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Complex Analysis and Operator Theory

Embeddings, Operator Ranges, and Dirac Operators

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Abstract We present a generalized operator range construction associated to an indefinite unbounded selfadjoint operator that yields closed embeddings of Kreĭn spaces. As an application we obtain an energy space representation, in the sense of Friedrichs, of a general free Dirac operator.

1 Introduction

In our article [9] a generalization of the continuous embedding of Hilbert spaces, that we call closed embedding and corresponds to unbounded kernel operators, has been obtained and we investigated its connection with operator ranges, its properties and especially uniqueness properties. These constructions have been exemplified by Hilbert spaces associated to certain multiplication or differentiation operators and to some Hilbert spaces of holomorphic functions. As an application it allows us to show

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the closed embedding for a certain homogenous Sobolev space that is associated to a singular integral operator defined by the Riesz potential. The approach starts from previous investigations on Hilbert spaces induced by unbounded operators, as considered in our article [7], and we show that closed embedding is a special representation of a Hilbert space induced by a positive selfadjoint operator.

On the other hand, an indefinite generalization of Hilbert spaces is that of Kreĭn spaces, see e.g. [4] and the rich bibliography cited there, and it is an interest in mathematical physics in connection with models in relativistic quantum physics involving operator theory in Kreĭn spaces, e.g. see [10]. Another motivation for these investigations comes from the works of de Branges [5] and Dritschel and Rovnyak [18] where operator spaces have been considered in the Kreĭn space setting. In [8] it was obtained an interpretation of the positive/negative energetic spaces associated to certain Dirac operators in terms of induced Kreĭn spaces, in the spirit of energy spaces of Friedrichs [20,21]. But induced Kreĭn spaces is a rather abstract notion and, thinking from the perspective of function spaces models, a more concrete representation, in the spirit of reproducing kernel Kreĭn spaces is needed. The new concept, closely embedded Kreĭn spaces, is a generalization of the notion of closely embedded Hilbert spaces but, in order to obtain the correct definition, a model is needed. Compared with its positive definite counter-part, this concept has an interesting feature concerning the splitting of the domain whose necessity can be obtained by considering a generalization of the de Branges space for an unbounded kernel. In this article we present this generalization and apply it to the construction of a Kreĭn space closely contained in $L^2(\mathbb{R}^n; \mathbb{C}^m)$ associated to a general free particle Dirac operator. It is interesting to note that the closely embedded Kreĭn space is of homogeneous Sobolev type, as well. There are other interesting questions that we leave out of this article (due to a constraint related to the size of the manuscript) such as uniqueness conditions, that we will consider in a forthcoming paper that will also contain an application to a description of an energy space representation of a Dirac operator corresponding to a massless free particle.

A few words on the prerequisites. Firstly, we assume that the reader is familiar with the basic notions of indefinite inner product spaces and their linear operators, e.g. see [4]. In this respect, our notation follows the one we used in [8] and we recall this briefly in Subsect. 2.1. Also, we assume that the reader has a good command of operator theory in Hilbert spaces, both bounded and unbounded. In particular, we will use freely the main concepts and results in the operator theory of unbounded selfadjoint operators, especially their spectral theory, borelian functional calculus, and polar decompositions. All these can be found in the classical textbooks of Birman and Solomyak [3], Kato [23], Reed and Simon [26,27]. Occasionally, we will also use freely the basic notions on Sobolev spaces, e.g. see Adams [1], and Maz'ja [25].

2 Induced Kreĭn Spaces

In this section we recall the notion of a Kreĭn space induced by a selfadjoint operator in Hilbert space, cf. [7]. We first recall some basic facts about Kreĭn spaces and their operator theory.

2.1 Kreĭn Spaces and their Linear Operators

We recall that a Kreĭn space $\mathcal{K}$ is a complex linear space on which it is defined an indefinite scalar product $[\cdot,\cdot]$ such that $\mathcal{K}$ is decomposed in a direct sum

$$\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_- \tag{2.1}$$

in such a way that $\mathcal{K}_\pm$ are Hilbert spaces with scalar products $\pm[\cdot,\cdot]$, respectively and the direct sum in (2.1) is orthogonal with respect to the indefinite scalar product $[\cdot,\cdot]$, i.e. $\mathcal{K}_+ \cap \mathcal{K}_- = \{0\}$ and $[x_+, x_-] = 0$ for all $x_\pm \in \mathcal{K}_\pm$. The decomposition (2.1) gives rise to a positive definite scalar product $\langle\cdot,\cdot\rangle$ by setting $\langle x, y\rangle := [x_+, y_+] - [x_-, y_-]$, where $x = x_+ + x_-$, $y = y_+ + y_-$, and $x_\pm, y_\pm \in \mathcal{K}_\pm$. The scalar product $\langle\cdot,\cdot\rangle$ defines on $\mathcal{K}$ a structure of Hilbert space. Subspaces $\mathcal{K}_\pm$ are orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle$, too. We denote by $P_\pm$ the corresponding orthogonal projections onto $\mathcal{K}_\pm$, and let $J = P_+ - P_-$. The operator J is a symmetry, i.e. a selfadjoint and unitary operator, $J^*J = JJ^* = J^2 = I$. The operator J is called a fundamental symmetry of the Kreĭn space $\mathcal{K}$. Note that $[x, y] = \langle Jx, y\rangle$, $(x, y \in \mathcal{K})$. If T is a densely defined operator from a Kreĭn space $\mathcal{K}_1$ to another $\mathcal{K}_2$, it can be defined the adjoint of T as an operator $T^\sharp$ defined on the set of all $y \in \mathcal{K}_2$ for which there exists $h_y \in \mathcal{K}_1$ such that $[Tx, y] = [x, h_y]$ for all $x \in \mathrm{Dom}(T)$, and $T^\sharp y = h_y$. We remark that $T^\sharp = J_1 T^* J_2$, where T^* denotes the adjoint operator of T with respect to the Hilbert spaces $(\mathcal{K}_1, \langle\cdot,\cdot\rangle_{J_1})$ and $(\mathcal{K}_2, \langle\cdot,\cdot\rangle_{J_2})$. We will use $\sharp$ to denote the adjoint when at least one of the spaces $\mathcal{K}_1$ or $\mathcal{K}_2$ is indefinite. In the case of an operator T defined on the Kreĭn space $\mathcal{K}$, T is called symmetric if $T \subset T^\sharp$, i.e. if the relation $[Tx, y] = [x, Ty]$ holds for each $x, y \in \mathrm{Dom}(T)$ and T is called selfadjoint if $T = T^\sharp$.

