LEBESGUE-RADON-NIKODYM DECOMPOSITIONS FOR OPERATOR VALUED COMPLETELY POSITIVE MAPS

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Bekir Danış

July, 2014

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Aurelian Gheondea(Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Alexander Goncharov

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Uğur Gül

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

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Bekir Danış
M.S. in Mathematics
Supervisor: Prof. Dr. Aurelian Gheondea
July, 2014

We discuss the notion of Radon-Nikodym derivatives for operator valued completely positive maps on C*-algebras, first introduced by Arveson [1969], and the notion of absolute continuity for completely positive maps, previously introduced by Parthasarathy [1996]. We begin with the definition and basic properties of positive and complete positive maps and we study the Stinespring dilation theorem which plays an essential role in the theory of Radon-Nikodym derivatives for completely positive maps, based on Poulsen [2002]. Then, the Radon-Nikodym derivative definition and basic properties belonging to Arveson is recorded and finally, we study the Lebesgue type decompositions defined by Parthasarathy in the light of the article Gheondea and Kavruk [2009].

Keywords: Radon-Nikodym derivatives, completely positive maps, C*-algebras, absolute continuity, stinespring representation, non-commutative lebesgue decompositions.

ÖZET

OPERATÖR DEĞERLİ TAMAMEN POZİTİF EŞLEMELER İÇİN LEBESGUE-RADON-NİKODYM AYRIŞMALARI

Bekir Danış
Matematik, Yüksek Lisans
Tez Yöneticisi: Prof. Dr. Aurelian Gheondea
Temmuz, 2014

C*-cebirleri üzerinde tanmlı operatör değerli tamamen pozitif eşlemeler için, Arveson [1969] tarafından matematiğe kazandırılmış Radon-Nikodym türevini ve Parthasaraty [1996] tarafından matematiğe kazandırılmş mutlak sürekliliği inceledik. Pozitif ve tamamen pozitif eşlemelerin tanımı ve temel özellikleri ile başladık ve Poulsen [2002] temel alınarak,Radon-Nikodym türevleri için önemli olan Stinespring genleşme teoremini çalıştık. Daha sonra, Arveson'a ait olan Radon-Nikodym türevinin tanımı ve temel özellikleri sunuldu. Son olarak, Gheondea ve Kavruk [2009] makalesinin ışığında, Parthasaraty tarafından tanımlanmış Lebesgue tipi ayrışmaları çalıştık.

Anahtar sözcükler: Radon-Nikodym türevleri, tamamen pozitif operatörler, C*-cebirleri, mutlak süreklilik, stinespring temsili, değişmeli olmayan lebesgue ayrışmaları.

Acknowledgement

Firstly, I would like to express my deepest gratitude to my mother Nezahat and my father Ünal for their great support, understanding and love.

I would like to express my gratitude to my advisor Prof. Dr. Aurelian Gheondea for his excellent guidance. I would like also to thank to my thesis jury members, Assoc. Prof. Dr. Alexander Goncharov and Assist. Prof. Uğur Gül for accepting to read and review the thesis.

My studies in the M.S. program was financially supported by TÜBİTAK through the graduate fellowship program, namely "TÜBİTAK-BİDEB 2210-Yurtiçi Yüksek Lisans Burs Programı". I am grateful to TÜBİTAK for their support.

Finally, I want to thank to my friends Abdullah, Alperen, Burak, Oğuz and Recep for the warm atmosphere that they create.

Contents

Chapter 1

Introduction

The aim of this thesis is to present rigorously a proof and all the necessary ingredients of some results concerning a noncommutative version of the Lebesgue decomposition for operator valued completely positive maps on C^*-algebras, following [1]. In order to fulfill this task, we include preliminary results concerning positivity in C^*-algebras, completely positive maps, the Arveson's Radon-Nikodym derivative and its basic properties, absolute continuity for completely positive maps, the parallel addition, shorted operators and the Lebesgue decomposition of positive bounded operators in a Hilbert space. The importance of this theorem relies on the fact that it permits to develop a comparison theory for operator valued completely positive maps on C^*-algebras that make the basic mathematical object used to model quantum operations, with various applications.

After Arveson defined a generalization of the Radon-Nikodym derivative for operator valued completely positive maps on C^*-algebra in [2], it was natural to follow investigations on the Lebesgue decompositions and, indeed, this was considered by Parthasarathy in [3]. The approach of Parthasarathy was to use an older idea of von Neumann of proving the classical Radon-Nikodym Theorem by techniques of Hilbert space operator theory. What was left unclear was the notion of absolute continuity for completely positive maps that was missing. For positive semidefinite maps on $*$-semigroups there is a notion of absolute continuity defined by Ando and Szymanski in [4] and, starting from here, Gheondea and

Kavruk have clarified the concept of absolute continuity in [1]. On the other hand, previous investigations of Ando [5] have shown that there is a very elegant theory of Radon-Nikodym derivatives and Lebesgue decompositions for bounded positive operators on Hilbert space. The basic approach of Gheondea and Kavruk in [1] is to make a connection, via the Arveson's Radon-Nikodym derivatives, between the comparison theory for completely positive maps and the comparison theory of Ando for bounded positive operators. It is interesting to note that the basic technical ingredients in the theory of Ando, referring to parallel addition and shorted operators, have appeared through some previous investigations of electrical engineers on linear electrical circuits.

In the following we briefly describe the contents of this thesis.

In Chapter 2 and Chapter 3, we discuss the results on positive and completely positive maps by closely following Poulsen [6] and Conway [7]. Then, we study their connections with dilation type results and Chapter 4 is dedicated to the Stinespring dilation theorem.

In Chapter 5, We give the definition and basic properties of Radon-Nikodym derivatives for operator valued completely positive maps, studied by Arveson. Then, we present the notion of absolute continuity for completely positive maps, defined by Gheondea and Kavruk. After the notion of absolute continuity for completely positive maps, we survey the Lebesgue decompositions in Chapter 6, defined by Parthasarathy in [3]. More specifically, we focus on the non-commutative Lebesgue decompositions. The main goal of the thesis is to present a proof of the noncommutative Lebesgue decomposition theorem for operator valued completely positive maps on C*-algebras with all necessary background material from the theory of C*-algebras.

We use a lot of results from the theory of C*-algebras. Hence, we make a collection of these results in Appendix without a proof.

Chapter 2

Positive Maps

Before focusing on completely positive maps, we give the definition and basic properties of positive maps, closely following [6] and [7].

Let $\mathcal{B}$ be a subset of a C*-algebra $\mathcal{A}$. We call $\mathcal{B}$ is self-adjoint if $\mathcal{B} = \mathcal{B}^*$ where $\mathcal{B}^* = \{b^* : b \in \mathcal{B}\}$. When $\mathcal{A}$ is a unital C*-algebra and $\mathcal{B}$ is a self-adjoint subspace of it containing 1, we call $\mathcal{B}$ by operator system.

Remark 2.1. The terminology may lead to the misunderstanding that C^*-algebras consist of only operators but this is not a problem since any C^*-algebra can be embedded into $\mathcal{B}(\mathcal{H})$ by Gelfand-Naimark Theorem C.2.

Example 2.2. Let $\mathcal{A}$ be a C*-algebra and $\mathcal{M}$ be a linear manifold in $\mathcal{A}$ that contains identity. Then $\mathcal{B} = \mathcal{M} + \mathcal{M}^*$ is an operator system.

In whole section, we assume all C*-algebras are unital. Thus, any C*-algebra itself is also an operator system. This enables us to have a plenty of positive elements. For example, if a is a hermitian element in an operator system $\mathcal{B}$, then $a = \frac{1}{2}(\|a\| \cdot 1 + a) - \frac{1}{2}(\|a\| \cdot 1 - a)$, that is, it can be written as difference of two positive elements. Here, it can be noticed that $\|a\| \cdot 1 \pm a \in \mathcal{B}_+$ ($\mathcal{B}_+$ is as defined in Appendix). This proves that linear span of positive elements is the set of hermitian elements. By combining the fact with cartesian decomposition of an element b of $\mathcal{B}$, we can see that the linear span of positive elements of $\mathcal{B}$ is $\mathcal{B}$.

