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Results in Mathematics

The Existence of Gradient Yamabe Solitons on Spacetimes

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Abstract. The main aim of the present article is to study generalized quasi-Yamabe gradient solitons on warped product manifolds. First, we obtain some necessary and sufficient conditions for the existence of generalized quasi-Yamabe gradient solitons equipped on a warped product structure. Then we study some important applications in the Lorentzian and the neutral settings for the particular class, called as gradient Yamabe soliton. More explicitly, we prove the existence of the non-trivial gradient Yamabe soliton on generalized Robertson–Walker spacetimes, standard static spacetimes, Walker manifolds and pp-wave spacetimes.

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1. Introduction

In recent years, self-similar solutions, referred as soliton solutions of some geometric flow equations, have been introduced and studied as they appear to be possible singularity models. The most famous and studied class of them is the Ricci solitons, defined as fixed points of the Ricci flow and significant progress has been made on this research area. Then, in the late 1980's, Hamilton introduced the Yamabe flow to prove the Yamabe problem, [1]. Basically, the Yamabe problem is about investigating a metric on an $n \geq 3$ dimensional manifold such that the underlying scalar curvature is constant. Thus, the Yamabe flow is defined as the metric $g(t)$ on a pseudo-Riemannian manifold (M^n, g) satisfying

$$\frac{\partial g(t)}{\partial t} = -\tau g(t)$$

where τ is the scalar curvature of M. The solution of this particular problem for two-dimensional case is already guaranteed by the Uniformization Theorem. For the further studies focused on this problem, we refer to [2–4].

Gradient Yamabe solitons are the solutions of this flow and defined as follows: an $n-$dimensional pseudo-Riemannian manifold (M^n, g) is said to be a gradient Yamabe soliton if M admits a smooth function φ and a constant λ satisfying

$$\mathrm{Hess}(\varphi) = (\tau - \lambda)g. \tag{1}$$

If $\lambda > 0, \lambda < 0$ or $\lambda = 0$, then (M^n, g) is called a shrinking, expanding or steady gradient Yamabe soliton respectively. One of the most important results for a solution to the Yamabe problem is given in [4] and it is proved that the scalar curvature of any compact Yamabe gradient soliton is constant.

In [5], some boundedness conditions of the potential function of gradient Yamabe solitons were investigated. In [6], it is proved that gradient Yamabe soliton admits a warped product structure, and this result enables to make possible studies in both Riemannian and Lorentzian settings. Moreover, in [7], the three-dimensional complete gradient Yamabe solitons with divergence free Cotton tensor (i.e., Bach-flat) has been classified in terms of having a warped product manifold structure. In [8] it is obtained that the fundamental group of a complete non-compact shrinking Yamabe soliton (M, g, V, λ) is finite provided that the scalar curvature is strictly bounded above by λ.

After the Ricci flow theory had been introduced and some substantial amount of progress had been made to classify the Riemannian manifolds and generalize the Ricci solitons, almost gradient Ricci solitons, quasi-Einstein manifolds and generalized quasi-Einstein manifolds were introduced and studied extensively. For further details, we refer to [9–13] and many others. Analogously, some generalizations of the self-similar solutions of Yamabe flow are also defined in the related literature. First, the notion of quasi-Yamabe gradient soliton in [14,15] and then notion of generalized quasi-Yamabe gradient soliton were introduced in [16]:

An n-dimensional pseudo-Riemannian manifold (M, g) is said to be a generalized quasi-Yamabe gradient soliton if there exist smooth functions φ and μ on M and also a constant λ satisfying

$$\mathrm{Hess}(\varphi) = (\tau - \lambda)g + \mu \mathrm{d}\varphi \otimes \mathrm{d}\varphi \tag{2}$$

where $\mathrm{d}\varphi$ is the dual 1-form of $\nabla\varphi$ and τ is the scalar curvature of M. Here, φ is called as a potential function and the underlying generalized quasi-Yamabe gradient soliton (briefly GQY) is denoted by $(M, g, \varphi, \mu, \lambda)$.

Assume that φ is a constant function, then (M, g) is called as a trivial generalized quasi-Yamabe gradient soliton. Otherwise, it will be called nontrivial. The restricted case where μ is constant is called quasi-Yamabe gradient soliton. Moreover, if $\mu = 0$, then the equation (2) reduces to the fundamental equation of gradient Yamabe soliton (1).

In [14], it is proved that compact quasi-Yamabe gradient soliton has constant scalar curvature and in [17], Wang studied the special generalized quasi-Yamabe gradient soliton in which $\mu = \frac{1}{m}$ for some constant $m > 0$ and proved that m-quasi-Yamabe gradient soliton has also a warped product structure in the region $||\nabla\varphi|| \neq 0$. In addition to that, it is established that the warping function is completely defined by the potential function of the soliton. Then in [16], Neto and Oliveira extended these results to the generalized quasi-Yamabe gradient solitons.

Inspired by these studies, we investigate some necessary and sufficient conditions for the existence of generalized quasi-Yamabe gradient solitons equipped with the warped product structure. Then we study some important applications in the Lorentzian and the neutral settings for the class of gradient Yamabe solitons. We prove the existence of the non-trivial gradient Yamabe soliton on generalized Robertson–Walker spacetimes, standard static spacetimes, Walker manifolds and pp-wave spacetimes.

