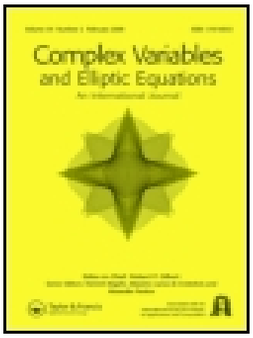

Complex Variables and Elliptic Equations

ISSN: 0278-1077 (Print) 1563-5066 (Online) Journal homepage: https://www.tandfonline.com/loi/gcov19

On the zero distributionof remainders of entire power series

I.V. Ostrovskll

To cite this article: I.V. Ostrovskll (2001) On the zero distributionof remainders of entire power series, Complex Variables and Elliptic Equations, 43:3-4, 391-397, DOI: 10.1080/17476930108815328

To link to this article: https://doi.org/10.1080/17476930108815328

Published online: 26 Jun 2007.

Complex Variables, Vol. 43, pp. 391–397
Reprints available directly from the publisher
Photocopying permitted by license only

On the Zero Distribution of Remainders of Entire Power Series

I. V. OSTROVSKII*

Department of Mathematics, Bilkent University, 06533 Bilkent, Ankara, Turkey and B. Verkin Institute for Low Temperature Physics and Engineering, 47 Lenin ave, 61164 Kharkov, Ukraine

Communicated by G. C. Wen

Dedicated to the memory of Chi-tai Chuang

(Received 4 January 2000)

It has been shown by the author that, if all remainders of the power series of an entire function f have only real positive zeros, then $\log M(r,f) = O((\log r)^2)$, $r \to \infty$. The main results of the paper are the following: (i) if at least two different remainders have only real positive zeros, then $\log M(r,f) = O(\sqrt{r})$, $r \to \infty$; (ii) this estimate cannot be improved even in the case if one replaces two by any given finite number of remainders.

Keywords: Angular zero distribution; Entire function; Power series; Remainder

AMS Classification Categories: 30D20, 30D10

1. INTRODUCTION

Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{1}$$

*Supported in part by INTAS Grant No. 96-0858.

be a power series with infinite radius of convergence. Let

$$r_n(z) = \sum_{k=n}^{\infty} a_k z^k \tag{2}$$

be its nth remainder.

In [3] the following theorem has been proved.

THEOREM A *If, for all sufficiently large n, the remainders r_n have only real nonnegative zeros, then*

$$\log M(r,f) = O((\log r)^2), \quad r \to \infty.$$

This bound is the best possible in the sense of order.

On the other hand, the following theorem holds.

THEOREM B *If there exist two values, n_1 and n_2, such that the remainders r_{n_1} and r_{n_2} are different and have only real nonnegative zeros, then*

$$\log M(r,f) = O(\sqrt{r}), \quad r \to \infty. \tag{3}$$

This bound is the best possible in the sense of order.

In a weaker form, this theorem has been proved in [3]; in the above formulation, we will prove it here.

The question arises what will happen when one considering other sets of values of n (than those mentioned in Theorems A and B), for which remainders r_n have only real nonnegative zeros. We are going to prove that any finite set of values of n does not imply any better bound, than (3). More precisely, we will prove the following theorem.

THEOREM C *Let $N \geq 2$ be an arbitrary integer. There exists an entire function f satisfying*

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{\sqrt{r}} > 0 \tag{4}$$

and such that the remainders $r_0, r_1, r_2, \ldots, r_N$ are different and have only real nonnegative zeros.

2. MAIN LEMMA

Denote by **T** the class of all real entire transcendental functions f satisfying the condition: there exists a constant $\alpha_f > 0$ such that all roots of the two equations:

$$f(z) - \alpha_f = 0, \quad f(z) + \alpha_f = 0 \tag{5}$$

are positive.