Definition 2.3. Let $\mathcal{B}$ be an operator system and $\mathcal{A}$ be a C*-algebra. If $f : \mathcal{B} \to \mathcal{A}$ is a linear map that sends positive elements of $\mathcal{B}$ to positive elements of $\mathcal{A}$, then we say f is a positive map.

If f is a positive linear functional on an operator system $\mathcal{B}$, then we know $\|f\| = f(1)$. This result is not valid anymore for positive maps since the image of a positive map is a C*-algebra. For positive maps, we have

Proposition 2.4. *Let $\mathcal{B}$ be an operator system and $\mathcal{A}$ be a C*-algebra. If $f : \mathcal{B} \to \mathcal{A}$ is a positive map, then f is bounded and $\|f\| \leq 2\|f(1)\|$.*

Proof. Let b be a self-adjoint element in $\mathcal{B}$. We know

$$b = \frac{1}{2}(\|b\| + b) - \frac{1}{2}(\|b\| - b).$$

Now we apply f to both sides and get

$$f(b) = \frac{1}{2}f(\|b\| + b) - \frac{1}{2}f(\|b\| - b).$$

This means $f(b)$ can be written as difference of two positive elements in $\mathcal{A}$. Hence,

$$\|f(b)\| \leq \frac{1}{2}\max\{\|f(\|b\| + b)\|, \|f(\|b\| - b)\|\} \leq \|b\| \cdot \|f(1)\|.$$

Take an element h from $\mathcal{B}$ and write h as $c + id$ where c and d are self adjoint elements of $\mathcal{B}$. Then,

$$\|f(h)\| \leq \|f(c)\| + \|f(d)\| \leq 2\|h\|\|f(1)\| \Rightarrow \|f\| \leq 2\|f(1)\|.$$

□

In Proposition 2.4, 2 is sharp. To see this, consider the following example.

Example 2.5. Take a manifold $\mathcal{N}$ in $\mathrm{C}(\partial\mathbb{D})$ such that $\mathcal{N} = \{a + bz + c\bar{z} : a, b, c \in \mathbb{C})$ where z is the coordinate function. Now we define a map ϕ from $\mathcal{N}$ to M_2 by

$$\phi(a + bz + c\bar{z}) = \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}.$$

It is easy to verify that $f = a + bz + c\bar{z}$ is a positive map on $\mathbb{D}$ if and only if $c = \bar{b}$ and $a \geq 2|b|$. If f is a positive element of $\mathcal{N}$, then $\phi(f)$ is self-adjoint and it is a positive matrix. In other words, ϕ is a positive map. The conditions in Proposition 2.4 are satisfied. But,

$$2\|\phi(1)\| = 2 = \|\phi(z)\| \leq \|\phi\|.$$

Combining this inequality with Proposition 2.4, we get $\|\phi\| = 2\|\phi(1)\|$.

When the range of a positive map is $\mathrm{C}(X)$ where X is a compact Hausdorff space, the result in positive linear functionals can be obtained since we deal with non-commutativity of the range.

Proposition 2.6. *Let $\mathcal{B}$ be an operator system and $f : \mathcal{B} \to \mathrm{C}(X)$ be a positive map. Then, we have $\|f\| = f(1)$.*

Proof. δ_x denotes the evaluation functional on $\mathrm{C}(X)$ for $x \in X$. Note that $\delta_x \circ f$ is a positive linear functional on $\mathcal{B}$. So, for an element $b \in \mathcal{B}$,

$$\begin{aligned}\|f(b)\| &= \sup\left(|f(b)(x)| : x \in X\right)\\ &= \sup\left(|\delta_x \circ f(b)| : x \in X\right)\\ &\leq \sup\left(\|b\|(\delta_x \circ f)(1) : x \in X\right)\\ &= \|b\|\|f(1)\|.\end{aligned}$$

□

If a positive map is defined on an abelian C*-algebra, we can show that its norm is also $\|f(1)\|$. Before proving the theorem, we need the following lemma.

Lemma 2.7. *Let $\mathcal{A}$ be a C*-algebra and a_i's be positive elements in $\mathcal{A}$ for $i = 1, 2, ..., n$ such that $\sum_i a_i \leq 1$. For any scalars $\lambda_1, \lambda_2, ..., \lambda_n$ with $|\lambda_i| \leq 1$ for $1 \leq i \leq n$, we have*

$$\|\sum_{i=1}^{n} \lambda_i a_i\| \leq 1.$$

Proof. Take the matrix B such that $B_{11} = \sum_{i=1}^{n} \lambda_i a_i$ and other entries are 0. Define M as:

$$M = \begin{bmatrix} \sqrt{a_1} & 0 & \cdots & 0 \\ \sqrt{a_2} & 0 & \cdots & 0 \\ . & . & \cdots & \\ \sqrt{a_n} & 0 & \cdots & 0 \end{bmatrix}$$

Now, we have $B = M^* \left[diag \left\{ \lambda_1, \lambda_2, ..., \lambda_n \right\} \right] M$. The norm of left hand side is $\sum_{i=1}^{n} \lambda_i a_i$ and the norm of each three matrices of right hand side is less than 1. Hence, by comparing the norms of both sides, we get

$$\| \sum_{i=1}^{n} \lambda_i a_i \| \leq 1.$$

□

Theorem 2.8. *Let $f : \mathrm{C}(X) \to \mathcal{A}$ be a positive map where $\mathcal{A}$ is a unital* C*-*algebra and X is a compact Hausdorff space. Then $\|f\| = f(1)$.*

Proof. We can assume $f(1) \leq 1$ by scaling. Our goal is to show $\|f\| \leq 1$. Let $\epsilon \geq 0$ be given and g be an element from $\mathrm{C}(X)$ with $\|g\| \leq 1$. By defining the open covering $\{U_x\}_{x \in X}$ where for each $x \in X$ we define $U_x = \{y \in X \mid |g(y) - g(x)| < \epsilon\}$ and then using the compactness of X, we get the finite open covering. Thus, there exist $U_{x_1}, \ldots, U_{x_n}$ an open covering for $x_i \in X$. Let $u_1, u_2, ..., u_n$ be a partition of unity on X subordinate to the covering. Set $\lambda_i = g(x_i)$ for simplicity. For any x,

$$\begin{aligned} |g(x) - \sum \lambda_i u_i(x)| =& | \sum (g(x) - \lambda_i) u_i(x)| \\ \leq& \sum |g(x) - \lambda_i| u_i(x) \\ <& \sum \epsilon \cdot u_i(x) = \epsilon. \end{aligned}$$

By preceding Lemma 2.7, $\| \sum \lambda_i f(u_i) \| \leq 1$ and this implies

$$\begin{aligned} \|f(g)\| \leq& \|f(g) - f(\sum \lambda_i u_i))\| + \|f(\sum \lambda_i (u_i))\| \\ \leq& \|f(g - \sum \lambda_i u_i) + \| \sum \lambda_i f(u_i)\| \\ <& 1 + \epsilon \cdot \|f\|. \end{aligned}$$

Thus, $\|f(g)\| \leq 1$, that is, $\|f\| \leq 1$. □

Lemma 2.9. *Let $\mathcal{A}$ be a* C**-algebra, $\mathcal{B}$ be an operator system in $\mathcal{A}$ and f be a linear functional on $\mathcal{B}$ such that $f(1) = 1$ and $\|f\| = 1$, that is, f is a unital contraction. When $a \in \mathcal{B}$ and a is a normal element of $\mathcal{A}$, $f(a)$ belongs to the closed convex hull of the spectrum of a.*

Proof. We prove the lemma by contradiction. Firstly, it should be noted that $\sigma(a)$ is a compact set. Additionally, we know the convex hull is the intersection of all closed disks containing the set. Therefore, there exist a λ and $\epsilon > 0$ such that $|f(a) - \lambda| > \epsilon$. The spectrum $\sigma(a)$ of a satisfies the following inclusion:

$$\sigma(a) \subseteq \{z : |z - \lambda| \leq \epsilon\}.$$

The inclusion gives us $\sigma(a - \lambda) \subseteq \{z : |z| \leq \epsilon\}$. We know norm and spectral radius agree for normal operators. Hence, we get while $|f(a - \lambda)| > \epsilon$,

$$\|a - \lambda\| \leq \epsilon.$$

By the inequality, we have $\|f\| > 1$ which is a contradiction. The proof is completed. □

Lemma 2.9 indicates that such f should be a positive map because the convex hull of the spectrum of a positive operator belongs to nonnegative reals.