2. Warped Product Generalized quasi-Yamabe Gradient Solitons

Assume that (B, g_B) and (F, g_F) are two pseudo-Riemannian manifolds of dimensions r and s, respectively. Let $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ be the natural projection maps of the Cartesian product $B \times F$ onto B and F, respectively. Also, let $b : B \to (0, \infty)$ be a positive real-valued smooth function. The warped product manifold $M = B \times_b F$ is the the product manifold $B \times F$ equipped with the metric tensor defined by

$$g = \pi^* (g_B) \oplus (b \circ \pi)^2 \sigma^* (g_F)$$

where * denotes the pull-back operator on tensors [18–20]. The function b is called the warping function of the warped product manifold $B \times_b F$, and the manifolds B and F are called base and fiber, respectively. In particular, if $b = 1$, then $B \times_1 F = B \times F$ is the usual Cartesian product manifold. For the sake of simplicity, throughout this paper, all relations will be written, without involving the projection maps from $B \times F$ to each component B and F as in $g = g_B \oplus b^2 g_F$.

Proposition 1. *Let (M, g) be an n-dimensional pseudo-Riemannian manifold. Then $(M, g, \varphi, \mu, \lambda)$ is a quasi-Yamabe gradient soliton if and only if*

$$\mathrm{Hess}(\theta) = -\frac{\theta}{m}(\tau - \lambda)g \tag{3}$$

where $\mu = 1/m$ and $\theta = \mathrm{e}^{-\varphi/m}$.

Proof. Introduce $\mu = 1/m$ and $\theta = \mathrm{e}^{-\varphi/m}$. Then $\nabla\theta = -\frac{\theta}{m}\nabla\varphi$ and $\mathrm{d}\theta = -\frac{\theta}{m}\mathrm{d}\varphi$. So, for vector fields X and Y on M, we have:

$$\begin{aligned}\mathrm{Hess}(\varphi)(X,Y) &= g(\nabla_X\nabla\varphi, Y)\\ &= g\Big(\frac{m}{\theta^2}X(\theta)\nabla\theta - \frac{m}{\theta}\nabla_X\nabla\theta, Y\Big)\\ &= \frac{m}{\theta^2}X(\theta)Y(\theta) - \frac{m}{\theta}\mathrm{Hess}(\theta)\\ &= \frac{1}{m}\mathrm{d}\varphi\otimes\mathrm{d}\varphi - \frac{m}{\theta}\mathrm{Hess}(\theta)\end{aligned}$$

Thus Eq. (2) can be reduced to (3). □

As a direct corollary of the above proposition, we have:

Corollary 1. *Every quasi-Yamabe gradient soliton is conformal gradient soliton.*

Now, we can apply one the results proved by Cheeger and Colding in [21] where the authors provided some characterization of warped product manifolds and established that any conformal gradient soliton satisfying $\mathrm{Hess}(\theta) = hg$, for some function h is isometric to a warped product with a base manifold of some open and connected interval. Thus, taking (3) into consideration, we conclude that (M^n, g) is isometric to the warped product $I \times_\rho N$, for some positive function ρ where $I \subseteq \mathbb{R}$ is an open and connected interval. Thus:

Corollary 2. *Every quasi-Yamabe gradient soliton admits the warped product structure $I \times_\rho N$, for some positive function ρ, where $I \subseteq \mathbb{R}$ is an open and connected interval.*

In what follows, we will present our main result:

Theorem 1. *Let $M = B \times_b F$ be a warped product manifold equipped with the metric $g = g_B \oplus b^2 g_F$. Then $(M, g, \varphi, \mu, \lambda)$ is a generalized quasi-Yamabe gradient soliton if and only if the followings hold:*

1. *the potential function φ depends only on the base manifold B,*
2. *the gradient vectors of the potential function φ and the warping function b cannot be orthogonal,*
3. *the base manifold B is also a conformal gradient soliton,*
4. *the scalar curvature τ_F of the fiber manifold (F, g_F) is constant.*

Proof. Assume that $(M, g, \varphi, \mu, \lambda)$ is a generalized quasi-Yamabe gradient soliton which is a also warped product. If X, Y are vector fields on B and V, W are vector fields on F, then apply the last proposition as well as Proposition 35 (pg. 206) of [19] and then obtain:

$\mathrm{Hess}(\theta)(X,V) = 0$ since $g(X,V) = 0$. On the other hand, by decomposing $\nabla\theta$ on the base and fiber,

$$\begin{aligned}\mathrm{Hess}(\theta)(X,V) &= g(\nabla_X\nabla\theta, V)\\ &= g(\nabla_X\tan(\nabla\theta), V) + g(\nabla_X\mathrm{nor}(\nabla\theta), V)\\ &= bX(b)g_F(\mathrm{nor}(\nabla\theta), V).\end{aligned}$$

Thus, $X(b)g_F(\mathrm{nor}(\nabla\theta), V) = 0$. If b is not constant, the last equation implies that $\mathrm{nor}(\nabla\theta) = 0$. So, $\theta = \mathrm{e}^{-\varphi/m}$ depends only on the base manifold B, that is, φ is only defined on B, $\varphi \in \mathcal{C}^\infty(B)$.

Moreover, $g(V,W) = b^2 g_F(V,W)$ and $\nabla\theta = \tan(\nabla\theta)$ since $\mathrm{nor}(\nabla\theta) = 0$.

$$\begin{aligned}\mathrm{Hess}(\theta)(V,W) &= g(\nabla_V\nabla\theta, W)\\ &= g(\nabla_V\tan(\nabla\theta), W)\\ &= b\tan(\nabla\theta)(b)g_F(V,W).\end{aligned}$$

Thus, by the last proposition,

$$b\tan(\nabla\theta)(b)g_F(V,W) = -\frac{\theta}{m}(\tau-\lambda)b^2 g_F(V,W).$$

Equivalently,

$$\Big(b\tan(\nabla\theta)(b) + \frac{b^2\theta}{m}(\tau-\lambda)\Big)g_F(V,W) = 0.$$

