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Dwell time-based stabilisation of switched delay systems using free-weighting matrices

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ABSTRACT

In this paper, we present a quasi-convex optimisation method to minimise an upper bound of the dwell time for stability of switched delay systems. Piecewise Lyapunov–Krasovskii functionals are introduced and the upper bound for the derivative of Lyapunov functionals is estimated by free-weighting matrices method to investigate non-switching stability of each candidate subsystems. Then, a sufficient condition for the dwell time is derived to guarantee the asymptotic stability of the switched delay system. Once these conditions are represented by a set of linear matrix inequalities, dwell time optimisation problem can be formulated as a standard quasi-convex optimisation problem. Numerical examples are given to illustrate the improvements over previously obtained dwell time bounds. Using the results obtained in the stability case, we present a nonlinear minimisation algorithm to synthesise the dwell time minimiser controllers. The algorithm solves the problem with successive linearisation of nonlinear conditions.

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1. Introduction

A switched system is a dynamical system that includes a set of subsystems and a discrete switching event between them. General behaviour of a switched system is governed by following differential equation:

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \quad \forall t > t_0,$$

where σ denotes the switching signal which is a piecewise constant map from time to an index set representing subsystems. See the survey of Lin and Antsaklis (2009) for a review of the recent results and further references.

The stability analysis encountered in switched systems can be classified into three categories (Mahmoud, 2010). The first one is to find common Lyapunov functions so that the switched systems are stable under any arbitrary switching signal (Fainshil, Margaliot, & Chigansky, 2009; Hou, Fu, & Duan, 2013; Shorten, Narendra, & Mason, 2003). The second one is to construct certain switching signals that make the switched system asymptotically stable (Liberzon & Morse, 1999). The third category is the slow switching strategies such as dwell time stability or average dwell time stability for which the system is asymptotically stable (Geromel & Colaneri, 2006; Hespanha, 2004; Hespanha & Morse, 1999; Mitra & Liberzon, 2004; Zhang, Han, Zhu, & Huang, 2013). The class of switching signals can be restricted to signals with

the property that the interval between any consecutive switching times is not less than a value called the *dwell time*. The switched delay system is asymptotically stable if all of the candidate subsystems are asymptotically stable and the dwell time is large enough (Morse, 1996). Most switched systems do not share a common Lyapunov function (Chen & Zheng, 2010). Furthermore, having a common Lyapunov function is a sufficient condition for the stability under arbitrary switching, so it can be found conservative (Lin & Antsaklis, 2009).

In this paper, we present some results on the dwell time stability analysis and stabilisation of the switched delay systems. A dwell time is observed in many switching system applications. The time intervals between the change in the road conditions among dry, wet and dirt for a car on the road can be considered as an example (Allerhand & Shaked, 2011). Also, the slow switching strategies with dwell time can avoid chattering problems which can damage the physical systems (Ishii & Francis, 2001). As a result, the stability analysis and stabilisation of switched systems with dwell time are increasingly popular.

The literature is abounded with various approaches for the stability analysis of time-delay systems, one can refer to Gu, Kharitonov, and Chen (2003) for a review on the topic. Common methods to deal with delay-dependent stability problems are model transformations. In this method, point wise delay system transferred into a dis-

tributed delay system. Stability of the transformed system is a sufficient condition for the stability of the original system. Hence, stability analysis with model transformations leads to a sort of conservatism since analysis operates on the transformed system instead of the original system (Gu et al., 2003). A less conservative approach to stability analysis is the free-weighting matrices method which does not include any model transformation of the original system (He, Wang, Xie, & Lin, 2007; Mahmoud, 2010; Wu, He, & She, 2010). In this paper, we present some results for switched delay systems with pointwise delays.

There are recent results on dwell time stability of the switched delay systems. In Sun, Zhao, and Hill (2006) and Li, Gao, Agarwal, and Kaynak (2013), stability conditions are presented for a given average dwell time. In those papers, the conditions involve exponential and bilinear terms when the dwell time is considered as a free parameter. Hence, the minimisation of the dwell time and synthesising the dwell time minimiser controllers with those methods are not tractable. There are some optimisation-based methods to minimise the upper bound for the dwell time (Çalışkan, Özbay, & Niculescu, 2013; Yan & Özbay, 2008). In Çalışkan et al. (2013), the calculation of dwell time is formulated as a semi-definite programming (SDP) in terms of linear matrix inequalities (LMIs). Piecewise Lyapunov–Krasovskii functionals are derived by model transformation methods. The upper bound of the derivative of the Lyapunov function is minimised which ends up with a sub-optimal solution to the dwell time minimisation problem. In Yan, Özbay, and Sansal (2011), parameter-varying systems with time-delays are stabilised by switching control. The resulting dwell time is minimised with iterative search methods. The present paper proposes a quasi-convex optimisation approach to directly minimise the dwell time and converges to global minimum of represented upper bound of the dwell time. To reduce conservatism due to model transformations, we derive the stability conditions by using free-weighting matrices.

The notation to be used in the paper is standard: \mathbb{R} (\mathbb{R}^+ , \mathbb{R}_0^+) stands for the set of real numbers (positive real numbers, non-negative real numbers), \mathcal{C} is used to denote the set of differentiable functions, \mathbb{Z}^+ symbolises the set of positive integers. The identity matrices are denoted by I . We use $X \succ 0$ (\succeq , \prec , $\preceq 0$) to denote a positive definite (positive-semidefinite, negative definite, negative-semidefinite) matrix. $\sigma_{\max}[X]$ and $\sigma_{\min}[X]$ denote the maximum and minimum singular values of X , respectively. The asterisk symbol ($*$) denotes complex conjugate transpose of a matrix and x_t denotes the translation operator acting on the trajectory such as $x_t(\theta) = x(t + \theta)$ for some non-zero interval $\theta \in [-\tau, 0]$. The

operator $\text{diag}[X_1, X_2, \dots, X_n]$ denotes a block diagonal matrix whose elements on the main block diagonal are X_1, X_2, \dots, X_n . The norm $\|\cdot\|$ is defined as the Euclidian norm for a vector in \mathbb{R}^n and the norm on \mathcal{C} is defined as follows:

$$\|f\|_{[a,b]} = \max \left\{ \sup_{t \in [a,b]} \|f(t)\|, \sup_{t \in [a,b]} \|\dot{f}(t)\| \right\}.$$

Rest of the paper is organised as follows. In Section 2, preliminaries and problem definition are introduced. In Section 3, dwell time stability condition is given. In Section 4, quasi-convex optimisation of the upper bound of the dwell time and some numerical examples are proposed. In Section 5, dwell time minimising controller synthesis is presented with some numerical examples to illustrate effectiveness of the proposed algorithm. Conclusions are summarised in Section 6.

