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Perfect fluid sources in 2+1 dimensions*

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Abstract. We show that a class of metrics of Einstein theory with perfect fluid sources are also the solutions of the Deser–Jackiw–Templeton (DJT) theory with a cosmological constant. Because of this analogy we interpret a recently found black hole solution of the DJT equations as a rotating fluid solution of the Einstein theory in three dimensions.

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Recently [1] it has been shown that topologically massive gravity (TMG) [2] can be obtained from three-dimensional string theory with constant dilaton and vanishing gauge fields. Hence the black hole solutions of TMG might bring some new insights to three-dimensional gravitational theories [3] and also to the string theory in three dimensions [4].

In this paper we first connect the solutions of the Einstein equations with perfect fluid source to the solutions of TMG and then interpret the recently found black hole solution [5] as a rotating fluid in three dimensions.

TMG equations with a cosmological constant are given as follows.

$$G^{\mu}_{\nu} + \frac{1}{\mu}\, \eta^{\mu\beta\alpha}\, \nabla_{\alpha}\, (R_{\nu\beta} - \tfrac{1}{4}\, R\, g_{\nu\beta}) = \lambda\, \delta^{\mu}_{\nu}. \tag{1}$$

Here $G_{\mu\nu}$ and $R_{\mu\nu}$ are the Einstein and Ricci tensors respectively, $\eta_{\mu\alpha\beta} = \sqrt{-g}\, \epsilon_{\mu\alpha\beta}$ is the Levi-Civita tensor in three dimensions, μ and λ are respectively the DJT parameter and the cosmological constant.

Einstein's equations with a perfect fluid source are given as follows

$$G_{\mu\,\nu} = T_{\mu\,\nu}, \tag{2}$$

where

$$T_{\mu\,\nu} = (p+\rho)\, u_{\mu}\, u_{\nu} + p\, g_{\mu\,\nu}, \tag{3}$$

and where the fluid equations are obtained through the conservation equation $\nabla_{\mu}\, T^{\mu\nu} = 0$, p and ρ are respectively the pressure and mass density of the fluid and u_{μ} is the fluid's timelike unit 4-velocity, i.e, $u^{\mu}\, u_{\mu} = -1$.

We have the following:

Lemma. If p, ρ are constants and

$$\nabla_{\mu}\, u_{\nu} = \frac{\mu}{3}\, \eta_{\mu\nu\alpha}\, u^{\alpha} \tag{4}$$

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then any solution of the Einstein equations $G_{\mu\nu} = T_{\mu\nu}$ is also a solution of the TMG (1) with a cosmological constant $\lambda = (2p - \rho)/3$.

Proof of the lemma is straightforward. The Einstein and Ricci tensors in (1) are first expressed in terms of the perfect fluid energy–momentum tensor through (2) and (3). Then DJT equations (1) reduce to coupled equations for p, ρ and u^{μ}. These equations are identically satisfied for the conditions given by the lemma. Here we assumed that the 4-velocity vector is timelike. One has a similar lemma for the null case.

As an application of the above lemma let us consider the following spacetime metric

$$\mathrm{d}s^2 = -\phi\,\mathrm{d}t^2 + 2q\,\mathrm{d}t\,\mathrm{d}\theta + \frac{-q^2 + h^2\psi}{\phi}\,\mathrm{d}\theta^2 + \frac{1}{\psi}\,\mathrm{d}r^2, \tag{5}$$

where

$$\phi = a_0 \qquad \psi = b_0 + \frac{b_1}{r^2} + \frac{3\lambda_0}{4} r^2$$
$$q = c_0 + \frac{e_0\mu}{3} r^2 \qquad h = e_0 r \qquad \lambda_0 = \lambda + \frac{\mu^2}{27}$$

and where a_0, b_0, b_1, c_0 and e_0 are arbitrary constants. This metric solves the Einstein equations (2) with the following fluid parameters

$$p = \frac{\mu^2}{9} \qquad \rho = \frac{\mu^2 - 9\lambda_0}{3}$$
$$u_\mu = \sqrt{a}_0(1, 0, \Omega) \qquad \Omega = -\frac{\mu e_0 r^2 + 3c_0}{3a_0}.$$

