. [Ten] For every $G \in Psh/_X$, there exist an étalé space (LF, P) such that $LF := \underset{x \in X}{\sqcup} F_x$ and $P : LF \to X$ is the natural projection i.e. $P^{-1}(x) = F_x$. Here LF is topologised as follows: for every open set $U \in X$, and $s \in F(U)$, the map $\bar{s} : U \to LF$ is defined by $x \mapsto s_x \in F_x$. And finally we define the open set in LF to be $\bar{s}(U) =: \{s_x |\ x \in U\}$.

Remark 1.2.14. $\widetilde{F} \cong \Gamma(LF)$ i.e. sheafification of F is isomorphic to the composition of the two functors

$$\Gamma \circ L : Psh/_X \xrightarrow{L} \acute{E}t/_X \xrightarrow{\Gamma} Sh/_X$$

$$F \mapsto LF \mapsto \Gamma(-, LF)$$

1.3 Direct And Inverse Image of Presheaves

Definition 1.3.1. Let $X, Y \in Ob(Top)$, $f : X \to Y$ a continuous map and $F \in Ob({}^{Psh}/_{X})$ then a presheaf f_*F on Y is defined as $(f_*F)(U) = F(f^{-1}(U))\ \forall U \in Open_Y$, and is called the direct image of F under f (aka pushforward sheaf). Restriction maps then will be $res_V^U = res_{f^{-1}(V)}^{f^{-1}(U)}$ for opens $V \subseteq U \subseteq Y$.

Remark 1.3.2. For $F_1, F_2 \in Ob({}^{Psh}/_{X})$ and a natural transformation $\eta : F_1 \longrightarrow F_2$, f_* induces a natural transformation $f_*(\eta) : f_*F_1 \longrightarrow f_*F_2$ $(f_*(\eta)_U := \eta_{f^{-1}(U)})$ where $f_*F_1, f_*F_2 \in Ob({}^{Psh}/_{X})$. So we get a functor $f_* : {}^{Psh}/_{X} \longrightarrow {}^{Psh}/_{Y}$.

$$\begin{tikzcd}
Psh/X \ni F_1 \arrow[r, "f_*"] \arrow[d, "\eta"'] & f_*F_1 \in Psh/Y \arrow[d, "F_*(\eta)"] \\
Psh/X \ni F_2 \arrow[r, "f_*"] & f_*F_2 \in Psh/Y
\end{tikzcd}$$

The dual notion of direct image of F is not exact, it is an approximation:

Definition 1.3.3. For a continuous map $f : X \to Y$ and $F \in Ob({}^{Psh}/_{Y})$, the inverse image of F under f (aka the pullback sheaf) is defined as the sheafification of the presheaf

$$f^-F : Open_X^{op} \longrightarrow \text{Set}$$

$$U \mapsto \varinjlim_{f(U) \subset V} F(V)$$

$(V \in Open_Y)$ the sheafified f^-F is denoted by f^*F or $f^{-1}F$.

1.4 (Co)kernel Presheaves

In this section, we exclusively consider the category of presheaves of abelian groups on topological space X, which we will be denoted it by ${}^{AbPsh}/_{X}$.

Definition 1.4.1. [Ten] Let $F, G \in {}^{AbPsh}/_{X}$, $f \in Hom_{{}^{AbPsh}/_{X}}(F, G)$. For open subset $U \subset X$, define $(kerf)(U) := \{s \in F(U) \mid f_U(s) = 0_{G(U)}\} \leq F(U)$. Since f is a natural transformation, then for opens $V \subseteq U \subseteq X$ and $s \in kerf(U)$, $f_V(res^U_V(s)) = res^U_V(f_U(s)) = 0$, which means that $res^U_V(s) \in kerf(V)$. So $kerf$ and $res^U_{V|kerf(U)}$ define a presheaf over X which is called the kernel (presheaf) of f and is denoted by $kerf$.

$$kerf : Open^{op}_X \longrightarrow Ab$$

$$U \mapsto kerf(U)$$

Remark 1.4.2. The composition $kerf \to F \xrightarrow{f} G$ is zero.

Remark 1.4.3. [Ten]The Relation Between $kerf$ and f: If $F, G \in Ob({}^{AbPsh}/_X)$ and $f \in Hom(F, G)$, then we will have the following : ($kerf = 0$) iff ($\forall$ open $U \subset X$, f_U is injective) iff (f is monomorphism i.e. $\forall$ presheaf E and morphisms $E \underset{\beta}{\overset{\alpha}{\rightrightarrows}} F \xrightarrow{f} G$ s.t. $f \circ \alpha = f \circ \beta$, we will have $\alpha = \beta$).

Having the same assumptions as above, we can define the cokernel presheaf:

Definition 1.4.4.

$$cokerf : Open^{op}_X \longrightarrow Ab$$

$$U \mapsto \frac{G(U)}{im(f_U)}$$

($im(f_U) = f_U(F(U)) \leq G(U)$) for opens $V \subseteq U \subseteq X$, if $s \in F(U)$, then $res^U_V(f_U(s)) = f_V(res^U_V(s)) \in im(f_Y)$. Hence the map $G(U) \to \frac{G(V)}{im(f_V)}$ nullifies $im(f_U)$ and induces a restriction map $\overline{res}^U_V : \frac{G(U)}{im(f_U)} \to \frac{G(V)}{im(f_V)}$. So $cokerf$ and $\overline{res}^U_V$ actually a presheaf.

Remark 1.4.5. [Ten] Cokernel sheaf is sheafification of the cokernel presheaf.

Remark 1.4.6. The composition $F \xrightarrow{f} G \to cokerf$ is zero.

Remark 1.4.7. [Ten] The relation between $cokerf$ and f: If $F, G \in Ob({}^{AbPsh}/_X)$ and $f \in Hom(F, G)$, then the following is true: ($cokerf = 0$) iff ($\forall$ open $U \subset X$, f_U is surjective) iff (f is an epimorphism i.e. $\forall$ presheaf E and morphisms $F \xrightarrow{f} G \underset{\beta}{\overset{\alpha}{\rightrightarrows}} E$ such that $\alpha \circ f = \beta \circ f$; $\alpha = \beta$).

1.5 The Yoneda Lemma

Definition 1.5.1. Let C be a category. We wil define $\widehat{C} := {}^{Psh}/_{C} := Fct(C^{op}, \text{Set})$ and

$$h : C \longrightarrow \widehat{C}$$

$$X \mapsto h_X := Hom_C(-, X)$$

The Yoneda Lemma: [KS]For $F \in \widehat{C}$ and $X \in C$; $Hom_{\widehat{C}}(h_X, F) \cong F(X)$.

Corollary: The functor h is fully faithful (i.e. the category C can fully faithfully embed in the category of presheaves over C).

Proof. Choosing h_Y for F and then applying the Yoneda lemma, we get $\forall X, Y \in C$, $Hom_{\widehat{C}}(h_X, h_Y) \cong h_Y(X) = Hom_C(X, Y)$. So h is a fully faithfully functor. $\square$

Definition 1.5.2. Due to above property, the functor h is called the Yoneda embedding functor.

Definition 1.5.3. A presheaf $F : C^{op} \to \text{Set}$ is called representable iff $\exists X \in C$ such that $F \cong h_X$ in $\widehat{C}$. X is called a representative of F.

Example 1.5.4. [KS] For commutative ring k, let A be $k-$algebra, N a right $A-$module, M a left $A-$module and L a $k-$module. By $Bil(N \times M, L)$ we mean the set of $A-$bilinear maps from $N \times M$ to L. Since $Bil(N \times M, L) \cong Hom_K(N \underset{A}{\otimes} M, L)$, then the functor $Bil(N \times M, -) : Module(k) \to \text{Set}$ is representable and $N \underset{A}{\otimes} M$ is its representative.

Chapter 2

Sites

2.1 Grothendieck Topologies, Sites, and Topos

Definition 2.1.1. Let C be a category. A *Grothendieck topology* τ on C is an assignment to each object $X \in C$, coverings of X denoted by $Cov(X)$ which is a collection of sets of morphisms $\{X_i \mapsto X\}_{i\in I}$ that satisfies the following axioms:

i. Isomorphism axiom: If $Y \xrightarrow{\sim} X$, then $\{Y \to X\} \in Cov(X)$.
ii. Change of base axiom: If $\{X_i \to X\}_{i\in I} \in Cov(X)$ and $Y \to X$ is a morphism, then the fibred products $X_i \underset{X}{\times} Y$ exist for all i, and $\{X_i \underset{X}{\times} Y \to Y\}_{i\in I} \in Cov(Y)$.
iii. Refinement axiom: If $\{X_i \xrightarrow{f_i} X\}_{i\in I} \in Cov(X)$ and for any fixed i, the covering $\{X_{ij} \xrightarrow{f'_{j_i}} X_i\}_{j\in J_i}$ exist, then the compositions $\{X_{ij} \xrightarrow{f_i \circ f'_{j_i}} X\}_{i\in I, j\in J_i} \in Cov(X)$.

Remark 2.1.2. In Grothendieck topology, open sets of a space X are replaced by maps into space X. Instead of intersections, we have fibred products, and union play no role at all.

Remark 2.1.3. The axioms describe the coverings of an object.

Definition 2.1.4. A category C that is equipped with a Grothendieck topology τ, (C, τ), is called a *(commutative) site*.

Remark 2.1.5. Applying the second and third axioms, if $\{X_i \to X\}_i \in Cov(X)$ and

$\{Y_j \to X\}_j \in Cov(X)$, then $\{X_i \underset{X}{\times} Y_j \to X\}_{ij} \in Cov(X)$.