Proposition 2.10. *Let $\mathcal{B}$ be an operator system, $\mathcal{A}$ be a* C**-algebra with unit and $\phi : \mathcal{B} \to \mathcal{A}$ be a linear map such that $\phi(1) = 1$ and $\|\phi\| = 1$. In other words, ϕ is a unital contraction. Then ϕ is a positive map.*

Proof. $\mathcal{A}$ can be represented on a Hilbert space $\mathcal{H}$. Thus, we can take $\mathcal{A} \subseteq \mathrm{B}(\mathcal{H})$ without loss of generality. Fix h in $\mathcal{H}$ such that $\|h\| = 1$. By defining $f(b) = \langle \phi(b)h, h\rangle$ for $b \in \mathcal{B}$, we may see that $f(1) = 1$ and $\|f\| \leq \|\phi\|$. By Lemma 2.9, f is a positive map. Thus, $\phi(b) \geq 0$ if $b \in \mathcal{B}_+$. □

Remark 2.11. Alternatively, we can prove the proposition without use Lemma 2.9 as follows: Take a state $\psi : \mathcal{A} \to \mathbb{C}$, then we know $\psi \circ \phi$ is a bounded linear funtional with $\psi \circ \phi(1) = 1$ and $\|\psi \circ \phi\| \leq 1$. By Remark C.5, $\psi \circ \phi \geq 0$. This proves ϕ is positive since ψ is arbitrary.

Chapter 3

Completely Positive Maps

In this section, we discuss completely positive maps in the sense of the structure of [6] and [7].

First, we explain how $\mathrm{M}_n(\mathcal{A})$ is a C*-algebra where $\mathcal{A}$ is a C*-algebra. It is clear that $\mathrm{M}_n(\mathcal{A})$ is an algebra since we can define addition and scalar multiplication as usual entry-wise matrix addition and matrix multiplication. Let $A = [a_{i,j}]$ be an element of $\mathrm{M}_n(\mathcal{A})$ and define the adjoint of A as the transpose of the matrix $[a^*_{i,j}]$. To make $\mathrm{M}_n(\mathcal{A})$ into a C*-algebra, it only remains to define a norm.

If $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathrm{M}_n(\mathcal{A})$ is a subalgebra of $\mathcal{B}(\mathcal{H}^{(n)})$ and so it has the norm. If $\mathcal{A}$ is not a subalgebra of $\mathcal{B}(\mathcal{H})$, we take an injective representation $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and using the representation, define $\phi_n : \mathrm{M}_n(\mathcal{A}) \to \mathcal{B}(\mathcal{H}^{(n)})$ as $\phi_n([a_{i,j}]) = [\phi(a_{i,j})]$. Note that existence of the representation is guaranteed by Gelfand and Naimark Theorem C.2. For $A = [a_{i,j}] \in \mathrm{M}_n(\mathcal{A})$, the norm of A is defined by $\|\phi_n([a_{i,j}])\|$. Observe that the norm is independent from choice of representation. Thus, it is unique.

Definition 3.1. For given an operator system $\mathcal{B}$ and a C*-algebra $\mathcal{A}$, $f : \mathcal{B} \to \mathcal{A}$ is completely positive if $f_n : \mathrm{M}_n(\mathcal{B}) \to \mathrm{M}_n(\mathcal{A})$, given by $f_n([b_{i,j}]) = [f(b_{i,j})]$, is a positive map for all positive integers n.

Remember that $\mathrm{M}_n(\mathcal{B})$ denotes the space of $n \times n$ matrices whose entries are in $\mathcal{B}$. To clarify above definition, it should be recorded that $[b_{i,j}]$ is a $n \times n$ matrix belonging to $\mathrm{M}_n(\mathcal{B})$.

Clearly, all completely positive maps are also positive but the converse implication does not hold in general.

Example 3.2. Recall that M_n represents $\mathrm{M}_n(\mathbb{C})$. Then, we define $f : \mathrm{M}_n \to \mathrm{M}_n$ as taking transpose, that is, $f(A) = A^t$ where $A \in \mathrm{M}_n$. The positive elements in M_n correspond the positive matrices and we know the transpose of a positive matrix is also a positive matrix. Thus, it is easy to show that f sends positive elements to positive elements. Now, we prove that f is not completely positive. Let $A = [E_{i,j}]$ be a $n \times n$ matrix where $E_{i,j}$'s are the matrix units in M_n. In other words, A belongs to $\mathrm{M}_n(\mathrm{M}_n)$. A is self-adjoint and direct calculation gives that $A^2 = nA$ which is equivalent to $A(A - nI) = 0$. This implies that the spectrum of A is contained in positive real numbers. Hence, A is a positive element of M_n. However, it is not difficult to observe $f_n(A)^2 = 1$ by simple calculation. We also have $f_n(A)$ is not equal to 1. Therefore, the spectrum of $f_n(A)$ is equal to $\{1, -1\}$. This shows that $f_n(A)$ is not positive so f is not completely positive.

For given an operator system $\mathcal{B}$, a C*-algebra $\mathcal{A}$ and a map $f : \mathcal{B} \to \mathcal{A}$, define $f_2 : \mathrm{M}_2(\mathcal{B}) \to \mathrm{M}_2(\mathcal{A})$ as $f_2([b_{i,j}]) = [f(b_{i,j})]$. We say that f is 2-positive if f_2 is positive. Observe that we can define the completely positivity as the n-positivity for all natural number n. Now, we give some properties of 2-positive maps. Before listing them, we need a lemma.

Lemma 3.3. *Given a unital* C**-algebra* $\mathcal{A}$*, fix an element* a *of* $\mathcal{A}$*. Then,* $\|a\| \leq 1$ *if and only if*

$$\begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}$$

is a positive element of $\mathrm{M}_2(\mathcal{A})$*.*

Proof. Let $(\pi, \mathcal{H})$ be a representation of $\mathcal{A}$ and set $X = \pi(a)$. If $\|X\| \leq 1$ holds, take $h_1, h_2 \in \mathcal{H}$ and we have

$$\left\langle \begin{bmatrix} I & X \\ X^* & I \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right\rangle = \langle h_1, h_1 \rangle + \langle X h_2, h_1 \rangle + \langle h_1, X h_2 \rangle + \langle h_2, h_2 \rangle$$
$$\geq \|h_1\|^2 - 2\|X\|\|h_1\|\|h_2\| + \|h_2\|^2 \geq 0.$$

If $\|X\| > 1$ holds, there exists h_1, h_2 such that $\langle X h_2, h_1 \rangle < -1$ and $\|h_1\| = 1$, $\|h_2\| = 1$. This implies that the inner product is negative. Thus, we also have the converse direction. □

Remark 3.4. It is possible to get the generalized version of the lemma as follows. For a C*-algebra $\mathcal{A}$, we fix two elements $a, b \in \mathcal{A}$. By similar arguments as in the Lemma 3.3, we can prove that $a^*a \leq b$ if and only if

$$\begin{bmatrix} 1 & a \\ a^* & b \end{bmatrix}$$

is a positive element of $\mathrm{M}_2(\mathcal{A})$.