A (closed) subspace $\mathcal{L}$ of a Kreĭn space $\mathcal{K}$ is called regular if $\mathcal{K} = \mathcal{L} + \mathcal{L}^\perp$, where $\mathcal{L}^\perp = \{x \in \mathcal{K} \mid [x, y] = 0 \text{ for all } y \in \mathcal{L}\}$. Regular spaces of Kreĭn spaces are important since they are exactly the analogue of Kreĭn subspaces, that is, if we want $\mathcal{L}$ be a Kreĭn space with the restricted indefinite inner product and the same strong topology, then it should be regular. In addition, let us recall that, given a subspace $\mathcal{L}$ of a Kreĭn space, we call $\mathcal{L}$ non-negative (positive) if the inequality $[x, x] \geq 0$ holds for $x \in \mathcal{L}$ (respectively, $[x, x] > 0$ for all $x \in \mathcal{L} \setminus \{0\}$). Similarly we define non-positive and negative subspaces. A subspace $\mathcal{L}$ is called degenerate if $\mathcal{L} \cap \mathcal{L}^\perp \neq \{0\}$. Regular subspaces are non-degenerate. As a consequence of the Schwarz inequality, if a subspace $\mathcal{L}$ is either positive or negative it is nondegenerate. A remarkable class of subspaces are those regular spaces that are either positive or negative, for which the terms uniformly positive, respectively, uniformly negative are used. These notions can be defined for linear manifolds also, that is, without assuming closedness.

A linear operator V defined from a subspace of a Kreĭn space $\mathcal{K}_1$ and valued into another Kreĭn space $\mathcal{K}_2$ is called isometric if $[Vx, Vy] = [x, y]$ for all x, y in the domain of V. A unitary operator between Kreĭn spaces means that it is a bounded isometric operator that has a bounded inverse. Also, a coisometric operator W between two Kreĭn spaces is a bounded operator such that its adjoint $W^\sharp$ is isometric.

2.2 Kreĭn Spaces Induced by Symmetric Operators.

If A is a symmetric densely defined linear operator in the Hilbert space $\mathcal{H}$ we can define a new inner product $[\cdot,\cdot]_A$ on $\mathrm{Dom}(A)$, the domain of A, by

$$[x, y]_A = \langle Ax, y\rangle_{\mathcal{H}}, \quad x, y \in \mathrm{Dom}(A). \tag{2.2}$$

In this subsection we recall the existence and the properties of some Kreĭn spaces associated to this kind of inner product space, cf. [8].

A pair $(\mathcal{K}, \Pi)$ is called a Kreĭn space induced by A if:

(iks1) $\mathcal{K}$ is a Kreĭn space;
(iks2) Π is a linear operator from $\mathcal{H}$ into $\mathcal{K}$ such that $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(\Pi)$;
(iks3) $\Pi\,\mathrm{Dom}(A)$ is dense in $\mathcal{K}$;
(iks4) $[\Pi x, \Pi y] = \langle Ax, y\rangle$ for all $x \in \mathrm{Dom}(A)$ and $y \in \mathrm{Dom}(\Pi)$.

The operator Π is called the canonical operator.

Remark 2.1 (1) $(\mathcal{K}, \Pi)$ is a Kreĭn space induced by A if and only if it satisfies the axioms (iks1)–(iks3) and

$$\Pi^\sharp\Pi \supseteq A, \tag{iks4'}$$

in the sense that $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(\Pi^\sharp\Pi)$ and $Ax = \Pi^\sharp\Pi x$ for all $x \in \mathrm{Dom}(A)$.

(2) If A is selfadjoint, hence maximal symmetric, the axiom (iks4') is equivalent with

$$\Pi^\sharp\Pi = A, \tag{iks4''}$$

in the sense that $\mathrm{Dom}(\Pi^\sharp\Pi) = \mathrm{Dom}(A)$ and $Ax = \Pi^\sharp\Pi x$ for all $x \in \mathrm{Dom}(A)$.

(3) Without loss of generality we can assume that Π is closed.

(4) If the symmetric densely defined operator A admits an induced Kreĭn space $(\mathcal{K}, \Pi)$ such that Π is bounded, then A is bounded. The converse is not true, in general, that is, if A is bounded then it may happen that Π is unbounded. However, if A is not only bounded but also everywhere defined (in particular, if A is bounded selfadjoint), then the operator Π is bounded as well.

