SIGNALING GAMES IN NETWORKED SYSTEMS

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By
Serkan Sarıtaş
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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

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We investigate decentralized quadratic cheap talk and signaling game problems when the decision makers (an encoder and a decoder) have misaligned objective functions. We first extend the classical results of Crawford and Sobel on cheap talk to multi-dimensional sources and noisy channel setups, as well as to dynamic (multi-stage) settings. Under each setup, we investigate the equilibria of both Nash (simultaneous-move) and Stackelberg (leader-follower) games. We show that for scalar cheap talk, the quantized nature of Nash equilibrium policies holds for arbitrary sources; whereas Nash equilibria may be of non-quantized nature, and even linear for multi-dimensional setups. All Stackelberg equilibria policies are fully informative, unlike the Nash setup. For noisy signaling games, a Gauss-Markov source is to be transmitted over a memoryless additive Gaussian channel. Here, conditions for the existence of affine equilibria, as well as informative equilibria are presented, and a dynamic programming formulation is obtained for linear equilibria. For all setups, conditions under which equilibria are non-informative are derived through information theoretic bounds. We then provide a different construction for signaling games in view of the presence of inconsistent priors among multiple decision makers, where we focus on binary signaling problems. Here, equilibria are analyzed, a characterization on when informative equilibria exist, and robustness and continuity properties to misalignment are presented under Nash and Stackelberg criteria. Lastly, we provide an analysis on the number of bins at equilibria for the quadratic cheap talk problem under the Gaussian and exponential source assumptions.

Our findings reveal drastic differences in signaling behavior under team and game setups and yield a comprehensive analysis on the value of information; i.e., for the decision makers, whether there is an incentive for information hiding, or not, which have practical consequences in networked control applications. Furthermore, we provide conditions on when affine policies may be optimal in decentralized multi-criteria control problems and for the presence of active information transmission even in strategic environments. The results also highlight that even when the decision makers have the same objective, presence of inconsistent priors among the decision makers may lead to a lack of robustness in equilibrium behavior.

Keywords: Networked control systems, game theory, signaling games, cheap talk, quantization, hypothesis testing, inconsistent priors, information theory.

ÖZET

AĞ TABANLI SİSTEMLERDE İŞARETLEME OYUNLARI

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Farklı hedeflere sahip karar vericilerin (kodlayıcı ve kod çözücü) yer aldığı merkezi olmayan karesel ucuz konuşma ve işaretleme oyunlarını incelemekteyiz. İlk olarak, Crawford ve Sobel'in ucuz konuşma hakkındaki önemli sonuçlarını, çok boyutlu, gürültülü kanallı ve dinamik (çok-aşamalı) kurgulara genişletmekteyiz. Her kurgu için, Nash (eş-zamanlı hamleli) ve Stackelberg (lider-takipçi) oyunlarının dengelerini incelemekteyiz. Tek boyutlu ucuz konuşma oyunlarında Nash dengesinin nicemlenmiş mizacının her türlü kaynak için korunduğunu, çok boyutlu kurgularda ise Nash dengesinin nicemlenmiş olmayabileceğini, hatta doğrusal olabileceğini göstermekteyiz. Tüm Stackelberg dengelerinde, Nash dengelerinden farklı olarak, kodlayıcı, elindeki bilgiyi kod çözücü ile gizlemeden paylaşmaktadır. Gürültülü işaretleme oyunlarında, Gaus-Markov dağılımlı kaynak, hafızasız eklemeli Gauss kanal üzerinden aktarılmaktadır. Bu kurguda, ilgin dengelerin bulunma koşullarının yanında bilgilendirici dengelerin bulunma koşulları da sunulmakta ve doğrusal dengeler için dinamik programlama formülasyonu elde edilmektedir. Çalışılan tüm kurgularda, hangi dengelerin bilgilendirici olmadığının koşulları, bilgi kuramsal sınırlar üzerinden türetilmektedir. Daha sonra, işaretleme oyunlarında karar vericilerin önsel bilgilerinde tutarsızlık olduğu durum göz önünde bulundurularak ikili işaretleme oyunlarını modellemekteyiz. Bu kısımda, Nash ve Stackelberg ölçütleri altında dengeler ve hangi durumlar altında bilgilendirici oldukları çözümlenmekte, tutarsız önsel bilgilere karşı gürbüzlük ve süreklilik özellikleri sunulmaktadır. Son olarak, karesel ucuz konuşma probleminde dengedeki nicemleme seviye sayısının Gauss ve üssel dağılımlı kaynaklar için analizini sağlamaktayız.

Bulgularımız, takım ve oyun kurguları altında işaretleme davranışlarındaki büyük farklılıkları ortaya koymakta ve bilginin değeri üzerine kapsamlı bir analiz sağlamaktadır; diğer bir deyişle, ağ tabanlı kontrol uygulamalarında pratik sonuçları olan, karar vericiler açısından bilginin gizlenmesi veya paylaşılması için bir teşvik olup olmadığı araştırılmaktadır. Ayrıca, merkezi olmayan çok ölçütlü kontrol problemlerinde ilgin politikaların ne zaman en iyi olabileceğinin ve stratejik ortamlarda bile aktif bilgi aktarımının ne zaman mevcut olabileceğinin koşullarını sağlamaktayız. Sonuçlarımız, karar vericiler aynı hedefe sahip olsalar bile, önsel bilgilerindeki tutarsızlığın dengede gürbüzlük eksikliğine yol açabileceğinin de altını çizmektedir.

Anahtar sözcükler: Ağ tabanlı kontrol sistemleri, oyun kuramı, işaretleme oyunları, ucuz konuşma, nicemleme, hipotez testi, tutarsız önsel bilgi, bilgi kuramı.

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To my one and only love, Tuğba

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Chapter 1

Introduction

1.1 Motivation

"*The more information the better*": This is commonly accepted both on the intuition level and more formally in decision theory. One of the earliest mathematical representation of this idea can be found in Frank Ramsey's study [1]. In order to represent and quantify the information more formally, the value of information was first introduced in decision theory for a decision-maker in a risky environment by Blackwell [2, 3]. The value of information is defined as the differential utility that the decision maker obtains by considering that information in addition to his initial beliefs. For one decision maker, the value of information is known to be positive: *more information is always at least as good*. As studied by Blackwell, there is a well-defined partial order of information structures which provide a general theory for the value of information [2, 3].

When multiple decision makers are considered, there are two different approaches depending on the objectives of the decision makers:

(i) *Team theory* is the field of study on the interaction dynamics among decentralized decision makers with identical objective functions. In the team, individual decision makers strive for the same goal, using the same (probabilistic) model of the underlying decision process, but not necessarily sharing the same online information (such as measurements) on the uncertainty.

(ii) *Game theory* deals with setups with misaligned objective functions, where each decision maker chooses a strategy to maximize its own utility which is determined by the joint strategies chosen by all decision makers.

Despite the difficulty to obtain solutions under general information structures, it is evident in team problems that more information provided to any of the decision makers does not negatively affect the utility of the players; i.e., the value of information is always positive for team problems. For a detailed account we refer the reader to [4].

However, usually accepted principle of decision theory that "*the more information the better*" seemingly breaks down in strategic contexts. More information can have negative effects on the utilities of some or even all of the players in a system. For example, [5] shows that public disclosure of information can make all decision makers worse off. In non-cooperative networks, it is possible that the addition of resources to the network is accompanied by a degradation of the performance, which is known as Braess paradox [6, 7]. Further examples of negative value of information were given in [8–10]. On the other hand, in game theory, it is also possible that more information does not hurt the decision makers. [11] shows that the value of information cannot be negative for a decision maker as long as the others are not aware of it. The value of information was proved to be positive in the case of a secret message by [11] or in the case of a private message in zero-sum games by [12]. Hence, as discussed above, informational aspects are very challenging to address for general non-zero sum game problems. For two-players games with incomplete information, in [13], it is shown that "*almost every situation is conceivable: information can be beneficial for all players, just for the one who does receive it, or, less intuitively, just for the one who does not receive it, or it could be bad for both*". Many examples exhibiting various effects of information can be found in [13]. Further intricacies on informational aspects in competitive setups have been discussed in [14, 15]. The main reason why one obtains such results is that, in games we cannot consider individual decision makers in isolation; we need to consider the equilibrium behavior and the effect of the additional information on the other decision makers.

Since the value of information may be negative in general strategic environments, the additional information is not always desirable. Although sharing information makes better utilities possible for the decision makers, it has also strategic effects that revealing all information to an opponent is not usually the most advantageous strategy. However, even a completely self-interested decision maker may prefer to reveal some information to get a higher utility. Within the scope of the above reasonings, Crawford and Sobel [16] probes "*how much and to which extent the information should be revealed in accordance with the similarity of agents' interests*".

In this dissertation, accordingly, we study the informational aspects in games in the context of 'signaling games': Signaling games and cheap talk are concerned with a class of Bayesian games where an informed decision maker (encoder or transmitter) transmits information to another decision maker (decoder or receiver). Unlike a team setup in the classical communication problems, however, the objective functions of the players are not aligned, and due to the Bayesian assumption, we have *games of incomplete information*; i.e., the decision makers may have private information abut their own utilities, about their type and preferences [17]. Such a study has been initiated by Crawford and Sobel [16], who obtained the surprising result that under some technical conditions on the utility functions of the decision makers, the cheap talk problem only admits equilibria that involve quantized encoding policies. This is in significant contrast to the usual communication/information theoretic case where the goals are aligned.

The cheap talk and signaling game problems are applicable in networked control systems when a communication channel exists among competitive and non-cooperative decision makers. For example, in a smart grid application, there may be strategic sensors in the system [18] that wish to change the equilibrium for their own interests through reporting incorrect measurement values.

In this dissertation,

(i) we consider both Nash equilibria and Stackelberg equilibria of the setup of Crawford and Sobel [16], and provided extensions to multi-dimensional and noisy setups. We showed that for all scalar sources, the quantized nature of all equilibrium policies holds under Nash equilibria, whereas policies are fully informative under Stackelberg equilibria. Single-stage signaling games were also considered, where Nash and Stackelberg equilibria were studied.

(ii) building on the static (one-stage) analysis, we extend the analysis of the setup in [16] to the multi-stage case and to the case where the priors may also be subjective.

(iii) we consider signaling games that refer to a class of two-player games of incomplete information in which an informed decision maker (encoder or transmitter) transmits information to another decision maker (decoder or receiver) in the hypothesis testing context.

(iv) we study the number of bins at the equilibrium under cheap talk setup with exponential and Gaussian sources as to whether there are finitely many bins or countably infinite number of bins in any equilibrium.

Even though in this dissertation we only consider quadratic criteria under a bias term leading to a misalignment, the contrast with the case where there is no bias (that has been heavily studied in the information theory literature) raises a number of sharp conclusions for system designers working on networked systems under competitive environments. Our findings provide further conditions on when affine policies may be optimal in decentralized multi-criteria control problems and lead to conditions for the presence of active information transmission in strategic environments.

In the following, we first provide the preliminaries and introduce the problems considered in the dissertation, and present the related literature briefly.

1.2 Preliminaries

Let there be two decision makers (players): An informed player (encoder or transmitter) knows the value of the $\mathbb{M}$-valued random variable M and transmits the $\mathbb{X}$-valued random variable X to another player (decoder or receiver), who generates his $\mathbb{M}$-valued optimal decision U upon receiving X. We allow for randomized decisions, therefore, we let the policy space of the encoder be the set of all stochastic kernels from $\mathbb{M}$ to $\mathbb{X}$. [1] Let Γ^e denote the set of all such policies. We let the policy space of the decoder be the set of all stochastic kernels from $\mathbb{X}$ to $\mathbb{M}$. Let Γ^d denote the set of all such stochastic kernels. Given $\gamma^e \in \Gamma^e$ and $\gamma^d \in \Gamma^d$, the goal in the classical communications theory is to minimize the expectation

$$J(\gamma^e, \gamma^d) = \int c(m,u)\gamma^e(\mathrm{d}x|m)\gamma^d(\mathrm{d}u|x)P(\mathrm{d}m),$$

where $c(m,u)$ is some cost function. One very common case is the setup with $c(m,u) = |m-u|^2$.

Recall that a collection of decision makers who have an agreement on the probabilistic description of a system and a cost function to be minimized, but who may have different on-line information is said to be a *team* (see, e.g. [4]). Hence, the classical communications setup may be viewed as a team of an encoder and a decoder.

In many applications (in networked systems, recommendation systems, and applications in economics) the objectives of the encoder and the decoder may not be aligned. For example, the encoder may aim to minimize

$$J^e(\gamma^e, \gamma^d) = \mathbb{E}\left[c^e(m,u)\right] ,$$

whereas the decoder may aim to minimize

$$J^d(\gamma^e, \gamma^d) = \mathbb{E}\left[c^d(m,u)\right] ,$$

where $c^e(m,u)$ and $c^d(m,u)$ denote the cost functions of the encoder and the decoder, respectively, when the action u is taken for the corresponding message m.

[1] P is a stochastic kernel from $\mathbb{M}$ to $\mathbb{X}$ if $P(\cdot|m)$ is a probability measure on $\mathcal{B}(\mathbb{X})$ for every $m \in \mathbb{M}$, and $P(A|\cdot)$ is a Borel measurable function of m for every $A \in \mathcal{B}(\mathbb{X})$.

Such a problem is known in the economics literature as *cheap talk* (the transmitted signal does not affect the cost, that is why the game is named as *cheap talk*). A more general formulation would be the case when the transmitted signal x is also an explicit part of the cost functions c^e and/or c^d, then the communication between the players is not costless and the formulation turns into a *signaling game* problem. We will consider both a noiseless communication setup as *cheap talk* and a noisy communication setup, where the problem may be viewed as a *signaling game* in this dissertation.

Such problems are studied under the tools and concepts provided by *game theory* since the goals are not aligned. We note that when $c^e = c^d$, the setup is a traditional communication theoretic setup. If $c^e = -c^d$, that is, if the setup is a zero-sum game, then an equilibrium is achieved when $\gamma^{*,e}$ is non-informative (e.g., a kernel with actions statistically independent of the source) and $\gamma^{*,d}$ uses only the prior information (since the received information is non-informative). We call such an equilibrium a *non-informative (babbling) equilibrium.* The following is a useful observation, which follows from [16, Theorem 1] and [17]:

Proposition 1.2.1. *A non-informative (babbling) equilibrium always exists for the cheap talk game.*

Although the encoder and decoder act sequentially in the game as described above, how and when the decisions are made and the nature of the commitments to the announced policies significantly affect the analysis of the equilibrium structure. Here, two different types of equilibria are investigated:

(i) *Nash game*: the encoder and the decoder make simultaneous decisions.

(ii) *Stackelberg game* : the encoder and the decoder make sequential decisions where the encoder is the leader and the decoder is the follower.

In this dissertation, the terms *Nash game* and the *simultaneous-move game* will be used interchangeably, and similarly, the *Stackelberg game* and the *leader-follower game* will be used interchangeably.

In the simultaneous-move game, the encoder and the decoder announce their policies at the same time, and a pair of policies $(\gamma^{*,e}, \gamma^{*,d})$ is said to be a *Nash equilibrium* [19] if

$$\begin{aligned} J^e(\gamma^{*,e}, \gamma^{*,d}) &\leq J^e(\gamma^{e}, \gamma^{*,d}) \quad \forall \gamma^e \in \Gamma^e\,, \\ J^d(\gamma^{*,e}, \gamma^{*,d}) &\leq J^d(\gamma^{*,e}, \gamma^{d}) \quad \forall \gamma^d \in \Gamma^d\,. \end{aligned} \tag{1.1}$$

As observed from the definition (1.1), under the Nash equilibrium, each individual player chooses an optimal strategy given the strategies chosen by the other players.

On the other hand, in a leader-follower game, the leader (encoder) commits to and announces his optimal policy before the follower (decoder) does, the follower observes what the leader is committed to before choosing and announcing his optimal policy, and a pair of policies $(\gamma^{*,e}, \gamma^{*,d})$ is said to be a *Stackelberg equilibrium* [19] if

$$J^e(\gamma^{*,e}, \gamma^{*,d}(\gamma^{*,e})) \leq J^e(\gamma^{e}, \gamma^{*,d}(\gamma^{e})) \quad \forall \gamma^e \in \Gamma^e\,,$$

where $\gamma^{*,d}(\gamma^e)$ satisfies (1.2)

$$J^d(\gamma^{e}, \gamma^{*,d}(\gamma^{e})) \leq J^d(\gamma^{e}, \gamma^{d}(\gamma^{e})) \quad \forall \gamma^d \in \Gamma^d\,.$$

As observed from the definition (1.2), the decoder takes his optimal action $\gamma^{*,d}(\gamma^e)$ after observing the policy of the encoder γ^e. Further, in the Stackelberg game, the leader cannot backtrack on his commitment, but has a leadership role since he can manipulate the follower by anticipating follower's actions.

Stackelberg games are commonly used to model attacker-defender scenarios in security domains [20]. In such setups, the defender (leader) acts first by committing to a strategy, and the attacker (follower) chooses how and where to attack after observing the defender's choice. However, in some situations, security measures may not be observable for the attacker; therefore, a simultaneous-move game is preferred to model such situations; i.e., the Nash equilibrium analysis is needed [21].

Heretofore, only *single-stage game*s are considered. If a game is played over a number of time periods, the game is called a *multi-stage game*. In this dissertation, with the term *dynamic*, we will refer to *multi-stage game* setups; even though strictly speaking a single stage setup may also be viewed to be dynamic [22] since the information available to the decoder is totally determined by encoder's actions. In the multi-stage version of the game, the encoder and the decoder aim to minimize the expected cost over the total horizon of the game as follows:

$$J^e\left(\gamma^e_{[0,N-1]}, \gamma^d_{[0,N-1]}\right) = \mathbb{E}\left[\sum_{k=0}^{N-1} c^e_k(m_k, u_k)\right] ,$$
$$J^d\left(\gamma^e_{[0,N-1]}, \gamma^d_{[0,N-1]}\right) = \mathbb{E}\left[\sum_{k=0}^{N-1} c^d_k(m_k, u_k)\right] .$$

The Nash and Stackelberg equilibria of the game are defined based on the total costs defined above.

Besides the static and multi-stage cheap talk and signaling game formulations, in this dissertation, the binary signaling problem is investigated under the hypothesis testing context. In this direction, the following binary hypothesis-testing problem is considered:

$$\mathcal{H}_0 : Y = S_0 + N ,$$
$$\mathcal{H}_1 : Y = S_1 + N ,$$

where Y is the observation (measurement) that belongs to the observation set $\Gamma = \mathbb{R}$, S_0 and S_1 denote the deterministic signals under hypothesis $\mathcal{H}_0$ and hypothesis $\mathcal{H}_1$, respectively, and N represents Gaussian noise; i.e., $N \sim \mathcal{N}(0, \sigma^2)$. In the conventional Bayesian framework, the aim of the receiver is to design the optimal decision rule (detector) based on Y in order to minimize the Bayes risk. However, in our game formulation, the transmitter and the receiver are considered as two decision makers with non-aligned Bayes risks; i.e., they have subjective priors and costs, and they aim to minimize their own Bayes risks. Based on the Bayes risks of the decision makers, Nash and Stackelberg equilibria of the binary hypothesis-testing game are investigated.

1.3 Literature Review

In many decentralized and networked control problems, decision makers have either misaligned criteria or have subjective priors, which necessitates solution concepts from game theory. For example, detecting attacks, anomalies, and malicious behavior with regard to security in networked control systems can be analyzed under a game theoretic perspective, see e.g., [23–34].

The cheap talk and signaling game problems find applications in networked control systems when a communication channel/network is present among competitive and non-cooperative decision makers [19]. For example, in a smart grid application, there may be strategic sensors in the system [18] that wish to alter the equilibrium decisions at a controller receiving data from the sensors to lead to a more desirable equilibrium, for example by enforcing an outcome to enhance its prolonged use in the system. One may also consider a utility company which wishes to inform users regarding pricing information; if the utility company and the users engage in selfish behavior, it may be beneficial for the utility company to hide certain information and the users to be strategic about how they interpret the given information. One further area of application is recommender systems (as in rating agencies) [35]. For further applications, see [18, 36]. All of these applications lead to a drastically new framework where the value of information and its utilization are very fragile to the system under consideration.

In game theory, Nash and Stackelberg equilibria are drastically different concepts. Both equilibrium concepts find applications depending on the assumptions on the leader, that is, the encoder, in view of the commitment conditions. Stackelberg games are commonly used to model attacker-defender scenarios in security domains [20]. In many frameworks, the defender (leader) acts first by committing to a strategy, and the attacker (follower) chooses how and where to attack after observing defender's choice. However, in some situations, security measures may not be observable for the attacker; therefore, a simultaneous-move game is preferred to model such situations; i.e., the Nash equilibrium analysis is needed [21].

Crawford and Sobel [16] have made foundational contributions to the study of cheap talk with misaligned objectives where the cost functions c^e and c^d satisfy certain monotonicity and differentiability properties but there is a bias term in the cost functions. Their result is that the number of bins at the equilibrium is upper bounded by a function which is negatively correlated to the bias. For the setup of Crawford and Sobel but when the source admits an exponentially distributed real random variable, [37] establishes the discrete-nature of equilibria, and obtains the equilibrium bins with *finite upper bounds on the number of bins* under any equilibrium in addition to some structural results on informative equilibria for general sources.

There have been a number of related contributions in the economics literature in addition to the seminal work by Crawford and Sobel, which we briefly review in the following: Reference [38] shows that even if the sender and the decoder have identical preferences, perfect communication may not be possible at the equilibrium because information transmission may be costly. Reference [39] studies the setup in [16] with two senders and shows that if senders transmit the messages sequentially once, then the equilibrium is always quantized and if senders transmit the messages simultaneously and their biases are either both positive or both negative, then a fully revealed equilibrium is possible. Reference [40] studies a scalar setup and proves that if multiple senders transmit the messages sequentially and their biases have opposite signs, then a fully revealed equilibrium is possible; this study also considers two-dimensional real valued sources, and shows that a fully revealed equilibrium occurs if and only if the multiple senders have perfectly opposing biases. For multi-dimensional cheap talk, [41] shows that it is possible to have a fully revealing equilibrium on a particular dimension on which the sender and the decoder agree on so that the interests of the sender and the decoder are aligned on that particular dimension. Moreover, multi-dimensional cheap talk with multiple senders is analyzed in [42] and [43] with unbounded and bounded state spaces, respectively. In [42], it is shown that full revelation of information is possible in multi-dimensional cheap talk with multiple encoders when the encoders send messages simultaneously; however, when the encoders send messages sequentially, fully revealing equilibria exist if they have perfectly opposing biases [44]. The study in [45] considers a special noisy channel setup between the sender and decoder, and shows that there may be infinitely many actions (countable or uncountable) induced at the equilibrium even though all equilibria are interval partitions in the noiseless case [16]. Conditions for Nash equilibria are investigated in [46] for a scenario in which there exists a discrete noisy channel between an informed sender and an uninformed decoder, and the source is finitely valued. Furthermore, there are some contributions which modify the information structure given in Crawford and Sobel's setup: In [47], the sender knows that the decoder has partial information about his/her private information; whereas the sender does not know this in [48,49]. For a detailed literature review on communication between informed experts and uninformed decision makers, we refer the reader to [50]. We note also that in the area of information theory, there exists a vast literature on security aspects of information transmission, see e.g., [51, 52]. Game theoretic analysis is also useful in various contexts involving security problems. For example, the security of the smart-grid infrastructure can be analyzed by considering the adversarial nature of the interaction between an attacker and a defender [25,26], and a game theoretic setup would be appropriate to analyze such interactions. For an overview of security and privacy problems in computer networks that are analyzed within a game-theoretic framework, [53] can be referred.

On the multi-stage side, much of the literature has focused on Stackelberg equilibria as we note below. A notable exception is [54], where the multi-stage extension of the setup of Crawford and Sobel is analyzed for a source which is a fixed random variable distributed according to some density on $[0, 1]$ (see Theorem 3.2.5 for a detailed discussion on this very relevant paper). These two concepts may have equilibria that are quite distinct: As discussed in [55,56], in the Nash equilibrium case, building on [16], equilibrium properties possess different characteristics as compared to team problems; whereas for the Stackelberg case, the leader agent is restricted to be committed to his announced policy, which leads to similarities with team problem setups [57, 58]. Since there is no such commitment in the Nash setup; the perturbation in the encoder does not lead to a functional perturbation in decoder's policy, unlike the Stackelberg setup.

However, in the context of binary signaling, we will see that the distinction is not as sharp as it is in the case of quadratic signaling games [55, 56]. [57] investigates a Gaussian cheap talk game under the Stackelberg assumption with quadratic cost functions for a class of single- and multi-terminal setups, and it is shown that the best response of the encoder is linear by restricting decoder strategies to be affine. In [59], the non-alignment between the cost functions of the encoder and the decoder is a function of a Gaussian random variable (r.v.) and secret to the decoder; whereas, it is fixed and known to the decoder in [16]. The multi-stage Gaussian signaling game is studied in [58] where the linearity of Stackelberg equilibria is investigated. [60–62] consider the information design and strategic source-channel coding problem between an encoder and a decoder with non-aligned utility functions under the Stackelberg equilibrium. [63] studies the central scheduling problem of allocating channels as a signaling game problem between the base station and mobile stations under the Stackelberg assumption. [64] investigates a multi-stage linear quadratic Gaussian game with asymmetric information and simultaneous moves, and it is shown that under certain conditions, players' strategies are linear in their private types.

Identifying when optimal policies are linear or affine for decentralized systems involving Gaussian variables under quadratic criteria is a recurring problem in control theory, starting perhaps from the seminal work of Witsenhausen [65], where sub-optimality of linear policies for such problems under *non-classical information structures* is presented. The reader is referred to *Chapters 3* and *11* of [4] for a detailed discussion on when affine policies are and are not optimal. These include the problem of communicating a Gaussian source over a Gaussian channel, variations of Witsenhausen's counterexample [66]; and game theoretic variations of such problems. For example if the noise variable is viewed as the maximizer and the encoders/decoders (or the controllers) act as the minimizer, then affine policies may be optimal for a class of settings, see [67–71]. [71] also provides a review on Linear Quadratic Gaussian (LQG) problems under non-classical information including Witsenhausen's counterexample. Our study provides further conditions on when affine policies may constitute equilibria for such decentralized quadratic Gaussian optimization problems.

Standard binary hypothesis testing has been extensively studied over several decades under different setups [72,73], which can also be viewed as a decentralized control/team problem involving a encoder and a decoder who wish to minimize a common objective function. However, there exist many scenarios in which the analysis falls within the scope of game theory; either because the goals of the decision makers are misaligned, or because the probabilistic model of the system is not common knowledge among the decision makers.

A game theoretic perspective can be utilized for hypothesis testing problem for a variety of setups. For example, detecting attacks, anomalies, and malicious behavior in network security can be analyzed under the game theoretic perspective [23–27]. In this direction, the hypothesis testing and the game theory approaches can be utilized together to investigate attacker-defender type applications [28–34], multimedia source identification problems [74], and inspection games [75–77]. In [29], a Nash equilibrium of a zero-sum game between Byzantine (compromised) nodes and the fusion center (FC) is investigated. The strategy of the FC is to set the local sensor thresholds that are utilized in the likelihood-ratio tests, whereas the strategy of Byzantines is to choose their flipping probability of the bit to be transmitted. In [30], a zero-sum game of a binary hypothesis testing problem is considered over finite alphabets. The attacker has control over the channel, and the randomized decision strategy is assumed for the defender. The dominant strategies in Neyman-Pearson and Bayesian setups are investigated under the Nash assumption. The authors of [76, 77] investigate both Nash and Stackelberg equilibria of a zero-sum inspection game where an inspector (environmental agency) verifies, with the help of randomly sampled measurements, whether the amount of pollutant released by the inspectee (management of an industrial plant) is higher than the permitted ones. The inspector chooses a false alarm probability α, and determines his optimal strategy over the set of all statistical tests with false alarm probability α to minimize the non-detection probability. On the other side, the inspectee chooses the signal levels (violation strategies) to maximize the non-detection probability. [31] considers a complete-information zero-sum game between a centralized detection network and a jammer equipped with multiple antennas and investigates pure strategy Nash equilibria for this game. The fusion center (FC) chooses the optimal threshold of a single-threshold rule in order to minimize his error probability based on the observations coming from multiple sensors, whereas the jammer disrupts the channel in order to maximize FC's error probability under instantaneous power constraints. However, unlike the setups described above, in this dissertation, we assume an additive Gaussian noise channel, and in the game setup, a Bayesian hypothesis testing setup is considered in which the encoder chooses signal levels to be transmitted and the decoder determines the optimal decision rule. Both players aim to minimize their individual Bayes risks, which leads to a nonzero-sum game.

1.4 Contributions and Organization of the Dissertation

1.4.1 Chapter 2

In this chapter, we study the decentralized quadratic cheap talk and signaling game problems when an encoder and a decoder, viewed as two decision makers, have misaligned objective functions. We investigate the extension of Crawford and Sobel's cheap talk formulation [16] to multi-dimensional sources and to noisy channel setups. We consider both (simultaneous-move) Nash equilibria and (leader-follower) Stackelberg equilibria. We show that for arbitrary scalar sources, in the presence of misalignment, the quantized nature of all equilibrium policies holds for Nash equilibria in the sense that all Nash equilibria are equivalent to those achieved by quantized encoder policies. On the other hand, all Stackelberg equilibria policies are fully informative. For multi-dimensional setups, unlike the scalar case, Nash equilibrium policies may be of non-quantized nature, and even linear. In the noisy setup, a Gaussian source is to be transmitted over an additive Gaussian channel. The goals of the encoder and the decoder are misaligned by a bias term and encoder's cost also includes a penalty term on signal power. Conditions for the existence of informative affine Nash equilibria are presented. For the noisy setup, the only Stackelberg equilibrium is the linear equilibrium when the variables are scalar. The results of Chapter 2 have appeared in part in [55, 78].

1.4.2 Chapter 3

In this chapter, dynamic (multi-stage) signaling games involving an encoder and a decoder who have subjective models on the cost functions or the probabilistic model are considered. Nash (simultaneous-move game) and Stackelberg (leader-follower game) equilibria of multi-stage cheap talk and signaling game problems are investigated under a perfect Bayesian formulation and quadratic criteria. For the multi-stage scalar cheap talk, a zero-delay communication setup is considered for i.i.d. and Markov sources; it is shown that the final stage equilibrium is always quantized and under further conditions the equilibria for all time stages must be quantized. In contrast, the Stackelberg equilibria are always fully revealing. In the multi-stage signaling game where the transmission of a Gauss-Markov source over a memoryless Gaussian channel is considered, affine policies constitute an invariant subspace under best response maps for Nash equilibria; whereas the Stackelberg equilibria always admit linear policies for scalar sources but such policies may be non-linear for multi-dimensional sources. We obtain an explicit dynamic recursion for optimal linear encoding policies for multi-dimensional sources, and derive conditions under which Stackelberg equilibria are non-informative. For the case where the encoder and the decoder have subjective priors on the source distribution, under identical costs, we show that there exist fully informative Nash and Stackelberg equilibria for the dynamic cheap talk as in the team theoretic setup under an absolute continuity condition. In particular, for the cheap talk problem, the equilibrium behavior is robust to a class of perturbations in the priors, but not to the perturbations in the cost models in general. For the signaling game, however, Stackelberg equilibrium policies are robust to perturbations in the cost but not to the priors considered in this chapter. The results of Chapter 3 have appeared in part in [79, 80].

1.4.3 Chapter 4

Many communication, sensor network, and networked control problems involve agents (decision makers) which have either misaligned objective functions or subjective probabilistic models. In the context of such setups, we consider binary signaling problems in which the decision makers (the transmitter and the receiver) have subjective priors and/or misaligned objective functions. Accordingly, the binary signaling problem investigated here can be motivated under different application contexts: subjective priors and the presence of a bias in the objective function of the encoder compared to that of the decoder. In the former setup, players have a common goal but subjective prior information, which necessarily alters the setup from a team problem to a game problem. The latter one is the adaptation of the biased utility function of the encoder in [16] to the binary signaling problem considered here. Depending on the commitment nature of the transmitter to his policies, we formulate the binary signaling problem as a Bayesian game under either Nash or Stackelberg equilibrium concepts and establish equilibrium solutions and their properties. It is shown that there can be informative or non-informative equilibria in the binary signaling game under the Stackelberg assumption, but there always exists an equilibrium. However, apart from the informative and non-informative equilibria cases, there may not exist a Nash equilibrium when the receiver is restricted to use deterministic policies. For the corresponding team setup, however, an equilibrium typically always exists and is always informative. Furthermore, we investigate the effects of small perturbations in priors and costs on equilibrium values around the team setup (with identical costs and priors), and show that the Stackelberg equilibrium behavior is not robust to small perturbations whereas the Nash equilibrium is. The results of Chapter 4 will appear in part in [81].

1.4.4 Chapter 5

In this chapter, we investigate Crawford and Sobel's cheap talk formulation [16] under the exponential and Gaussian source assumptions and derive the upper bounds on the number of the quantization bins (if any) are derived depending on the misalignment between the objective functions of the encoder and the decoder. Firstly, for a uniform source, we verify the upper bound on the number of the quantization bins, obtain the total cost at the equilibrium, and show that the equilibrium with more bins is preferable for both the encoder and the decoder. Then, it is shown that, for an exponential source, at the equilibrium, the number of bins can be bounded or unbounded; i.e., infinitely many, depending on the misalignment between the objective functions of the decision makers. For the Gaussian case, it is always possible to have an equilibrium with two bins.

1.5 Notation and Conventions

We denote random variables with capital letters, e.g., Y, whereas possible realizations are shown by lower-case letters, e.g., y. The absolute value of scalar y is denoted by $|y|$. The vectors are denoted by bold-faced letters, e.g., $\mathbf{y}$. For vector $\mathbf{y}$, $\mathbf{y}^T$ denotes the transpose and $\|\mathbf{y}\|$ denotes the Euclidean (L_2) norm. $\mathbb{1}_{\{D\}}$ represents the indicator function of an event D, $\oplus$ stands for the exclusive-or operator, $\mathcal{Q}$ denotes the standard $\mathcal{Q}$-function; i.e., $\mathcal{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\{-\frac{t^2}{2}\}\mathrm{d}t$, and the sign of x is defined as

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} .$$

Chapter 2

Static (One-Stage) Quadratic Cheap Talk and Signaling Games

In this chapter, Nash and Stackelberg equilibria of static (one-stage) scalar and multi-dimensional quadratic cheap talk and signaling games are investigated. For all setups, conditions under which equilibria are non-informative are derived.

The main contributions of this chapter can be summarized as follows:

(i) We prove that for any scalar source, all Nash equilibrium policies at the encoder are equivalent to some quantized policy, but all Stackelberg equilibrium policies are fully informative. That is, there is some information hiding for the Nash setup, as opposed to the Stackelberg setup.

(ii) We show that for multi-dimensional setups, however, unlike the scalar case, Nash equilibrium policies may be non-quantized and can in fact be linear.

(iii) In the noisy setup, a Gaussian source is to be transmitted over an additive Gaussian channel. The goals of the encoder and the decoder are misaligned by a bias term and encoder's cost also includes a penalty term of the transmitted signal. Conditions for the existence of affine Nash equilibrium policies are presented.

(iv) We compare the results with socially optimal costs and information theoretic lower bounds, and discuss the effects of the bias term on equilibria. Furthermore, we prove that the only equilibrium in the Stackelberg noisy setup is the linear equilibrium for the scalar case.

2.1 Problem Formulation

A single-stage *cheap talk* problem, which is depicted in Fig. 2.1, can be formulated as follows: An informed player (encoder) knows the value of the $\mathbb{M}$-valued random variable M and transmits the $\mathbb{X}$-valued random variable X to another player (decoder), who generates his $\mathbb{M}$-valued optimal decision U upon receiving X. Let $c^e(m,u)$ and $c^d(m,u)$ denote the cost functions of the encoder and the decoder, respectively, when the action u is taken for the corresponding message m. Then, given the encoding and decoding policies, the encoder's induced expected cost is

$$J^e\left(\gamma^e,\gamma^d\right)=\mathbb{E}\left[c^e(m,u)\right],$$

whereas, the decoder's induced expected cost is

$$J^d\left(\gamma^e,\gamma^d\right)=\mathbb{E}\left[c^d(m,u)\right].$$

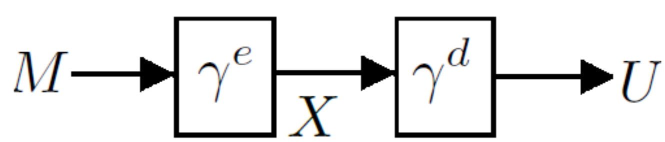

Figure 2.1: System model for static cheap talk.

2.2 Static Scalar Quadratic Cheap Talk

We will first consider the scalar setting by taking the cost functions as $c^e\left(m,u\right)=\left(m-u-b\right)^2$ and $c^d\left(m,u\right)=\left(m-u\right)^2$ where b denotes the bias term. The motivation for such functions stems from the fields of information theory, communication theory and LQG control; for these fields quadratic criteria are extremely important. Recall that for the case with $b = 0$, the cost functions simply reduce to those for a minimum mean-square estimation (MMSE) problem.

2.2.1 Nash Equilibrium Analysis

Some existence and deterministic properties of the equilibrium policies of the encoder and the decoder are stated in [37] and [4, Chp.4].

Theorem 2.2.1. *[37] (i) For any γ^e, there exists an optimal γ^d, which is deterministic. (ii) For any γ^d, any randomized encoding policy can be replaced with a deterministic γ^e without any loss to the encoder. (iii) Suppose γ^e is an M-cell quantizer with bins B_i for $i = 1, 2, \ldots, M$, then there exists an optimal deterministic γ^d, which is the conditional expectation of the respective bin; i.e., the optimal action of the decoder is $\mathbb{E}[m|m \in B_k]$ for the k-th bin.*

We first review the following classical result from [16, Lemma 1]:

Theorem 2.2.2. *[16, Lemma 1] Let there be two players, a Sender (**S**) and a Receiver (**R**). **S** observes the value of a random variable m (private to **S**), then sends a signal x which may be random, and can be viewed as a noisy estimate of m, to **R**. Then, **R** processes the information in **S**'s signal and chooses an action u, which determines players' payoffs. Here, m, which is supported on $[0, 1]$, has differentiable probability distribution function, $F(m)$, with density $f(m)$, and the utility functions of the players $U^S(m, u, b)$ and $U^R(m, u)$, where b is a scalar parameter to measure how nearly agents' interests coincide, have some technical properties. Then, the set of actions induced in any equilibrium is finite. Thus, information is not fully revealed.*

As observed from [16, Lemma 1] above, only sources on $[0, 1]$ that admit densities are considered. However, we note that the analysis here applies to arbitrary scalar valued random variables. The proof essentially follows from [16].

Theorem 2.2.3. *Let m be a real-valued random variable with an arbitrary probability measure. Let the strategy set of the encoder consist of the set of all measurable (deterministic) functions from $\mathbb{M}$ to $\mathbb{X}$. Then,* *(i) an equilibrium encoder policy has to be quantized almost surely (that is, it is equivalent to a quantized policy for the encoder in the sense that the performance of any equilibrium encoder policy is equivalent to the performance of a quantized encoder policy),*

(ii) the quantization bins are convex.

Remark 2.2.1. *Recall that encoder prefers to transmit everything if $b = 0$. However, if $b \neq 0$, encoder prefers the quantized policy. Misalignment changes the nature of the solutions drastically.*

Recall again that for the case when the source admits density on $[0, 1]$, Crawford and Sobel established the discrete nature of the equilibrium policies. For the case when the source is exponential, [37] (also, Chapter 5 of this dissertation) established the discrete-nature, and obtained the equilibrium bins with *finite upper bounds on the number of bins* in any equilibrium.

To facilitate our analysis to handle certain intricacies that arise due to the multi-stage setup in this dissertation, in the following, we state that the result in Theorem 2.2.3 also holds when the encoder is allowed to adapt randomized encoding policies by extending [16, Lemma 1] as follows:

Theorem 2.2.4. *The conclusion of Theorem 2.2.3, i.e., that an equilibrium policy of the encoder is equivalent to a quantized policy, also holds if the policy space of the encoder is extended to the set of all stochastic kernels from $\mathbb{M}$ to $\mathbb{X}$ for any arbitrary source. That is, even when the encoder is allowed to use private randomization, all equilibria are equivalent to those that are attained by quantized equilibria.*

Proof. [16, Lemma 1] proves that all equilibria have finitely many partitions when the source has bounded support. Theorem 2.2.3 extends this result to a countable number of partitions for deterministic equilibria for any source with an arbitrary probability measure. The result follows by utilizing Theorem 2.2.3 and [16, Lemma 1]. □

Theorem 2.2.4 will be used crucially to analyze the multi-stage setups; since in a multi-stage game, at a given time stage, the source variables from the earlier stages can serve as private randomness for the encoder.

2.2.2 Stackelberg Equilibrium Analysis

We will now observe that the Stackelberg setup is less interesting.

Theorem 2.2.5. *The Stackelberg equilibrium is unique and corresponds to a fully revealing (fully informative) encoder policy.*

Proof. Due to the Stackelberg assumption, the encoder knows that the decoder will use $\gamma^d(x) = u = \mathbb{E}[m|x]$ as an optimal decoder policy to minimize its cost. Then the goal of the encoder is to minimize the following:

$$\begin{aligned}\min_{x=\gamma^e(m)} \mathbb{E}[(m-u-b)^2] &= \min_{x=\gamma^e(m)} \mathbb{E}[(m-\mathbb{E}[m|x]-b)^2] \\ &\overset{(a)}{=} \min_{x=\gamma^e(m)} \mathbb{E}[(m-\mathbb{E}[m|x])^2] + b^2 \\ &= \min_{x=\gamma^e(m)} \mathbb{E}[(m-u)^2] + b^2 \,.\end{aligned}$$

Here, (a) follows from the law of the iterated expectations. Since the goal of the decoder is to minimize $\min_{u=\gamma^d(x)} \mathbb{E}[(m-u)^2]$, the goals of the encoder and the decoder become essentially the same in the Stackelberg game setup, which effectively reduces the *game* setup to a *team* setup. In the team setup, the equilibrium is fully informative; i.e. the encoder reveals all of its information. □

2.3 Static Multi-Dimensional Quadratic Cheap Talk

The scalar setup considered in Section 2.2 can be extended to the multi-dimensional cheap talk setup by defining the cost functions of the encoder and the decoder as $c^e(\mathbf{m},\mathbf{u}) = \|\mathbf{m}-\mathbf{u}-\mathbf{b}\|^2$ and $c^d(\mathbf{m},\mathbf{u}) = \|\mathbf{m}-\mathbf{u}\|^2$, respectively, where the lengths of the vectors are defined in L_2 norm and $\mathbf{b}$ is the bias vector.

2.3.1 Nash Equilibrium Analysis

Although the Nash equilibrium is always quantized in a scalar setup, the equilibrium structure changes drastically in a multi-dimensional setting as follows:

Theorem 2.3.1. *In the multi-dimensional cheap talk, the Nash equilibrium cannot be fully revealing in the single-stage multi-dimensional cheap talk when the source has positive measure for every non-empty open set. An equilibrium policy, unlike the scalar case, can be non-discrete and even linear.*

From the discussion in the proof of Theorem 2.3.1, it can be deduced that if $\mathbf{b}$ is orthogonal to the basis vectors or satisfies certain symmetry conditions, then non-discrete or linear equilibria exist. This approach applies also to the n-dimensional setup for any $n \in \mathbb{N}$. For example, if the bias vector involves only one nonzero coordinate component and if the source distribution is uniform over an n-dimensional unit cube, then full information revelation in all the other coordinates will lead to a non-discrete equilibrium. In particular, if nonzero component of the bias is greater than 0.25, then there is only one bin in that coordinate and the full information is sent in other coordinates. Furthermore, if the encoder only sends the 0 variable for the value of the only bin in the coordinate for which the bias has nonzero component, then what we have is indeed a linear policy.

2.3.2 Stackelberg Equilibrium Analysis

The Stackelberg equilibria in the multi-dimensional cheap talk can be obtained by extending its scalar case; i.e., it is unique and corresponds to a fully revealing (fully informative) encoder policy as in the scalar case.

Theorem 2.3.2. *In the multi-dimensional cheap talk, the Stackelberg equilibrium is unique and corresponds to a fully revealing (fully informative) encoder policy.*

Proof. Due to the Stackelberg assumption, the encoder knows that the decoder will use $\gamma^d(\mathbf{x}) = \mathbf{u} = \mathbb{E}[\mathbf{m}|\mathbf{x}]$ as an optimal decoder policy to minimize its cost. Then the goal of the encoder is to minimize the following:

$$\begin{aligned}\min_{\mathbf{x}=\gamma^e(\mathbf{m})} \mathbb{E}[(\mathbf{m}-\mathbf{u}-\mathbf{b})^2] &= \min_{\mathbf{x}=\gamma^e(\mathbf{m})} \mathbb{E}[(\mathbf{m}-\mathbb{E}[\mathbf{m}|\mathbf{x}]-\mathbf{b})^2] \\ &\overset{(a)}{=} \min_{\mathbf{x}=\gamma^e(\mathbf{m})} \mathbb{E}[(\mathbf{m}-\mathbb{E}[\mathbf{m}|\mathbf{x}])^2] + b^2 \\ &= \min_{\mathbf{x}=\gamma^e(\mathbf{m})} \mathbb{E}[(\mathbf{m}-\mathbf{u})^2] + b^2\,.\end{aligned}$$

Here, (a) follows from the law of the iterated expectations. Since the goal of the decoder is to minimize $\min_{\mathbf{u}=\gamma^d(\mathbf{x})} \mathbb{E}[(\mathbf{m}-\mathbf{u})^2]$, the goals of the encoder and the decoder become essentially the same in the Stackelberg game setup, which effectively reduces the game setup to a team setup. In the team setup, the equilibrium is fully informative; i.e. the encoder reveals all of its information. □

2.4 Static Scalar Quadratic Quadratic Signaling Games

The noisy game setup is similar to the noiseless case except that there exists an additive Gaussian noise channel between the encoder and decoder, as depicted in Fig. 2.2, and the encoder has a *soft* power constraint.

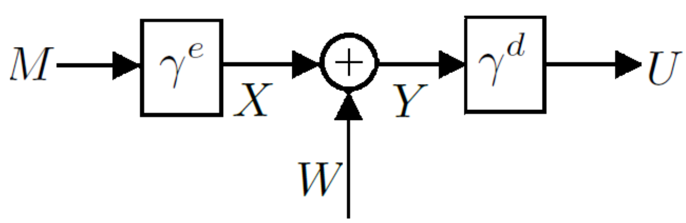

Figure 2.2: System model for static signaling game.

The encoder encodes a zero-mean Gaussian random variable M and sends the real-valued random variable X. During the transmission, the zero mean Gaussian noise with a variance of σ^2 is added to X; hence, the decoder receives $Y =$

$X + W$, where $W \sim \mathcal{N}(0, \sigma^2)$. Here, the signaling game problem is investigated where the encoder and the decoder are deterministic rather than randomized; i.e., $\gamma^e(\mathrm{d}x|m) = \mathbb{1}_{\{f^e(m) \in \mathrm{d}x\}}$ and $\gamma^d(\mathrm{d}u|y) = \mathbb{1}_{\{f^d(y) \in \mathrm{d}u\}}$ where $f^e(m)$ and $f^d(y)$ are some deterministic functions of the encoder and decoder, respectively. The encoder aims to minimize

$$J^e(\gamma^e, \gamma^d) = \mathbb{E}\left[c^e(m, x, u)\right] ,$$

whereas the decoder aims to minimize

$$J^d(\gamma^e, \gamma^d) = \mathbb{E}\left[c^d(m, u)\right] .$$

The cost functions are modified as $c^e(m, x, u) = (m - u - b)^2 + \lambda x^2$ and $c^d(m, u) = (m - u)^2$. Note that a power constraint with an associated multiplier is appended to the cost function of the encoder, which corresponds to power limitation for transmitters in practice. If $\lambda = 0$, this corresponds to the setup with no power constraint at the encoder.

2.4.1 Nash Equilibrium Analysis

2.4.1.1 A Supporting Result

Suppose that there is an equilibrium with an arbitrary policy leading to finite (at least two), countably infinite or uncountably infinite equilibrium bins. Let two of these bins be $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$. Also let m^α indicate any point in $\mathcal{B}^\alpha$; i.e., $m^\alpha \in \mathcal{B}^\alpha$; and the encoder encodes m^α to x^α and sends to the decoder. Similarly, let m^β represent any point in $\mathcal{B}^\beta$; i.e., $m^\beta \in \mathcal{B}^\beta$; and the encoder encodes m^β to x^β and sends to the decoder. Without any loss of generality, we can assume that $m^\alpha < m^\beta$. The decoder chooses the action $u = \mathbb{E}[m|y]$ (MMSE rule). Let $F(m, x)$ be the encoder cost when message m is encoded as x; i.e.,

$$F(m, x) = \int_y p\left(\gamma^d(y) = u \middle| \gamma^e(m) = x\right) \left((m - u - b)^2 + \lambda x^2\right) \mathrm{d}y \,.$$

Then, the equilibrium definitions from the view of the encoder require $F(m^\alpha, x^\alpha) \leq F(m^\alpha, x^\beta)$ and $F(m^\beta, x^\beta) \leq F(m^\beta, x^\alpha)$. Now let $G(m) =$

$F(m, x^\alpha) - F(m, x^\beta)$. If it can be shown that $G(m)$ is a continuous function of m on the interval $[m^\alpha, m^\beta]$, then it can be deduced that $\exists\, \overline{m} \in [m^\alpha, m^\beta]$ such that $G(\overline{m}) = 0$ by the Mean Value Theorem since $G(m^\alpha) \leq 0$ and $G(m^\beta) \geq 0$.