Proposition 3.5. *Let $\mathcal{B}$ be an operator system and $\mathcal{A}$ be a unital* C*-*algebra. If $f : \mathcal{B} \to \mathcal{A}$ is a 2-positive map such that $f(1) = 1$, f is a contractive mapping.*

Proof. Take an element b from $\mathcal{B}$ such that $\|b\| \leq 1$. By the Lemma 3.3 and 2-positivity of f, we have

$$f_2 \begin{bmatrix} 1 & b \\ b^* & 1 \end{bmatrix} = \begin{bmatrix} 1 & f(b) \\ f(b)^* & 1 \end{bmatrix}$$

is a positive element. This shows, again by Lemma 3.3, $\|f(b)\| \leq 1$. Hence, f is contractive. □

We prove the Cauchy-Schwarz inequality for 2-positive maps.

Proposition 3.6. *For given two unital* C*-*algebras namely $\mathcal{C}$ and $\mathcal{D}$, assume there is a unital, 2-positive map $f : \mathcal{C} \to \mathcal{D}$. Then, we have*

$$f(x)^* f(x) \leq f(x^* x) \text{ for all } x \in \mathcal{C}.$$

Proof. By the matrix multiplication, we obtain

$$\begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & x \\ x^* & x^*x \end{bmatrix} \geq 0.$$

Combining this with the 2-positivity of f, we get

$$\begin{bmatrix} 1 & f(x) \\ f(x)^* & f(x^*x) \end{bmatrix} \geq 0.$$

By the generalized version of the Lemma 3.3 (see the Remark 3.4), we get

$$f(x)^*f(x) \leq f(x^*x) \text{ for all } x \in \mathcal{C}.$$

□

We focus on the relations between positivity and complete positivity. Let $\mathcal{A}$ be a C*-algebra and $B = [b_{i,j}]$ be a $n \times n$ matrix with complex entries. Take an element a from $\mathcal{A}$ and recall that $a \otimes B = [b_{i,j}a]$ belons to $\mathrm{M}_n(\mathcal{A})$.

Proposition 3.7. *Let $\mathcal{A}$ be a* C**-algebra and X be a compact space. If f : $\mathrm{C}(X) \to \mathcal{A}$ is a positive map, then f is completely positive.*

Proof. We know that $\mathrm{M}_n(\mathrm{C}(X))$ can be perceived as the space of continuous functions from X to M_n. Thus, take a positive element B from $\mathrm{C}(X, \mathrm{M}_n)$. Our aim is to show $f_n(B)$ is positive for fixed n.

For given ϵ, take an open covering $\{U_1, U_2, ..., U_m\}$ and a set of points $\{y_1, y_2, ..., y_m\}$ such that $\|B(y) - B(y_k)\| < \epsilon$ for $y, y_k \in U_k$. The finiteness of the open covering comes from the compactness of X.

Set $E_k = B(y_k)$ and note that the matrix E_k is positive. Consider the partition of unity $\{u_k\}$ which subordinates the open cover. As we did in the Theorem 2.8, we have $\|B - \sum u_k \otimes E_k\| < \epsilon$ and so we get

$$\|f_n(B) - \sum f_n(u_k \otimes E_k)\| = \|f_n(B) - f_n(\sum u_k \otimes E_k) < \epsilon\|f_n\|.$$

Observe that $f_n(u_k \otimes E_k)$ is positive because $u_k \geq 0$ and $E_k \geq 0$. This observation shows that it is possible to approximate $f_n(B)$ by positive elements.

□

In above proposition, positivity implies complete positiviy since we have the domain of f is commutative. If the range of f is commutative, the result still holds.

Proposition 3.8. *Let X be a compact space and $\mathcal{B}$ be an operator system. If $f : \mathcal{B} \to \mathrm{C}(X)$ is positive, f is also completely positive.*

Proof. Take an element G from $\mathrm{M}_n(\mathcal{B})$ such that $G \geq 0$. Our goal is to prove $f_n(G)$ is a positive element of $\mathrm{M}_n(\mathrm{C}(X))$. By the identification of $\mathrm{M}_n(\mathrm{C}(X))$ with $\mathrm{C}(X, \mathrm{M}_n)$, we need to show that $f_n(G)(a)$ is a positive matrix in M_n for $a \in X$.

Denoting the evaluation functional on $\mathrm{C}(X)$ at the point $a \in X$ by λ, we obtain

$$f_n(G)(a) = \lambda_n \circ f_n(G) = (\lambda \circ f)_n(G).$$

We know that positive linear functionals on an operator system are completely positive maps. In above equation, observe that $\lambda \circ f$ is a positive linear functional and $(\lambda \circ f)_n$ is positive. This finishes the proof. $\square$

Chapter 4

Stinespring's Dilation Theorem

In this section, we recall the Stinespring representation and the minimality of it by closely following [6].

Theorem 4.1 (Stinespring's Dilation Theorem)**.** *For given a unital* C**-algebra* $\mathcal{C}$ *and a completely positive map* $f : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ *where* $\mathcal{H}$ *denotes a Hilbert space, we can find a triple* $(\pi, \mathcal{K}, V)$ *such that* $\mathcal{K}$ *is a Hilbert space,* $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{K})$ *is a unital *-homomorphism,* $V : \mathcal{H} \to \mathcal{K}$ *is an operator belonging to* $\mathcal{B}(\mathcal{H})$*,* $\|f(1)\| = \|V\|^2$ *and for* $c \in \mathcal{C}$*,*

$$f(c) = V^*\pi(c)V.$$

Proof. We define a map $g : (\mathcal{C} \otimes \mathcal{H}) \times (\mathcal{C} \otimes \mathcal{H}) \to \mathbb{C}$ as follows:

$$g(c_1 \otimes h_1, c_2 \otimes h_2) = \langle f(c_2^* c_1)h_1, h_2\rangle.$$

Observe that g is a symmetric, bilinear map. To make the definition clear, it can be noted that the inner product is taken on $\mathcal{H}$. Complete positivity of f leads to positive semi-definiteness of g. To see this, consider

$$\left\langle \sum_{i=1}^{k} c_i \otimes h_i, \sum_{j=1}^{k} c_j \otimes h_j \right\rangle = \left\langle f_n([c_j^* c_i])h, h\right\rangle \geq 0.$$

In above equation, $h = (h_1, h_2, ..., h_k)$ and the inner product is taken on $\mathcal{H}^{(n)}$.

Recall that the inner product on $\mathcal{H}^{(n)}$ is the sum of the inner products on $\mathcal{H}$ for each component.

Additionaly, g can be perceived as an inner product on $\mathcal{C} \otimes \mathcal{H}$ since it satisfies the conditions of being an inner product. Thus, g will be used as an inner product on $\mathcal{C} \otimes \mathcal{H}$ while producing a bounded operator on $\mathcal{K}$.

Symmetric-bilinear map g satisfies the Cauchy-Schwarz inequality since it is a positive semi-definite map. Thus, for $x, y \in \mathcal{C} \otimes \mathcal{H}$,

$$|g(x,y)|^2 \leq g(x,x)\ g(y,y).$$

Define the subspace $\mathcal{B}$ as $\{x \in \mathcal{C} \otimes \mathcal{H} : g(x,x) = 0\}$. This definition is equivalent to following definition because of the Cauchy-Schwarz inequality.

$$\mathcal{B} := \{x \in \mathcal{C} \otimes \mathcal{H} : g(x,y) = 0\ \forall y \in \mathcal{C} \otimes \mathcal{H}\}.$$

Inner product on $\mathcal{C} \otimes \mathcal{H}/\mathcal{B}$ is given by

$$\langle x + \mathcal{B}, y + \mathcal{B} \rangle = g(x,y).$$

According to the inner product, $\mathcal{C} \otimes \mathcal{H}/\mathcal{B}$ is an inner product space and we define $\mathcal{K}$ as the completion of $\mathcal{C} \otimes \mathcal{H}/\mathcal{B}$.