Contracting the last equation over V and W, we obtain:

$$sb\big(\tan\big(\nabla\theta)(b) + \frac{b\theta}{m}(\tau-\lambda)\big) = 0.$$

Noting that $\tan(\nabla\theta) = \nabla^B(\theta)$ since $\theta \in \mathcal{C}^\infty(B)$, we have:

$$g_B(\nabla^B\theta, \nabla^B b) = (\lambda-\tau)\frac{b\theta}{m}.$$

Finally,

$$\begin{aligned}\mathrm{Hess}(\theta)(X,Y) &= g(\nabla_X\nabla\theta, Y)\\ &= g_B(\nabla_X\tan(\nabla\theta), Y)\\ &= \mathrm{Hess}^B(\theta)(X,Y)\end{aligned}$$

since $\theta \in \mathcal{C}^\infty(B)$, i.e, $\tan(\nabla\theta) = \nabla^B(\theta)$. By the last proposition, we can have:

$$\mathrm{Hess}^B(\theta) = \frac{\theta}{m}(\lambda-\tau)g_B.$$

Hence, B is conformal gradient soliton. It is obvious that $\lambda-\tau$ is defined only on B. Now applying the scalar curvature τ formula of a warped product (see Exercise 13 (pg. 214) of [19]), one can easily deduce that the scalar curvature τ_F of the fiber manifold (F, g_F) is constant. The converse statement is straightforward. □

Combining the above results, we can state that:

Corollary 3. *Let $(M, g, \varphi, \mu, \lambda)$ be a generalized quasi-Yamabe gradient soliton satisfying the conditions (1–4) of Theorem 1. Then it admits the multiply warped product structure $I \times_\rho N \times_b F$, for some positive functions ρ and b, where $I \subseteq \mathbb{R}$ is an open and connected interval.*

3. Applications

3.1. Gradient Yamabe Soliton on Generalized Robertson–Walker Spacetimes

We first define generalized Robertson–Walker spacetimes. Assume that (F, g_F) is an $s-$dimensional Riemannian manifold and $b : I \to (0, \infty)$ is a smooth function. Then the $(s+1)-$dimensional product manifold $I \times_b F$ equipped with the metric tensor

$$g = -\mathrm{d}t^2 \oplus b^2 g_F$$

is called a generalized Robertson–Walker spacetime and is denoted by $M = I \times_b F$ where I is an open and connected interval in $\mathbb{R}$ and $\mathrm{d}t^2$ is the usual Euclidean metric tensor on I. This structure was introduced to extend Robertson–Walker spacetimes [22,23] and have been studied by many authors, such as [24–26]. From now on, we will denote $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ by ∂_t to state our results in compact forms.

We will apply our main result Theorem 1. Assume that $\varphi \in \mathcal{C}^\infty(I)$ is a potential function for a generalized Robertson–Walker spacetime of the form $M = I \times_b F$.

The equation $(\tau - \lambda)g_{ij} = \mathrm{Hess}(\varphi)_{ij}$ yields

$$\begin{cases} \varphi'' = -(\tau - \lambda), \\ b'\varphi' = (\tau - \lambda)b. \end{cases}$$

Thus $b\varphi'' = -b'\varphi'$. By solving the last ODE, as $b \neq 0$, we have:

$$\varphi(t) = \alpha \int_{t_0}^{t} \frac{1}{b(\bar{t})} \mathrm{d}\bar{t} \quad \text{for some} \quad \alpha \in \mathbb{R}.$$

Hence, we can state that:

Theorem 2. *A generalized Robertson–Walker spacetime of the form $M = I \times_b F$ is a gradient Yamabe soliton with the potential function φ given by*

$$\varphi(t) = \alpha \int_{t_0}^{t} \frac{1}{b(\bar{t})} \mathrm{d}\bar{t} \quad \textit{for some} \quad \alpha \in \mathbb{R}.$$

3.2. Gradient Yamabe Soliton on Standard Static Spacetimes

We begin by defining standard static spacetimes. Let (F, g_F) be an $s-$dimensional Riemannian manifold and $f : F \to (0, \infty)$ be a smooth function. Then the $(s+1)-$dimensional product manifold ${}_fI \times F$ furnished with the metric tensor

$$g = -f^2 \mathrm{d}t^2 \oplus g_F$$

is called a standard static spacetime and is denoted by $M =_f I \times F$ where I is an open and connected subinterval in $\mathbb{R}$ and dt^2 is the usual Euclidean metric tensor on I.

Note that standard static spacetimes can be considered as a generalization of the Einstein static universe [27–30] and many spacetime models that characterize the universe and the solutions of Einstein's field equations are known to have this structure.

Again we apply Theorem 1. Suppose that $\varphi \in \mathcal{C}^\infty(F)$ is a potential function for a standard static spacetime of the form $M =_f I \times F$.

The equation $(\tau - \lambda)g_{ij} = \mathrm{Hess}(\varphi)_{ij}$ yields

$$\begin{cases} \nabla\varphi(f) = (\tau - \lambda)f, \\ \mathrm{Hess}_F(\varphi) = (\tau - \lambda)g_F. \end{cases}$$

By contracting the last equation over F, we have $\Delta_F(f) = s(\tau - \lambda)$. Then we obtain:

$$\Delta_F(\varphi) = \frac{s}{f}\nabla\varphi(f).$$

Hence, we conclude that:

Theorem 3. *A standard static spacetime of the form $M =_f I \times F$ is a gradient Yamabe soliton with the potential function φ given by*

$$\Delta_F(\varphi) = \frac{s}{f}\nabla\varphi(f).$$

Example 1. The exterior Schwarzschild spacetime [31,32] can be expressed as a standard static spacetime of the form $\mathbb{R}_f \times (2m, \infty) \times \mathbb{S}^2$ where $\mathbb{S}^2$ is the 2-dimensional Euclidean sphere and the warping function $f \colon (2m, \infty) \times \mathbb{S}^2 \to (0, \infty)$ is given by $f(r, \theta, \phi) = \sqrt{1 - 2m/r}$, $r > 2m$ and also the line element on $(2m, \infty) \times \mathbb{S}^2$ is

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + sin^2\theta \mathrm{d}\phi^2).$$

It is known that on a standard static spacetime, $\tau = \tau_F - 2\frac{\Delta_F(f)}{f}$ holds. Note that Schwarzschild spacetime is Ricci-flat, i.e, $\tau = 0$ and so $\tau_F = 2\frac{\Delta_F(f)}{f}$. Also, the fiber $F = (2m, \infty) \times \mathbb{S}^2$ is noncompact. Then by (3.2), we have $g_F(\nabla\varphi, \nabla f) = \frac{\tau_F}{2s}$ this implies that $\lambda = -\frac{\Delta_F(f)}{s} < 0$. Thus the exterior Schwarzschild metric can admit an expanding gradient Yamabe soliton.