Definition 2.1.6. A set of functions or morphisms $\{U_i \xrightarrow{f_i} U\}_i$ on topological spaces, schemes etc, is called jointly surjective if and only if the (set-theoratic) union of their images be equal to U.

2.2 Examples

Example 2.2.1. *The site of usual topology.* Let $X \in Ob(Top)$ and $Open_X$ be the category of open subsets of X. Then $\forall\ U \in Ob(Open_X)$, τ will assign the coverings $Cov(U)$, consist of set of open coverings of U. Here $U_1 \underset{U}{\times} U_2$ and $U_1 \cap U_2$ coincide.

Example 2.2.2. *The site of global topology.* Let $C = Top$. If $X \in Top$, then any covering of $Cov(X)$ is a jointly surjective family of open embeddings $\{X_i \to X\}_i$. Note that by open embedding, we mean an open continuous injective map, not the inclusion because otherwise, the isomorphism axiom will be violated.

Example 2.2.3. [FGI+] *The site of global étalé topology.* Let $C = Top$ and $X \in Ob(C)$. A covering of X then will be a jointly surjective family of local homeomorphism $\{X_i \xrightarrow{f_i} X\}_i$.

The last two examples which are going to be introduced, are two of the most common sites in algebraic geometry. But before we proceed to those examples, we need to review some notions from algebraic geometry.

Definition 2.2.4. Let X be a topological space. The structure sheaf of X, denoted by O_X, is a sheaf of commutative rings on X.

Definition 2.2.5. [Gat] A morphism $f : X \to Y$ of schemes is called a closed immersion (embedding) if and only if

1. $X \approx f(X) \subset Y$, where $f(X)$ is a closed subset.
2. The induced morphism $O_Y \to f_*O_X$ is a surjection (O_X and O_Y are structure sheaves on X and Y respectively).

Definition 2.2.6. [Gat] The kernel sheaf of the morphism $O_Y \to f_*O_X$ is called the ideal sheaf of the embedding and is denoted by $I_{X/Y}$.

Definition 2.2.7. A morphism of commutative rings $f : S \to R$ (which makes R to be an $S - algebra$) is said to be of finite type if and only if $\exists\ n \in \mathbb{N}$ and a surjective S-algebras morphism $S[x_1, \ldots, x_n] \twoheadrightarrow R$.

Definition 2.2.8. [Gat] Let R and S be commutative rings, and $f : S \to R$ a ring homomorphism . Then the R-module of Kähler differentials (a.k.a module of relative differentials) is defined to be the free R-module, generated by formal differentials $\{dr | r \in R\}$ mod out the relations $d(r_1 + r_2) = dr_1 + dr_2$, $\forall r_1, r_2 \in R$, $d(r_1 r_2) = r_1 dr_2 + r_2 dr_1$, $\forall r_1, r_2 \in R$, and $ds = \circ$, $\forall s \in S$. This module is denoted by $\Omega_{R/S}$.

Definition 2.2.9. A commutative ring morphism $f : S \to R$ is called unramified if and only if

1. f is of finite type,
2. $\Omega_{R/S} = 0$.

Definition 2.2.10. A morphism of schemes $f : X \to Y$ is called unramified at $x \in X$ if and only if $\exists\ U$ an open neighbourhood of x, $\mathrm{Spec}(R) = U \subset X$ and an open $\mathrm{Spec}(S) = V \subset Y$, such that $f(U) \subset V$, and the corresponding induced ring morphism $S \to R$ is unramified.

Definition 2.2.11. $f : X \to Y$ is unramified if and only if it is unramified $\forall x \in X$.

Definition 2.2.12. A scheme (X, O_x) is called regular if and only if for every $x \in X$, $O_{X,x}$ (stalk of O_x at x) is regular local ring, i.e., a Noetherian local ring whose maximal ideal has the minimal number of generators, equal to its Krull dimension.

Definition 2.2.13. A morphism of schemes $f : X \to Y$ is called smooth if and only if

1. f is locally of finite presentation, i.e., for every $x \in X$, $\exists$ an affine neighbourhood $U_x \subset X$ and $V_{f(x)} \subset Y$ such that $O_x(U_x) = \frac{O_Y(V_{f(x)}[x_1,\ldots,x_n]}{I}$, where I is finitely generated.

2. f is flat, i.e. for every $x \in X$, the local ring $O_{X,x}$ is a flat module over $O_{Y,f(x)}$.

3. For every geometric point $\mathrm{Spec}k$ (k is an algebraically closed field), and morphism $\mathrm{Spec}k \to Y$, the fiber product $X \underset{Y}{\times} \mathrm{Spec}k$ is regular (scheme).

Definition 2.2.14. $f : X \to Y$ is called étalé if and only if f is unramified and smooth.

Example 2.2.15. *Zariski Site.* Let $C =$ Scheme, the category of schemes, and $X \in$ Scheme. A (Zariski) covering for X is a family of morphisms of schemes $\{f_i : X_i \to X\}_i$ such that $f_i s$ are open embeddings and $\bigcup\limits_i f_i(X_i) = X$ (i.e. f_i's are jointly surjective).

Example 2.2.16. *Étale Site.* Let $X \in Ob(Scheme)$. An étale covering of X, is a family of étale morphisms of schemes $\{f_i : X_i \to X\}_i$, such that $X = \bigcup\limits_i f_i(x_i)$.

Definition 2.2.17. [Kat] A morphism of two sites (C, τ) and (C', τ'), h, is a functor $h : C \to C'$ such that for $\{X_i \xrightarrow{f_i} X\}_i \in Cov(X)$, we have $\{h(X_i) \xrightarrow{h(f_i)} h(x)\}_i \in Cov(h(X))$, and for $Y \to X$, $h(X_i \underset{X}{\times} Y) \to h(X_i) \underset{h(X)}{\times} h(Y)$ is an isomorphism.

Remark 2.2.18. [Kat] A presheaf $F \in Ob(\widehat{C})$ is said to be a sheaf on (C, τ) if and only if the following diagram is exact for all i and j:

$$F(X) \to \prod_i F(X_i) \rightrightarrows \prod_{i,j} F(X_i \underset{X}{\times} X_j)$$

Applying the Yoneda lemma, the diagram can be written as

$$Hom_{\widehat{C}}(h_X, F) \to \prod_i Hom_{\widehat{C}}(h_{X_i}, F) \rightrightarrows \prod_{i,j} Hom_{\widehat{C}}(h_{X_i \underset{X}{\times} X_j}, F).$$

Definition 2.2.19. The full subcategory $Sh(C, \tau)$ of $Psh(C, \tau)$ (the subcategory of sheaves on sites) is called a topos.

Definition 2.2.20. Topology τ is called sub-canonical if and only if $\forall X \in C$, $Hom(-, X) : C^{op} \to$ Set is a sheaf for the given topology.

From now on, we will always assume that the topology τ is sub-canonical.

2.3 Nerves

Definition 2.3.1. [Pau] Let (C, τ) be a site, and $\Phi := \{f_i : X_i \to Y\}_i \in$Cov$(Y)$. Then the nerve of Φ, denoted by $N(\Phi)$, is the simplicial object in Build, with n-vertices defined as

$$N(\Phi)_n := \bigsqcup_{i_1,\ldots,i_n} X_{i_1} \underset{Y}{\times} \ldots \underset{Y}{\times} X_{i_n}.$$

Restrictions and inclusions will be faces and degeneracies respectively.

Proposition 2.3.2. *[Pau] Let $(Build, \tau)$ be a site. $F : Build^{op} \to Set$ is a sheaf if and only if F commutes with colimits along nerves of coverings, i.e.,*

$$F(U) \cong F(\mathrm{Colim}_{[n]\in\Delta} N(\Phi)_n) \cong \lim_{[n]\in\Delta} F(N(\Phi)_n) \ \forall \Phi \in Cov(U)),$$

Δ is a Simplex category.

Theorem 2.3.3. [Pau] *Let (C, τ) be a site where τ is sub-canonical. The Yoneda embedding $C \hookrightarrow Sh(C, \tau)$ preserve limits, and preserve colimits that has been taken along nerves of coverings. So in contrast to embedding $C \hookrightarrow \widehat{C}$, the embedding $C \hookrightarrow Sh(C, \tau)$ preserve coverings.*

Chapter 3

Monoidal Categories

Monoidal categories are somehow generalization of the algebraic structures which behave like modules.

3.1 Modoidal Categories

Definition 3.1.1. A monoidal category $(C, \otimes)$ is a tuple $(C, \otimes, \mathbb{1}, Un^r, Un^l, as)$ consists of following datum:

1. A category C,
2. A functor $\otimes : C \times C \to C$ called tensor product,
3. A unit object $\mathbb{1}$ in C ,
4. $\forall X \in Ob(C)$ two natural isomorphisms $Un^r_X : X \otimes \mathbb{1} \xrightarrow{\simeq} X$ and $Un^l_X : \mathbb{1} \otimes X \xrightarrow{\simeq} X$ called right and left unitors respectively,
5. $\forall X, Y, Z \in Ob(C)$ a natural isomorphism $as_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\simeq} (X \otimes Y) \otimes Z$ called a associator, such that the following diagrams commute:

(a) Associators' diagram

$$\begin{tikzcd}
 & (X \otimes (Y \otimes Z) \otimes W) \arrow[dr, "as_{X,Y\otimes Z,W}"] & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[ur, "as_{X,Y,Z}\otimes id_W"] \arrow[d, "as_{X\otimes Y,Z,W}"'] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[d, "id_X \otimes as_{Y,Z,W}"] \\
(X \otimes Y) \otimes (Z \otimes W) \arrow[rr, "as_{X,Y,Z\otimes W}"] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

(b) Compatibility of unitors and associator

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "as_{X,\mathbb{1},Y}"] \arrow[dr, "Un^r_X \otimes id_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "id_X \otimes Un^l_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

Definition 3.1.2. For two objects $X, Y \in Ob(C)$, if $Hom(X,Y) \in Ob(C)$ as well, then we call this object a hom-object and denote it by $\underline{Hom}(X,Y)$.