For a densely defined symmetric operator A in a Hilbert space, various necessary and sufficient conditions of existence of Hilbert spaces induced by A are available (see [8]). In this paper we are interested mainly in the case of selfadjoint operators, when the existence is guaranteed by the spectral theorem.

Two Kreĭn spaces $(\mathcal{K}_i, \Pi_i)$ $i = 1, 2$, induced by the same symmetric operator A, are called unitary equivalent if there exists a bounded unitary operator $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that $U\Pi_1 x = \Pi_2 x$, for all $x \in \mathrm{Dom}(A)$.

3 Closely Embedded Kreĭn Spaces

In this section we make the connection between induced Kreĭn spaces and de Branges-Rovnayk [6] and Schwartz [28] theory of Hilbert/Kreĭn spaces continuously contained. It was shown in [12] that for bounded selfadjoint operators, continuously embedded Kreĭn spaces are particular cases of induced Kreĭn spaces. To a certain extent we replace "continuously embedded" by "closely embedded" but, once again, the intricate geometry of Kreĭn spaces makes additional difficulties.

3.1 The Induced Kreĭn Space $(\mathcal{B}_A, \Pi_{\mathcal{B}_A})$

In this subsection we start with a selfadjoint operator A and present a particular construction of a Kreĭn space induced by A, in the spirit of de Branges Kreĭn space of operator range type.

Let A be a selfadjoint operator in a Hilbert space $\mathcal{H}$ and consider its polar decomposition $A = S_A|A|$, where $|A| = (A^2)^{1/2}$ is a positive selfadjoint operator in $\mathcal{H}$ and $S_A \in \mathcal{B}(\mathcal{H})$ is a selfadjoint partial isometry, $S_A = S_A^*$ and $S_A^2 = P_{\mathcal{H}\ominus\mathrm{Ker}(A)}$, in particular $\mathrm{Ker}(S_A) = \mathrm{Ker}(A)$. As in [9], let us consider the Hilbert space $\mathcal{H}_+ = \mathcal{R}(|A|^{1/2})$, continuously embedded in $\mathcal{H}$ and having the kernel operator $|A|$, that is, with the closed embedding $j_+\colon \mathrm{Ran}(|A|^{1/2}) \to \mathcal{H}$, and such that $|A| = j_+ j_+^*$. Then the positive definite inner product of $\mathcal{H}_+$ is

$$\left\langle |A|^{1/2}x, |A|^{1/2}y\right\rangle_+ = \left\langle P_{\mathcal{H}\ominus\mathrm{Ker}(A)}x, y\right\rangle_{\mathcal{H}}, \quad x, y \in \mathrm{Dom}\left(|A|^{1/2}\right). \tag{3.1}$$

We define an indefinite inner product $[\cdot,\cdot]_{\mathcal{B}_A}$ on $\mathrm{Ran}(|A|^{1/2})$ by

$$\left[|A|^{1/2}x, |A|^{1/2}y\right]_{\mathcal{B}_A} := \langle S_A x, y\rangle_{\mathcal{H}}, \quad x, y \in \mathrm{Dom}\left(|A|^{1/2}\right). \tag{3.2}$$

Denote $\mathcal{B}_A := (\mathcal{H}_+; [\cdot,\cdot]_{\mathcal{B}_A})$ as an indefinite inner product and let $\Pi_{\mathcal{B}_A}$ be the linear operator $\mathcal{H} \to \mathcal{B}_A$ defined by $\mathrm{Dom}(\Pi_{\mathcal{B}_A}) = \mathrm{Dom}(|A|)$ and $\Pi_{\mathcal{B}_A}x = |A|x$ for all $x \in \mathrm{Dom}(|A|)$.

Proposition 3.1 *With the notation as before, $(\mathcal{B}_A; \Pi_{\mathcal{B}_A})$ is a Kreĭn space induced by the selfadjoint operator A having, in addition, the property that $\mathrm{Ran}(\Pi_{\mathcal{B}_A}^\sharp) = \mathcal{D}_+[+]\mathcal{D}_-$ for some positive/negative uniformly definite linear manifolds $\mathcal{D}_\pm$ of the Kreĭn space $\mathcal{B}_A$.*

Proof We verify the axioms (iks1)–(iks4) from the definition of the induced Kreĭn space.

(iks1) Since the bounded operator S_A commutes with $|A|^{1/2}$, it leaves invariant the linear manifold $\mathrm{Ran}(|A|^{1/2})$. We prove that $S_A|\mathrm{Ran}(|A|^{1/2})$ extends uniquely to a symmetry in the Hilbert space $\mathcal{H}_+(=\mathcal{B}_A)$. Indeed, for any $x, y \in \mathrm{Dom}(|A|^{1/2})$ we have

$$\begin{aligned}\left\langle S_A|A|^{1/2}x, |A|^{1/2}y\right\rangle_+ &= \langle S_A x, y\rangle_{\mathcal{H}} \\ &= \langle P_{\mathcal{H}\ominus \mathrm{Ker}(A)} x, S_A y\rangle_{\mathcal{H}} \\ &= \left\langle |A|^{1/2}x, S_A|A|^{1/2}y\right\rangle_+,\end{aligned}$$