3.3. Gradient Yamabe Soliton on 3-dimensional Walker Manifolds

In general, a 3-dimensional manifold admitting a parallel degenerate line field is said to be a Walker manifold, [33,34]. Suppose that (M, g) is a 3-dimensional Walker Manifold then there exist local coordinates (t, x, y) such that the Lorentzian metric tensor with respect to the local frame fields $\{\partial_t, \partial_x, \partial_y\}$ takes the form given as:

$$g = 2\mathrm{d}t\mathrm{d}y + \mathrm{d}x^2 + \phi(t, x, y)\mathrm{d}y^2, \tag{4}$$

for some function $\phi(t, x, y)$. The restricted case of Walker manifolds where ϕ described as a function of only x and y is called as a strictly Walker manifold and in particular strictly Walker manifolds are geodesically complete. Also, it is known that a Walker manifold is Einstein if and only if it is flat, [33].

The non-zero components of the Levi-Civita connection of (M, g) are determined by:

$$\Gamma^t_{ty} = -\Gamma^y_{yy} = \frac{1}{2}\phi_t, \quad \Gamma^t_{xy} = -\Gamma^x_{yy} = \frac{1}{2}\phi_x, \quad \Gamma^t_{yy} = \frac{1}{2}(\phi\phi_t + \phi_y). \tag{5}$$

Now, we will investigate conditions on this particular class of manifolds to have gradient Yamabe solitons, that is,

$$\mathrm{Hess}(f)_{ij} = (\tau - \lambda)g_{ij} \tag{6}$$

where f is a potential function.

Note that this equation implies that

$$\frac{\Delta(f)}{3} = \tau - \lambda.$$

By using the metric (4), (5) and straightforward computations, we have:

$$\begin{cases} \mathrm{Hess}(f)_{tt} = f_{tt}, \\ \mathrm{Hess}(f)_{tx} = f_{tx}, \\ \mathrm{Hess}(f)_{ty} = f_{ty} - \frac{1}{2}\phi_t f_t, \\ \mathrm{Hess}(f)_{xx} = f_{xx}, \\ \mathrm{Hess}(f)_{xy} = f_{xy} - \frac{1}{2}\phi_x f_t, \\ \mathrm{Hess}(f)_{yy} = f_{yy} - \frac{1}{2}(\phi\phi_t + \phi_y) f_t + \frac{1}{2}\phi_x f_x + \frac{1}{2}\phi_t f_y. \end{cases} \tag{7}$$

Moreover,

$$\Delta(f) = -\phi f_{tt} + 2f_{ty} - \phi_t f_t + f_{xx}. \tag{8}$$

By combining these, we get:

$$\tau - \lambda = \frac{1}{3}\Big(-\phi f_{tt} + 2f_{ty} - \phi_t f_t + f_{xx}\Big). \tag{9}$$

By applying Eqs. (6), (7) and (9), we obtain the following system of PDEs:

$$\begin{cases} f_{tt} = 0, \\ f_{tx} = 0, \\ 2f_{xy} = \phi_x f_t, \\ 2f_{ty} = 2f_{xx} + \phi_t f_t, \\ 2f_{yy} - \phi_y f_t + \phi_x f_x + \phi_t f_y = 2f_{ty}\phi. \end{cases} \tag{10}$$

Notice first that $f_{tt} = 0$ and $f_{tx} = 0$ imply that

$$f(t, x, y) = tb(y) + c(x, y), \tag{11}$$

for some differentiable functions b and c.

Thus the system (10) turns out to be

$$\begin{cases} 2c_{xx}(x, y) + \phi_t b(y) = 2b'(y), \\ 2c_{xy}(x, y) = \phi_x b(y), \\ 2[tb''(y) + c_{yy}(x, y)] - \phi_y b(y) + \phi_x c_x(x, y) \\ +\phi_t[tb'(y) + c_y(x, y)] = 2\phi b'(y). \end{cases} \tag{12}$$

Differentiating the first two equations of the system (12) with respect to t, we have $\phi_{tt} b(y) = \phi_{xt} b(y) = 0$. Also, differentiating the last equation of the system (12) with respect to t, we get

$$2b''(y) + \phi_{xt} c_x(x, y) + \phi_{tt} c_y(x, y) = [\phi_t b(y)]_y. \tag{13}$$

In this case, two cases should be examined: We define $\mathcal{U} = \{p \in M : \; b(y) \neq 0 \text{ at } p\}$ and $\mathcal{V} = \{p \in M : \; b(y) = 0 \text{ in a neighborhood of } p\}$. Then $\mathcal{U} \cup \mathcal{V}$ is a dense open subset in M. If we assume that M is connected and $\mathcal{U} \neq \emptyset$ (respectively, $\mathcal{V} \neq \emptyset$), then by the continuity argument $M = \mathcal{U}$ (respectively, $M = \mathcal{V}$). Thus, we can construct the gradient Yamabe soliton on $\mathcal{U}$ and $\mathcal{V}$ separately.