Proposition 2.4.1. *$G(m)$ is a continuous function of m on the interval $[m^\alpha, m^\beta]$.*

Proof. It suffices to show that $F(m, x)$ is continuous in m. Let $\{m_n\}$ be a sequence which converges to m. Recall that $(m_n - u - b)^2 \leq 2m_n^2 + 2(u + b)^2 < \infty$ since m is bounded from above and below ($m \in [m^\alpha, m^\beta]$), b is a finite bias and $\mathbb{E}[u^2] = \mathbb{E}[(\gamma^d(y))^2] < \infty$ (note that any finite cost $\mathbb{E}[(m - u^2)]$ inevitably leads to a finite $\mathbb{E}[u^2]$ since $\mathbb{E}[u^2] = \mathbb{E}[(m + u - m)^2] \leq 2\sigma_M^2 + 2\mathbb{E}[(m - u)^2] < \infty$). Then, by the dominated convergence theorem,

$$\lim_{n\to\infty} F(m_n, x) = \lim_{n\to\infty} \mathbb{E}[(m_n - u - b)^2 + \lambda x^2] = \mathbb{E}[(m - u - b)^2 + \lambda x^2] = F(m, x)\,,$$

which shows the continuity of $F(\cdot, x)$ in the interval (m^α, m^β). □

From Proposition 2.4.1, $\exists\, \overline{m} \in [m^\alpha, m^\beta]$ such that $G(\overline{m}) = 0$ which implies $F(\overline{m}, x^\alpha) = F(\overline{m}, x^\beta)$, or equivalently,

$$\mathbb{E}[(\overline{m} - u - b)^2 + \lambda(x^\alpha)^2] = \mathbb{E}[(\overline{m} - u - b)^2 + \lambda(x^\beta)^2]\,.$$

Then,

$$\overline{m} = \frac{\mathbb{E}[u^2|x^\beta] - \mathbb{E}[u^2|x^\alpha]}{2\left(\mathbb{E}[u|x^\beta] - \mathbb{E}[u|x^\alpha]\right)} + \frac{\lambda\left((x^\beta)^2 - (x^\alpha)^2\right)}{2\left(\mathbb{E}[u|x^\beta] - \mathbb{E}[u|x^\alpha]\right)} + b \qquad (2.1)$$

is obtained. Recall that the arguments in Theorem 2.2.3 cannot be applied here because of the presence of noise. However, when there is noise in a communication channel, the relation between $\mathbb{E}[u|x]$, $\mathbb{E}[u^2|x]$ and $\overline{m}$ can be constructed as in (2.1).

2.4.1.2 Existence and Uniqueness of Informative Affine Equilibria

We first note that Proposition 1.2.1 is valid also in the noisy formulation; i.e. a non-informative (babbling) equilibrium is an equilibrium for the noisy signaling game, since the appended power constraint is always positive. The following holds:

Theorem 2.4.1. *(i) If $\lambda \geq \frac{\sigma_M^2}{\sigma_W^2}$, there does not exist an informative affine equilibrium. The only affine equilibrium is the non-informative one.*

(ii) Let $0 < \lambda < \frac{\sigma_M^2}{\sigma_W^2}$. For any $b \in \mathbb{R}$, there exists a unique informative affine equilibrium.

(iii) If $\lambda = 0$, there exists no informative equilibrium with affine policies.

Remark 2.4.1. *The expression $\frac{\sigma_M^2}{\sigma_W^2}$ defines a quantity which determines the Shannon-theoretic capacity of the channel given a signal energy constraint at the encoder. This can be interpreted as Signal-to-Noise Ratio (SNR) of the received signal, which is related to the channel attenuation coefficient. If the multiplier of the signal λ in the cost function is greater than $\frac{\sigma_M^2}{\sigma_W^2}$, it will not be rational for the encoder to send any signal at all under any equilibrium.*

Corollary 2.4.1. *If either $\lambda = 0$ or $\sigma_W^2 = 0$, an affine equilibrium exists only if $\lambda = \sigma_W^2 = b = 0$.*

Proof. Note that, from (2.9) and (2.11), we have $A = \frac{K}{K^2+\lambda}$, $K = \frac{A\sigma_M^2}{A^2\sigma_M^2+\sigma_W^2}$, $L = -KC$ and $C = -A(L+b)$. From these equalities, we observe the following:

1. When $\lambda = 0$, it is shown in Theorem 2.4.1 that there is not any fixed point solution to (2.14). However, if there is not a noisy channel between the encoder and the decoder; i.e., the noise variance is zero ($\sigma_W^2 = 0$), then (2.14) has a fixed point solution. Even when (2.14) has a fixed point solution A, (2.9) and (2.11) cannot hold together unless $b = 0$.

2. when the noise variance is zero ($\sigma_W^2 = 0$), there is not any fixed point solution to (2.14) unless $\lambda = 0$. Even when (2.14) has a fixed point solution A, (2.9) and (2.11) cannot hold together unless $b = 0$.

3. when $\lambda = 0$ and the noise variance is zero ($\sigma_W^2 = 0$); the consistency of (2.9) and (2.11) can be satisfied if only if $b = 0$. Hence, if $b \neq 0$, there

cannot be a affine equilibrium; the equilibrium has to be discrete due to Theorem 2.2.3.

□

2.4.1.3 Price of Anarchy and Comparison with Socially Optimal Cost

In a game theoretic setup, the encoder and the decoder try to minimize their individual costs, thus the game theoretic cost can be found as $\min_{\gamma^e} J^e + \min_{\gamma^d} J^d$. If the encoder and the decoder work together to minimize the total cost, then the problem can be regarded as a team problem and the resulting cost is a socially optimal cost, which is $\min_{\gamma^e,\gamma^d}(J^e + J^d)$. In the game theoretic setup, because of the selfish behavior of the players, there is some loss from the socially optimal cost, and this loss is measured by the ratio between the game theoretic cost and the socially optimal cost, which was proposed as a *price of anarchy* [82]. In this part, it will be shown that the game theoretic cost is higher than the socially optimal cost as expected.

Theorem 2.4.2. *(i) Let g_i and g_u represent the informative and the non-informative equilibrium game costs, respectively. Then, $g_i = 3\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2\sqrt{\frac{\sigma_M^2}{\lambda\sigma_W^2}} - \lambda\sigma_W^2$ and $g_u = 2\sigma_M^2 + b^2$. Further, the total cost in the game equilibrium is the following*

$$J^{*,g} = \begin{cases} \min\{g_i,\ g_u\} & \lambda < \sigma_M^2/\sigma_W^2 \\ g_u & \lambda \geq \sigma_M^2/\sigma_W^2 \end{cases}.$$

(ii) Let t_i and t_u represent the informative and the non-informative team costs, respectively. Then, $t_i = 2\sqrt{2\lambda\sigma_M^2\sigma_W^2} + \frac{b^2}{2} - \lambda\sigma_W^2$ and $t_u = 2\sigma_M^2 + \frac{b^2}{2}$. Further, the socially optimal cost (the total cost in the team setup) is the following

$$J^{*,t} = \begin{cases} \min\{t_i,\ t_u\} & \lambda < 2\sigma_M^2/\sigma_W^2 \\ t_u & \lambda \geq 2\sigma_M^2/\sigma_W^2 \end{cases}.$$

After investigating the game theoretic cost and the socially optimal cost in Theorem 2.4.2, the price of anarchy can be obtained as follows:

Theorem 2.4.3. *The price of anarchy is always larger than 1, i.e., the sum of the costs under any Nash equilibria is always larger than the socially optimal cost.*

Proof. By Theorem 2.4.2, we have the following

$$J^{*,g} = \begin{cases} \min\{g_i,\ g_u\} & \lambda < \sigma_M^2/\sigma_W^2 \\ g_u & \lambda \geq \sigma_M^2/\sigma_W^2 \end{cases},$$

$$J^{*,t} = \begin{cases} \min\{t_i,\ t_u\} & \lambda < 2\sigma_M^2/\sigma_W^2 \\ t_u & \lambda \geq 2\sigma_M^2/\sigma_W^2 \end{cases}.$$

Notice that we have $t_i < g_i$ for $\lambda < \sigma_M^2/\sigma_W^2$, and $t_u < g_u$ always. Consider the following cases:

1. $\underline{0 < \lambda < \sigma_M^2/\sigma_W^2}$: There are four cases to be considered:

   (a) $\underline{\min\{g_i,\ g_u\} = g_i \text{ and } \min\{t_i,\ t_u\} = t_i}$: Since $t_i < g_i$, $J^{*,t} < J^{*,g}$ is satisfied.

   (b) $\underline{\min\{g_i,\ g_u\} = g_i \text{ and } \min\{t_i,\ t_u\} = t_u}$: Since $t_u < t_i < g_i$, $J^{*,t} < J^{*,g}$ is satisfied.

   (c) $\underline{\min\{g_i,\ g_u\} = g_u \text{ and } \min\{t_i,\ t_u\} = t_i}$: Since $t_i < t_u < g_u < g_i$, $J^{*,t} < J^{*,g}$ is satisfied.

   (d) $\underline{\min\{g_i,\ g_u\} = g_u \text{ and } \min\{t_i,\ t_u\} = t_u}$: Since $t_u < g_u$, $J^{*,t} < J^{*,g}$ is satisfied.

2. $\underline{\sigma_M^2/\sigma_W^2 \leq \lambda < 2\sigma_M^2/\sigma_W^2}$: There are two cases to be considered:

   (a) $\underline{\min\{t_i,\ t_u\} = t_i}$: Since $t_i < t_u < g_u$, $J^{*,t} < J^{*,g}$ is satisfied.

   (b) $\underline{\min\{t_i,\ t_u\} = t_u}$: Since $t_u < g_u$, $J^{*,t} < J^{*,g}$ is satisfied.

3. $\underline{\lambda \geq 2\sigma_M^2/\sigma_W^2}$: Since $t_u < g_u$, $J^{*,t} < J^{*,g}$ is satisfied.

Hence, one can observe that $J^{*,g} > J^{*,t}$ always holds, which shows that the price of anarchy is always greater than 1, i.e., the game theoretic cost is always larger than the socially optimal cost. □

As discussed in Theorem 2.4.2, even though $\lambda < \frac{\sigma_M^2}{\sigma_W^2}$, the non-informative equilibrium may be preferred over the informative one. The following theorem clarifies the conditions under which the informative equilibrium is preferred over the non-informative one, or vice-versa.

Theorem 2.4.4. *There exists informative affine equilibria in the single-stage signaling game if and only if* $\frac{\sigma_M^2-2b^2-\sqrt{\sigma_M^2}\sqrt{\sigma_M^2-4b^2}}{2\sigma_W^2} < \lambda < \frac{\sigma_M^2-2b^2+\sqrt{\sigma_M^2}\sqrt{\sigma_M^2-4b^2}}{2\sigma_W^2}$ *and* $\sigma_M^2 \geq 4b^2$.

2.4.2 Stackelberg Equilibrium Analysis[1]

In this section, the Stackelberg equilibrium of the static scalar signaling game is investigated; i.e. the encoder knows the policy of the decoder, and it is shown that the only equilibrium is the linear equilibrium.

Theorem 2.4.5. *For* $0 < \lambda < \sigma_M^2/\sigma_W^2$*, the only equilibrium (affine or not) in the Stackelberg setup of the signaling game is the linear equilibrium. Otherwise, the equilibrium is non-informative.*

2.4.3 Information Theoretic Lower Bounds and Nash Equilibria

In the following, we investigate the performance of Nash equilibria and socially optimal strategies by comparing their costs with the information theoretic lower bounds derived in Theorem 2.4.5, and comment on their achievability.

Theorem 2.4.6. *(i) For the game setup, if* $\lambda \geq \frac{\sigma_M^2}{\sigma_W^2}$ *(i.e., non-informative equilibria), the information theoretic lower bounds on the costs are achievable.*

[1]In our corresponding paper [55, Theorem 4.1], mistakenly, we have used the information theoretic lower bounds for the Nash equilibrium analysis. However, due to the assumption on the optimal decoder action; i.e., $u = \mathbb{E}[m|y]$, the information theoretic arguments are valid for the Stackelberg case.

(ii) *For the game setup, if* $\lambda < \frac{\sigma_M^2}{\sigma_W^2}$ *and* $b = 0$*, the information theoretic lower bounds on the costs are achievable by linear policies.*

(iii) *For the game setup, if* $\lambda < \frac{\sigma_M^2}{\sigma_W^2}$ *and* $b \neq 0$*, the information theoretic lower bounds on the costs are not achievable by affine policies.*

(iv) *For the team setup, the information theoretic lower bounds on the costs are always (both in the informative and non-informative equilibria) achievable by affine policies.*

2.4.4 The Encoder with a Hard Power Constraint

Now consider the case in which the encoder has a hard power constraint instead of a soft power constraint. Under this assumption, the goal of the encoder is to minimize

$$J^e(\gamma^e, \gamma^d) = \mathbb{E}\left[c^e(m,u)\right] \\ \text{s.t. } \mathbb{E}\left[\gamma^e(m)\right] \leq \overline{P},$$

whereas the decoder aims to minimize

$$J^d(\gamma^e, \gamma^d) = \mathbb{E}\left[c^d(m,u)\right],$$

where $c^e(m,u) = (m-u-b)^2$ and $c^d(m,u) = (m-u)^2$.

Unlike the soft power constrained case, there always exist informative Nash and Stackelberg equilibria for the hard power constrained case as follows:

Theorem 2.4.7. *There always exists an informative affine Nash equilibrium in the hard power constrained scalar quadratic signaling game in contrast to the soft power constrained scalar quadratic signaling game.*

Theorem 2.4.8. *The only equilibrium (affine or not) in the Stackelberg setup of the signaling game is the linear equilibrium under the hard power constraint, and contrary to the soft power constrained scalar quadratic signaling game, the Stackelberg equilibrium is always informative.*

2.5 Static Multi-Dimensional Quadratic Signaling Games

The scalar setup considered in Section 2.4 can be extended to the multi-dimensional Gaussian noisy signaling game problem setup as follows: The encoder encodes an n-dimensional zero-mean Gaussian random variable $\mathbf{M}$ with the covariance matrix $\Sigma_{\mathbf{M}}$ and sends the real-valued n-dimensional random variable $\mathbf{X}$. During the transmission, the n-dimensional zero-mean Gaussian noise with the covariance matrix $\Sigma_{\mathbf{W}}$ is added to $\mathbf{X}$ and the decoder receives $\mathbf{Y}=\mathbf{X}+\mathbf{W}$, where $\mathbf{W}\sim\mathcal{N}(\mathbf{0},\Sigma_{\mathbf{W}})$. The encoder aims to minimize

$$J^e(\gamma^e,\gamma^d)=\mathbb{E}\left[c^e(\mathbf{m},\mathbf{x},\mathbf{u})\right]\,,$$

whereas the decoder aims to minimize

$$J^d(\gamma^e,\gamma^d)=\mathbb{E}\left[c^d(\mathbf{m},\mathbf{u})\right]\,.$$

The cost functions are $c^e\left(\mathbf{m},\mathbf{x},\mathbf{u}\right)=\|\mathbf{m}-\mathbf{u}-\mathbf{b}\|^2+\lambda\|\mathbf{x}\|^2$ and $c^d\left(\mathbf{m},\mathbf{u}\right)=\|\mathbf{m}-\mathbf{u}\|^2$ where the lengths of the vectors are defined in L_2 norm and $\mathbf{b}$ is the bias vector. Note that we have appended a power constraint and an associated multiplier. If $\lambda=0$, this corresponds to the setup with no power constraint at the encoder.

2.5.1 Nash Equilibrium Analysis

Theorem 2.5.1. *(i) If the encoder is linear (affine), the decoder, as an MMSE decoder for a Gaussian source over a Gaussian channel, is linear (affine).*

(ii) If the decoder is linear (affine), then an optimal encoder policy for a multi-dimensional Gaussian source over a multi-dimensional Gaussian channel is an affine policy.

(iii) An equilibrium encoder policy $\gamma^e(\mathbf{m})=A\mathbf{m}+\mathbf{C}$ satisfies the equation $A=T(A)$ where $T(A)=\left(FF^T+\lambda I\right)^{-1}F$ and $F=\left(A\Sigma_{\mathbf{M}}A^T+\Sigma_{\mathbf{W}}\right)^{-1}A\Sigma_{\mathbf{M}}$.

(iv) There exists at least one equilibrium.

We note, however, that there always exist a non-informative equilibrium (see Proposition 1.2.1, which also applies to the signaling game discussed in this section). However, there exist games with informative affine equilibria as we state in Theorem 2.5.2.

Proposition 2.5.1. *If either λ or $\Sigma_{\mathbf{W}}$ is zero, an informative affine equilibrium exists only if λ, $\Sigma_{\mathbf{W}}$ and $\mathbf{b}$ are all zero.*

Proof. Note that, from (2.43) and (2.45), assuming $|A| \neq 0$, we have $A = \left(K^T K + \lambda I\right)^{-1} K^T$, $K = \Sigma_{\mathbf{M}} A^T \left(A \Sigma_{\mathbf{M}} A^T + \Sigma_{\mathbf{W}}\right)^{-1}$, $\mathbf{L} = -K\mathbf{C}$ and $\mathbf{C} = -A(\mathbf{L} + \mathbf{b})$. From these equalities, we can analyze the equilibrium as in the scalar case:

1. when $\lambda = 0$ and the noise is zero ($\Sigma_{\mathbf{W}} = 0$), then $A = K^{-1}$ and $K = A^{-1}$ are obtained. Then $\mathbf{C} = -A(\mathbf{L} + \mathbf{b}) = -A(-K\mathbf{C} + \mathbf{b}) = AK\mathbf{C} - A\mathbf{b}$, thus the consistency of the equalities can be satisfied if only if $\mathbf{b} = \mathbf{0}$. Hence, if $\mathbf{b} \neq \mathbf{0}$, there cannot exist an informative affine equilibrium. Recall that in the multi-dimensional noiseless cheap talk, the linearity of the equilibrium is shown for the uniform source; here the source is Gaussian.

2. when $\lambda = 0$, then $A = K^{-1}$ and $A \Sigma_{\mathbf{M}} A^T + \Sigma_{\mathbf{W}} = K^{-1} \Sigma_{\mathbf{M}} A^T$ are obtained. There does not exist a solution to (2.14) unless the noise is zero ($\Sigma_{\mathbf{W}} = 0$). Even when (2.14) has a fixed point solution A, (2.9) and (2.9) cannot hold together unless $\mathbf{b} = \mathbf{0}$.

3. when the noise is zero ($\Sigma_{\mathbf{W}} = 0$), then $K = A^{-1}$ and $K^T K + \lambda I = K^T A^{-1}$ are obtained. There does not exist a solution to (2.14) unless $\lambda = 0$. Even when (2.14) has a fixed point solution A, (2.9) and (2.9) cannot hold together unless $\mathbf{b} = \mathbf{0}$.

□

Remark 2.5.1. *Unlike the scalar setting, in the multi-dimensional case, fixed points may not be unique: with* $\lambda = 1.0311$ *and*

$$\Sigma_{\mathbf{M}} = \begin{bmatrix} 1.6421 & 0.1299 & 0.5713 & 0.2305 \\ 0.1299 & 1.4803 & 0.6810 & 0.4749 \\ 0.5713 & 0.6810 & 1.7312 & 0.4292 \\ 0.2305 & 0.4749 & 0.4292 & 1.3515 \end{bmatrix},$$

$$\Sigma_{\mathbf{W}} = \begin{bmatrix} 1.2742 & 0.1868 & 0.2318 & 0.0559 \\ 0.1868 & 1.8266 & 0.5955 & 0.3091 \\ 0.2318 & 0.5955 & 1.2377 & 0.4951 \\ 0.0559 & 0.3091 & 0.4951 & 1.5336 \end{bmatrix},$$

we can obtain two fixed points with different absolute-valued elements as follows (recall that if A *is a fixed point,* $-A$ *is also a fixed point):*

$$A = \begin{bmatrix} -0.1543 & 0.1762 & 0.0606 & 0.1117 \\ 0.1602 & 0.0159 & 0.1036 & 0.0279 \\ -0.2000 & -0.1879 & -0.2700 & -0.1565 \\ 0.0603 & 0.1052 & 0.1221 & 0.0824 \end{bmatrix},$$

$$A = \begin{bmatrix} -0.2431 & 0.0738 & -0.0752 & 0.0285 \\ 0.0293 & -0.1351 & -0.0966 & -0.0948 \\ 0.1520 & 0.2181 & 0.2682 & 0.1735 \\ -0.1003 & -0.0801 & -0.1236 & -0.0683 \end{bmatrix}.$$

Theorem 2.5.2. *Let source* $\mathbf{M}$ *be a zero-mean* n*-dimensional Gaussian random variable with covariance matrix* $\Sigma_{\mathbf{M}} = diag\{\sigma^2_{m_1}, \ldots, \sigma^2_{m_n}\}$ *where diag indicates a diagonal matrix, and noise* $\mathbf{W}$ *be a zero-mean* n*-dimensional Gaussian random variable with covariance matrix* $\Sigma_{\mathbf{W}} = diag\{\sigma^2_{w_1}, \ldots, \sigma^2_{w_n}\}$*. Then an informative affine equilibrium exists if* $\lambda < \max\{\frac{\sigma^2_{m_1}}{\sigma^2_{w_1}}, \ldots, \frac{\sigma^2_{m_n}}{\sigma^2_{w_n}}\}$*.*

Note that, from (2.43) and (2.45), by assuming $|A| \neq 0$, we have $\lambda A \Sigma_{\mathbf{M}} A^T = K^T K \Sigma_{\mathbf{W}}$ which is equivalent to

$$\lambda (A^T)^{-1} \Sigma_{\mathbf{M}} A^T = (K^T K + \lambda I)(K^T K + \lambda I) \Sigma_{\mathbf{W}} . \tag{2.2}$$

Remark 2.5.2. *Assuming all channels are informative, i.e.,* $|A| \neq 0$*, we make the following observations.*

(i) *If the source is i.i.d.; i.e.,* $\Sigma_{\mathbf{M}} = \sigma_m^2 I$*, then* (2.2) *becomes*

$$\begin{aligned}
&\lambda(A^T)^{-1}\sigma_m^2 I A^T = (K^T K + \lambda I)(K^T K + \lambda I)\Sigma_{\mathbf{W}} \\
&\Rightarrow \lambda\sigma_m^2(\Sigma_{\mathbf{W}})^{-1} = (K^T K + \lambda I)(K^T K + \lambda I) \\
&\Rightarrow \lambda\sigma_m^2(\Sigma_{\mathbf{W}})^{-1} \geq \lambda^2 I \\
&\Rightarrow \lambda I \leq \sigma_m^2(\Sigma_{\mathbf{W}})^{-1}.
\end{aligned}$$

This result implies that λ *must satisfy the inequality* $\lambda I \leq \sigma_m^2(\Sigma_{\mathbf{W}})^{-1}$ *for the i.i.d. source; otherwise, there must be at least one non-informative channel; i.e.,* $|A|$ *must be* 0.

(ii) *If the channel noise is i.i.d.; i.e.,* $\Sigma_{\mathbf{W}} = \sigma_w^2 I$*, (since* $\Sigma_{\mathbf{M}}$ *is real-symmetric, it has the eigenvalue decomposition as* $\Sigma_{\mathbf{M}} = Q\Lambda Q^T$*), then* (2.2) *becomes*

$$\begin{aligned}
&\lambda(A^T)^{-1}\Sigma_{\mathbf{M}} A^T = (K^T K + \lambda I)(K^T K + \lambda I)\sigma_w^2 I \\
&\Rightarrow \frac{\lambda}{\sigma_w^2}(A^T)^{-1}Q\Lambda Q^T A^T = (K^T K + \lambda I)(K^T K + \lambda I) \\
&\Rightarrow (A^T)^{-1}Q\Lambda Q^T A^T \geq \lambda\sigma_w^2.
\end{aligned}$$

This result implies that for each eigenvalue $\lambda_{\mathbf{M}}$ *of* $\Sigma_{\mathbf{M}}$*,* λ *must satisfy* $\lambda \leq \lambda_{\mathbf{M}}/\sigma_w^2$ *for the i.i.d. channel noise; otherwise, there must be at least one non-informative channel; i.e.,* $|A|$ *must be* 0.

(iii) *For the general case, recall the Minkowski determinant theorem,* $|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$*, which holds for any non-negative* $n \times n$ *Hermitian matrix* A *and* B*. This implies* $|A + B| \geq |A| + |B|$*. By using this inequality and* (2.45)*,*

$$|A| = \frac{|K|}{|K^T K + \lambda I|} \leq \frac{|K|}{|K|^2 + \lambda^n}.$$

Assuming $|A| \neq 0$*, recall the equality* $\lambda A\Sigma_{\mathbf{M}} A^T = K^T K\Sigma_{\mathbf{W}}$*. Taking the determinant of both sides,*

$$\begin{aligned}
&|K|^2|\Sigma_{\mathbf{W}}| = \lambda^n|A|^2|\Sigma_{\mathbf{M}}| \leq \lambda^n\left(\frac{|K|}{|K|^2 + \lambda^n}\right)^2|\Sigma_{\mathbf{M}}| \leq \lambda^n\frac{|K|^2}{\lambda^{2n}}|\Sigma_{\mathbf{M}}| \\
&\Rightarrow \lambda^n \leq \frac{|\Sigma_{\mathbf{M}}|}{|\Sigma_{\mathbf{W}}|}.
\end{aligned}$$

The result can be interpreted as follows: If $\lambda > \left(\frac{|\Sigma_{\mathbf{M}}|}{|\Sigma_{\mathbf{W}}|}\right)^{1/n}$, then $|A| = |K| = 0$ at the equilibrium; i.e., there must be at least one non-informative channel.

2.5.2 Stackelberg Equilibrium Analysis[2]

In this section, the Stackelberg equilibrium of the static multi-dimensional signaling game is investigated; i.e. the encoder knows the policy of the decoder, $\mathbf{u} = \mathbb{E}[\mathbf{m}|\mathbf{y}]$. We provide an information theoretic lower bound for the encoder cost; this serves us also to obtain condition for the existence of an informative equilibrium.

Let $\mathbf{e} = \mathbf{m} - \mathbf{u} = \mathbf{m} - \mathbb{E}[\mathbf{m}|\mathbf{y}]$, then we have $\Sigma_{\mathbf{e}} = \mathbb{E}[\mathbf{e}\mathbf{e}^T] = \mathbb{E}[(\mathbf{m} - \mathbb{E}[\mathbf{m}|\mathbf{y}])(\mathbf{m} - \mathbb{E}[\mathbf{m}|\mathbf{y}])^T]$. Since the differential entropy is $h(\mathbf{m}) = \frac{1}{2}\log_2((2\pi e)^n|\Sigma_{\mathbf{m}}|)$ for a Gaussian source $\mathbf{M}$, by following similar information theoretic arguments to those in (2.23);

$$
\begin{aligned}
I(\mathbf{m};\mathbf{y}) &= h(\mathbf{m}) - h(\mathbf{m}|\mathbf{y}) = h(\mathbf{m}) - h(\mathbf{m} - \mathbb{E}[\mathbf{m}|\mathbf{y}]|\mathbf{y}) \\
&\geq h(\mathbf{m}) - h(\mathbf{m} - \mathbb{E}[\mathbf{m}|\mathbf{y}]) \\
&\geq \frac{1}{2}\log_2((2\pi e)^n|\Sigma_{\mathbf{m}}|) - \frac{1}{2}\log_2((2\pi e)^n|\Sigma_{\mathbf{e}}|) \\
&= \frac{1}{2}\log_2(|\Sigma_{\mathbf{m}}|/|\Sigma_{\mathbf{e}}|)\,.
\end{aligned}
$$

Also from the rate-distortion theorem, the data processing theorem and the channel capacity theorem:

$$
\begin{aligned}
R(D) &\leq \min_{f(\mathbf{u}|\mathbf{m}):\mathbb{E}[\|\mathbf{m}-\mathbf{u}\|^2]\leq D} I(\mathbf{m};\mathbf{u}) \leq I(\mathbf{m};\mathbf{u}) \\
&\leq I(\mathbf{x};\mathbf{y}) \leq \max_{f(\mathbf{x}):\mathbb{E}[\|\mathbf{x}\|^2]\leq P} I(\mathbf{x};\mathbf{y}) \leq C(P)\,.
\end{aligned}
$$

If we combine these, we obtain the following:

$$
|\Sigma_{\mathbf{e}}| \geq |\Sigma_{\mathbf{m}}|2^{-2R(D)} \geq |\Sigma_{\mathbf{m}}|2^{-2I(\mathbf{m};\mathbf{u})} \geq |\Sigma_{\mathbf{m}}|2^{-2I(\mathbf{x};\mathbf{y})} \geq |\Sigma_{\mathbf{m}}|2^{-2C(P)}\,. \tag{2.3}
$$

[2]In our corresponding paper [55, Section V.B], mistakenly, we have used the information theoretic lower bounds for the Nash equilibrium analysis. However, due to the assumption on the optimal decoder action; i.e., $\mathbf{u} = \mathbb{E}[\mathbf{m}|\mathbf{y}]$, the information theoretic arguments are valid for the Stackelberg case.

Now consider the following:

$$\mathbb{E}[\|\mathbf{m}-\mathbf{u}\|^2] = \mathbb{E}[\|\mathbf{e}\|^2] = \operatorname{tr}\Sigma_{\mathbf{e}} \overset{(a)}{\geq} n\Big(\prod_{i=1}^{n}\Sigma_{\mathbf{e}}(i,i)\Big)^{1/n}$$
$$\overset{(b)}{\geq} n\Big(|\Sigma_{\mathbf{e}}|\Big)^{1/n}$$
$$\overset{(c)}{\geq} n\Big(|\Sigma_{\mathbf{m}}|2^{-2C(P)}\Big)^{1/n}. \tag{2.4}$$

Here, (a) follows from the inequality for the arithmetic and geometric mean where $\Sigma_{\mathbf{e}}(i,i)$ stands for ith diagonal element of $\Sigma_{\mathbf{e}}$, (b) follows from the Hadamard inequality (since $\Sigma_{\mathbf{e}}$ is a positive semi-definite matrix), and (c) follows from (2.3). Now we will rewrite [83, Eq. (9.166)] which presents the capacity of the additive colored Gaussian noise channel with typo corrected:

$$C(P) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}\log_2\Big(1+\frac{\max(\nu-\lambda_i,0)}{\lambda_i}\Big),$$

where $P = \mathbb{E}[\|\mathbf{x}\|^2]$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\Sigma_{\mathbf{w}}$ and ν is chosen so that $\sum_{i=1}^{n}\max(\nu-\lambda_i,0) = nP$. Then we can obtain the following:

$$\begin{aligned}
2^{-2C(P)} &= 2^{-2\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}\log_2\Big(1+\frac{\max(\nu-\lambda_i,0)}{\lambda_i}\Big)} \\
&= \prod_{i=1}^{n}\Big(1+\frac{\max(\nu-\lambda_i,0)}{\lambda_i}\Big)^{-1/n} \\
&= \prod_{i=1}^{n}\Big(\frac{\max(\nu,\lambda_i)}{\lambda_i}\Big)^{-1/n} \\
&= \frac{(\prod_{i=1}^{n}\lambda_i)^{1/n}}{(\prod_{i=1}^{n}\max(\nu,\lambda_i))^{1/n}} \\
&\overset{(a)}{\geq} \frac{(|\Sigma_{\mathbf{w}}|)^{1/n}}{(P+\sum_{i=1}^{n}\frac{\lambda_i}{n})} \\
&= \Big(|\Sigma_{\mathbf{w}}|\Big)^{1/n}\Big(P+\frac{\operatorname{tr}\Sigma_{\mathbf{w}}}{n}\Big)^{-1}.
\end{aligned} \tag{2.5}$$

Here, (a) holds, since our assumption $\sum_{i=1}^{n}\max(\nu-\lambda_i,0) = nP$ implies $\sum_{i=1}^{n}\max(\nu,\lambda_i) = nP+\sum_{i=1}^{n}\lambda_i$ and $(\prod_{i=1}^{n}\max(\nu,\lambda_i))^{1/n} \leq \sum_{i=1}^{n}\max(\nu,\lambda_i)/n = P+\sum_{i=1}^{n}\lambda_i/n$ holds by the inequality for the arithmetic and geometric mean. If we insert (2.5) to (2.4),

$$\mathbb{E}[\|\mathbf{m}-\mathbf{u}\|^2] \geq n\Bigg(|\Sigma_{\mathbf{m}}|\Big(|\Sigma_{\mathbf{w}}|\Big)^{1/n}\Big(P+\frac{\operatorname{tr}\Sigma_{\mathbf{w}}}{n}\Big)^{-1}\Bigg)^{1/n}$$

$$=n(|\Sigma_{\mathbf{m}}|)^{1/n}(|\Sigma_{\mathbf{w}}|)^{1/n^2}\left(P+\frac{\operatorname{tr}\Sigma_{\mathbf{w}}}{n}\right)^{-1/n}. \tag{2.6}$$

The encoder costs reduces to $J^e = \mathbb{E}[\|\mathbf{m}-\mathbf{u}\|^2] + \lambda\mathbb{E}[\|\mathbf{x}\|^2] + \|\mathbf{b}\|^2$ since the decoder always chooses $\mathbf{u} = \mathbb{E}[\mathbf{m}|\mathbf{y}]$. Then, by (2.6),

$$\begin{aligned}J^e &= \|\mathbf{b}\|^2 + \lambda\mathbb{E}[\|\mathbf{x}\|^2] + \mathbb{E}[\|\mathbf{m}-\mathbf{u}\|^2]\\ &\geq \|\mathbf{b}\|^2 + \lambda P + n(|\Sigma_{\mathbf{m}}|)^{1/n}(|\Sigma_{\mathbf{w}}|)^{1/n^2}\left(P+\frac{\operatorname{tr}\Sigma_{\mathbf{w}}}{n}\right)^{-1/n}. \end{aligned}\tag{2.7}$$

The minimizer of this function can be found by the local perturbation condition:

$$\begin{aligned}&\lambda-(|\Sigma_{\mathbf{m}}|)^{1/n}(|\Sigma_{\mathbf{w}}|)^{1/n^2}\left(P+\frac{\operatorname{tr}\Sigma_{\mathbf{w}}}{n}\right)^{-\frac{1}{n}-1}=0\\ &\Rightarrow \lambda = (|\Sigma_{\mathbf{m}}|)^{1/n}(|\Sigma_{\mathbf{w}}|)^{1/n^2}\left(P+\frac{\operatorname{tr}\Sigma_{\mathbf{w}}}{n}\right)^{-\frac{1}{n}-1}\\ &\quad\overset{(a)}{\leq} (|\Sigma_{\mathbf{m}}|)^{1/n}(|\Sigma_{\mathbf{w}}|)^{1/n^2}\left((|\Sigma_{\mathbf{w}}|)^{1/n}\right)^{-\frac{1}{n}-1}\\ &\quad= (|\Sigma_{\mathbf{m}}|)^{1/n}(|\Sigma_{\mathbf{w}}|)^{-1/n}.\end{aligned}$$

Here, (a) follows from the nonnegativeness of P and the inequality for the arithmetic and geometric mean and the Hadamard inequality, similar to (2.4). Hence, if $\lambda < (|\Sigma_{\mathbf{m}}|)^{1/n}(|\Sigma_{\mathbf{w}}|)^{-1/n}$, the lower bound is minimized at a nonzero P value, but if $\lambda \geq (|\Sigma_{\mathbf{m}}|)^{1/n}(|\Sigma_{\mathbf{w}}|)^{-1/n}$, the minimizer P becomes zero. Finally, if channels and source are assumed to be i.i.d.; i.e., $\Sigma_{\mathbf{m}} = \sigma_m^2\mathbb{I}$ and $\Sigma_{\mathbf{w}} = \sigma_w^2\mathbb{I}$ where $\mathbb{I}$ is $n\times n$ identity matrix, and the encoder and the decoder use linear policies, then (2.7) becomes tight and can be interpreted as follows: If $\lambda > (|\Sigma_{\mathbf{m}}|)^{1/n}(|\Sigma_{\mathbf{w}}|)^{-1/n} = \sigma_m^2/\sigma_w^2$, then (2.7) is minimized at $P=0$; that is, the encoder does not signal any output. Hence, the encoder engages in an non-informative equilibrium and the minimum cost becomes $E\left[\|\mathbf{m}\|^2\right] + \|\mathbf{b}\|^2$ at this non-informative equilibrium. Recall that this is analogous to the analysis in the scalar setup (2.24).

2.6 Conclusion

For a strategic information transmission problem under quadratic criteria with a non-zero bias term leading to a mismatch in the encoder and the decoder objective functions, Nash and Stackelberg equilibria have been investigated in a number of setups. It has been proven that for any scalar source, the quantized nature of Nash equilibrium policies hold, whereas all Stackelberg equilibrium policies are fully informative. Further, it has been shown that the Nash equilibrium policies may be non-discrete and even linear for a multi-dimensional cheap talk problem, unlike the scalar case. The additive noisy channel setup with Gaussian statistics has also been studied, such a case leads to a signaling game due to the communication constraints in the transmission. Conditions for the existence of affine Nash equilibrium policies are presented for both the scalar and multi-dimensional setups. Lastly, we proved that the only equilibrium in the Stackelberg noisy setup is the linear equilibrium. Table 2.1 summarizes the results of this chapter.

Table 2.1: Static (one-stage) cheap talk and signaling games

SETUP	**SOURCE**	**Nash Equilibrium**	**Stackelberg Equilibrium**
STATIC CHEAP TALK	scalar	quantized	fully revealing
	multi-dimensional	may be of non-quantized nature, even linear	fully revealing
STATIC SIGNALING GAMES	scalar	affine policies constitute invariant subspace under best response maps	always linear
	multi-dimensional	affine policies constitute invariant subspace under best response maps	no general structure

2.7 Proofs

2.7.1 Proof of Theorem 2.2.3

Let there be an equilibrium in the game (with possibly uncountably infinitely many bins, countably many bins or finitely many bins). Let two bins be $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$. Also let m^α indicate any point in $\mathcal{B}^\alpha$; i.e., $m^\alpha \in \mathcal{B}^\alpha$. Similarly, let m^β represent any point in $\mathcal{B}^\beta$; i.e., $m^\beta \in \mathcal{B}^\beta$. The decoder chooses action $u^\alpha = \mathbb{E}[m|m \in \mathcal{B}^\alpha]$ when the encoder sends $m^\alpha \in \mathcal{B}^\alpha$ and action $u^\beta = \mathbb{E}[m|m \in \mathcal{B}^\beta]$ when the encoder sends $m^\beta \in \mathcal{B}^\beta$ in order to minimize its total cost. Without loss of generality, we can assume that $u^\alpha < u^\beta$. Let $F(m,u) \triangleq (m-u-b)^2$. Because of the equilibrium definitions from the view of the encoder; $F(m^\alpha, u^\alpha) < F(m^\alpha, u^\beta)$

and $F(m^\beta, u^\beta) < F(m^\beta, u^\alpha)$. Hence, $\exists\ \overline{m}$ that satisfies $F(\overline{m}, u^\alpha) = F(\overline{m}, u^\beta)$ which reduces to

$$\overline{m} = \frac{u^\alpha + u^\beta}{2} + b \iff (\overline{m} - u^\alpha) = (u^\beta - \overline{m}) + 2b\,. \tag{2.8}$$

Since $F(\overline{m} + \Delta, u^\alpha) > F(\overline{m} + \Delta, u^\beta)$ for any $\Delta > 0$, $\mathcal{B}^\beta$ and $\{m | m < \overline{m}\}$ are disjoint sets. Similarly, $\mathcal{B}^\alpha$ and $\{m | m > \overline{m}\}$ are disjoint sets, too. Thus, from the definitions of u^α and u^β, we have $u^\alpha < \overline{m} < u^\beta$ which implies $\overline{m} - u^\alpha > 0$ and $u^\beta - \overline{m} > 0$. Then, from (2.8),

$$u^\beta - u^\alpha = (u^\beta - \overline{m}) + (\overline{m} - u^\alpha) = 2(u^\beta - \overline{m}) + 2b > 2b$$

and

$$u^\beta - u^\alpha = (u^\beta - \overline{m}) + (\overline{m} - u^\alpha) = 2(\overline{m} - u^\alpha) - 2b > -2b$$

are obtained. Hence, $u^\beta - u^\alpha > 2|b|$, which implies that there must be at least $2|b|$ distance between the equilibrium points (decoder's actions, centroids of the bins). Further, from the encoder's point of view, given any two bins $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$, there exists a point $\overline{m}$ which lies between these two bins. This assures that each bin must be a single interval; i.e., convex cell except for a possible insignificant set of points with measure zero. Since there is an injective and monotonic relation between the convex cells of the encoder and decoder's actions, the equilibrium policy must be quantized almost surely.

2.7.2 Proof of Theorem 2.3.1

Similar to the single-stage scalar case in Theorem 2.2.3, at the equilibrium, define two cells C^α and C^β, any points in those cells as $\mathbf{m}^\alpha \in C^\alpha$ and $\mathbf{m}^\beta \in C^\beta$, and the actions of the decoder as $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$ when the encoder transmits $\mathbf{m}^\alpha$ and $\mathbf{m}^\beta$, respectively. Let $F(\mathbf{m}, \mathbf{u}) \triangleq \|\mathbf{m} - \mathbf{u} - \mathbf{b}\|^2$. Due to the equilibrium definitions from the view of the encoder; $F(\mathbf{m}^\alpha, \mathbf{u}^\alpha) < F(\mathbf{m}^\alpha, \mathbf{u}^\beta)$ and $F(\mathbf{m}^\beta, \mathbf{u}^\beta) < F(\mathbf{m}^\beta, \mathbf{u}^\alpha)$. Hence, there exists a hyperplane defined by $F(\mathbf{z}, \mathbf{u}^\alpha) = F(\mathbf{z}, \mathbf{u}^\beta)$ which is equivalent to $\|(\mathbf{z} - \mathbf{b}) - \mathbf{u}^\alpha\|^2 = \|(\mathbf{z} - \mathbf{b}) - \mathbf{u}^\beta\|^2$. It can be seen that $\mathbf{z} - \mathbf{b}$ defines a hyperplane which is a perpendicular bisector of $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$; i.e., the hyperplane defined by the points $\mathbf{z}$ is a perpendicular bisector of $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$ shifted by $\mathbf{b}$. The hyperplane defined by the points $\mathbf{z}$ divides the space into two subspaces: let Z^α that contains $\mathbf{u}^\alpha$ and Z^β that contains $\mathbf{u}^\beta$ be those subspaces. $\mathcal{C}^\beta$ and Z^α are disjoint subspaces since $F(\mathbf{z} + \delta(\mathbf{u}^\beta - \mathbf{u}^\alpha), \mathbf{u}^\alpha) \geq F(\mathbf{z} + \delta(\mathbf{u}^\beta - \mathbf{u}^\alpha), \mathbf{u}^\beta)$ for any positive scalar δ. Similarly, $\mathcal{C}^\alpha$ and Z^β are disjoint subspaces, too. Thus, the hyperplane defined by the points $\mathbf{z}$ must lie between $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$ which implies that the length of $\mathbf{b}$ along the $\mathbf{d} \triangleq \mathbf{u}^\beta - \mathbf{u}^\alpha$ direction should not exceed half of the distance between $\mathbf{u}^\alpha$ and $\mathbf{u}^\beta$; i.e., $\|\mathbf{b_d}\| \leq \|\mathbf{d}\|/2$, where $\mathbf{b_d}$ is the projection of $\mathbf{b}$ along the direction of $\mathbf{d}$. Since $\mathbf{d}$ can be any vector at a fully revealing equilibrium by the assumption on the source, $\|\mathbf{b_d}\| \leq \|\mathbf{d}\|/2$ cannot be satisfied unless $\mathbf{b} = \mathbf{0}$. Thus, there cannot be a fully revealing equilibrium in the single-stage multi-dimensional cheap talk.

For a linear equilibrium, it suffices to provide an example. Let the source be uniform on $[0,1] \times [0,1]$ and consider $\mathbf{b} = [0.3\ 0]$. Then, as a (properly interpreted) limit case of the equilibrium in Fig. 2.3, the following encoder and decoder policies form an equilibrium:

$$\begin{aligned} \gamma^e(m_1, m_2) &= (x_1, x_2) = (0, m_2)\,, \\ \gamma^d(x_1, x_2) &= (u_1, u_2) = (0.5, m_2)\,. \end{aligned}$$

Here, the scalar setup is applied on the x-dimension with one quantization bin (recall that $u_1 = \mathbb{E}[m_1|x_1]$), and a fully-informative equilibrium exists on the y-dimension since there is no bias on that dimension. It is observed that the encoder policy is linear due to the unbiased property of the y-dimension.

Besides linear equilibria, there may be multiple (hence, non-unique) quantized equilibria with finite regions in the multi-dimensional case as illustrated in Fig. 2.4.