For fixed $c \in \mathcal{C}$, consider $\pi_c : \mathcal{C} \otimes \mathcal{H} \to \mathcal{C} \otimes \mathcal{H}$ given by

$$\pi_c(\sum c_i \otimes h_i) = \sum (cc_i) \otimes h_i.$$

Our aim is to show that π_c belongs to $\mathcal{B}(\mathcal{K})$.

$$\begin{aligned} g\left(\pi_c(\sum c_i \otimes h_i), \pi_c(\sum c_j \otimes h_j)\right) =& \sum \langle f(c_j^* c^* c c_i) h_i, h_j \rangle \\ \leq& \|c^* c\| \ \sum \langle f(c_j^* c_i) h_i, h_j \rangle \\ =& \|c\|^2\ g\left(\sum c_i \otimes h_i, \sum c_j \otimes h_j\right). \end{aligned}$$

Thus, π_c is bounded and $\mathcal{B}$-invariant. Taking it's restriction to $\mathcal{C} \otimes \mathcal{H}/\mathcal{B}$ and extending it to $\mathcal{K}$, we get a bounded operator on $\mathcal{K}$.

We construct $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{K})$ as $\pi(c) = \pi_c$ for $c \in \mathcal{C}$. It is easy to see that π is a unital *-homomorphism.

After constructing $\mathcal{K}$ and π, it remains to define V. We define $V : \mathcal{H} \to \mathcal{K}$ as

$$V(h) = 1 \otimes h + \mathcal{B}.$$

To show that V is bounded, notice that

$$\|V(h)\|^2 = g\,(1 \otimes h, 1 \otimes h) = \langle f(1)h, h\rangle \leq \|f(1)\| \cdot \|h\|^2.$$

By the equation, we get $\|V\|^2 = f(1)$. Finally, we have

$$\langle V^*\pi_c V a, b\rangle = g(\pi_c 1 \otimes a, 1 \otimes b) = \langle f(c)a, b\rangle.$$

The equation is true for all a and b in $\mathcal{H}$. Thus, for all $c \in \mathcal{C}$, we have

$$f(c) = V^*\pi(c)V.$$

$\square$

The Stinespring dilation theorem characterizes the completely positive maps since any map which has the form $V^*\pi(c)V$ is completely positive.

Remark 4.2. If we assume f is unital, V becomes an isometry. Using the notation $P_{\mathcal{H}}$ as the projection of $\mathcal{K}$ onto $\mathcal{H}$, we reach

$$f(c) = P_{\mathcal{H}}\pi(c)|_{\mathcal{H}}.$$

It also should be noted that the separability of $\mathcal{H}$ and $\mathcal{C}$ implies the separability of $\mathcal{K}$.

Definition 4.3. For given the Stinespring representation $(\pi, \mathcal{K}, V)$ associated to a C*-algebra $\mathcal{C}$ and a completely positive map f, we call the triple is minimal if $\mathcal{K}$ is the closure of the linear span of $\pi(\mathcal{C})V\mathcal{K}$.

Proposition 4.4. *For given a* C*-*algebra* $\mathcal{C}$ *and a completely positive map* $f : \mathcal{C} \to \mathrm{B}(\mathcal{H})$ *where* $\mathcal{H}$ *is a Hilbert space, assume* $(\pi_1, V_1, \mathcal{H}_1)$ *and* $(\pi_2, V_2, \mathcal{H}_2)$ *are two minimal Stinespring representations corresponding to* f. *We can find a unitary map* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *such that* $UV_1 = V_2$, $U\pi_1 U^* = \pi_2$.

Proof. Define U by

$$U\left(\sum_i \pi_1(c_i)V_1h_i\right) = \sum_i \pi_2(c_i)V_2h_i.$$

Then, we show U is an isometry. To being an isometry finishes the proof since the minimality implies the dense range and ontoness. Note that it is clear that U satisfies $UV_1 = V_2$ and $U\pi_1U^* = \pi_2$. To see U is an isometry, consider

$$\begin{aligned}\|\sum_i \pi_1(c_i)V_1h_i\|^2 &= \sum \langle (V_1^*\pi_1(c_i^*c_j)V_1)h_j, h_i\rangle \\ &= \sum \langle f(c_i^*c_j)h_j, h_i\rangle = \|\sum_i \pi_2(c_i)V_2h_i\|^2.\end{aligned}$$

□

The proposition shows that the minimal Stinespring representation is unique up to unitary transformations.

Chapter 5

Radon-Nikodym Derivatives

In this section, we study how to calculate the Radon-Nikodym derivative for completely positive maps, introduced by Arveson in [2]. Then, we discuss the absolute continuity for operator valued completely positive maps by closely following [1].

CP($\mathcal{A}$; $\mathcal{H}$) denotes the set of all completely positive maps from $\mathcal{A}$ into B($\mathcal{H}$). We define $f \leq g$ as $g - f \in$ CP($\mathcal{A}$; $\mathcal{H}$). Let f, $g \in$ CP($\mathcal{A}$; $\mathcal{H}$) be such that $f \leq g$ and $(\pi_f; \mathcal{K}_f; V_f)$, $(\pi_g; \mathcal{K}_g; V_g)$ be the minimal Stinespring representations for f and g respectively. Now consider the identity operator $I_{f,g} : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$ with $I_{f,g}(N_g) \subseteq N_f$ and then consider $I_{f,g} : (\mathcal{A} \otimes H)/N_g \to (\mathcal{A} \otimes H)/N_f$. Now, we can extend the operator by continuity, $I_{f,g} \in \mathcal{B}(\mathcal{K}_g, \mathcal{K}_f)$. We know that the following equalities hold by the definition of the operator and Stinespring representations of f, g:

$$I_{g,f} V_g = V_f, \tag{0.1}$$

and

$$I_{g,f} \pi_g(a) = \pi_f(a) I_{g,f}. \tag{0.2}$$

Note that the equation (0.2) is valid for all $a \in \mathcal{A}$. We define the Radon-Nikodym derivative of f associated to g as follows:

$$D_g(f) = I^*_{g,f} I_{g,f}. \tag{0.3}$$

Main focus after the definiton is on well-definedness of it. Firstly, we know $D_g(f)$ commutes with all $\pi_g(a)$ for all a. Thus, $D_g(f)$ belongs to the commutant of

$\pi_g(\mathcal{A})$. By combining the fact with Stinespring's dilation theorem and (0.1), we get

$$f(a) = V_g^* D_g(f)\pi_g(a)V_g = V_g^* D_g(f)^{\frac{1}{2}}\pi_g(a)D_g(f)^{\frac{1}{2}}V_g. \tag{0.4}$$

This equation shows the uniqueness of the Radon-Nikodym derivative since we know the minimal Stinespring representation is unique up to unitary maps.