LEMMA *If $f \in \boldsymbol{T}$, then there exists a real constant $\beta \neq 0$ such that $zf(z) + \beta \in \boldsymbol{T}$.*

Proof We will use the following properties of a function $f \in \mathbf{T}$:

(i) f is of order not greater than 1/2.
(ii) Zeros $\{a_k\}_{k=1}^{\infty}$ of f and zeros $\{b_k\}_{k=1}^{\infty}$ of f' are positive simple and interlace, *i.e.*,

$$0 < a_1 < b_1 < a_2 < b_2 < a_3 < \cdots$$

(iii) $|f(b_k)| \geq \alpha_f$ for $k = 1, 2, \ldots$

These properties can be found in [2, §1] where reality instead of positivity of roots of (5) is considered.

If $f \in \mathbf{T}$, then $f(z^2)$ belongs to the class considered in [2] therefore (i)–(iii) follow easily.

Without loss of generality we can assume that $f(0) > 0$. Then f is decreasing on

$$(-\infty, b_1] \bigcup [b_2, b_3] \bigcup [b_4, b_5] \bigcup \cdots$$

and is increasing on

$$[b_1, b_2] \bigcup [b_3, b_4] \bigcup [b_5, b_6] \bigcup \cdots$$

Evidently (see Fig. 1), for sufficiently small $\varepsilon > 0$, the number of roots of the equation

$$f(x) = \frac{\varepsilon}{x}$$

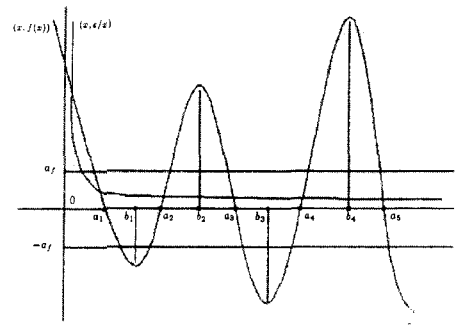

FIGURE 1

lying on the interval $[0, b_k]$, $k = 1, 2, \ldots$, is not less than $k+1$. Hence the function $zf(z) - \varepsilon$ has at least $k+1$ zeros on $[0, b_k]$, $k = 1, 2, \ldots$.

Let us consider the rectangle

$$Q_{k,R} = \{z : |\mathrm{Re} z| < b_k, |\mathrm{Im} z| < R\}, \quad k = 1, 2, \ldots$$

and show that, for k large enough, we can choose R so large that the following inequality holds

$$|zf(z)| > \varepsilon, \quad \text{for } z \in \partial Q_{k,R}. \tag{6}$$

The Hadamard product representation

$$f(z) = f(0) \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \tag{7}$$

implies that $|f(x+iy)|$ is an increasing function of $|y|$ for any fixed x. Since $|f(b_k)| \geq \alpha_f$, $|f(-b_k)| > f(0)$, this implies that $|f(\pm b_k + iy)| \geq \min(\alpha_f, f(0))$ for all real y and hence $|zf(z)| > \varepsilon$ on the lines $\{z : \mathrm{Re} z = \pm b_k\}$ for large enough k. Since (7) implies also that $|f(x+iy)| \to \infty$ as $|y| \to \infty$ uniformly on any compact set of values of x, we can choose R so large that $|zf(z)| > \varepsilon$ on the horizontal sides of $Q_{k,R}$. Thus, (6) holds.

Since the number of zeros of $zf(z)$ in $Q_{k,R}$ is equal to $k+1$, Rouché's theorem implies that all zeros of $zf(z) - \varepsilon$ in $Q_{k,R}$ are positive. We conclude that all zeros of $zf(z) - \varepsilon$ are positive for all sufficiently small ε.

Let us fix such an ε and consider $F(z)=zf(z)+\beta$ with $\beta=-\varepsilon/2$. Then all zeros of both functions $F(z)-|\beta|/2=zf(z)-3\varepsilon/4$ and $F(z)+|\beta|/2=zf(z)-\varepsilon/4$ are positive. Thus, $F\in\mathbf{T}$. ■

3. PROOF OF THEOREM C

Firstly, let us do two following remarks. (i) If $f(z)$ and $r_n(z)$ are defined by (1) and (2) respectively, then the 'normalized' remainders

$$t_n(z) := z^{-n}r_n(z), \quad n=0,1,2,\ldots$$

satisfy the equation

$$t_{n-1}(z) = zt_n(z) + a_{n-1}, \quad n=1,2,\ldots$$

(ii) Theorem C will be proved if we construct an entire function f satisfying (4) and such that $t_0, t_1, t_2, \ldots, t_N$ have only nonnegative zeros.