2.7.3 Proof of Theorem 2.4.1

(i) If the encoder is linear (affine), the decoder, as an MMSE decoder for a Gaussian source over a Gaussian channel, is linear (affine); this follows from

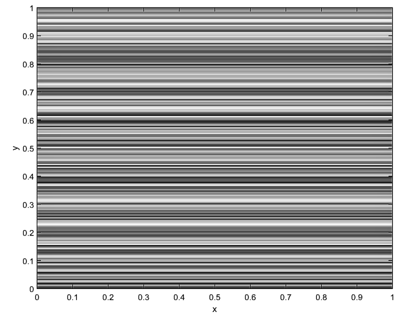

Figure 2.3: Sample linear equilibrium for $\mathbf{b}_x = 0.2$ *and* $\mathbf{b}_y = 0$. *Note that the number of quantization levels on the* y*-dimension can be arbitrarily chosen (since* $\mathbf{b}$ *is orthogonal to that dimension).*

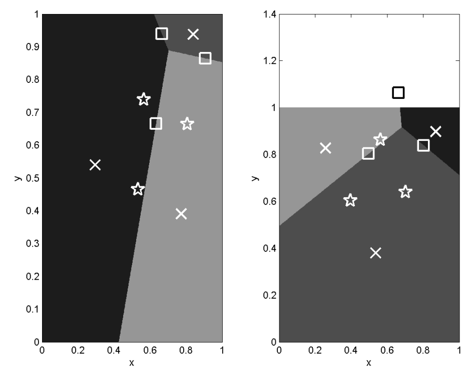

Figure 2.4: Sample discrete equilibria in 2D with $\mathbf{b}_x = 0.1$ *and* $\mathbf{b}_y = 0.2$ *where the crosses indicate the centroids of the bins, the star indicates the middle point and the square indicates the shifted middle point. Note that the equilibrium is not unique for a given number of bins as in the scalar case.*

the property of the conditional expectation for jointly Gaussian random variables. In particular, for the given affine encoding policy $x = \gamma^e(m) = Am + C$, the optimal decoder policy would be

$$\gamma^d(y) = \frac{A\sigma_M^2}{A^2\sigma_M^2 + \sigma_W^2}(y - C) \triangleq Ky + L\,. \tag{2.9}$$

Suppose on the other hand that the decoder is affine so that $u = \gamma^d(y) = Ky + L$ and the encoder policy is $x = \gamma^e(m)$. We will show that the encoder is also affine in this case: With $y = \gamma^e(m) + w$, it follows that $u = K\gamma^e(m) + Kw + L$. By completing the square, the optimal cost of the encoder can be written as

$$J^{*,e} = \min_{x=\gamma^e(m)} \mathbb{E}[(m - u - b)^2 + \lambda x^2]$$

$$= \min_{\gamma^e(m)} \mathbb{E}\Big[(m - K\gamma^e(m) - Kw - L - b)^2 + \lambda(\gamma^e(m))^2\Big]$$

$$= \min_{\gamma^e(m)} \mathbb{E}\Big[(m - K\gamma^e(m) - L - b)^2 + \lambda(\gamma^e(m))^2\Big] + K^2\sigma_W^2$$

$$= \min_{\gamma^e(m)} \mathbb{E}\Big[(K^2 + \lambda)(\gamma^e(m))^2 + (m - L - b)^2 - 2(m - L - b)K\gamma^e(m)\Big] + K^2\sigma_W^2$$

$$= \min_{\gamma^e(m)} (K^2 + \lambda)\mathbb{E}\Big[(\gamma^e(m))^2 + \frac{(m - L - b)^2}{K^2 + \lambda} - \frac{2(m - L - b)K}{K^2 + \lambda}\gamma^e(m)\Big] + K^2\sigma_W^2$$

$$= \min_{\gamma^e(m)} (K^2 + \lambda)\mathbb{E}\Big[\Big(\gamma^e(m) - \frac{(m - L - b)K}{K^2 + \lambda}\Big)^2 - \Big(\frac{(m - L - b)K}{K^2 + \lambda}\Big)^2 + \frac{(m - L - b)^2}{K^2 + \lambda}\Big] + K^2\sigma_W^2$$

$$= \min_{\gamma^e(m)} (K^2 + \lambda)\mathbb{E}\Big[\Big(\gamma^e(m) - \frac{(m - L - b)K}{K^2 + \lambda}\Big)^2 + \lambda\Big(\frac{(m - L - b)}{K^2 + \lambda}\Big)^2\Big] + K^2\sigma_W^2$$

$$= \min_{\gamma^e(m)} (K^2 + \lambda)\mathbb{E}\Big[\Big(\gamma^e(m) - \frac{(m - L - b)K}{K^2 + \lambda}\Big)^2\Big] + \frac{\lambda}{K^2 + \lambda}\Big(\sigma_M^2 + (L + b)^2\Big) + K^2\sigma_W^2 \,. \tag{2.10}$$

In (2.10), only the $\mathbb{E}\Big[\Big(\gamma^e(m) - \frac{(m-L-b)K}{K^2+\lambda}\Big)^2\Big]$ part depends on the minimization parameter $\gamma^e(m)$. Hence, the optimal $\gamma^e(m)$ can be chosen as

$$\gamma^{*,e}(m) = \Big(\frac{1}{K + \lambda/K}\Big)m + \Big(\frac{-L - b}{K + \lambda/K}\Big) \triangleq Am + C \,, \tag{2.11}$$

and the minimum encoder cost is obtained as

$$J^{*,e} = \frac{\lambda}{K^2 + \lambda}\Big(\sigma_M^2 + (L + b)^2\Big) + K^2\sigma_W^2 \,. \tag{2.12}$$

Recall that (2.11) implies that an optimal encoder policy for a Gaussian source over a Gaussian channel is an affine policy if the decoder policy is chosen as affine. We now wish to see if the sets of optimal policies of the encoder and the decoder satisfy a fixed point equation. By combining (2.9) and (2.11), we have $A = \frac{K}{K^2+\lambda}$, $K = \frac{A\sigma_M^2}{A^2\sigma_M^2+\sigma_W^2}$, $L = -KC$ and $C =$

$-A(L+b)$. Then, for nonzero A, by utilizing $A^2\sigma_M^2+\sigma_W^2=\frac{A\sigma_M^2}{K}=\frac{\sigma_M^2}{K^2+\lambda}$, we have

$$AK=\frac{K^2}{K^2+\lambda}=\frac{A^2\sigma_M^2}{A^2\sigma_M^2+\sigma_W^2}\Rightarrow\frac{\lambda}{K^2+\lambda}=\frac{\sigma_W^2}{A^2\sigma_M^2+\sigma_W^2}=\frac{\sigma_W^2}{\sigma_M^2}\left(K^2+\lambda\right). \tag{2.13}$$

Hence, we have $(K^2+\lambda)^2\sigma_W^2=\lambda\sigma_M^2\Rightarrow K^2=\sqrt{\frac{\lambda\sigma_M^2}{\sigma_W^2}}-\lambda\geq 0\Rightarrow\lambda<\sigma_M^2/\sigma_W^2$ for nonzero A. Thus, if $\lambda\geq\sigma_M^2/\sigma_W^2$, the solution is $A=0$; in other words, there does not exist an informative affine equilibrium, the only affine equilibrium is the non-informative one; i.e., $A=K=C=L=0$.

(ii) For $\lambda<\frac{\sigma_M^2}{\sigma_W^2}$, if we combine the fixed point equations (2.9) and (2.11) by using A, and define the resulting mapping as $T(A)$, we obtain

$$A=\frac{K}{K^2+\lambda}=\frac{\frac{A}{A^2+\sigma_W^2/\sigma_M^2}}{\left(\frac{A}{A^2+\sigma_W^2/\sigma_M^2}\right)^2+\lambda}\triangleq T(A). \tag{2.14}$$

Note now that

$$A\geq 1\Rightarrow\frac{A}{A^2+\frac{\sigma_W^2}{\sigma_M^2}}<1\Rightarrow T(A)<\frac{1}{\lambda},$$

$$A<1\Rightarrow\frac{A}{A^2+\frac{\sigma_W^2}{\sigma_M^2}}<\frac{\sigma_M^2}{\sigma_W^2}\Rightarrow T(A)<\frac{\frac{\sigma_M^2}{\sigma_W^2}}{\lambda},$$

which implies that the mapping defined by $T(A)=A$ can be viewed as a continuous function mapping the compact convex set $[0,\max(\sigma_M^2/\sigma_W^2,1)/\lambda]$ to itself. Therefore, by Brouwer's fixed point theorem [84], there exists $A=T(A)$. Indeed, we can find nonzero A,K,C,L for every $0<\lambda<\frac{\sigma_M^2}{\sigma_W^2}$ as in (2.15).

For the uniqueness of an informative fixed point, suppose that there are two different nonzero fixed points: $A_1=T(A_1)$ and $A_2=T(A_2)$ and let $\gamma=\sigma_W^2/\sigma_M^2$ for simplicity. Then $A_1/T(A_1)=A_2/T(A_2)$ implies

$$\frac{A_1^2}{A_1^2+\gamma}+\lambda(A_1^2+\gamma)=\frac{A_2^2}{A_2^2+\gamma}+\lambda(A_2^2+\gamma)$$
$$\Rightarrow(A_1^2-A_2^2)\Big(\frac{\gamma}{(A_1^2+\gamma)(A_2^2+\gamma)}+\lambda\Big)=0.$$

Hence, $|A_1| = |A_2|$ is obtained, and since the mapping is defined from $[0, \max(\sigma_M^2/\sigma_W^2, 1)/\lambda]$ to itself, the nonzero fixed point is unique. Then the encoder may choose the nonzero fixed point for the informative equilibrium if it results in a lower cost than the non-informative equilibrium (due to the cost of communication, an informative equilibrium is not always beneficial to the encoder compared to the non-informative one as shown in Theorem 2.4.4).

(iii) It is proved in (2.11) that an optimal encoder is affine such that $x = \gamma^e(m) = Am + C$ when the decoder is affine, that is, $u = \gamma^d(y) = Ky + L$. Then, by inserting $\lambda = 0$ to (2.1), $\overline{m}$ is obtained as $\overline{m} = KA\frac{(m^\alpha + m^\beta)}{2} + KC + L + b$. This holds for all m^α and m^β with $m^\alpha \leq \overline{m} \leq m^\beta$. Thus, if the distance between m^α and m^β is made arbitrarily small, then it must be that $KA = 1$ and $KC + L + b = 0$. On the other hand, it was shown that an optimal decoder policy is affine if an encoder is affine in (2.9). By combining $KA = 1$ and $K = \frac{A\sigma_M^2}{A^2\sigma_M^2 + \sigma_W^2}$, it follows that a real-valued solution does not exist for any given affine coding parameter.

2.7.4 Proof of Theorem 2.4.2

(i) From (2.9) and (2.11), we have the following equalities for $\lambda < \frac{\sigma_M^2}{\sigma_W^2}$:

$$K = \pm\sqrt{\sqrt{\frac{\lambda\sigma_M^2}{\sigma_W^2}} - \lambda},$$

$$A = \frac{K}{K^2 + \lambda} = \pm\sqrt{\sqrt{\frac{\sigma_W^2}{\lambda\sigma_M^2}} - \frac{\sigma_W^2}{\sigma_M^2}},$$

$$AK = \sqrt{\left(\sqrt{\frac{\sigma_W^2}{\lambda\sigma_M^2}} - \frac{\sigma_W^2}{\sigma_M^2}\right)\left(\sqrt{\frac{\lambda\sigma_M^2}{\sigma_W^2}} - \lambda\right)} = \left|1 - \sqrt{\frac{\lambda\sigma_W^2}{\sigma_M^2}}\right| = 1 - \sqrt{\frac{\lambda\sigma_W^2}{\sigma_M^2}},$$

$$C = -A(L + b) = -A(-KC + b) = AKC - Ab$$

$$\Rightarrow C = \frac{Ab}{AK - 1} = \frac{\pm b\sqrt{\sqrt{\frac{\sigma_W^2}{\lambda\sigma_M^2}} - \frac{\sigma_W^2}{\sigma_M^2}}}{-\sqrt{\frac{\lambda\sigma_W^2}{\sigma_M^2}}} = \mp b\sqrt{\frac{1}{\lambda}\left(\sqrt{\frac{\sigma_M^2}{\lambda\sigma_W^2}} - 1\right)},$$

$$L + b = -\frac{C}{A} \Rightarrow (L+b)^2 = \frac{C^2}{A^2} = b^2 \frac{1}{\lambda} \frac{\sigma_M^2}{\sigma_W^2} \,. \tag{2.15}$$

Utilizing (2.15) in the optimal encoder cost (2.12) results in:

$$\begin{aligned}
J^{*,e} &= \frac{\lambda}{K^2 + \lambda} \Big(\sigma_M^2 + (L+b)^2 \Big) + K^2 \sigma_W^2 \\
&= \frac{\lambda}{\sqrt{\frac{\lambda \sigma_M^2}{\sigma_W^2}}} \Big(\sigma_M^2 + b^2 \frac{1}{\lambda} \frac{\sigma_M^2}{\sigma_W^2} \Big) + \Big(\sqrt{\frac{\lambda \sigma_M^2}{\sigma_W^2}} - \lambda \Big) \sigma_W^2 \\
&= \sqrt{\lambda \sigma_M^2 \sigma_W^2} + b^2 \sqrt{\frac{\sigma_M^2}{\lambda \sigma_W^2}} + \sqrt{\lambda \sigma_M^2 \sigma_W^2} - \lambda \sigma_W^2 \\
&= 2 \sqrt{\lambda \sigma_M^2 \sigma_W^2} + b^2 \sqrt{\frac{\sigma_M^2}{\lambda \sigma_W^2}} - \lambda \sigma_W^2 \,.
\end{aligned}$$

Now recall that the optimal decoder policy is $u^* = \mathbb{E}[m|(y = Am + C + w)] = \frac{A\sigma_M^2}{A^2\sigma_M^2 + \sigma_W^2}(y - C)$. Then, the optimal decoder cost becomes

$$\begin{aligned}
J^{*,d} &= \min_{u = \gamma^d(y)} \mathbb{E}[(m-u)^2] = \mathbb{E}[(m - \mathbb{E}[m|y])^2] \\
&= \sigma_M^2 - \frac{\sigma_{MY}^2}{\sigma_Y^2} = \sigma_M^2 - \frac{A^2 (\sigma_M^2)^2}{A^2 \sigma_M^2 + \sigma_W^2} \\
&= \frac{\sigma_M^2 \sigma_W^2}{A^2 \sigma_M^2 + \sigma_W^2} = \frac{\sigma_M^2 \sigma_W^2}{\Big(\sqrt{\frac{\sigma_W^2}{\lambda \sigma_M^2}} - \frac{\sigma_W^2}{\sigma_M^2} \Big) \sigma_M^2 + \sigma_W^2} \\
&= \sqrt{\lambda \sigma_M^2 \sigma_W^2} \,.
\end{aligned}$$

As a result, the game theoretic cost at the equilibrium is found as

$$J^{*,g} = J^{*,e} + J^{*,d} = 3 \sqrt{\lambda \sigma_M^2 \sigma_W^2} + b^2 \sqrt{\frac{\sigma_M^2}{\lambda \sigma_W^2}} - \lambda \sigma_W^2 \,. \tag{2.16}$$

Recall that, if $\lambda \geq \sigma_M^2/\sigma_W^2$, since $A = K = C = L = 0$, $J^{*,e} = \sigma_M^2 + b^2$ and $J^{*,d} = \sigma_M^2$ are obtained; hence, the game theoretic cost becomes $J^{*,g} = 2\sigma_M^2 + b^2$. If there were no cost of communication (consider the cheap talk; i.e., remove λx^2 from the encoder cost function), then one could say that the informative equilibria would always be beneficial to both the encoder and the decoder; however, due to the cost of communication, an informative equilibrium is not always beneficial to the encoder when compared with the non-informative one (i.e., for $\lambda < \sigma_M^2/\sigma_W^2$, it does not always hold that

$2\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2\sqrt{\frac{\sigma_M^2}{\lambda\sigma_W^2}} - \lambda\sigma_W^2 < \sigma_M^2 + b^2$, see Theorem 2.4.4 for details). For the receiver, however, information never hurts the performance and the informative equilibria are more desirable (i.e., for $\lambda < \sigma_M^2/\sigma_W^2$, the inequality $\sqrt{\lambda\sigma_M^2\sigma_W^2} < \sigma_M^2$ always holds). As a result, one can expect a non-informative equilibrium even if $\lambda < \sigma_M^2/\sigma_W^2$, as shown in Theorem 2.4.4.

(ii) The part below aims to construct the socially optimal affine setup. In this part, $J^{e,t}$ represents the team cost minimized over the encoder policies for a given decoder policy, $J^{d,t}$ represents the team cost minimized over the decoder policies for a given encoder policy, and $J^{*,t}$ represents the optimum team cost; i.e., minimization over all affine encoding and decoding policies as follows:

$$J^{*,t} = \min_{x=\gamma^e(m),u=\gamma^d(y)} \mathbb{E}[(m-u-b)^2 + \lambda x^2 + (m-u)^2]\,.$$

Similar to the game theoretic analysis above, with the given affine encoding policy $x = \gamma^e(m) = Am + C$ (then $y = x + w = Am + C + w$), the optimal decoder policy can be found as follows (by completing the square):

$$\begin{aligned} J^{d,t} &= \min_{u=\gamma^d(y)} \mathbb{E}[(m-u-b)^2 + \lambda x^2 + (m-u)^2] \\ &= \min_{u=\gamma^d(y)} \mathbb{E}[2(m-u)^2 - 2(m-u)b + \lambda x^2] \\ &= \min_{u=\gamma^d(y)} 2\mathbb{E}\Big[(m-u-\frac{b}{2})^2 + \frac{b^2}{4} + \lambda\frac{x^2}{2}\Big]\,. \end{aligned}$$

Hence the optimal decoder policy can be chosen as $\gamma^{d,t}(y) = \mathbb{E}[m - \frac{b}{2}\,|\,y]$. Due to the joint Gaussanity of m and y, the minimizer decoder policy is affine:

$$\gamma^{d^*,t}(y) = \frac{A\sigma_M^2}{A^2\sigma_M^2 + \sigma_W^2}(y - C) - \frac{b}{2} \triangleq Ky + L\,. \tag{2.17}$$

Similar to the game theoretic analysis above, for any affine decoder policy $\gamma^d(y) = Ky + L$ with $y = \gamma^e(m) + w$, the optimal encoder policy for the team setup can be obtained as follows (by completing the square):

$$J^{e,t} = \min_{x=\gamma^e(m)} \mathbb{E}[(m-u-b)^2 + \lambda x^2 + (m-u)^2]$$

$$
\begin{aligned}
&= \min_{\gamma^e(m)} \mathbb{E}\Big[(m - K\gamma^e(m) - Kw - L - b)^2 + \lambda(\gamma^e(m))^2 \\
&\qquad + (m - K\gamma^e(m) - Kw - L)^2\Big] \\
&= \min_{\gamma^e(m)} \mathbb{E}\Big[(m - K\gamma^e(m) - L - b)^2 + \lambda(\gamma^e(m))^2 + (m - K\gamma^e(m) - L)^2\Big] \\
&\qquad + 2K^2\sigma_W^2 \\
&= \min_{\gamma^e(m)} \mathbb{E}\Big[(2K^2 + \lambda)(\gamma^e(m))^2 + (m - L - b)^2 + (m - L)^2 \\
&\qquad - 2(2m - 2L - b)K\gamma^e(m)\Big] + 2K^2\sigma_W^2 \\
&= \min_{\gamma^e(m)} (2K^2 + \lambda)\mathbb{E}\left[(\gamma^e(m))^2 + \frac{(m - L - b)^2}{2K^2 + \lambda} + \frac{(m - L)^2}{2K^2 + \lambda}\right. \\
&\qquad \left. - \frac{2(2m - 2L - b)K}{2K^2 + \lambda}\gamma^e(m)\right] + 2K^2\sigma_W^2 \\
&= \min_{\gamma^e(m)} (2K^2 + \lambda)\mathbb{E}\left[\Big(\gamma^e(m) - \frac{(2m - 2L - b)K}{2K^2 + \lambda}\Big)^2 - \Big(\frac{(2m - 2L - b)K}{2K^2 + \lambda}\Big)^2\right. \\
&\qquad \left. + \frac{(m - L - b)^2}{2K^2 + \lambda} + \frac{(m - L)^2}{2K^2 + \lambda}\right] + 2K^2\sigma_W^2 \\
&= \min_{\gamma^e(m)} (2K^2 + \lambda)\mathbb{E}\left[\Big(\gamma^e(m) - \frac{(2m - 2L - b)K}{2K^2 + \lambda}\Big)^2\right. \\
&\qquad \left. + \frac{b^2K^2 + \lambda\left((m - L - b)^2 + (m - L)^2\right)}{(2K^2 + \lambda)^2}\right] + 2K^2\sigma_W^2 \\
&= \min_{\gamma^e(m)} (2K^2 + \lambda)\mathbb{E}\left[\Big(\gamma^e(m) - \frac{(2m - 2L - b)K}{2K^2 + \lambda}\Big)^2\right] \\
&\qquad + \frac{b^2K^2 + \lambda\left(2\sigma_M^2 + (L + b)^2 + L^2\right)}{2K^2 + \lambda} + 2K^2\sigma_W^2\,.
\end{aligned}
$$

Hence, the optimal encoder $\gamma^e(m)$ is

$$
\gamma^{e^*,t}(m) = \frac{(2m - 2L - b)}{2K + \lambda/K} \triangleq Am + C\,, \tag{2.18}
$$

and the minimum team cost is obtained as

$$
J^{*,t} = \frac{b^2K^2 + \lambda\left(2\sigma_M^2 + (L + b)^2 + L^2\right)}{2K^2 + \lambda} + 2K^2\sigma_W^2\,. \tag{2.19}
$$

This implies that, in the team setup, an optimal encoder policy for a Gaussian source over a Gaussian channel is a affine policy if the decoder policy is chosen as affine.

In order to achieve the socially optimal cost $J^{*,t}$, the optimal encoder policy $\gamma^{e^*,t}(m)$ and the optimal decoder policy $\gamma^{d^*,t}(y)$ must satify the following equalities by (2.17) and (2.18):

$$\begin{aligned}
&A = \frac{2}{2K + \lambda/K}, \qquad K = \frac{A\sigma_M^2}{A^2\sigma_M^2 + \sigma_W^2}, \\
&C = \frac{A}{2}(-2L - b) = -AL - \frac{Ab}{2}, \quad L = -KC - \frac{b}{2} \\
&\Rightarrow C = -A\left(-KC - \frac{b}{2}\right) - \frac{Ab}{2} = AKC\,.
\end{aligned}$$

Here, either $AK = 1$ or $C = 0$. If $AK = 1$, then it becomes that $1 = AK = \frac{A^2\sigma_M^2}{A^2\sigma_M^2+\sigma_W^2} \Rightarrow \sigma_W^2 = 0$, which contradicts with the noise assumption. Then $C = 0$ and $L = -b/2$. By using the equalities for A and K above, one can obtain $2(K^2 + \lambda/2)^2\sigma_W^2 = \lambda\sigma_M^2$ by assuming $A \neq 0$; which implies $K^2 = \sqrt{\frac{\lambda\sigma_M^2}{2\sigma_W^2}} - \frac{\lambda}{2}$. Since K^2 is positive, λ cannot be greater than $\frac{2\sigma_M^2}{\sigma_W^2}$; otherwise, because of our assumption, A must be equal to 0 which implies that $K = 0$, and there does not exist an informative affine team setup. Then $K^2 = \sqrt{\frac{\lambda\sigma_M^2}{2\sigma_W^2}} - \frac{\lambda}{2}$ and $\lambda < \frac{2\sigma_M^2}{\sigma_W^2}$ for nonzero A. Thus, for $\lambda < \frac{2\sigma_M^2}{\sigma_W^2}$, by using $K^2 = \sqrt{\frac{\lambda\sigma_M^2}{2\sigma_W^2}} - \frac{\lambda}{2}$, $A = \frac{2K}{2K^2+\lambda}$, $C = 0$ and $L = -\frac{b}{2}$ in (2.19), we have

$$\begin{aligned}
J^{*,t} &= \frac{b^2K^2 + \lambda\left(2\sigma_M^2 + (L+b)^2 + L^2\right)}{2K^2 + \lambda} + 2K^2\sigma_W^2 \\
&= \frac{b^2\left(\sqrt{\frac{\lambda\sigma_M^2}{2\sigma_W^2}} - \frac{\lambda}{2}\right) + 2\lambda\sigma_M^2 + \lambda\left((\frac{b}{2})^2 + (-\frac{b}{2})^2\right)}{2\sqrt{\frac{\lambda\sigma_M^2}{2\sigma_W^2}}} + 2\left(\sqrt{\frac{\lambda\sigma_M^2}{2\sigma_W^2}} - \frac{\lambda}{2}\right)\sigma_W^2 \\
&= \frac{b^2}{2} + \sqrt{2\lambda\sigma_M^2\sigma_W^2} + \sqrt{2\lambda\sigma_M^2\sigma_W^2} - \lambda\sigma_W^2 \\
&= 2\sqrt{2\lambda\sigma_M^2\sigma_W^2} + \frac{b^2}{2} - \lambda\sigma_W^2\,. \qquad (2.20)
\end{aligned}$$

Recall that, if $\lambda \geq 2\sigma_M^2/\sigma_W^2$, then $J^{*,t} = 2\sigma_M^2 + \frac{b^2}{2}$. Since $2\sqrt{2\lambda\sigma_M^2\sigma_W^2} + \frac{b^2}{2} - \lambda\sigma_W^2 < 2\sigma_M^2 + \frac{b^2}{2}$ always holds (this can be shown by using the fact $\left(2\sigma_M^2 - \lambda\sigma_W^2\right)^2 > 0$), which implies that the informative equilibrium should always be preferred over the non-informative one. However, for $\lambda \geq 2\sigma_M^2/\sigma_W^2$, the team equilibrium is always non-informative as shown above.

2.7.5 Proof of Theorem 2.4.4

Recall that in the single-stage Nash signaling game, the optimal affine encoder cost is obtained as $J^{*,e} = \sigma_M^2 + b^2$ for $\lambda \geq \sigma_M^2/\sigma_W^2$ (at the non-informative equilibrium) and $J^{*,e} = 2\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2\sqrt{\frac{\sigma_M^2}{\lambda\sigma_W^2}} - \lambda\sigma_W^2$ for $\lambda < \sigma_M^2/\sigma_W^2$ (at the informative affine equilibrium). Even though $\sigma_M^2 + b^2 \geq 2\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2\sqrt{\frac{\sigma_M^2}{\lambda\sigma_W^2}} - \lambda\sigma_W^2$ for $\lambda \geq \sigma_M^2/\sigma_W^2$ (since $\left(\sqrt{\sigma_M^2} - \sqrt{\lambda\sigma_W^2}\right)^2 \geq 0$), the non-informative equilibrium is preferred over the informative affine equilibrium since there does not exist any informative affine equilibrium for $\lambda \geq \sigma_M^2/\sigma_W^2$ as shown in Theorem 2.4.1. However, the non-informative equilibrium can also be preferred even if $\lambda < \sigma_M^2/\sigma_W^2$. Now, we analyze the conditions under which the informative affine equilibrium has strictly lower cost than the non-informative one for $\lambda < \sigma_M^2/\sigma_W^2$:

$$
\begin{aligned}
&\sigma_M^2 + b^2 > 2\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2\sqrt{\frac{\sigma_M^2}{\lambda\sigma_W^2}} - \lambda\sigma_W^2 \\
&\Rightarrow \left(\sqrt{\sigma_M^2} - \sqrt{\lambda\sigma_W^2}\right)^2 + b^2\left(1 - \sqrt{\frac{\sigma_M^2}{\lambda\sigma_W^2}}\right) > 0 \\
&\Rightarrow \left(\sqrt{\sigma_M^2} - \sqrt{\lambda\sigma_W^2}\right)\left(\sqrt{\lambda\sigma_W^2}\left(\sqrt{\sigma_M^2} - \sqrt{\lambda\sigma_W^2}\right) - b^2\right) > 0\,. \qquad (2.21)
\end{aligned}
$$

Since $\lambda < \sigma_M^2/\sigma_W^2$, (2.21) reduces to $\sqrt{\lambda\sigma_W^2}\left(\sqrt{\sigma_M^2} - \sqrt{\lambda\sigma_W^2}\right) - b^2 > 0$. Let $t \triangleq \sqrt{\lambda\sigma_W^2}$, then the inequality becomes $t^2 - \sqrt{\sigma_M^2}t + b^2 < 0$. If $\sigma_M^2 < 4b^2$, then $t^2 - \sqrt{\sigma_M^2}t + b^2$ is always positive, which implies that the non-informative equilibrium has strictly less cost than the informative affine equilibrium. Otherwise; i.e., if $\sigma_M^2 \geq 4b^2$, then $t^2 - \sqrt{\sigma_M^2}t + b^2 < 0$ holds when $\frac{\sqrt{\sigma_M^2} - \sqrt{\sigma_M^2 - 4b^2}}{2} < t < \frac{\sqrt{\sigma_M^2} + \sqrt{\sigma_M^2 - 4b^2}}{2}$. Then, the result follows through inserting $t \triangleq \sqrt{\lambda\sigma_W^2}$:

$$
\begin{aligned}
\frac{\sqrt{\sigma_M^2} - \sqrt{\sigma_M^2 - 4b^2}}{2} &< \sqrt{\lambda\sigma_W^2} < \frac{\sqrt{\sigma_M^2} + \sqrt{\sigma_M^2 - 4b^2}}{2} \\
\frac{\sigma_M^2 - 2b^2 - \sqrt{\sigma_M^2}\sqrt{\sigma_M^2 - 4b^2}}{2} &< \lambda\sigma_W^2 < \frac{\sigma_M^2 - 2b^2 + \sqrt{\sigma_M^2}\sqrt{\sigma_M^2 - 4b^2}}{2} \\
0 < \frac{\sigma_M^2 - 2b^2 - \sqrt{\sigma_M^2}\sqrt{\sigma_M^2 - 4b^2}}{2\sigma_W^2} &< \lambda < \frac{\sigma_M^2 - 2b^2 + \sqrt{\sigma_M^2}\sqrt{\sigma_M^2 - 4b^2}}{2\sigma_W^2} < \frac{\sigma_M^2}{\sigma_W^2}\,.
\end{aligned}
\qquad (2.22)
$$

Thus, the encoder prefers the informative affine equilibrium when $\frac{\sigma_M^2-2b^2-\sqrt{\sigma_M^2}\sqrt{\sigma_M^2-4b^2}}{2\sigma_W^2} < \lambda < \frac{\sigma_M^2-2b^2+\sqrt{\sigma_M^2}\sqrt{\sigma_M^2-4b^2}}{2\sigma_W^2}$ and $\sigma_M^2 \geq 4b^2$; otherwise, the non-informative equilibrium is preferred.

2.7.6 Proof of Theorem 2.4.5

Due to the Stackelberg assumption, the encoder knows that the optimal decoder policy will be $u = \mathbb{E}[m|y]$. Then, by the law of the iterated expectations, the encoder cost $J^e = \mathbb{E}[(m-u-b)^2+\lambda x^2]$ reduces to $J^e = \mathbb{E}[(m-u)^2]+\lambda\mathbb{E}[x^2]+b^2$. Further, let $P \triangleq \mathbb{E}[x^2]$, and D be defined as the squared error distortion; i.e., $D \triangleq \mathbb{E}[(m-u)^2] = \mathbb{E}[(m-\mathbb{E}[m|y])^2]$. Then,

$$\begin{aligned}
I(M;Y) =& h(M) - h(M|Y) = h(M) - h(M-\mathbb{E}[M|Y]|Y) \\
\geq& h(M) - h(M-\mathbb{E}[M|Y]) \stackrel{(a)}{\geq} \frac{1}{2}\log_2(2\pi e\sigma_M^2) - \frac{1}{2}\log_2(2\pi e D) = \frac{1}{2}\log_2(\frac{\sigma_M^2}{D}) \\
\Rightarrow& D \geq \sigma_M^2 2^{-2I(m;y)} \stackrel{(b)}{\geq} \sigma_M^2 2^{-2I(x;y)} \geq \sigma_M^2 2^{-2\sup I(x;y)} \stackrel{(c)}{=} \sigma_M^2 2^{-2\frac{1}{2}\log_2\left(1+\frac{P}{\sigma_W^2}\right)} \\
\Rightarrow& D = \mathbb{E}[(m-u)^2] \geq \frac{\sigma_M^2}{1+P/\sigma_W^2}\,. \qquad (2.23)
\end{aligned}$$

Here, (a) holds since the differential entropy is $h(M) = \frac{1}{2}\log_2(2\pi e\sigma_M^2)$ for a Gaussian source M, (b) follows from the data-processing inequality, and (c) follows from the channel capacity of the Gaussian channel (see [4, p. 96] for another example of a rate-distortion theoretic bound through the data-processing inequality). Then, the following leads to a lower bound on the encoder cost:

$$\begin{aligned}
J^e &= b^2 + \lambda\mathbb{E}[x^2] + \mathbb{E}[(m-u)^2] \\
&\geq b^2 + \lambda P + \frac{\sigma_M^2}{1+P/\sigma_W^2}\,. \qquad (2.24)
\end{aligned}$$

Let $h(P) \triangleq \lambda P + \frac{\sigma_M^2}{1+P/\sigma_W^2}$, then $\frac{\mathrm{d}h(P)}{\mathrm{d}P} = \lambda - \frac{\sigma_M^2}{\sigma_W^2}\left(1+\frac{P}{\sigma_W^2}\right)^{-2}$. Since $\frac{\mathrm{d}h(P)}{\mathrm{d}P} \geq 0$ when $\lambda \geq \sigma_M^2/\sigma_W^2$, (2.24) is minimized at $P = 0$; that is, the encoder does not signal any output. Hence, the encoder engages in a non-informative equilibrium and the minimum encoder cost becomes $\sigma_M^2 + b^2$ at this non-informative equilibrium. Otherwise; i.e., $\lambda < \sigma_M^2/\sigma_W^2$, $h(P)$ is minimized when $\frac{\mathrm{d}h(P)}{\mathrm{d}P} = 0 \Rightarrow P = \sqrt{\frac{\sigma_M^2\sigma_W^2}{\lambda}} -$

σ_W^2 since $\frac{\mathrm{d}^2 h(P)}{\mathrm{d}P^2} = \frac{2\sigma_M^2}{\sigma_W^4}\left(1+\frac{P}{\sigma_W^2}\right)^{-3} > 0$. In this case, the lower bound on the encoder cost in (2.24) becomes $2\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2 - \lambda\sigma_W^2$.

After obtaining the information theoretic lower bounds on the encoder cost above, now we assume the linear encoding policy and show that the lower bounds are achieved, then we conclude that the encoder policy must be linear. Let the encoder policy be $x = \gamma^e(m) = Am + C$. Due to the Stackelberg assumption, the encoder knows that the decoder will use $\gamma^d(y) = u = \mathbb{E}[m|y]$ as an optimal decoder policy to minimize the decoder cost, thus $u = \gamma^d(y) = \frac{A\sigma_M^2}{A^2\sigma_M^2+\sigma_W^2}(y - C)$ where $y = Am+C+w$. Then the goal of the encoder is to minimize the following:

$$\begin{aligned}
J^{*,e} &= \min_{x=\gamma^e(m)=Am+C} \mathbb{E}[(m-u-b)^2 + \lambda x^2] \\
&= \min_{A,\,C} \mathbb{E}\Big[\left(\frac{m\sigma_W^2 - A\sigma_M^2 w}{A^2\sigma_M^2+\sigma_W^2} - b\right)^2 + \lambda(Am+C)^2\Big] \\
&= \min_{A,\,C} \frac{\sigma_M^2(\sigma_W^2)^2 + A^2(\sigma_M^2)^2\sigma_W^2}{(A^2\sigma_M^2+\sigma_W^2)^2} + b^2 + \lambda A^2\sigma_M^2 + \lambda C^2 \\
&= \min_{A,\,C} \frac{\sigma_M^2\sigma_W^2}{A^2\sigma_M^2+\sigma_W^2} + b^2 + \lambda A^2\sigma_M^2 + \lambda C^2 \,. \qquad (2.25)
\end{aligned}$$

The optimal encoder cost in (2.25) is achieved for $C^* = 0$, and $A^* = 0$ for $\lambda \geq \sigma_M^2/\sigma_W^2$ and $A^* = \sqrt{\sqrt{\frac{\sigma_W^2}{\lambda\sigma_M^2}} - \frac{\sigma_W^2}{\sigma_M^2}}$ for $\lambda < \sigma_M^2/\sigma_W^2$. Then, the optimal encoder cost is obtained as $J^{*,e} = \sigma_M^2+b^2$ for $\lambda \geq \sigma_M^2/\sigma_W^2$ and $J^{*,e} = 2\sqrt{\lambda\sigma_M^2\sigma_W^2}+b^2-\lambda\sigma_W^2$ for $\lambda < \sigma_M^2/\sigma_W^2$. Note that these are the information theoretic lower bounds above, and these lower bounds are achieved when the encoder and the decoder use linear policies jointly, which is valid for the current case.

Recall that, in the static Stackelberg signaling game, the optimal encoder cost is obtained as $J^{*,e} = \sigma_M^2 + b^2$ for $\lambda \geq \sigma_M^2/\sigma_W^2$ and $J^{*,e} = 2\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2 - \lambda\sigma_W^2$ for $\lambda < \sigma_M^2/\sigma_W^2$. Notice that $\sigma_M^2 + b^2 \geq 2\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2 - \lambda\sigma_W^2$ holds always (since $\left(\sqrt{\sigma_M^2} - \sqrt{\lambda\sigma_W^2}\right)^2 \geq 0$); i.e., the informative equilibrium should always be preferred over the non-informative one. However, as proved above, the equilibrium is always non-informative for $\lambda \geq \sigma_M^2/\sigma_W^2$.

2.7.7 Proof of Theorem 2.4.6

(i) From (2.24), we have a bound on the encoder cost $J^e \geq b^2 + \lambda P + \frac{\sigma_M^2}{1+P/\sigma_W^2}$ where $P \triangleq \mathbb{E}[x^2]$ represents the power. This bound is tight when the encoder and the decoder use linear policies leading to jointly Gaussian random variables. For $\lambda < \sigma_M^2/\sigma_W^2$, a minimizer of this cost is $P^* = \sqrt{\frac{\sigma_M^2\sigma_W^2}{\lambda}} - \sigma_W^2$. If we insert this value into (2.24), we have $J^e \geq 2\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2 - \lambda\sigma_W^2$. By the same reasoning above, we also have $J^d = \mathbb{E}[(m-u)^2] \geq \frac{\sigma_M^2}{1+\frac{P}{\sigma_W^2}} \geq \sqrt{\lambda\sigma_M^2\sigma_W^2}$. Hence, the information theoretic lower bound on the game cost $J^g = J^e + J^d$ is found as

$$J^g \geq 3\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2 - \lambda\sigma_W^2 \,. \tag{2.26}$$

Through an analysis similar to the one in [4], one can see that when $\lambda \geq \sigma_M^2/\sigma_W^2$, (2.24) is minimized at $P = 0$ (the encoder does not signal any output); thus we obtain a non-informative equilibrium: The encoder and the decoder do not engage in communications; i.e., $A = 0$ and $K = 0$ is an equilibrium. In this case, the encoder may be considered to be linear, but this is a degenerate coding policy. This implies $J^g \geq 2\sigma_M^2 + b^2$, and remember that $J^{*,g} = 2\sigma_M^2 + b^2$ when $\lambda \geq \sigma_M^2/\sigma_W^2$, hence the information theoretic lower bound is achievable at the non-informative equilibria.

(ii) From (2.16) and (2.26), it can be deduced that when $b = 0$, the lower bound of the encoder cost is achievable by linear policies; i.e., $C = 0$ and $L = 0$. When $b = 0$, the problem corresponds to what is known as a *soft-constrained version of the quadratic signaling problem* where we append the constraint to the cost functional (see [4, p. 96]).

(iii) If $b \neq 0$, then, from (2.16) and (2.26), one can observe that the lower bound becomes unachievable by affine policies since the power constraint related part of the cost function, λx^2, contains b^2 related parameters (recall $C = \frac{Ab}{AK-1}$). In this case, by modifying the power from P to $P - C^2$ (which must be positive) in the information theoretic inequalities; i.e., $J^e \geq b^2 + \lambda P + \frac{\sigma_M^2}{1+(P-C^2)/\sigma_W^2}$, then the minimum game cost is obtained as $J^g \geq 3\sqrt{\lambda\sigma_M^2\sigma_W^2} + b^2\sqrt{\frac{\sigma_M^2}{\lambda\sigma_W^2}} - \lambda\sigma_W^2$, which is the same cost that is achieved by affine policies.

(iv) By following a similar approach to (2.24) for finding the lower bound on the socially optimal cost, we can obtain:

$$\begin{aligned} J^t &= \mathbb{E}[(m-u-b)^2 + \lambda x^2 + (m-u)^2] \\ &= \frac{b^2}{2} + \lambda \mathbb{E}[x^2] + 2\mathbb{E}\left[\left(m-u-\frac{b}{2}\right)^2\right] \\ &\overset{(a)}{\geq} \frac{b^2}{2} + \lambda P + \frac{2\sigma_M^2}{1+P/\sigma_W^2}. \end{aligned}$$

Here (a) holds since the decoder chooses $u = \mathbb{E}[m - \frac{b}{2}|y]$ and shifting does not affect the differential entropy. Similar to the previous analysis, a minimizer of this cost is $P^* = \sqrt{\frac{2\sigma_M^2\sigma_W^2}{\lambda}} - \sigma_W^2$ for $\lambda < 2\sigma_M^2/\sigma_W^2$. If we insert this value into the total cost, we have

$$J^t \geq 2\sqrt{2\lambda\sigma_M^2\sigma_W^2} + \frac{b^2}{2} - \lambda\sigma_W^2. \tag{2.27}$$

Recall that, if $\lambda \geq 2\sigma_M^2/\sigma_W^2$, then $P=0$ becomes the minimizer, hence $J^t \geq 2\sigma_M^2 + \frac{b^2}{2}$ in the non-informative equilibrium. For this case, remember that $J^{*,t} = 2\sigma_M^2 + \frac{b^2}{2}$, thus the information theoretic lower bound is achievable at the non-informative equilibria. In addition, from (2.20) and (2.27), for $\lambda < 2\sigma_M^2/\sigma_W^2$ (which implies the informative equilibria), it can easily be seen that the information theoretic lower bound is achievable by affine policies (actually the encoder policy is linear and the decoder policy is affine).

2.7.8 Proof of Theorem 2.4.7

For an affine encoder; i.e., $x = \gamma^e(m) = Am + C$ which satisfies $\mathbb{E}[x^2] = A^2\sigma_M^2 + C^2 \leq \overline{P}$, the optimal decoder is affine :

$$\gamma^d(y) = \frac{A\sigma_M^2}{A^2\sigma_M^2 + \sigma_W^2}(y - C).$$

For an affine decoder; i.e., $u = \gamma^d(y) = Ky + L$, we will investigate the optimal encoder. With $y = \gamma^e(m) + w$, it follows that $u = K\gamma^e(m) + Kw + L$. Then, under the hard power constraint $\mathbb{E}\left[(\gamma^e(m))^2\right] \leq \overline{P}$, the optimal cost of the encoder can be written as

$$\begin{aligned} J^{*,e} &= \min_{x=\gamma^e(m)} \mathbb{E}\left[(m-u-b)^2\right] \\ &= \min_{\gamma^e(m)} \mathbb{E}\Big[(m-K\gamma^e(m)-Kw-L-b)^2\Big] \\ &= \min_{\gamma^e(m)} \mathbb{E}\Big[(m-K\gamma^e(m)-L-b)^2\Big] + K^2\sigma_W^2 \,. \end{aligned} \tag{2.28}$$

For the optimization problem in (2.28), the corresponding Lagrangian function is expressed as

$$\begin{aligned} \mathcal{L}\left(\gamma^e(m),\mu\right) &= \mathbb{E}\left[(m-K\gamma^e(m)-L-b)^2\right] + K^2\sigma_W^2 + \mu\left(\mathbb{E}\left[(\gamma^e(m))^2\right]-\overline{P}\right) \\ &= \mathbb{E}\left[\left(K^2+\mu\right)(\gamma^e(m))^2 - 2(m-L-b)K\gamma^e(m) + (m-L-b)^2\right] \\ &\qquad + K^2\sigma_W^2 - \mu\overline{P} \\ &= \left(K^2+\mu\right)\mathbb{E}\left[\left(\gamma^e(m) - \frac{(m-L-b)K}{K^2+\mu}\right)^2\right] \\ &\qquad + \frac{\mu}{K^2+\mu}\left(\sigma_M^2+(L+b)^2\right) + K^2\sigma_W^2 - \mu\overline{P}\,, \end{aligned} \tag{2.29}$$

and the dual function is given by

$$g(\mu) \triangleq \inf_{\gamma^e(m)} \quad \mathcal{L}\left(\gamma^e(m),\mu\right)\,, \tag{2.30}$$

and the Lagrangian dual problem of (2.28) is defined as

$$\min_{\mu} g\left(\mu\right) \text{ s.t. } \mu \geq 0\,. \tag{2.31}$$

Since the optimization problem is convex, the duality gap between the solutions of the primal and the dual problem is zero.

It is observed from (2.29) that the Lagrangian function $\mathcal{L}\left(\gamma^e(m),\mu\right)$ can be decomposed into

$$\begin{aligned} \mathcal{L}\left(\gamma^e(m),\mu\right) = &\int_{m\in\mathbb{R}} \mathcal{L}_m\left(\gamma^e(m),\mu\right)p(m)\,\mathrm{d}m \\ &\qquad + \frac{\mu}{K^2+\mu}\left(\sigma_M^2+(L+b)^2\right) + K^2\sigma_W^2 - \mu\overline{P}\,, \end{aligned} \tag{2.32}$$

where $\mathcal{L}_m\left(\gamma^e(m),\mu\right) \triangleq \left(\gamma^e(m)-\frac{(m-L-b)K}{K^2+\mu}\right)^2$. Evidently, the optimal encoder policy that minimizes $\mathcal{L}\left(\gamma^e(m),\mu\right)$ obtained from (2.30) should also minimize $\mathcal{L}_m\left(\gamma^e(m),\mu\right)$ for each given value of m. This is known as dual decomposition and it facilitates the decomposition of the dual problem into sub-optimization problems which are coupled only through m. More explicitly, we need the compute

$$\min_{\gamma^e(m)} \mathcal{L}_m\left(\gamma^e(m),\mu\right) = \min_{\gamma^e(m)} \left(\gamma^e(m)-\frac{(m-L-b)K}{K^2+\mu}\right)^2 \tag{2.33}$$

for each value of $m\in\mathbb{R}$.

The Karush-Kuhn-Tucker (KKT) conditions can be obtained for the optimization problem in (2.28) as follows:

$$\frac{\partial \mathcal{L}_m\left(\gamma^e(m),\mu\right)}{\partial\left(\gamma^e(m)\right)} = 0\,, \tag{2.34}$$

$$\mu\left(\mathbb{E}\left[\left(\gamma^e(m)\right)^2\right]-\overline{P}\right) = 0\,, \tag{2.35}$$

$$\mu \geq 0\,, \tag{2.36}$$

$$\mathbb{E}\left[\left(\gamma^e(m)\right)^2\right]-\overline{P} \leq 0\,. \tag{2.37}$$

From (2.34), the optimal encoder policy is $\gamma^e(m) = \frac{K}{K^2+\mu}m - \frac{K}{K^2+\mu}(L+b)$. By (2.35), we must have either $\mu = 0$ or $\mathbb{E}\left[\left(\gamma^e(m)\right)^2\right] = \overline{P}$. If $\mu = 0$, for an informative affine equilibrium, $K = \frac{A\sigma_M^2}{A^2\sigma_M^2+\sigma_W^2}$ and $A = \frac{K}{K^2+\mu} = \frac{1}{K}$ must be satisfied simultaneously, which is not possible. Thus, we must investigate $\mathbb{E}\left[\left(\gamma^e(m)\right)^2\right] = \overline{P}$ case with $\mu > 0$ to obtain the conditions for the informative affine equilibria.

$$\begin{aligned}
\mathbb{E}\left[\left(\gamma^e(m)\right)^2\right] &= \mathbb{E}\left[\left(\frac{K}{K^2+\mu}m - \frac{K}{K^2+\mu}(L+b)\right)^2\right] \\
&= \frac{K^2}{(K^2+\mu)^2}\left(\sigma_M^2+(L+b)^2\right) = \overline{P} \\
\Rightarrow \mu &= \sqrt{\frac{K^2\left(\sigma_M^2+(L+b)^2\right)}{\overline{P}}} - K^2 > 0 \qquad (2.38)\\
\Rightarrow \overline{P} &< \frac{\sigma_M^2+(L+b)^2}{K^2}\,. \qquad (2.39)
\end{aligned}$$

At the equilibrium, we have

$$K = \frac{A\sigma_M^2}{A^2\sigma_M^2 + \sigma_W^2} \,,$$

$$L = -KC = -\frac{AC\sigma_M^2}{A^2\sigma_M^2 + \sigma_W^2} \,,$$

$$A = \frac{K}{K^2+\mu} = \frac{K}{\sqrt{\frac{K^2\left(\sigma_M^2+(L+b)^2\right)}{\overline{P}}}} = \mathrm{sgn}(K)\sqrt{\frac{\overline{P}}{\sigma_M^2+(L+b)^2}} \,,$$

$$C = -A(L+b) = -\frac{K(L+b)}{K^2+\mu} = -\mathrm{sgn}(K)(L+b)\sqrt{\frac{\overline{P}}{\sigma_M^2+(L+b)^2}} \,.$$

Since A and K must have the same sign, we can assume the positive (A, K) pair. Then,

$$A = \sqrt{\frac{\overline{P}}{\sigma_M^2+(L+b)^2}} \,,$$

$$K = \frac{A\sigma_M^2}{A^2\sigma_M^2+\sigma_W^2} = \frac{\sqrt{\frac{\overline{P}}{\sigma_M^2+(L+b)^2}}\sigma_M^2}{\frac{\overline{P}}{\sigma_M^2+(L+b)^2}\sigma_M^2+\sigma_W^2} = \frac{\sqrt{\overline{P}(\sigma_M^2+(L+b)^2)}\sigma_M^2}{\overline{P}\sigma_M^2+\sigma_W^2(\sigma_M^2+(L+b)^2)} \,,$$

$$\begin{aligned} L = -KC &= \frac{\sqrt{\overline{P}(\sigma_M^2+(L+b)^2)}\sigma_M^2}{\overline{P}\sigma_M^2+\sigma_W^2(\sigma_M^2+(L+b)^2)}(L+b)\sqrt{\frac{\overline{P}}{\sigma_M^2+(L+b)^2}} \\ &= \frac{\overline{P}(L+b)\sigma_M^2}{\overline{P}\sigma_M^2+\sigma_W^2(\sigma_M^2+(L+b)^2)} \\ \Rightarrow & L\left(\overline{P}\sigma_M^2+\sigma_W^2(\sigma_M^2+(L+b)^2)\right) = \overline{P}(L+b)\sigma_M^2 \\ \Rightarrow & L\sigma_W^2(\sigma_M^2+(L+b)^2) = \overline{P}b\sigma_M^2 \,. \end{aligned} \tag{2.40}$$

By solving the cubic equation (2.40), we can find a real solution for L, then A, K and C can be obtained.

For the informative equilibria, (2.39) must be satisfied.

$$\overline{P} \overset{?}{<} \frac{\sigma_M^2+(L+b)^2}{K^2} = \frac{\sigma_M^2+(L+b)^2}{\frac{\overline{P}(\sigma_M^2+(L+b)^2)\sigma_M^4}{\left(\overline{P}\sigma_M^2+\sigma_W^2(\sigma_M^2+(L+b)^2)\right)^2}} = \frac{\left(\overline{P}\sigma_M^2+\sigma_W^2(\sigma_M^2+(L+b)^2)\right)^2}{\overline{P}\sigma_M^4} > \overline{P}$$

Thus, the condition (2.39) for the informative equilibria always holds. Also we can find always at least one real L which implies that the equilibrium is informative.