2. Preliminaries and problem definition

Consider a class of switched delay system given by

$$\Sigma_{\sigma(t)} : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) \\ \quad + \bar{A}_{\sigma(t)}x(t - r_{\sigma(t)}(t)), & t \geq 0 \\ x(\theta) = \varphi(\theta), & \forall \theta \in [-\tau_{\max}, 0] \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the pseudo-state and $\sigma(t)$ is the piecewise switching signal such that $\sigma(t) : \mathbb{R}^+ \rightarrow \mathcal{P}$, $\mathcal{P} := \{1, 2, \dots, m\}$ is an index set, $m \in \mathbb{Z}^+$ is the number of subsystems and initial condition belongs to Banach space of continuous functions such that $\varphi(\cdot) \in \mathcal{C}$. Time-delay, $r_{\sigma(t)}(t)$, is a time-varying differentiable function that satisfies

$$0 \leq r_{\sigma(t)}(t) \leq \tau_{\sigma(t)}, \quad (2)$$

$$|\dot{r}_{\sigma(t)}(t)| \leq d_{\sigma(t)}, \quad (3)$$

where $\tau_{\sigma(t)}$, $d_{\sigma(t)} > 0$ are piecewise constants. We introduce the quadropole

$$\Sigma_i := (A_i, \bar{A}_i, \tau_i, d_i) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R} \times \mathbb{R}$$

to describe the i th candidate subsystem of Equation (1) and $\tau_{\max} = \max_{i \in \mathcal{P}} \tau_i$.

Definition 2.1: A switched delay system is *stable* if there exists a function β of class \mathcal{K} such that

$$\|x(t)\| \leq \beta(\|x\|_{[t_0 - \tau_{\max}, t_0]})$$

along every solution of Equation (1). Furthermore, a switched delay system is *asymptotically stable* when it is stable and $\lim_{t \rightarrow +\infty} x(t) = 0$.

Lemma 2.1: (See Gu et al., 2003). Consider the non-switched linear subsystem Σ_i of the system (1) for an $i \in \mathcal{P}$. Suppose $u_i, v_i, w_i : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ are continuous, non-decreasing functions satisfying $u_i(0) = v_i(0) = 0, w_i(s) > 0$ for $s > 0$. If there exists a continuous functional V , such that

$$u_i(\|x(t)\|) \leq V_i(t, x_t) \leq v_i(|x|_{[t-\tau_i, t]}), \quad \forall t \geq t_0 \quad (4)$$

$$\dot{V}_i(t, x_t) \leq -w_i(\|x(t)\|), \quad \forall t \geq t_0 \quad (5)$$

then the solution $x = 0$ of the subsystem Σ_i is uniformly asymptotically stable.

Let us construct the following piecewise Lyapunov function:

$$V_i(t, x_t) := x^T(t)P_i x(t) + \int_{t-\tau_i(t)}^t x^T(s)Q_i x(s)ds + \int_{-\tau_i}^0 \int_{t+\theta}^t \dot{x}^T(s)Z_i \dot{x}(s)dsd\theta, \quad \forall i \in \mathcal{P} \quad (6)$$

Lemma 2.2: (See Wu, He, & She, 2010). Consider the non-switched linear subsystem Σ_i for an $i \in \mathcal{P}$ of the switched system (1) with varying delays, $r_i(t)$. Given scalar $\tau_i > 0$ and $d_i > 0$ for which both Equations (2) and (3) hold, the i th subsystem is asymptotically stable if there exist symmetric matrices $P_i > 0, Q_i \geq 0, Z_i > 0$, and

$$X_i := \begin{bmatrix} X_{11i} & X_{12i} \\ * & X_{22i} \end{bmatrix} \geq 0, \quad (7)$$

and any appropriately dimensioned matrices N_{1i} and N_{2i} such that the following LMIs hold:

$$\phi_i := \begin{bmatrix} \phi_{11i} & \phi_{12i} & \tau_i A_i^T Z_i \\ * & \phi_{22i} & \tau_i \bar{A}_i^T Z_i \\ * & * & -\tau_i Z_i \end{bmatrix} < 0, \quad (8)$$

$$\psi_i := \begin{bmatrix} X_{11i} & X_{12i} & N_{1i} \\ * & X_{22i} & N_{2i} \\ * & * & Z_i \end{bmatrix} \geq 0, \quad (9)$$

where

$$\phi_{11i} = P_i A_i + A_i^T P_i + N_{1i} + N_{1i}^T + Q_i + \tau_i X_{11i},$$

$$\phi_{12i} = P_i \bar{A}_i - N_{1i} + N_{2i}^T + \tau_i X_{12i},$$

$$\phi_{22i} = -N_{2i} - N_{2i}^T - (1 - d_i)Q_i + \tau_i X_{22i}.$$

3. Main results

The following proposition is a modified version of a result obtained in Çalışkan et al. (2013). In the corresponding proposition, the time T_* is calculated as the time instant after which norm of the states does not exceed the pre-defined parameter ρ for the non-switched case. Furthermore, after the dwell time $T_* + \tau_{\max}$ the norm of the state functional does not exceed ρ . Note that the norm of the state functional is computed as $|x|_{[t-\tau_{\max}, t]}$. The time T_* is related to the upper bounds defined in Lemma 2.1.

Proposition 3.1: For any non-switching linear subsystem Σ_i satisfying Lemma 2.1 with $\lim_{s \rightarrow \infty} u_i(s) \rightarrow \infty$, assume there exists a non-decreasing function u_{d_i} such that

$$u_{d_i}(\|\dot{x}(t)\|) \leq V_i(t, x_t).$$

For an arbitrary $\rho, 0 < \rho < \delta_2, |x|_{[t_0-\tau_i, t_0]} \leq \delta_1$ implies

$$|x|_{[t-\tau_{\max}, t]} \leq \rho, \quad \forall t > t_0 + \tau_{\max} + T_i(\delta_1, \rho)$$

where v_i and w_i are defined as in Lemma 2.1, $u(\delta_2) = v(\delta_1)$ and

$$T_i(\delta_1, \rho) = \frac{v_i(\delta_1)}{w_i(\rho)}.$$

Proof: Let $T_* > 0$ and let $\|x(t_1)\| > \rho$ for a time instant $t_1 > t_0 + T_*$. Function w_i is non-decreasing by definition, as a result $\inf_{\rho < s < \delta_2} w_i(s) = w_i(\rho)$. Since the subsystem Σ_i is stable and V_i is a Lyapunov–Krasovskii functional, from Lemma 2.1, we have the following:

$$\dot{V}_i(t, x_t) \leq -w_i(\rho), \quad t_0 \leq t \leq t_1.$$

This implies

$$V_i(t, x_t) \leq V_i(t_0, x_0) - (t - t_0)w_i(\rho) \leq v_i(\delta_1) - (t - t_0)w_i(\rho).$$