hence S_A is symmetric in the Hilbert space $\mathcal{H}_+$ as well, and

$$\begin{aligned}\left\langle S_A|A|^{1/2}x, S_A|A|^{1/2}y\right\rangle_+ &= \langle S_A x, S_A y\rangle_{\mathcal{H}} \\ &= \langle P_{\mathcal{H}\ominus \mathrm{Ker}(A)} x, y\rangle_{\mathcal{H}} \\ &= \left\langle |A|^{1/2}x, |A|^{1/2}y\right\rangle_+,\end{aligned}$$

and hence S_A is isometric in $\mathcal{H}_+$ as well. Thus, $S_A|\operatorname{Ran}(|A|^{1/2})$ extends uniquely to a symmetry in the Hilbert space $\mathcal{H}_+(= cB_A)$. Taking into account (3.2), this shows that $\mathcal{B}_A$ is a Kreĭn space and that the extended operator, still denoted by S_A, is a fundamental symmetry.

(iks2) By definition, $\mathrm{Dom}(\Pi_{\mathcal{B}_A}) = \mathrm{Dom}(|A|) = \mathrm{Dom}(A)$.

(iks3) We have to prove that $\Pi_{\mathcal{B}_A}\mathrm{Dom}(A) = \operatorname{Ran}(|A|)$ is dense in $\mathcal{B}_A = \mathcal{H}_+$ with respect to the norm $\|\cdot\|_+$, which follows from Theorem 3.4.(c) in [9].

(iks4) We have to prove that for all $x \in \mathrm{Dom}(\Pi_{\mathcal{B}_A}) = \mathrm{Dom}(|A|)$ and all $y \in \mathrm{Dom}(A)$ we have

$$[\Pi_{\mathcal{B}_A}x, \Pi_{\mathcal{B}_A}y]_{\mathcal{B}_A} = \langle Ax, y\rangle_{\mathcal{H}}. \tag{3.3}$$

Indeed,

$$\begin{aligned}\left[\Pi_{\mathcal{B}_A}x, \Pi_{\mathcal{B}_A}y\right]_{\mathcal{B}_A} &= \left[|A|^{1/2}|A|^{1/2}x, |A|^{1/2}|A|^{1/2}y\right]_{\mathcal{B}_A} \\ &= \left\langle S_A|A|^{1/2}x, |A|^{1/2}y\right\rangle_{\mathcal{H}} \\ &= \left\langle |A|^{1/2}S_A|A|^{1/2}x, y\right\rangle_{\mathcal{H}} \\ &= \langle Ax, y\rangle_{\mathcal{H}}.\end{aligned}$$

Finally, we prove the additional property. This follows due to the fact that S_A, which is a fundamental symmetry of the Kreĭn space $\mathcal{B}_A$, commutes with $|A|^{1/2}$ and hence $\operatorname{Ran}(\Pi_{\mathcal{B}_A}) = \operatorname{Ran}(|A|^{1/2}) = \mathcal{D}_+[+]\mathcal{D}_-$, where $\mathcal{D}_\pm = (I \pm S_A)\operatorname{Ran}(|A|^{1/2})$ is positive/negative uniformly definite in the Kreĭn space $\mathcal{B}_A$. □

Remark 3.2 With the notation as in Proposition 3.1 let us observe that the closure of $\Pi_{\mathcal{B}_A}$ is $j_+^\sharp$.

The Kreĭn space $\mathcal{B}_A$ may not be uniquely determined by the selfadjoint operator A for the indefinite case, even for bounded kernel operators A, as observed by Schwartz [28] and de Branges [5]. This phenomenon is reflected also in the lack of uniqueness, modulo unitary equivalence, of Kreĭn spaces induced by general selfadjoint operators; see Hara [22] (as well as equivalent results in [11] and [17]) for the bounded case, and [8] for the unbounded case. For this reason it may be interesting to point out the unitary equivalence of the induced Kreĭn spaces $(\mathcal{B}_A; \Pi_{\mathcal{B}_A})$ and $(\mathcal{K}_A; \Pi_A)$. Briefly, with the notation as before, we can consider the seminorm $\||A|^{1/2}\cdot\|$ on $\operatorname{Dom}(A)$ and make the quotient completion with respect to this seminorm in order to get a Hilbert space $(\mathcal{K}_A, \Pi_A)$ induced by $|A|$, cf. [7]. Further on, S_A can be lifted to $\mathcal{K}_A$ and it yields an indefinite inner product $[\cdot,\cdot]_A$ with respect to which $\mathcal{K}_A$ becomes a Kreĭn space.

Proposition 3.3 *Given a selfadjoint operator A in a Hilbert space $\mathcal{H}$, the induced Kreĭn spaces $(\mathcal{B}_A; \Pi_{\mathcal{B}_A})$ and $(\mathcal{K}_A; \Pi_A)$ are unitary equivalent.*