2.7.9 Proof of Theorem 2.4.8

Due to the Stackelberg assumption, the encoder knows that the optimal decoder policy will be $u = \mathbb{E}[m|y]$. Then, by the law of the iterated expectations, the encoder cost $J^e = \mathbb{E}[(m-u-b)^2 + \lambda x^2]$ reduces to $J^e = \mathbb{E}[(m-u)^2] + \lambda\mathbb{E}[x^2] + b^2$. Let $P \triangleq \mathbb{E}[x^2]$, then, by following a similar approach to (2.23), the lower bound on the encoder cost can be achieved as follows:

$$J^e \geq b^2 + \frac{\sigma_M^2}{1 + P/\sigma_W^2}\,. \tag{2.41}$$

Since $\frac{\sigma_M^2}{1+P/\sigma_W^2}$ is a decreasing function of P, in order to minimize the lower bound in (2.41), P should be chosen as maximum; i.e., $P = \overline{P}$. Further, the lower bound (2.41) can be achieved when the encoder and the decoder use linear policies jointly, thus there always exist a linear equilibrium in the static quadratic signaling game with a hard-power constraint.

As a further demonstration, we will show that, the linear policies achieve the lower bound in (2.41) indeed. Let the encoder policy be $x = \gamma^e(m) = Am + C$. Then, the hard power constraint becomes $\mathbb{E}[x^2] = A^2\sigma_M^2 + C^2 \leq \overline{P}$. Due to the Stackelberg assumption, the encoder knows that the decoder will use $\gamma^d(y) = u = \mathbb{E}[m|y]$ as an optimal decoder policy to minimize the decoder cost, thus $u = \gamma^d(y) = \frac{A\sigma_M^2}{A^2\sigma_M^2+\sigma_W^2}(y - C)$ where $y = Am + C + w$. Then, by following a similar approach to (2.25), the goal of the encoder can be found as follows:

$$J^{*,e} = \min_{A,\,C} \frac{\sigma_M^2\sigma_W^2}{A^2\sigma_M^2 + \sigma_W^2} + b^2 \text{ s.t. } A^2\sigma_M^2 + \sigma_W^2 \leq \overline{P}\,. \tag{2.42}$$

$J^{*,e}$ is minimized for $C^* = 0$ and maximum A^*, which can be chosen as $(A^*)^2\,\sigma_M^2 + (C^*)^2 = \overline{P} \Rightarrow A^* = \sqrt{\frac{\overline{P}}{\sigma_M^2}}$. Then, the encoder cost becomes $J^{*,e} = \frac{\sigma_M^2\sigma_W^2}{\overline{P}+\sigma_W^2} = \frac{\sigma_M^2}{1+\overline{P}/\sigma_W^2}$, which is the lower bound in (2.41), as expected.

2.7.10 Proof of Theorem 2.5.1

(i) Let the affine encoding policy be $\mathbf{x} = \gamma^e(\mathbf{m}) = A\mathbf{m} + \mathbf{C}$ where A is an $n \times n$ matrix and $\mathbf{C}$ is an $n \times 1$ vector. Then $\mathbf{y} = \mathbf{x} + \mathbf{w} = A\mathbf{m} + \mathbf{C} + \mathbf{w}$. The optimal cost of the decoder, by the law of the iterated expectations, can be expressed as $J^{*,d} = \min_{\mathbf{u}=\gamma^d(\mathbf{y})} \mathbb{E}\left[\|\mathbf{m}-\mathbf{u}\|^2|\mathbf{y}\right]$. Hence, a minimizer policy of the decoder is $\mathbf{u} = \gamma^{*,d}(\mathbf{y}) = \mathbb{E}\left[\mathbf{m}|\mathbf{y}\right]$. Since both $\mathbf{m}$ and $\mathbf{y}$ are Gaussian, then the optimal decoder is

$$\begin{aligned} E\left[\mathbf{m}|\mathbf{y}\right] &= \mathbb{E}[\mathbf{m}] + \Sigma_{\mathbf{MY}}\Sigma_{\mathbf{YY}}^{-1}(\mathbf{y}-\mathbb{E}[\mathbf{y}]) \\ &= \Sigma_{\mathbf{M}}A^T\left(A\Sigma_{\mathbf{M}}A^T + \Sigma_{\mathbf{W}}\right)^{-1}(\mathbf{y}-\mathbf{C}) \triangleq Ky + L\,. \end{aligned} \tag{2.43}$$

(ii) Let the affine decoding policy be $\mathbf{u} = \gamma^d(\mathbf{y}) = K\mathbf{y} + \mathbf{L}$ where K is an $n \times n$ matrix and $\mathbf{L}$ is an $n \times 1$ vector. Then $\mathbf{u} = K\mathbf{y} + \mathbf{L} = K(\mathbf{x}+\mathbf{w}) + \mathbf{L} = K\gamma^e(\mathbf{m}) + K\mathbf{w} + \mathbf{L}$. By using the completion of the squares method, the optimal cost is

$$\begin{aligned} J^{*,e} &= \min_{\mathbf{x}=\gamma^e(\mathbf{m})} \mathbb{E}\left[\|\mathbf{m}-\mathbf{u}-\mathbf{b}\|^2 + \lambda\|x\|^2\right] \\ &= \min_{\mathbf{x}=\gamma^e(\mathbf{m})} \mathbb{E}\left[\|\mathbf{m}-\mathbf{u}-\mathbf{b}\|^2 + \lambda\|x\|^2\Big|\mathbf{m}\right] \\ &= \min_{\gamma^e(\mathbf{m})} \mathbb{E}\Big[\|\mathbf{m}-K\gamma^e(\mathbf{m})-K\mathbf{w}-\mathbf{L}-\mathbf{b}\|^2 + \lambda\|\gamma^e(\mathbf{m})\|^2\Big|\mathbf{m}\Big] \\ &= \min_{\gamma^e(\mathbf{m})} \mathbb{E}\Bigg[\Big(\mathbf{m}-K\gamma^e(\mathbf{m})-K\mathbf{w}-\mathbf{L}-\mathbf{b}\Big)^T\Big(\mathbf{m}-K\gamma^e(\mathbf{m})-K\mathbf{w}-\mathbf{L}-\mathbf{b}\Big) \\ &\qquad + \lambda\Big(\gamma^e(\mathbf{m})\Big)^T\Big(\gamma^e(\mathbf{m})\Big)\Big|\mathbf{m}\Bigg] \\ &= \min_{\gamma^e(\mathbf{m})} \mathbb{E}\Bigg[\Big(\mathbf{m}-K\gamma^e(\mathbf{m})-\mathbf{L}-\mathbf{b}\Big)^T\Big(\mathbf{m}-K\gamma^e(\mathbf{m})-\mathbf{L}-\mathbf{b}\Big) \\ &\qquad + \lambda\Big(\gamma^e(\mathbf{m})\Big)^T\Big(\gamma^e(\mathbf{m})\Big)\Big|\mathbf{m}\Bigg] + \mathbb{E}\Big[\mathbf{w}^TK^TK\mathbf{w}\Big] \\ &= \min_{\gamma^e(\mathbf{m})} \mathbb{E}\Bigg[\Big(\mathbf{m}-\mathbf{L}-\mathbf{b}\Big)^T\Big(\mathbf{m}-\mathbf{L}-\mathbf{b}\Big) - 2\Big(\mathbf{m}-\mathbf{L}-\mathbf{b}\Big)^TK\gamma^e(\mathbf{m}) \\ &\qquad + \Big(\gamma^e(\mathbf{m})\Big)^TK^TK\Big(\gamma^e(\mathbf{m})\Big) + \lambda\Big(\gamma^e(\mathbf{m})\Big)^T\Big(\gamma^e(\mathbf{m})\Big)\Big|\mathbf{m}\Bigg] \\ &\qquad + \mathbb{E}\Big[\mathbf{w}^TK^TK\mathbf{w}\Big] \\ &= \min_{\gamma^e(\mathbf{m})} \mathbb{E}\Bigg[\Big((K^TK+\lambda I)\gamma^e(\mathbf{m}) - K^T(\mathbf{m}-\mathbf{L}-\mathbf{b})\Big)^T\Big(K^TK+\lambda I\Big)^{-1} \\ &\qquad \Big((K^TK+\lambda I)\gamma^e(\mathbf{m}) - K^T(\mathbf{m}-\mathbf{L}-\mathbf{b})\Big) + \Big(\mathbf{m}-\mathbf{L}-\mathbf{b}\Big)^T \end{aligned}$$

$$\Big(I - K(K^TK + \lambda I)^{-1}K^T\Big)\Big(\mathbf{m} - \mathbf{L} - \mathbf{b}\Big)\Big|\mathbf{m}\Big] + \mathbb{E}\Big[\mathbf{w}^T K^T K \mathbf{w}\Big]. \tag{2.44}$$

Hence, the optimal $\gamma^e(m)$ can be chosen as follows:

$$\gamma^{*,e}(\mathbf{m}) = \Big(K^TK + \lambda I\Big)^{-1} K^T \Big(\mathbf{m} - \mathbf{L} - \mathbf{b}\Big) \triangleq A\mathbf{m} + \mathbf{C}. \tag{2.45}$$

(iii) We have $K = \Sigma_{\mathbf{M}} A^T \left(A \Sigma_{\mathbf{M}} A^T + \Sigma_{\mathbf{W}}\right)^{-1}$ and $A = \Big(K^TK + \lambda I\Big)^{-1} K^T$ from (2.43) and (2.45). By combining these, $A = T(A) = \left(FF^T + \lambda I\right)^{-1} F$ can be obtained.

(iv) Since FF^T is a real and symmetric matrix, it is diagonalizable and can be written as $FF^T = Q\Upsilon Q^{-1}$ for a diagonal Υ. Now consider $\|T(A)\|_F$ where $\|\cdot\|_F$ denotes the Frobenius norm:

$$\begin{aligned}
\|T(A)\|_F &= \mathrm{tr}\left(\left(\left(FF^T + \lambda I\right)^{-1} F\right)\left(\left(FF^T + \lambda I\right)^{-1} F\right)^T\right) \\
&= \mathrm{tr}\left(\left(FF^T + \lambda I\right)^{-1} FF^T \left(FF^T + \lambda I\right)^{-1}\right) \\
&= \mathrm{tr}\left(\left(Q\Upsilon Q^{-1} + \lambda I\right)^{-1} Q\Upsilon Q^{-1} \left(Q\Upsilon Q^{-1} + \lambda I\right)^{-1}\right) \\
&= \mathrm{tr}\left(\left(Q(\Upsilon + \lambda I)Q^{-1}\right)^{-1} Q\Upsilon Q^{-1} \left(Q(\Upsilon + \lambda I)Q^{-1}\right)^{-1}\right) \\
&= \mathrm{tr}\left(Q(\Upsilon + \lambda I)^{-1}Q^{-1}Q\Upsilon Q^{-1}Q(\Upsilon + \lambda I)^{-1}Q^{-1}\right) \\
&= \mathrm{tr}\left(Q(\Upsilon + \lambda I)^{-1}\Upsilon(\Upsilon + \lambda I)^{-1}Q^{-1}\right) \\
&= \mathrm{tr}\left((\Upsilon + \lambda I)^{-1}\Upsilon(\Upsilon + \lambda I)^{-1}Q^{-1}Q\right) \\
&= \mathrm{tr}\left((\Upsilon + \lambda I)^{-1}\Upsilon(\Upsilon + \lambda I)^{-1}\right) \\
&= \sum_{i=1}^{n} \frac{\upsilon_i}{(\upsilon_i + \lambda)^2}\,,
\end{aligned} \tag{2.46}$$

where $\upsilon_i, i = 1, \ldots, n$ are the eigenvalues of FF^T and since FF^T is positive semi-definite, all these eigenvalues are nonnegative. Since $\lambda > 0$, we observe the following:

$$\upsilon_i \in [0,1] \Rightarrow \frac{\upsilon_i}{(\upsilon_i+\lambda)^2} < \frac{1}{\lambda^2},$$
$$\upsilon_i \in (1,\infty) \Rightarrow \frac{\upsilon_i}{(\upsilon_i+\lambda)^2} < \frac{\upsilon_i}{\upsilon_i^2} = \frac{1}{\upsilon_i} < 1.$$

Hence, $\upsilon_i/(\upsilon_i+\lambda)^2 < \max(1, 1/\lambda^2)$ always holds. Then, by (2.46), we have $\|T(A)\|_F < n\max(1, 1/\lambda^2)$, which implies that $T(A)$ can be viewed as a continuous function mapping the compact convex set $\|A\|_F \in [0, n\max(1, 1/\lambda^2)]$ to itself. Therefore, by Brouwer's fixed point theorem [84], there exists $A = T(A)$.

2.7.11 Proof of Theorem 2.5.2

Since the source components are independent and the noise components are independent, the n-dimensional noisy signaling game problem turns into n independent scalar noisy signaling game problems as follows:

(i) If the decoder uses the channels independently; i.e., $u_i = \gamma_i^d(y_i)$ for $i = 1, \ldots, n$, then the optimal cost of the encoder will be

$$\begin{aligned} J^{*,e} &= \min_{\mathbf{x}=\gamma^e(\mathbf{m})} \mathbb{E}\left[\|\mathbf{m}-\mathbf{u}-\mathbf{b}\|^2 + \lambda\|\mathbf{x}\|^2\right] \\ &= \min_{\mathbf{x}=\gamma^e(\mathbf{m})} \sum_{i=1}^{n} \mathbb{E}[(m_i - \gamma_i^d(y_i) - b_i)^2 + \lambda x_i^2] \\ &= \sum_{i=1}^{n} \min_{\mathbf{x}=\gamma^e(\mathbf{m})} \mathbb{E}[(m_i - \gamma_i^d(y_i) - b_i)^2 + \lambda x_i^2]. \end{aligned}$$

Since, $y_i = x_i + w_i$ for each $i = 1, \ldots, n$, the optimal encoder also uses the channels independently; i.e., $x_i = \gamma_i^e(m_i)$ for $i = 1, \ldots, n$.

(ii) Similarly, if the encoder uses the channels independently; i.e., $x_i = \gamma_i^e(m_i)$ for $i = 1, \ldots, n$, then the optimal cost of the decoder will be

$$J^{*,d} = \min_{\mathbf{u}=\gamma^d(\mathbf{y})} \mathbb{E}\left[\|\mathbf{m}-\mathbf{u}\|^2\right]$$

$$= \sum_{i=1}^{n} \min_{\mathbf{u}=\gamma^d(\mathbf{y})} \mathbb{E}[(m_i - u_i)^2] \, .$$

Since, $y_i = \gamma_i^e(m_i) + w_i$ for each $i = 1, \ldots, n$, the optimal decoder will also use channels independently; i.e., $u_i = \gamma_i^d(y_i)$ for $i = 1, \ldots, n$.

Thus, we have, in each dimension i ($i = 1, \ldots, n$);

(i) the source M_i is a zero-mean Gaussian with variance $\sigma^2_{m_i}$,

(ii) the channel has the Gaussian noise W_i with zero-mean and variance $\sigma^2_{w_i}$,

(iii) the encoder's goal is to find the optimal policy which minimizes its cost $\min_{x_i=\gamma^e(m_i)} \mathbb{E}[(m_i - u_i - b_i)^2 + \lambda x_i^2]$,

(iv) the decoder's goal is to find the optimal policy which minimizes its cost $\min_{u_i=\gamma^d(y_i)} \mathbb{E}[(m_i - u_i)^2]$.

For each dimension, the informative affine equilibrium exists if $\lambda < \sigma^2_{m_i}/\sigma^2_{w_i}$. For the multi-dimensional setup, the existence of the informative equilibrium in at least one dimension implies the existence of the informative equilibrium for the whole sytem. Hence, it is sufficient that the inequality $\lambda < \sigma^2_{m_i}/\sigma^2_{w_i}$ is valid for at least one dimension. As a result, the condition for the existence of the informative affine equilibrium becomes $\lambda < \max\{\frac{\sigma^2_{m_1}}{\sigma^2_{w_1}}, \ldots, \frac{\sigma^2_{m_n}}{\sigma^2_{w_n}}\}$.

Chapter 3

Multi-Stage Quadratic Cheap Talk and Signaling Games under Subjective Models

In this chapter Nash and Stackelberg equilibria of multi-stage scalar and multi-dimensional quadratic cheap talk and signaling games are investigated under the consistent and subjective priors assumptions. A characterization on when informative equilibria exist, and robustness properties to misalignment are presented under Nash and Stackelberg criteria.

The main contributions of this chapter can be summarized as follows:

(i) We show that in the multi-stage cheap talk game under Nash equilibria, the last stage equilibria are quantized for i.i.d. and Markov sources with arbitrary conditional probability measures, whereas the equilibrium must be fully revealing in the multi-stage scalar cheap talk game under Stackelberg equilibria. Further, for i.i.d. sources, the quantized nature of the Nash equilibrium for all stages is established under mild conditions. We further show that the equilibria are fully revealing in the multi-stage multi-dimensional cheap talk under Stackelberg equilibria whereas the equilibrium cannot be fully revealing under Nash equilibria.

(ii) For the multi-stage signaling game under Nash equilibria, it is shown that affine encoder and decoder policies constitute an invariant subspace under best response dynamics. We provide conditions for the existence of informative Stackelberg equilibria for scalar Gauss-Markov sources and scalar Gaussian channels where we also show that Stackelberg equilibria are always linear for scalar sources and channels, which is not always the case for multi-dimensional setups. For multi-dimensional setups, a dynamic programming formulation is presented for Stackelberg equilibria when the encoders are linear.

(iii) For the case where the encoder and the decoder have subjective priors on the source distribution, under identical costs, provided that the priors are mutually absolutely continuous (that is, both measures agree the set of sets with zero measure; i.e. the Radon-Nikodym derivative of either measure with respect to the other exists), we show that there exist fully informative Nash and Stackelberg equilibria for the dynamic cheap talk as in the team theoretic setup. Thus, the equilibrium behavior is robust to perturbations in the priors, which is not necessarily the case for the perturbations in the cost models. On the other hand, for the signaling game, Stackelberg equilibrium policies are robust to perturbations in the cost but not to the priors considered in this chapter.

3.1 Problem Formulation

In this chapter, the problems are investigated where the encoder and the decoder are deterministic rather than randomized; i.e., $\gamma^e(\mathrm{d}x|m) = \mathbb{1}_{\{f^e(m)\in \mathrm{d}x\}}$ and $\gamma^d(\mathrm{d}u|x) = \mathbb{1}_{\{f^d(x)\in \mathrm{d}u\}}$, and $f^e(m)$ and $f^d(x)$ are some deterministic functions of the encoder and decoder, respectively. The policies of the encoder and decoder are assumed to be deterministic; i.e., $x = \gamma^e(m)$ and $u = \gamma^d(x) = \gamma^d(\gamma^e(m))$.

In Chapter 2, only *single-stage game*s are considered. If a game is played over a number of time periods, the game is called a *multi-stage game*. In this dissertation, with the term *dynamic*, we will refer to *multi-stage game* setups, which also has been the usage in the prior literature [54]; even though strictly speaking a single stage setup may also be viewed to be dynamic [22] since the information available to the decoder is totally determined by encoder's actions.

Let $m_{[0,N-1]} = \{m_0, m_1, \ldots, m_{N-1}\}$ be a collection of random variables to be encoded sequentially (causally) to a decoder. At the k-th stage of an N-stage game, the encoder knows the values of $\mathcal{I}_k^e = \{m_{[0,k]}, x_{[0,k-1]}\}$ with $\mathcal{I}_0^e = \{m_0\}$, and transmits x_k to the decoder who generates his optimal decision by knowing the values of $\mathcal{I}_k^d = \{x_{[0,k]}\}$. Thus, under the policies considered, $x_k = \gamma_k^e(\mathcal{I}_k^e)$ and $u_k = \gamma_k^d(\mathcal{I}_k^d)$. The encoder's goal is to minimize

$$J^e\left(\gamma^e_{[0,N-1]}, \gamma^d_{[0,N-1]}\right) = \mathbb{E}\left[\sum_{k=0}^{N-1} c_k^e(m_k, u_k)\right], \tag{3.1}$$

whereas the decoder's goal is to minimize

$$J^d\left(\gamma^e_{[0,N-1]}, \gamma^d_{[0,N-1]}\right) = \mathbb{E}\left[\sum_{k=0}^{N-1} c_k^d(m_k, u_k)\right] \tag{3.2}$$

by finding the optimal policy sequences $\gamma^{*,e}_{[0,N-1]} = \{\gamma_0^{*,e}, \gamma_1^{*,e}, \cdots, \gamma_{N-1}^{*,e}\}$ and $\gamma^{*,d}_{[0,N-1]} = \{\gamma_0^{*,d}, \gamma_1^{*,d}, \cdots, \gamma_{N-1}^{*,d}\}$, respectively. Using the encoder cost in (3.1) and the decoder cost in (3.2), the Nash equilibrium and the Stackelberg equilibrium for multi-stage games can be defined similarly as in (1.1) and (1.2), respectively.

Under both equilibria concepts, we consider the setups where the decision makers act optimally for each history path of the game (available to each decision maker) and the updates are Bayesian; thus the equilibria are to be interpreted under a *perfect Bayesian equilibria* concept. Since we assume such a (perfect Bayesian) framework, the equilibria lead to sub-game perfection and each decision maker performs optimal Bayesian decisions for every realized play path. For example, more general Nash equilibrium scenarios such as non-credible threats [17] or equilibria that are not strong time-consistent [85], [4, Definition 2.4.1] may not be considered.

Subjectivity in the model can also manifest itself in the probability measures, in particular the encoder and the decoder may have subjective/inconsistent priors with respect to the probability measure which defines the random source $\{m_k\}$. We note that decentralized stochastic control with subjective priors has been studied extensively in the literature [81,86–88]. In this setup, players may have a common goal but subjective prior information, which necessarily alters the setup from a team problem to a game problem involving strategy/policy spaces. Building on this motivation, in Section 3.6, we investigate such a setup in the context of quadratic cost criteria where the encoder and the decoder have subjective priors.

In this chapter, the quadratic cost functions are assumed; i.e., $c_k^e(m_k, u_k) = (m_k - u_k - b)^2$ and $c_k^d(m_k, u_k) = (m_k - u_k)^2$ where b is the bias term as in the previous chapter.

3.2 Multi-Stage Scalar Quadratic Cheap Talk

For the purpose of illustration, the system model of the 2-stage cheap talk is depicted in Fig. 3.1.

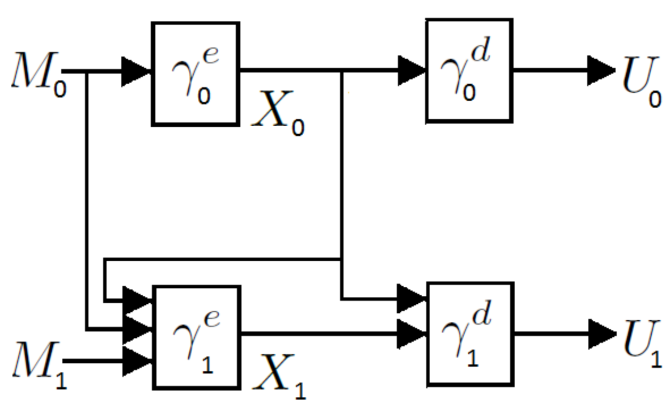

Figure 3.1: 2-stage cheap talk.

3.2.1 Nash Equilibrium Analysis

As a prelude to the more general Markov source setup, we first analyze the multi-stage cheap talk game with an i.i.d. scalar source.

3.2.1.1 Multi-Stage Game with an i.i.d. Source

Theorem 3.2.1. *In the N-stage repeated cheap talk game, the equilibrium policies for the final stage encoder must be quantized for any collection of policies $\left(\gamma^e_{[0,N-2]}, \gamma^d_{[0,N-2]}\right)$ and for any real-valued source model with arbitrary probability measure $P(\mathrm{d}m_{N-1})$.*

Proof. Here, we prove the result for the 2-stage setup, the extension to multiple stages is merely technical, as we comment on at the end of the proof.

Let $c^e_1(m_1, u_1)$ be the second stage cost function of the encoder. Then the expected cost of the second stage encoder J^e_1 can be written as follows:

$$\begin{aligned} J^e_1 &= \int P(\mathrm{d}m_0, \mathrm{d}m_1, \mathrm{d}x_0, \mathrm{d}x_1)\, c^e_1(m_1, u_1) \\ &= \int P(\mathrm{d}x_0)\, P(\mathrm{d}m_1|x_0)\, P(\mathrm{d}m_0|m_1, x_0) P(\mathrm{d}x_1|m_0, m_1, x_0)\, c^e_1(m_1, u_1) \\ &\overset{(a)}{=} \int P(\mathrm{d}x_0) \int P(\mathrm{d}m_1)\, P(\mathrm{d}m_0|x_0) \mathbb{1}_{\{x_1 = \gamma^e_1(m_0, m_1, x_0)\}}\, c^e_1(m_1, u_1) \\ &= \int P(\mathrm{d}x_0) \int P(\mathrm{d}m_1)\, P(\mathrm{d}m_0|x_0) c^e_1(m_1, \gamma^d_1(x_0, \gamma^e_1(m_0, m_1, x_0)))\ . \end{aligned} \tag{3.3}$$

Here, (a) holds due to the i.i.d. source and the deterministic encoder assumptions. The inner integral of (3.3) can be considered as an expression for a given x_0. Thus, given the second stage encoder and decoder policies $\gamma^e_1(m_0, m_1, x_0)$ and $\gamma^d_1(x_0, x_1)$, it is possible to define policies which are parametrized by the common information x_0 almost surely so that $\widehat{\gamma}^e_{x_0}(m_0, m_1) \triangleq \gamma^e_1(m_0, m_1, x_0)$ and $\widehat{\gamma}^d_{x_0}(x_1) \triangleq \gamma^d_1(x_0, x_1)$.

Now fix the first stage policies γ^e_0 and γ^d_0. Suppose that the second stage encoder does not use m_0; i.e., $\widehat{\gamma}^{e\prime}_{x_0}(m_1)$ is the policy of the second stage encoder. For

the policies $\widehat{\gamma}^{e\prime}_{x_0}(m_1)$ and $\widehat{\gamma}^{d}_{x_0}(x_1)$, by using the second stage encoder cost function $H_{x_0}(m_1, u_1) \triangleq \mathbb{E}[(m_1 - u_1 - b)^2 | x_0]$ and the bin arguments from Theorem 2.2.3, it can be deduced that, due to the continuity of $H_{x_0}(m_1, u_1)$ in m_1, the equilibrium policies for the second stage must be quantized for any collection of policies (γ^e_0, γ^d_0) and for any given x_0. Now let the second stage encoder use m_0; i.e., $\widehat{\gamma}^{e}_{x_0}(m_0, m_1)$ is the policy of the second stage encoder. Here, even if $\widehat{\gamma}^{e}_{x_0}(m_0, m_1)$ is a deterministic policy, it can be regarded as an equivalent randomized encoder policy (as a stochastic kernel from $\mathbb{M}_1$ to $\mathbb{X}_1$) where m_0 is a real valued random variable independent of the source, m_1. From Theorem 2.2.4, the equilibrium is achievable with an encoder policy which uses only m_1; i.e., $\widehat{\gamma}^{e*}_{x_0}(m_1)$ is an encoder policy at the equilibrium and thus the equilibria are quantized.
For the N-stage game, the common information of the final stage encoder and decoder becomes $x_{[0,N-2]}$, and $m_{[0,N-2]}$ is a vector valued random variable independent of the final stage source m_{N-1}. □

Before studying the equilibrium structure of the other stages, we make the following assumption on the source structure.

Assumption 3.2.2. *The source m_k is so that the single-stage cheap-talk game satisfies the following:*

(i) There exists a finite upper bound on the number of quantization bins that any equilibrium admits.

(ii) There exists finitely many equilibria corresponding to a given number of quantization bins.

A number of comments on Assumption 3.2.2 is in order: This assumption is not unrealistic, e.g., it will hold if the source is a bounded uniform source, or if the source is exponentially distributed [37]. A sufficient condition for Assumption 3.2.2 is that the source admits a bounded support (which would require by [16, Lemma 1] that there exists an upper bound on the number of bins in any equilibrium), and that a monotonicity condition (M) (or equivalently (M′)) in [16] holds, which characterize the behavior of equilibrium policies. Note though that this condition is much more than what is needed in Assumption 3.2.2, since it entails the uniqueness of equilibria for a given number of bins: The uniqueness of equilibria even for team problems with $b = 0$ requires restrictive log-concavity conditions [89, p. 1475], [90].

Theorem 3.2.3. *Suppose Assumption 3.2.2 holds. Then, all stages must have quantized equilibria in the N-stage repeated cheap talk game.*

Proof. Consider first the 2-stage setup; i.e., given that the second stage has a quantized equilibrium by Theorem 3.2.1, the quantized nature of the first stage will be established. Let $F(m_0, x_0)$ be a cost function for the first stage encoder if it encodes message m_0 as x_0. Since the second stage equilibrium cost does not depend on m_0 (since m_0 is a random variable independent of the source m_1 as shown in Theorem 3.2.1), $F(m_0, x_0)$ can be written as $F(m_0, x_0) = \big(m_0 - \gamma_0^d(x_0) - b\big)^2 + G(x_0)$ where $G(x_0) \triangleq \mathbb{E}_{m_1}\left[\left(m_1 - \gamma_1^{*,d}(x_0, \gamma_1^{*,e}(m_1, x_0)) - b\right)^2 \middle| x_0\right]$ is the expected cost of the second stage encoder, and $\gamma_1^{*,e}$ and $\gamma_1^{*,d}$ are the second stage encoder and decoder policies at the equilibrium, respectively. Note that the second stage encoder cost can take finitely many different values by Assumption 3.2.2-(ii); i.e., $G(x_0)$ can take finitely many values.

Now define the equivalence classes T_{x_p} for every $x_p \in \mathbb{X}$ as $T_{x_p} = \{x \in \mathbb{X} : G(x) = G(x_p)\}$. As it can be seen from the definition, $G(x)$ takes the first stage encoding value as input, and gives the expected cost of the second stage encoder. Thus, the equivalence classes T_{x_p} keep the first stage encoder actions that result in the same second stage cost in the same set. Note that there are finitely many equivalence classes T_{x_p} since $G(x_0)$ can take finitely many different values.

If the number of bins of the first stage equilibrium is less than or equal to the number of the equivalence classes T_{x_p}, then the proof is complete; i.e., the first stage equilibrium is already quantized with finitely may bins. Otherwise; i.e., the number of bins of the first stage equilibrium is greater than the number of the equivalence classes T_{x_p}, then one of the equivalence classes T_{x_p} has at least two elements; say x_0^α and x_0^β, that implies $G(x_0^\alpha) = G(x_0^\beta)$. Let corresponding bins of the actions x_0^α and x_0^β be $\mathcal{B}_0^\alpha$ and $\mathcal{B}_0^\beta$, respectively. Also let m_0^α indicate any point in $\mathcal{B}_0^\alpha$; i.e., $m_0^\alpha \in \mathcal{B}_0^\alpha$. Similarly, let m_0^β represent any point in $\mathcal{B}_0^\beta$; i.e., $m_0^\beta \in \mathcal{B}_0^\beta$. The decoder chooses an action $u_0^\alpha = \gamma_0^d(x_0^\alpha)$ when the encoder sends $x_0^\alpha = \gamma_0^e(m_0^\alpha)$, and an action $u_0^\beta = \gamma_0^d(x_0^\beta)$ when the encoder sends $x_0^\beta = \gamma_0^e(m_0^\beta)$ in order to minimize his total cost; further, we can assume that $u_0^\alpha < u_0^\beta$ without loss of generality. Due to the equilibrium definitions from the view of the encoder, $F(m_0^\alpha, x_0^\alpha) < F(m_0^\alpha, x_0^\beta)$ and $F(m_0^\beta, x_0^\beta) < F(m_0^\beta, x_0^\alpha)$. These inequalities imply that

$$\begin{aligned}
&(m_0^\alpha - u_0^\alpha - b)^2 + G(x_0^\alpha) < (m_0^\alpha - u_0^\beta - b)^2 + G(x_0^\beta) \\
&\Rightarrow (u_0^\alpha - u_0^\beta)(u_0^\alpha + u_0^\beta - 2(m_0^\alpha - b)) < 0 \ , \\
&(m_0^\beta - u_0^\beta - b)^2 + G(x_0^\beta) < (m_0^\beta - u_0^\alpha - b)^2 + G(x_0^\alpha) \\
&\Rightarrow (u_0^\beta - u_0^\alpha)(u_0^\alpha + u_0^\beta - 2(m_0^\beta - b)) < 0 \ .
\end{aligned}$$

Thus, we have $u_0^\alpha + u_0^\beta - 2(m_0^\alpha - b) > 0$ and $u_0^\alpha + u_0^\beta - 2(m_0^\beta - b) < 0$ that reduce to $m_0^\alpha < \frac{u_0^\alpha + u_0^\beta}{2} + b < m_0^\beta$. Since $u^\alpha = \mathbb{E}[m | m \in \mathcal{B}^\alpha]$ and $u^\beta = \mathbb{E}[m | m \in \mathcal{B}^\beta]$ at the equilibrium, $u_0^\alpha < \frac{u_0^\alpha + u_0^\beta}{2} + b < u_0^\beta$ is obtained. After simplifications, the last inequality becomes $u_0^\beta - u_0^\alpha > 2|b|$. Hence, there must be at least $2|b|$ distance between the actions of the first stage decoder that are in the same equivalence class. Therefore, the cardinality of any equivalence class T_{x_p} is finite due to Assumption 3.2.2-(i). Further, there are finitely many equivalence classes T_{x_p} as shown above. These two results imply that the first stage equilibrium must be quantized with finitely many bins. Due to Assumption 3.2.2, the quantized equilibrium with finitely many bins in the first stage implies that there are finitely many equilibria in the first stage; i.e., the first stage encoder cost can take finitely many values.

For the N-stage game, we apply the similar recursion from the final stage to the first stage. It is already proven that the last two stage encoder cost can take finitely many values; thus, the same methods can be applied to show the quantized structure (with finitely many bins) of the equilibria for all stages. □

Remark 3.2.1. *It is important to note that the first stage encoder minimizes his expected cost $J_0^e = \mathbb{E}[F(m_0, x_0)]$ by minimizing his cost $F(m_0, x_0)$ for every realizable m_0; this property will be later used as well.*

3.2.1.2 Multi-Stage Game with a Markov Source

Here, the source M_k is assumed to be real valued Markovian for $k = 0, 1, \ldots, N-1$. The following result generalizes Theorem 3.2.1, which only considered i.i.d. sources.

Theorem 3.2.4. *In the N-stage cheap talk game with a Markov source, the equilibrium policies for the final stage encoder must be quantized for any collection of policies $\left(\gamma^e_{[0,N-2]}, \gamma^d_{[0,N-2]}\right)$ and for any real-valued source model with arbitrary probability measure.*

Proof. Here, we prove the results for the 2-stage games as the extension is merely technical. The expected cost of the second stage encoder J^e_1 can be written as follows similar to that in Theorem 3.2.1:

$$\begin{aligned} J^e_1 &= \int P(\mathrm{d}m_0, \mathrm{d}m_1, \mathrm{d}x_0, \mathrm{d}x_1)\, c^e_1(m_1, u_1) \\ &= \int P(\mathrm{d}x_0) \int P(\mathrm{d}m_1|x_0)\, P(\mathrm{d}m_0|m_1, x_0) c^e_1(m_1, \gamma^d_1(x_0, \gamma^e_1(m_0, m_1, x_0)))\,. \end{aligned}$$

After following similar arguments to those in the proof of Theorem 3.2.1, the second stage encoder policy becomes $\widehat{\gamma}^e_{x_0}(m_0, m_1) \overset{(a)}{=} \widehat{\gamma}^e_{x_0}(g(m_1, r), m_1) = \widetilde{\gamma}^e_{x_0}(m_1, r)$ where (a) holds since any stochastic kernel from a complete, separable and metric space to another one, $P(\mathrm{d}m_0|m_1)$, can be realized by some measurable function $m_0 = g(m_1, r)$ where r is a $[0, 1]$-valued independent random variable (see [91, Lemma 1.2], or in [92, Lemma 3.1]). Hence, the equilibria are quantized by Theorem 3.2.1. □

As it can be observed from Theorem 3.2.3, to be able to claim that the equilibria for all stages are quantized, we require very strong conditions. In fact, in the absence of such conditions, the equilibria for a Markov source can be quite counterintuitive and even fully revealing as we observe in the following theorem due to [54].

Theorem 3.2.5. *[54] There exist multi-stage cheap talk games with a Markov source which admit fully revealing equilibria.*

Proof. An example is presented in Golosov et. al. [54], where an individual source is transmitted repeatedly (thus the Markov source is a constant source) for a sufficiently small bias value. For such a source, the terminal stage conditional measure can be made atomic via a careful construction of equilibrium policies for earlier time stages; i.e., the defined separable groups/types and discrete/quantized stage-wise equilibria through multiple stages can lead to a fully informative equilibrium for the complete game. □

3.2.2 Stackelberg Equilibrium Analysis

In this part, the cheap talk game is analyzed under the Stackelberg assumption; i.e., the encoder knows the policy of the decoder. In this case, admittedly the problem is less interesting.

Theorem 3.2.6. *An equilibrium has to be fully revealing in the multi-stage Stackelberg cheap talk game regardless of the source model.*

Proof. This result follows as a special case of Theorem 3.3.2, but the proof is provided for completeness.

We will use the properties of iterated expectations in the analysis. Recall that the total decoder cost is $J^d(\gamma^e_{[0,N-1]}, \gamma^d_{[0,N-1]}) = \mathbb{E}\left[\sum_{k=0}^{N-1}(m_k - u_k)^2\right]$. Considering the last stage, the goal of the decoder is to minimize $J^d_{N-1}(\gamma^e_{N-1}, \gamma^d_{N-1}) = \mathbb{E}[(m_{N-1} - u_{N-1})^2 | \mathcal{I}^d_{N-1}]$ by choosing the optimal action $u^*_{N-1} = \gamma^{*,d}_{N-1}(\mathcal{I}^d_{N-1}) = \mathbb{E}[m_{N-1}|\mathcal{I}^d_{N-1}]$. For the previous stage, the goal of the decoder is to minimize $J^d_{N-2}(\gamma^{*,e}_{N-1}, \gamma^e_{N-2}, \gamma^{*,d}_{N-1}, \gamma^d_{N-2}) = \mathbb{E}[(m_{N-2} - u_{N-2})^2 + J^{*,d}_{N-1}(\gamma^{*,e}_{N-1}, \gamma^{*,d}_{N-1})|\mathcal{I}^d_{N-2}]$ by choosing the optimal action $u^*_{N-2} = \gamma^{*,d}_{N-2}(\mathcal{I}^d_{N-2})$. Since $J^{*,d}_{N-1}(\gamma^{*,e}_{N-1}, \gamma^{*,d}_{N-1})$ is not affected by the choice of γ^d_{N-2}, the goal of the decoder is equivalent to the minimization of $\mathbb{E}[(m_{N-2} - u_{N-2})^2|\mathcal{I}^d_{N-2}]$ at this stage. Thus, the optimal policy is $u^*_{N-2} = \gamma^{*,d}_{N-2}(\mathcal{I}^d_{N-2}) = \mathbb{E}[m_{N-2}|\mathcal{I}^d_{N-2}]$. Similarly, since the actions taken by the decoder do not affect the future states and encoder policies, the optimal decoder actions can be found as $u^*_k = \gamma^{*,d}_k(\mathcal{I}^d_k) = \mathbb{E}[m_k|\mathcal{I}^d_k] = \mathbb{E}[m_k|x_{[0,k]}]$ for $k = 0, 1, \ldots, N-1$.

Due to the Stackelberg assumption, the encoder knows that the decoder will use $u_k^* = \gamma_k^{*,d}(\mathcal{I}_k^d) = \mathbb{E}[m_k|\mathcal{I}_k^d]$ for each stage $k = 0, 1, \ldots, N-1$. By using this assumption and the smoothing property of the expectation, the total encoder cost can be written as $J^e(\gamma^e_{[0,N-1]}, \gamma^{*,d}_{[0,N-1]}) = \mathbb{E}\left[\sum_{k=0}^{N-1}(m_k - u_k - b)^2\right] = \mathbb{E}\left[\sum_{k=0}^{N-1}(m_k - u_k)^2\right] + Nb^2$. Thus, as in the one-stage game setup [55, Theorem 3.3], the goals of the encoder and the decoder become essentially the same in the Stackelberg game setup, which effectively reduces the game setup to a team setup, resulting in fully informative equilibria; i.e., the encoder reveals all of his information. □

3.3 Multi-Stage Multi-Dimensional Quadratic Cheap Talk

In this section, Nash and Stackelberg equilibria of the multi-stage multi-dimensional cheap talk are analyzed.

3.3.1 Nash Equilibrium Analysis

Now, Nash equilibria of the multi-stage multi-dimensional cheap talk are analyzed. Since there may be discrete, non-discrete or even linear Nash equilibria in the single-stage multi-dimensional cheap talk by Theorem 2.3.1, the equilibrium policies are more difficult to characterize; however, we state the following:

Theorem 3.3.1. *The final stage Nash equilibria cannot be fully revealing in the multi-stage multi-dimensional cheap talk for i.i.d. and Markov sources when the conditional distribution* $P(\mathrm{d}\mathbf{m}_{N-1}|\mathbf{m}_{N-2})$ *has positive measure for every non-empty open set.*

Proof. The proof is the multi-dimensional extension of Theorem 3.2.1 for i.i.d. sources, and Theorem 3.2.4 for Markov sources. □

3.3.2 Stackelberg Equilibrium Analysis

Unlike the different characteristics between Nash equilibria of the multi-stage scalar and multi-dimensional cheap talk, fully revealing characteristics of the Stackelberg equilibrium still hold for the multi-stage multi-dimensional cheap talk, as for the scalar case:

Theorem 3.3.2. *The Stackelberg equilibria in the multi-stage multi-dimensional cheap talk can be obtained by extending its scalar case; i.e., it is unique and corresponds to a fully revealing encoder policy as in the scalar case.*

Proof. We will use the properties of iterated expectations in the analysis. Recall that the total decoder cost is $J^d(\gamma^e_{[0,N-1]}, \gamma^d_{[0,N-1]}) = \mathbb{E}\left[\sum\limits_{k=0}^{N-1} \|\mathbf{m}_k - \mathbf{u}_k\|^2\right]$. Considering the last stage, the goal of the decoder is to minimize $J^d_{N-1}(\gamma^e_{N-1}, \gamma^d_{N-1}) = \mathbb{E}[\|\mathbf{m}_{N-1} - \mathbf{u}_{N-1}\|^2 | \mathcal{I}^d_{N-1}]$ by choosing the optimal action $\mathbf{u}^*_{N-1} = \gamma^{*,d}_{N-1}(\mathcal{I}^d_{N-1}) = \mathbb{E}[\mathbf{m}_{N-1}|\mathcal{I}^d_{N-1}]$. For the previous stage, the goal of the decoder is to minimize $J^d_{N-2}(\gamma^{*,e}_{N-1}, \gamma^e_{N-2}, \gamma^{*,d}_{N-1}, \gamma^d_{N-2}) = \mathbb{E}[\|\mathbf{m}_{N-2} - \mathbf{u}_{N-2}\|^2 + J^{*,d}_{N-1}(\gamma^{*,e}_{N-1}, \gamma^{*,d}_{N-1}) | \mathcal{I}^d_{N-2}]$ by choosing the optimal action $\mathbf{u}^*_{N-2} = \gamma^{*,d}_{N-2}(\mathcal{I}^d_{N-2})$. Since $J^{*,d}_{N-1}(\gamma^{*,e}_{N-1}, \gamma^{*,d}_{N-1})$ is not affected by the choice of γ^d_{N-2}, the goal of the decoder is equivalent to the minimization of $\mathbb{E}[\|\mathbf{m}_{N-2} - \mathbf{u}_{N-2}\|^2 | \mathcal{I}^d_{N-2}]$ at this stage. Thus, the optimal policy is $\mathbf{u}^*_{N-2} = \gamma^{*,d}_{N-2}(\mathcal{I}^d_{N-2}) = \mathbb{E}[\mathbf{m}_{N-2}|\mathcal{I}^d_{N-2}]$. Similarly, since the actions taken by the decoder do not affect the future states and encoder policies, the optimal decoder actions can be found as $\mathbf{u}^*_k = \gamma^{*,d}_k(\mathcal{I}^d_k) = \mathbb{E}[\mathbf{m}_k|\mathcal{I}^d_k] = \mathbb{E}[\mathbf{m}_k|\mathbf{x}_{[0,k]}]$ for $k = 0, 1, \ldots, N-1$.

Due to the Stackelberg assumption, the encoder knows that the decoder will use $\mathbf{u}^*_k = \gamma^{*,d}_k(\mathcal{I}^d_k) = \mathbb{E}[\mathbf{m}_k|\mathcal{I}^d_k]$ for each stage $k = 0, 1, \ldots, N-1$. By using this assumption and the smoothing property of the expectation, the total encoder cost can be written as $J^e(\gamma^e_{[0,N-1]}, \gamma^{*,d}_{[0,N-1]}) = \mathbb{E}\left[\sum\limits_{k=0}^{N-1} \|\mathbf{m}_k - \mathbf{u}_k - \mathbf{b}\|^2\right] = \mathbb{E}\left[\sum\limits_{k=0}^{N-1} \|\mathbf{m}_k - \mathbf{u}_k\|^2\right] + N\|b\|^2$. Thus, as in the scalar setup Theorem 3.2.6, the goals of the encoder and the decoder become essentially the same in the Stackelberg game setup, which effectively reduces the game setup to a team setup, resulting in fully informative equilibria; i.e., the encoder reveals all of his information. $\square$

As in the scalar case, the equilibria under the Nash and Stackelberg assumptions are drastically different: There cannot be fully revealing Nash equilibria in the multi-stage multi-dimensional cheap talk whereas the equilibrium is always fully revealing under the Stackelberg assumption.

Remark 3.3.1. *Note that there is no constraint on the source and channel dimensions in Section 3.3: the source and channel dimensions can be different, and further, the source and channel dimensions can change at every stage; i.e., Theorem 3.3.1 and Theorem 3.3.2 hold for an n_k-dimensional source and an r_k-dimensional channel at the k-th stage for any $n_k, r_k \in \mathcal{Z}$ and $k \in \{0, \ldots, N-1\}$.*

3.4 Multi-Stage Scalar Quadratic Signaling Games

The multi-stage signaling game setup is similar to the multi-stage cheap talk setup except that there exists an additive Gaussian noise channel between the encoder and the decoder at each stage, and the encoder has a *soft* power constraint. For the purpose of illustration, the system model of the 2-stage signaling game is depicted in Fig. 3.2.

Here, source is assumed to be a Markov source with an initial Gaussian distribution; i.e., $M_0 \sim \mathcal{N}(0, \sigma^2_{M_0})$ and $M_{k+1} = gM_k + V_k$ where $g \in \mathbb{R}$ and $V_k \sim \mathcal{N}(0, \sigma^2_{V_k})$ is an i.i.d. Gaussian noise sequence for $k = 0, 1, \ldots, N-2$. The channels between the encoder and the decoder are assumed to be i.i.d. additive Gaussian channels; i.e., $W_k \sim \mathcal{N}(0, \sigma^2_{W_k})$, and W_k and V_l are independent for $k = 0, 1, \ldots, N-1$ and $l = 0, 1, \ldots, N-2$. Since the messages transmitted by the encoder and received by the decoder are not the same due to the noisy channel, the information available to the encoder and the decoder slightly changes compared to that in the cheap talk setup. At the k-th stage of the N-stage game, the

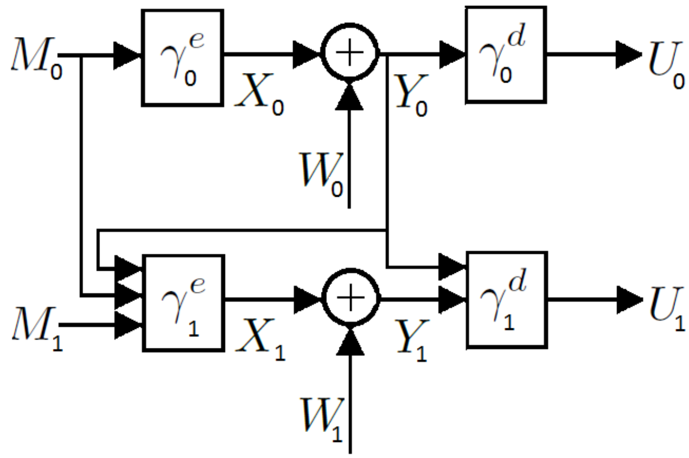

Figure 3.2: *2-stage signaling game.*

encoder knows the values of $\mathcal{I}_k^e = \{m_{[0,k]}, y_{[0,k-1]}\}$ (a noiseless feedback channel is assumed) and the decoder knows the values of $\mathcal{I}_k^d = \{y_{[0,k]}\}$ with $y_k = x_k + w_k$. Thus, under the policies considered, $x_k = \gamma_k^e(\mathcal{I}_k^e)$ and $u_k = \gamma_k^d(\mathcal{I}_k^d)$. The encoder's goal is to minimize

$$J^e\left(\gamma^e_{[0,N-1]}, \gamma^d_{[0,N-1]}\right) = \mathbb{E}\left[\sum_{k=0}^{N-1} c_k^e(m_k, x_k, u_k)\right],$$

whereas, the decoder's goal is to minimize

$$J^d\left(\gamma^e_{[0,N-1]}, \gamma^d_{[0,N-1]}\right) = \mathbb{E}\left[\sum_{k=0}^{N-1} c_k^d(m_k, u_k)\right].$$

by finding the optimal policy sequences $\gamma^e_{[0,N-1]}$ and $\gamma^d_{[0,N-1]}$, respectively. The cost functions are modified as $c_k^e(m_k, x_k, u_k) = (m_k - u_k - b)^2 + \lambda x_k^2$ and $c_k^d(m_k, u_k) = (m_k - u_k)^2$. Note that a power constraint with an associated multiplier is appended to the cost function of the encoder, which corresponds to power limitation for transmitters in practice. If $\lambda = 0$, this corresponds to the setup with no power constraint at the encoder.