Let us choose as t_N any entire function belonging to $\mathbf{T}$ and satisfying (4). For instance, we can choose $t_N(z)=\cos\sqrt{z}$. Then, by the main lemma, we choose a real constant $a_{N-1}\neq 0$ such that $t_{N-1}(z)=zt_N(z)+a_{N-1}$ belongs to $\mathbf{T}$. Using the lemma once again, we choose $a_{N-2}\neq 0$ such that $t_{N-2}(z)=zt_{N-1}(z)+a_{N-2}$ belongs to $\mathbf{T}$ and so on. Repeating these reasonings, we conclude that the function

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{N-1}z^{N-1} + z^N t_N(z)$$

has the desired properties. ■

4. PROOF OF THEOREM B

We will prove a more general result. To formulate it, let us consider a finite system of rays

$$D = \bigcup_{j=1}^{p}\{z : \arg z = \alpha_j, 0\le |z| < \infty\},$$
$$0\le \alpha_1 < \alpha_2 < \cdots < \alpha_p < 2\pi,$$

and set

$$\gamma = \min\{\alpha_{j+1} - \alpha_j : j = 1, 2, \ldots, p\}, \quad \alpha_{p+1} = \alpha_1 + 2\pi.$$

THEOREM D *Let $f(z)$ be an entire function. Assume that there exist two different polynomials, P_1 and P_2, such that all but a finite number of zeros of functions*

$$f - P_m, \quad m = 1, 2,$$

are located on the system D. Then

$$\log M(r, f) = O(r^{\pi/\gamma}), \quad r \to \infty.$$

We obtain Theorem B by applying Theorem D to the system D consisting of one positive ray ($\gamma = 2\pi$) and to

$$P_m(z) = \sum_{k=0}^{n_m - 1} a_k z^k, \quad m = 1, 2.$$

To prove Theorem D, we apply the following result (see [1, Ch. VI, §2, Theorem 2.4]):

THEOREM *Let F be a meromorphic function such that all but a finite number of its zeros and poles are located on a system D. If F has a finite nonzero Borel exceptional value, then*

$$T(r, F) = O(r^{\pi/\gamma}), \quad r \to \infty.$$

If f is a transcendental entire function satisfying conditions of Theorem D, then all but a finite number of zeros and poles of the meromorphic function

$$F(z) = \frac{f(z) - P_1(z)}{f(z) - P_2(z)}$$

are located on the system D. Moreover, since

$$F(z) - 1 = \frac{P_2(z) - P_1(z)}{f(z) - P_2(z)},$$

the value 1 is Borel exceptional for F. Hence, by the theorem quoted above, we have

$$T(r,F) = O(r^{\pi/\gamma}), \quad r \to \infty.$$

Since $T(r,f) = T(r,F) + O(\log r)$, $r \to \infty$, we get

$$\log M(r,f) \leq 3T(2r,f) = O(r^{\pi/\gamma}), \quad r \to \infty. \qquad \blacksquare$$

References

[1] Goldberg, A. A. and Ostrovskii, I. V., *Distribution of values of meromorphic functions*, Moscow, Nauka, 1970 (Russian).

[2] Marchenko, V. A. and Ostrovskii, I. V. (1975). A characterization of the spectrum of Hill's operator. *Math. USSR Sbornik*, **26**, 493–554.

[3] Ostrovskii, I. V. (1997). Les séries de puissances dont les restes ont seulement des zéros non-positifs. *Comptes Rendus Acad. Sci. Paris*, **325**, 1257–1262.