Let $T_* > v_i(\delta_1)/w_i(\rho)$. Then for every $t > t_0 + T_*$, we have $V_i(t, x_t) \leq 0$. However, we assume that there is a time instant $t_1 > t_0 + T_*$ such that $\|x(t_1)\| > \rho$. This implies that

$$V_i(t_1, x_{t_1}) \geq u_i(\|x(t_1)\|) > u_i(\rho) > 0$$

This is a contradiction. Therefore, time instant t_1 cannot exist and this implies

$$\|x(t)\| \leq \rho, \quad \forall t > t_0 + \frac{v_i(\delta_1)}{w_i(\rho)}.$$

Similarly, assuming there is a time instant $t_1 > t_0 + T_*$ such that $\|\dot{x}(t_1)\| \geq \rho$

$$V_i(t_1, x_{t_1}) \geq u_{d_i}(\|\dot{x}(t_1)\|) > u_{d_i}(\rho) > 0$$

which is also a contradiction. Hence,

$$\|x(t)\| < \rho, \quad \|\dot{x}(t)\| < \rho, \quad \forall t > t_0 + T_*$$

If we wait for a period of maximum time-delay such that $t > t_0 + T_* + \tau_{\max}$, the inequality $|x(t)|_{[t-\tau_{\max}, t]} \leq \rho$ holds, which concludes the proof. \blacksquare

Now, some specific upper and lower bounds for the Lyapunov function (6) can be given as $v_i(s) := \mu_i s^2$ with

$$\mu_i := \sigma_{\max} [P_i] + \tau_i^2 \sigma_{\max} [Q_i] + \frac{1}{2} \tau_i^2 \sigma_{\max} [Z_i] \quad (10)$$

and

$$u_i(s) = \sigma_{\min} [P_i] s^2, \quad (11)$$

respectively. Another lower bound of the Lyapunov function with respect to norm of $\dot{x}(t)$ can be defined as

$$u_{d_i}(s) := \frac{1}{2} \tau_i^2 \sigma_{\min} [Z_i] s^2$$

where $u_{d_i}(\|\dot{x}(t)\|) \leq V_i(t, x_t)$.

The upper bounds $v_i(s)$ can be calculated via LMI conditions defined in Lemma 2.2 due to Equation (10). In order to formulate the upper bounds of the derivative of the Lyapunov functions $w_i(s)$ as an LMI feasibility problem, we need a new result, stated as Proposition 3.2.

Remark 3.1: In the proof of Proposition 3.2, an inequality from the proof of Lemma 2.2 in Wu et al. (2010) will be used, specifically:

$$\dot{V}_i(t, x_t) \leq \eta_1^T(t) \Xi_i \eta_1(t) - \int_{t-r_i(t)}^t \eta_2^T(t, s) \psi_i \eta_2(t, s) ds \quad (12)$$

where ψ_i is defined in Equation (9) and

$$\eta_2(t, s) = \begin{bmatrix} x^T(t), & x^T(t - r_i(t)), & \dot{x}^T(s) \end{bmatrix}^T$$

$$\Xi_i = \begin{bmatrix} \phi_{11i} + \tau_i A_i^T Z_i A_i & \phi_{12i} + \tau_i \bar{A}_i^T Z_i \bar{A}_i \\ * & \phi_{22i} + \tau_i \bar{A}_i^T Z_i \bar{A}_i \end{bmatrix}.$$

Note that Equation (8) is the Schur complement of Ξ_i . For more information about the proof, we refer to Wu et al. (2010).

Proposition 3.2: Consider the system (1) with each Σ_i satisfying Lemma 2.2, if there exist matrices $W_i^T = W_i \geq 0$

such that following LMIs hold:

$$\bar{\phi}_i := \begin{bmatrix} \phi_{11i} + W_i & \phi_{12i} & \tau_i A_i^T Z_i \\ & \phi_{22i} & \tau_i \bar{A}_i^T Z_i \\ & * & -\tau_i Z_i \end{bmatrix} < 0, \quad \forall i \in \mathcal{P} \quad (13)$$

then $\dot{V}_i(t, x_t) \leq -x^T(t) W_i x(t)$ for all $i \in \mathcal{P}$.

Proof: Consider the inequality (12). Since $\psi_i \geq 0$, we know that $\dot{V}_i(t, x_t) \leq \eta_1^T(t) \Xi_i \eta_1(t)$. Bounding this inequality,

$$\eta_1^T(t) \Xi_i \eta_1(t) \leq -x^T(t) W_i x(t)$$

yields $\eta_1^T(t) D_i \eta_1(t) \leq 0$ where

$$D_i := \begin{bmatrix} \phi_{11i} + W_i + \tau_i A_i^T Z_i A_i & \phi_{12i} + \tau_i A_i^T Z_i \bar{A}_i \\ & \phi_{22i} + \tau_i \bar{A}_i^T Z_i \bar{A}_i \end{bmatrix}.$$

Since $\bar{\phi}_i$ is the Schur complement of D_i , if Equation (13) holds, then $\dot{V}_i(t, x_t) \leq -x^T(t) W_i x(t)$. \blacksquare

After defining a new variable

$$\lambda_i := \sigma_{\min} [W_i], \quad (14)$$

we can select the upper bound function for the derivative of the Lyapunov function as $w_i(s) = \lambda_i s^2$.

Assume that the Lemma 2.1 is satisfied for the system (1). There exists a $\delta_2 > \delta_1 > 0$ such that $u(\delta_2) = v(\delta_1)$. For such δ_2 , Lemma 2.1 implies that $\|x(t)\| \leq \delta_2$ for all $t > t_0$ if $|x|_{[t_0-\tau_i, t_0]} \leq \delta_1$. Hence, for $u(s)$ and $v(s)$ defined in Equations (10) and (11), following inequality holds:

$$\|x(t)\| \leq \beta |x|_{[t_0-\tau_i, t_0]}, \quad \forall i \in \mathcal{P} \quad (15)$$

where

$$\beta = \max_{i \in \mathcal{P}} \sqrt{\frac{\mu_i}{\sigma_{\min} [P_i]}}.$$

Consider the k th switching instant t_k . The dwell time τ_D is defined as the time instant after which the norm of the state functionals for any $t_k > t_{k-1} + \tau_D$ does not exceed the norm of the state functional at time t_{k-1} . Hence, ρ in Proposition 3.1 is defined as a fraction of the norm of the state functional at the switching instant t_{k-1} . As a result, the dwell time defined in this paper should be strictly greater than the maximum of all the possible delays. The fraction is a pre-defined number $\alpha \in (0, 1)$.