3.4.1 Nash Equilibrium Analysis

In multi-stage scalar signaling games, affine policies constitute an invariant subspace under best response maps for Nash equilibria which is stated as follows:

Theorem 3.4.1. *(i) If the encoder uses affine policies at all stages, then the decoder will also be affine at all stages.*

(ii) If the decoder uses affine policies at all stages, then the encoder will also be affine at all stages.

While it provides a structural result on the plausibility of affine equilibria, Theorem 3.4.1 does not lead to a conclusion about the existence of an affine informative equilibrium. It may be tempting to apply fixed point theorems (such as Brouwer's fixed point theorem [84]) to establish the existence of informative equilibria; however, that there always exists a non-informative equilibrium for the cheap talk game also applies to the signaling game as mentioned in Section 2.4.1.2. Later on, we will make information theoretic arguments (in Theorem 3.4.3) for the existence of informative equilibria for the Stackelberg setup, but this is not feasible for the Nash setup.

Even though Theorem 3.4.1 does not provide a result on the existence of an informative affine equilibrium for the n-stage signaling game, a conclusive argument can be established for the 2-stage signaling game. Theorem 2.4.4 will be utilized in the following theorem where informative affine equilibria for a 2-stage signaling game are analyzed.

Theorem 3.4.2. *For the* 2*-stage signaling game setup under affine encoder and decoder assumptions,*

(i) if $\lambda > \max\left\{\frac{(g^2+1)\sigma^2_{M_0}}{\sigma^2_{W_0}}, \frac{\sigma^2_{M_1}}{\sigma^2_{W_1}}\right\}$*, then there does not exist an informative affine equilibrium,*

(ii) if $\min\left\{\frac{(g^2+1)\sigma^2_{M_0}}{\sigma^2_{W_0}}, \frac{\sigma^2_{M_1}}{\sigma^2_{W_1}}\right\} < \lambda \leq \max\left\{\frac{(g^2+1)\sigma^2_{M_0}}{\sigma^2_{W_0}}, \frac{\sigma^2_{M_1}}{\sigma^2_{W_1}}\right\}$*, then;*

(a) *for* $\frac{(g^2+1)\sigma^2_{M_0}}{\sigma^2_{W_0}} < \frac{\sigma^2_{M_1}}{\sigma^2_{W_1}}$, *the equilibrium is informative if and only if* $\sigma^2_{M_1} \geq 4b^2$ *and* $\max\left\{\frac{\sigma^2_{M_1}-2b^2-\sqrt{\sigma^2_{M_1}}\sqrt{\sigma^2_{M_1}-4b^2}}{2\sigma^2_{W_1}}, (g^2+1)\frac{\sigma^2_{M_0}}{\sigma^2_{W_0}}\right\} < \lambda < \frac{\sigma^2_{M_1}-2b^2+\sqrt{\sigma^2_{M_1}}\sqrt{\sigma^2_{M_1}-4b^2}}{2\sigma^2_{W_1}}$.

(b) *for* $\frac{\sigma^2_{M_1}}{\sigma^2_{W_1}} < \frac{(g^2+1)\sigma^2_{M_0}}{\sigma^2_{W_0}}$, *the second stage message* m_1 *is not used in the game.*

The analysis in Theorem 3.4.2 can be carried over to the N-stage signaling game; however, for an N-stage problem, this would involve $(3N^2+5N)/2$ equations and as many unknowns.

3.4.2 Stackelberg Equilibrium Analysis

In this section, the signaling game is analyzed under the Stackelberg concept. The bibliographical notes regarding the optimality of the linear policies are included at the beginning of Section 3.5.2. In the following, we provide an information theoretic analysis to establish the existence of informative equilibria:

Theorem 3.4.3. *If* $\lambda \geq \max_{k=0,1,\dots,N-1} \frac{\sigma^2_{M_k}}{\sigma^2_{W_k}} \sum_{i=0}^{N-k-1} g^{2i}$, *there does not exist an informative (affine or non-linear) equilibrium in the* N*-stage scalar signaling game under the Stackelberg assumption; i.e., the only equilibrium is the non-informative one. Otherwise, an equilibrium has to be always linear.*

Remark 3.4.1. *Notice that, in Theorem 3.4.3, due to the Stackelberg assumption and the smoothing property of the expectation, the bias term in the total encoder cost can be decoupled. Thus, the bound is independent of the bias terms, which is the exact same bound when there is no bias term. However, when the priors are subjective, even for the single-stage game, it may not possible to construct optimal encoder policies without considering the bias term, as shown in Section 3.6.2.2 and Remark 3.6.2.*

Now consider the multi-stage Stackelberg signaling game with a discounted infinite horizon and a discount factor $\beta \in (0,1)$; i.e., $J^e(\gamma^e,\gamma^d) = \mathbb{E}\left[\sum_{i=0}^{\infty}\beta^i\left((m_i-u_i-b)^2+\lambda x_i^2\right)\right]$ and $J^d(\gamma^e,\gamma^d) = \mathbb{E}\left[\sum_{i=0}^{\infty}\beta^i\left(m_i-u_i\right)^2\right]$.

Corollary 3.4.1. *If* $\lambda \geq \max_{k=0,1,\ldots} \frac{\sigma^2_{M_k}}{\sigma^2_{W_k}}\frac{1}{1-\beta g^2}$ *where* $\beta g^2<1$*, there does not exist an informative (affine or non-linear) equilibrium in the infinite horizon discounted multi-stage Stackelberg signaling game for scalar Gauss-Markov sources; i.e., the only equilibrium is the non-informative one,*

Proof. Firstly, consider the finite horizon discounted case; i.e., $J^e(\gamma^e_{[0,N-1]},\gamma^d_{[0,N-1]}) = \sum_{i=0}^{N-1}\beta^k\mathbb{E}\left[(m_i-u_i-b)^2+\lambda x_i^2\right]$. Then, applying the similar steps as in Theorem 3.4.3, we will have $J_0^{e,lower} = \sum_{i=0}^{N-1}\beta^i\left(\Delta_i+\lambda P_i+b^2\right)$. Then, the critical value of λ becomes $\lambda > \max_{k=0,1,\ldots,N-1}\frac{\sigma^2_{M_k}}{\sigma^2_{W_k}}\sum_{i=0}^{N-k-1}\beta^i g^{2i}$ for the non-informative equilibrium.

For the infinite horizon case, it can be observed

$$
\begin{aligned}
&\inf_{\gamma^e_{[0,N-1]}}\lim_{N\to\infty} J^e(\gamma^e_{[0,N-1]},\gamma^d_{[0,N-1]})\\
&\qquad= \inf_{\gamma^e_{[0,N-1]}}\lim_{N\to\infty}\sum_{i=0}^{N-1}\beta^i\mathbb{E}\left[(m_i-u_i-b)^2+\lambda x_i^2\right]\\
&\qquad\geq \limsup_{N\to\infty}\inf_{\gamma^e_{[0,N-1]}}\sum_{i=0}^{N-1}\beta^i\mathbb{E}\left[(m_i-u_i-b)^2+\lambda x_i^2\right]\\
&\qquad\geq \limsup_{N\to\infty}\inf_{\gamma^e_{[0,N-1]}}\sum_{i=0}^{N-1}\beta^i\left(\Delta_i+\lambda P_i+b^2\right)\,. \qquad (3.4)
\end{aligned}
$$

As discussed above, $\inf_{\gamma^e}\sum_{i=0}^{N-1}\beta^i\left(\Delta_i+\lambda P_i+b^2\right)$ is achieved at non-informative equilibrium if $\lambda > \max_{k=0,1,\ldots,N-1}\frac{\sigma^2_{M_k}}{\sigma^2_{W_k}}\sum_{i=0}^{N-k-1}\beta^i g^{2i}$. Thus, the lower bound in (3.4) is achieved at a non-informative equilibrium if

$$
\lambda > \limsup_{N\to\infty}\max_{k=0,1,\ldots,N-1}\frac{\sigma^2_{M_k}}{\sigma^2_{W_k}}\sum_{i=0}^{N-k-1}\beta^i g^{2i} = \frac{\sigma^2_{M_k}}{\sigma^2_{W_k}}\frac{1}{1-\beta g^2}\ \text{for } \beta g^2<1\,.
$$

Hence, if $\lambda \geq \max_{k=0,1,\ldots}\frac{\sigma^2_{M_k}}{\sigma^2_{W_k}}\frac{1}{1-\beta g^2}$, then the lower bound $J_0^{e,lower}$ of the encoder costs J_0^e is minimized by choosing $P_0=P_1=\cdots=0$, and the minimum cost becomes $J_0^e = J_0^{e,lower} = \sum_{i=0}^{\infty}\beta^i\left(\sigma^2_{M_i}+b^2\right)$ at this non-informative equilibrium. □

3.5 Multi-Stage Multi-Dimensional Quadratic Signaling Games

In this section, the scalar setup considered in Section 3.4 is extended to the n-dimensional setup as follows: The source is assumed to be an n-dimensional Markovian source with initial Gaussian distribution; i.e., $\mathbf{M}_0 \sim \mathcal{N}(\mathbf{0}, \Sigma_{\mathbf{M}_0})$ and $\mathbf{M}_{k+1} = G\mathbf{M}_k + \mathbf{V}_k$ where G is an $n \times n$ matrix and $\mathbf{V}_k \sim \mathcal{N}(\mathbf{0}, \Sigma_{\mathbf{V}_k})$ is an i.i.d. Gaussian noise sequence for $k = 0, 1, \ldots, N-2$. The channels between the encoder and the decoder are assumed to be i.i.d. additive Gaussian channels; i.e., $\mathbf{W}_k \sim \mathcal{N}(\mathbf{0}, \Sigma_{\mathbf{W}_k})$, and $\mathbf{W}_k$ and $\mathbf{V}_l$ are independent for $k = 0, 1, \ldots, N-1$ and $l = 0, 1, \ldots, N-2$. At the k-th stage of the N-stage game, the encoder knows the values of $\mathcal{I}_k^e = \{\mathbf{m}_{[0,k]}, \mathbf{y}_{[0,k-1]}\}$ (a noiseless feedback channel is assumed) and the decoder knows the values of $\mathcal{I}_k^d = \{\mathbf{y}_{[0,k]}\}$ with $\mathbf{y}_k = \mathbf{x}_k + \mathbf{w}_k$. Thus, under the policies considered, $\mathbf{x}_k = \gamma_k^e(\mathcal{I}_k^e)$ and $\mathbf{u}_k = \gamma_k^d(\mathcal{I}_k^d)$, the encoder's goal is to minimize

$$J^e\left(\gamma^e_{[0,N-1]}, \gamma^d_{[0,N-1]}\right) = \mathbb{E}\left[\sum_{k=0}^{N-1} c_k^e(\mathbf{m}_k, \mathbf{x}_k, \mathbf{u}_k)\right],$$

whereas, the decoder's goal is to minimize

$$J^d\left(\gamma^e_{[0,N-1]}, \gamma^d_{[0,N-1]}\right) = \mathbb{E}\left[\sum_{k=0}^{N-1} c_k^d(\mathbf{m}_k, \mathbf{u}_k)\right]$$

by finding the optimal policy sequences $\gamma^{*,e}_{[0,N-1]}$ and $\gamma^{*,d}_{[0,N-1]}$, respectively. The cost functions are $c^e\left(\mathbf{m}_k, \mathbf{x}_k, \mathbf{u}_k\right) = \|\mathbf{m}_k - \mathbf{u}_k - \mathbf{b}\|^2 + \lambda\|\mathbf{x}_k\|^2$ and $c^d\left(\mathbf{m}_k, \mathbf{u}_k\right) = \|\mathbf{m}_k - \mathbf{u}_k\|^2$, where the lengths of the vectors are defined in L_2 norm and $\mathbf{b}$ is the bias vector.

3.5.1 Nash Equilibrium Analysis

Similar to the scalar source case, affine policies constitute an invariant subspace under the best response maps for Nash equilibria when the source is multi-dimensional in the multi-stage signaling games as shown below:

Theorem 3.5.1. *(i) If the encoder uses affine policies at all stages, then the decoder will be affine at all stages.*

(ii) If the decoder uses affine policies at all stages, then the encoder will be affine at all stages.

Please see the discussion following Theorem 3.4.1: The existence of an informative equilibrium cannot be deduced by utilizing the Brouwer's fixed point theorem [84] since there always exist a non-informative equilibrium.

Remark 3.5.1. *Note that there is no constraint on the source and channel dimensions for Theorem 3.5.1: the source and channel dimensions can be different, and further, the source and channel dimensions can change at every stage; i.e., Theorem 3.5.1 holds for an n_k-dimensional source and an r_k-dimensional channel at the k-th stage for any $n_k, r_k \in \mathcal{Z}$ and $k \in \{0, \ldots, N-1\}$.*

3.5.2 Stackelberg Equilibrium Analysis

Linear policies are optimal for scalar sources as shown in Section 3.4.2. Before analyzing the Stackelberg equilibrium for the multi-dimensional setup, it will be appropriate to review the optimality of linear policies in Gaussian setups for the classical communication theoretic setup when the bias term is absent:

Optimality of linear coding policies for scalar Gaussian source-channel pairs with noiseless feedback has been known since 1960s, see e.g. [93–95], [96, Chapter 16]. Further, the transmission over scalar Gaussian channels has been studied also in [96, Chapter 16], [97] and [98], where the error exponents have been shown to be unbounded (and it has been shown that the error probability decreases at least doubly exponentially in the block-length). However, when there is a noisy feedback, this result does not hold [99]. There is also a recent work for Gaussian channels with memory, where the aim is to design controllers (encoders and decoders) that both stabilize the unstable dynamical channels and achieve the capacity [100].

Even when the encoder and the decoder have identical (non-biased) quadratic cost functions, when the source and the channel are multi-dimensional, linear policies may not be optimal. The case where there is a mismatch in the source and the channel is a challenging one. Partial results are known regarding optimal zero-delay policies. Matching essentially requires that the capacity achieving source probabilities and the rate-distortion achieving channel probabilistic characteristics are simultaneously realized for a given system; this is precisely the case for a scalar Gaussian source transmitted over a scalar additive Gaussian channel. One special case where such a matching holds is the case when the noise and signal power levels are identical in every channel and the distortion criterion is identical for all scalar components [101]. In particular, except for settings where matching between the source and the channel exists (building on [102], [103]), the optimality of linear policies is quite rare [4,101,104]. Optimal linear encoders for single-stage setups have been studied [105–107]. Partially observed settings have been considered in [108–111]. Gastpar [112] has considered various settings of multi-access and broadcast channels. For further discussions on multi-dimensional Gaussian source and channel pairs, we refer the reader to [101,103–105,113–116]. A comprehensive discussion and literature review is available at [105, 117] and [4, Chapter 11].

It is evident from Theorem 3.5.1 that when the encoder is linear, the optimal decoder is linear. In this case, a relevant problem is to find the optimal Stackelberg policy among the linear or affine class. In the following, a dynamic programming approach is adapted to find such Stackelberg equilibria. Building on the optimality of linear innovation encoders, we restrict the analysis to such encoders; i.e., we consider a sub-optimal scenario. Our analysis builds on and generalizes the arguments in [117, Theorem 3] and [118].

Theorem 3.5.2. *Suppose that G, $\Sigma_{\mathbf{M}_0}$ and $\Sigma_{\mathbf{v}_k}$ are diagonal. Suppose further that the innovation is given by $\widetilde{\mathbf{m}}_k \triangleq \mathbf{m}_k - \mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k-1]}]$ with $\widetilde{\mathbf{m}}_0 = \mathbf{m}_0$, and that the encoder linearly encodes the innovation. Then, an optimal such linear policy can be computed through dynamic programming with value functions $V_k(\Sigma_{\widetilde{\mathbf{m}}_k}) \triangleq \operatorname{tr}(K_k\Sigma_{\widetilde{\mathbf{m}}_k} + L_k)$ that satisfy the terminal condition $V_N(\Sigma_{\widetilde{\mathbf{m}}_N}) = 0$ with diagonal K_k matrices for $k = 0, 1, \ldots, N-1$.*

Remark 3.5.2. *In Theorem 3.5.2, similar to the scalar case (Theorem 3.4.3), the bias term in the total encoder cost can be decoupled and the optimal encoder parameters become independent of the bias terms. However, when the priors are subjective, decoupling the bias related parameters may not be possible (see Remark 3.6.2 in Section 3.6.2.2, and Remark 3.4.1). Further, Theorem 3.5.2 provides the vector channel extension of [117, Theorem 2.3], which considers a scalar channel.*

3.6 Quadratic Cheap Talk and Signaling Games with Subjective Priors

The setup in decentralized decision making where the priors of the decision makers may be different is a practically important and researched area: For example, [87, 88] have investigated optimal decentralized decision making with subjective priors, a team problem is converted into a game problem as mappings on policies/strategies (see [4, Section 12.2.3] for a literature review on subjective priors also from a statistical decision making perspective). In this section, we consider such a setup where the source is perceived to be different from the perspectives of the encoder and the decoder; i.e., $\mathbf{M} \sim f_e(\mathbf{m})$ and $\mathbf{M} \sim f_d(\mathbf{m})$ from encoder's and decoder's perspectives, respectively. The expected cost of the encoder and the decoder, under given policies, become different even if the costs $c^e(\mathbf{m}, \mathbf{u})$ and $c^d(\mathbf{m}, \mathbf{u})$ are the same, i.e., the expectation is taken over different probability measures, and in order to reflect this, $\mathbb{E}_e$ and $\mathbb{E}_d$ will be used to denote the expectations from the perspective of the encoder and the decoder, respectively. Hence, the discrepancy in the perception of the source alters the problem to the game problem, and, in this direction, the effects of inconsistent priors in cheap talk and signaling game will be investigated in the following.

3.6.1 Quadratic Cheap Talk with Subjective Priors

In this section, we consider encoders and decoders with subjective priors, and to reflect mainly the effects of the subjectivity in the priors we assume that the costs are identical with $c^e(\mathbf{m},\mathbf{u}) = c^d(\mathbf{m},\mathbf{u}) = \|\mathbf{m}-\mathbf{u}\|^2$. Let $\mathbf{M} \sim f_e$ and $\mathbf{M} \sim f_d$ from encoder's and decoder's perspectives, respectively. We assume mutual absolute continuity of f_e and f_d; that is, for any Borel B, $f_e(B) = 0 \implies f_d(B) = 0$ and $f_d(B) = 0 \implies f_e(B) = 0$.

Theorem 3.6.1. *(i) If the priors are mutually absolutely continuous, there exists a fully informative Nash equilibrium.*

(ii) If the priors are mutually absolutely continuous, there exists a fully informative Stackelberg equilibrium.

Proof. (i) Let the encoder and the decoder use fully informative policies; i.e., the encoder transmits every individual message distinctly as $\mathbf{x} = \gamma^e(\mathbf{m}) = \mathbf{m}$, and the decoder takes unique actions for each distinct message he receives as $\mathbf{u} = \gamma^d(\mathbf{x}) = \mathbf{x}$. Then, the cost of the encoder and the decoder is zero almost surely (due to the mutual absolute continuity assumption, the set of $\mathbf{m}$ values in the support of both the encoder and the decoder priors has measure 1 under either the encoder and the decoder prior); and thus $J^e = \mathbb{E}_{f_e}[\|\mathbf{m}-\mathbf{u}]\|^2] = 0$ and $J^d = \mathbb{E}_{f_d}[\|\mathbf{m}-\mathbf{u}]\|^2] = 0$. Since both the encoder and the decoder achieves the minimum possible cost, none of the players deviates from their current choices; i.e., they prefer to stick at the fully informative policies, which implies that there exists a fully informative equilibrium.

(ii) Under the Stackelberg assumption, the optimal decoder action is $\mathbf{u}^* = \gamma^{*,d}(\mathbf{x}) = \mathbb{E}_{f_d}[\mathbf{m}|\mathbf{x}]$. Then, the encoder aims to choose the optimal encoding policy $\gamma^{*,e}(\mathbf{m}) = \mathbf{x}^* = \arg\min_{\mathbf{x}} \mathbb{E}_{f_e}[\|\mathbf{m} - \mathbb{E}_{f_d}[\mathbf{m}|\mathbf{x}]\|^2]$. Thus, for every possible realization of $\mathbf{m}$, the encoder can choose $\mathbf{x} = \gamma^e(\mathbf{m})$ such that $\mathbf{m} = \mathbb{E}_{f_d}[\mathbf{m}|\mathbf{x}]$, and this is achievable at fully informative equilibria; i.e., $\gamma^{*,e}(\mathbf{m}) = \mathbf{x}^* = \mathbf{m}$. Under this encoding policy and due to the mutual absolute continuity assumption, the optimal encoder cost is zero almost surely, and the optimal decoder policy is $\mathbf{u}^* = \gamma^{*,d}(\mathbf{x}) = \mathbf{x} = \mathbf{m}$, which entails the zero decoder cost almost surely.

$\square$

Remark 3.6.1. *The subjective priors assumption does not make a difference when the priors are mutually absolutely continuous; i.e., both the team setup and game setup result in fully informative equilibria.*

Theorem 3.6.1 and Remark 3.6.1 also apply to the multi-stage case; i.e., if the priors are mutually absolutely continuous, there exists a fully informative Nash and Stackelberg equilibria in multi-stage cheap talk as in the team theoretic setup.

3.6.2 Quadratic Signaling Games with Subjective Priors

Here, we will adapt the subjective priors assumption to the single-stage signaling games as follows: Let the Gaussian source have different mean and variance from the perspectives of the encoder and the decoder; i.e., the source is $M \sim \mathcal{N}(\mu_e, \sigma_e^2)$ and $M \sim \mathcal{N}(\mu_d, \sigma_d^2)$ from encoder's and decoder's perspective, respectively. The channel between the encoder and the decoder is assumed to be additive Gaussian channel with distribution $W \sim \mathcal{N}(0, \sigma_W^2)$ and independent of the source M.

3.6.2.1 Nash Equilibria Analysis

We first study the single-stage Nash equilibria.

Theorem 3.6.2. *If $\lambda > \frac{\sigma_d^2}{\sigma_W^2}$, the affine equilibrium is non-informative; otherwise, the affine equilibrium is unique and informative.*

Proof. If the encoder is affine; i.e., $x = \gamma^e(m) = Am + C$, then the optimal decoder becomes

$$u^* = \gamma^{*,d}(y) = \mathbb{E}_d[m|y] = \mathbb{E}_d[m|Am + C + w]$$

$$= \mu_d + \frac{A\sigma_d^2}{A^2\sigma_d^2 + \sigma_W^2}(y - A\mu_d - C)$$
$$= \frac{A\sigma_d^2}{A^2\sigma_d^2 + \sigma_W^2} y + \frac{\sigma_W^2 \mu_d - AC\sigma_d^2}{A^2\sigma_d^2 + \sigma_W^2} .$$

Now suppose that the decoder is affine; i.e., $u = \gamma^d(y) = Ky + L$, then the optimal encoder is (2.11):

$$x^* = \gamma^{*,e}(m) = \frac{K}{K^2 + \lambda} m - \frac{K(L + b)}{K^2 + \lambda} .$$

We now wish to see if these optimal sets of policies satisfy a fixed point equation. By combining the optimal policies, we get

$$A = \frac{K}{K^2 + \lambda} , \quad C = -A(L + b) ,$$
$$K = \frac{A\sigma_d^2}{A^2\sigma_d^2 + \sigma_W^2} , \quad L = \mu_d \frac{\sigma_W^2}{A^2\sigma_d^2 + \sigma_W^2} - KC .$$

Similar to Theorem 2.4.1, we obtain $(K^2 + \lambda)^2\sigma_W^2 = \lambda\sigma_d^2$ by assuming $A \neq 0$. Here, for $\lambda > \frac{\sigma_d^2}{\sigma_W^2}$, $(K^2 + \lambda)^2\sigma_W^2 = \lambda\sigma_d^2$ cannot be satisfied, thus $A = 0$ and the affine equilibrium is non-informative. Otherwise, the affine equilibrium is informative and the optimal policy parameters can be determined uniquely (actually, if $\{A^*, C^*, K^*, L^*\}$ constitutes optimal policies of the encoder and the decoder, then $\{-A^*, -C^*, -K^*, L^*\}$ is also a parameter set for optimal policies):

$$A = \sqrt{\sqrt{\frac{\sigma_W^2}{\lambda\sigma_d^2}} - \frac{\sigma_W^2}{\sigma_d^2}} ,$$
$$C = -\left(\mu_d \sqrt{\sqrt{\frac{\sigma_W^2}{\lambda\sigma_d^2}} - \frac{\sigma_W^2}{\sigma_d^2}} + b\sqrt{\frac{1}{\lambda}\left(\sqrt{\frac{\sigma_d^2}{\lambda\sigma_W^2}} - 1\right)}\right) ,$$
$$K = \sqrt{\sqrt{\frac{\lambda\sigma_d^2}{\sigma_W^2}} - \lambda} ,$$
$$L = \mu_d + b\left(\sqrt{\frac{\sigma_d^2}{\lambda\sigma_W^2}} - 1\right) .$$

Note that none of the parameters depend on the parameters of the perspective of the encoder since the encoder minimizes his cost for every realization m of source M without considering its distribution. □

It can be shown that, when the priors are subjective, the encoder (decoder) is affine for an affine decoder (encoder) in the multi-dimensional signaling game by using a similar analysis to that of Theorem 2.4.1 as follows:

Theorem 3.6.3. *Let the source be* $\mathbf{M} \sim \mathcal{N}(\boldsymbol{\mu}_e, \Sigma_e)$ *and* $\mathbf{M} \sim \mathcal{N}(\boldsymbol{\mu}_d, \Sigma_d)$ *from encoder's and decoder's perspective, respectively, and let the channel noise be* $\mathbf{W} \sim \mathcal{N}(\mathbf{0}, \Sigma_{\mathbf{W}})$*. Then, encoder (decoder) is affine for an affine decoder (encoder) in multi-dimensional signaling game when the priors are inconsistent.*

Proof. For the affine encoder $\mathbf{x} = \gamma^e(\mathbf{m}) = A\mathbf{m} + \mathbf{C}$, the optimal decoder will be

$$
\begin{aligned}
\mathbf{u}^* &= \gamma^{*,d}(\mathbf{y}) = \mathbb{E}_d[\mathbf{m}|\mathbf{y}] = \mathbb{E}_d[\mathbf{m}|A\mathbf{m} + \mathbf{C} + \mathbf{w}] \\
&= \boldsymbol{\mu}_d + \Sigma_d A^T (A\Sigma_d A^T + \Sigma_{\mathbf{W}})^{-1}(\mathbf{y} - A\boldsymbol{\mu}_d - \mathbf{C}) \\
&= \Sigma_d A^T (A\Sigma_d A^T + \Sigma_{\mathbf{W}})^{-1}(\mathbf{y} - \mathbf{C}) + (I - \Sigma_d A^T (A\Sigma_d A^T + \Sigma_{\mathbf{W}})^{-1} A)\boldsymbol{\mu}_d \\
&\overset{(a)}{=} \Sigma_d A^T (A\Sigma_d A^T + \Sigma_{\mathbf{W}})^{-1}(\mathbf{y} - \mathbf{C}) + (I + \Sigma_d A^T \Sigma_{\mathbf{W}}^{-1} A)^{-1}\boldsymbol{\mu}_d \\
&= \Sigma_d A^T (A\Sigma_d A^T + \Sigma_{\mathbf{W}})^{-1}(\mathbf{y} - \mathbf{C}) + (A^{-1}(\Sigma_{\mathbf{W}} + A\Sigma_d A^T)\Sigma_{\mathbf{W}}^{-1} A)^{-1}\boldsymbol{\mu}_d \\
&= \Sigma_d A^T (A\Sigma_d A^T + \Sigma_{\mathbf{W}})^{-1}(\mathbf{y} - \mathbf{C}) + A^{-1}\Sigma_{\mathbf{W}}(\Sigma_{\mathbf{W}} + A\Sigma_d A^T)^{-1} A\boldsymbol{\mu}_d \,.
\end{aligned}
$$

Here, (a) is due to the matrix inversion lemma, $(I + UWV)^{-1} = I - U(W^{-1} + VU)^{-1}V$, where $U = \Sigma_d A^T$, $W = \Sigma_{\mathbf{W}}^{-1}$, and $V = A$. Now suppose that the decoder is affine; i.e., $\mathbf{u} = \gamma^d(\mathbf{y}) = K\mathbf{y} + \mathbf{L}$, then the optimal encoder is (2.45):

$$
\mathbf{x}^* = \gamma^{*,e}(\mathbf{m}) = \left(K^T K + \lambda I\right)^{-1} K^T \left(\mathbf{m} - \mathbf{L} - \mathbf{b}\right).
$$

□

As it can be seen from Theorem 3.6.2 and Theorem 3.6.3, the following conclusion can be made by utilizing Theorem 3.4.1 and Theorem 3.5.1:

Corollary 3.6.1. *Even if the priors are inconsistent from the perspectives of the encoder and the decoder in the multi-stage signaling game, affine policies constitute an invariant subspace under best response maps for scalar and multi-dimensional sources under Nash equilibria.*

3.6.2.2 Stackelberg Equilibria Analysis

Theorem 3.6.4. *The optimal encoder policy $\gamma^{*,e}(m) = A^* m + C^*$ among the affine class can be characterized by the following:*

$$A^* = \arg\min_A \left(\frac{\sigma_W^2(\mu_d - \mu_e)}{A^2\sigma_d^2 + \sigma_W^2} + b\right)^2 + \frac{\sigma_W^4}{(A^2\sigma_d^2+\sigma_W^2)^2}(\sigma_e^2 - \sigma_d^2) + \frac{\sigma_d^2\sigma_W^2}{A^2\sigma_d^2+\sigma_W^2} + \lambda A^2\sigma_e^2\,,$$
$$C^* = -A^*\mu_e\,. \tag{3.5}$$

Proof. Assuming the affine encoder; i.e., $x = \gamma^e(m) = Am + C$, by Theorem 3.6.2, we know that the optimal decoder is $u^* = \gamma^{*,d}(y) = \frac{A\sigma_d^2}{A^2\sigma_d^2+\sigma_W^2}y + \frac{\sigma_W^2\mu_d - AC\sigma_d^2}{A^2\sigma_d^2+\sigma_W^2} \triangleq Ky + L$, where $y = Am + C + w$. Then the goal of the encoder is to minimize the following:

$$\begin{aligned}
J^{*,e} &= \min_{x=\gamma^e(m)=Am+C} \mathbb{E}_e\Big[(m-u-b)^2 + \lambda x^2\Big] \\
&= \min_{A,\,C} \mathbb{E}_e\Big[(m - AKm - KC - Kw - L - b)^2 + \lambda(Am+C)^2\Big] \\
&= \min_{A,\,C} \mathbb{E}_e\Big[m^2(1-AK)^2 - 2(1-AK)(KC+L+b)m + (KC+L+b)^2 \\
&\qquad + \lambda A^2m^2 + 2\lambda ACm + \lambda C^2\Big] + K^2\sigma_W^2 \\
&= \min_{A,\,C} (\mu_e^2+\sigma_e^2)\frac{\sigma_W^4}{(A^2\sigma_d^2+\sigma_W^2)^2} - 2\frac{\sigma_W^2}{A^2\sigma_d^2+\sigma_W^2}\left(\frac{\sigma_W^2\mu_d}{A^2\sigma_d^2+\sigma_W^2} + b\right)\mu_e \\
&\qquad + \left(\frac{\sigma_W^2\mu_d}{A^2\sigma_d^2+\sigma_W^2} + b\right)^2 + \lambda A^2(\mu_e^2+\sigma_e^2) + 2\lambda AC\mu_e \\
&\qquad + \lambda C^2 + \frac{A^2\sigma_d^4}{(A^2\sigma_d^2+\sigma_W^2)^2}\sigma_W^2 \\
&= \min_{A,\,C} \frac{(\mu_e^2+\sigma_e^2)\sigma_W^4 - 2\sigma_W^4\mu_e\mu_d + \sigma_W^4\mu_d^2 + A^2\sigma_W^2\sigma_d^4}{(A^2\sigma_d^2+\sigma_W^2)^2} \\
&\qquad + \frac{-2\sigma_W^2\mu_e b + 2\sigma_W^2\mu_d b}{A^2\sigma_d^2+\sigma_W^2} + b^2 + \lambda A^2\sigma_e^2 + \lambda(A\mu_e + C)^2 \\
&= \min_{A,\,C} \frac{\sigma_W^4(\mu_d-\mu_e)^2 + \sigma_W^4\sigma_e^2 + A^2\sigma_W^2\sigma_d^4 + \sigma_W^4\sigma_d^2 - \sigma_W^4\sigma_d^2}{(A^2\sigma_d^2+\sigma_W^2)^2}
\end{aligned}$$

$$+ \frac{2\sigma_W^2 b}{A^2\sigma_d^2 + \sigma_W^2}(\mu_d - \mu_e) + b^2 + \lambda A^2\sigma_e^2 + \lambda(A\mu_e + C)^2$$

$$= \min_{A,\,C} \left(\frac{\sigma_W^2(\mu_d - \mu_e)}{A^2\sigma_d^2 + \sigma_W^2} + b\right)^2 + \frac{\sigma_W^4}{(A^2\sigma_d^2 + \sigma_W^2)^2}(\sigma_e^2 - \sigma_d^2)$$

$$+ \frac{\sigma_d^2\sigma_W^2}{A^2\sigma_d^2 + \sigma_W^2} + \lambda A^2\sigma_e^2 + \lambda(A\mu_e + C)^2 \,. \tag{3.6}$$

Here, the optimal encoder cost is achieved when $C^* = -A^*\mu_e$, and A^* can be found by analyzing the 6th order equation of A. □

Remark 3.6.2. *When the consistent priors and the zero-mean Gaussian source are assumed; i.e.,* $\mu_e = \mu_d = 0$ *and* $\sigma_e^2 = \sigma_d^2$*, then* (3.6) *turns into* (2.25) *as expected. Further, note that, in* (3.6)*, unless* $\mu_e \neq \mu_d$*, the optimal encoder policy is independent of the bias term* b*; i.e., the bias term has no effect on the optimal policies of the encoder and the decoder.*

The analysis in Theorem 3.6.4 can be carried over to the N-stage signaling game: the encoder searches over the affine class to find his optimal policy by anticipating the best response of the decoder, and this would involve the optimization over $N^2 + N$ parameters for an N-stage problem.

Remark 3.6.3. *Note that, under the Nash assumption, the agents do not need to know their subjective priors; they know only their policies as they (simultaneously) announce to each other. On the other hand, for the Stackelberg case, the encoder must know the decoder's subjective prior so that he, as a leader, can anticipate the decoder's optimal actions.*

3.7 Conclusion

In this chapter, we studied Nash and Stackelberg equilibria for multi-stage quadratic cheap talk and signaling games. We established qualitative (e.g. on full revelation, quantization nature, linearity, informativeness and non-informativeness) and quantitative properties (on linearity or explicit computation) of Nash and Stackelberg equilibria under either subjective/inconsistent cost models or priors.

For the cheap talk problem under Nash equilibria, we have shown that the last stage equilibria are quantized for any scalar source with an arbitrary distribution, and all stages must be quantized under certain assumptions on i.i.d. sources whereas for the multi-stage Stackelberg cheap talk game, the equilibria must be fully revealing regardless of the source model. We have also proved that the equilibria are fully revealing in the multi-stage multi-dimensional cheap talk under Stackelberg equilibria whereas the equilibria cannot be fully revealing under a Nash concept. In the multi-stage signaling game where the transmission of a Gaussian source over a Gaussian channel is considered, affine policies constitute an invariant subspace under best response maps for scalar and multi-dimensional sources under Nash equilibria. However, for multi-stage Stackelberg signaling games involving Gauss-Markov sources and memoryless Gaussian channels, we have proved that, for scalar setups, linear policies are optimal and the only equilibrium is the linear one, whereas this is not the case for general multi-dimensional setups. Further, the conditions under which the equilibrium is non-informative under the Stackelberg assumption are derived for scalar Gauss-Markov sources, and the dynamic programming formulation is presented for a class of Stackelberg equilibria when the encoders are restricted to be linear for multi-dimensional Gauss-Markov sources. When the source is perceived to admit different probability measures from the perspectives of the encoder and the decoder, under identical cost functions and mutual absolute continuity, we show that there exist fully informative Nash and Stackelberg equilibria for the dynamic cheap talk as in the usual team theoretic setup. Thus, the equilibrium behavior is robust to a class of perturbations in the prior models, which is not necessarily the case for the perturbations in the cost models. On the other hand, for the signaling game, Stackelberg equilibrium policies are robust to a class of perturbations in the cost models but not to the perturbations in the prior models considered in this chapter. Table 3.1 summarizes the results of this chapter.

Table 3.1: Multi-stage cheap talk and signaling games

SETUP	SOURCE	Nash Equilibrium	Stackelberg Equilibrium
MULTI-STAGE CHEAP TALK	scalar	final stage must be quantized for Markov sources, and all stages are quantized for i.i.d. sources	fully revealing
	multi-dimensional	final stage cannot be fully revealing	fully revealing
MULTI-STAGE SIGNALING GAMES	scalar	affine policies constitute invariant subspace under best response maps	always linear
	multi-dimensional	affine policies constitute invariant subspace under best response maps	no general structure

3.8 Proofs

3.8.1 Proof of Theorem 3.4.1

This result follows as a special case of Theorem 3.5.1, but the proof is provided for completeness.

(i) Let the encoder policies be $x_k = \gamma_k^e(m_{[0,k]}, y_{[0,k-1]}) = \sum_{i=0}^{k} A_{k,i}\, m_i + \sum_{i=0}^{k-1} B_{k,i}\, y_i + C_k$ where $A_{k,i}$, $B_{k,i}$ and C_k are scalars for $k \leq N-1$ and $i \leq k$. Similar to the dynamic Stackelberg cheap talk analysis in Theorem 3.2.6, the optimal decoder actions can be found as $u_k^* = \gamma_k^{*,d}(\mathcal{I}_k^d) = \mathbb{E}[m_k|\mathcal{I}_k^d] = \mathbb{E}[m_k|y_{[0,k]}]$ for $k \leq N-1$. Notice that $y_{[0,k]}$ is multivariate Gaussian for $k \leq N-1$ since $y_k = x_k + w_k$. This proves that $\gamma_k^{*,d}(\mathcal{I}_k^d)$ is an affine function of $y_{[0,k]}$ due to the joint Gaussianity.

(ii) Let the decoder policies be $u_k = \gamma_k^d(y_{[0,k]}) = \sum_{i=0}^{k} K_{k,i}\, y_i + L_k$ where $K_{k,i}$ and L_k are scalars for $k \leq N-1$ and $i \leq k$. With $y_{N-1} = x_{N-1} + w_{N-1}$, it follows that $u_{N-1} = \sum_{i=0}^{N-2} K_{N-1,i}\, y_i + K_{N-1,N-1}\, x_{N-1} + K_{N-1,N-1}\, w_{N-1} + L_{N-1}$. Then, by a dynamic programming approach, the final stage encoder cost can be written as

$$
\begin{aligned}
J_{N-1}^{*,e} &= \min_{x_{N-1} = \gamma_{N-1}^e(m_{[0,N-1]}, y_{[0,N-2]})} \mathbb{E}\Big[(m_{N-1} - u_{N-1} - b)^2 + \lambda x_{N-1}^2 \Big] \\
&= \min_{x_{N-1}} \mathbb{E}\Bigg[\Bigg(m_{N-1} - \sum_{i=0}^{N-2} K_{N-1,i}\, y_i - K_{N-1,N-1}\, x_{N-1} - K_{N-1,N-1}\, w_{N-1} \\
&\qquad - L_{N-1} - b \Bigg)^2 + \lambda x_{N-1}^2 \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \min_{x_{N-1}} \mathbb{E}\Bigg[\Bigg(m_{N-1} - \sum_{i=0}^{N-2} K_{N-1,i}\, y_i - K_{N-1,N-1}\, x_{N-1} - L_{N-1} - b\Bigg)^2 \\
&\qquad + \lambda x_{N-1}^2\Bigg] + K_{N-1,N-1}^2\, \sigma_{W_{N-1}}^2 \\
&= \min_{x_{N-1}} \mathbb{E}\Bigg[\left(K_{N-1,N-1}^2 + \lambda\right) x_{N-1}^2 - 2K_{N-1,N-1} \\
&\qquad\qquad \times \Bigg(m_{N-1} - \sum_{i=0}^{N-2} K_{N-1,i}\, y_i - L_{N-1} - b\Bigg) x_{N-1} \\
&\qquad + \Bigg(m_{N-1} - \sum_{i=0}^{N-2} K_{N-1,i}\, y_i - L_{N-1} - b\Bigg)^2\Bigg] + K_{N-1,N-1}^2\, \sigma_{W_{N-1}}^2 \\
&= \min_{x_{N-1}} \left(K_{N-1,N-1}^2 + \lambda\right) \mathbb{E}\Bigg[\Bigg(x_{N-1} - \frac{K_{N-1,N-1}}{K_{N-1,N-1}^2 + \lambda} \\
&\qquad\qquad \times \Bigg(m_{N-1} - \sum_{i=0}^{N-2} K_{N-1,i}\, y_i - L_{N-1} - b\Bigg)\Bigg)^2\Bigg] \\
&\qquad + \mathbb{E}\Bigg[\frac{\lambda}{K_{N-1,N-1}^2 + \lambda}\Bigg(m_{N-1} - \sum_{i=0}^{N-2} K_{N-1,i}\, y_i - L_{N-1} - b\Bigg)^2\Bigg] \\
&\qquad + K_{N-1,N-1}^2\, \sigma_{W_{N-1}}^2 .
\end{aligned}
$$

Hence, the optimal $\gamma_{N-1}^e(m_{[0,N-1]}, y_{[0,N-2]})$ can be chosen as

$$
\begin{aligned}
\gamma_{N-1}^e(m_{[0,N-1]}, y_{[0,N-2]}) &= \frac{K_{N-1,N-1}}{K_{N-1,N-1}^2 + \lambda} \\
&\qquad \times \Bigg(m_{N-1} - \sum_{i=0}^{N-2} K_{N-1,i}\, y_i - L_{N-1} - b\Bigg) ,
\end{aligned}
$$

and the minimum final stage encoder cost is obtained as

$$
\begin{aligned}
J_{N-1}^{*,e} = \mathbb{E}\Bigg[&\frac{\lambda}{K_{N-1,N-1}^2 + \lambda}\Bigg(m_{N-1} - \sum_{i=0}^{N-2} K_{N-1,i}\, y_i - L_{N-1} - b\Bigg)^2\Bigg] \\
&+ K_{N-1,N-1}^2\, \sigma_{W_{N-1}}^2 .
\end{aligned} \tag{3.7}
$$

Notice that even though $m_{[0,N-1]}$ and $y_{[0,N-2]}$ are available to the encoder, the encoder uses only m_{N-1} and $y_{[0,N-2]}$ at the final stage; i.e., the encoder does not need $m_{[0,N-2]}$.

Then, by a dynamic programming approach, the cost of the encoder at $N-1$st stage becomes

$$J^{*,e}_{N-2} = \min_{x_{N-2}=\gamma^e_{N-2}(m_{[0,N-2]},y_{[0,N-3]})} \mathbb{E}\left[(m_{N-2}-u_{N-2}-b)^2 + \lambda x^2_{N-2} + J^{*,e}_{N-1}\right] . \tag{3.8}$$

By using the relation between the sources $m_{N-1} = gm_{N-2} + v_{N-2}$ and $y_{N-2} = x_{N-2} + w_{N-2}$, (3.7) can be refined and inserted into (3.8). Further, with $y_{N-2} = x_{N-2} + w_{N-2}$, it follows that $u_{N-2} = \sum_{i=0}^{N-3} K_{N-2,i}\, y_i + K_{N-2,N-2}\, x_{N-2} + K_{N-2,N-2}\, w_{N-2} + L_{N-2}$, and the completion of the squares method can be applied to the previous step. By using the relation between the sources $m_{N-1} = gm_{N-2} + v_{N-2}$ and $y_{N-2} = x_{N-2} + w_{N-2}$, the final stage encoder cost can be simplified as

$$\begin{aligned}
J^{*,e}_{N-1} &= \mathbb{E}\Bigg[\frac{\lambda}{K^2_{N-1,N-1}+\lambda}\Bigg(gm_{N-2} + v_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, y_i - K_{N-1,N-2}\, x_{N-2} \\
&\qquad - K_{N-1,N-2}\, w_{N-2} - L_{N-1} - b\Bigg)^2\Bigg] + K^2_{N-1,N-1}\,\sigma^2_{W_{N-1}} \\
&= \mathbb{E}\Bigg[\frac{\lambda}{K^2_{N-1,N-1}+\lambda}\Bigg(gm_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, y_i - K_{N-1,N-2}\, x_{N-2} \\
&\qquad - L_{N-1} - b\Bigg)^2\Bigg] + \frac{\lambda}{K^2_{N-1,N-1}+\lambda}\left(\sigma^2_{V_{N-2}} + K^2_{N-1,N-2}\,\sigma^2_{W_{N-2}}\right) \\
&\qquad + K^2_{N-1,N-1}\,\sigma^2_{W_{N-1}} .
\end{aligned}$$

With $y_{N-2} = x_{N-2} + w_{N-2}$, it follows that $u_{N-2} = \sum_{i=0}^{N-3} K_{N-2,i}\, y_i + K_{N-2,N-2}\, x_{N-2} + K_{N-2,N-2}\, w_{N-2} + L_{N-2}$. Then, by a dynamic programming approach, the cost of the encoder at $N-1$st stage can be written as

$$\begin{aligned}
J^{*,e}_{N-2} &= \min_{x_{N-2}=\gamma^e_{N-2}(m_{[0,N-2]},y_{[0,N-3]})} \mathbb{E}\left[(m_{N-2}-u_{N-2}-b)^2 + \lambda x^2_{N-2} + J^{*,e}_{N-1}\right] \\
&= \min_{x_{N-2}} \mathbb{E}\Bigg[\Bigg(m_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, y_i - K_{N-2,N-2}\, x_{N-2} - K_{N-2,N-2}\, w_{N-2} \\
&\qquad - L_{N-2} - b\Bigg)^2 + \lambda x^2_{N-2} + J^{*,e}_{N-1}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \min_{x_{N-2}} \mathbb{E}\Bigg[\left(m_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, y_i - K_{N-2,N-2}\, x_{N-2} - L_{N-2} - b \right)^2 \\
&\qquad + \lambda x_{N-2}^2 \Bigg] + \mathbb{E}\Bigg[\frac{\lambda}{K_{N-1,N-1}^2 + \lambda} \Bigg(g m_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, y_i \\
&\qquad - K_{N-1,N-2}\, x_{N-2} - L_{N-1} - b \Bigg)^2 \Bigg] + \frac{\lambda}{K_{N-1,N-1}^2 + \lambda} \\
&\qquad\quad \times \left(\sigma_{V_{N-2}}^2 + K_{N-1,N-2}^2\, \sigma_{W_{N-2}}^2 \right) + K_{N-1,N-1}^2\, \sigma_{W_{N-1}}^2 \\
&\qquad + K_{N-2,N-2}^2\, \sigma_{W_{N-2}}^2 \\
&= \min_{x_{N-2}} \mathbb{E}\Bigg[\left(K_{N-2,N-2}^2 + \lambda + \frac{\lambda K_{N-1,N-2}^2}{K_{N-1,N-1}^2 + \lambda} \right) x_{N-2}^2 \\
&\qquad - 2K_{N-2,N-2} \left(m_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, y_i - L_{N-2} - b \right) x_{N-2} \\
&\qquad - \frac{2\lambda K_{N-1,N-2}}{K_{N-1,N-1}^2 + \lambda} \left(g m_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, y_i - L_{N-1} - b \right) x_{N-2} \\
&\qquad + \left(m_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, y_i - L_{N-2} - b \right)^2 \\
&\qquad + \frac{\lambda}{K_{N-1,N-1}^2 + \lambda} \left(g m_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, y_i - L_{N-1} - b \right)^2 \Bigg] \\
&\qquad + \frac{\lambda}{K_{N-1,N-1}^2 + \lambda} \left(\sigma_{V_{N-2}}^2 + K_{N-1,N-2}^2\, \sigma_{W_{N-2}}^2 \right) \\
&\qquad + K_{N-1,N-1}^2\, \sigma_{W_{N-1}}^2 + K_{N-2,N-2}^2\, \sigma_{W_{N-2}}^2 \\
&= \min_{x_{N-2}} \left(K_{N-2,N-2}^2 + \lambda + \frac{\lambda K_{N-1,N-2}^2}{K_{N-1,N-1}^2 + \lambda} \right) \\
&\qquad\quad \times \mathbb{E}\Bigg[\Bigg(x_{N-2} - \frac{K_{N-2,N-2}}{K_{N-2,N-2}^2 + \lambda + \frac{\lambda K_{N-1,N-2}^2}{K_{N-1,N-1}^2+\lambda}} \\
&\qquad\quad \times \left(m_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, y_i - L_{N-2} - b \right) \\
&\qquad - \frac{\frac{\lambda K_{N-1,N-2}}{K_{N-1,N-1}^2+\lambda}}{K_{N-2,N-2}^2 + \lambda + \frac{\lambda K_{N-1,N-2}^2}{K_{N-1,N-1}^2+\lambda}}
\end{aligned}
$$

$$
\times \left(gm_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, y_i - L_{N-1} - b \right) \Bigg)^2 \Bigg]
$$

$$
- \mathbb{E}\Bigg[\frac{K_{N-2,N-2}^2}{K_{N-2,N-2}^2 + \lambda + \frac{\lambda K_{N-1,N-2}^2}{K_{N-1,N-1}^2 + \lambda}}
$$

$$
\times \left(m_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, y_i - L_{N-2} - b \right)^2 \Bigg]
$$

$$
- \mathbb{E}\Bigg[\frac{\left(\frac{\lambda K_{N-1,N-2}}{K_{N-1,N-1}^2 + \lambda}\right)^2}{K_{N-2,N-2}^2 + \lambda + \frac{\lambda K_{N-1,N-2}^2}{K_{N-1,N-1}^2 + \lambda}}
$$

$$
\times \left(gm_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, y_i - L_{N-1} - b \right)^2 \Bigg]
$$

$$
+ \mathbb{E}\Bigg[\left(m_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, y_i - L_{N-2} - b \right)^2
$$

$$
+ \frac{\lambda}{K_{N-1,N-1}^2 + \lambda} \left(gm_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, y_i - L_{N-1} - b \right)^2 \Bigg]
$$

$$
+ \frac{\lambda}{K_{N-1,N-1}^2 + \lambda} \left(\sigma_{V_{N-2}}^2 + K_{N-1,N-2}^2\, \sigma_{W_{N-2}}^2 \right)
$$

$$
+ K_{N-1,N-1}^2\, \sigma_{W_{N-1}}^2 + K_{N-2,N-2}^2\, \sigma_{W_{N-2}}^2 .
$$

Then, the optimal $\gamma_{N-2}^e(m_{[0,N-2]}, y_{[0,N-3]})$ is obtained as

$$
\gamma_{N-2}^e(m_{[0,N-2]}, y_{[0,N-3]}) =
$$

$$
\frac{K_{N-2,N-2}}{K_{N-2,N-2}^2 + \lambda + \frac{\lambda K_{N-1,N-2}^2}{K_{N-1,N-1}^2 + \lambda}} \left(m_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, y_i - L_{N-2} - b \right)
$$

$$
+ \frac{\frac{\lambda K_{N-1,N-2}}{K_{N-1,N-1}^2 + \lambda}}{K_{N-2,N-2}^2 + \lambda + \frac{\lambda K_{N-1,N-2}^2}{K_{N-1,N-1}^2 + \lambda}} \left(gm_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, y_i - L_{N-1} - b \right) .
$$

Notice that even though $m_{[0,N-2]}$ and $y_{[0,N-3]}$ are available to the encoder, the encoder uses only m_{N-2} and $y_{[0,N-3]}$ at the $N-1$st stage; i.e., the encoder does not need $m_{[0,N-3]}$.