Proposition 5.1. *Let f, g and h be elements of* $\mathrm{CP}(\mathcal{A};\mathcal{H})$ *such that $f, h \leq g$. Then, $f \leq h$ if and only if $D_g(f) \leq D_g(h)$.*

Proof. Take a finite set of elements from the C*-algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$, $(a_i)_{i=1}^n \in \mathcal{A}$ and $(h_j)_{j=1}^n \in H$. Our aim is to reach the following formula:

$$\sum_{i,j=1}^n \langle f(a_i^* a_j)h_j, h_i\rangle = \|D_g(f)^{\frac{1}{2}} \sum_{i=1}^n \pi_g(a_i)V_g h_i\|^2 \text{ If } f \leq g. \tag{0.5}$$

If this formula holds, it can be easily seen that the proof finishes. By using (0.4), it can be observed that the formula is true as follows:

$$\begin{aligned}
\sum_{i,j=1}^n \langle f(a_i^* a_j)h_j, h_i\rangle &= \sum_{i,j=1}^n \langle (V_g^* D_g(f)^{\frac{1}{2}}\pi_g(a_i^* a_j)D_g(f)^{\frac{1}{2}}V_g)h_j, h_i\rangle \\
&= \sum_{i,j=1}^n \langle (D_g(f)^{\frac{1}{2}}\pi_g(a_i^* a_j)D_g(f)^{\frac{1}{2}}V_g)h_j, V_g h_i\rangle \\
&= \sum_{i,j=1}^n \langle (\pi_g(a_i^* a_j)D_g(f)^{\frac{1}{2}}V_g)h_j, D_g(f)^{\frac{1}{2}}V_g h_i\rangle \\
&= \sum_{i,j=1}^n \langle (\pi_g(a_j)D_g(f)^{\frac{1}{2}}V_g)h_j, \pi_g(a_i)D_g(f)^{\frac{1}{2}}V_g h_i\rangle \\
&= \|D_g(f)^{\frac{1}{2}} \sum_{i=1}^n \pi_g(a_i)V_g h_i\|^2.
\end{aligned}$$

□

Now we give the definition of the notion of absolute continuity in completely positivity sense as in [1]. This definition is analog of the absolute continuity for positive definite functions given by Ando and Szymański in [4].

Definition 5.2. Let f and g be completely positive maps, $f, g \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$. We call f is g-absolutely continuous and use $f \ll g$ as notation if there exists a sequence f_n in $\mathrm{CP}(\mathcal{A}; \mathcal{H})$ satisfying the following three conditions:

1. $f_n \leq f_{n+1}$ for all natural number n.

2. In strong operator limit sense, $\lim f_n(a) = f(a)$ for any element $a \in \mathcal{A}$.

3. For every natural number n there exists c_n such that $f_n \leq c_n g$.

Remark 5.3. We write g uniformly dominates f and denote this by $f \leq_u g$ if there exists some $c \geq 0$ such that $f \leq cg$. Thus, the third condition in above definition corresponds that g uniformly dominates f_n for all n.

Lemma 5.4. *Take $f_n, f, g \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$ such that $f_n \leq f \leq_u g$ and f_n is a non-decreasing sequence. Then, the following conditions are equivalent. Note that SO-*lim *denotes the strong operator limit.*

(a). SO-$\lim D_g(f_n) = D_g(f)$ *as* $n \to \infty$.

(b). Fix $a \in \mathcal{A}$, *then SO-*$\lim_{n\to\infty} f_n(a) = f(a)$.

(c). For fixed $a \in \mathcal{A}$, *WO-*$\lim f_n(a) = f(a)$ *for all* $a \in \mathcal{A}$ *where WO-*lim *denotes the weak operator limit.*

Proof. Firstly, (a) implies (b) by equation (0.4). Then, we also know that the strong convergence implies the weak convergence. Hence, we have (b) implies (c). If we prove (c) $\Rightarrow$ (a), we finishes the proof.

We assume $f = 0$ without loss of generality because we can replace f_n by $f - f_n$. Then, by considering the formula (0.5), we have $D_g(f_n)^{\frac{1}{2}}$ strongly converges to 0. Observe that the strong convergence is valid in a subspace which is dense in $\mathcal{K}_g$. (Note that $(\pi_g, V_g, \mathcal{K}_g)$ is the minimal stinespring representation of g). To clarify, the subspace is the span of all elements in $\pi(\mathcal{A})V_g\mathcal{H}$. It is clear that the density comes from the minimality. Now, by using the fact that all Radon-Nikodym derivatives for operator valued completely positive maps are contractions, we get the condition (a), that is, $D_g(f_n)$ converges strongly to 0. $\square$

By Lemma 5.4 and Proposition 5.1, we can prove that the absolute continuity remains stable with respect to taking the Radon-Nikodym derivative.

Proposition 5.5. *Let $f, g, h \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$ be such that $f, h \leq_u g$. Then, f is h-absolutely continuous $\Leftrightarrow$ $D_g(f)$ is $D_g(h)$-absolutely continuous.*

Proof. Assume f is h-absolutely continuous. Then, there exists a non-decreasing sequence $f_n \in \mathrm{CP}(\mathcal{A})$ such that SO-lim $f_n(a) = f(a)$ for all $a \in \mathcal{A}$ and $f_n \leq_u h$ by definition of absolute continuity. By Proposition 5.1, we have a non-decreasing sequence $D_g(f_n)$ and SO-lim of the sequence is $D_g(f)$ by Lemma 5.4. This implies that $D_g(f)$ is $D_g(h)$-absolutely continuous. Similarly, it is not difficult to prove the reverse implication. $\square$

Chapter 6

Lebesque Decompositions

In this section, firstly we give the definition of the parallel sum and the shorted operators then explain how to write the Lebesque Decomposition of operator valued completely positive maps. We follow closely [1] in whole section.

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. If $A, B \in \mathcal{B}(\mathcal{H})$ are selfadjoint we write $A \leq B$ if $\langle Ah, h\rangle \leq \langle Bh, h\rangle$ for all $h \in \mathcal{H}$, the natural order relation (reflexive, antisymmetric, and transitive).

Take two positive elements namely A, B from $\mathcal{B}(\mathcal{H})$. Let C and D be the minimal bounded operators satisfying $A^{1/2} = (A+B)^{1/2}C$, $B^{1/2} = (A+B)^{1/2}D$.

Definition 6.1. The parallel sum of A, B is denoted by $A : B$ and defined as $A : B = A^{1/2}C^*DB^{1/2}$.

Note that the definition of the parallel sum for positive bounded operators was taken from Fillmore and Williams [8]. Then, Ando showed the following useful formula holds in [5].

$$\langle (A : B)h, h\rangle = \inf_{h\in\mathcal{H}} \left(\langle Ak, k\rangle + \langle B(h-k), h-k\rangle \mid k \in \mathcal{H}\right). \qquad (0.1)$$

By using the parallel sum, Ando introduced the definition of the shorted operator for positive bounded operators in [5].

Definition 6.2. For given positive elements A, B in $\mathcal{B}(\mathcal{H})$, the shorted operator is the strong operator limit of $(nA : B)$ as n goes to ∞. Remember that the strong operator limit means that the limit is calculated in the strong operator topology.

$$\big([A]B\big)h := \mathrm{SO}-\lim_{n\to\infty}(nA : B). \tag{0.2}$$

Remark 6.3. We need to check whether the definition is well defined or not. To see that the SO$-$ lim exists, we use two facts about the parallel sum. The first one is $A \leq E$ implies $A : B \leq E : B$ and the second one is $0 \leq A : B \leq A, B$ where $E \in \mathrm{B}(\mathcal{H})$. By the two facts, we have $(nA : B) \leq B$ and $(nA : B \leq (n+1A : B))$. The two inequalities show that the SO$-$ lim exists.

Before passing to the Lebesque Decomposition, we need to state the theorem which was proved by Kosaki. For proof of the theorem, see [9].

Theorem 6.4 (Kosaki)**.** *For given $A, B \in \mathcal{B}(\mathcal{H})^+$, we have $[A]\,B = B^{1/2}P_{A,B}A^{1/2}$ where $P_{A,B}$ denotes the orthogonal projection onto the smallest closed set containing $\big\{h \in H \mid \exists k \in H : B^{1/2}h = B^{1/2}k\big\}$.*

Now, we prove the following lemma by using the theorem.