Definition 3.1.3. A monoidal category is said to be

1. Closed iff it has inner homomorphism i.e. if $\forall X, Y \in Ob(C)$, the functor $(Hom(- \otimes X, Y)) \cong h_{\underline{Hom}(X,Y)} := Hom(-, \underline{Hom}(X,Y))$.
2. Strict iff associator and unitors are equalities.

Example 3.1.4. 1. For commutative (unital) ring k , the category of modules over k, $Mod(k)$ is a closed monoidal category.

2. Any category C with finite products is monoidal where product is the tensor product.
3. The category of categories with their usual product is a monoidal category.

Definition 3.1.5. A monoid in a monoidal category $(C, \otimes)$ is a triple $(X, \mu, 1)$ consists of:

1. An object $X \in Ob(C)$,
2. A multiplication morphism $\mu : X \otimes X \to X$,

3. A unit morphism $1 : \mathbb{1} \longrightarrow X$ satisfying:

   (a) Associativity condition

$$\begin{tikzcd}[column sep=small]
 & X \otimes (X \otimes X) \arrow[rr, "as_{X,X,X}"] \arrow[dl, "id_X \otimes \mu"'] & & (X \otimes X) \otimes X \arrow[dr, "\mu \otimes id_X"] & \\
X \otimes X \arrow[rr, "\mu"] & & X & & X \otimes X \arrow[ll, "\mu"']
\end{tikzcd}$$

   (b) Unit condition

$$\begin{tikzcd}
X \otimes \mathbb{1} \arrow[r, "id_X \otimes 1"] \arrow[dr, "Un^r_X"'] & X \otimes X \arrow[r, "\mathbb{1} \otimes id_X"] \arrow[d, "\mu"] & 1 \otimes X \arrow[dl, "Un^l_X"] \\
 & X &
\end{tikzcd}$$

Definition 3.1.6. A (left) module over a monoid $(X, \mu, 1)$ in C is a pair (M, μ_M) such that $M \in Ob(C)$ and μ_M is a scalar multiplication map $\mu_M : X \otimes M \to M$ and μ_M is compatible with μ and 1 i.e. we have the following two commutative diagrams:

$$\begin{tikzcd}
X \otimes X \otimes M \arrow[r, "id_X \otimes \mu_M"] \arrow[d, "\mu \otimes id_M"'] & X \otimes M \arrow[d, "\mu_M"] \\
X \otimes M \arrow[r, "\mu_M"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \otimes M \arrow[rr, "1 \otimes id_M"] \arrow[dr, "Un^l_M"'] & & X \otimes M \arrow[dl, "\mu_M"] \\
 & M &
\end{tikzcd}$$

Example 3.1.7. 1. Monoid objects in the monoidal category (Set, $\times$) (the tensor product is the Cartesian product, ant the unit object $\mathbb{1}$ is a set with one element) are just ordinary monoids.

2. A monoid A in the monoidal category $(Mod(k), \otimes)$ is a $k-$algebra.

3.2 Symmetric Monoidal Category

Definition 3.2.1. [Pau] A braided monoidal category is a monoidal category $(C, \otimes, \mathbb{1}, Un^r, Un^l, as)$ that is equipped with a natural isomorphism $Com_{X,Y} : X \otimes Y \xrightarrow{\simeq} Y \otimes X,\ \forall X, Y \in C$ that is called a commutator (aka a braiding) such that the following diagrams commute:

1. Compatibility of commutator and associator:

$$\begin{tikzcd} & X \otimes (Y \otimes Z) \arrow[r, "Com_{X,Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[dr, "as_{Y,Z,X}"] & \\ (X \otimes Y) \otimes Z \arrow[ur, "as_{X,Y,Z}"] \arrow[dr, "Com_{X,Y}\otimes id_Z"'] & & & Y \otimes (Z \otimes X) \arrow[dl, "id_Y\otimes Com_{Z,X}"] \\ & (Y \otimes X) \otimes Z \arrow[r, "as_{Y,X,Z}"] & Y \otimes (X \otimes Z) & \end{tikzcd}$$

2. Compatibility of unitors and associator: $Un^r \circ Com = Un^l$

$$\begin{tikzcd} \mathbb{1} \otimes X \arrow[rr, "Com_{\mathbb{1},X}"] \arrow[dr, "Un^l_X"'] & & X \otimes \mathbb{1} \arrow[dl, "Un^r_X"] \\ & X & \end{tikzcd}$$

Definition 3.2.2. A braided monoidal category that satisfies the condition $Com_{X,Y} \circ Com_{Y,X} = id_{Y\otimes X}$ $(\forall X, Y \in Ob(C))$ is called a symmetric monoidal category.

Example 3.2.3.
1. Categories that accept finite products are symmetric monoidal.

2. The category of modules over (unital) rings, $(Mod(k), \otimes)$ is also a closed symmetric monoidal.

3. [Fai] The category of graded modules (or $\mathbb{Z}-$graded modules) over a commutative (unital) ring k is denoted by $gMod(k)$ $(or Mod_g(k))$. $\forall V \in Ob(gMod(k)); V = \bigoplus\limits_{i\in\mathbb{Z}} V_i$ where $\forall i \in \mathbb{Z},\ V_i \in Ob(Mod(k))$. Also $\forall V, W \in Ob(gMod(k));\ T \in Hom(V, W)$ such that $T(V_i) \subset W_i\ \forall i$ every element $x \in V_i$ is called a homogeneous element of degree i $(deg(x) := |x| = i)$.
   $\forall\, V, W \in Ob(gMod(k))$, their tensor product defined as $V \otimes W$ such that $(V \otimes W)_k =$

$\bigoplus\limits_{i+j=k}(V_i \otimes W_j)$. $\otimes$ is associative. The unit object of $gMod(k)$ is $\mathbb{1} = k$, and is of degree zero. The commutator defined as

$$Com_{V,W} : V \otimes W \to V \to W$$

$$V \otimes W \mapsto (-1)^{|V||W|} W \otimes V$$

$gMod(k)$ admits inner homomorphism objects. $\forall\ V, W \in Ob(gMod(k)$, $\underline{Hom}(V,W) := \underline{Hom}_{Mod(k)}(V,W)$ (i.e. the module of all linear maps from V to W) with the grading $\underline{Hom}(V,W) = \bigoplus\limits_{i\in\mathbb{Z}} Hom_i(V,W)$ where $\forall\ f_i \in Hom_i(V,W)$ and $V_k \in V$; $f_i(V_k) \subset V_{i+k}$. So the category $gMod(k)$ is a closed symmetric monoidal.

4. [CCF] A special case of the above example which is the main category of study in super geometry, is the category of super modules (or $\mathbb{Z}_2-$ graded modules) over k and is denoted by $SMod(k)$ (or $Mod_s(k)$). Here $\forall\ V \in Ob(SMod(k))$; $V = \bigoplus\limits_{i\in\mathbb{Z}_2} V_i = V_0 \oplus V_1$, with the same tensor product as $gMod(k)$.

Definition 3.2.4. [Pau] In a symmetric monoidal category $(C, \otimes)$, a commutative monoid (aka a commutative algebras) is a monoid $(X, \mu, 1)$ that satisfies the commutativity condition $\mu \circ Com_{X,X} = \mu$, i.e. we have the following commutative diagram:

$$\begin{tikzcd} X \otimes X \arrow[rr, "Com_{X,X}"] \arrow[dr, "\mu"'] & & X \otimes X \arrow[dl, "\mu"] \\ & X & \end{tikzcd}$$

The category of monoids (algebras) and commutative monoids (commutative algebras) in $(C, \otimes)$ denoted by ALG_C and $CALG_C$ respectively.

Example 3.2.5. 1. In the symmetric monoidal category $(Mod(\mathbb{Z}), \otimes)$, commutative monoids are usual commutative rings.

2. In the the symmetric monoidal category $(gMod(k), \otimes)$, commutative monoids are graded commutative $k-$algebras.

3. In the the symmetric monoidal category $(\text{Set}, \times)$, usual commutative monoids are commutative monoids.

Chapter 4

Categorical Calculus

4.1 Abelian Group Objects and Torsors

Before we start our main topics, we give a brief introduction to the notion of torsors in the context of sheaves.

Definition 4.1.1. [Wed] Let $X \in Ob(Top)$ and $H : Open_X^{op} \to Grp$ an H-sheaf on X is a pair (F, h) where $F : Open_X^{op} \to \text{Set}$ and $h : H \times F \to F$ (natural transformation of sheaves of sets) such that $\forall U \in Open_X; h_U : H(U) \times F(U) \to F(U)$ is a group $H(U)$ action on the set $F(U)$.