It can be seen that the optimal x_k can be obtained as an affine function of m_k and $y_{[0,k-1]}$ for each stage, $k = 0, 1, \ldots, N-1$ by completing the square, since the cost of the current stage and the next stages consist of the quadratic function of x_k after using the proper identities; i.e., $m_k = gm_{k-1} + v_{k-1}$ and $y_k = x_k + w_k$.

3.8.2 Proof of Theorem 3.4.2

If the decoder policies are $u_0 = \gamma_0^d(y_0) = Ky_0 + L$ and $u_1 = \gamma_1^d(y_0, y_1) = M_0 y_0 + M_1 y_1 + N$ where K, L, M_0, M_1 and N are scalars, then the optimal encoder policies are in (3.9).

$$\begin{aligned}
\gamma_0^{*,e}(m_0) &= \frac{KM_1^2 + \lambda K + \lambda g M_0}{(\lambda + K^2)(\lambda + M_1^2) + \lambda M_0^2}\, m_0 - \frac{K(M_1^2 + \lambda)(L + b) + \lambda M_0(N + b)}{(\lambda + K^2)(\lambda + M_1^2) + \lambda M_0^2}\,, \\
\gamma_1^{*,e}(m_0, m_1, y_0) &= \frac{M_1}{M_1^2 + \lambda} m_1 - \frac{M_0 M_1}{M_1^2 + \lambda} y_0 - \frac{M_1(N + b)}{M_1^2 + \lambda}\,.
\end{aligned} \tag{3.9}$$

If the encoder policies are $x_0 = \gamma_0^e(m_0) = Am_0 + C$ and $x_1 = \gamma_1^e(m_0, m_1, y_0) = Fm_0 + B_0 y_0 + B_1 m_1 + D$ where A, C, F, B_0, B_1 and D are scalars, then the optimal decoder policies are in (3.10).

$$\begin{aligned}
&\gamma_0^{*,d}(y_0) = \frac{A\sigma_{M_0}^2}{A^2\sigma_{M_0}^2 + \sigma_{W_0}^2}\, y_0 - \frac{AC\sigma_{M_0}^2}{A^2\sigma_{M_0}^2 + \sigma_{W_0}^2}\,, \\
&\gamma_1^{*,d}(y_0, y_1) = \\
&\frac{-(g^2 B_0 B_1 + gB_0 F)\sigma_{M_0}^2\sigma_{W_0}^2 - (A^2 B_0 B_1 + AB_1 F)\sigma_{M_0}^2\sigma_{V_0}^2 + gA\sigma_{M_0}^2\sigma_{W_1}^2 - B_0 B_1 \sigma_{V_0}^2\sigma_{W_0}^2}{(gB_1 + F)^2\sigma_{M_0}^2\sigma_{W_0}^2 + A^2\sigma_{M_0}^2\sigma_{W_1}^2 + A^2 B_1^2\sigma_{M_0}^2\sigma_{V_0}^2 + \sigma_{W_0}^2\sigma_{W_1}^2 + B_1^2\sigma_{V_0}^2\sigma_{W_0}^2}\, y_0 \\
&+ \frac{(g^2 B_1 + gF)\sigma_{M_0}^2\sigma_{W_0}^2 + A^2 B_1 \sigma_{M_0}^2\sigma_{V_0}^2 + B_1\sigma_{V_0}^2\sigma_{W_0}^2}{(gB_1 + F)^2\sigma_{M_0}^2\sigma_{W_0}^2 + A^2\sigma_{M_0}^2\sigma_{W_1}^2 + A^2 B_1^2\sigma_{M_0}^2\sigma_{V_0}^2 + \sigma_{W_0}^2\sigma_{W_1}^2 + B_1^2\sigma_{V_0}^2\sigma_{W_0}^2}\, y_1 \\
&- \frac{gAC\sigma_{M_0}^2\sigma_{W_1}^2 + (g^2 B_1 + gF)D\sigma_{M_0}^2\sigma_{W_0}^2 + (A^2 B_1 D - AB_1 CF)\sigma_{M_0}^2\sigma_{V_0}^2 + B_1 D\sigma_{V_0}^2\sigma_{W_0}^2}{(gB_1 + F)^2\sigma_{M_0}^2\sigma_{W_0}^2 + A^2\sigma_{M_0}^2\sigma_{W_1}^2 + A^2 B_1^2\sigma_{M_0}^2\sigma_{V_0}^2 + \sigma_{W_0}^2\sigma_{W_1}^2 + B_1^2\sigma_{V_0}^2\sigma_{W_0}^2}\,.
\end{aligned} \tag{3.10}$$

By using the policy equations above, the equations for the parameters can be written as follow:

$$\begin{aligned}
A &= \frac{KM_1^2 + \lambda K + \lambda g M_0}{(\lambda + K^2)(\lambda + M_1^2) + \lambda M_0^2}\,, \\
C &= -\frac{K(M_1^2 + \lambda)(L + b) + \lambda M_0(N + b)}{(\lambda + K^2)(\lambda + M_1^2) + \lambda M_0^2}\,,
\end{aligned}$$

$$
\begin{aligned}
F &= 0 \\
B_0 &= -\frac{M_0 M_1}{M_1^2+\lambda} = -M_0 B_1\,, \\
B_1 &= \frac{M_1}{M_1^2+\lambda}\,, \\
D &= -\frac{M_1(N+b)}{M_1^2+\lambda} = -(N+b)B_1\,, \\
K &= \frac{A\sigma_{M_0}^2}{A^2\sigma_{M_0}^2+\sigma_{W_0}^2}\,, \\
L &= -\frac{AC\sigma_{M_0}^2}{A^2\sigma_{M_0}^2+\sigma_{W_0}^2} = -KC\,, \\
M_0 &= \frac{-g^2B_0B_1\sigma_{M_0}^2\sigma_{W_0}^2 - A^2B_0B_1\sigma_{M_0}^2\sigma_{V_0}^2 + gA\sigma_{M_0}^2\sigma_{W_1}^2 - B_0B_1\sigma_{V_0}^2\sigma_{W_0}^2}{g^2B_1^2\sigma_{M_0}^2\sigma_{W_0}^2 + A^2\sigma_{M_0}^2\sigma_{W_1}^2 + A^2B_1^2\sigma_{M_0}^2\sigma_{V_0}^2 + \sigma_{W_0}^2\sigma_{W_1}^2 + B_1^2\sigma_{V_0}^2\sigma_{W_0}^2}\,, \\
M_1 &= \frac{g^2B_1\sigma_{M_0}^2\sigma_{W_0}^2 + A^2B_1\sigma_{M_0}^2\sigma_{V_0}^2 + B_1\sigma_{V_0}^2\sigma_{W_0}^2}{g^2B_1^2\sigma_{M_0}^2\sigma_{W_0}^2 + A^2\sigma_{M_0}^2\sigma_{W_1}^2 + A^2B_1^2\sigma_{M_0}^2\sigma_{V_0}^2 + \sigma_{W_0}^2\sigma_{W_1}^2 + B_1^2\sigma_{V_0}^2\sigma_{W_0}^2}\,, \\
N &= -\frac{gAC\sigma_{M_0}^2\sigma_{W_1}^2 + g^2B_1D\sigma_{M_0}^2\sigma_{W_0}^2 + A^2B_1D\sigma_{M_0}^2\sigma_{V_0}^2 + B_1D\sigma_{V_0}^2\sigma_{W_0}^2}{g^2B_1^2\sigma_{M_0}^2\sigma_{W_0}^2 + A^2\sigma_{M_0}^2\sigma_{W_1}^2 + A^2B_1^2\sigma_{M_0}^2\sigma_{V_0}^2 + \sigma_{W_0}^2\sigma_{W_1}^2 + B_1^2\sigma_{V_0}^2\sigma_{W_0}^2}\,.
\end{aligned}
$$

By inserting $B_0 = -M_0B_1$ into M_0,

$$
\begin{aligned}
M_0 &= \frac{g^2B_1^2M_0\sigma_{M_0}^2\sigma_{W_0}^2 + A^2B_1^2M_0\sigma_{M_0}^2\sigma_{V_0}^2 + gA\sigma_{M_0}^2\sigma_{W_1}^2 + B_1^2M_0\sigma_{V_0}^2\sigma_{W_0}^2}{g^2B_1^2\sigma_{M_0}^2\sigma_{W_0}^2 + A^2\sigma_{M_0}^2\sigma_{W_1}^2 + A^2B_1^2\sigma_{M_0}^2\sigma_{V_0}^2 + \sigma_{W_0}^2\sigma_{W_1}^2 + B_1^2\sigma_{V_0}^2\sigma_{W_0}^2} \\
&\Rightarrow M_0\left(A^2\sigma_{M_0}^2+\sigma_{W_0}^2\right)\sigma_{W_1}^2 = gA\sigma_{M_0}^2\sigma_{W_1}^2 = gK\left(A^2\sigma_{M_0}^2+\sigma_{W_0}^2\right)\sigma_{W_1}^2 \\
&\Rightarrow M_0 = gK
\end{aligned}
$$

is obtained. Similarly, by inserting $B_1 = \frac{M_1}{M_1^2+\lambda}$ into M_1,

$$
\begin{aligned}
&M_1 = \\
&\frac{g^2M_1(M_1^2+\lambda)\sigma_{M_0}^2\sigma_{W_0}^2 + A^2M_1(M_1^2+\lambda)\sigma_{M_0}^2\sigma_{V_0}^2 + M_1(M_1^2+\lambda)\sigma_{V_0}^2\sigma_{W_0}^2}{g^2M_1^2\sigma_{M_0}^2\sigma_{W_0}^2 + A^2(M_1^2+\lambda)^2\sigma_{M_0}^2\sigma_{W_1}^2 + A^2M_1^2\sigma_{M_0}^2\sigma_{V_0}^2 + (M_1^2+\lambda)^2\sigma_{W_0}^2\sigma_{W_1}^2 + M_1^2\sigma_{V_0}^2\sigma_{W_0}^2} \\
&\Rightarrow g^2M_1\lambda\sigma_{M_0}^2\sigma_{W_0}^2 + A^2M_1\lambda\sigma_{M_0}^2\sigma_{V_0}^2 + M_1\lambda\sigma_{V_0}^2\sigma_{W_0}^2 = \\
&\qquad A^2M_1(M_1^2+\lambda)^2\sigma_{M_0}^2\sigma_{W_1}^2 + M_1(M_1^2+\lambda)^2\sigma_{W_0}^2\sigma_{W_1}^2
\end{aligned}
$$

is obtained. For nonzero M_1,

$$
\begin{aligned}
&(M_1^2+\lambda)^2(A^2\sigma_{M_0}^2+\sigma_{W_0}^2)\sigma_{W_1}^2 = \lambda\left(g^2\sigma_{M_0}^2\sigma_{W_0}^2 + (A^2\sigma_{M_0}^2+\sigma_{W_0}^2)\sigma_{V_0}^2\right) \\
&\Rightarrow (M_1^2+\lambda)^2\sigma_{W_1}^2 = \lambda\left(g^2\frac{\sigma_{M_0}^2\sigma_{W_0}^2}{A^2\sigma_{M_0}^2+\sigma_{W_0}^2} + \sigma_{V_0}^2\right) \qquad (3.11)
\end{aligned}
$$

is found. Further, by inserting $D = -(N+b)B_1$ into N,

$$N = -\frac{gAC\sigma_{M_0}^2\sigma_{W_1}^2 - g^2B_1^2(N+b)\sigma_{M_0}^2\sigma_{W_0}^2 - A^2B_1^2(N+b)\sigma_{M_0}^2\sigma_{V_0}^2 - B_1^2(N+b)\sigma_{V_0}^2\sigma_{W_0}^2}{g^2B_1^2\sigma_{M_0}^2\sigma_{W_0}^2 + A^2\sigma_{M_0}^2\sigma_{W_1}^2 + A^2B_1^2\sigma_{M_0}^2\sigma_{V_0}^2 + \sigma_{W_0}^2\sigma_{W_1}^2 + B_1^2\sigma_{V_0}^2\sigma_{W_0}^2}$$

$$\Rightarrow A^2N\sigma_{M_0}^2\sigma_{W_1}^2 + N\sigma_{W_0}^2\sigma_{W_1}^2 = - gAC\sigma_{M_0}^2\sigma_{W_1}^2 + b\left(g^2B_1^2\sigma_{M_0}^2\sigma_{W_0}^2 + A^2B_1^2\sigma_{M_0}^2\sigma_{V_0}^2 + B_1^2\sigma_{V_0}^2\sigma_{W_0}^2\right)$$

$$\Rightarrow N = -\frac{AC\sigma_{M_0}^2}{A^2\sigma_{M_0}^2 + \sigma_{W_0}^2}\,g + \frac{bB_1^2}{\sigma_{W_1}^2}\left(\frac{g^2\sigma_{M_0}^2\sigma_{W_0}^2}{A^2\sigma_{M_0}^2 + \sigma_{W_0}^2} + \sigma_{V_0}^2\right)$$

is obtained. Since $\left(\frac{g^2\sigma_{M_0}^2\sigma_{W_0}^2}{A^2\sigma_{M_0}^2+\sigma_{W_0}^2} + \sigma_{V_0}^2\right) = \frac{(M_1^2+\lambda)^2\sigma_{W_1}^2}{\lambda}$ for nonzero M_1 by (3.11), we will have $N = gL + \frac{bM_1^2}{\lambda}$. Finally, by inserting $L = -KC$ and $N = -M_0C + \frac{bM_1^2}{\lambda}$ into C for nonzero M_1,

$$C = -\frac{K(M_1^2+\lambda)(-KC+b) + \lambda M_0(-M_0C + \frac{bM_1^2}{\lambda} + b)}{(\lambda+K^2)(\lambda+M_1^2) + \lambda M_0^2}$$

$$\Rightarrow \lambda(\lambda+M_1^2)C = -b\left(K(M_1^2+\lambda) + M_0M_1^2 + \lambda M_0\right) = -b(M_1^2+\lambda)(K+M_0)$$

$$\Rightarrow C = -\frac{b}{\lambda}(K+M_0) = -\frac{b}{\lambda}(g+1)K\,.$$

After some algebraic manipulations, consider the following cases:

(i) $(g^2+1)\sigma_{M_0}^2/\sigma_{W_0}^2 < \sigma_{M_1}^2/\sigma_{W_1}^2$: By the relation between A and K ($A = \frac{KM_1^2+\lambda K+\lambda gM_0}{(\lambda+K^2)(\lambda+M_1^2)+\lambda M_0^2}$ and $K = \frac{A\sigma_{M_0}^2}{A^2\sigma_{M_0}^2+\sigma_{W_0}^2}$), and assuming nonzero A,

$$A = \frac{KM_1^2+\lambda K+\lambda gM_0}{(\lambda+K^2)(\lambda+M_1^2)+\lambda M_0^2} = \frac{KM_1^2+\lambda K+\lambda g^2K}{(\lambda+K^2)(\lambda+M_1^2)+\lambda g^2K^2}$$

$$= \frac{K(\lambda+M_1^2+\lambda g^2)}{K^2(\lambda+M_1^2+\lambda g^2)+\lambda(\lambda+M_1^2)}$$

$$\Rightarrow \frac{A^2\sigma_{M_0}^2}{A^2\sigma_{M_0}^2+\sigma_{W_0}^2} = \frac{K^2(\lambda+M_1^2+\lambda g^2)}{K^2(\lambda+M_1^2+\lambda g^2)+\lambda(\lambda+M_1^2)}$$

$$\Rightarrow \frac{\sigma_{W_0}^2}{A^2\sigma_{M_0}^2+\sigma_{W_0}^2} = \frac{\lambda(\lambda+M_1^2)}{K^2(\lambda+M_1^2+\lambda g^2)+\lambda(\lambda+M_1^2)}$$

$$\Rightarrow \lambda A^2(\lambda+M_1^2)\sigma_{M_0}^2 = K^2(\lambda+M_1^2+\lambda g^2)\sigma_{W_0}^2$$

$$\Rightarrow (\lambda+M_1^2)(\lambda A^2\sigma_{M_0}^2 - K^2\sigma_{W_0}^2) = \lambda g^2K^2\sigma_{W_0}^2$$

$$\Rightarrow M_1^2 = \frac{\lambda g^2K^2\sigma_{W_0}^2}{\lambda A^2\sigma_{M_0}^2 - K^2\sigma_{W_0}^2} - \lambda \geq 0$$

$$\Rightarrow g^2K^2\sigma^2_{W_0} \geq \lambda A^2\sigma^2_{M_0} - K^2\sigma^2_{W_0}$$
$$\Rightarrow \lambda \leq \frac{K^2}{A^2}\frac{g^2\sigma^2_{W_0}+\sigma^2_{W_0}}{\sigma^2_{M_0}} \overset{(a)}{\leq} \left(\frac{\sigma^2_{M_0}}{\sigma^2_{W_0}}\right)^2 \frac{g^2\sigma^2_{W_0}+\sigma^2_{W_0}}{\sigma^2_{M_0}} = \left(g^2+1\right)\frac{\sigma^2_{M_0}}{\sigma^2_{W_0}}$$

is obtained. Here, (a) follows from $\frac{K}{A} = \frac{\sigma^2_{M_0}}{A^2\sigma^2_{M_0}+\sigma^2_{W_0}} \leq \frac{\sigma^2_{M_0}}{\sigma^2_{W_0}}$ for nonzero A. Hence, for $\lambda \geq (g^2+1)\frac{\sigma^2_{M_0}}{\sigma^2_{W_0}}$, there does not exist an informative affine equilibrium at the first stage since $A = K = 0$.

Now suppose that $\lambda > (g^2+1)\frac{\sigma^2_{M_0}}{\sigma^2_{W_0}}$. Then the two-stage game setup reduces to the one-stage game setup; i.e., $A = C = B_0 = K = L = M_0 = 0$, and the encoder and the decoder policies are $\gamma^e_1(m_1) = B_1 m_1 + D$ and $\gamma^d_1(y_1) = M_1 y_1 + N$, respectively. Thus, the equilibrium is informative if and only if $\max\left\{\frac{\sigma^2_{M_1}-2b^2-\sqrt{\sigma^2_{M_1}}\sqrt{\sigma^2_{M_1}-4b^2}}{2\sigma^2_{W_1}}, (g^2+1)\frac{\sigma^2_{M_0}}{\sigma^2_{W_0}}\right\} < \lambda < \frac{\sigma^2_{M_1}-2b^2+\sqrt{\sigma^2_{M_1}}\sqrt{\sigma^2_{M_1}-4b^2}}{2\sigma^2_{W_1}}$ and $\sigma^2_{M_1} \geq 4b^2$ by Theorem 2.4.4, and $M_1 = \mp\sqrt{\sqrt{\frac{\lambda\sigma^2_{M_1}}{\sigma^2_{W_1}}}-\lambda}$, $B_1 = \mp\sqrt{\sqrt{\frac{\sigma^2_{W_1}}{\lambda\sigma^2_{M_1}}}-\frac{\sigma^2_{W_1}}{\sigma^2_{M_1}}}$, $D = \pm b\sqrt{\frac{1}{\lambda}\left(\sqrt{\frac{\sigma^2_{M_1}}{\lambda\sigma^2_{W_1}}}-1\right)}$, and $N = b\left(\sqrt{\frac{\sigma^2_{M_1}}{\lambda\sigma^2_{W_1}}}-1\right)$ at this informative equilibrium [55]. Otherwise, the equilibrium is non-informative.

(ii) $\underline{\sigma^2_{M_1}/\sigma^2_{W_1} < (g^2+1)\sigma^2_{M_0}/\sigma^2_{W_0}}$: By utilizing (3.11) and assuming nonzero M_1,

$$M_1^2 = \sqrt{\frac{\lambda}{\sigma^2_{W_1}}\left(g^2\frac{\sigma^2_{M_0}\sigma^2_{W_0}}{A^2\sigma^2_{M_0}+\sigma^2_{W_0}}+\sigma^2_{V_0}\right)} - \lambda \geq 0$$
$$\Rightarrow \sqrt{\frac{\lambda}{\sigma^2_{W_1}}\left(g^2\sigma^2_{M_0}+\sigma^2_{V_0}\right)} - \lambda \geq \sqrt{\frac{\lambda}{\sigma^2_{W_1}}\left(g^2\frac{\sigma^2_{M_0}\sigma^2_{W_0}}{A^2\sigma^2_{M_0}+\sigma^2_{W_0}}+\sigma^2_{V_0}\right)} - \lambda \geq 0$$
$$\Rightarrow \lambda \leq \frac{g^2\sigma^2_{M_0}+\sigma^2_{V_0}}{\sigma^2_{W_1}} = \frac{\sigma^2_{M_1}}{\sigma^2_{W_1}}$$

is obtained. Hence, for $\lambda > \frac{\sigma^2_{M_1}}{\sigma^2_{W_1}}$, $M_1 = 0$ and the second stage message m_1 will not be used in the game.

Now suppose that $\lambda > \frac{\sigma^2_{M_1}}{\sigma^2_{W_1}}$. Then $B_0 = B_1 = D = M_1 = 0$, $M_0 = gK$, $N = gL$, $C = -\frac{b}{\lambda}(g+1)K$, $L = \frac{b}{\lambda}(g+1)K^2$, and $A = \frac{(g^2+1)K}{(g^2+1)K^2+\lambda}$. By using the relation between A and K, and assuming nonzero K, $\left((g^2+1)K^2+\lambda\right)^2\sigma^2_{W_0} = \lambda(g^2+1)\sigma^2_{M_0}$ is obtained, which implies $K = \sqrt{\sqrt{\frac{\lambda}{g^2+1}\frac{\sigma^2_{M_0}}{\sigma^2_{W_0}}}-\frac{\lambda}{g^2+1}}$ and

$A = \sqrt{\sqrt{\frac{g^2+1}{\lambda}\frac{\sigma_{W_0}^2}{\sigma_{M_0}^2}} - \frac{\sigma_{W_0}^2}{\sigma_{M_0}^2}}$ for $0 < \lambda \le (g^2+1)\frac{\sigma_{M_0}^2}{\sigma_{W_0}^2}$. Hence, if $\frac{\sigma_{M_1}^2}{\sigma_{W_1}^2} < \lambda \le \frac{(g^2+1)\sigma_{M_0}^2}{\sigma_{W_0}^2}$, the second stage message m_1 will not be used in the game. If $\lambda > \max\left\{\frac{(g^2+1)\sigma_{M_0}^2}{\sigma_{W_0}^2}, \frac{\sigma_{M_1}^2}{\sigma_{W_1}^2}\right\}$, then $A = K = L = C = N = M_0 = 0$, which leads to a non-informative equilibrium. Then, for $\lambda < (g^2+1)\frac{\sigma_{M_0}^2}{\sigma_{W_0}^2}$, we have $K = \sqrt{\sqrt{\frac{\lambda}{g^2+1}\frac{\sigma_{M_0}^2}{\sigma_{W_0}^2}} - \frac{\lambda}{g^2+1}}$ and $A = \sqrt{\sqrt{\frac{g^2+1}{\lambda}\frac{\sigma_{W_0}^2}{\sigma_{M_0}^2}} - \frac{\sigma_{W_0}^2}{\sigma_{M_0}^2}}$.

Hence, for the 2-stage dynamic game with affine encoder and decoder, the results can be summarized as follows:

(i) If $\lambda > \max\left\{\frac{(g^2+1)\sigma_{M_0}^2}{\sigma_{W_0}^2}, \frac{\sigma_{M_1}^2}{\sigma_{W_1}^2}\right\}$, then there does not exist an informative affine equilibrium.

(ii) If $\frac{\sigma_{M_1}^2}{\sigma_{W_1}^2} < \lambda \le \frac{(g^2+1)\sigma_{M_0}^2}{\sigma_{W_0}^2}$, then the second stage message m_1 is not used in the game.

(iii) If $\frac{(g^2+1)\sigma_{M_0}^2}{\sigma_{W_0}^2} < \lambda \le \frac{\sigma_{M_1}^2}{\sigma_{W_1}^2}$, the equilibrium is informative if and only if $\sigma_{M_1}^2 \ge 4b^2$ and $\max\left\{\frac{\sigma_{M_1}^2 - 2b^2 - \sqrt{\sigma_{M_1}^2}\sqrt{\sigma_{M_1}^2 - 4b^2}}{2\sigma_{W_1}^2}, (g^2+1)\frac{\sigma_{M_0}^2}{\sigma_{W_0}^2}\right\} < \lambda < \frac{\sigma_{M_1}^2 - 2b^2 + \sqrt{\sigma_{M_1}^2}\sqrt{\sigma_{M_1}^2 - 4b^2}}{2\sigma_{W_1}^2}$.

3.8.3 Proof of Theorem 3.4.3

Similar to the multi-stage Stackelberg cheap talk analysis in Theorem 3.2.6, The optimal decoder actions are given as $u_k^* = \gamma_k^{*,d}(\mathcal{I}_k^d) = \mathbb{E}[m_k|\mathcal{I}_k^d] = \mathbb{E}[m_k|y_{[0,k]}]$ for $k = 0, 1, \ldots, N-1$.

Due to the Stackelberg assumption, the encoder knows that the decoder will use $u_k^* = \gamma_k^{*,d}(\mathcal{I}_k^d) = \mathbb{E}[m_k|\mathcal{I}_k^d]$ at each stage $k = 0, 1, \ldots, N-1$. Based on this assumption and the smoothing property of the expectation, the total encoder cost can be written as $J^e(\gamma_{[0,N-1]}^e, \gamma_{[0,N-1]}^d) = \mathbb{E}[\sum_{k=0}^{N-1}(m_k - u_k - b)^2 + \lambda x_k^2] =$

$\mathbb{E}\left[\sum_{k=0}^{N-1}\mathbb{E}[(m_k-\mathbb{E}[m_k|\mathcal{I}_k^d])^2+b^2+\lambda x_k^2|\mathcal{I}_k^d]\right]$. This problem is an instance of problems studied in [119], and can be reduced to a team problem where both the encoder and the decoder are minimizing the same cost. The linearity of the optimal encoder and decoder can be deduced from [119]. Here, we follow [119] and adapt the proof to our setup.

For the second part of the proof, the lower bound for the encoder cost will be obtained and analyzed. From the chain rule, $I(m_k;y_{[0,k]}) = I(m_k;y_{[0,k-1]}) + I(m_k;y_k|y_{[0,k-1]})$. By following similar arguments to those in [119] and [4, Theorem 11.3.1],

$$
\begin{aligned}
I(m_k;y_k|y_{[0,k-1]}) &= h(y_k|y_{[0,k-1]}) - h(y_k|m_k,y_{[0,k-1]})\\
&= h(y_k|y_{[0,k-1]}) - h(y_k|m_k,y_{[0,k-1]},\gamma_k^e(m_k,y_{[0,k-1]}))\\
&= h(y_k|y_{[0,k-1]}) - h(y_k|\gamma_k^e(m_k,y_{[0,k-1]}))\\
&\le h(y_k) - h(y_k|\gamma_k^e(m_k,y_{[0,k-1]}))\\
&= I\left(\gamma_k^e(m_k,y_{[0,k-1]});y_k\right) = I(x_k;y_k)\\
&\le \sup I(x_k;y_k)\\
&= \frac{1}{2}\log_2\left(1+\frac{P_k}{\sigma_{W_k}^2}\right) \triangleq \widehat{C}_k \text{ where } P_k = \mathbb{E}[x_k^2]\,.
\end{aligned}
$$

It can be seen that $m_k - \mathbb{E}[m_k|m_{k-1}]$ is orthogonal to the random variables $m_{k-1}, y_{[0,k-1]}$ where $y_{[0,k-1]}$ is included due to the Markov chain $m_k \leftrightarrow m_{k-1} \leftrightarrow (y_{[0,k-1]})$. By using this orthogonality, it follows that

$$
\begin{aligned}
&\mathbb{E}[(m_k-\mathbb{E}[m_k|y_{[0,k-1]}])^2]\\
&\quad= \mathbb{E}[(m_k-\mathbb{E}[m_k|m_{k-1}])^2] + \mathbb{E}[(\mathbb{E}[m_k|m_{k-1}]-\mathbb{E}[m_k|y_{[0,k-1]}])^2]\\
&\quad\overset{(a)}{=} \mathbb{E}[(m_k-\mathbb{E}[m_k|m_{k-1}])^2] + \mathbb{E}\left[\left(\mathbb{E}[m_k|m_{k-1}]-\mathbb{E}[\mathbb{E}[m_k|m_{k-1},y_{[0,k-1]}]|y_{[0,k-1]}]\right)^2\right]\\
&\quad\overset{(a)}{=} \mathbb{E}[(m_k-\mathbb{E}[m_k|m_{k-1}])^2] + \mathbb{E}\left[\left(\mathbb{E}[m_k|m_{k-1}]-\mathbb{E}[\mathbb{E}[m_k|m_{k-1}]|y_{[0,k-1]}]\right)^2\right]\\
&\quad\overset{(b)}{=} \sigma_{V_{k-1}}^2 + g^2\mathbb{E}[(m_{k-1}-\mathbb{E}[m_{k-1}|y_{[0,k-1]}])^2]\\
&\quad\overset{(c)}{\ge} \sigma_{V_{k-1}}^2 + g^2\sigma_{M_{k-1}}^2\,2^{-2C_{k-1}}\,,
\end{aligned}
\tag{3.12}
$$

where $C_k \triangleq \sup I(m_k;y_{[0,k]})$. Here, (a) holds due to the iterated expectation rule and the Markov chain property, (b) holds since $\mathbb{E}[m_k|m_{k-1}] =$

$\mathbb{E}[gm_{k-1}+v_{k-1}|m_{k-1}] = gm_{k-1}$, and (c) holds due to [4, Lemma 11.3.1]. From [4, Lemma 11.3.2], $I(m_k; y_{[0,k-1]})$ is maximized with linear policies, and the lower bound of (3.12), $\mathbb{E}[(m_k - \mathbb{E}[m_k|y_{[0,k-1]}])^2] \geq \sigma^2_{V_{k-1}} + g^2\sigma^2_{M_{k-1}} 2^{-2C_{k-1}} \triangleq \sigma^2_{M_k} 2^{-2\widetilde{C}_k}$, is achievable through linear policies where $\sup I(m_k; y_{[0,k-1]}) \triangleq \widetilde{C}_k = \frac{1}{2}\log_2\left(\frac{\sigma^2_{M_k}}{\sigma^2_{V_{k-1}}+g^2\sigma^2_{M_{k-1}}2^{-2C_{k-1}}}\right)$. Thus, we have the following recursion on upper bounds on mutual information for the N-stage signaling game:

$$
\begin{aligned}
C_k &= \sup I(m_k; y_{[0,k]}) = \widetilde{C}_k + \widehat{C}_k \\
&= \sup I(m_k; y_{[0,k-1]}) + \sup I(m_k; y_k|y_{[0,k-1]}) \\
&= \widetilde{C}_k + \widehat{C}_k \\
&= \frac{1}{2}\log_2\left(\frac{\sigma^2_{M_k}}{\sigma^2_{V_{k-1}} + g^2\sigma^2_{M_{k-1}}2^{-2C_{k-1}}}\right) + \frac{1}{2}\log_2\left(1+\frac{P_k}{\sigma^2_{W_k}}\right)
\end{aligned}
$$

for $k = 1, 2, \ldots, N-1$ with $C_0 = \frac{1}{2}\log_2\left(1+\frac{P_0}{\sigma^2_{W_0}}\right)$. Let the lower bound on $\mathbb{E}\left[\left(m_k - \mathbb{E}[m_k|y_{[0,k]}]\right)^2\right]$ be Δ_k; i.e., $\mathbb{E}\left[\left(m_k - \mathbb{E}[m_k|y_{[0,k]}]\right)^2\right] \geq \sigma^2_{M_k} 2^{-2C_k} \triangleq \Delta_k$. Then the following recursion can be obtained for the N-stage signaling game:

$$
\Delta_k = \frac{\sigma^2_{V_{k-1}} + g^2\Delta_{k-1}}{1+\frac{P_k}{\sigma^2_{W_k}}} \text{ for } k = 1, 2, \ldots, N-1\,,
$$

with $\Delta_0 = \frac{\sigma^2_{M_0}}{1+\frac{P_0}{\sigma^2_{W_0}}}$. Since $\Delta_k = \sigma^2_{M_k} 2^{-2C_k}$ by definition, $\Delta_k \leq \sigma^2_{M_k}$ for $k = 0, 1, \ldots, N-1$. At the equilibrium, since the decoder always chooses $u_k = \mathbb{E}[m_k|y_{[0,k]}]$ for $k = 0, 1, \ldots, N-1$, the total encoder cost for the first stage can be lower bounded by $J_0^{e,lower} = \sum_{i=0}^{N-1}(\Delta_i + \lambda P_i + b^2)$. Now observe the following:

$$
\frac{\partial \Delta_l}{\partial P_k} = \begin{cases} 0 & \text{if } l < k \\ g^2\left(1+\frac{P_l}{\sigma^2_{W_l}}\right)^{-1}\frac{\partial \Delta_{l-1}}{\partial P_k} - \frac{1}{\sigma^2_{W_l}}\frac{\partial P_l}{\partial P_k} & \\ \times\left(\sigma^2_{V_{l-1}} + g^2\Delta_{l-1}\right)\left(1+\frac{P_l}{\sigma^2_{W_l}}\right)^{-2} & \text{if } l \geq k \end{cases},
$$

where $\frac{\partial P_l}{\partial P_k} = 0$ for $l < k$ due to the information structure of the encoder. Then we obtain the following:

$$
\frac{\partial J_0^{e,lower}}{\partial P_{N-1}} = \lambda - \frac{\sigma^2_{V_{N-2}} + g^2\Delta_{N-2}}{\sigma^2_{W_{N-1}}}\left(1+\frac{P_{N-1}}{\sigma^2_{W_{N-1}}}\right)^{-2}
$$

$$
\begin{aligned}
&\geq \lambda - \left(\sigma^2_{V_{N-2}} + g^2\sigma^2_{M_{N-2}}\right)\frac{1}{\sigma^2_{W_{N-1}}} \\
&\geq \lambda - \frac{\sigma^2_{M_{N-1}}}{\sigma^2_{W_{N-1}}}\,.
\end{aligned}
$$

If $\lambda > \frac{\sigma^2_{M_{N-1}}}{\sigma^2_{W_{N-1}}}$, then $\frac{\partial J_0^{e,lower}}{\partial P_{N-1}} > 0$, which implies that $J_0^{e,lower}$ is an increasing function of P_{N-1}. For this case, in order to minimize $J_0^{e,lower}$, P_{N-1} must be chosen as 0; i.e., $P^*_{N-1} = 0$. Then, for $\lambda > \frac{\sigma^2_{M_{N-1}}}{\sigma^2_{W_{N-1}}}$, we have the following:

$$
\begin{aligned}
\frac{\partial J_0^{e,lower}}{\partial P_{N-2}} &= \lambda\left(1 + \frac{\partial P_{N-1}}{\partial P_{N-2}}\right) + \sum_{i=N-2}^{N-1}\frac{\partial \Delta_i}{\partial P_{N-2}} \\
&= \lambda\left(1 + \frac{\partial P_{N-1}}{\partial P_{N-2}}\right) + \left(g^2\left(1 + \frac{P_{N-1}}{\sigma^2_{W_{N-1}}}\right)^{-1} + 1\right)\frac{\partial \Delta_{N-2}}{\partial P_{N-2}} \\
&\qquad - \left(\sigma^2_{V_{N-2}} + g^2\Delta_{N-2}\right)\left(1 + \frac{P_{N-1}}{\sigma^2_{W_{N-1}}}\right)^{-2}\frac{1}{\sigma^2_{W_{N-1}}}\frac{\partial P_{N-1}}{\partial P_{N-2}} \\
&\overset{(a)}{=} \lambda + \frac{\partial \Delta_{N-2}}{\partial P_{N-2}}\left(g^2+1\right) \\
&= \lambda - \left(\sigma^2_{V_{N-3}} + g^2\Delta_{N-3}\right)\left(1 + \frac{P_{N-2}}{\sigma^2_{W_{N-2}}}\right)^{-2}\frac{g^2+1}{\sigma^2_{W_{N-2}}} \\
&\geq \lambda - \left(\sigma^2_{V_{N-3}} + g^2\sigma^2_{M_{N-3}}\right)\frac{1}{\sigma^2_{W_{N-2}}}\left(g^2+1\right) \\
&= \lambda - \frac{\sigma^2_{M_{N-2}}}{\sigma^2_{W_{N-2}}}\left(g^2+1\right)\,.
\end{aligned}
$$

Here, (a) holds since $P^*_{N-1} = 0$ for $\lambda > \frac{\sigma^2_{M_{N-1}}}{\sigma^2_{W_{N-1}}}$. If $\lambda > \max\left\{\frac{\sigma^2_{M_{N-1}}}{\sigma^2_{W_{N-1}}}, \frac{\sigma^2_{M_{N-2}}}{\sigma^2_{W_{N-2}}}\left(g^2+1\right)\right\}$, then $\frac{\partial J_0^{e,lower}}{\partial P_{N-2}} > 0$, which implies that $J_0^{e,lower}$ is an increasing function of P_{N-2}. For this case, in order to minimize $J_0^{e,lower}$, P_{N-2} must be chosen as 0. By following the similar approach and assumptions on λ, since $P^*_{N-1} = P^*_{N-2} = \cdots = P^*_{k+1} = 0$, we have the following:

$$
\begin{aligned}
\frac{\partial J_0^{e,lower}}{\partial P_k} &= \lambda + \sum_{i=k}^{N-1}\frac{\partial \Delta_i}{\partial P_k} = \lambda + \frac{\partial \Delta_k}{\partial P_k}\sum_{i=k}^{N-1}\prod_{j=k+1}^{i} g^2 \\
&= \lambda - \left(\sigma^2_{V_{k-1}} + g^2\Delta_{k-1}\right)\left(1 + \frac{P_k}{\sigma^2_{W_k}}\right)^{-2}\frac{1}{\sigma^2_{W_k}}\sum_{i=k}^{N-1}\prod_{j=k+1}^{i} g^2
\end{aligned}
$$

$$
\begin{aligned}
&\geq \lambda - \left(\sigma^2_{V_{k-1}} + g^2\sigma^2_{M_{k-1}}\right) \frac{1}{\sigma^2_{W_k}} \sum_{i=k}^{N-1} \prod_{j=k+1}^{i} g^2 \\
&= \lambda - \frac{\sigma^2_{M_k}}{\sigma^2_{W_k}} \sum_{i=k}^{N-1} g^{2(i-k)} \\
&= \lambda - \frac{\sigma^2_{M_k}}{\sigma^2_{W_k}} \sum_{i=0}^{N-k-1} g^{2i} \,,
\end{aligned}
$$

where $\prod_{i=k}^{l} = 1$ if $k > l$. If $\lambda > \frac{\sigma^2_{M_k}}{\sigma^2_{W_k}} \sum_{i=0}^{N-k-1} g^{2i}$, then $\frac{\partial J_0^{e,lower}}{\partial P_k} > 0$, which implies that $J_0^{e,lower}$ is an increasing function of P_k. For this case, in order to minimize $J_0^{e,lower}$, P_k must be chosen as 0.

By combining all the results above, it can be deduced that if $\lambda > \max_{k=0,1,\dots,N-1} \frac{\sigma^2_{M_k}}{\sigma^2_{W_k}} \sum_{i=0}^{N-k-1} g^{2i}$, then the lower bound $J_0^{e,lower}$ of the encoder costs J_0^e is minimized by choosing $P_0^* = P_1^* = \cdots = P_{N-1}^* = 0$; that is, the encoder does not signal any output. Hence, the encoder engages in a non-informative equilibrium and the minimum cost becomes $J_0^e = J_0^{e,lower} = \left(\sum_{i=0}^{N-1} \sigma^2_{M_i}\right) + Nb^2$ at this non-informative equilibrium.

3.8.4 Proof of Theorem 3.5.1

(i) Let the encoder policies be $\mathbf{x}_k = \gamma_k^e(\mathbf{m}_{[0,k]}, \mathbf{y}_{[0,k-1]}) = \sum_{i=0}^{k} A_{k,i}\,\mathbf{m}_i + \sum_{i=0}^{k-1} B_{k,i}\,\mathbf{y}_i + \mathbf{C}_k$ where $A_{k,i}$ and $B_{k,i}$ are $n \times n$ matrices, and $\mathbf{C}_k$ is $n \times 1$ vector for $k \leq N-1$ and $i \leq k$. Similar to the multi-stage multi-dimensional Stackelberg cheap talk analysis in Theorem 3.3.2, the optimal decoder actions can be found as $\mathbf{u}_k^* = \gamma_k^{*,d}(\mathcal{I}_k^d) = \mathbb{E}[\mathbf{m}_k|\mathcal{I}_k^d] = \mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k]}]$ for $k \leq N-1$. Notice that $\mathbf{y}_{[0,k]}$ is multivariate Gaussian for $k \leq N-1$ since $\mathbf{y}_k = \mathbf{x}_k + \mathbf{w}_k$. This proves that $\gamma_k^{*,d}(\mathcal{I}_k^d)$ is an affine function of $\mathbf{y}_{[0,k]}$ due to the joint Gaussianity of $\mathbf{m}_k$ and $\mathbf{y}_{[0,k]}$.