Theorem 3.1: Consider the switched delay system described in Equation (1). Assume all of the candidate subsystems satisfy Lemma 2.2. Then, the switched delay system is asymptotically stable for all switching signals

satisfying dwell time requirement τ_D

$$\tau_D = \frac{1}{\alpha^2} \max_{i \in \mathcal{P}} \frac{\mu_i}{\lambda_i} + \tau_{\max}, \quad \text{for any } \alpha \in (0, 1) \quad (16)$$

with μ_i and λ_i being defined in Equations (10) and (14), respectively.

Proof: Let us choose $\rho = \alpha \delta_{k-1}$ where δ_k denotes norm of the state at the k th switching instant such that $\delta_k = |x|_{[t_k - \tau_{\max}, t_k]}$. Let us restrict ourselves to switching signals to signals for which the time interval between two consecutive switching instants is larger than dwell time such that $t_k - t_{k-1} > \tau_D$. Introducing this dwell time as

$$\tau_D = \max_{i \in \mathcal{P}} T_i(\delta_{k-1}, \alpha \delta_{k-1}) + \tau_{\max},$$

leads us to an inequality from Proposition 3.1 as

$$|x|_{[t_k - \tau_{\max}, t_k]} \leq \alpha |x|_{[t_{k-1} - \tau_{\max}, t_{k-1}]}, \quad \forall t_k > t_{k-1} + \tau_D, \quad (17)$$

where

$$T_i(\delta_{k-1}, \alpha \delta_{k-1}) = \frac{v_i(\delta_{k-1})}{w_i(\alpha \delta_{k-1})} = \frac{\mu_i}{\alpha^2 \lambda_i}.$$

From Equations (15) and (17),

$$\begin{aligned} \|x(t)\| &\leq \beta |x|_{[t_k - \tau_{\max}, t_k]} \\ &\leq \beta \alpha |x|_{[t_{k-1} - \tau_{\max}, t_{k-1}]} \\ &\vdots \\ &\leq \beta \alpha^k |x|_{[t_0 - \tau_{\max}, t_0]} \\ &\leq \beta \alpha |x|_{[t_0 - \tau_{\max}, t_0]}, \quad \forall \alpha \in (0, 1), \end{aligned}$$

which is satisfying the stability condition described in Definition 2.1. \blacksquare

Remark 3.2: The parameter α is the ratio of the norms of the state functionals at the consecutive switching instants as in Equation (17). Hence, it can be regarded as a measure of the decay rate. This parameter quantifies a trade-off between the dwell time and the decay rate, i.e. the larger α , the smaller dwell time but the slower decay rate (it should be strictly less than 1 for stability of the switched system).

4. Minimum dwell time via quasi-convex optimisation

In order to minimise the dwell time in Equation (16), we can define the optimisation problem with a cost function $f(\mu_i, \lambda_i) := \max_{i \in \mathcal{P}} \mu_i / \lambda_i$ for a given α . This is a quasi-convex function since it is the composition of a convex

function with a nondecreasing function (Bullo & Liberzon, 2006). It is known that an optimisation problem with a quasi-convex cost function and convex constraints can be solved by iterative methods such as bisection algorithm (Boyd & Vandenberghe, 2004).

We define a new free variable t to bound the cost function:

$$\frac{\mu_i}{\lambda_i} \leq t, \quad \forall i \in \mathcal{P} \quad (18)$$

The parameters μ_i and λ_i are related with the eigenvalues of P_i , Q_i , Z_i and W_i as in Equations (10) and (14). We define p_i , q_i and z_i to define maximum eigenvalues of P_i , Q_i and Z_i , respectively. So, the inequality (18) can be rewritten as $p_i + \tau_i q_i + \frac{1}{2} \tau_i^2 z_i - t \lambda_i < 0$.

With respect to the free parameters P_i , Q_i , Z_i , W_i , X_{11i} , X_{12i} , X_{22i} , N_{1i} , N_{2i} , p_i , q_i , z_i , w_i , t_u , the upper bound of the dwell time is minimised via following optimisation problem:

$$\begin{aligned} &\text{minimise} && t && (19) \\ &\text{subject to} && \text{diag}[P_i, Q_i, Z_i, W_i, X_i] > 0, \quad \forall i \in \mathcal{P} \\ &&& \text{diag}[P_i, Q_i, Z_i, -W_i] \\ &&& < \text{diag}[p_i I, q_i I, z_i I, -\lambda_i I], \quad \forall i \in \mathcal{P} \\ &&& \psi_i \geq 0, \quad \bar{\phi}_i < 0, \quad \forall i \in \mathcal{P} \\ &&& p_i + \tau_i q_i + \frac{1}{2} \tau_i^2 z_i - t \lambda_i < 0, \quad \forall i \in \mathcal{P} \end{aligned}$$

where X_i , ψ_i and $\bar{\phi}_i$ are defined in Equations (7), (9) and (13), respectively. Then, the dwell time can be chosen as $\tau_D = \alpha t + \tau_{\max}$ for any $\alpha \in (0, 1)$. However, the optimisation problem involves a bilinear matrix inequality when t is considered as a free parameter.

Searching for minimum t with bisection algorithm generates a sequence of linear SDP feasibility problems which can easily be solved by SeDuMi Sturm (1999).

In this section, the examples are taken from published papers for comparison purposes. Examples 4.1–4.3 can be found in Çalıřkan et al. (2013), Yan and Özbay (2008) and Chen and Zheng (2010), respectively.

Example 4.1: Let Σ_1 and Σ_2 be

$$\begin{aligned} A_1 &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, & \bar{A}_1 &= \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix}, \\ & & \tau_1 &= 0.3s, & d_1 &= 0, \\ A_2 &= \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, & \bar{A}_2 &= \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}, \\ & & \tau_2 &= 0.6s, & d_2 &= 0. \end{aligned}$$

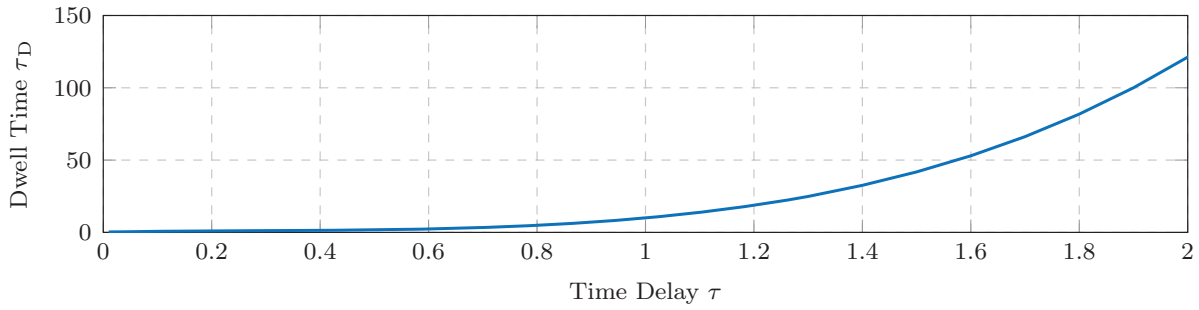


Figure 1. Dwell time results for different delay values of the switched delay system described in the Example 4.1 where delays are fixed such that $d_1 = d_2 = 0$ with upper bounds $\tau_1 = \tau$ and $\tau_2 = 2\tau$.