For two H-sheaves F and G, a morphism $f : F \to G$ is a natural transformation such that $\forall\ U \in Open_X$ the map $f_U : F(U) \to G(U)$ is $H(U)$-equvariant (i.e. $\forall\ t \in H(U)$ and $x \in F(U)$, $f_U(tx) = tf_U(x)$). So H-sheaves on X form a category which is denoted by $H - Sh_{/X}$.

Definition 4.1.2. [Wed] If for $T \in Ob(H - Sh_{/X})$ and every $U \in Open_X$ the action $H(U) \times T(U) \to T(U)$ is simply transitive (i.e. a transitive action that $\forall\ x, y \in T(U)$, $\exists!\ g \in H(U)$ such that $gx = y$) then T is called and $H-$Pseduotorsor.

Definition 4.1.3. If there exist an open covering $X = \bigcup_i U_i$ such that an $H-$Pseduotorsor T, $T(U_i) \neq \emptyset$ $\forall i$, then T is called an $H-$torsor.

Example 4.1.4. [Wed] For all holomorphic functions $f : W \to \mathbb{C}$ on open $W \subseteq \mathbb{C}$, define the sheaf $\int f : Open_W^{op} \to \mathbb{C}$ such that for $U \subset W$ $\int f(U) := \{F : U \to \mathbb{C} \mid F \text{ holomorphic and } F' = f_{|U}\}$ then the constant sheaf $W_{\mathbb{C}}$, defined as $W_{\mathbb{C}}(U) = \{k : U \to \mathbb{C} \mid k \text{ is locally constant}\}$ acts simply transitively on $\int f$ by addition i.e.

$$W_{\mathbb{C}}(U) \times \int f(U) \to \int f(U)$$

$$(k, F) \mapsto k + F$$

and therefore $\int f$ is a $W_{\mathbb{C}}-$Pseudotorsor and since we can cover W by convex opens $U_i, \subseteq \mathbb{C}$ such that $\int f(U_i) \neq \emptyset$ $\forall i$, $\int f$ is a $W_{\mathbb{C}}-$torsor.

Remark 4.1.5. $H-$ torsors form a full subcategory of $H - Sh_{/X}$ which is denoted by $H-$torsors.

Categorical calculus is heavily rely on the notion of tangent categories, which will be the subject of our study in Section 2. In order to define this category, we need to know what abelian group objects and torsors are, in categorical context.

For any category C with terminal object Pt_C we have the following definitions:

Definition 4.1.6. [Pau] A triple $(X, \mu, 0)$ composed of an object X, a multiplication morphism $\mu : X \times X \to X$ and an identity morphism $0 : Pt_C \to X$, such that X induces a functor $Hom(-, X) : C \to Ab$ is called an abelian group object. Here Ab denotes the category of abelian groups.

The collection of all abelian group objects in C will be denoted by $Ab(C)$.

Definition 4.1.7. [Pau] For an abelian group object $(X, \mu, 0)$ a pair (Y, ρ) made of $Y \in Ob(C)$ and action morphism $\rho : X \times Y \to Y$ that induces an action $Hom(-, X) \times Hom(-, Y) \to Hom(-, Y) \times Hom(-, Y)$ is a set isomorphism which is called a torsor over $(X, \mu, 0)$.

Remark 4.1.8. In the above definition, $Hom(-, X) : C \to Ab$, $Hom(-, Y) : C \to Set$, and the map is defined by pair $(\bar{\rho}, id)$ where $\bar{\rho}$ denotes the morphism induced by ρ, and id, the morphism induced by id_Y.

4.2 Tangent Categories

Let I be the category with two objects and an arrow between them, and C any category with pullbacks.

Definition 4.2.1. The arrow category which is denoted by $[I, C]$, consists of $Ob([I, C]) = \{[X \to Y] \mid X, Y \in Ob(C)\}$ and $\forall$ $[X \xrightarrow{f} Y]$, $[X' \xrightarrow{g} Y'] \in Ob([I, C])$;

$$\mathrm{Hom}([X \xrightarrow{f} Y], [X' \xrightarrow{g} Y']) = \left\{ \begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\ X' \arrow[r, "g"'] & Y' \end{tikzcd} \right\}.$$

Definition 4.2.2. The tangent category which is denoted by TC, is the category whose objects are abelian group objects in the arrow category $[I, C]$, i.e. $Ob(TC) = \{(Y \to X, Y \underset{X}{\times} Y \to Y, \underline{0} : X \to Y)\}$, and whose morphisms are commutative diagrams, such that the arrow $Y \to X \underset{X}{\times} Y$ is a morphism between abelian group objects in the over category C_X i.e.

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & \circlearrowright & X \underset{X'}{\times} Y' \arrow[dl] \\ & X & \end{tikzcd}.$$

Definition 4.2.3. [Pau] For object A in C, the category of modules over A (a.k.a Beck modules) which is denoted by $Mod(A)$, is the fiber of TC at A, i.e. $Ob(Mod(A)) = \{[B \to A] \in Ab(C_A)\}$.

Definition 4.2.4. [Pau] For the domain functor

$$dom : TC \to C$$

$$[Y \to X] \mapsto Y$$

a left adjoint functor $\Omega^1 : C \to TC$ is called a cotangent functor.

Definition 4.2.5. Let $R \in Ob(CRing)$, and A be an $R-$algebra. A square-zero extension of A is a pair (A', P) where $P : A' \to A$ is an $R-$algebra surjection whose kernel ideal $ker(P) =: I$ is nilpotent of degree two i.e. $I^2 = 0$.

Definition 4.2.6. A morphism between two square-zero extensions (A', P) and (A'', q) of $R-$algebra A is an $R-$algebras morphism $\varphi : A' \to A''$ such that diagram
$$\begin{tikzcd} A' \arrow[rr, "\phi"] \arrow[dr, "p"'] & & A'' \arrow[dl, "q"] \\ & A & \end{tikzcd}$$
is commutative.

Definition 4.2.7. Let R be a commutative ring, A an $R-$algebra and J a $A-$module. A square-zero extension of A by J is a triple (A', P, σ) where (A', P) is a square-zero extension of A and $\sigma : ker(P) \xrightarrow{\cong} J$ is an isomorphism.

A morphism of two square-zero extensions of A by J (A', P, σ) and (A', P', ρ) is a $\varphi : (A', P') \to (A'', P'')$ such that $\varphi\sigma = \rho$

$$\begin{tikzcd} & A' \arrow[dd, "\varphi"] \arrow[dr, "P'"] & \\ Ker(p'') \cong Ker(p') \cong J \arrow[ur, "\sigma"] \arrow[dr, "\rho"'] & & A. \\ & A'' \arrow[ur, "P''"'] & \end{tikzcd}$$

Definition 4.2.8. [Pau] A symmetric monoidal category $(C, \otimes)$ is called pre-additive iff C has zero object 0, finite (co)products, and for every $X, Y \in Ob(C)$, $X \oplus Y \cong X \times Y$ such that the monoidal structure commutes with (finite direct) sums.

Lemma 4.2.9. *Let $(C, \otimes)$ be a pre-additive symmetric monoidal category, $A \in (CALG_C)$(i.e. A is a commutative monoid), and M is a module over A. Then $A \oplus M \in CALG_C$ and the projection $A \oplus M \to A$ is a monoid morphism.*

Proof. The multiplication morphism on $A \oplus M$ is defined by $\mu' : (A \oplus M) \otimes (A \oplus M) \cong (A \otimes A) \oplus (A \otimes M) \oplus (M \otimes A) \oplus (M \otimes M) \to A \oplus M$ where μ' is the combination of the following four morphisms: $\mu \oplus 0 : A \otimes A \to A \oplus M$, $0 \oplus \mu^l_M : A \otimes M \to A \oplus M$, $0 \oplus \mu^r_M : M \otimes A \to A \oplus M$, and $0 : M \otimes M \to M$. The unit morphism $1 : \mathbb{1} \to A \oplus M$ is going to be $1_A \oplus 0$ ($1_A : \mathbb{1} \to A$, and $0 : \mathbb{1} \to M$). Both morphisms satisfy the associativity and unit conditions therefore, $A \oplus M \in CALG_C$ and $Pr_1 : A \oplus M \to A$ is a monoid morphism.

□

Definition 4.2.10. Having the same assumptions as above, we will define $A+\varepsilon J := \{a+\varepsilon x \mid a \in A,\ \varepsilon \in J,\ x \in J\}$. A ring structure can be defined on $A+\varepsilon J : 0 = 0+\varepsilon 0,\ 1 = 1+\varepsilon 0$

$$(a + \varepsilon x) + (a' + \varepsilon x') = (a + a') + \varepsilon(x + x')$$

$$(a + \varepsilon x)(a' + \varepsilon x') = (aa') + \varepsilon(ax' + a'x)$$

then

$$A + \varepsilon J \to A$$

$$a + \varepsilon x \mapsto a$$

and $\forall\ x \in J,\ \sigma(x) = 0 + \varepsilon x = ker(P)$ so $(A + \varepsilon J, P, \sigma)$ becomes a square-zero extension of A by J which is called the trivial square-zero extension of A by J.

Definition 4.2.11. For monoidal category $(C, \otimes)$ and $X \in Ob(ALG_C)$ (i.e. X being a monoidal object in C), the category of (left) modules over X will be denoted by Mod_X or $XMod$.

Definition 4.2.12. The category of modules over (all) monoidal objects which will denoted by Mod_{ALG_C} or just Mod consist of $Ob(Mod) = \{(X, M) \mid X \in ALG_C,\ M \in Mod_X\}$ and $\forall\ (X, M), (X', M') \in Ob(Mod),\ Hom_{Mod}((X, M), (X', M')) = \{(f, f_*) \mid f \in Hom_{ALG_C}(X, X'),\ f_* \in Hom_{Mod_X}(M, f^*(M'))\}$.