Lemma 6.5. *Take an element A in $\mathcal{B}(\mathcal{H})$. Assume $0 \leq A \leq I$. Then,*

$$[A](I - A) = P_{\mathcal{H}\ominus\ker(A)}(I - A).$$

Proof. It can be observed that

$$\big\{h \in \mathcal{H} \mid (I - A)^{1/2}h \in R(A^{1/2})\big\} \subseteq \mathcal{H} \ominus \ker(A).$$

Thus, we have $P_{A,I-A} = P_{\mathcal{H}\ominus\ker(A)}$. Using Theorem 6.4, we get

$$[A](I - A) = (I - A)^{1/2}P_{\mathcal{H}\ominus\ker(A)}(I - A)^{1/2} = P_{\mathcal{H}\ominus\ker(A)}(I - A).$$

□

Let f, g be completely positive maps. We say f is g-singular if there is not any non-zero completely positive map h such that $h \leq f, g$.

Definition 6.6. We call $f = f_0 + f_1$ is a h-lebesque decomposition of f if $f_0, f_1 \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$, f_0 is h-absolutely continuous and f_1 is h-singular.

If we change $\mathrm{CP}(\mathcal{A}; \mathcal{H})$ with $\mathcal{B}(\mathcal{H})^+$, we get the similar definition for positive bounded maps. Before jumping the conclusion, we need the following theorem which was showed by Ando in [5].

Theorem 6.7. *For given $A, B \in \mathcal{B}(\mathcal{H})^+$, we have the following two properties:*

(i) $[A]B$ is A-absolutely continuous and $B - [A]B$ is A-singular.

(ii) $[A]B$ is maximal when considered to all maps $C \in \mathcal{B}(\mathcal{H})^+$ satisfying $h \leq g$.

After this theorem, we explain the lebesque decomposition for completely positive maps as in [1] that is obtained by A.Gheondea and A.Ş.Kavruk.

Theorem 6.8. *Given $f, g \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$, we can find a g-lebesque decomposition of $f = f_0 + f_1$ such that f_0 is maximum when considered to all g-absolutely continuous maps $\phi \leq g$.*

Proof. Define $h = f + g$ and take the minimal Stinespring representation $(\pi_h, \mathcal{K}_h, V_h)$ of h. We know that the Radon-Nikodym derivatives $D_h(f)$, $D_h(g)$ are bounded operators on $\mathcal{K}_h$ and their sum is equal to identity map. In other words, we have the following formula by the definition of the Radon-Nikodym derivative and h:

$$D_h(f) + D_h(g) = I. \tag{0.3}$$

Our aim is to obtain f_0, f_1. Define f_0, f_1 as follows:

$$f_0(a) = V_h^* D_h(f) P_{\mathcal{K} \ominus \ker(D_h(g))} \pi_h(a) V_h \quad \text{and} \quad f_1(a) = V_h^* P_{\ker(D_h(g))} \pi_h(a) V_h. \tag{0.4}$$

Direct calculation gives that $f = f_0 + f_1$ and observe f_0, f_1 are completely positive maps. By (0.3) and Lemma 6.5, we have

$$[D_h(g)] D_h(f) = D_h(f) P_{\mathcal{K} \ominus \ker(D_h(g))}.$$

Then, we also have by (0.4),

$$D_h(f_0) = D_h(f) P_{\mathcal{K} \ominus \ker(D_h(g))}.$$

By using the uniqueness of the Radon-Nikodym derivative, we get the following:

$$D_h(f_0) = [D_h(g)]D_h(f). \tag{0.5}$$

Now our interest is turning on how to show f_0 is g-absolutely continuous. By Proposition 5.5, it is enough to prove that $D_h(f_0)$ is $D_h(g)$-absolutely continuous. To see this, we combine the first part of the Theorem 6.7 with the relation (0.5). For maximality, we pay attention only the Radon-Nikodym derivatives since we know the partial order is preserved when we take the Radon-Nikodym derivative by Proposition 5.1. Then, we have the following about the singular part:

$$D_h(f_1) = D_h(f) - [D_h(g)]D_h(f) = P_{\ker(D_h(g))}. \tag{0.6}$$

Then, we use similar arguments to get desired result about the singular part. (Use Proposition 5.5, (0.6) and the Theorem 6.7) □

In proof, we define $h = f + g$ but we can replace it with any h which uniformly dominates f, g. Thus, we can extend the theorem to the generalized version. The generalized theorem was obtained by A.Gheondea and A.Ş.Kavruk in [1]. Before this generalization, we prove the following lemma.

Lemma 6.9. *Given $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with the fact that Ran(C) is dense in $\mathcal{H}_2$ and A, B belong to $\mathcal{B}(\mathcal{H}_2)^+$. The following relations hold:*

$$(C^*AC) : (C^*BC) = C^*(A : B)C \text{ and } [C^*AC]C^*BC = C^*([A]B)C.$$

Proof. Fix $a \in \mathcal{H}_1$ then we use the formula (0.1) to prove the first relation. By the formula,

$$\begin{aligned}
\langle C^*(A : B)Ca, a\rangle =& \langle (A : B)Ca, Ca\rangle \\
=& \inf_{b \in \mathcal{H}_2} \{\langle Ab, b\rangle + \langle B(Ca - b), Ca - b\rangle\} \\
=& \inf_{c \in \mathcal{H}_1} \{\langle ACc, Cc\rangle + \langle B(Ca - Cc), Ca - Cc\rangle\} \\
=& \inf_{c \in \mathcal{H}_1} \{\langle C^*ACc, c\rangle + \langle C^*B(Ca - Cc), a - c\rangle\} \\
=& \langle (C^*AC) : (C^*BC)a, a\rangle.
\end{aligned}$$

For the second relation,

$$[C^*AC]C^*BCa = lim_{n\to\infty}(nC^*AC : C^*BC)a = lim_{n\to\infty}C^*(nA : B)Ca = C^*([A]B)Ca.$$

$\square$

After the lemma, we can prove the generalized theorem as in [1].

Theorem 6.10. *For given $f, g, h \in \mathrm{CP}(\mathcal{A};\mathcal{H})$ with $f, g \leq_u h$ and the g-lebesque decomposition of f as $f_0 + f_1$, it is possible to find the Radon-Nikodym derivatives as follows:*

$$D_h(f_0) = [D_h(g)]D_h(f) \text{ and } D_h(f_1) = D_h(f) - [D_h(g)]D_h(f). \tag{0.7}$$

Let $(\pi_h, \mathcal{K}_h, V_h)$ be the minimal Stinespring representation of h, for arbitrary element $a \in \mathcal{A}$

$$f_0(a) = V_h^*[D_h(g)]D_h(f)\pi_h(a)V_h, \tag{0.8}$$

and

$$f_1(a) = V_h^*(D_h(f) - [D_h(g)]D_h(f))\pi_h(a)V_h. \tag{0.9}$$

Proof. If we get the formula (0.7), it is easy to show (0.8), (0.9) is true as we did in the Theorem 6.8. Thus, we prove only how to find the Radon-Nikodym derivatives in (0.7). Consider the $\mathcal{K}_h$ as the direct sum of $\ker(D_h(f) + D_h(g)) \oplus \mathcal{K}_h \ominus \ker(D_h(f) + D_h(g))$ and represent $D_h(f) + D_h(g)$ by 2×2 matrix (according to the direct sum, also consider the representations of π_h, V_h) :

$$D_h(f) + D_h(g) = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}, \quad \pi_h(a) = \begin{bmatrix} \pi_1(a) & 0 \\ 0 & \pi_2(a) \end{bmatrix} \quad \text{and} \quad V_h = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$

where C denotes a positive injective bounded map on $\mathcal{K}_h \ominus \ker(D_h(f) + D_h(g))$. Note that C belongs the commutant of $\pi_2(\mathcal{A})$. Now, we get the relation about how to calculate $g(a)$.