Table 1. Dwell time for different τ_i and d_i values of Example 4.1, $\alpha = 0.99$.

τ_1	τ_2	d_1	d_2	τ_D
0.15 s	0.3 s	0 s	0 s	0.69 s
0.15 s	0.3 s	0.15 s	0.3 s	0.69 s
0.3 s	0.6 s	0 s	0 s	1.11 s
0.3 s	0.6 s	0.3 s	0.3 s	1.11 s
0.3 s	0.6 s	0.6 s	0.6 s	1.11 s
0.6 s	1.2 s	0 s	0 s	2.54 s
0.6 s	1.2 s	0.3 s	0.3 s	2.76 s
0.6 s	1.2 s	0.6 s	0.6 s	3.51 s

Table 2. Dwell time for different τ_i and d_i values of Example 4.2, $\alpha = 0.99$.

τ_1	τ_2	d_1	d_2	τ_D
0.08 s	0.1 s	0 s	0 s	0.46 s
0.16 s	0.2 s	0.15 s	0.15 s	0.58 s
0.3 s	0.4 s	0.2 s	0.2 s	0.84 s
0.6 s	0.8 s	0 s	0 s	1.38 s
0.9 s	1.2 s	0 s	0 s	2.39 s
0.9 s	1.2 s	0.3 s	0.3 s	2.58 s
0.9 s	1.2 s	0.6 s	0.6 s	3.15 s
0.9 s	1.2 s	0.9 s	0.9 s	176.70 s

Dwell time results for different delay values for this example can be seen in Figure 1. Corresponding minimum dwell times for different τ_i and d_i values are illustrated in Table 1.

Example 4.2: Let Σ_1 and Σ_2 be

$$A_1 = \begin{bmatrix} -1.799 & -0.814 \\ 0.2 & -0.714 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -1 & 0 \\ -0.45 & -1 \end{bmatrix}, \\ \tau_1 = 0.155s, \quad d_1 = 0. \\ A_2 = \begin{bmatrix} -1.853 & -0.093 \\ -0.853 & -1.1593 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} -1 & 0 \\ 0.05 & -1 \end{bmatrix}, \\ \tau_2 = 0.2s, \quad d_2 = 0.$$

Corresponding minimum dwell times for different τ_i and d_i values are illustrated in Table 2. Note that second subsystem of Example 4.2 is unstable for $d_2 > 0.905$. As d_2

Table 3. Dwell time for $\alpha = 0.99$.

Example	1	2	3
Chen and Zheng (2010)	–	–	43.99 s
Yan and Özbay (2008)	6.51 s	–	–
Çalışkan et al. (2013)	3.4 s	0.72 s	–
Present work	1.11 s	0.58 s	16.60 s

coming closer to the stability limits, dwell time increases dramatically, that sits in shaded row of Table 2.

Example 4.3: This example is the Case 3 of Example 4.1 from the paper (Chen & Zheng, 2010). Let Σ_1 be

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad \bar{A}_1 = 0.9 \cdot \begin{bmatrix} 0.1 & 0 \\ -0.01 & 0.05 \end{bmatrix}, \\ \tau_1 = 1.82, \quad d_1 = 0, \\ A_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.5 \end{bmatrix}, \quad \bar{A}_2 = 0.7 \cdot \begin{bmatrix} 0.02 & 0 \\ -0.01 & 0.02 \end{bmatrix}, \\ \tau_2 = 1.82, \quad d_2 = 0,$$

Comparison of present paper with previous works for Examples 4.1–4.3 can be seen in Table 3.

Example 1.1: This example is a slightly modified version of Example 4.1 of Sun and Ge (2011) with $a = 50$, where a is a parameter used in corresponding example. In the example, system is not guaranteed stable under arbitrary switching.

Let the subsystems be:

$$A_1 = \begin{bmatrix} -0.1 & 1.1 \\ -0.9 & -1 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} 0.05 & -0.1 \\ -0.1 & 0 \end{bmatrix}, \\ \tau_1 = 0.01, \quad d_1 = 0.2 \\ A_2 = \begin{bmatrix} -0.1 & 1 \\ -150 & -50 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 0.05 & 0 \\ -1 & -1 \end{bmatrix}, \\ \tau_2 = 0.05, \quad d_2 = 0.1.$$

Resulting dwell time is $\tau_D = 1.698$ with $T_* = 1.648$.

5. Dwell time minimising controller synthesis

Consider a class of switched delay systems given by

$$\tilde{\Sigma}_{\sigma(t)} : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) \\ \quad + A_{\sigma(t)}x(t - r_{\sigma(t)}(t)) \\ \quad + B_{\sigma(t)}u(t), & t \geq 0 \\ x(\theta) = \varphi(\theta), & \forall \theta \in [-\tau_{\max}, 0] \end{cases} \quad (20)$$

We introduce the quintet

$$\tilde{\Sigma}_i := (A_i, \bar{A}_i, B_i, \tau_i, d_i) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R} \times \mathbb{R}$$

to describe the i th candidate subsystem of Equation (20) and $\tau_{\max} = \max_{i \in \mathcal{P}} \tau_i$.