Example 4.2.13. For monoidal category of abelian group $(Mod(\mathbb{Z}), \otimes)$, $ALG_{Mod(\mathbb{Z})} = CRing$, $Mod_{CRing} = \{(R, M) \mid R \in Ob(CRing),\ M \in Mod_R\}$. For a fixed ring $R \in Ob(CRing)$, a module over R, $M \in Ob(Mod_R)$ is an abelian group equipped with linear map $\mu : R \otimes M \to M,\ (r,m) \to rm$ such that the following diagrams commute:

$$\begin{tikzcd}
R \otimes R \otimes M \arrow[r, "id_R \otimes \mu_M"] \arrow[d, "\mu \otimes id_M"'] & R \otimes M \arrow[d, "\mu_M"] \\
R \otimes M \arrow[r, "\mu_M"'] & M
\end{tikzcd}$$

$$\begin{tikzcd}
1_R \otimes M \arrow[rr, "1 \otimes id_M"] \arrow[dr, "Un^l_R"'] & & R \otimes M \arrow[dl, "\mu_M"] \\
& M &
\end{tikzcd}$$

Proposition 4.2.14. *Let $R \in Ob(CRing)$. Then $Mod_R \simeq Ab(CRing_R) = Mod(R)$.*

Proof. Any $R-$module $R \to B$ is unit of the abelian group object $CRing_R$ i.e. a diagram

$$\begin{tikzcd}
R \arrow[rr, "e"] \arrow[dr, "id_R"'] & & B \arrow[dl, "p"] \\
& R &
\end{tikzcd}$$

where e is the section of P.

The unit diagram also identifies B with a ring such that its underlying abelian group is $R \oplus ker(P) =: R \oplus M \cong B$ and $P = Pr_1$. The product of $R \oplus M \to R$ with itself is the fiber product over R. So, $(R \oplus M) \underset{R}{\times} (R \oplus M) = R \oplus M \oplus M$.

The unit axiom of group objects on $R \oplus M \cong B$ i.e. the commutative diagram

$$\begin{tikzcd}
(R \oplus M) \times R \arrow[rr, "id_{R\oplus M} \times e"] \arrow[dr, "Pr_1"'] & & (R \oplus M) \times (R \oplus M) \arrow[dl, "\mu"] \\
& R \oplus M &
\end{tikzcd}$$

$$\begin{tikzcd}
((r,m),r) \arrow[rr, mapsto, "id_{R\oplus M} \times e"] \arrow[dr, mapsto, "Pr_1"'] & & ((r,m),(r,o)) \arrow[dl, mapsto, "\mu"] \\
& (r,m) &
\end{tikzcd}$$

defines the morphism

$$\begin{tikzcd}
R \oplus M \oplus M \arrow[rr] \arrow[dr] & & R \oplus M \arrow[dl] \\
& R &
\end{tikzcd}$$

to be $id_R \otimes (id_M + id_M)$. Finally, using both sided unit axiom of group objects we get $\forall r \in R$ and $m, m' \in M;\ \mu((r,m),(r,m')) = (r, m+m')$ and $\mu \circ (e \times id_{R\otimes M}) = Pr_2 \Rightarrow \mu(e \times id_{R\oplus M}(r,(r,m))) = \mu((r,0),(r,m)) = (r,m)$.

The same way we get $\mu((r,m),(r,0)) = (r,m') = Pr_1((r,m),(r,0))$. So, for $m \in M,\ m \cong (0,m)$ and $\mu((r,0),(0,m)) = (0,m) \cong m$. $\mu((0,m'),(r,0)) = (0,m') \cong m'$.
$mm' = (0,m)(0,m') = (0,0(m+m')) = (0,0) = 0 \implies (ker(Pr_1))^2 = M^2 = 0 \to R \oplus M \xrightarrow{Pr_1} R$ is a square-zero extension of R. So every abelian group object related to $R-$module B is a square-zero extension of R, and vice versa. ☐

Proposition 4.2.15. *$Mod_{CRing} \simeq TCRing$.*

Proof. The natural isomorphism map $R_1 \oplus f^*M_2 \xrightarrow{\simeq} R \underset{R_2}{\times} (R_2 \oplus M_2)$ together with the following diagram shows that $Mod_{CRing} \simeq TCRing$ iff $Mod_R \simeq Ab(CRing_R)$ for a fixed R, which is what we showed in the previous proposition.

$$\begin{tikzcd}
\text{Mod} \arrow[rrr, "F"] & & & \text{TCRing} & & \\
(R_1, M_1) \arrow[d, "(f, f_*)"] & & & R_1 \oplus M_1 \arrow[d] \arrow[drr] & & \\
(R_2, f^*M_2) \arrow[d, equal] & & & R_1 \oplus f^*M_2 \arrow[rr] \arrow[dd] \arrow[dr, "\simeq"] & & R_2 \oplus M_2 \arrow[dd] \\
R_1 \otimes M_2 & & & & R_1 \times_{R_2} (R_1 \oplus M_2) \arrow[ur] \arrow[dl] & \\
 & & & R_1 \arrow[rr] & & R_2
\end{tikzcd}$$

. ☐

Remark 4.2.16. For ring homomorphism $S \to R$ and the map $\delta : R \underset{S}{\otimes} R \to R$, $(r_1 \otimes r_2) \to r_1 r_2$, if $I := ker\delta$ then ${}^{I}/{}_{I^2} \cong \Omega_{R/S}$.

Example 4.2.17. In the category $C = CRing$, $TC \simeq Mod$ where every $(R,M) \in Ob(Mod)$ corresponds to $[R \oplus M \xrightarrow{P} R] \in Ob(TC)$. Moreover $[R \oplus M \to R] \xmapsto{dom} R \oplus M$, and $R \xmapsto{\Omega^1} [R \oplus {}^{I}/{}_{I^2} \to R]$.

Definition 4.2.18. [Pau] In category C, let $[M \to A] \in Ob(Mod(A))$. A section $D : A \to M$ is usually called a ($M-$valued) derivation and the set of $M-$valued derivations is denoted by $Der(A,M)$.

Remark 4.2.19. If there exist a cotangent module Ω^1_A over A (i.e. the universal module over A such that for every module M over A, $M \to A$, there exist a unique map $M \to \Omega^1_A$ such that the diagram
$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & \Omega^1_A \arrow[dl] \\ & A & \end{tikzcd}$$
commutes) then $Der(A, M) \cong Hom_{Mod(A)}(\Omega^1_A, M)$.

Cotangent modules are very general objects in the sense that, some of the familiar objects of study in differential and alegebraic geometry are just particular cotangent modules. The following two examples will provide more clarification:

Example 4.2.20. In differential geometry, Ω^1_A usually appears as a cotangent bundle with following construction: Take three copies of a smooth manifold M. Form the Cartesian product $M \times M$. Let the diagonal map $M \xrightarrow{\triangle} M \times M$ act on M. Define I to be the sheaf of germs of smooth maps. $f : \triangle(M) \to M \times M$ which vanishes on $\triangle(M)$. Form the quotient sheaf ${}^{I}/_{I^2}$ (i.e. the equivalence classes of smooth maps that vanish on diagonal modulo all higher order terms). Pullback ${}^{I}/_{I^2}$ along $\triangle$ i.e. $\triangle^*({}^{I}/_{I^2})$. $\triangle^*({}^{I}/_{I^2})$ is smooth section of the cotangent bundle T^*M which is isomorphic to the differential one-forms i.e. $\Omega^1_M := \triangle^*({}^{I}/_{I^2}) = \Gamma(T^*M) \cong \Omega^1(M)$.

Example 4.2.21. In algebraic geometry, Ω^1_A is a cotangent sheaf with following construction: Take schemes X and S and a morphism $f : X \to S$. Form the fiber product $X \underset{S}{\times} X$. Act the diagonal map on X i.e. $\triangle : X \to X \underset{S}{\times} X$. Define I to be the ideal sheaf of $\triangle(X)$ (i.e. the kernel sheaf of the morphism $O_{X \underset{S}{\times} X} \to f_* O_{\triangle(X)}$ that is induced from morphism $\triangle(X) \to X \underset{S}{\times} X$). Form the quotient sheaf ${}^{I}/_{I^2}$. The pullback sheaf $\triangle^*({}^{I}/_{I^2})$ is the cotangent sheaf $\Omega_{{}^{X}/_{S}}$. If X and S are affine schemes, then $\Omega_{{}^{X}/_{S}}$ is the module of Kahler differentials.

4.3 Thickening and Jet Functors

Definition 4.3.1. [Pau] For category C, the first thickening category which is denoted by Th^1C is the category of torsos $(X \to Z,\ Y \underset{Z}{\times} X \xrightarrow{\rho} X)$ in the arrow category $[I, C]$ over abelian group objects $(Y \to Z, \mu, \underline{0}) \in Ob(TC)$.

Definition 4.3.2. [Pau] The $n-$th thickening category, Th^nC, is the category of objects $X' \to Z$ in $[I, C]$ such that there exist a sequence $X' = X_n \to X_{n-1} \to \ . \ . \ . \ \to X_0 = Z$ where $X_i \to X_{i-1} \in Ob(Th^1C)$.