- *Case 1*: If $p \in \mathcal{U}$, then $\phi_{tt} = \phi_{xt} = 0$ which implies that

$$\phi(t, x, y) = tB(y) + D(x, y)$$

for some differentiable functions B and D. Then by (13), we obtain

$$B(y) = \frac{2b'(y) + \alpha_0}{b(y)}, \quad \alpha_0 \in \mathbb{R}. \tag{14}$$

Then from the Eqs. (11), (14) and integrating the first equation of the system (12), we get

$$c(x, y) = -\frac{\alpha_0}{4} x^2 + xh(y) + k(y),$$

for some differentiable functions h and k. Thus, integrating the second equation of the system (12), we get

$$D(x, y) = xE(y) + F(y),$$

for some differentiable functions E and F. Thus, in so far we found the potential function

$$f(t,x,y) = tb(y) - \frac{\alpha_0}{4}x^2 + xh(y) + k(y)$$

and the relevant function ϕ of the 3-dimensional Walker metric

$$\phi(t,x,y) = t\Big(\frac{2b'(y)+\alpha_0}{b(y)}\Big) + xE(y) + F(y).$$

- *Case 2*: If $p \in \mathcal{V}$, then $b(y)=0$, i.e, $f(t,x,y)=c(x,y)$ and $\phi_{tt} \neq 0$, $\phi_{xt} \neq 0$. Thus from the system (12), we have $c_{xx}=0$ and $c_{xy}=0$ imply that $c(x,y)=\kappa_0 x + \eta(y)$, for some $\kappa_0 \in \mathbb{R}$. Also, from the last equation of the system (12), we have $2f_{yy}(x,y)+\phi_x f_x(x,y)+\phi_t f_y(x,y)=0$, which implies that

$$\eta(y) = \kappa_1 - \kappa_0 \int \frac{\phi_{xt}}{\phi_{tt}} dy,$$

for some constant $\kappa_1 \in \mathbb{R}$. Therefore, in this case, the potential function is given by:

$$f(x,y) = \kappa_1 + \Big(x - \int \frac{\phi_{xt}}{\phi_{tt}} dy\Big)\kappa_0.$$

Hence we obtain the main result of this section:

Theorem 4. *Let (M,g) be a connected 3-dimensional Lorenztian Walker manifold equipped with metric:*

$$g = 2\mathrm{d}t\mathrm{d}y + \mathrm{d}x^2 + \phi(t,x,y)\mathrm{d}y^2.$$

Then (M,g) is a gradient Yamabe soliton if and only if one of the following cases occurs:

1. *the potential function of the soliton structure is given by*

$$f(t,x,y) = tb(y) - \frac{\alpha_0}{4}x^2 + xh(y) + k(y)$$

and the function ϕ defining Walker metric is given by

$$\phi(t,x,y) = t\Big(\frac{2b'(y)+\alpha_0}{b(y)}\Big) + xE(y) + F(y)$$

for some differentiable functions $b(y) \neq 0$, $h(y)$, $k(y)$, $E(y)$, $F(y)$ and constant $\alpha_0 \in \mathbb{R}$,

2. *the potential function of the soliton structure is given by*

$$f(x,y) = \kappa_1 + \Big(x - \int \frac{\phi_{xt}}{\phi_{tt}} dy\Big)\kappa_0,$$

for some constants $\kappa_0, \kappa_1 \in \mathbb{R}$, where the relevant function ϕ satisfies the conditions $\phi_{tt} \neq 0$, $\phi_{xt} \neq 0$ and

$$2f_{yy}(x,y) + \phi_x f_x(x,y) + \phi_t f_y(x,y) = 0.$$

Note that by Eq. (8) and system (10), we can get: $\Delta(f) = 3f_{xx}$. In above cases, we conclude that $\Delta(f)$ is a non-zero constant and $\Delta(f) = 0$, respectively. In order to have a non-trivial potential function we may assume that M is non-compact due to Hopf's Lemma. Thus, we have:

Corollary 4. *Let (M, g) be a 3-dimensional Lorenztian Walker manifold equipped with metric (4). Then the gradient Yamabe soliton structure on (M, g) characterized by the Case (1) and Case (2) of Theorem 4 is non-compact.*

3.4. Gradient Yamabe Soliton on 4-dimensional Walker Manifolds

Let us now construct the gradient Yamabe soliton structure on 4-dimensional Walker manifolds.

A 4-dimensional Walker manifold is a triple (M, g, D) consisting of an indefinite metric g and a 2-dimensional parallel null plane D and in this case g has neutral signature $(-,-,+,+)$ and in suitable coordinates (x, y, z, t) such that with respect to the local frame fields $\{\partial_x, \partial_y, \partial_z, \partial_t\}$ it can be given by [33]

$$g = 2\mathrm{d}x\mathrm{d}z + 2\mathrm{d}y\mathrm{d}t + a(x, y, z, t)\mathrm{d}z^2 + 2c(x, y, z, t)\mathrm{d}z\mathrm{d}t + b(x, y, z, t)\mathrm{d}t^2, \tag{15}$$

for some functions $a(x, y, z, t), b(x, y, z, t), c(x, y, z, t)$ and $D =< \partial_x, \partial_y >$. The case where $c(x, y, z, t) = 0$ was studied and locally conformally flatness of this special metric was investigated in [35]. Moreover, some characterization of gradient Ricci solitons for a 4-dimensional Lorentzian Walker manifold is provided in [36].