Lemma 5.1: (See Wu et al., 2010). Consider any non-switching linear subsystem $\tilde{\Sigma}_i$ of the switched delay system (20) with a delay, $r_i(t)$. For given scalar τ_i and d_i which both Equations (2) and (3) hold, if there exist matrices $L_i > 0$, $T_i \geq 0$, $R_i > 0$, and

$$Y_i := \begin{bmatrix} Y_{11i} & Y_{12i} \\ * & Y_{22i} \end{bmatrix} \succeq 0,$$

and any appropriately dimensioned matrices M_{1i} , M_{2i} and V_i such that the following matrix inequalities hold:

$$\Pi_i = \begin{bmatrix} \Pi_{11i} & \Pi_{12i} & \tau_i(L_i A_i^T + V_i^T B_i^T) \\ * & \Pi_{22i} & \tau_i L_i \bar{A}_i^T \\ * & * & -\tau_i R_i \end{bmatrix} \prec 0, \quad (21)$$

$$\Lambda_i = \begin{bmatrix} Y_{11i} & Y_{12i} & M_{1i} \\ * & Y_{22i} & M_{2i} \\ * & * & L_i R_i^{-1} L_i \end{bmatrix} \succeq 0, \quad (22)$$

where

$$\begin{aligned} \Pi_{11i} &= L_i A_i^T + A_i L_i + B_i V_i + V_i^T B_i^T + M_{1i} \\ &\quad + M_{1i}^T + T_i + \tau_i Y_{11i} \\ \Pi_{12i} &= \bar{A}_i L_i - M_{1i} + M_{2i}^T + \tau_i Y_{12i} \\ \Pi_{22i} &= -M_{2i} - M_{2i}^T - (1 - d_i) T_i + \tau_i Y_{22i} \end{aligned}$$

then the subsystem Σ_i can be stabilised by control law $u(t) = K_i x(t)$, and the controller gain is $K_i = V_i L_i^{-1}$.

Proof: After applying memoryless state-feedback controller to closed-loop system

$$\dot{x}(t) = (A_i + B_i K_i) x(t) + \bar{A}_i x(t - r_i(t)),$$

let us replace the A_i with $A_i + B_i K_i$, pre- and post-multiply (8) by $\text{diag}[P_i^{-1}, P_i^{-1}, Z_i^{-1}]$, pre- and post-multiply (9) by $\text{diag}[P_i^{-1}, P_i^{-1}, P_i^{-1}]$, and make the following change of variables:

$$\begin{aligned} L_i &:= P_i^{-1}, & T_i &:= P_i^{-1} Q_i P_i^{-1}, & R_i &:= Z_i^{-1} \\ M_{1i} &:= P_i^{-1} N_{1i} P_i^{-1}, & M_{2i} &:= P_i^{-1} N_{2i} P_i^{-1}, & V_i &= K_i P_i^{-1} \\ Y_i &:= \text{diag}[P_i^{-1}, P_i^{-1}] \cdot X_i \cdot \text{diag}[P_i^{-1}, P_i^{-1}] \end{aligned} \quad (23)$$

These operations end up with Equations (21) and (22) which complete the proof. \blacksquare

Due to the term $L_i R_i^{-1} L_i$, condition (22) in Lemma 5.1 is not an LMI. In order to handle this term, let us define a new variable, S_i , for which

$$L_i R_i^{-1} L_i \succeq S_i, \quad (24)$$

and replace Equation (22) with

$$\bar{\Lambda}_i := \begin{bmatrix} Y_{11i} & Y_{12i} & M_{1i} \\ * & Y_{22i} & M_{2i} \\ * & * & S_i \end{bmatrix} \succeq 0. \quad (25)$$

Inequality (24) is equivalent to $L^{-1} R L^{-1} \preceq S^{-1}$, which the Schur complement allows us to write as

$$\begin{bmatrix} S_i^{-1} & L_i^{-1} \\ * & R_i^{-1} \end{bmatrix} \succeq 0. \quad (26)$$

We introduce new variables

$$J_i = L_i^{-1}, \quad U_i = S_i^{-1}, \quad H_i = R_i^{-1} \quad (27)$$

so that we can re-write the condition (26) as

$$\begin{bmatrix} U_i & J_i \\ * & H_i \end{bmatrix} \succeq 0. \quad (28)$$

This lifting provides us to use LMIs (28) and (25) instead of (22) in order to make the condition LMI.

Proposition 5.1: Consider the system (20) with each $\tilde{\Sigma}_i$ satisfying Lemma 5.1, if there exists a matrix $W_i = W_i^T \succeq 0$ such that following LMIs hold:

$$\begin{bmatrix} \Pi_{11i} + L_i W_i L_i & \Pi_{12i} & \tau_i(L_i A_i^T + V_i^T B_i^T) \\ * & \Pi_{22i} & \tau_i L_i \bar{A}_i^T \\ * & * & -\tau_i R_i \end{bmatrix} \prec 0, \quad \forall i \in \mathcal{P} \quad (29)$$

then $\dot{V}_i(t, x_t) \leq -x^T(t) W_i x(t)$ for all $i \in \mathcal{P}$.

Proof: Let us pre- and post-multiply the LMI (13) in Proposition 3.2 by $\text{diag}[P_i^{-1}, P_i^{-1}, P_i^{-1}]$ and make the change of variables defined in Equation (23). \blacksquare

Similar to the condition in Lemma 5.1, Equation (29) is also not an LMI and we handle this term with the same procedure. By defining the new variables C_i and O_i , where $C_i - L_i W_i L_i \geq 0$, whose Schur complement is

$$\begin{bmatrix} C_i & L_i \\ * & O_i \end{bmatrix} \succeq 0, \quad (30)$$

and assuming $O_i = W_i^{-1}$, then we can replace the non-convex representation in Equation (29) with C_i as a convex one

$$\bar{\Pi}_i := \begin{bmatrix} \Pi_{11i} + C_i & \Pi_{12i} & \tau_i(L_i A_i^T + V_i^T B_i^T) \\ * & \Pi_{22i} & \tau_i L_i \bar{A}_i^T \\ * & * & -\tau_i R_i \end{bmatrix} < 0. \quad (31)$$

Now, we define lower and upper bounds for the Lyapunov functions:

$$\begin{aligned} \tilde{\mu}_i &:= \sigma_{\max} [L_i^{-1}] + \tau_i \sigma_{\max} [L_i^{-1} T_i L_i^{-1}] + \frac{1}{2} \tau_i^2 \sigma_{\max} [R_i^{-1}] \\ \tilde{\lambda}_i &:= \sigma_{\min} [W_i]. \end{aligned}$$

Repeating the same procedure with new variables of F_i and E_i , ($F_i \succeq L_i^{-1} T_i L_i^{-1}$, $E_i := T_i^{-1}$),

$$\begin{bmatrix} F_i & J_i \\ * & E_i \end{bmatrix} \succeq 0,$$

and re-writing $\tilde{\mu}_i$

$$\tilde{\mu}_i = \sigma_{\max} [J_i] + \tau_i \sigma_{\max} [F_i] + \frac{1}{2} \tau_i^2 \sigma_{\max} [H_i]. \quad (32)$$

we obtain LMI conditions.