Definition 4.3.3. [Pau] For $X \in Ob(C)$ and forgetful functor

$$\text{forg} : Th^nC_X \longrightarrow C_X$$

$$[X_n \to X_{n-1} \to \ . \ . \ . \ \to X_0 = X] \to [X_n \to X]$$

a left adjoint functor to forg is called an infinitesimal functor and is denoted by $Th^n : C_X \to Th^nC_X$.

Definition 4.3.4. Let $\triangle^*$ denotes the codiagonal map i.e. for every $X \in Ob(C)$, $\triangle^*(X) := [X \sqcup X \to X]$. The functor $Jet^n := Th^n o \triangle^* : C \to Th^nC$ is called the jet functor. We assume that the category C accepts finite (co)products.

Proposition 4.3.5. *Suppose that C is a category that admits pullback and $X \in On(C)$. Then there exist a natural isomorphism $Jet^1(X) \xrightarrow{\simeq} \Omega^1_X$ in the category Th^1C_X.*

Definition 4.3.6. Let X and Y be two generalized spaces (i.e. $X, Y \in SH(Build)-, \tau)$ where $Build$ is a category of building blocks for a given geometry) and f a morphism $f : X \to Y$. Then the relative $k-$th jet space, $Jet^k(X/_Y)$ is defined as the space $Jet^k(X/_Y) := X \underset{Y}{\times} Jet^k(Y)$.

Definition 4.3.7. For every two objects $U, V \in Ob(Build)$, an infinitesimal thickening object is a morphism $U \to V$ is an object in Th^nBuild, for some $n \geq 1$.

Remark 4.3.8. The usual notion of Jets in differential geometry is the following: Let $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ denote the vector space of smooth functions $f : \mathbb{R}^m \to \mathbb{R}^m$ for $k \geq 0$ and $p \in \mathbb{R}^n$, $(f \sim g)$ if $(f - g \equiv 0$ to the $k-$th order, i.e. f and g have the same value at p and all their partial derivates agree at p up to $k-$th order derivations). The $k-$th order Jet space of $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ is them defined as the set of equivalence classes of $\sim$ and is denoted by $J^\infty_p(\mathbb{R}^n, \mathbb{R}^m)$.

Example 4.3.9. In algebraic geometry, a thickening usually refers to a closed immersion of schemes $X \hookrightarrow X'$ whose ideal is nilideal.

One of the prototype examples of thickenings which shows up in algebro-geometric context, is the the formal spectrum:

Example 4.3.10. [FGI$^+$] Formal spectrum of an $I-$adic Noetherian ring A, $SpfA$, is another geometric object whose construction relies on the concept pf thickening: an $I-$adic Noetherian ring is a Noetherian ring A equipped with the powers I^n $(n > 0)$ of an ideal I of A as a fundamental system of open neighbourhoods of zero in A. For $n \in \mathbb{N}$, let $X_n := SpecA_n$ where $A_n := {}^{A}/_{I^{n+1}}$. Affine schemes X_n form an increasing sequence of closed immersions.

$$X_0 := Spec\ {}^{A}/_{I} \hookrightarrow X_1 := {}^{A}/_{I^2} \hookrightarrow\ \ldots\ \hookrightarrow X_n \hookrightarrow\ \ldots$$

They all have the same underlying space, namely $|X_0|$ which will be denoted by $\mathcal{X}$: $\underset{n}{colim}\ Spec\ {}^{A}/_{I^n}$. The colimit is taken in the category of topologically ringed spaces, i.e. objects (X, O_X) where $X \in Ob(Top)$ and O_X is a sheaf of topological rings. The family of structure sheaves $\{O_{X_n}\}$ (each O_{X_n} is the structure sheaf of X_n) is a projective system. Hence we define $O_{\mathcal{X}} := \underset{\substack{\leftarrow \\ n}}{\lim}\ O_{X_n}$. The formal spectrum of A then is defined to be the (topological) ringed space $SpfA := (\mathcal{X}, O_{\mathcal{X}})$. $SpfA$ is an example of a formal scheme.

Definition 4.3.11. [Pau] A sheaf $X \in SH(Build, \tau)$ is called:

1. Formally smooth (resp. formally unramified, resp. formally étale) iff for every infinitesimal thickening $U \to V$, the map $X(V) \to X(U)$ is surjective (resp. injective, resp. bijective);

2. Locally finitely presented iff X commutes with directed limits;

3. Smooth (resp. unramified, resp. étale) iff it is locally finitely presented and formally smooth. (resp. formally unramified, resp. formally étale).

Chapter 5

Categorical Geometric Invariants

Equipped with some categorical calculus tools from the previous chapter, we are now able to introduce some categorical differential geometric notions.

Theorem 5.0.1. *[Pau]Let $C = (Mod(\mathbb{Z}), \otimes)$ be the monoidal category of abelian groups. Then:*

1. *The category $Th^n CALG_C$ is the subcategory of $[I, C]$ whose objects are quotient morphisms $[A \to {}^{A}/_{J}]$ i.e. $[A \to {}^{A}/_{J^n} \to {}^{A}/_{J^{n-1}} \to \ \ldots \ \to {}^{A}/_{J}]$ where $A \in CALG_{Mod(\mathbb{Z})} = CRing$, with kernel being the nilpotent ideal J of order $n+1$.*
2. *The jet functor $Jet^n : CALG_C \to Th^n CALG_C$ is given by the jet algebra $Jet^n(X) = \frac{X \otimes X}{J^{n+1}}$ where $X \in Mod(\mathbb{Z})$ and J is the kernel of the multiplication map $X \otimes X \to X$.*

Definition 5.0.2. [Pau] Let A be a commutative monoid. Then the $A-$module of vector fields is defined by $\theta_A := \underline{Hom}_{Mod_A}(\Omega^1_A, A)$.

Definition 5.0.3. [Pau] The A-module of n-th order differential operators is defined by $D^n_A := \underline{Hom}_{Mod_A}(Jet^n(A), A)$.

The tensor structure on D^n_A is defined as follows: For every $D_1, D_2 \in D^n_A,\ D_1 \otimes D_2 \mapsto D_1\ o\ D_2 := D_1\ o\ d_1\ o\ D_2$ where $d_1 : A \to Jet^n(A)$ is the section of $Jet^n(A) \to A$ and $Jet^n(A) \xrightarrow{D_2} A \xrightarrow{d_1} Jet^n(A) \xrightarrow{D_1} A$.

Definition 5.0.4. The $A-$module of differential operators is defined by $D_A : \varinjlim_n D_A^n$.

Definition 5.0.5. [Pau] Let $M \in Mod_A$ where $A \in CALG_C$ for a category C. The inner derivation object denoted by $\underline{Der}(A, M)$, is defines as the equilizer of morphisms $\underline{Hom}(A, M) \overset{\beta}{\underset{\alpha}{\rightrightarrows}} \underline{Hom}(A \otimes A, M)$ where $\alpha(D) = D\ o\ \mu$ and $\beta(D) = \mu_M^l\ o\ (id_A \otimes D) + \mu_M^r\ o\ (D \otimes id_A)$ for every $\underline{Hom}(A, M)$.

Note that in the above definition, $\mu : A \otimes A \to A$, $\mu_M^l : A \otimes M \to M$ and $\mu_M^r : M \otimes A \to M$.

In what follows, we will assume that $(C, \otimes) = (Mod(K), \otimes)$ where K is a commutative (unital) ring and $A \in CALG_C$.

Definition 5.0.6. A lie bracket on module θ_A is defined by $[-,-] : \theta_A \otimes \theta_A \to \theta_A,\ (f, g) \mapsto [f, g] := fg - gf$ for $f, g \in \theta_A$.

Note that $Hom(\Omega_A^1, A) \cong Der(A, A) \subset Hom(A, A)$. So the above definition is meaningful.

There is a natural action $[-,-] : \theta_A \otimes A \to A$ defined by $(f, a) \mapsto [f, a] := f(a)$ for $a \in A$ and $f \in \theta_A$.

Every $f \in \theta_A$ induces a derivation $\partial \in \underline{Der}(A, A)$ defined by $\partial := [f, -] : A \to A$.

Definition 5.0.7. A Lie algebroid over A is an $A-$module L equipped with a Lie bracket $[-,-] : L \otimes L \to L$ and an anchor map $\tau : L \to \theta_A$ such that for every $x, y, z \in L$ and $a \in A$, the following conditions are satisfied:

1. $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ (Jacobi's identity).
2. $[x, y] = -[y, x]$ (anti-commutativity).
3. $[x, ay] = (\tau(x))(a)y + a[x, y]$ (compatibility of anchor with bracket).

Definition 5.0.8. For any Lie algebroid L, an $L-$module is defined as an $A-$module M with an action $L \times A \to A$ in the following sense: for every $a \in A$, $m \in M$ and $x \in L$; $x(am) = x(a)m + a(x(m))$.

Remark 5.0.9. The Lie bracket $[-,-] : \theta_A \otimes \theta_A \to \theta_A$ defines a Lie algebroid structure on θ_A.

Definition 5.0.10. Commutative monoid A is called smooth if Ω^1_A is a projective $A-$module of finite type.

Proposition 5.0.11. *[Pau] Let A be smooth. Then D_A is the enveloping algebra of the Lie algebroid θ_A i.e. if B is an $A-$algebra in $(C, \otimes)$ equipped with and $A-$linear map $i : \theta_A \to B$, then there exist a unique morphism $D_A \to B$ that extends the map i.*

Definition 5.0.12. A left D_A-module in $(C, \otimes)$ is an object $M \in C$ that is equipped with a left multiplication morphism $\mu^l_{D_A} : D_A \otimes M \to M$ which is compatible with multiplication in D_A.