Here, we consider the restricted case of Walker manifold (M, g) where $a = c = 0$ and b described as a function of only t so the metric (15) reduces to the form

$$g = 2\mathrm{d}x\mathrm{d}z + 2\mathrm{d}y\mathrm{d}t + b(t)\mathrm{d}t^2. \tag{16}$$

The non-zero components of the Levi-Civita connection of (M, g) are determined by:

$$\begin{cases} \Gamma^y_{xt} = -\Gamma^z_{tt} = \frac{1}{2}b_x, & \Gamma^y_{yt} = -\Gamma^t_{tt} = \frac{1}{2}b_y, \\ \Gamma^y_{zt} = -\Gamma^x_{tt} = \frac{1}{2}b_z, & \Gamma^y_{tt} = \frac{1}{2}(bb_y + b_t). \end{cases} \tag{17}$$

Now, we will investigate conditions on this particular class of manifolds to have gradient Yamabe solitons, that is,

$$\mathrm{Hess}(f)_{ij} = (\tau - \lambda)g_{ij} \tag{18}$$

where f is a potential function.

Note that this equation implies that

$$\frac{\Delta(f)}{4} = \tau - \lambda.$$

By using the metric (16), (17) and straightforward computations, we have:

$$\begin{cases} \mathrm{Hess}(f)_{xx} = f_{xx}, & \mathrm{Hess}(f)_{xy} = f_{xy}, \\ \mathrm{Hess}(f)_{xz} = f_{xz}, & \mathrm{Hess}(f)_{xt} = f_{xt}, \\ \mathrm{Hess}(f)_{yy} = f_{yy}, & \mathrm{Hess}(f)_{yz} = f_{yz}, \\ \mathrm{Hess}(f)_{yt} = f_{yt}, & \mathrm{Hess}(f)_{zz} = f_{zz}, \\ \mathrm{Hess}(f)_{zt} = f_{zt}, & \mathrm{Hess}(f)_{tt} = f_{tt} - \frac{1}{2}b_t f_y. \end{cases} \tag{19}$$

Moreover,

$$\Delta(f) = 2f_{xz} - bf_{yy} + 2f_{yt}.$$

By combining these, we get:

$$\tau - \lambda = \frac{1}{4}\Big(2f_{xz} - bf_{yy} + 2f_{yt}\Big). \tag{20}$$

By applying Eqs. (18), (19) and (20), we obtain the following system of PDEs:

$$\begin{cases} f_{xx} = f_{xy} = f_{yy} = f_{yz} = f_{zz} = 0, \\ f_{xt} = f_{zt} = 0, \\ f_{xz} = f_{yt} = \frac{\Delta(f)}{4}, \\ f_{tt} - \frac{1}{2}b_t f_y = b\frac{\Delta(f)}{4}. \end{cases} \tag{21}$$

Notice first that $f_{xy} = 0$, $f_{xx} = 0$, $f_{yz} = 0$ and $f_{yy} = 0$ imply that $f(x, y, z, t) = x\beta(z, t) + yA(t) + B(z, t)$ for some functions β, A and B. Thus from $f_{xz} = f_{yt}$, we have $\beta_z(z, t) = A'(t)$. Since $f_{zz} = 0$ and $f_{xt} = 0$, $A'(t) = C'(z)$ which yields

$$A(t) = c_0 t + c_1, \quad C(z) = c_0 z + c_2, \quad \text{where } c_0, c_1, c_2 \in \mathbb{R}.$$

Also, from $f_{xt} = 0$ and $f_{zt} = 0$, we obtain $\beta(y, z) = \kappa z + \mu$ and $B(z, t) = c_3 z + E(t)$, respectively where $c_0 \in \mathbb{R}$. As a result, the potential function is given by

$$f(x, y, z, t) = x(c_0 z + c_2) + y(c_0 z + c_1) + c_3 z + E(t).$$

Substituting this in the last equation of the system (21), we obtain

$$2E''(t) - b_t(c_0 t + c_1) = 2c_0 b. \tag{22}$$

Integrating (22) with respect to t and using integration by parts for the second term, we get $2E'(t) - b(c_0 t + c_1) = c_0 \int_{t_0}^{t} b(\bar{t})\mathrm{d}\bar{t}$.

Hence we obtain the main result of this section:

Theorem 5. *Let (M, g) be a 4-dimensional Walker manifold equipped with metric (16). Then (M, g) is a gradient Yamabe soliton if and only if its potential function is defined by*

$$f(x, y, z, t) = x(c_0 z + c_2) + y(c_0 z + c_1) + c_3 z + E(t)$$

where $c_i \in \mathbb{R}$, *(*$i = 0, 1, 2, 3$*) and the metric function* $b(t)$ *and* $E(t)$ *are related by*

$$2E'(t) - b(c_0 t + c_1) = c_0 \int_{t_0}^{t} b(\bar{t}) \mathrm{d}\bar{t}.$$

Moreover, $\Delta(f) = 4c_0$.

If in additionally, M is compact, then by applying the Divergence Theorem, $c_0 = 0$ and we conclude that:

Corollary 5. *Let* (M, g) *be a 4-dimensional Walker manifold equipped with metric* (16). *Then* (M, g) *is a compact gradient Yamabe soliton if and only if its potential function is defined by*

$$f(x, y, z, t) = c_2 x + c_1 y + c_3 z + \frac{c_1}{2} \int_{t_0}^{t} b(\bar{t}) \mathrm{d}\bar{t}.$$

where $c_i \in \mathbb{R}$, *(*$i = 1, 2, 3$*).*

3.5. Gradient Yamabe Soliton on pp-Wave Spacetimes

Before finishing our investigation, finally we consider 4-dimensional *pp*-wave (i.e. plane-fronted waves with parallel rays) metric introduced by Brinkmann [37].

A *pp*-wave spacetime is a Lorentzian manifold with the local coordinates $\{u, v, x, y\}$ equipped with the metric tensor given by

$$\mathrm{d}s^2 = H(u, x, y)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}v + \mathrm{d}y^2.$$

In this spacetime, $\mathcal{D} = \mathrm{Span}\{\partial_v\}$ is the degenerate parallel line field.