Proof Let U be the operator defined on $\operatorname{Dom}(A) = \operatorname{Dom}(|A|) \subseteq \mathcal{K}_A$ and valued in $\mathcal{B}_A$ by

$$Ux := |A|x, \quad x \in \operatorname{Dom}(A).$$

Then, for all $x, y \in \operatorname{Dom}(A)$ we have

$$\begin{aligned}[Ux, Uy]_{\mathcal{B}_A} &= [|A|x, |A|y]_{\mathcal{B}_A} = \left\langle S_A|A|^{1/2}x, |A|^{1/2}y\right\rangle_{\mathcal{H}} \\ &= \left\langle |A|^{1/2}S_A|A|^{1/2}x, y\right\rangle_{\mathcal{H}} = \langle Ax, y\rangle_{\mathcal{H}} = [x, y]_A\end{aligned}$$

and hence U is isometric with respect to the indefinite inner products. In addition,

$$\|Ux\|_+ = \||A|x\|_+ = \left\| |A|^{1/2}x \right\|_{\mathcal{H}}$$

and hence U is isometric with respect to the underlying positive definite inner products on $\mathcal{K}_A$ and respectively $\mathcal{B}_A$, hence it it bounded. Since U has both dense domain and dense range, it is uniquely extended to a bounded unitary operator between the Kreĭn spaces $\mathcal{K}_A$ and $\mathcal{B}_A$. By the definitions of Π_A and $\Pi_{\mathcal{B}_A}$ we have $U\Pi_A x = |A|x = \Pi_{\mathcal{B}_A}x$ for all $x \in \operatorname{Dom}(A) = \operatorname{Dom}(|A|)$ and hence U is the required unitary equivalence. ◻

3.2 Closely Embedded Kreĭn Spaces

In view of Proposition 3.1 we can now introduce the definition of a closely embedded Kreĭn space. Without loss of generality the ambient space $\mathcal{H}$ will be considered a Hilbert space. Thus, a Kreĭn space $\mathcal{K}$ is called closely embedded in $\mathcal{H}$, equivalently we say that there exists a closed embedding of $\mathcal{K}$ in $\mathcal{H}$, if:

(cek1) There exists a linear manifold $\mathcal{D}$ in $\mathcal{K} \cap \mathcal{H}$ that is dense in $\mathcal{K}$.
(cek2) The canonical embedding $j\colon \mathcal{D}(\subseteq \mathcal{K}) \to \mathcal{H}$ is closed, as an operator from $\mathcal{K}$ to $\mathcal{H}$.

(cek3) There exist positive/negative uniformly definite linear manifolds $\mathcal{D}_\pm$ in $\mathcal{K}$ such that $\mathrm{Dom}(j) = \mathcal{D}_+[+]\mathcal{D}_-$.

This definition is a generalization of the concept of closely embedded Hilbert space that allows us to establish the connection with induced Kreĭn spaces. Again, the meaning of the axiom (cek1) is that on $\mathcal{D}$ the algebraic structures of $\mathcal{K}$ and $\mathcal{H}$ coincide.

Proposition 3.4 *If $\mathcal{H}$ is a Hilbert space and $\mathcal{K}$ is a Kreĭn space closely embedded in $\mathcal{H}$, with embedding operator j, then $A = jj^\sharp$ is a selfadjoint operator in $\mathcal{H}$ and $(\mathcal{K}; j^\sharp)$ is a Kreĭn space induced by A.*

Proof By the Phillips Theorem, e.g. see [4], there exists $\mathcal{K} = \mathcal{K}^+[+]\mathcal{K}^-$ a fundamental decomposition of the Kreĭn space $\mathcal{K}$ such that $\mathcal{D}_\pm \subseteq \mathcal{K}^\pm$, and let J be the associated fundamental decomposition. Then $A = jj^\sharp = jJj^*$ is a selfadjoint operator in the Hilbert space $\mathcal{H}$, where j^* is the adjoint of j with respect to the Hilbert space $\mathcal{H}_+ := (\mathcal{K}; \langle\cdot,\cdot\rangle_J)$. Also, $|A| = jj^*$ and we can apply Proposition 3.1 in [9] in order to conclude that $(\mathcal{H}_+; j^*)$ is a Hilbert space induced by $|A|$. Since $j^\sharp = Jj^*$ this implies that $(\mathcal{K}; j^\sharp)$ is a Kreĭn space induced by A. □

Given $\mathcal{K}$, a Kreĭn space closely embedded in the Hilbert space $\mathcal{H}$, with the closed embedding $j\colon \mathrm{Dom}(j)(\subseteq \mathcal{K}) \to \mathcal{H}$, we call $A := jj^\sharp$ the kernel operator of $\mathcal{K}$. The axiom (cek3) in the definition of a closely embedded Kreĭn space is justified by the anomaly in the indefinite setting that allows closed densely defined operators T between Kreĭn spaces such that $TT^\sharp$ may not be densely defined.

Example 3.5 Let (w_n) be a real sequence with $w_n \neq 0$ for all $n \in \mathbb{N}$. Let $|w| = (|w_n|)$. On the Hilbert space $\ell^2_{|w|}$, of complex sequences x with $\sum_{n=1}^\infty |w_n| x_n^2 < \infty$, consider the inner product

$$[x, y]_w = \sum_{n=1}^\infty w_n x_n \overline{y}_n, \quad x, y \in \ell^2_{|w|}.$$

Then $(\ell^2_{|w|}; [\cdot,\cdot]_w)$ is a Kreĭn space.

We split the components of the sequence (w_n) in two components according to the signs, (w_n^+) and (w_n^-) (one of them may be a finite sequence). If either $\inf_n w_n^+ = 0$ or $\inf_n w_n^- = 0$ then ℓ^2_w is closely embedded, but not continuously, in ℓ^2, with kernel operator $M_{w^{-1}}$, the operator of multiplication with $w^{-1} = (w_n^{-1})$ in ℓ^2.