Definition 5.0.13. A graded $A-$module M equipped with a (D_A-)linear morphism $d : M \otimes D_A \to M \otimes D_A[1]$ is called a differential complex.

Proposition 5.0.14. *[Pau] Let $A \in CALG_C$ be smooth. Then the category of (left) D_A-modules is equipped with a symmetric monoidal structure defined by $M \otimes N := M \underset{A}{\otimes} N$ and the D_A- module structure is induced by the action of derivations $\partial \in \underline{Der}(A, A)$ by $\partial(m \otimes n) = \partial(m) \otimes n + m \otimes \partial(n)$ for every $m \in M$ and $n \in N$.*

The symmetric monoidal category of differential complexes is denotes by $(DiffMod_g(A), \otimes)$

Definition 5.0.15. The algebra of differential forms on A is a free algebra in $(DiffMod_g(A), \otimes)$ on the differential $d : A \to \Omega^1_A$, given by the symmetric algebra $\Omega^*_A := Sym_{DiffMod_g(A)}([A \xrightarrow{d} \Omega^1_A])$.

Proposition 5.0.16. *[Pau] The natural map $\theta_A \to \underline{Hom}(\Omega^1_A, A)$ extends to a morphism $i : \theta_A \to Hom_{DiffMod_g(A)}(\Omega^*_A, \Omega^*_A[1])$ which is called the inner product map. The map i can be depicted diagrammatically as below:*

$$(\Omega^1_A \xrightarrow{f} A) \quad \xmapsto{\ i\ } \quad \left(\begin{array}{ccccccccc} A & \longrightarrow & \Omega^1_A & \longrightarrow & S^2\Omega^1_A & \longrightarrow & S^3\Omega^1_A & \longrightarrow & \dots \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} & & \\ 0 & \longrightarrow & A & \longrightarrow & S^1\Omega^1_A & \longrightarrow & S^2\Omega^1_A & \longrightarrow & \dots \end{array}\right)$$

Proposition 5.0.17. *[Pau] The natural map $\theta_A \to \underline{Hom}(A, A)$ extends to a morphism $L : \theta_A \to Hom_{DiffMod_g(A)}(\Omega_A^*, \Omega_A^*)$ which is called the Lie derivative. The map L diagrammatically looks as follows:*

$$(A \xrightarrow{f} A) \qquad \xmapsto{\mathcal{L}} \qquad \left(\begin{tikzcd} A \arrow[r] \arrow[d, "\mathcal{L}_0"'] & \Omega_A^1 \arrow[r] \arrow[d, "\mathcal{L}_1"'] & S^2\Omega_A^1 \arrow[r] \arrow[d, "\mathcal{L}_2"'] & S^3\Omega_A^1 \arrow[r] \arrow[d, "\mathcal{L}_3"'] & \dots \\ A \arrow[r] & \Omega_A^1 \arrow[r] & S^2\Omega_A^1 \arrow[r] & S^3\Omega_A^1 \arrow[r] & \dots \end{tikzcd} \right)$$

Definition 5.0.18. For $k \leq \infty$, $C^k - AFF$ denotes the category whose objects are the affine spaces $\mathbb{R}^n$ (for varying $n \geq 0$), and morphisms being the C^k-maps between them. $C^k - AFF$ has finite products.

Definition 5.0.19. [Pau] A product-preserving functor $F : C^k - AFF \to \text{Set}$ is called C^k-algebra. The category of C^k-algebras is denoted by Alg_{C^k}.

Note that the category $Open_{C^k}$ (the category of open subsets of $\mathbb{R}^n$ for varying n with C^k-maps between them as morphisms) fully-faithfully embed into Alg_{C^k} by

$$Open_{C^k}^{op} \longrightarrow Alg_{C^k}$$

$$U \mapsto Hom(U, -) : C^k - AFF \to \text{Set}.$$

Definition 5.0.20. Let M be a smooth manifold. Then the smooth algebra of functions on M is defined by $C^\infty(M) := C^\infty(M, -)$:

$$C^\infty - AFF \to \text{Set}$$

$$\mathbb{R}^n \mapsto C^\infty(M, \mathbb{R}^n).$$

Theorem 5.0.21. *[Pau] $C^\infty(J^nM, -) \cong Jet^n(C^\infty(M, -))$ where J^nM is the jet space (bundle) of smooth functions M, and $Jet^n(C^\infty(M, -))$ is the $n-th$ jet functor on smooth algebra of functions on M.*

Chapter 6

Geometry of Sheaves on Sites

6.1 Sheaves on Sites, and Varieties

By *Build*, we mean a category where its objects are building blocks for a given geometry which we want to study. For example, for differential geometry, $Ob(Build) = Ob(Open_{C^\infty})$ with smooth maps as morphisms.

Some notations:

- $(Build, \tau)$ denotes a site,
- $\widehat{Build} \coloneqq Psh_{Build} \coloneqq Fct(Build^{op}, Set)$,

The category of $Open_{C^\infty}$ is neither complete nor cocomplete, i.e., it lacks enough limits and colimits. The first step towards resolving this issue was to use the Yoneda embedding, $Build \overset{\mathrm{h}}{\hookrightarrow} Fct(Build^{op}, Set)$ where the latter category is (co)complete. However, $Fct(Build^{op}, Set)$ still has one drawback: The image of Yoneda embedding functor does not preserve finite colimits. For example, in category $Open_{C^\infty}$, $V_1 \cup V_2 = V_1 \sqcup_{V_1 \cap V_2} V_2$ for $V_1, V_2 \in Open_{C^\infty}$ but the morphism $\underline{V_1} \sqcup_{\underline{V_1 \cap V_2}} \underline{V_2} = \underline{V_1 \sqcup_{V_1 \cap V_2} V_2}$ in $\widehat{Build}$ is not an isomorphism in general. The notion of sheaf on sites was invented to resolve this issue.

Definition 6.1.1. The category of sheaves on site $(Build, \tau)$ which is denoted by $Sh(Build, \tau)$ is called a (Grothendieck) topos.

Definition 6.1.2. An object $X \in Ob(Sh(Build, \tau))$ is called a generalized space.

Let $(Build, \tau)$ be a site, $X, Y \in Ob(Sh(Build, \tau))$, and $f : X \to Y$ a morphism between them. Then:

Definition 6.1.3. [Pau] f is called an open embedding iff f is pointwise injective (i.e. $f_U : X(U) \rightarrowtail Y(U)$, $\forall U \in Ob(Build)$), and if X and Y are representable, then there exists an open covering of Y, $\{U_i \xrightarrow{f_i} Y\}_i \in Cov(Y)$ in τ such that f is the image of a morphism $\sqcup_i \underline{U_i} \xrightarrow{\psi} Y$. Otherwise, if X is not representable or Y is not representable, then for every map $\underline{U} \to Y$ ($\underline{U}$ is representable), the fiber $X \underset{Y}{\times} \underline{U_X}$ is isomorphic to an embedding $\underline{W} \subset \underline{U}$.

Diagrammatically the above definition can be seen as below when X and Y are representable:

$$\begin{tikzcd}
\bigsqcup_i U_i \arrow[r, "\psi"] & Y \arrow[r, "\phi"] & Coker(\psi). \\
 & X \cong Im(\psi) = Ker(\phi) \arrow[u, "f"'] &
\end{tikzcd}$$

Now we introduce an important class of a topos $Sh(Build, \tau)$, where its objects correspond to objects of a usual category in which we do geometry:

Definition 6.1.4. [Pau] (Generalized) space X is called a variety iff X can be covered by a family of open embeddings, i.e., there exists a family of open embeddings $f_i : U_i \to X$ where $U_i \in Sh(Build, \tau)$ are representable, and the map $\psi : \sqcup_i U_i \to X$ is a sheaf epimorphism (i.e., ψ can be canceled from the right).

The category of varieties is denoted by $VAR(Build, \tau) \subset Sh(Build, \tau)$.

Definition 6.1.5. If $Build = Open_{C^k}$ for $k \leq \infty$ (i.e., the category of open subsets of $\mathbb{R}^n$ for varying n with C^k-maps between as morphisms), and τ is usual topology, then $VAR(Open_{C^k}, \tau)$ is called C^k-manifolds.

Definition 6.1.6. If $Build = CRing^{op}$ (i.e., the category of affine schemes), and τ is the Zariski covering $\{Spec\, A_i \to Spec\, R\}_i \in Cov(Spec\, R)$ where $R \in CRing$, $A_i = R[r_i^{-1}]$ for $r_i \in R$, $Spec\, R \in CRing^{op}$, then $VAR(CRing^{op}, \tau)$ is called schemes.