Consider the upper bound of the term $\tilde{\mu}_i / \tilde{\lambda}_i \leq t$. For a constant t , feasibility of the dwell time $\tau_D = t + \tau_{\max}$ is the following nonlinear SDP minimisation problem:

$$\min \text{trace} \left[\sum_{i \in \mathcal{P}} (L_i J_i + S_i U_i + R_i H_i + T_i E_i + O_i W_i) \right]$$

$$\begin{aligned} \text{subject to } & \text{diag} [L_i, T_i, R_i, Y_i, W_i] \succ 0, & \forall i \in \mathcal{P}, \\ & \text{diag} [J_i, F_i, H_i, -W_i] \\ & < \text{diag} [j_i I, f_i I, h_i I, -\tilde{\lambda}_i I], & \forall i \in \mathcal{P}, \\ & \bar{\Lambda}_i \succeq 0, \bar{\Pi}_i < 0, & \forall i \in \mathcal{P}, \\ & \begin{bmatrix} U_i & J_i \\ * & H_i \end{bmatrix} \succeq 0, \begin{bmatrix} C_i & L_i \\ * & O_i \end{bmatrix} \succeq 0, \\ & \begin{bmatrix} F_i & J_i \\ * & E_i \end{bmatrix} \succeq 0, & \forall i \in \mathcal{P}, \\ & \begin{bmatrix} L_i & I \\ * & J_i \end{bmatrix} \succeq 0, \begin{bmatrix} S_i & I \\ * & U_i \end{bmatrix} \succeq 0, \\ & \begin{bmatrix} R_i & I \\ * & H_i \end{bmatrix} \succeq 0, & \forall i \in \mathcal{P}, \\ & \begin{bmatrix} T_i & I \\ * & E_i \end{bmatrix} \succeq 0, \begin{bmatrix} O_i & I \\ * & W_i \end{bmatrix} \succeq 0, & \forall i \in \mathcal{P}, \\ & j_i + \tau_i f_i + \frac{1}{2} \tau_i^2 h_i - t \tilde{\lambda}_i < 0, & \forall i \in \mathcal{P}. \end{aligned} \quad (33)$$

The cost function in Equation (33) is minimised to satisfy the inequality constraints (22) and (29). During the minimisation procedure, J_i , U_i , H_i , O_i , E_i converge to L_i^{-1} , S_i^{-1} , R_i^{-1} , W_i^{-1} , T_i^{-1} , respectively. We overcome the nonlinearity of the cost function of Equation (33) by using linearisation method provided in Ghaoui, Oustry, and AitRami (1997). The linearisation of the cost function is

$$\begin{aligned} \underline{f}_i = \text{constant} + \text{trace} & \left[\sum_{i \in \mathcal{P}} (L_i J_i^0 + L_i^0 J_i + S_i U_i^0 + S_i^0 U_i \right. \\ & \left. + R_i H_i^0 + R_i^0 H_i + T_i E_i^0 + T_i^0 E_i + O_i W_i^0 + O_i^0 W_i) \right]. \end{aligned}$$

The linearised cost function \underline{f}_i is minimised iteratively. The cost function is re-linearised around new point $(L_i^k, J_i^k, S_i^k, U_i^k, R_i^k, H_i^k, T_i^k, E_i^k, O_i^k, W_i^k)$ in each step. The nonlinear conditions (22) and (29) are checked in each iteration up to a pre-defined number of maximum iterations. If the conditions (22), (29) and (33) are satisfied, t is a proper dwell time.

The minimisation of the dwell time problem is a nested optimisation problem, where the outer loop (Steps 1 and 4) is a bisection algorithm with the cost function $\bar{f}_i := t$ and the inner loop (Steps 2–4) is the optimisation problem defined in Equation (33) with the cost function \underline{f}_i . If the inner loop is concluded successively, t is halved in its bisection interval, otherwise doubled (Step 4).

- Step 1. Choose a sufficiently large initial $t_u > 0$ such that there exists a solution. Set $t_l = 0$.
- Step 2. Set the iteration index k to 0 and $t = (t_u + t_l)/2$. Find a feasible set for the free parameters

$(L_i, J_i, S_i, U_i, R_i, H_i, T_i, E_i, O_i, W_i, Y_i, M_{1i}, M_{2i}, V_i, j_i, h_i, f_i, w_i)$ subject to conditions in Equation (33). Set $L_i^0 = L_i, J_i^0 = J_i, S_i^0 = S_i, U_i^0 = U_i, R_i^0 = R_i, H_i^0 = H_i, T_i^0 = T_i, E_i^0 = E_i, O_i^0 = O_i$ and $W_i^0 = W_i$.

Step 3. Solve the following convex optimisation problem for the same free parameters in Step 2:

$$\begin{aligned} \min \quad & \text{trace} \left[\sum_{i \in \mathcal{P}} (L_i J_i^k + L_i^k J_i + S_i U_i^k + S_i^k U_i \right. \\ & + R_i H_i^k + R_i^k H_i + T_i E_i^k + T_i^k E_i \\ & \left. + O_i W_i^k + O_i^k W_i) \right] \\ \text{s.t.} \quad & \text{conditions in Equation (33)}. \end{aligned}$$

Set $L_i^{k+1} = L_i, J_i^{k+1} = J_i, S_i^{k+1} = S_i, U_i^{k+1} = U_i, R_i^{k+1} = R_i, H_i^{k+1} = H_i, T_i^{k+1} = T_i, E_i^{k+1} = E_i, O_i^{k+1} = O_i$ and $W_i^{k+1} = W_i$.

Step 4. If specified tolerance, such that $t_u - t_l < \text{tol}$, is satisfied, then set $K_i = V_i L_i^{-1}$ for all $i \in \mathcal{P}$ and exit.

The dwell time is $\tau_D = t + \tau_{\max}$.

Else if Equations (22) and (29) are satisfied, then set $t_u = t$, and return to Step 2.

Otherwise, set $k = k + 1$ and go to Step 3.

If there is no feasible solutions after specified number of iterations, then set $t_l = t$ and return to Step 2.

Example 5.1: This example is from Yan, Özbay, and Şansal (2014). In the corresponding paper, stabilisation of a linear time-varying system guaranteed with a switching controller. In order to achieve that, linear parameter varying (LPV) system is represented as a switching delay system with two nominal subsystems and uncertainty bounds are determined. Then, controllers are designed with robust stability conditions. Synthesised controllers are

$$K_1 = [0.9681, 0.0465], \quad K_2 = [-0.2708, 0.3715]$$

and the resulting dwell time is found to be 0.92 seconds. In this paper, we only considered the nominal subsystems of the switching delay system representation. The two nominal subsystems are defined as

$$\begin{aligned} A_1 &= \begin{bmatrix} -3 & -1 \\ -1 & -1.9 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -1 & 0 \\ -0.45 & -1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \tau_1 = 0.2, \quad d_1 = 0.01 \\ A_2 &= \begin{bmatrix} -2 & -0.5 \\ -1 & -2 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} -1 & 0 \\ 0.05 & -1 \end{bmatrix}, \end{aligned}$$

$$B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \tau_2 = 0.155, \quad d_2 = 0.01$$

Resulting controllers of our algorithm are

$$K_1 = [0.5527, -0.5036], \quad K_2 = [-0.6483, -0.6561]$$

and the dwell time is $\tau_d = 0.49$ seconds.