The nonzero Christoffel symbols of this metric are given by:

$$\begin{cases} \Gamma^v_{uu} = \frac{1}{2} H_u, \quad \Gamma^v_{ux} = -\Gamma^x_{uu} = \frac{1}{2} H_x, \\ \\ \Gamma^v_{uy} = -\Gamma^y_{uu} = \frac{1}{2} H_y. \end{cases} \tag{23}$$

Moreover, from (23), we have

$$\begin{cases} \mathrm{Hess}(\varphi)_{uu} = \varphi_{uu} - \frac{1}{2} H_u \varphi_v + \frac{1}{2} H_x \varphi_x + \frac{1}{2} H_y \varphi_y, \\ \mathrm{Hess}(\varphi)_{uv} = \varphi_{uv}, \\ \mathrm{Hess}(\varphi)_{ux} = \varphi_{ux} - \frac{1}{2} H_x \varphi_v, \\ \mathrm{Hess}(\varphi)_{uy} = \varphi_{uy} - \frac{1}{2} H_y \varphi_v, \\ \mathrm{Hess}(\varphi)_{vv} = \varphi_{vv}, \quad \mathrm{Hess}(\varphi)_{vx} = \varphi_{vx}, \quad \mathrm{Hess}(\varphi)_{vy} = \varphi_{vy}, \\ \mathrm{Hess}(\varphi)_{xx} = \varphi_{xx}, \quad \mathrm{Hess}(\varphi)_{xy} = \varphi_{xy}, \quad \mathrm{Hess}(\varphi)_{yy} = \varphi_{yy}. \end{cases} \tag{24}$$

Thus,

$$\Delta\varphi = 2\varphi_{uv} - H\varphi_{vv} + \varphi_{xx} + \varphi_{yy}.$$

The fundamental equation of gradient Yamabe soliton

$$\mathrm{Hess}\varphi = (\tau - \lambda)g \quad \text{and} \quad \tau - \lambda = \frac{1}{4}\Delta\varphi$$

and (24) imply that

$$\begin{cases} \varphi_{uu} - \frac{1}{2}\mathrm{H}_u\varphi_v + \frac{1}{2}H_x\varphi_x + \frac{1}{2}H_y\varphi_y = \frac{\Delta\varphi}{4}H, \\ \varphi_{uv} = \varphi_{xx} = \varphi_{yy} = \frac{\Delta\varphi}{4}, \\ \varphi_{ux} = \frac{1}{2}H_x\varphi_v, \\ \varphi_{uy} = \frac{1}{2}H_y\varphi_v, \\ \varphi_{vv} = \varphi_{vx} = \varphi_{vy} = \varphi_{xy} = 0. \end{cases} \tag{25}$$

The last item of the system (25) implies that

$$\varphi = vA(u) + B(u, x, y), \tag{26}$$

for two smooth functions A and B. Thus, we have

$$\Delta\varphi = 4\varphi_{xx} = 4\varphi_{yy} = 4A'(u) \tag{27}$$

where

$$\varphi_{xx} = B_{xx}, \quad \varphi_{yy} = B_{yy}, \quad B_{xy} = 0. \tag{28}$$

Integrating the third and the forth equations of the system (25) with respect to x and y, respectively and comparing the resulting equations we get

$$B_u(u, x, y) = \frac{1}{2}HA(u) + E(u). \tag{29}$$

for some function E. Now, using the Eqs. (26)–(29),we get

$$vA''(u) + E'(u) + \frac{1}{2}H_xB_x(u, x, y) + \frac{1}{2}H_yB_y(u, x, y) = \frac{1}{2}HA'.$$

By differentiating the both sides of the last equation with respect to v, we obtain that $A''(u) = 0$. Thus $A(u) = au + b$ for some a and b. Using this into (28), we obtain $B_{xx} = B_{yy} = a$ and $B_{xy} = 0$ imply that

$$B(u, x, y) = \frac{ax^2}{2} + xK(u) + \frac{ay^2}{2} + yM(u),$$

for some smooth functions K and M. Consequently, the potential function takes the form

$$\varphi = (au + b)v + \frac{ax^2}{2} + \frac{ay^2}{2} + xK(u) + yM(u). \tag{30}$$

Now, using (30) into the third and the forth equations of the system (25), we obtain

$$K'(u) = \frac{1}{2}(au + b)H_x \quad \text{and} \quad M'(u) = \frac{1}{2}(au + b)H_y.$$

Integrating the first and the second equations of above with respect to x and y, respectively and comparing the resulting equations, we get the function H related with the *pp*-wave metric as follows

$$\frac{1}{2}(au + b)H = xK'(u) + yM'(u). \tag{31}$$

From (31), we also have

$$xK''(u) + yM''(u) = \frac{a}{2}H + \frac{1}{2}(au+b)H_u.$$

By virtue of (30) and the last relation, the first equation of the system (25) reduces to

$$[ax + K(u)]H_x + [ay + M(u)]H_y = aH. \tag{32}$$

Therefore, (31) satisfies (32), provided $K(u)K'(u) + M(u)M'(u) = 0$ that implies that

$$K^2(u) + M^2(u) \equiv constant.$$

Hence we can state the following:

Theorem 6. *Let (M, g) be a 4-dimensional pp-wave spacetime endowed with the metric*

$$\mathrm{d}s^2 = H(u, x, y)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}v + \mathrm{d}y^2.$$

Then (M, g) is a gradient Yamabe soliton if and only if

(1) its potential function is given by

$$\varphi = (au+b)v + \frac{ax^2}{2} + \frac{ay^2}{2} + xK(u) + yM(u),$$

(2) the metric function H is given by

$$H(u, x, y) = \frac{2[xK'(u) + yM'(u)]}{au+b},$$

where a and b are nonzero constants and K and M are two smooth functions such that $K^2(u) + M^2(u) \equiv constant$.