6.2 Generalized Spaces of Symmetric Monoidal Categories

We know that an affine scheme is an object in category $CRing^{op}$, and $CRing$ is the category of commutative monoids (algebras) in the symmetric monoidal category $(Mod(\mathbb{Z}), \otimes)$, i.e. $CRing = CALG_{Mod(\mathbb{Z})}$. This fact can leads to an idea, which generalize the notion of schemes for every symmetric monoidal category:

Definition 6.2.1. Let $(C, \otimes)$ be a symmetric monoidal category, $Build_C \coloneqq CALG_C^{op}$ and $X \in Ob(CALG_C^{op})$. The spectrum of X, denoted by $\underline{\text{Spec}}$(X) is a functor defined by

$$\begin{aligned} \underline{Spec}(X) : Build_C^{op} = & CALG_C \to Set \\ & Y \to Hom(X, Y). \end{aligned}$$

Definition 6.2.2. [Pau] For $X, Y \in Ob(CALG_C)$, an algebra morphism $f : X \to Y$ or its corresponding morphism $\underline{Spec}(f) : \underline{Spec}(Y) \to \underline{Spec}(X)$ is called:

1. monomorphism iff $\forall Z \in Ob(CALG_C)$, $\underline{Spec}(Y)(Z) \subset \underline{Spec}(X)(Z)$;
2. flat iff the base change functor $- \otimes_X Y : Mod_X \to Mod_Y$ is left exact, i.e., commutes with finite limits;
3. finitely presented iff $\underline{Spec}_X(Y)$ denotes $\underline{Spec}(Y)$ restricted to X-algebra commutes with filtered colimits.

Definition 6.2.3. [Pau] Morphism $\underline{Spec}(f) : \underline{Spec}(Y) \to \underline{Spec}(X)$ is called Zariski open iff it is a flat, finitely presented monomorphism.

Definition 6.2.4. [Pau] A family of morphisms $\{\underline{Spec}(X_i) \xrightarrow{f_i} \underline{Spec}(X)\}$ is called a Zariski covering iff f_i is Zariski open for every i, and there exists a finite subset $J \subset I$ such that the functor $\sqcap_{j\in J} - \otimes_X X_j : Mod_X \to \sqcap_{j\in J} Mod_{X_j}$ preserves isomorphisms.

Definition 6.2.5. The Grothendieck topology τ that is generated by Zariski coverings on $Build_C$ is called Zariski topology.

Definition 6.2.6. [Pau] $VAR(CALG_C^{op}, \tau)$ is called schemes.

Remark 6.2.7. [Pau] Usual schemes are schemes of the symmetic monoidal category $(Mod(\mathbb{Z}), \otimes)$, i.e., $X \in VAR(CRing^{op}, \tau)$.

6.3 Diffeologies and Differential Geometric Constructions

Definition 6.3.1. A usual smooth manifold is a topological space X, equipped with an atlas (i.e., a family of open embeddings $\{f_i : U_i \to X\}_i$ with $U_i \subseteq \mathbb{R}^n$ open subsets) such that for every U_i, U_j, the transition map $\psi_{U_iU_j} := f_j^{-1} \circ f_i : U_j \cap f_i^{-1}\big(f_j(U_j)\big) \to U_i \cap f_j^{-1}\big(f_i(U_i)\big)$ is smooth (i.e., $\psi_{U_iU_j} \in Mor(Open_{C^\infty})$).

Each atlas is included a maximal one. A morphism of manifolds X and Y is a continuous map which induces a morphism of maximal atlases as can be seen form the following diagram:

$$\begin{tikzcd} U_i \arrow[r, "f_i"] \arrow[d, "\phi_i := g_i^{-1}\phi f_i"'] & X \arrow[d, "\phi."] \\ V_i \arrow[r, "g_i"'] & Y \end{tikzcd}$$

We will denote the category of (usual) smooth manifolds by Mfd_{C^∞}.

Definition 6.3.2. The topos $Sh(Open_{C^\infty}, \tau)$ is called diffeologies.

Now we show how a differential geometric construction on generalized spaces in $Sh(Build, \tau)$ can be derived from a construction on site $(Build, \tau)$.

Definition 6.3.3. Let C be a category and $(Build, \tau)$ a site. A (differential geometric) construction on $Build$ is a sheaf of Ω on site $(Build, \tau)$ with values in C, $\Omega : Build^{op} \to C$.

Construction on $Sh(Build, \tau)$**:**[Pau] Let $X \in Ob(Sh(Build, \tau))$ be a generalized space, $Build_X$ denotes the category whose objects are $Ob(Build_X) = \{x : \underline{U_x} \to X | U_x \in Ob(Build)\}$, and $\Omega : Build^{op} \to C$ be a construction. Then the differential geometric construction on $Sh(Build, \tau)$ is defined as $\Omega(-) : Sh(Build, \tau) \to C$, $X \mapsto \Omega(X) \coloneqq \varprojlim_{x \in Build_X} \Omega(U_x)$. We assumed that the limit exists.

In general, if $\underline{U} \to X$ is an open embedding of generalized spaces, $\Omega(U) \in C$ can be defined.

Example 6.3.4. Suppose $(Build, \tau) = (Open_{C^\infty}, \tau)$ where τ is the usual topology. Let $\Omega \coloneqq \Omega^1 : Open_{C^\infty}^{op} \to \mathbb{R}-Vect$, $U \mapsto \Omega^1(U) = \{\omega : U \to T^\star U\}$ be the sheaf of differential one-forms field (i.e., sections of $P : T^\star U \to U$). For diffeology $X \in Ob\big(Sh(Open_{C^\infty}, \tau)\big)$, $X : Open_{C^\infty} \to$Set, a differential one-form field on X is $\Omega^1(X) \coloneqq \varprojlim_{x} \Omega^1(U_x)$, denoted by $x^\star\omega$ such that if $f : x \to y$ is a morphism in $Build_X$, then we have $f^\star(y^\star\omega) = x^\star\omega$. f is shown in the following commutative diagram

$$\begin{tikzcd} \underline{U_x} \arrow[rr, "f"] \arrow[dr, "x"'] & & \underline{U_y}. \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

Finally, there is a relationship between the category of smooth manifolds, and the category of diffeologies.

Theorem 6.3.5. *The category Mfd_{C^∞} fully faithfully embed into category $Sh(Open_{C^\infty}, \tau)$ via the map*

$$\begin{aligned} Mfd_{C^\infty} &\to Sh(Open_{C^\infty}, \tau) \\ M &\to Hom(-, M). \end{aligned}$$

Moreover, this map induces an equivalence between categories Mfd_{C^∞} and $VAR(Open_{C^\infty}, \tau)$.

Proof. The category Mfd_{C^∞} is equipped with the global topology τ and form the site $(\text{Mfd}_{C^\infty}, \tau)$. The Yoneda embedding functor $h : \text{Mfd}_{C^\infty} \to Sh(\text{Mfd}_{C^\infty}, \tau)$ embeds Mfd_{C^∞} fully faithfully into $Sh(\text{Mfd}_{C^\infty}, \tau)$. Let $N \in Ob(\text{Mfd}_{C^\infty})$, and $\Phi := \{U_i \to N\}_i \in Cov(N)$ where $U_i \in Ob(Open_{C^\infty})$. We form the nerve $N(\Phi)$ and note that for $P_j := U_{i_1} \times_N U_{i_2} \times_N \cdots \times_N U_{i_n}$, $j := (i_1, \cdots, i_n)$; $P_j \in Ob(Open_{C^\infty})$ for all j. Moreover, $colim_j P_j = N$. The inclusion map $Open_{C^\infty} \hookrightarrow \text{Mfd}_{C^\infty}$ induces the natural functor $\eta : Sh(\text{Mfd}_{C^\infty}, \tau) \to Sh(Open_{C^\infty}, \tau)$ given by $X \mapsto X_{|Open_{C^\infty}}$, where $X_{|Open_{C^\infty}}$ is the restriction of sheaf X to $Open_{C^\infty}$. Applying the Yoneda lemma and above facts, for every $N \in \text{Mfd}_{C^\infty}$, we get $X(N) \cong Hom(\underline{N}, X) \cong Hom(\underline{colim_j P_j}, X) \cong lim_j Hom(\underline{P_j}, X) \cong lim_j X(P_j)$. This means that $X(N)$ is determined by values of X at P_j, and η is fully faithful. The composition functor $\eta h : \text{Mfd}_{C^\infty} \hookrightarrow Sh(Open_{C^\infty}, \tau)$ is the required map.

For the second part of the proof, we first observe that the fact in both categories $Sh(\text{Mfd}_{C^\infty}, \tau)$ and $Sh(Open_{C^\infty}, \tau)$, varieties are colimits of nerves for the same topology with same covering objects, i.e., objects in $Open_{C^\infty}$. Therefore, $Sh(\text{Mfd}_{C^\infty}, \tau) \supset VAR(\text{Mfd}_{C^\infty}, \tau) \simeq VAR(Open_{C^\infty}, \tau) \subset Sh(Open_{C^\infty}, \tau)$. Using the above equivalent of categories and replace $VAR(Open_{C^\infty}, \tau)$ with $VAR(\text{Mfd}_{C^\infty}, \tau)$, we finish the proof by showing that the map $\psi : (\text{Mfd}_{C^\infty}, \tau) \to VAR(\text{Mfd}_{C^\infty}, \tau)$ is an equivalent. We already showed that ψ is fully faithful. So it only remains to prove that ψ is essentially surjective.

Suppose that $X \in VAR(\text{Mfd}_{C^\infty}, \tau)$. By definition of variety, there exists a family $\{\underline{U_i} \to X\}_i$ of open embeddings where $U_i \in Open_{C^\infty}$ and $\sqcup_i \underline{U_i} \to X$ is a sheaf epimorphism. So $X = colim_i \underline{U_i}$ in $VAR(\text{Mfd}_{C^\infty}, \tau)$. We define $Y := colim_j P_j \in Ob(\text{Mfd}_{C^\infty})$. Using the fact that nerves of coverings in $VAR(\text{Mfd}_{C^\infty}, \tau)$ correspond to the colimits in the category Mfd_{C^∞}, we conclude that X will be correspond to Y, and $\underline{Y} \cong X$. So ψ is essentially surjective.

$\square$

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