As a consequence of Proposition 3.3, Proposition 3.4 and the Lifting Theorem as in [8], we have a generalization, to the unbounded case, of the indefinite variant of the Lifting Theorem in [14] in the formulation of [18],

Theorem 3.6 *Let A and B be two selfadjoint operators in the Hilbert spaces $\mathcal{H}_1$ and respectively $\mathcal{H}_2$. We consider the Kreĭn spaces $\mathcal{B}_A$ and $\mathcal{B}_B$, closely embedded in $\mathcal{H}_1$ and respectively $\mathcal{H}_2$, as well as the closed embeddings $j_A\colon \mathrm{Dom}(j_A)(\subseteq \mathcal{B}_A) \to \mathcal{H}_1$ and respectively, $j_B\colon \mathrm{Dom}(j_B)(\subseteq \mathcal{B}_B) \to \mathcal{H}_2$. Then, for any operators $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that*

$$\langle Bx, Ty\rangle_{\mathcal{H}_2} = \langle Sx, Ay\rangle_{\mathcal{H}_1}, \quad x \in \mathrm{Dom}(B),\ y \in \mathrm{Dom}(A),$$

there exist uniquely determined operators $\tilde{T} \in \mathcal{L}(\mathcal{B}_A, \mathcal{B}_B)$ *and* $\tilde{S} \in \mathcal{L}(\mathcal{B}_B, \mathcal{B}_A)$ *such that* $\widetilde{T} j_A^\sharp x = j_B^\sharp T x$ *for all* $x \in \mathrm{Dom}(A)$*,* $\widetilde{S} j_B^\sharp y = j_A^\sharp S y$*, for all* $y \in \mathrm{Dom}(B)$*, and*

$$\langle \widetilde{S}h, k \rangle_{\mathcal{B}_A} = \langle h, \widetilde{T}k \rangle_{\mathcal{B}_B}, \quad h \in \mathcal{B}_B,\ k \in \mathcal{B}_A.$$

4 Closely Embedded Kreĭn Spaces Associated to Dirac Operators

Our motivation for introducing the concept of closely embedded Kreĭn space comes from the energy space representation, in the sense of Friedrichs [20,21], of the Dirac operators. In this section we will use the definitions and basic properties of Sobolev spaces, as in Adams [1] and Maz'ja [25]. In addition, some basic facts on Dirac operators and their spectral theory that will be used can be found in Thaller [30].

Below the following notations are systematically used. We let $L_2(\mathbb{R}^n; \mathbb{C}^m) = \mathbb{C}^m \otimes L_2(\mathbb{R}^n)$ the space of all square summable $\mathbb{C}^m$-valued functions on $\mathbb{R}^n$. By $\widehat{u}(\xi)$ we denote the Fourier transform of $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$:

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int u(x) e^{\mathrm{i}\langle x, \xi \rangle} \,\mathrm{d}\, x,$$

in which $\langle x, \xi \rangle$ designates the scalar product of all elements $x, \xi \in \mathbb{R}^n$. Here and in what follows $\int := \int_{\mathbb{R}^n}$. The norm in $\mathbb{R}^n$ (or $\mathbb{C}^m$) will be denoted as simply by $|\cdot|$. The operator norm of $m \times m$ matrices corresponding to the norm $|\cdot|$ in $\mathbb{C}^m$ will be denoted by $|\cdot|$,as well. We will also need two more Hilbert spaces. $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ is defined as the completion of $L_2(\mathbb{R}^n; \mathbb{C}^m)$ with respect to the norm

$$\|u\|^2_{W_2^{-1/2}} := \int (1 + |\xi|^2)^{-1/2} |\widehat{u}(\xi)|^2 \,\mathrm{d}\,\xi. \tag{4.1}$$

In addition, $W_2^{1/2}(\mathbb{R}^n; \mathbb{C}^m)$ is defined to be the Sobolev space of all $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$ for which the norm

$$\|u\|^2_{W_2^{1/2}} := \int (1 + |\xi|^2)^{1/2} |\widehat{u}(\xi)|^2 \,\mathrm{d}\,\xi < \infty. \tag{4.2}$$

Let H denote the free Dirac operator defined in the space $L_2(\mathbb{R}^n; \mathbb{C}^m) = \mathbb{C}^m \otimes L_2(\mathbb{R}^n)$ by

$$H = \sum_{k=1}^{n} \alpha_k \otimes D_k + \alpha_0 \otimes I, \tag{4.3}$$

where $D_k = \mathrm{i}\partial/\partial x_k$ for $(k = 1, \dots, n)$, α_k for $(k = 0, 1, \dots, n)$ are $m \times m$ Hermitian matrices which satisfy the Clifford's anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_m, \quad (j, k = 0, 1, \dots, n), \tag{4.4}$$

$m = 2^{n/2}$ for n even and $m = 2^{(n+1)/2}$ for n odd, δ_{jk} denotes the Kronecker symbol, I_m is the $m \times m$ unit matrix, and I is the identity operator on $L_2(\mathbb{R}^n)$.

We consider the operator H defined on its maximal domain, the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$, and it is a selfadjoint operator. Note that

$$\begin{aligned} H^2 &= \sum_{k=1}^{n} \alpha_k^2 \otimes D_k^2 + \sum_{j \neq k} (\alpha_j \alpha_k + \alpha_k \alpha_j) \otimes D_j D_k \\ &\quad + \sum_{k=1}^{n} (\alpha_0 \alpha_k + \alpha_k \alpha_0) \otimes D_k + \alpha_0^2 \otimes I \\ &= \sum_{k=1}^{n} I_m \otimes D_k^2 + I_m \otimes I = I_m \otimes (-\Delta + I), \end{aligned}$$

that is,

$$H^2 = I_m \otimes (-\Delta + I), \tag{4.5}$$

where Δ denotes the Laplace operator on $\mathbb{R}^n$.