(ii) Let the decoder policies be $\mathbf{u}_k = \gamma_k^d(\mathbf{y}_{[0,k]}) = \sum_{i=0}^{k} K_{k,i}\,\mathbf{y}_i + \mathbf{L}_k$ where $K_{k,i}$ is $n \times n$ matrix and $\mathbf{L}_k$ is $n \times 1$ vector for $k \leq N-1$ and $i \leq k$. With $\mathbf{y}_{N-1} = \mathbf{x}_{N-1} + \mathbf{w}_{N-1}$, it follows that $\mathbf{u}_{N-1} = \sum_{i=0}^{N-2} K_{N-1,i}\,\mathbf{y}_i + K_{N-1,N-1}\,\mathbf{x}_{N-1} + K_{N-1,N-1}\,\mathbf{w}_{N-1} + \mathbf{L}_{N-1}$. Then, the encoder aims to find an optimal policy that minimizes the following at the final stage:

$$
\begin{aligned}
J^{*,e}_{N-1} &= \min_{\mathbf{x}_{N-1}=\gamma^e_{N-1}(\mathbf{m}_{[0,N-1]},\mathbf{y}_{[0,N-2]})} \mathbb{E}\left[\|\mathbf{m}_{N-1}-\mathbf{u}_{N-1}-\mathbf{b}\|^2+\lambda\|\mathbf{x}_{N-1}\|^2\right] \\
&= \min_{\mathbf{x}_{N-1}} \mathbb{E}\Bigg[\Bigg(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i - K_{N-1,N-1}\,\mathbf{x}_{N-1} - K_{N-1,N-1}\,\mathbf{w}_{N-1} \\
&\qquad - \mathbf{L}_{N-1}-\mathbf{b}\Bigg)^T\Bigg(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i - K_{N-1,N-1}\,\mathbf{x}_{N-1} \\
&\qquad - K_{N-1,N-1}\,\mathbf{w}_{N-1}-\mathbf{L}_{N-1}-\mathbf{b}\Bigg)+\lambda\mathbf{x}_{N-1}^T\mathbf{x}_{N-1}\Bigg] \\
&= \min_{\mathbf{x}_{N-1}} \mathbb{E}\Bigg[\Bigg(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\Bigg)^T \\
&\qquad\quad \times\Bigg(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\Bigg) \\
&\qquad - \Bigg(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\Bigg)^T K_{N-1,N-1}\,\mathbf{x}_{N-1} \\
&\qquad - \mathbf{x}_{N-1}^T K_{N-1,N-1}^T\Bigg(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\Bigg) \\
&\qquad + \mathbf{x}_{N-1}^T K_{N-1,N-1}^T K_{N-1,N-1}\,\mathbf{x}_{N-1} \\
&\qquad + \mathbf{w}_{N-1}^T K_{N-1,N-1}^T K_{N-1,N-1}\,\mathbf{w}_{N-1}+\lambda\mathbf{x}_{N-1}^T\mathbf{x}_{N-1}\Bigg] \\
&\overset{(a)}{=} \min_{\mathbf{x}_{N-1}} \mathbb{E}\Bigg[\Bigg((K_{N-1,N-1}^T K_{N-1,N-1}+\lambda I)\mathbf{x}_{N-1} \\
&\qquad - K_{N-1,N-1}^T\Bigg(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\Bigg)\Bigg)^T \\
&\qquad\quad \times\Bigg(K_{N-1,N-1}^T K_{N-1,N-1}+\lambda I\Bigg)^{-1} \\
&\qquad\quad \times\Bigg((K_{N-1,N-1}^T K_{N-1,N-1}+\lambda I)\mathbf{x}_{N-1} \\
&\qquad - K_{N-1,N-1}^T\Bigg(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\Bigg)\Bigg)
\end{aligned}
$$

$$+\left(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\right)^T$$
$$\times\left(I-K_{N-1,N-1}(K_{N-1,N-1}^TK_{N-1,N-1}+\lambda I)^{-1}K_{N-1,N-1}^T\right)$$
$$\times\left(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\right)$$
$$\left.+\,\mathbf{w}_{N-1}^TK_{N-1,N-1}^TK_{N-1,N-1}\,\mathbf{w}_{N-1}\right],$$

and by completing the square, the optimal encoder policy can be obtained as

$$\gamma_{N-1}^e(\mathbf{m}_{[0,N-1]},\mathbf{y}_{[0,N-2]})=\left(K_{N-1,N-1}^TK_{N-1,N-1}+\lambda I\right)^{-1}K_{N-1,N-1}^T$$
$$\times\left(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\right)\ ,$$

with the final stage encoder cost

$$J_{N-1}^{*,e}=\mathbb{E}\left[\left(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\right)^T\right.$$
$$\times\left(I-K_{N-1,N-1}(K_{N-1,N-1}^TK_{N-1,N-1}+\lambda I)^{-1}K_{N-1,N-1}^T\right)$$
$$\times\left(\mathbf{m}_{N-1}-\sum_{i=0}^{N-2}K_{N-1,i}\,\mathbf{y}_i-\mathbf{L}_{N-1}-\mathbf{b}\right)$$
$$\left.+\,\mathbf{w}_{N-1}^TK_{N-1,N-1}^TK_{N-1,N-1}\,\mathbf{w}_{N-1}\right]\ . \tag{3.13}$$

Notice that even though $\mathbf{m}_{[0,N-1]}$ and $\mathbf{y}_{[0,N-2]}$ are available to the encoder, the optimal encoder uses only $\mathbf{m}_{N-1}$ and $\mathbf{y}_{[0,N-2]}$ at the final stage; i.e., the encoder does not need $\mathbf{m}_{[0,N-2]}$. Then, the cost of the encoder at $(N-1)$st stage becomes

$$J_{N-2}^{*,e}=$$
$$\min_{\mathbf{x}_{N-2}=\gamma_{N-2}^e(\mathbf{m}_{[0,N-2]},\mathbf{y}_{[0,N-3]})}\mathbb{E}\left[\|\mathbf{m}_{N-2}-\mathbf{u}_{N-2}-\mathbf{b}\|^2+\lambda\|\mathbf{x}_{N-2}\|^2+J_{N-1}^{*,e}\right]\ . \tag{3.14}$$

By using the relation between the sources $\mathbf{m}_{N-1} = G\mathbf{m}_{N-2} + \mathbf{v}_{N-2}$ and $\mathbf{y}_{N-2} = \mathbf{x}_{N-2} + \mathbf{w}_{N-2}$, and defining $\Omega_{N-1} = (I - K_{N-1,N-1}(K^T_{N-1,N-1}K_{N-1,N-1} + \lambda I)^{-1}K^T_{N-1,N-1})$, the final stage encoder cost can be simplified as

$$
\begin{aligned}
J^{*,e}_{N-1} = \mathbb{E}\Bigg[\Bigg(& G\mathbf{m}_{N-2} + \mathbf{v}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\,\mathbf{y}_i - K_{N-1,N-2}\mathbf{x}_{N-2} \\
& - K_{N-1,N-2}\mathbf{w}_{N-2} - \mathbf{L}_{N-1} - \mathbf{b}\Bigg)^T \Omega_{N-1} \\
& \times \Bigg(G\mathbf{m}_{N-2} + \mathbf{v}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\,\mathbf{y}_i - K_{N-1,N-2}\mathbf{x}_{N-2} \\
& - K_{N-1,N-2}\mathbf{w}_{N-2} - \mathbf{L}_{N-1} - \mathbf{b}\Bigg) \\
& + \mathbf{w}^T_{N-1}K^T_{N-1,N-1}K_{N-1,N-1}\,\mathbf{w}_{N-1}\Bigg] \\
= \mathbb{E}\Bigg[& \Bigg(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\,\mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b}\Bigg)^T \Omega_{N-1} \\
& \times \Bigg(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\,\mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b}\Bigg) \\
& - \Bigg(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\,\mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b}\Bigg)\Omega_{N-1}K_{N-1,N-2}\mathbf{x}_{N-2} \\
& - \mathbf{x}^T_{N-2}K^T_{N-1,N-2}\Omega_{N-1}\Bigg(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\,\mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b}\Bigg) \\
& + \mathbf{x}^T_{N-2}K^T_{N-1,N-2}\Omega_{N-1}K_{N-1,N-2}\mathbf{x}_{N-2} \\
& + \mathbf{v}^T_{N-2}\Omega_{N-1}\mathbf{v}_{N-2} + \mathbf{w}^T_{N-2}K^T_{N-1,N-2}\Omega_{N-1}K_{N-1,N-2}\mathbf{w}_{N-2} \\
& + \mathbf{w}^T_{N-1}K^T_{N-1,N-1}K_{N-1,N-1}\,\mathbf{w}_{N-1}\Bigg].
\end{aligned}
$$

With $\mathbf{y}_{N-2} = \mathbf{x}_{N-2} + \mathbf{w}_{N-2}$, it follows that $\mathbf{u}_{N-2} = \sum_{i=0}^{N-3} K_{N-2,i}\,\mathbf{y}_i + K_{N-2,N-2}\,\mathbf{x}_{N-2} + K_{N-2,N-2}\,\mathbf{w}_{N-2} + \mathbf{L}_{N-2}$. Then, by completing the squares method, (3.14) becomes

$$
J^{*,e}_{N-2} = \min_{\mathbf{x}_{N-2}} \mathbb{E}\Bigg[\Bigg(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\,\mathbf{y}_i - K_{N-2,N-2}\,\mathbf{x}_{N-2}
$$

$$
\begin{aligned}
&\qquad\qquad - K_{N-2,N-2}\,\mathbf{w}_{N-2} - \mathbf{L}_{N-2} - \mathbf{b}\Bigg)^T \\
&\qquad \times \Bigg(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\,\mathbf{y}_i - K_{N-2,N-2}\,\mathbf{x}_{N-2} \\
&\qquad\qquad - K_{N-2,N-2}\,\mathbf{w}_{N-2} - \mathbf{L}_{N-2} - \mathbf{b}\Bigg) \\
&+ \lambda \mathbf{x}_{N-2}^T \mathbf{x}_{N-2} + J_{N-1}^{*,e}\Bigg] \\
= \min_{\mathbf{x}_{N-2}} \mathbb{E}\Bigg[&\Bigg(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\,\mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b}\Bigg)^T \\
&\qquad \times \Bigg(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\,\mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b}\Bigg) \\
&- \Bigg(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\,\mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b}\Bigg)^T K_{N-2,N-2}\,\mathbf{x}_{N-2} \\
&- \mathbf{x}_{N-2}^T K_{N-2,N-2}^T \Bigg(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\,\mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b}\Bigg) \\
&+ \mathbf{x}_{N-2}^T K_{N-2,N-2}^T K_{N-2,N-2}\,\mathbf{x}_{N-2} + \mathbf{w}_{N-2}^T K_{N-2,N-2}^T K_{N-2,N-2}\,\mathbf{w}_{N-2} \\
&+ \lambda \mathbf{x}_{N-2}^T \mathbf{x}_{N-2} + \Bigg(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\,\mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b}\Bigg)^T \Omega_{N-1} \\
&\qquad \times \Bigg(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\,\mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b}\Bigg) \\
&- \Bigg(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\,\mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b}\Bigg) \Omega_{N-1} K_{N-1,N-2} \mathbf{x}_{N-2} \\
&- \mathbf{x}_{N-2}^T K_{N-1,N-2}^T \Omega_{N-1} \Bigg(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\,\mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b}\Bigg) \\
&+ \mathbf{x}_{N-2}^T K_{N-1,N-2}^T \Omega_{N-1} K_{N-1,N-2} \mathbf{x}_{N-2} \\
&+ \mathbf{v}_{N-2}^T \Omega_{N-1} \mathbf{v}_{N-2} + \mathbf{w}_{N-2}^T K_{N-1,N-2}^T \Omega_{N-1} K_{N-1,N-2} \mathbf{w}_{N-2} \\
&+ \mathbf{w}_{N-1}^T K_{N-1,N-1}^T K_{N-1,N-1}\,\mathbf{w}_{N-1}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \min_{\mathbf{x}_{N-2}} \mathbb{E}\Bigg[\Bigg(& (K_{N-2,N-2}^T K_{N-2,N-2} + \lambda I + K_{N-1,N-2}^T \Omega_{N-1} K_{N-1,N-2}) \mathbf{x}_{N-2} \\
& - K_{N-2,N-2}^T \left(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, \mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b} \right) \\
& - K_{N-1,N-2}^T \Omega_{N-1} \left(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, \mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b} \right) \Bigg)^T \\
\times & (K_{N-2,N-2}^T K_{N-2,N-2} + \lambda I + K_{N-1,N-2}^T \Omega_{N-1} K_{N-1,N-2})^{-1} \\
\times & \Bigg((K_{N-2,N-2}^T K_{N-2,N-2} + \lambda I + K_{N-1,N-2}^T \Omega_{N-1} K_{N-1,N-2}) \mathbf{x}_{N-2} \\
& - K_{N-2,N-2}^T \left(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, \mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b} \right) \\
& - K_{N-1,N-2}^T \Omega_{N-1} \left(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, \mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b} \right) \Bigg) \\
& - \Bigg(K_{N-2,N-2}^T \left(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, \mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b} \right) \\
& + K_{N-1,N-2}^T \Omega_{N-1} \left(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, \mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b} \right) \Bigg)^T \\
\times & (K_{N-2,N-2}^T K_{N-2,N-2} + \lambda I + K_{N-1,N-2}^T \Omega_{N-1} K_{N-1,N-2})^{-1} \\
\times & \Bigg(K_{N-2,N-2}^T \left(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, \mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b} \right) \\
& + K_{N-1,N-2}^T \Omega_{N-1} \left(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, \mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b} \right) \Bigg) \\
& + \left(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, \mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b} \right)^T \\
& \qquad \times \left(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, \mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b} \right) \\
& + \left(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, \mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b} \right)^T \Omega_{N-1} \\
& \qquad \times \left(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, \mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b} \right)
\end{aligned}
$$

$$+ \mathbf{w}_{N-2}^T K_{N-2,N-2}^T K_{N-2,N-2}\, \mathbf{w}_{N-2} + \mathbf{v}_{N-2}^T \Omega_{N-1} \mathbf{v}_{N-2}$$
$$+ \mathbf{w}_{N-2}^T K_{N-1,N-2}^T \Omega_{N-1} K_{N-1,N-2} \mathbf{w}_{N-2}$$
$$\left. + \mathbf{w}_{N-1}^T K_{N-1,N-1}^T K_{N-1,N-1}\, \mathbf{w}_{N-1} \right] .$$

Hence, the optimal $\gamma_{N-2}^e(\mathbf{m}_{[0,N-2]}, \mathbf{y}_{[0,N-3]})$ is obtained as

$$\gamma_{N-2}^e(\mathbf{m}_{[0,N-2]}, \mathbf{y}_{[0,N-3]}) =$$
$$\left(K_{N-2,N-2}^T K_{N-2,N-2} + \lambda I + K_{N-1,N-2}^T \Omega_{N-1} K_{N-1,N-2} \right)^{-1}$$
$$\times \left(K_{N-2,N-2}^T \left(\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-2,i}\, \mathbf{y}_i - \mathbf{L}_{N-2} - \mathbf{b} \right) \right.$$
$$\left. + K_{N-1,N-2}^T \Omega_{N-1} \left(G\mathbf{m}_{N-2} - \sum_{i=0}^{N-3} K_{N-1,i}\, \mathbf{y}_i - \mathbf{L}_{N-1} - \mathbf{b} \right) \right) .$$

Notice that even though $\mathbf{m}_{[0,N-2]}$ and $\mathbf{y}_{[0,N-3]}$ are available to the encoder, the encoder uses only $\mathbf{m}_{N-2}$ and $\mathbf{y}_{[0,N-3]}$ at the $(N-1)$st stage; i.e., the encoder does not need $\mathbf{m}_{[0,N-3]}$. We observe that the optimal $\mathbf{x}_k$ can be obtained as an affine function of $\mathbf{m}_k$ and $\mathbf{y}_{[0,k-1]}$ at each stage $k = 0, 1, \ldots, N-1$ by completing the square, since the cost of the current stage and the next stages consist of the quadratic function of $\mathbf{x}_k$ after using the proper identities; i.e., $\mathbf{m}_k = G\mathbf{m}_{k-1} + \mathbf{v}_{k-1}$ and $\mathbf{y}_k = \mathbf{x}_k + \mathbf{w}_k$.

3.8.5 Proof of Theorem 3.5.2

We will follow an approach similar to that in [117] which restricted the analysis to a team problem and a scalar channel; [117] in turn builds on [118], which considers continuous time systems. Since the $(k+1)$st stage encoder policy only transmits the linearly encoded innovation by assumption, $\mathbf{x}_k = \gamma_k^e(\mathcal{I}_k^e) = A_k \widetilde{\mathbf{m}}_k$ where A_k is an $n \times n$ matrix for $k = 0, 1, \ldots, N-1$. Then the decoder receives $\mathbf{y}_k = \mathbf{x}_k + \mathbf{w}_k = A_k \widetilde{\mathbf{m}}_k + \mathbf{w}_k$ and applies the action $\mathbf{u}_k = \gamma_k^d(\mathcal{I}_k^d) = \mathbb{E}[\mathbf{m}_k | \mathbf{y}_{[0,k]}]$ to minimize his stage-wise cost $\|\mathbf{e}_k\|^2 \triangleq \mathbb{E}[\|\mathbf{m}_k - \mathbf{u}_k\|^2] = \mathbb{E}[(\mathbf{m}_k - \mathbf{u}_k)^T(\mathbf{m}_k - \mathbf{u}_k)] = \operatorname{tr}(\Sigma_{\mathbf{e}_k})$ for $k = 0, 1, \ldots, N-1$ where $\Sigma_{\mathbf{R}}$ stands for the covariance matrix of the random variable $\mathbf{R}$; i.e., $\Sigma_{\mathbf{R}} \triangleq \mathbb{E}[(\mathbf{R}-\mathbb{E}[\mathbf{R}])(\mathbf{R}-\mathbb{E}[\mathbf{R}])^T]$. Due to the orthogonality of $\widetilde{\mathbf{m}}_k$ and $\mathbf{y}_{[0,k-1]}$, and the iterated expectations rule, $\mathbf{u}_k = \mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k]}] = \mathbb{E}\left[\widetilde{\mathbf{m}}_k + \mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k-1]}]|\mathbf{y}_{[0,k]}\right] = \mathbb{E}[\widetilde{\mathbf{m}}_k|\mathbf{y}_k] + \mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k-1]}]$, and it follows that $\mathbf{e}_k = \mathbf{m}_k - \mathbf{u}_k = \mathbf{m}_k - \mathbb{E}[\widetilde{\mathbf{m}}_k|\mathbf{y}_k] - \mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k-1]}] = \widetilde{\mathbf{m}}_k - \mathbb{E}[\widetilde{\mathbf{m}}_k|\mathbf{y}_k]$. Since $\mathbb{E}[\widetilde{\mathbf{m}}_k|\mathbf{y}_k] = \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \left(\Sigma_{\mathbf{y}_k}\right)^{-1} \mathbf{y}_k$, the stage-wise cost of the decoder becomes the trace of the following:

$$
\begin{aligned}
\Sigma_{\mathbf{e}_k} &= \mathbb{E}[\mathbf{e}_k\mathbf{e}_k^T] = \mathbb{E}\left[(\mathbf{m}_k - \mathbf{u}_k)(\mathbf{m}_k - \mathbf{u}_k)^T\right] \\
&= \mathbb{E}\Big[\left(\widetilde{\mathbf{m}}_k - \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \left(\Sigma_{\mathbf{y}_k}\right)^{-1} \mathbf{y}_k\right) \left(\widetilde{\mathbf{m}}_k - \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \left(\Sigma_{\mathbf{y}_k}\right)^{-1} \mathbf{y}_k\right)^T \Big] \\
&= \Sigma_{\widetilde{\mathbf{m}}_k} - \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \left(\Sigma_{\mathbf{y}_k}\right)^{-1} A_k \Sigma_{\widetilde{\mathbf{m}}_k} \\
&= \Sigma_{\widetilde{\mathbf{m}}_k} - \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \left(A_k \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T + \Sigma_{\mathbf{w}_k}\right)^{-1} A_k \Sigma_{\widetilde{\mathbf{m}}_k} \\
&= \Sigma_{\widetilde{\mathbf{m}}_k} - \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \Sigma_{\mathbf{w}_k}^{-1/2} \left(\Sigma_{\mathbf{w}_k}^{-1/2} A_k \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \Sigma_{\mathbf{w}_k}^{-1/2} + I\right)^{-1} \Sigma_{\mathbf{w}_k}^{-1/2} A_k \Sigma_{\widetilde{\mathbf{m}}_k} \\
&= \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \left(I - \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} A_k^T \Sigma_{\mathbf{w}_k}^{-1/2} \left(\Sigma_{\mathbf{w}_k}^{-1/2} A_k \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} A_k^T \Sigma_{\mathbf{w}_k}^{-1/2} + I\right)^{-1} \Sigma_{\mathbf{w}_k}^{-1/2} A_k \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \right) \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \\
&\overset{(a)}{=} \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \left(I - H_k^T \left(H_k H_k^T + I\right)^{-1} H_k\right) \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \\
&\overset{(b)}{=} \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \left(I + H_k^T H_k\right)^{-1} \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2},
\end{aligned} \tag{3.15}
$$

where (a) follows from $H_k \triangleq \Sigma_{\mathbf{w}_k}^{-1/2} A_k \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}$, and (b) follows by utilizing the matrix inversion lemma, $(I + UWV)^{-1} = I - U(W^{-1} + VU)^{-1}V$, where $U = H_k^T$, $W = I$, and $V = H_k$.

Observe the following identity:

$$
\begin{aligned}
\mathbb{E}[\mathbf{m}_k|\mathbf{y}_k] &= \mathbb{E}[\mathbf{m}_k] + \mathbb{E}[\mathbf{m}_k\mathbf{y}_k^T] \left(\Sigma_{\mathbf{y}_k}\right)^{-1} \mathbf{y}_k \\
&= \mathbb{E}\left[(\widetilde{\mathbf{m}}_k + \mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k-1]}])(A_k\widetilde{\mathbf{m}}_k + \mathbf{w}_k)^T\right] \left(\Sigma_{\mathbf{y}_k}\right)^{-1} \mathbf{y}_k \\
&= \left(\Sigma_{\widetilde{\mathbf{m}}_k} A_k^T + \mathbb{E}\left[\mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k-1]}]\, \widetilde{\mathbf{m}}_k^T\right] A_k^T\right) \left(\Sigma_{\mathbf{y}_k}\right)^{-1} \mathbf{y}_k \\
&\overset{(a)}{=} \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \left(\Sigma_{\mathbf{y}_k}\right)^{-1} \mathbf{y}_k \,,
\end{aligned}
$$

where (a) due to the orthogonality of $\mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k-1]}]$ and $\widetilde{\mathbf{m}}_k$. Then the innovation can be expressed recursively as follows:

$$
\begin{aligned}
\widetilde{\mathbf{m}}_{k+1} &= \mathbf{m}_{k+1} - \mathbb{E}[\mathbf{m}_{k+1}|\mathbf{y}_{[0,k]}] \\
&= G\mathbf{m}_k + \mathbf{v}_k - \mathbb{E}[\mathbf{m}_{k+1}|\mathbf{y}_{[0,k-1]}] - \mathbb{E}[\mathbf{m}_{k+1}|\mathbf{y}_k]
\end{aligned}
$$

$$
\begin{aligned}
&= G\mathbf{m}_k + \mathbf{v}_k - G\mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k-1]}] - G\mathbb{E}[\mathbf{m}_k|\mathbf{y}_k] \\
&= G\widetilde{\mathbf{m}}_k + \mathbf{v}_k - G\mathbb{E}[\mathbf{m}_k|\mathbf{y}_k] \\
&= G\widetilde{\mathbf{m}}_k + \mathbf{v}_k - G\Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \left(\Sigma_{\mathbf{y}_k}\right)^{-1} \mathbf{y}_k \,.
\end{aligned}
$$

Then the covariance matrices of the innovations can be expressed as

$$
\begin{aligned}
\Sigma_{\widetilde{\mathbf{m}}_{k+1}} &= \mathbb{E}[\widetilde{\mathbf{m}}_{k+1}\widetilde{\mathbf{m}}_{k+1}^T] \\
&= G\Sigma_{\widetilde{\mathbf{m}}_k} G^T + \Sigma_{\mathbf{v}_k} - G\Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \left(\Sigma_{\mathbf{y}_k}\right)^{-1} A_k \Sigma_{\widetilde{\mathbf{m}}_k} G^T \\
&= G\Sigma_{\widetilde{\mathbf{m}}_k} G^T + \Sigma_{\mathbf{v}_k} - G\Sigma_{\widetilde{\mathbf{m}}_k} A_k^T \left(A_k \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T + \Sigma_{\mathbf{w}_k}\right)^{-1} A_k \Sigma_{\widetilde{\mathbf{m}}_k} G^T \\
&= G\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \Bigg(I - \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} A_k^T \Sigma_{\mathbf{w}_k}^{-1/2} \left(\Sigma_{\mathbf{w}_k}^{-1/2} A_k \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} A_k^T \Sigma_{\mathbf{w}_k}^{-1/2} + I\right)^{-1} \\
&\qquad \times \Sigma_{\mathbf{w}_k}^{-1/2} A_k \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \Bigg) \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} G^T + \Sigma_{\mathbf{v}_k} \\
&= G\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \left(I - H_k^T \left(H_k H_k^T + I\right)^{-1} H_k \right) \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} G^T + \Sigma_{\mathbf{v}_k} \\
&= G\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \left(I + H_k^T H_k\right)^{-1} \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} G^T + \Sigma_{\mathbf{v}_k} \,. \qquad (3.16)
\end{aligned}
$$

The optimal encoder chooses A_k in order to minimize his stage-wise cost

$$
\begin{aligned}
\mathbb{E}\big[\|\mathbf{m}_k - \mathbf{u}_k - \mathbf{b}\|^2 + \lambda\|\mathbf{x}_k\|^2\big] &= \mathbb{E}\left[\|\mathbf{m}_k - \mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k]}] - \mathbf{b}\|^2 + \lambda\|\mathbf{x}_k\|^2\right] \\
&= \mathbb{E}\left[\|\mathbf{m}_k - \mathbb{E}[\mathbf{m}_k|\mathbf{y}_{[0,k]}]\|^2 + \|\mathbf{b}\|^2 + \lambda\|\mathbf{x}_k\|^2\right] \\
&= \mathbb{E}\left[\|\mathbf{m}_k - \mathbf{u}_k\|^2 + \|\mathbf{b}\|^2 + \lambda\|\mathbf{x}_k\|^2\right] \\
&= \mathrm{tr}\left(\Sigma_{\mathbf{e}_k}\right) + \mathrm{tr}\left(\lambda\Sigma_{\mathbf{x}_k}\right) + \|\mathbf{b}\|^2 \\
&\overset{(a)}{=} \mathrm{tr}\left(\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(I + H_k^T H_k\right)^{-1}\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\right) + \mathrm{tr}\left(\lambda A_k \Sigma_{\widetilde{\mathbf{m}}_k} A_k^T\right) + \|\mathbf{b}\|^2 \\
&= \mathrm{tr}\left(\Sigma_{\widetilde{\mathbf{m}}_k}\left(I + H_k^T H_k\right)^{-1}\right) + \mathrm{tr}\left(\lambda\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} A_k^T A_k \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\right) + \|\mathbf{b}\|^2 \\
&\overset{(a)}{=} \mathrm{tr}\left(\Sigma_{\widetilde{\mathbf{m}}_k}\left(I + H_k^T H_k\right)^{-1}\right) + \mathrm{tr}\left(\lambda H_k^T \Sigma_{\mathbf{w}_k} H_k\right) + \|\mathbf{b}\|^2, \qquad (3.17)
\end{aligned}
$$

where (a) is obtained by using (3.15).

Let the value functions be $V_k\left(\Sigma_{\widetilde{\mathbf{m}}_k}\right) = \mathrm{tr}\left(K_k\Sigma_{\widetilde{\mathbf{m}}_k} + L_k\right)$ with K_k being diagonal. In the following we show that there exist such V_k that satisfy Bellman's principle of optimality [120, Theorem 3.2.1]. Here $V_k\left(\Sigma_{\widetilde{\mathbf{m}}_k}\right) \triangleq \min_{H_k} \Big(\mathcal{C}_k\left(\Sigma_{\widetilde{\mathbf{m}}_k}, H_k\right) +$

$V_{k+1}\left(\Sigma_{\widetilde{\mathbf{m}}_{k+1}}\right)\Big)$ and $\mathcal{C}_k\left(\Sigma_{\widetilde{\mathbf{m}}_k}, H_k\right) \triangleq \operatorname{tr}\left(\Sigma_{\mathbf{e}_k}\right) + \operatorname{tr}\left(\lambda \Sigma_{\mathbf{x}_k}\right) + \|\mathbf{b}\|^2$ is the stage-wise cost of the k-th stage encoder. Then,

$$\begin{aligned}
V_k\left(\Sigma_{\widetilde{\mathbf{m}}_k}\right) &= \min_{H_k}\left(\mathcal{C}_k\left(\Sigma_{\widetilde{\mathbf{m}}_k}, H_k\right) + V_{k+1}\left(\Sigma_{\widetilde{\mathbf{m}}_{k+1}}\right)\right) \\
&\overset{(a)}{=} \min_{H_k}\Bigg(\operatorname{tr}\left(\Sigma_{\widetilde{\mathbf{m}}_k}\left(I + H_k^T H_k\right)^{-1}\right) + \operatorname{tr}\left(\lambda H_k^T \Sigma_{\mathbf{w}_k} H_k\right) + \|\mathbf{b}\|^2 \\
&\qquad + \operatorname{tr}\left(K_{k+1}\Sigma_{\widetilde{\mathbf{m}}_{k+1}} + L_{k+1}\right)\Bigg) \\
&\overset{(b)}{=} \min_{H_k}\Bigg(\operatorname{tr}\left(\Sigma_{\widetilde{\mathbf{m}}_k}\left(I + H_k^T H_k\right)^{-1}\right) + \operatorname{tr}\left(\lambda H_k^T \Sigma_{\mathbf{w}_k} H_k\right) + \|\mathbf{b}\|^2 \\
&\qquad + \operatorname{tr}\left(K_{k+1} G \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(I + H_k^T H_k\right)^{-1}\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} G^T + K_{k+1}\Sigma_{\mathbf{v}_k} + L_{k+1}\right)\Bigg) \\
&\overset{(c)}{=} \operatorname{tr}\left(K_{k+1}\Sigma_{\mathbf{v}_k} + L_{k+1}\right) + \|\mathbf{b}\|^2 \\
&\qquad + \min_{H_k}\Bigg(\operatorname{tr}\left(\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(I + H_k^T H_k\right)^{-1}\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\right) + \operatorname{tr}\left(\lambda H_k^T \Sigma_{\mathbf{w}_k} H_k\right) \\
&\qquad\quad + \operatorname{tr}\left(G^T K_{k+1} G \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(I + H_k^T H_k\right)^{-1}\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\right)\Bigg) \\
&= \operatorname{tr}\left(K_{k+1}\Sigma_{\mathbf{v}_k} + L_{k+1}\right) + \|\mathbf{b}\|^2 \\
&\qquad + \min_{H_k}\Bigg(\operatorname{tr}\left(\left(G^T K_{k+1} G + I\right)\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(I + H_k^T H_k\right)^{-1}\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\right) \\
&\qquad\quad + \operatorname{tr}\left(\lambda H_k^T \Sigma_{\mathbf{w}_k} H_k\right)\Bigg) \\
&= \operatorname{tr}\left(K_{k+1}\Sigma_{\mathbf{v}_k} + L_{k+1}\right) + \|\mathbf{b}\|^2 \\
&\qquad + \min_{H_k}\Bigg(\operatorname{tr}\left(\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(G^T K_{k+1} G + I\right)\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(I + H_k^T H_k\right)^{-1}\right) \\
&\qquad\quad + \operatorname{tr}\left(\lambda H_k^T \Sigma_{\mathbf{w}_k} H_k\right)\Bigg),
\end{aligned} \tag{3.18}$$

where (a) follows by substituting $\mathcal{C}_k\left(\Sigma_{\widetilde{\mathbf{m}}_k}, H_k\right)$ using (3.17), (b) follows by employing (3.16), and (c) follows from the fact that K_{k+1} and L_{k+1} do not depend on H_k. The equivalent problem of the minimization of

$\mathrm{tr}\left(\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}\left(G^TK_{k+1}G+I\right)\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}\left(I+H_k^TH_k\right)^{-1}\right)$ over H_k under the constraint $\mathrm{tr}\left(\lambda H_k^T\Sigma_{\mathbf{w}_k}H_k\right)=\mu_k$ is considered in [106], and the solution technique can be adapted as follows:

Let $\nu_{k_1}\geq\nu_{k_2}\geq\ldots\geq\nu_{k_n}>0$ and $\tau_{k_n}\geq\tau_{k_{n-1}}\geq\ldots\geq\tau_{k_1}>0$ be the eigenvalues of $\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}\left(G^TK_{k+1}G+I\right)\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}$ and $\lambda\Sigma_{\mathbf{w}_k}$, respectively, and $\mu_{k_p}\triangleq\sum_{i=1}^{p}\left(\sqrt{\tau_{k_i}\nu_{k_i}}-\tau_{k_i}\right)$. If μ_{k_p} values are non-positive for $p=1,2,\ldots,n$, then the optimal H_k becomes zero; i.e., $H_k^*=0$. Otherwise, check if $\tau_{k_p}/\nu_{k_p}<1$ for p that makes μ_{k_p} positive. If the inequality is not satisfied, again the optimal H_k becomes zero; i.e., $H_k^*=0$. Finally, pick p and corresponding μ_{k_p} which give the minimum of $\frac{\left(\sum_{i=1}^{p}\sqrt{\tau_{k_i}\nu_{k_i}}\right)^2}{\mu_{k_p}+\sum_{i=1}^{p}\tau_{k_i}}+\sum_{i=p+1}^{n}\nu_{k_i}+\mu_{k_p}$. If that minimum is greater than $\mathrm{tr}\left(\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}\left(G^TK_{k+1}G+I\right)\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}\right)$, then the optimal H_k becomes zero; i.e., $H_k^*=0$. Otherwise, the optimal H_k is found as $H_k^*=\Pi_k\zeta_kP_k^T$ where Π_k is a unitary matrix such that $\Pi_k^T\left(\lambda\Sigma_{\mathbf{w}_k}\right)\Pi_k=\mathrm{diag}\left(\tau_{k_1},\tau_{k_2},\ldots,\tau_{k_n}\right)\triangleq\widetilde{\Pi}_k$, P_k is a unitary matrix such that $P_k^T\left(\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}\left(G^TK_{k+1}G+I\right)\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}\right)P_k=\mathrm{diag}\left(\nu_{k_1},\nu_{k_2},\ldots,\nu_{k_n}\right)$, and ζ_k is a diagonal matrix such that $\zeta_k=\mathrm{diag}\left(\sqrt{\alpha_{k_1}},\sqrt{\alpha_{k_2}},\ldots,\sqrt{\alpha_{k_{p_k^*}}},0,\ldots,0\right)$ with $\alpha_{k_i}=-1+\left(\frac{\sqrt{\nu_{k_i}/\tau_{k_i}}}{\sum_{j=1}^{p_k^*}\sqrt{\tau_{k_j}\nu_{k_j}}}\right)\left(1+\sum_{j=1}^{p_k^*}\tau_{k_j}\right)$. For $p\leq n$, let $f_k(p)\triangleq\sqrt{\tau_{k_p}/\nu_{k_p}}\sum_{i=1}^{p}\sqrt{\tau_{k_i}\nu_{k_i}}-\sum_{i=1}^{p}\tau_{k_i}$, then p_k^* is defined by

$$p_k^*=n \text{ if } f_k(n)<1\ ,$$
$$f_k(p_k^*)<1\leq f_k(p_k^*+1) \text{ if } f_k(n)\geq 1\ .$$

Since the optimal H_k always has the form of $H_k^*=\Pi_k\zeta_kP_k^T$ for every $\mu_k=\mathrm{tr}\left(\lambda H_k^T\Sigma_{\mathbf{w}_k}H_k\right)$ as described above, then the recursion of the innovation's covariance matrix (3.16) can be expressed as

$$\begin{aligned}\Sigma_{\widetilde{\mathbf{m}}_{k+1}}&=G\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}\left(I+H_k^TH_k\right)^{-1}\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}G^T+\Sigma_{\mathbf{v}_k}\\&=G\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}\left(I+P_k\zeta_k^T\Pi_k^T\Pi_k\zeta_kP_k^T\right)^{-1}\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}G^T+\Sigma_{\mathbf{v}_k}\\&=G\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}\left(I+P_k\zeta_k^T\zeta_kP_k^T\right)^{-1}\Sigma^{1/2}_{\widetilde{\mathbf{m}}_k}G^T+\Sigma_{\mathbf{v}_k}\ .\end{aligned}\tag{3.19}$$

Then, by utilizing $H_k^*=\Pi_k\zeta_kP_k^T$ in (3.18), Then (3.18) becomes

$$V_k\left(\Sigma_{\widetilde{\mathbf{m}}_k}\right)=\mathrm{tr}\left(K_{k+1}\Sigma_{\mathbf{v}_k}+L_{k+1}\right)+\|\mathbf{b}\|^2$$

$$
\begin{aligned}
&\quad + \operatorname{tr}\left(\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(G^T K_{k+1} G + I\right)\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(I + P_k \zeta_k^T \Pi_k^T \Pi_k \zeta_k P_k^T\right)^{-1}\right) \\
&\quad + \operatorname{tr}\left(\lambda P_k \zeta_k^T \Pi_k^T \Sigma_{\mathbf{w}_k} \Pi_k \zeta_k P_k^T\right) \\
&\overset{(a)}{=} \operatorname{tr}\left(K_{k+1}\Sigma_{\mathbf{v}_k} + L_{k+1}\right) + \operatorname{tr}\left(\mathbf{b}\mathbf{b}^T\right) + \operatorname{tr}\left(\zeta_k^T \widetilde{\Pi}_k \zeta_k\right) \\
&\quad + \operatorname{tr}\left(\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(G^T K_{k+1} G + I\right)\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(I + P_k \zeta_k^T \zeta_k P_k^T\right)^{-1}\right) \\
&\overset{(b)}{=} \operatorname{tr}\left(K_{k+1}\Sigma_{\mathbf{v}_k} + L_{k+1}\right) + \operatorname{tr}\left(\mathbf{b}\mathbf{b}^T\right) + \operatorname{tr}\left(\zeta_k^T \widetilde{\Pi}_k \zeta_k\right) \\
&\quad + \operatorname{tr}\Bigg(\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(G^T K_{k+1} G + I\right)\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \\
&\qquad \times \left(I - P_k \zeta_k^T \left(I + \zeta_k P_k^T P_k \zeta_k^T\right)^{-1} \zeta_k P_k^T\right)\Bigg) \\
&\overset{(c)}{=} \operatorname{tr}\left(K_{k+1}\Sigma_{\mathbf{v}_k} + L_{k+1}\right) + \operatorname{tr}\left(\mathbf{b}\mathbf{b}^T\right) + \operatorname{tr}\left(\zeta_k^T \widetilde{\Pi}_k \zeta_k\right) \\
&\quad + \operatorname{tr}\Bigg(\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\left(G^T K_{k+1} G + I\right)\Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} \\
&\qquad \times \left(I - P_k \zeta_k^T \left(I + \zeta_k \zeta_k^T\right)^{-1} \zeta_k P_k^T\right)\Bigg) \\
&= \operatorname{tr}\left(K_{k+1}\Sigma_{\mathbf{v}_k} + L_{k+1}\right) + \operatorname{tr}\left(\mathbf{b}\mathbf{b}^T\right) + \operatorname{tr}\left(\zeta_k^T \widetilde{\Pi}_k \zeta_k\right) \\
&\quad + \operatorname{tr}\Bigg(\left(G^T K_{k+1} G + I\right) \\
&\qquad \times \left(\Sigma_{\widetilde{\mathbf{m}}_k} - \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2} P_k \zeta_k^T \left(I + \zeta_k \zeta_k^T\right)^{-1} \zeta_k P_k^T \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}\right)\Bigg) \\
&\overset{(d)}{=} \operatorname{tr}\left(K_{k+1}\Sigma_{\mathbf{v}_k} + L_{k+1}\right) + \operatorname{tr}\left(\mathbf{b}\mathbf{b}^T\right) + \operatorname{tr}\left(\zeta_k^T \widetilde{\Pi}_k \zeta_k\right) \\
&\quad + \operatorname{tr}\left(\left(G^T K_{k+1} G + I\right)\left(\Sigma_{\widetilde{\mathbf{m}}_k} - \zeta_k^T \left(I + \zeta_k \zeta_k^T\right)^{-1} \zeta_k \Sigma_{\widetilde{\mathbf{m}}_k}\right)\right) \\
&= \operatorname{tr}\left(K_{k+1}\Sigma_{\mathbf{v}_k} + L_{k+1}\right) + \operatorname{tr}\left(\mathbf{b}\mathbf{b}^T\right) + \operatorname{tr}\left(\zeta_k^T \widetilde{\Pi}_k \zeta_k\right) \\
&\quad + \operatorname{tr}\left(\left(G^T K_{k+1} G + I\right)\left(I - \zeta_k^T \left(I + \zeta_k \zeta_k^T\right)^{-1} \zeta_k\right)\Sigma_{\widetilde{\mathbf{m}}_k}\right),
\end{aligned}
\tag{3.20}
$$

where (a) follows from $\Pi_k^T \Pi_k = I$, $P_k^T P_k = I$, $\widetilde{\Pi}_k = \Pi_k^T\left(\lambda \Sigma_{\mathbf{w}_k}\right)\Pi_k$, and the properties of the trace operator, (b) is due to the matrix inversion lemma by choosing $U = P_k \zeta_k^T$, $W = I$, and $V = \zeta_k P_k^T$ in $(I + UWV)^{-1} = I - U(W^{-1} + VU)^{-1}V$,

(c) is due to $P_k^T P_k = I$, (d) follows from the diagonality of $\Sigma_{\widetilde{\mathbf{m}}_k}$, P_k and ζ_k: Since G, $\Sigma_{\widetilde{\mathbf{m}}_0}$, K_k and $\Sigma_{\mathbf{v}_k}$ are diagonal for $k = 0, 1, \ldots, N-1$, it is always possible to find a unitary diagonal P_0 such that $P_0^T \left(\Sigma_{\widetilde{\mathbf{m}}_0}^{1/2} \left(G^T K_1 G + I \right) \Sigma_{\widetilde{\mathbf{m}}_0}^{1/2} \right) P_0 = \operatorname{diag}\left(\nu_{0_1}, \nu_{0_2}, \ldots, \nu_{0_n}\right)$, which makes $\Sigma_{\widetilde{\mathbf{m}}_1}$ diagonal by (3.19). By following the same approach, $\Sigma_{\widetilde{\mathbf{m}}_k}$ and P_k are diagonal for $k = 0, 1, \ldots, N-1$.

In order to satisfy (3.20), since $V_N\left(\Sigma_{\widetilde{\mathbf{m}}_N}\right) = 0$, we choose $K_N = L_N = 0$, and $\{K_{k+1}, L_{k+1}\}$ according to

$$
\begin{aligned}
K_k &= \left(G^T K_{k+1} G + I\right)\left(I - \zeta_k^T \left(I + \zeta_k \zeta_k^T\right)^{-1} \zeta_k\right) , \\
L_k &= K_{k+1} \Sigma_{\mathbf{v}_k} + L_{k+1} + \zeta_k^T \widetilde{\Pi}_k \zeta_k + \mathbf{b}\mathbf{b}^T , \qquad (3.21)
\end{aligned}
$$

for $k = 0, 1, \ldots, N-1$. Now we verify that the diagonal K_k matrices satisfy the dynamic programming recursion.

Now consider the special case when the channel is scalar instead of multi-dimensional. Then x_k, w_k and y_k become scalar random variables rather than vectors. A_k is now $1 \times n$ matrix for $k = 0, 1, \ldots, N-1$. We will use $\sigma_{y_k}^2$ and $\sigma_{w_k}^2$ for the variances of y_k and w_k, respectively. Then $\Pi = 1$, $\widetilde{\Pi} = \tau_1 = \lambda \sigma_{w_k}^2$, $\zeta = \left[\sqrt{\alpha_1}, 0, 0, \cdots, 0\right]$ is a $1 \times n$ row vector with $\alpha_1 = -1 + \left(\frac{\sqrt{\nu_1/\tau_1}}{\sqrt{\tau_1 \nu_1}}\right)\left(1 + \tau_1\right) = 1/\tau_1$ since $p^* = 1$. Then, the optimal linear encoder policy is found as $A_k^* = \Sigma_{\mathbf{w}_k}^{1/2} \zeta_k P_k^T \Sigma_{\widetilde{\mathbf{m}}_k}^{-1/2}$ since the optimal H_k^* is $H_k^* = \Pi_k \zeta_k P_k^T$, the H_k is defined as $H_k \triangleq \Sigma_{\mathbf{w}_k}^{-1/2} A_k \Sigma_{\widetilde{\mathbf{m}}_k}^{1/2}$, and $\Pi_k = 1$ and $\zeta_k = \left[\frac{1}{\sqrt{\lambda \sigma_{W_k}^2}}, 0, \ldots, 0\right]$ for the scalar channel. Further, (3.21) reduces to the following for the scalar channel:

$$
\begin{aligned}
K_k &= \left(G^T K_{k+1} G + I\right) \times \operatorname{diag}\left(\frac{\lambda \sigma_{W_k}^2}{1 + \lambda \sigma_{W_k}^2}, 1, 1, \ldots, 1\right) \\
L_k &= K_{k+1} \Sigma_{\mathbf{v}_k} + L_{k+1} + \operatorname{diag}(1, 0, 0, \ldots, 0) + \mathbf{b}\mathbf{b}^T
\end{aligned}
$$

for $k = 0, 1, \ldots, N-1$.

Chapter 4

Hypothesis Testing under Subjective Priors and Costs as a Signaling Game

In this chapter, the binary signaling problem is investigated under the hypothesis testing context. Nash and Stackelberg equilibria of the binary hypothesis-testing game are analyzed, a characterization on when informative equilibria exist, and robustness and continuity properties to misalignment are presented under Nash and Stackelberg criteria.

The main contributions of this chapter can be summarized as follows:

(i) A game theoretic formulation of the binary signaling problem is established under subjective priors and/or subjective costs.

(ii) The corresponding Stackelberg and Nash equilibrium policies are obtained, and their properties (such as uniqueness and informativeness) are investigated. It is proved that an equilibrium is almost always informative for a team setup, whereas in the case of subjective priors and/or costs, it may cease to be informative. It is shown that Stackelberg equilibria always exist, whereas there are setups under which Nash equilibria may not exist.

(iii) Furthermore, robustness of equilibrium solutions to small perturbations in the priors or costs are established. It is shown that, the game equilibrium behavior around the team setup is robust under the Nash assumption, whereas it is not robust under the Stackelberg assumption.

(iv) For each of the results, applications to two motivating setups (involving subjective priors and the presence of a bias in the objective function of the transmitter) are presented.

4.1 Problem Formulation

Consider a binary hypothesis-testing problem:

$$\begin{aligned} \mathcal{H}_0 &: Y = S_0 + N \ , \\ \mathcal{H}_1 &: Y = S_1 + N \ , \end{aligned} \tag{4.1}$$

where Y is the observation (measurement) that belongs to the observation set $\Gamma = \mathbb{R}$, S_0 and S_1 denote the deterministic signals under hypothesis $\mathcal{H}_0$ and hypothesis $\mathcal{H}_1$, respectively, and N represents Gaussian noise; i.e., $N \sim \mathcal{N}(0, \sigma^2)$. In the Bayesian setup, it is assumed that the prior probabilities of $\mathcal{H}_0$ and $\mathcal{H}_1$ are available, which are denoted by π_0 and π_1, respectively, with $\pi_0 + \pi_1 = 1$.

In the conventional Bayesian framework, the aim of the receiver is to design the optimal decision rule (detector) based on Y in order to minimize the Bayes risk, which is defined as [72]

$$r(\delta) = \pi_0 R_0(\delta) + \pi_1 R_1(\delta) \ , \tag{4.2}$$

where δ is the decision rule, and $R_i(\cdot)$ is the conditional risk of the decision rule when hypothesis $\mathcal{H}_i$ is true for $i \in \{0, 1\}$. In general, a decision rule corresponds to a partition of the observation set Γ into two subsets Γ_0 and Γ_1, and the decision becomes $\mathcal{H}_i$ if the observation y belongs to Γ_i, where $i \in \{0, 1\}$.

The conditional risks in (4.2) can be calculated as

$$R_i(\delta) = C_{0i}\mathsf{P}_{0i} + C_{1i}\mathsf{P}_{1i} \ , \tag{4.3}$$

for $i \in \{0,1\}$, where $C_{ji} \geq 0$ is the cost of deciding for $\mathcal{H}_j$ when $\mathcal{H}_i$ is true, and $\mathsf{P}_{ji} = \mathsf{Pr}(y \in \Gamma_j | \mathcal{H}_i)$ represents the conditional probability of deciding for $\mathcal{H}_j$ given that $\mathcal{H}_i$ is true, where $i, j \in \{0,1\}$ [72].

It is well-known that the optimal decision rule δ which minimizes the Bayes risk is the following test, known as the likelihood ratio test (LRT):

$$\delta : \begin{cases} \pi_1(C_{01} - C_{11})p_1(y) \overset{\mathcal{H}_1}{\underset{\mathcal{H}_0}{\gtrless}} \pi_0(C_{10} - C_{00})p_0(y) \ , \end{cases} \tag{4.4}$$

where $p_i(y)$ represents the probability density function (PDF) of Y under $\mathcal{H}_i$ for $i \in \{0,1\}$ [72].

If the transmitter and the receiver have the same objective function specified by (4.2) and (4.3), then the signals can be designed to minimize the Bayes risk corresponding to the decision rule in (4.4). This leads to a conventional formulation which has been studied intensely in the literature [72, 73].

On the other hand, it may be the case that the transmitter and the receiver can have non-aligned Bayes risks. In particular, the transmitter and the receiver may have different objective functions or priors: Let C^t_{ji} and C^r_{ji} represent the costs from the perspective of the transmitter and the receiver, respectively, where $i, j \in \{0,1\}$. Also let π^t_i and π^r_i for $i \in \{0,1\}$ denote the priors from the perspective of the transmitter and the receiver, respectively, with $\pi^j_0 + \pi^j_1 = 1$, where $j \in \{t, r\}$. Here, from transmitter's and receiver's perspectives, the priors are assumed to be mutually absolutely continuous with respect to each other; i.e., $\pi^t_i \pi^r_i = 0 \Leftrightarrow \pi^t_i = \pi^r_i = 0$ for $i \in \{0,1\}$. This condition assures that the impossibility of any hypothesis holds for both the transmitter and the receiver simultaneously. The aim of the transmitter is to perform the optimal design of signals $\mathcal{S} = \{S_0, S_1\}$ to minimize his Bayes risk; whereas, the aim of the receiver is to determine the optimal decision rule δ over all possible decision rules Δ to minimize his Bayes risk.

The Bayes risks are defined as follows for the transmitter and the receiver:

$$r^j(\mathcal{S}, \delta) = \pi^j_0 R^j_0(\mathcal{S}, \delta) + \pi^j_1 R^j_1(\mathcal{S}, \delta) \ ,$$

where

$$R_i^j(\mathcal{S}, \delta) = C_{0i}^j \mathsf{P}_{0i} + C_{1i}^j \mathsf{P}_{1i} \ ,$$

for $i \in \{0, 1\}$ and $j \in \{t, r\}$. Here, the transmitter performs the optimal signal design problem under the power constraint below:

$$\mathbb{S} \triangleq \{\mathcal{S} = \{S_0, S_1\} : |S_0|^2 \leq P_0 \, , \ |S_1|^2 \leq P_1\} \ ,$$

where P_0 and P_1 denote the power limits.

According to the formulation above, the extensive form of the game is depicted in Figure 4.1.