To have a physically reasonable solution we must have $\mu^2 > 9\lambda_0$ or $\lambda > \frac{2}{27}\mu^2$. When $p + \rho = 0$ the solution reduces to the Banados–Teitelboim–Zanelli (BTZ) metric. This limit is $\mu^2 = 9\lambda$. After performing the transformation $\theta = -\theta' + \omega t$,the parameters a_0, b_0, b_1, c_0, e_0 and ω can be arranged in such a way that the above metric (5) becomes

$$\begin{aligned}\mathrm{d}s^2 = &-\left[-M + \frac{\mu^2}{9} r^2 + \frac{\mu^2}{18}\,\frac{\mu^2 - 9\lambda}{(2J\mu - 6M)} r^4\right] \mathrm{d}t^2 \\ &+ \left(J - \frac{\mu}{3}\,\frac{\mu^2 - 9\lambda}{2J\mu - 6M} r^4\right) \mathrm{d}t\,\mathrm{d}\theta + r^2\left[1 - \frac{\mu^2 - 9\lambda}{2(2J\mu - 6M)} r^2\right] \mathrm{d}\theta^2 \\ &+ \frac{\mathrm{d}r^2}{-M + (J^2/4r^2) + [(\mu^2 + 27\lambda) r^2/36]}\end{aligned} \tag{6}$$

with $-\infty < t < \infty$, $0 < r < \infty$, and $0 \leqslant \theta \leqslant 2\pi$. Here the constants are chosen as

$$\begin{aligned} &a_0 = (J\mu - 3M)/3, \qquad b_0 = -M, \qquad c_0 = -J/2, \\ &e_0 = 1, \qquad b_1 = J^2/4, \qquad \omega = -\mu/3. \end{aligned} \tag{7}$$

The metric given in equation (6) describes a black hole with horizons r_+ and r_- defined through the following relations

$$M = \frac{r_+^2 + r_-^2}{l^2}, \qquad J = \frac{2r_+ r_-}{l} \tag{8}$$

where

$$\frac{1}{l^2} = \frac{\mu^2 + 27\lambda}{36}$$

and $r_\pm$ are given by

$$r_\pm^2 = \tfrac{1}{2} l\,(M\,l \pm \sqrt{M^2 l^2 - J^2}). \tag{9}$$

Here we assume $\lambda_0 \neq 0$ ($l \neq \infty$) and $M^2 l^2 - J^2 \geqslant 0$. Two horizons coincide when $J^2 = M^2 l^2$, which is the extremal limit. There is also another extremal limit which corresponds to $l \to \infty$ or $\lambda_0 = 0$. In this case the horizon is located at $r^2 = J^2/4M$. The curvature invariants $R = -6\,\lambda$ and $R_{\mu\nu}\,R^{\mu\nu} = \frac{2}{27}\,(\mu^4 - 18\,\mu^2\lambda + 243\lambda^2)$ are regular everywhere. Further properties of the solution (6) will be discussed elsewhere.

Fluid parameters p and ρ remain the same but the velocity 4-vector becomes

$$u_\mu = \sqrt{a}_0(1 + \tfrac{1}{3}\mu\Omega', 0, \Omega') \tag{10}$$

where Ω' becomes

$$\Omega' = \frac{2\,\mu\,r^2 - 3\,J}{2\mu\,J\, - 6\,M}$$

and a_0 is given in (7). In the limit $\mu^2 = 9\,\lambda$ the above metric reduces exactly to the form given by BTZ. In that form M and J are respectively interpreted as the mass and angular momentum of the solution.

References

[1] Kaloper N 1993 Topological mass generation in three dimensional string theory *Preprint* Alberta-Thy-46-93, September 1993 (to appear in *Phys. Lett.* **B**)

[2] Deser S, Jackiw R and Templeton S 1982 *Phys. Rev. Lett.* **48** 975; 1982 *Ann.Phys., NY* **140** 372

[3] Banados M, Teitelboim C and Zanelli J 1992 *Phys. Rev. Lett.* **69** 1849

[4] Horowitz G and Welch D L 1993 *Phys. Rev. Lett.* **71** 328

[5] Gürses M 1993 Black hole solutions of DJT equations with cosmological constant (unpublished). The solution given in equation (5) has been utilized by Y Nutku in his recent publication 1993 *Class. Quantum. Grav.* **10** 2657