In the following we want to construct the space $\mathcal{B}_H$ as in Subsect. 3.1. One of the difficulties encountered in pursuing this way is related to making explicit and computable the operator $|H|^{1/2}$. Thus, we consider the polar decomposition of the Dirac operator H writing $H = S|H|$ with the selfadjoint and positive operator $|H|$ (the modulus of H) defined on $\mathrm{Dom}(|H|) = \mathrm{Dom}(H)$ and $S = \mathrm{sgn}(H)$. By (4.5) we have

$$|H| = I_m \otimes (-\Delta + I)^{1/2} \ \text{ and } \ S = H\left(I_m \otimes (-\Delta + I)^{-1/2}\right).$$

Further on, we let

$$T = |H|^{1/2} = I_m \otimes (-\Delta + I)^{1/4} \tag{4.6}$$

by considering T defined in $L_2(\mathbb{R}^n; \mathbb{C}^m)$ with domain $\mathrm{Dom}(T) := W_2^{1/2}(\mathbb{R}^n; \mathbb{C}^m)$. The operator T represents on this domain a positive definite selfadjoint operator. In particular, T is a boundedly invertible operator, and its inverse T^{-1} is the (vector-valued) Bessel potential $I_m \otimes (I - \Delta)^{-1/4}$ of order $l = 1/2$ (cf. Stein [29]).

We consider on $\mathrm{Ran}(T) = L_2(\mathbb{R}^n; \mathbb{C}^m)$ an inner product by setting

$$\langle Tf, Tg \rangle := \langle f, g \rangle_{L_2}, \quad f, g \in W_2^{1/2}(\mathbb{R}^n; \mathbb{C}^m).$$

We can choose for the completion of $L_2(\mathbb{R}^n; \mathbb{C}^m)$ with respect to the corresponding norm $\| \cdot \|_T$ the space $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ that is not entirely made up of functions, but at least of $\mathbb{C}^m$-valued distributions. Keeping the notations made in Subsect. 3.1 we have $\mathcal{R}(T) = \mathcal{B}_H = W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$, for T defined as in (4.6). Since S commutes with H, it follows from Theorem 3.6 that the operator S extends uniquely to a symmetry J_T

in the space $\mathcal{R}(T)$, and hence $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ can be regarded as a Krein space with respect to the fundamental symmetry J_T. The corresponding indefinite inner product is defined by

$$[u, v]_T = \langle J_T u, v \rangle_{W_2^{-1/2}}, \quad u, v \in W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m), \tag{4.7}$$

and $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ is closely (but not continuously) embedded in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$. The canonical embedding operator j_T of $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ in $L_2(\mathbb{R}^n; \mathbb{C}^m)$ is defined on the domain $\mathrm{Dom}(j_T) = L_2(\mathbb{R}^n; \mathbb{C}^m)$, and since the kernel operator of this closed embedding is H (cf. Proposition 3.4), we get that for the Dirac operator there holds the following factorization

$$H = j_T j_T^{\sharp} = j_T J_T j_T^{*}. \tag{4.8}$$

Concerning the symmetry S, the space $\mathcal{H} := L_2(\mathbb{R}^n; \mathbb{C}^m)$ can be decomposed into an orthogonal direct sum

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

where $\mathcal{H}_\pm = S_\pm \mathcal{H}$ and $S_\pm = \frac{1}{2}(I \pm S)$, that is, $S = S_+ - S_-$ is the Jordan decomposition of S. This provides the Jordan decomposition of $H = H_+ - H_-$, where

$$H_+ := S_+ H S_+ = S_+ S|H|S_+ = S_+|H|S_+ \geq 0,$$

and

$$H_- := -S_- H S_- = -S_- S|H|S_- = S_-|H|S_- \geq 0$$

on $\mathrm{Dom}(H)$. In this respect, we note that both operators H_+ and H_- are positive definite selfadjoint in $\mathcal{H}$, and that $\sigma(H_-) = (-\infty, -1]$ and $\sigma(H_+) = [1, +\infty)$, (cf. (4.5)) and, of course, $\sigma(H) = \sigma(H_-) \cup \sigma(H_+) = (-\infty, -1] \cup [1, +\infty)$.

Summing up, we proved the following

Theorem 4.1 (i) *The space $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ defined by (4.1) can be organized as a Kreĭn space by extending uniquely the symmetry S to a fundamental symmetry J_T on the space $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$.*

(ii) *The space $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ endowed with the indefinite inner product (4.7) is a Kreĭn space closely, but not continuously, embedded in $L_2(\mathbb{R}^n; \mathbb{C}^m)$, with canonical embedding operator j_T having the domain $L_2(\mathbb{R}^n; \mathbb{C}^m)$, and the kernel operator of this canonical embedding j_T is the Dirac operator H.*