$$g(a) = \left(C^{1/2}V_2\right)^* \, D_{f+g}(g)\pi_2(a) \, \left(C^{1/2}V_2\right)$$

$$= V_2^* C^{1/2} D_{f+g}(g) C^{1/2} \pi_2(a) V_2$$

$$= \begin{bmatrix} V_1^* & V_2^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & C^{1/2} D_{f+g}(g) C^{1/2} \end{bmatrix} \begin{bmatrix} \pi_1(a) & 0 \\ 0 & \pi_2(a) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$= V_h^* \begin{bmatrix} 0 & 0 \\ 0 & C^{1/2} D_{f+g}(g) C^{1/2} \end{bmatrix} \pi_h(a) V_h.$$

In above calculations, it can be observed that we use the fact that $(\pi_2, \mathcal{K}_h \ominus \ker(D_h(f) + D_h(g), C^{1/2}V_h)$ is the minimal Stinespring representation for $f + g$. We know the Radon-Nikodym derivative is unique so we have

$$D_h(f) = \begin{bmatrix} 0 & 0 \\ 0 & C^{1/2} D_{f+g}(g) C^{1/2} \end{bmatrix}.$$

By following same process, we get

$$D_h(g) = \begin{bmatrix} 0 & 0 \\ 0 & C^{1/2} D_{f+g}(f) C^{1/2} \end{bmatrix}.$$

Apply same procedure to $f_0(a)$ by using Lemma 6.5 and get the relation:

$$f_0(a) = V_h^* \begin{bmatrix} 0 & 0 \\ 0 & C^{1/2}[D_{f+g}(g)](D_{f+g}(f)) C^{1/2} \end{bmatrix} \pi_h(a) V_h.$$

Again, considering the fact that the Radon-Nikodym derivative is unique, we have

$$D_h(f_0) = \begin{bmatrix} 0 & 0 \\ 0 & C^{1/2}[D_{f+g}(g)](D_{f+g}(f)) C^{1/2} \end{bmatrix}.$$

Now, we start to calculate $[D_h(f)]D_h(g)$ by using above formulas. In this calculations, we use the Lemma 6.9.

$$
\begin{aligned}
[D_h(f)]D_h(g) &= \left[\begin{bmatrix} 0 & 0 \\ 0 & C^{1/2}D_{f+g}(g)C^{1/2} \end{bmatrix}\right] \begin{bmatrix} 0 & 0 \\ 0 & C^{1/2}(D_{f+g}(f))C^{1/2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & [C^{1/2}D_{f+g}(g)C^{1/2}](C^{1/2}(D_{f+g}(f))C^{1/2}) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & C^{1/2}\ [D_{f+g}(g)](D_{f+g}(f))\ C^{1/2} \end{bmatrix} = D_h(f_0).
\end{aligned}
$$

Hence, it is clear that $D_h(f_1) = D_h(g) - [D_h(f)]D_h(g)$ holds. $\square$

Remark 6.11. $C^{1/2}$ is also a positive bounded injective operator and so it's range is dense. Thus, to use the Lemma 6.9 is meaningful.

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Appendix A

Introduction to C*-algebras

We use a lot of results from the theory of C*-algebras. Thus, we make a collection of definitions and facts from the theory of C*-algebras by closely following [7].

Let $\mathcal{A}$ be a Banach algebra and define an involution which is a map $a \to a^*$ of $\mathcal{A}$ into itself satisfying the following conditions: (i) $(a^*)^* = a$; (ii) $(ab)^* = b^*a^*$; (iii) $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$ where $a, b \in \mathcal{A}$ and α is a scalar.

Definition A.1. If $\mathcal{A}$ is a Banach algebra with involution such that $\|a^*a\| = \|a\|^2$ for $a \in \mathcal{A}$, $\mathcal{A}$ is a C*-algebra.

If a C*-algebra $\mathcal{A}$ has an identity, $\mathcal{A}$ is called unital.

Proposition A.2. *For an element $a \in \mathcal{A}$ where $\mathcal{A}$ is a* C**-algebra,*

(a) $\|a^*\| = \|a\|$.

(b) $\|aa^*\| = \|a\|^2$.

(c) $\|a\| = \sup\{\|ax\| : x \in ball\mathcal{A}\} = \sup\{\|xa\| : x \in ball\mathcal{A}\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. We say a map $\phi : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism if ϕ is an algebraic homomorphism such that $\phi(a^*) = \phi(a)^*$.

If ϕ is a bijective *-homomorphism, we say ϕ is a *-isomorphism.

Definition A.3. For given a C*-algebra $\mathcal{A}$ and $a \in \mathcal{A}$, a is hermitian if $a^* = a$; a is normal if $aa^* = a^*a$; when $\mathcal{A}$ is a unital, a is unitary if $a^*a = aa^* = 1$.

Appendix B

Positive elements in a C*-algebra

Definition B.1. Let $\mathcal{A}$ be a C*-algebra and a be an element from $\mathcal{A}$. If a is hermitian element and $\sigma(a)$ belongs to non-negative real numbers, a is a positive element and it is denoted by $a \geq 0$.

$\mathcal{A}_+$ denotes the set of positive elements in $\mathcal{A}$.

Proposition B.2. *For a* C*-*algebra* $\mathcal{A}$*,*

(a) If a is a hermitian element of $\mathcal{A}$, then a can be written as difference of two positive elements.

(b) If $a \in \mathcal{A}_+$, there is a unique $b \in \mathcal{A}_+$ such that $a = b^n$ where n is a natural number and b is called the n-th root of a.

Proposition B.3. *For a* C*-*algebra* $\mathcal{A}$*,* $\mathcal{A}_+$ *is closed cone in* $\mathcal{A}$*.*

Definition B.4. For an element a in a C*-algebra, the absolute value of a is defined by $|a| = (a^*a)^{1/2}$.

Proposition B.5. *Let $\mathcal{H}$ be a Hilbert space. C is a positive element of $\mathcal{B}(\mathcal{H})$ if and only if $\langle Ch, h\rangle \geq 0$.*

Appendix C

Representations of a C^*-algebra and Positive Linear Functionals

Definition C.1. For a C^*-algebra $\mathcal{A}$, a representation is a pair $(\pi, \mathcal{H})$ where $\mathcal{H}$ is a Hilbert Space and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism. If $\mathcal{A}$ is unital, $\pi(1)$ should be 1.

Theorem C.2 (Theorem of Gelfand and Naimark)**.** *Any* C^**-algebra can be embedded into* $\mathcal{B}(\mathcal{H})$ *where* $\mathcal{H}$ *is a Hilbert space.*

A state on a C^*-algebra is a positive linear functional with norm 1.

Proposition C.3. *If* f *is a positive linear functional on a unital* C^**-algebra* $\mathcal{A}$*, then* f *is bounded and* $\|f\| = f(1)$*.*

Proposition C.4. *If* $\mathcal{A}$ *is a unital* C^**-algebra and* $f : \mathcal{A} \to \mathbb{C}$ *is a bounded linear functional such that* $\|f\| = f(1)$*, then* f *is positive.*

Proof. If $\mathcal{A} = \mathrm{C}(X)$ for a compact space X, f corresponds a measure ν and $\nu(X) = \|\nu\|$. Thus, the measure ν is positive and this implies f is a positive linear functional.

If $\mathcal{A} \neq \mathrm{C}(X)$, take a positive element a in $\mathcal{A}$ and consider the C^*-algebra $\mathcal{B}$ generated by a and 1. By functional calculus, $\mathcal{B} \cong \mathrm{C}(\sigma(a))$. If f_0 is the restriction of f to $\mathcal{B}$, then $f_0(1) \leq \|f_0\| \leq \|f\| = f(1) = f_0(1)$. This implies $f(a) \geq 0$. Hence, f is positive. $\square$

Remark C.5. If we change $\mathcal{A}$ by an operator system $\mathcal{S}$ in Proposition C.4, then the result still holds and proof is identical with the proof of Proposition C.4.

Definition C.6. If $A \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space, the commutant of A is denoted by A' and defined by

$$A' \equiv \{B \in \mathcal{B}(\mathcal{H}) : BA = AB \text{ for all } B \in A\}.$$