Example 5.2: This example is a slightly modified version of the example of Yuan and Wu (2015), where the switched system in question is a non-delayed system which does not admit a common Lyapunov function. In the corresponding example, switched linear plant is in the form:

$$\dot{x} = A_{0,\sigma(t)}x + B_{0,\sigma(t)}w + B_{1,\sigma(t)}u$$

where w is the disturbance,

$$\begin{aligned} A_{0,1} &= \begin{bmatrix} 0.5108 & -0.9147 & -0.2 \\ -0.6563 & 0.1798 & 0.113 \\ 0.881 & -0.7841 & 0.1 \end{bmatrix}, \\ B_{1,1} &= \begin{bmatrix} 0.3257 \\ 1.2963 \\ 2.43 \end{bmatrix} \\ A_{0,2} &= \begin{bmatrix} -0.125 & -0.9833 & -0.34 \\ -0.5305 & 0.3848 & 0.58 \\ 1.0306 & 0.6521 & 0.1 \end{bmatrix}, \\ B_{1,2} &= \begin{bmatrix} 1.0992 \\ 0.6532 \\ 3.5 \end{bmatrix} \end{aligned}$$

By using $A_{0,i}$ and $B_{1,i}$, we generated our example. Let Σ_1 and Σ_2 be

$$\begin{aligned} A_1 &= (1 - \lambda) \cdot A_{0,1}, \quad \bar{A}_1 = \lambda \cdot A_{0,1}, \quad B_1 = B_{1,1}, \\ \tau_1 &= \tilde{\tau}, \quad d_1 = 0.01, \\ A_2 &= (1 - \lambda) \cdot A_{0,2}, \quad \bar{A}_2 = \lambda \cdot A_{0,2}, \quad B_2 = B_{1,2}, \\ \tau_2 &= 1.6 \cdot \tilde{\tau}, \quad d_2 = 0.01. \end{aligned}$$

For $\lambda = 0.9$ and $\tilde{\tau} = 0.05$, resulting dwell time is 13.78 seconds and controllers are

$$\begin{aligned} K_1 &= [27.78, -25.94, 1.70], \\ K_2 &= [1.09, -3.08, -1.48] \end{aligned}$$

In Figure 2, minimum upper bounds for the dwell times can be seen for various $\tilde{\tau}$ and λ values. Dwell time grows linearly for large delay values whereas an exponential growth is observed in Figure 1. The key difference between two examples is that the subsystems in Example 5.2 can be stabilised independent of delays but

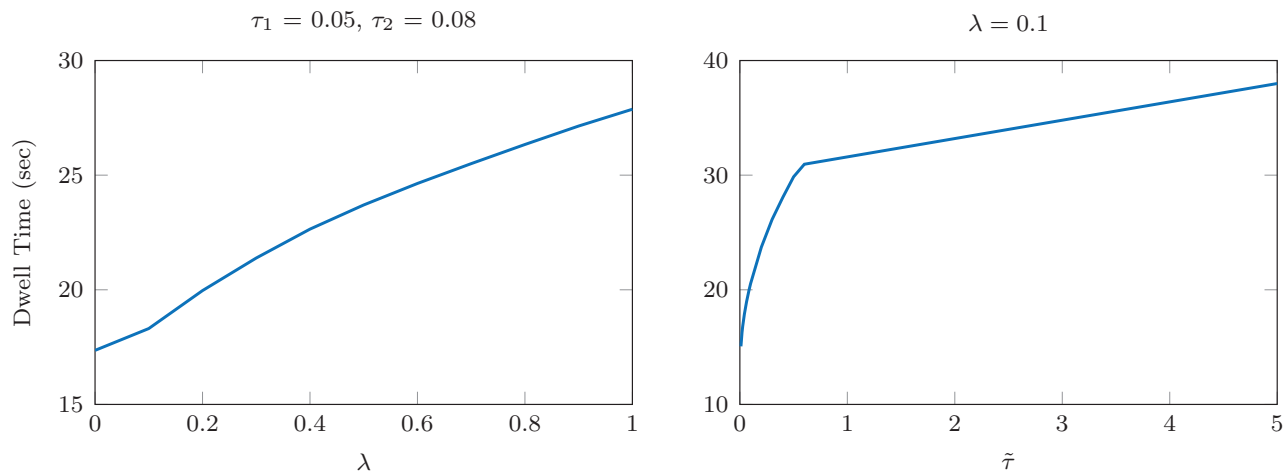


Figure 2. Dwell time results for different $\tilde{\tau}$ and λ values of Example 5.2.

the stability of the subsystems in Example 4.1 depends on delays.

6. Conclusions

We performed the calculation of an upper bound of dwell time by quasi-convex optimisation methods to ensure stability of linear switched delay system. LMI conditions of free-weighting matrices method are used to find appropriate Lyapunov–Krasovskii functionals for non-switching subsystems. By combining these conditions with a cost function, which represents the upper bound of dwell time, the upper bound is optimised using a bisection algorithm where each step is a linear SDP feasibility problem. By the numerical examples, it is shown that the results obtained in Çalıřkan et al. (2013) and Yan and Özbay (2008) can be improved using the method proposed in the present paper. In addition to this, a dwell time minimising controller synthesis algorithm is also developed in this work. Although the conditions are non-linear and the corresponding set is non-convex, this algorithm successively linearise the conditions and turn the problem into a linear SDP. The numerical examples are given to illustrate the efficiency of the proposed method.


Less conservative conditions for the stability of the delayed switching systems can be found in papers presenting average dwell time methods (see Sun et al., 2006; Chesi, Colaneri, Geromel, Middleton, & Shorten, 2012). However, the average dwell time conditions are non-convex due to exponential and bilinear terms when the dwell time is considered as a free parameter in optimisation. Representation of the dwell time in the present paper is more conservative, but the dwell time minimiser controller synthesis problem is tractable due to convex nature of the conditions.

A typical application of the switched control scheme is the network congestion control systems (Zhao, Zhang, Shi, & Liu, 2012). Due to the time-delay nature of the network systems, the method presented in this paper can contribute to the research on application of the network congestion control systems.

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No potential conflict of interest was reported by the authors.

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