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References

[1] Hamilton, R.S.: The Ricci flow on surfaces, in: Mathematics and General Relativity. Contemp. Math., **71**, 237–262 (1988)

[2] Brendle, S.: Convergence of the Yamabe flow for arbitrary initial energy. J. Diff. Geom. **69**(2), 217–278 (2005)

[3] Brendle, S.: Convergence of the Yamabe flow in dimension 6 and higher. Invent. Math. **170**(3), 541–576 (2007)

[4] Daskalopoulosa, P., Sesum, N.: The classification of locally conformally flat Yamabe solitons. Adv. Math. **240**, 346–369 (2013)

[5] Yang, F., Zhang, L.: Geometry of gradient Yamabe solitons. Ann. Glob. Anal. Geom. **50**, 367–379 (2016)

[6] Cao, H.-D., Sun, X., Zhang, Y.: On the structure of gradient Yamabe solitons. Math. Res. Lett. **19**, 767–774 (2012)

[7] Maeta, S.: Three-dimensional complete gradient Yamabe solitons with divergence-free Cotton tensor. Ann. Glob. Anal. Geom. **58**, 227–237 (2020)

[8] Bidabad, B., Yar Ahmadi, M.: On complete Yamabe solitons. Adv. Geom. **18**(1), 101–104 (2018)

[9] Bejan, C.-L., Güler, S.: Laplace, Einstein and related equations on D-General Warping. Mediter. J. Math. **16**, 19 (2019). https://doi.org/10.1007/s00009-018-1283-9

[10] Catino, G.: Generalized quasi-Einstein manifolds with harmonic Weyl tensor. Math. Z. **271**(3), 751–756 (2012)

[11] Altay Demirbağ, S., Güler, S.: Rigidity of (m, ρ)-quasi Einstein manifolds. Math. Nachr., **290**(14–15), 2100–2110 (2017)

[12] Naik, D.M.: Ricci solitons on Riemannian manifolds admitting certain vector field. Ricerche Mat. https://doi.org/10.1007/s11587-021-00622-z (2021)

[13] Bejan, C.-L., Crasmareanu, M.: Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry. Ann. Glob. Anal. Geom. **46**, 117–127 (2014)

[14] Huang, G., Li, H.: On a classification of the quasi-Yamabe gradient solitons. Methods Appl. Anal. **21**, 379–390 (2014)

[15] Leandro Neto, B.: A note on (anti-)self dual quasi-Yamabe gradient solitons, Results Math. 1-7 (2015)

[16] Leandro Neto, B., Oliveira, H.P.: Generalized quasi-Yamabe gradient solitons. Diff. Geo. Appl. **49**, 167–175 (2016)

[17] Wang, L.F.: On noncompact quasi-Yamabe gradient solitons. Differ. Geom. Appl. **31**, 337–348 (2013)

[18] Bishop, R.L., O'Neill, B.: Manifolds of negative curvature. Trans. Amer. Math. Soc. **145**, 1–49 (1969)

[19] O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press Limited, London (1983)

[20] Shenawy, S., Ünal, B.: 2−Killing vector fields on warped product manifolds. Int. J. Math. **26**, 1550065 (2015)

[21] Cheeger, J., Colding, T.: Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. Math. **144**, 189–237 (1996)

[22] Sánchez, M.: On the Geometry of Generalized Robertson-Walker Spacetimes: geodesics. Gen. Relat. Gravit. **30**, 915–932 (1998)

[23] Sánchez, M.: On the geometry of generalized Robertson-Walker spacetimes: curvature and killing fields. J. Geom. Phys. **31**, 1–15 (1999)

[24] Mantica, C.A., Molinari, L.G., De, U.C.: A condition for a perfect fluid spacetime to be a generalized Robertson-Walker spacetime. J. Math. Phys. **57**(2), 022508 (2016)

[25] Mantica, C.A., Suh, Y.J., De, U.C.: A note on generalized Robertson-Walker spacetimes. Int. J. Geom. Meth. Mod. Phys. **13**, 1650079 (2016)

[26] Chen, B.-Y.: A simple characterization of generalized Robertson-Walker spacetimes. Gen. Relativ. Gravit. **46**, 1833 (2014)

[27] Allison, D.E.: Energy conditions in standard static spacetimes. Gen. Relativ. Gravit. **20**, 115–122 (1988)

[28] Allison, D.E.: Geodesic completeness in static spacetimes. Geom. Dedicata **26**, 85–97 (1988)

[29] Allison, D.E., Ünal, B.: Geodesic structure of standard static spacetimes. J. Geo. Phys. **46**, 193–200 (2003)

[30] Besse, A.L.: Einstein Manifolds, Classics in Mathematics. Springer, Berlin (2008)

[31] Beem, J. K., Ehrlich, P. E., Easley, K. L.: Global Lorentzian Geometry. Marcel Dekker, Second Edition, NewYork, (1996)

[32] Dobarro, F., Ünal, B.: Special standard static spacetimes. Nonlinear Anal. Theory Methods Appl. **59**(5), 759–770 (2004)

[33] Walker, A.G.: Canonical form for a Riemannian space with a parallel field of null planes. Quart. J. Math. Oxford **21**, 69–79 (1950)

[34] Chaichi, M., Garcia-Rio, E., Vazquez-Abal, M.E.: Three-dimensional Lorentz manifolds admitting a parallel null vector field. J. Phys. A Math. Gen. **38**, 841–850 (2005)

[35] Chaichi, M., Garcia-Rio, E., Matsushita, Y.: Curvature properties of four-dimensional Walker metrics. Class. Quant. Gravity **22**, 559–577 (2005)

[36] Sidhoumi, N., Batat, W.: Ricci solitons on four-dimensional Lorentzian Walker manifolds Adv. Geom. **17**(4), 397–406 (2017)

[37] Brinkmann, H.W.: Einstein spaces which are mapped conformally on each other. Math. Ann. **18**, 119–145 (1925)

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