(iii) *The Dirac operator H admits the factorization (4.8).*

According to the Friedrichs interpretation of the energy space associated to a Hamiltonian, the Kreĭn space $\mathcal{K} = W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ can be regarded as the energy space associated to the Dirac operator H. This space consists of distributions in which the function space $L_2(\mathbb{R}^n; \mathbb{C}^m)$ is dense. The Kreĭn space structure of $\mathcal{K}$ shows that there exist some vectors u of positive energy $[u, u]_{\mathcal{K}} > 0$, some vectors v of negative energy $[v, v]_{\mathcal{K}} < 0$, as well as nontrivial vectors w of null energy $[w, w]_{\mathcal{K}} = 0$. The fundamental symmetry J_T defined as the lifting of the symmetry S from $\mathcal{H} = L_2(\mathbb{R}^n; \mathbb{C}^m)$ to $\mathcal{K}$ through the lifting Theorem 3.6, has a special role, because the associated fundamental symmetry $\mathcal{K} = \mathcal{K}_-[+]\mathcal{K}_+$ has the remarkable property that $\mathcal{H}_\pm$ are, respectively, dense in $\mathcal{K}_\pm$. Thus, even though some of the elements in $\mathcal{K}_\pm$ are distributions, they can be normally approximated by functions in $\mathcal{H} = L_2(\mathbb{R}^n; \mathbb{C}^m)$, of the same type (that is, positive or, respectively, negative).

References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Aronszajn, N.: Theory of reproducing kernels. Trans. Am. Math. Soc. **68**, 337–404 (1950)
3. Birman, M.S., Solomjak, M.Z.: Spectral Theory of Self-Adjoint Operators in Hilbert Space. D. Reidel Publishing Company, Dordrecht (1987)
4. Bognar, J.: Indefinite Inner Product Spaces. Springer, Berlin (1976)
5. de Branges, L.: Complementation in Kreĭn spaces. Trans. Am. Math. Soc. **305**, 277–291 (1988)
6. de Branges, L., Rovnyak, J.: Canonical models in quantum scattering theory. In: Perturbation Theory and its Applications in Quantum Mechanics. Proc. Adv. Sem. Math. Res. Center, US Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, 1965, pp. 295–392. Wiley, New York (1966)
7. Cojuhari, P.A., Gheondea, A.: On lifting of operators to Hilbert spaces induced by positive selfadjoint operators. J. Math. Anal. Appl. **304**, 584–598 (2005)
8. Cojuhari, P.A., Gheondea, A.: Kreĭn spaces induced by symmetric operators. J. Oper. Theory **61**, 347–367 (2009)
9. Cojuhari, P.A., Gheondea, A.: Closed embeddings of Hilbert spaces. J. Math. Anal. Appl. (to appear)
10. Constantinescu, F.: Definite and indefinite inner products on superspace (Hilbert-Krein superspace). J. Phys. A **38**(6), 1385–1402 (2005)
11. Constantinescu, T., Gheondea, A.: Elementary rotations of linear operators in Kreĭn spaces. J. Oper. Theory **29**, 167–203 (1993)
12. Constantinescu, T., Gheondea, A.: Representations of Hermitian kernels by means of Kreĭn spaces. Publ. RIMS Kyoto Univ. **33**, 917–951 (1997)
13. Davies, E.B.: A review of Hardy inequalities. In: The Maz'ya Anniversary Collection, vol. 2 (Rostock, 1998), pp. 55–67. Oper. Theory Adv. Appl., 110, Birkhäuser, Basel (1999)
14. Dijksma, A., Langer, H., de Snoo, H.S.V.: Unitary colligations in Kreĭn spaces and their role in extension theory of isometries and symmetric linear relations in Hilbert spaces. In: Functional Analysis. II, Lecture Notes in Math., vol. 1242, pp. 1–42. Springer, Berlin (1987)
15. Dixmier, J.: Étude sur les variétés et les operateurs de Julia. Bull. Soc. Math. France **77**, 11–101 (1949)
16. Dixmier, J.: Sur les varietés J d'un espace de Hilbert. J. Math. Pures Appl. **28**, 321–358 (1949)
17. Dritschel, M.A.: The essential uniqueness property for operators on Kreĭn spaces. J. Funct. Anal. **118**, 198–248 (1993)
18. Dritschel, M.A., Rovnyak, J.: Operators on indefinite inner product spaces. In: Lectures on Operator Theory and its Applications (Waterloo, ON, 1994), Fields Inst. Monogr., vol. 3, pp. 141–232. Am. Math. Soc., Providence (1996)
19. Fillmore, P.A., Williams, J.P.: On operator ranges. Adv. Math. **7**, 254–281 (1971)
20. Friedrichs, K.O.: Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren. I. Math. Ann. **109**, 465–487 (1934)
21. Friedrichs, K.O.: Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren. II. Math. Ann. **109**, 685–713 (1934)
22. Hara, T.: Operator inequalities and construction of Kreĭn spaces. Integral Equation Oper Theory **15**, 551–567 (1992)
23. Kato, T.: Perturbation Theory for Linear Operators. Springer, Berlin (1966)
24. Mackey, G.W.: On the domains of closed linear transformations in Hilbert spaces. Bull. Am. Math. Soc. **52**, 1009 (1946)

25. Maz'ja, V.G.: Sobolev Spaces. Springer, Berlin (1985)
26. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-adjointness. Academic Press, New York (1975)
27. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. IV. Analysis of Operators. Academic Press, New York (1978)
28. Schwartz, L.: Sous espace Hilbertiens d'espaces vectoriel topologiques et noyaux associés (noyaux reproduisants). J. Anal. Math. **13**, 115–256 (1964)
29. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
30. Thaller, B.: The Dirac Equation. Springer, Berlin (1992)