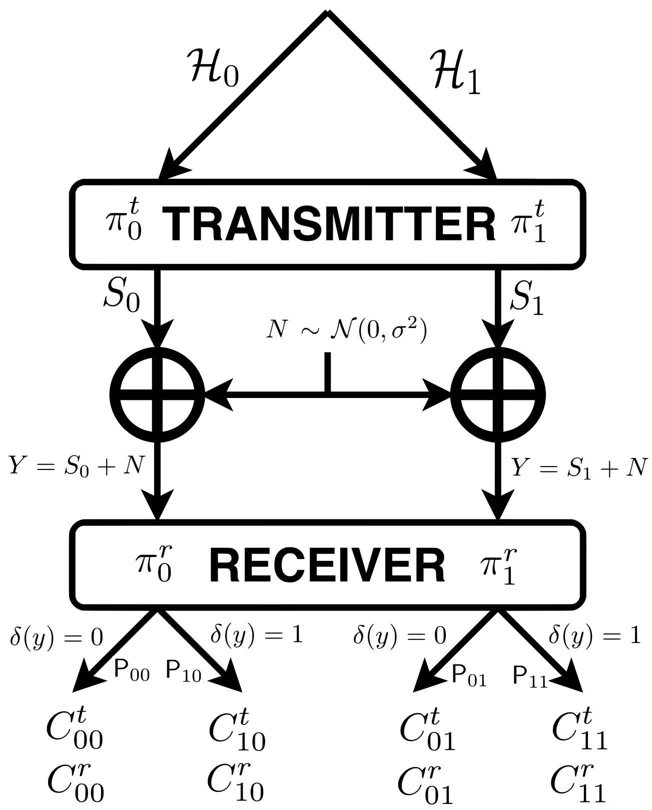

Figure 4.1: The extensive form of the binary signaling game.

In the current formulation, and a pair of policies $(\mathcal{S}^*, \delta^*)$ is said to be a Nash equilibrium [19] if

$$\begin{aligned} r^t(\mathcal{S}^*, \delta^*) &\leq r^t(\mathcal{S}, \delta^*) \quad \forall\, \mathcal{S} \in \mathbb{S}\,, \\ r^r(\mathcal{S}^*, \delta^*) &\leq r^r(\mathcal{S}^*, \delta) \quad \forall\, \delta \in \Delta\,, \end{aligned} \tag{4.5}$$

whereas, a pair of policies $(\mathcal{S}^*, \delta^*_{\mathcal{S}^*})$ is said to be a Stackelberg equilibrium [19] if

$$\begin{aligned} &r^t(\mathcal{S}^*, \delta^*_{\mathcal{S}^*}) \leq r^t(\mathcal{S}, \delta^*_{\mathcal{S}}) \quad \forall\, \mathcal{S} \in \mathbb{S}\,, \\ &\text{where } \delta^*_{\mathcal{S}} \text{ satisfies} \\ &r^r(\mathcal{S}, \delta^*_{\mathcal{S}}) \leq r^r(\mathcal{S}, \delta_{\mathcal{S}}) \quad \forall\, \delta_{\mathcal{S}} \in \Delta\,. \end{aligned} \tag{4.6}$$

4.1.1 Two Motivating Setups

We present two different scenarios that fit into the binary signaling context discussed here and revisit these setups throughout the chapter.

4.1.1.1 Subjective Priors

In almost all practical applications, there is some mismatch between the true and an assumed probabilistic system/data model, which results in performance degradation. This performance loss due to the presence of mismatch has been studied extensively in various setups (see e.g., [121], [122], [123] and references therein). In this chapter, we have a further salient aspect due to decentralization, where the transmitter and the receiver have a mismatch. We note that in decentralized decision making, there have been a number of studies on the presence of a mismatch in the priors of decision makers [86–88]. In such setups, even when the objective functions to be optimized are identical, the presence of subjective priors alters the formulation from a team problem to a game problem (see [4, Section 12.2.3] for a comprehensive literature review on subjective priors also from a statistical decision making perspective).

With this motivation, we will consider a setup where the transmitter and the receiver have different priors on the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$, and the costs of the transmitter and the receiver are identical. In particular, from transmitter's perspective, the priors are π_0^t and π_1^t, whereas the priors are π_0^r and π_1^r from receiver's perspective, and $C_{ji} = C_{ji}^t = C_{ji}^r$ for $i, j \in \{0, 1\}$. We will investigate equilibrium solutions for this setup throughout the chapter.

4.1.1.2 Biased Transmitter Cost[1]

A further application will be for a setup where the transmitter and the receiver have misaligned objective functions. Consider a binary signaling game in which the transmitter encodes a random binary signal $x = i$ as $\mathcal{H}_i$ by choosing the corresponding signal level S_i for $i \in \{0, 1\}$, and the receiver decodes the received signal y as $u = \delta(y)$. Let the priors from the perspectives of the transmitter and the receiver be the same; i.e., $\pi_i = \pi_i^t = \pi_i^r$ for $i \in \{0, 1\}$, and the Bayes risks of the transmitter and the receiver be defined as $r^t(\mathcal{S}, \delta) = \mathbb{E}[\mathbb{1}_{\{1=(x\oplus u\oplus b)\}}]$ and $r^r(\mathcal{S}, \delta) = \mathbb{E}[\mathbb{1}_{\{1=(x\oplus u)\}}]$, respectively, where b is a random variable with a Bernoulli distribution; i.e., $\alpha \triangleq \mathsf{Pr}(b = 0) = 1 - \mathsf{Pr}(b = 1)$, and α can be translated as the probability that the Bayes risks (objective functions) of the transmitter and the receiver are aligned. Then, the following relations can be observed:

$$
\begin{aligned}
r^t(\mathcal{S}, \delta) &= \mathbb{E}[\mathbb{1}_{\{1=(x\oplus u\oplus b)\}}] = \alpha(\pi_0 \mathsf{P}_{10} + \pi_1 \mathsf{P}_{01}) + (1-\alpha)(\pi_0 \mathsf{P}_{00} + \pi_1 \mathsf{P}_{11}) \\
&\Rightarrow \quad C_{01}^t = C_{10}^t = \alpha \text{ and } C_{00}^t = C_{11}^t = 1 - \alpha \ , \\
r^r(\mathcal{S}, \delta) &= \mathbb{E}[\mathbb{1}_{\{1=(x\oplus u)\}}] = \pi_0 \mathsf{P}_{10} + \pi_1 \mathsf{P}_{01} \\
&\Rightarrow \quad C_{01}^r = C_{10}^r = 1 \text{ and } C_{00}^r = C_{11}^r = 0 \ .
\end{aligned}
$$

Note that, in the formulation above, the misalignment between the Bayes risks of the transmitter and the receiver is due to the presence of the bias term b (i.e., the discrepancy between the Bayes risks of the transmitter and the receiver) in the Bayes risk of the transmitter. This can be viewed as an analogous setup to what was studied in a seminal work due to Crawford and Sobel [16], who obtained the striking result that such a bias term in the objective function of the transmitter

[1] Here, the *cost* refers to the objective function (Bayes risk), not the cost of a particular decision, C_{ji}. Note that, throughout the chapter, the *cost* refers to C_{ji} except when it is used in the phrase *Biased Transmitter Cost*.

may have a drastic effect on the equilibrium characteristics; in particular, under regularity conditions, all equilibrium policies under a Nash formulation involve information hiding.

4.2 Team Theoretic Analysis: Classical Setup with Identical Costs and Priors

Now consider the team setup where the costs and the priors are assumed to be the same for both the transmitter and the receiver; i.e., $C_{ji} = C^t_{ji} = C^r_{ji}$ and $\pi_i = \pi^t_i = \pi^r_i$ for $i, j \in \{0, 1\}$. Thus the common Bayes risk becomes $r^t(\mathcal{S}, \delta) = r^r(\mathcal{S}, \delta) = \pi_0(C_{00}\mathsf{P}_{00} + C_{10}\mathsf{P}_{10}) + \pi_1(C_{01}\mathsf{P}_{01} + C_{11}\mathsf{P}_{11})$. The arguments for the proof of the following result follow from the standard analysis in the detection and estimation literature [72, 73]. However, for completeness, and for the relevance of the analysis in the following sections, a proof is included.

Theorem 4.2.1. *Let* $\tau \triangleq \frac{\pi_0(C_{10}-C_{00})}{\pi_1(C_{01}-C_{11})}$. *If* $\tau \leq 0$ *or* $\tau = \infty$, *the team solution of the binary signaling setup is non-informative. Otherwise; i.e., if* $0 < \tau < \infty$, *the team solution is always informative.*

4.3 Stackelberg Game Analysis

Under the Stackelberg assumption, first the transmitter (the leader agent) announces and commits to a particular policy, and then the receiver (the follower agent) acts accordingly. In this direction, first the transmitter chooses optimal signals $\mathcal{S} = \{S_0, S_1\}$ to minimize his Bayes risk $r^t(\mathcal{S}, \delta)$, then the receiver chooses an optimal decision rule δ accordingly to minimize his Bayes risk $r^r(\mathcal{S}, \delta)$. Due to the sequential structure of the Stackelberg game, the transmitter knows the priors and the costs of the receiver so that he can adjust his optimal policy accordingly. On the other hand, the receiver knows only the policy and the action (signals $\mathcal{S} = \{S_0, S_1\}$) of the transmitter as he announces during the game-play.

4.3.1 Equilibrium Solutions

Under the Stackelberg assumption, the equilibrium structure of the binary signaling game can be characterized as follows:

Theorem 4.3.1. *If* $\tau \triangleq \frac{\pi_0^r(C_{10}^r - C_{00}^r)}{\pi_1^r(C_{01}^r - C_{11}^r)} \leq 0$ *or* $\tau = \infty$*, the Stackelberg equilibrium of the binary signaling game is non-informative. Otherwise; i.e., if* $0 < \tau < \infty$*, let* $d \triangleq \frac{|S_1 - S_0|}{\delta}$, $d_{\max} \triangleq \frac{\sqrt{P_0}+\sqrt{P_1}}{\sigma}$, $\zeta \triangleq \text{sgn}(C_{01}^r - C_{11}^r)$, $k_0 \triangleq \pi_0^t \zeta (C_{10}^t - C_{00}^t)\tau^{-\frac{1}{2}}$*, and* $k_1 \triangleq \pi_1^t \zeta (C_{01}^t - C_{11}^t)\tau^{\frac{1}{2}}$*. Then, the Stackelberg equilibrium structure can be characterized as in Table 4.1, where* $d^* = 0$ *stands for a non-informative equilibrium, and a nonzero* d^* *corresponds to an informative equilibrium.*

Table 4.1: Stackelberg equilibrium analysis for $0 < \tau < \infty$*.*

	$\ln\tau\ (k_0 - k_1) < 0$	$\ln\tau\ (k_0 - k_1) \geq 0$
$k_0 + k_1 < 0$	$d^* = \min\left\{d_{\max}, \sqrt{\left\lvert\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right\rvert}\right\}$	$d^* = 0$, non-informative
$k_0 + k_1 \geq 0$	$d^* = d_{\max}$	$d_{\max}^2 < \left\lvert\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right\rvert \Rightarrow d^* = 0$, non-informative $d_{\max}^2 \geq \left\lvert\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right\rvert \Rightarrow \left(\frac{k_1}{k_0\tau}\right)^{\text{sgn}(\ln(\tau))} \mathcal{Q}\left(\frac{\lvert\ln(\tau)\rvert}{d_{\max}} - \frac{d_{\max}}{2}\right) - \mathcal{Q}\left(\frac{\lvert\ln(\tau)\rvert}{d_{\max}} + \frac{d_{\max}}{2}\right) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtrless}} 0$

Before proving Theorem 4.3.1, we make the following remark:

Remark 4.3.1. *As we observed in Theorem 4.2.1, for a team setup, an equilibrium is almost always informative (practically,* $0 < \tau < \infty$*), whereas in the case of subjective priors and/or costs, it may cease to be informative.*

Similar to the team setup analysis, for every possible case in Table 4.1, there are more than one equilibrium points, and they are essentially unique since the Bayes risks of the transmitter and the receiver depend on d. For example, for $d^* = d_{max}$, $(S_0^*, S_1^*) = \left(-\sqrt{P_0}, \sqrt{P_1}\right)$ and $(S_0^*, S_1^*) = \left(\sqrt{P_0}, -\sqrt{P_1}\right)$ are the only possible choices for the transmitter, and the decision rule of the receiver is chosen based on the rule in (4.9). Similarly, for $d^* = 0$ or $d^* = \sqrt{\left\lvert\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right\rvert}$, there are infinitely many choices for the transmitter and the receiver, and all of them are essentially unique; i.e., they result in the same Bayes risks for the transmitter and the receiver.

4.3.2 Continuity and Robustness to Perturbations around the Team Setup

We now investigate the effects of small perturbations in priors and costs on equilibrium values. In particular, we consider the perturbations around the team setup; i.e., at the point of identical priors and costs.

Define the perturbation around the team setup as $\boldsymbol{\epsilon} = \{\epsilon_{\pi 0}, \epsilon_{\pi 1}, \epsilon_{00}, \epsilon_{01}, \epsilon_{10}, \epsilon_{11}\} \in \mathbb{R}^6$ such that $\pi_i^t = \pi_i^r + \epsilon_{\pi i}$ and $C_{ji}^t = C_{ji}^r + \epsilon_{ji}$ for $i, j \in \{0, 1\}$ (note that the transmitter parameters are perturbed around the receiver parameters which are assumed to be fixed). Then, for $0 < \tau < \infty$, at the point of identical priors and costs, small perturbations in both priors and costs imply $k_0 = (\pi_0^r + \epsilon_{\pi 0})\zeta(C_{10}^r - C_{00}^r + \epsilon_{10} - \epsilon_{00})\tau^{-\frac{1}{2}}$ and $k_1 = (\pi_1^r + \epsilon_{\pi 1})\zeta(C_{01}^r - C_{11}^r + \epsilon_{01} - \epsilon_{11})\tau^{\frac{1}{2}}$. Since, for $0 < \tau < \infty$, $k_0 = k_1 = \sqrt{\pi_0^r \pi_1^r}\sqrt{(C_{10}^r - C_{00}^r)(C_{01}^r - C_{11}^r)} > 0$ at the point of identical priors and costs, it is possible to obtain both positive and negative $(k_0 - k_1)$ by choosing the appropriate perturbation $\boldsymbol{\epsilon}$ around the team setup. Then, as it can be observed from Table 4.1, even the equilibrium may alter from an informative one to a non-informative one; hence, under the Stackelberg equilibrium, the policies are not continuous with respect to small perturbations around the point of identical priors and costs, and the equilibrium behavior is not robust to small perturbations in both priors and costs.

4.3.3 Application to the Motivating Examples

4.3.3.1 Subjective Priors

Referring to Section 4.1.1.1, for $0 < \tau < \infty$, the related parameters can be found as follows (note that the equilibrium is non-informative if $\tau \leq 0$ or $\tau = \infty$):

$$\tau = \frac{\pi_0^r(C_{10} - C_{00})}{\pi_1^r(C_{01} - C_{11})},$$

$$k_0 = \pi_0^t \sqrt{\frac{\pi_1^r}{\pi_0^r}} \sqrt{(C_{10} - C_{00})(C_{01} - C_{11})},$$

$$k_1 = \pi_1^t \sqrt{\frac{\pi_0^r}{\pi_1^r}} \sqrt{(C_{10} - C_{00})(C_{01} - C_{11})} \, .$$

Since $k_0 + k_1 > 0$, depending on the values of $\ln \tau \ (k_0 - k_1)$, $d^2_{\max}$, and $\left|\frac{2 \ln \tau (k_0 - k_1)}{(k_0 + k_1)}\right|$, Case-1, Case-5 or Case-6 of Theorem 4.3.1 may hold as depicted in Table 4.2. Here, the decision rule in Case-6 is the same as (4.14).

Table 4.2: Stackelberg equilibrium analysis of subjective priors case for $0 < \tau < \infty$.

	$0 < \tau < 1$	$1 \leq \tau < \infty$
$\frac{\pi_0^t}{\pi_1^t} < \frac{\pi_0^r}{\pi_1^r}$	$d^2_{\max} < \left\|\frac{2 \ln \tau (k_0 - k_1)}{(k_0 + k_1)}\right\| \Rightarrow$ Case-5 applies, $d^* = 0$, non-informative $d^2_{\max} \geq \left\|\frac{2 \ln \tau (k_0 - k_1)}{(k_0 + k_1)}\right\| \Rightarrow$ Case-6 applies	Case-1 applies, $d^* = d_{\max}$
$\frac{\pi_0^t}{\pi_1^t} \geq \frac{\pi_0^r}{\pi_1^r}$	Case-1 applies, $d^* = d_{\max}$	$d^2_{\max} < \left\|\frac{2 \ln \tau (k_0 - k_1)}{(k_0 + k_1)}\right\| \Rightarrow$ Case-5 applies, $d^* = 0$, non-informative $d^2_{\max} \geq \left\|\frac{2 \ln \tau (k_0 - k_1)}{(k_0 + k_1)}\right\| \Rightarrow$ Case-6 applies

4.3.3.2 Biased Transmitter Cost

Based on the arguments in Section 4.1.1.2, the related parameters can be found as follows:

$$\tau = \frac{\pi_0}{\pi_1} \, , \; k_0 = \sqrt{\pi_0 \pi_1} (2\alpha - 1) \, , \; k_1 = \sqrt{\pi_0 \pi_1} (2\alpha - 1) \, .$$

Then, $\ln \tau \ (k_0 - k_1) = 0$ and $k_0 + k_1 = 2\sqrt{\pi_0 \pi_1}(2\alpha - 1)$; hence, either Case-4 or Case-6 of Theorem 4.3.1 applies. Namely, if $\alpha < 1/2$ (Case-4 of Theorem 4.3.1 applies), the transmitter chooses $S_0 = S_1$ to minimize d and the equilibrium is non-informative; i.e., he does not send any meaningful information to the transmitter and the receiver considers only the priors. If $\alpha = 1/2$, the transmitter has no control on his Bayes risk, hence the equilibrium is non-informative. Otherwise; i.e., if $\alpha > 1/2$ (Case-6 of Theorem 4.3.1 applies), the equilibrium is always informative. In other words, if $\alpha > 1/2$, the players act like a team. As it can be seen, the informativeness of the equilibrium depends on $\alpha = \mathsf{Pr}(b = 0)$, the probability that the Bayes risks of the transmitter and the receiver are aligned.

4.4 Nash Game Analysis

Under the Nash assumption, the transmitter chooses optimal signals $\mathcal{S} = \{S_0, S_1\}$ to minimize $r^t(\mathcal{S}, \delta)$, and the receiver chooses optimal decision rule δ to minimize $r^r(\mathcal{S}, \delta)$ simultaneously. In this Nash setup, the transmitter and the receiver do not know the priors and the costs of each other; they know only their policies as they announce to each other. Further, there is no commitment between the transmitter and the receiver; hence, the perturbation in the transmitter does not lead to a functional perturbation in receiver's policy, unlike the Stackelberg setup. Due to this difference, the equilibrium structure and robustness properties of the Nash equilibrium show significant differences from the ones in the Stackelberg equilibrium, as stated in the following.

In the analysis, we restrict the receiver to use only the single-threshold rules. Although a single-threshold rule is suboptimal for the receiver in general, it is always optimal for Gaussian densities, and always optimal for unimodal densities under the maximum likelihood decision rule [72, 124].

4.4.1 Equilibrium Solutions

Under the Nash assumption, the equilibrium structure of the binary signaling game can be characterized as follows:

Theorem 4.4.1. *Let* $\tau \triangleq \frac{\pi_0^r (C_{10}^r - C_{00}^r)}{\pi_1^r (C_{01}^r - C_{11}^r)}$ *and* $\zeta \triangleq \operatorname{sgn}(C_{01}^r - C_{11}^r)$, $\xi_0 \triangleq \frac{C_{10}^t - C_{00}^t}{C_{10}^r - C_{00}^r}$, *and* $\xi_1 \triangleq \frac{C_{01}^t - C_{11}^t}{C_{01}^r - C_{11}^r}$. *If* $\tau \leq 0$ *or* $\tau = \infty$, *then the Nash equilibrium of the binary signaling game is non-informative. Otherwise; i.e., if* $0 < \tau < \infty$, *the Nash equilibrium structure is as depicted in Table 4.3.*

Table 4.3: *Nash equilibrium analysis for* $0 < \tau < \infty$.

	$\xi_0 > 0$	$\xi_0 = 0$	$\xi_0 < 0$
$\xi_1 > 0$	unique informative equilibrium	non-informative equilibrium	$P_0 > P_1 \Rightarrow$ no equilibrium $P_0 = P_1 \Rightarrow$ non-informative equilibrium $P_0 < P_1 \Rightarrow$ unique informative equilibrium
$\xi_1 = 0$	non-informative equilibrium	non-informative equilibrium	non-informative equilibrium
$\xi_1 < 0$	$P_0 > P_1 \Rightarrow$ unique informative equilibrium $P_0 = P_1 \Rightarrow$ non-informative equilibrium $P_0 < P_1 \Rightarrow$ no equilibrium	non-informative equilibrium	no equilibrium

As it can be deduced from Table 4.3, as the costs related to both hypotheses are aligned[2] for the transmitter and the receiver, the Nash equilibrium is informative. If the power limit corresponding to the hypothesis that has aligned costs for the transmitter and receiver is greater than the power limit of the other hypothesis, again, there exists an informative equilibrium. For the other cases, there may exist non-informative equilibrium; further, the misalignment between the costs can even induce a scenario, in which there exists no equilibrium.

The main reason for the absence of a non-informative (babbling) equilibrium under the Nash assumption is that in the binary signaling game setup, the receiver is forced to make a decision. Using only the prior information, the receiver always chooses one of the hypothesis. By knowing this, the transmitter can manipulate his signaling strategy for his own benefit. However, after this manipulation, the receiver no longer keeps his decision rule the same; namely, the best response of the receiver alters based on the signaling strategy of the transmitter, which entails another change of the best response of the transmitter. Due to such an infinite recursion, the optimal policies of the transmitter and the receiver keep changing, and thus, there does not exist a pure Nash equilibrium.

As shown in Theorem 4.4.1, at the Nash equilibrium, the transmitter selects $S_0 = -\text{sgn}(a)\text{sgn}(C^t_{10} - C^t_{00})\sqrt{P_0}$ and $S_1 = \text{sgn}(a)\text{sgn}(C^t_{01} - C^t_{11})\sqrt{P_1}$, and the decision rule of the receiver is $\delta : \begin{cases} ay \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta \end{cases}$ where $a = \zeta(S_1 - S_0)$ and $\eta =$

[2] ξ_i is the indicator that the transmitter and the receiver have similar preferences about hypothesis $\mathcal{H}_i$; i.e., if $\xi_i > 0$, then both the transmitter and the receiver aim to transmit and decode the hypothesis $\mathcal{H}_i$ correctly (or incorrectly). If $\xi_i < 0$, then the transmitter and the receiver have conflicting goals over hypothesis $\mathcal{H}_i$; i.e., one of them tries to achieve the correct transmission and decoding, whereas the goal of the other player is the opposite.

$\zeta\left(\sigma^2\ln(\tau)+\frac{S_1^2-S_0^2}{2}\right)$. Similar to the team and Stackelberg setup analysis, the equilibrium is essentially unique in Nash case, too; i.e., if $(S_0^*, S_1^*, a^*, \eta^*)$ is an equilibrium point, then $(-S_0^*, -S_1^*, -a^*, \eta^*)$ is another equilibrium point, and they both result in the same Bayes risks for the transmitter and the receiver.

4.4.2 Continuity and Robustness to Perturbations around the Team Setup

Similar to that in Section 4.3.2 for the Stackelberg setup, the effects of small perturbations in priors and costs on equilibrium values around the team setup are investigated for the Nash setup as follows:

Define the perturbation around the team setup as $\boldsymbol{\epsilon}=\{\epsilon_{\pi 0},\epsilon_{\pi 1},\epsilon_{00},\epsilon_{01},\epsilon_{10},\epsilon_{11}\}\in\mathbb{R}^6$ such that $\pi_i^t=\pi_i^r+\epsilon_{\pi i}$ and $C_{ji}^t=C_{ji}^r+\epsilon_{ji}$ for $i,j\in\{0,1\}$ (note that the transmitter parameters are perturbed around the receiver parameters which are assumed to be fixed). Then, for $0<\tau<\infty$, at the point of identical priors and costs, small perturbations in priors and costs imply $\xi_0=\frac{C_{10}^r-C_{00}^r+\epsilon_{10}-\epsilon_{00}}{C_{10}^r-C_{00}^r}$ and $\xi_1=\frac{C_{01}^r-C_{11}^r+\epsilon_{01}-\epsilon_{11}}{C_{01}^r-C_{11}^r}$. As it can be seen, the Nash equilibrium is not affected by small perturbations in priors. Further, since $\xi_0=\xi_1=1$ at the point of identical priors and costs for $0<\tau<\infty$, as long as the perturbation $\boldsymbol{\epsilon}$ is chosen such that $|\frac{\epsilon_{10}-\epsilon_{00}}{C_{10}^r-C_{00}^r}|<1$ and $|\frac{\epsilon_{01}-\epsilon_{11}}{C_{01}^r-C_{11}^r}|<1$, we always obtain positive ξ_0 and ξ_1 in Table 4.3. Thus, under the Nash assumption, the equilibrium behavior is robust to small perturbations in both priors and costs.

For the continuity analysis, consider the following: if the priors and costs are perturbed around the team setup, $S_0=-\operatorname{sgn}(a)\operatorname{sgn}(C_{10}^r-C_{00}^r+\epsilon_{10}-\epsilon_{00})\sqrt{P_0}$ and $S_1=\operatorname{sgn}(a)\operatorname{sgn}(C_{01}^r-C_{11}^r+\epsilon_{01}-\epsilon_{11})\sqrt{P_1}$ are obtained. As long as the perturbation $\boldsymbol{\epsilon}$ is chosen such that $|\frac{\epsilon_{10}-\epsilon_{00}}{C_{10}^r-C_{00}^r}|<1$ and $|\frac{\epsilon_{01}-\epsilon_{11}}{C_{01}^r-C_{11}^r}|<1$, the changes in η, S_0 and S_1 are continuous with respect to perturbations; actually, the values of the equilibrium parameters remain constant; i.e., either $(S_0^*, S_1^*, a^*, \eta^*)=\left(-\zeta\sqrt{P_0},\zeta\sqrt{P_1},(\sqrt{P_0}+\sqrt{P_1}),\zeta\left(\sigma^2\ln(\tau)+\frac{S_1^2-S_0^2}{2}\right)\right)$ or the essentially equivalent one $(S_0^*, S_1^*, a^*, \eta^*)=\left(\zeta\sqrt{P_0},-\zeta\sqrt{P_1},-(\sqrt{P_0}+\sqrt{P_1}),\zeta\left(\sigma^2\ln(\tau)+\frac{S_1^2-S_0^2}{2}\right)\right)$

holds. Thus, the policies are continuous with respect to small perturbations around the point of identical priors and costs.

4.4.3 Application to the Motivating Examples

4.4.3.1 Subjective Priors

The related parameters are $\tau = \frac{\pi_0^r(C_{10}-C_{00})}{\pi_1^r(C_{01}-C_{11})}$, $\xi_0 = 1$, and $\xi_1 = 1$. Thus, if $\tau < 0$ or $\tau = \infty$, the equilibrium is non-informative; otherwise, there always exists a unique informative equilibrium. As it is shown in Section 4.4.2, as long as the priors are mutually absolutely continuous, the subjectivity in the priors does not affect the equilibrium.

4.4.3.2 Biased Transmitter Cost

Based on the arguments in Section 4.1.1.2, the related parameters can be found as follows:

$$
\begin{aligned}
C_{01}^t &= C_{10}^t = \alpha \text{ and } C_{00}^t = C_{11}^t = 1-\alpha\,, \\
C_{01}^r &= C_{10}^r = 1 \text{ and } C_{00}^r = C_{11}^r = 0\,, \\
\tau &= \frac{\pi_0(C_{10}^r - C_{00}^r)}{\pi_1(C_{01}^r - C_{11}^r)} = \frac{\pi_0}{\pi_1}\,, \\
\xi_0 &= \frac{C_{10}^t - C_{00}^t}{C_{10}^r - C_{00}^r} = 2\alpha - 1\,, \\
\xi_1 &= \frac{C_{01}^t - C_{11}^t}{C_{01}^r - C_{11}^r} = 2\alpha - 1\,.
\end{aligned}
$$

If $\alpha > 1/2$ (Case-3-d of Theorem 4.4.1 applies), the players act like a team and the equilibrium is informative. If $\alpha = 1/2$ (Case-2 of Theorem 4.4.1 applies), the equilibrium is non-informative. Otherwise; i.e., if $\alpha < 1/2$ (Case-3-a of Theorem 4.4.1 applies), there exists no equilibrium. As it can be seen, the existence of the equilibrium depends on $\alpha = \mathsf{Pr}(b = 0)$, the probability that the Bayes risks of the transmitter and the receiver are aligned.

4.5 Extension to the Multi-Dimensional Case

When the transmitter sends a multi-dimensional signal over a multi-dimensional channel, or the receiver takes multiple samples from the observed waveform, the scalar analysis considered heretofore is not applicable anymore; thus, the vector case can be investigated. In this direction, the binary hypothesis-testing problem aforementioned can be modified as

$$\mathcal{H}_0 : \mathbf{Y} = \mathbf{S}_0 + \mathbf{N} \,,$$
$$\mathcal{H}_1 : \mathbf{Y} = \mathbf{S}_1 + \mathbf{N} \,,$$

where $\mathbf{Y}$ is the observation (measurement) vector that belongs to the observation set $\Gamma = \mathbb{R}^n$, $\mathbf{S}_0$ and $\mathbf{S}_1$ denote the deterministic signals under hypothesis $\mathcal{H}_0$ and hypothesis $\mathcal{H}_1$, such that $\mathbb{S} \triangleq \{\mathcal{S} : \|\mathbf{S}_0\|^2 \leq P_0 \,,\ \|\mathbf{S}_1\|^2 \leq P_1\}$, respectively, and $\mathbf{N}$ represents a zero-mean Gaussian noise vector with the positive definite covariance matrix Σ; i.e., $\mathbf{N} \sim \mathcal{N}(\mathbf{0}, \Sigma)$. All the other parameters (π_i^k and C_{ji}^k for $i, j \in \{0, 1\}$ and $k \in \{t, r\}$) and their definitions remain unchanged.

4.5.1 Team Setup Analysis

Theorem 4.5.1. *Theorem 4.2.1 also holds for the vector case: if $0 < \tau < \infty$, the team solution is always informative; otherwise, there exist only non-informative equilibria.*

4.5.2 Stackelberg Game Analysis

Theorem 4.5.2. *Let $d \triangleq \sqrt{(\mathbf{S}_1 - \mathbf{S}_0)^T \Sigma^{-1} (\mathbf{S}_1 - \mathbf{S}_0)}$ and $d_{\max}^2 \triangleq \frac{(\sqrt{P_0} + \sqrt{P_1})^2}{\lambda_{\min}}$, where $\lambda_{\min}$ is the minimum eigenvalue of Σ. Then Theorem 4.3.1 also holds for the vector case.*

4.5.3 Nash Game Analysis

Theorem 4.5.3. *Theorem 4.4.1 also holds for the vector case.*

4.6 Conclusion

In this chapter, we considered binary signaling problems in which the decision makers (the transmitter and the receiver) have subjective priors and/or misaligned objective functions. Depending on the commitment nature of the transmitter to his policies, we formulated the binary signaling problem as a Bayesian game under either Nash or Stackelberg equilibrium concepts and established equilibrium solutions and their properties.

We showed that there can be informative or non-informative equilibria in the binary signaling game under the Stackelberg assumption, but there always exists an equilibrium. However, apart from the informative and non-informative equilibria cases, there may not be a Nash equilibrium when the receiver is restricted to use deterministic policies. We also studied the effects of small perturbations at the point of identical priors and costs and showed that the game equilibrium behavior around the team setup is robust under the Nash assumption, whereas it is not robust under the Stackelberg assumption.

The binary setup considered here can be extended to the M-ary hypothesis testing setup, and the corresponding signaling game structure can be formed in order to model a game between players with a multiple-bit communication channel. The extension to more general noise distributions is possible: the Nash equilibrium analysis holds identically when the noise distribution leads to a single-threshold test. Finally, in addition to the Bayesian approach considered here, different cost structures and parameters can be introduced by investigating the game under Neyman-Pearson and minimax criteria.

4.7 Proofs

4.7.1 Proof of Theorem 4.2.1

The players adjust S_0, S_1, and δ so that $r^t(\mathcal{S},\delta) = r^r(\mathcal{S},\delta)$ is minimized. The Bayes risk of the transmitter and the receiver can be written as follows[3]:

$$\begin{aligned} r^j(\mathcal{S},\delta) &= \pi_0^j \left(C_{00}^j \mathsf{P}_{00} + C_{10}^j \mathsf{P}_{10}\right) + \pi_1^j \left(C_{01}^j \mathsf{P}_{01} + C_{11}^j \mathsf{P}_{11}\right) \\ &= \pi_0^j C_{00}^j + \pi_1^j C_{11}^j + \pi_0^j (C_{10}^j - C_{00}^j)\mathsf{P}_{10} + \pi_1^j (C_{01}^j - C_{11}^j)\mathsf{P}_{01} \ , \end{aligned} \tag{4.7}$$

for $j \in \{t, r\}$.

Here, first the receiver chooses the optimal decision rule $\delta^*_{S_0,S_1}$ for any given signal levels S_0 and S_1, and then the transmitter chooses the optimal signal levels S_0^* and S_1^* depending on the optimal receiver policy $\delta^*_{S_0,S_1}$.

Assuming non-zero priors $\pi_0^t, \pi_0^r, \pi_1^t$, and π_1^r, the different cases for the optimal receiver decision rule can be investigated by utilizing (4.4) as follows:

(i) If $C_{01}^r > C_{11}^r$,

- (a) if $C_{10}^r > C_{00}^r$, the LRT in (4.4) must be applied to determine the optimal decision.
- (b) if $C_{10}^r \le C_{00}^r$, the left-hand side (LHS) of the inequality in (4.4) is always greater than the right-hand side (RHS); thus, the receiver always chooses $\mathcal{H}_1$.

(ii) If $C_{01}^r = C_{11}^r$,

- (a) if $C_{10}^r > C_{00}^r$, the LHS of the inequality in (4.4) is always less than the RHS; thus, the receiver always chooses $\mathcal{H}_0$.
- (b) if $C_{10}^r = C_{00}^r$, the LHS and RHS of the inequality in (4.4) are equal; hence, the receiver is indifferent of deciding $\mathcal{H}_0$ or $\mathcal{H}_1$.

[3]Note that we are still keeping the parameters of the transmitter and the receiver as distinct in order to be able to utilize the expressions for the game formulations.

(c) if $C^r_{10} < C^r_{00}$, the LHS of the inequality in (4.4) is always greater than the RHS; thus, the receiver always chooses $\mathcal{H}_1$.

(iii) If $C^r_{01} < C^r_{11}$,

(a) if $C^r_{10} \geq C^r_{00}$, the LHS of the inequality in (4.4) is always less than the RHS; thus, the receiver always chooses $\mathcal{H}_0$.

(b) if $C^r_{10} < C^r_{00}$, the LRT in (4.4) must be applied to determine the optimal decision.

The analysis on the optimal receiver decision rule above is summarized in Table 4.4:

Table 4.4: Optimal decision rule analysis for the receiver.

	$\boldsymbol{C^r_{10} > C^r_{00}}$	$\boldsymbol{C^r_{10} = C^r_{00}}$	$\boldsymbol{C^r_{10} < C^r_{00}}$
$\boldsymbol{C^r_{01} > C^r_{11}}$	LRT	always $\mathcal{H}_1$	always $\mathcal{H}_1$
$\boldsymbol{C^r_{01} = C^r_{11}}$	always $\mathcal{H}_0$	indifferent ($\mathcal{H}_0$ or $\mathcal{H}_1$)	always $\mathcal{H}_1$
$\boldsymbol{C^r_{01} < C^r_{11}}$	always $\mathcal{H}_0$	always $\mathcal{H}_0$	LRT

As it can be observed from Table 4.4, the LRT is needed only when $\tau \triangleq \frac{\pi^r_0(C^r_{10}-C^r_{00})}{\pi^r_1(C^r_{01}-C^r_{11})}$ takes a finite positive value; i.e., $0 < \tau < \infty$. Otherwise; i.e., $\tau \leq 0$ or $\tau = \infty$, since the receiver does not consider any message sent by the transmitter, the equilibrium is non-informative.

For $0 < \tau < \infty$, let $\zeta \triangleq \mathrm{sgn}(C^r_{01} - C^r_{11})$ (notice that $\zeta = \mathrm{sgn}(C^r_{01} - C^r_{11}) = \mathrm{sgn}(C^r_{10} - C^r_{00})$ and $\zeta \in \{-1, 1\}$). Then, the optimal decision rule for the receiver in (4.4) becomes

$$\delta : \begin{cases} \zeta \frac{p_1(y)}{p_0(y)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta \frac{\pi^r_0(C^r_{10} - C^r_{00})}{\pi^r_1(C^r_{01} - C^r_{11})} = \zeta\tau \ . \end{cases} \tag{4.8}$$

Let the transmitter choose optimal signals $\mathcal{S} = \{S_0, S_1\}$. Then the measurements in (4.1) become

$$\mathcal{H}_0 : Y \sim \mathcal{N}(S_0, \sigma^2) \ ,$$
$$\mathcal{H}_1 : Y \sim \mathcal{N}(S_1, \sigma^2) \ ,$$

as $N \sim \mathcal{N}(0, \sigma^2)$, and the optimal decision rule for the receiver is obtained by utilizing (4.8) as

$$\begin{aligned}
\delta^*_{S_0,S_1} &: \left\{ \zeta \frac{\exp\left\{-\frac{1}{2\sigma^2}(y-S_1)^2\right\}}{\exp\left\{-\frac{1}{2\sigma^2}(y-S_0)^2\right\}} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\tau \right. \\
&: \left\{ \zeta \exp\left\{\frac{S_0^2 - S_1^2}{2\sigma^2}\right\} \exp\left\{\frac{y}{\sigma^2}(S_1 - S_0)\right\} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\tau \right. \\
&: \left\{ \zeta y(S_1 - S_0) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\left(\sigma^2 \ln(\tau) + \frac{S_1^2 - S_0^2}{2}\right) \right. . \qquad (4.9)
\end{aligned}$$

Since

$$\zeta Y(S_1 - S_0) \sim \begin{cases} \mathcal{N}\Big(\zeta(S_1 - S_0)S_0, (S_1 - S_0)^2\sigma^2\Big) \text{ under } \mathcal{H}_0 \\ \mathcal{N}\Big(\zeta(S_1 - S_0)S_1, (S_1 - S_0)^2\sigma^2\Big) \text{ under } \mathcal{H}_1 \end{cases},$$

the conditional probabilities can be written based on (4.9) as follows:

$$\begin{aligned}
\mathsf{P}_{10} &= \mathsf{Pr}(y \in \Gamma_1 | \mathcal{H}_0) = \mathsf{Pr}(\delta(y) = 1|\mathcal{H}_0) = 1 - \mathsf{Pr}(\delta(y) = 0|\mathcal{H}_0) = 1 - \mathsf{P}_{00} \\
&= \mathcal{Q}\left(\frac{\zeta\left(\sigma^2\ln(\tau) + \frac{S_1^2 - S_0^2}{2} - (S_1 - S_0)S_0\right)}{|S_1 - S_0|\,\sigma}\right) \\
&= \mathcal{Q}\left(\zeta\left(\frac{\sigma\ln(\tau)}{|S_1 - S_0|} + \frac{|S_1 - S_0|}{2\sigma}\right)\right) , \\
\mathsf{P}_{01} &= \mathsf{Pr}(y \in \Gamma_0 | \mathcal{H}_1) = \mathsf{Pr}(\delta(y) = 0|\mathcal{H}_1) = 1 - \mathsf{Pr}(\delta(y) = 1|\mathcal{H}_1) = 1 - \mathsf{P}_{11} \\
&= 1 - \mathcal{Q}\left(\frac{\zeta\left(\sigma^2\ln(\tau) + \frac{S_1^2 - S_0^2}{2} - (S_1 - S_0)S_1\right)}{|S_1 - S_0|\,\sigma}\right) \\
&= 1 - \mathcal{Q}\left(\zeta\left(\frac{\sigma\ln(\tau)}{|S_1 - S_0|} - \frac{|S_1 - S_0|}{2\sigma}\right)\right) \\
&= \mathcal{Q}\left(\zeta\left(-\frac{\sigma\ln(\tau)}{|S_1 - S_0|} + \frac{|S_1 - S_0|}{2\sigma}\right)\right) . \qquad (4.10)
\end{aligned}$$

By defining $d \triangleq \frac{|S_1 - S_0|}{\delta}$, $\mathsf{P}_{10} = 1 - \mathsf{P}_{00} = \mathcal{Q}\left(\zeta\left(\frac{\ln(\tau)}{d} + \frac{d}{2}\right)\right)$ and $\mathsf{P}_{01} = 1 - \mathsf{P}_{11} = \mathcal{Q}\left(\zeta\left(-\frac{\ln(\tau)}{d} + \frac{d}{2}\right)\right)$ can be obtained. Then, the optimum behavior of the transmitter can be found by analyzing the derivative of the Bayes risk of the transmitter in (4.7) with respect to d:

$$\frac{\mathrm{d}\, r^t(\mathcal{S}, \delta)}{\mathrm{d}\, d} = \pi_0^t(C_{10}^t - C_{00}^t)\frac{-1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln\tau}{d} + \frac{d}{2}\right)^2\right\} \zeta\left(-\frac{\ln\tau}{d^2} + \frac{1}{2}\right)$$

$$
\begin{aligned}
&+ \pi_1^t (C_{01}^t - C_{11}^t) \frac{-1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(-\frac{\ln \tau}{d} + \frac{d}{2} \right)^2 \right\} \zeta \left(\frac{\ln \tau}{d^2} + \frac{1}{2} \right) \\
= &-\frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(\ln \tau)^2}{2d^2} \right\} \exp\left\{ -\frac{d^2}{8} \right\} \\
\times &\left(\pi_0^t \zeta (C_{10}^t - C_{00}^t) \tau^{-\frac{1}{2}} \left(-\frac{\ln \tau}{d^2} + \frac{1}{2} \right) + \pi_1^t \zeta (C_{01}^t - C_{11}^t) \tau^{\frac{1}{2}} \left(\frac{\ln \tau}{d^2} + \frac{1}{2} \right) \right) .
\end{aligned}
\tag{4.11}
$$

In (4.11), if we utilize $C_{ji} = C_{ji}^t = C_{ji}^r$, $\pi_i = \pi_i^t = \pi_i^r$ and $\tau = \frac{\pi_0(C_{10}-C_{00})}{\pi_1(C_{01}-C_{11})}$, we obtain the following: we obtain $\frac{\mathrm{d}\, r^t(\mathcal{S},\delta)}{\mathrm{d}\, d} < 0$, as expected [72].

$$
\begin{aligned}
\frac{\mathrm{d}\, r^t(\mathcal{S},\delta)}{\mathrm{d}\, d} &= -\frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(\ln \tau)^2}{2d^2} \right\} \exp\left\{ -\frac{d^2}{8} \right\} \\
&\times \left(\pi_0 |C_{10} - C_{00}| \tau^{-\frac{1}{2}} \left(-\frac{\ln \tau}{d^2} + \frac{1}{2} \right) + \pi_1 |C_{01} - C_{11}| \tau^{\frac{1}{2}} \left(\frac{\ln \tau}{d^2} + \frac{1}{2} \right) \right) \\
&= -\frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(\ln \tau)^2}{2d^2} \right\} \exp\left\{ -\frac{d^2}{8} \right\} \sqrt{\pi_0 \pi_1 (C_{10} - C_{00})(C_{01} - C_{11})} < 0 \, .
\end{aligned}
$$

Thus, in order to minimize the Bayes risk, the transmitter always prefers the maximum d, i.e., $d^* = \frac{\sqrt{P_0}+\sqrt{P_1}}{\sigma}$, and the equilibrium is informative. Further, there are two equilibrium points: $(S_0^*, S_1^*) = \left(-\sqrt{P_0}, \sqrt{P_1}\right)$ and $(S_0^*, S_1^*) = \left(\sqrt{P_0}, -\sqrt{P_1}\right)$, and the decision rule of the receiver is chosen based on the rule in (4.9) accordingly. Actually, the equilibrium points are essentially unique; i.e., they result in the same Bayes risks for the transmitter and the receiver.

4.7.2 Proof of Theorem 4.3.1

By applying the same case analysis as in the proof of Theorem 4.2.1, it can be deduced that the equilibrium is non-informative if $\tau \leq 0$ or $\tau = \infty$ (see Table 4.4). Thus, $0 < \tau < \infty$ can be assumed. Then, from (4.11), $r^t(\mathcal{S},\delta)$ is a monotone decreasing (increasing) function of d if $k_0 \left(-\frac{\ln \tau}{d^2} + \frac{1}{2}\right) + k_1 \left(\frac{\ln \tau}{d^2} + \frac{1}{2}\right)$, or equivalently $d^2(k_0 + k_1) - 2 \ln \tau \, (k_0 - k_1)$ is positive (negative) $\forall d$, where k_0 and k_1 are as defined in the theorem statement. Therefore, one of the following cases is applicable:

1. if $\ln\tau\ (k_0 - k_1) < 0$ and $k_0 + k_1 \geq 0$, then $d^2(k_0 + k_1) > 2\ln\tau(k_0 - k_1)$ is satisfied $\forall d$, which means that $r^t(\mathcal{S}, \delta)$ is a monotone decreasing function of d. Therefore, the transmitter tries to maximize d; i.e., chooses the maximum of $|S_1 - S_0|$ under the constraints $|S_0|^2 \leq P_0$ and $|S_1|^2 \leq P_1$, hence $d^* = \max \frac{|S_1 - S_0|}{\sigma} = \frac{\sqrt{P_0}+\sqrt{P_1}}{\sigma} = d_{\max}$, which entails an informative equilibrium.

2. if $\ln\tau\ (k_0 - k_1) < 0$, $k_0 + k_1 < 0$, and $d_{\max}^2 < \left|\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right|$, then $r^t(\mathcal{S}, \delta)$ is a monotone decreasing function of d. Therefore, the transmitter maximizes d as in the previous case.

3. if $\ln\tau\ (k_0 - k_1) < 0$, $k_0 + k_1 < 0$, and $d_{\max}^2 \geq \left|\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right|$, since $d^2(k_0 + k_1) - 2\ln\tau\,(k_0 - k_1)$ is initially positive then negative, $r^t(\mathcal{S}, \delta)$ is first decreasing and then increasing with respect to d. Therefore, the transmitter chooses the optimal d^* such that $(d^*)^2 = \left|\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right|$ which results in a minimal Bayes risk $r^t(\mathcal{S}, \delta)$ for the transmitter. This is depicted in Figure 4.2.

4. if $\ln\tau\ (k_0 - k_1) \geq 0$ and $k_0 + k_1 < 0$, then $d^2(k_0 + k_1) < 2\ln\tau(k_0 - k_1)$ is satisfied $\forall d$, which means that $r^t(\mathcal{S}, \delta)$ is a monotone increasing function of d. Therefore, the transmitter tries to minimize d; i.e., chooses $S_0 = S_1$ so that $d^* = 0$. In this case, the transmitter does not provide any information to the receiver and the decision rule of the receiver in (4.8) becomes $\delta : \zeta \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\tau$; i.e., the receiver uses only the prior information, thus the equilibrium is non-informative.

5. if $\ln\tau\ (k_0 - k_1) \geq 0$, $k_0 + k_1 \geq 0$, and $d_{\max}^2 < \left|\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right|$, then $r^t(\mathcal{S}, \delta)$ is a monotone increasing function of d. Therefore, the transmitter chooses $S_0 = S_1$ so that $d^* = 0$. Similar to the previous case, the equilibrium is non-informative.

6. if $\ln\tau\ (k_0 - k_1) \geq 0$, $k_0 + k_1 \geq 0$, and $d_{\max}^2 \geq \left|\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right|$, $r^t(\mathcal{S}, \delta)$ is first an increasing then a decreasing function of d, which makes the transmitter choose either the minimum d or the maximum d; i.e., he chooses the one that results in a lower Bayes risk $r^t(\mathcal{S}, \delta)$ for the transmitter. If the minimum Bayes risk is achieved when $d^* = 0$, then the equilibrium is non-informative;

otherwise (i.e., when the minimum Bayes risk is achieved when $d^* = d_{\max}$), the equilibrium is an informative one. There are three possible cases:

(a) $\underline{\zeta(1-\tau) > 0}$:

i. If $d^* = 0$, since $\delta : \zeta \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\tau$, the receiver always chooses $\mathcal{H}_1$, thus $\mathsf{P}_{10} = \mathsf{P}_{11} = 1$ and $\mathsf{P}_{00} = \mathsf{P}_{01} = 0$. Then, from (4.7), $r^t(\mathcal{S},\delta) = \pi_0^t C_{00}^t + \pi_1^t C_{11}^t + \pi_0^t(C_{10}^t - C_{00}^t)$.

ii. If $d^* = d_{\max}$, by utilizing (4.7) and (4.10), $r^t(\mathcal{S},\delta) = \pi_0^t C_{00}^t + \pi_1^t C_{11}^t + \pi_0^t(C_{10}^t - C_{00}^t)\mathcal{Q}\left(\zeta\left(\frac{\ln(\tau)}{d_{\max}} + \frac{d_{\max}}{2}\right)\right) + \pi_1^t(C_{01}^t - C_{11}^t)\mathcal{Q}\left(\zeta\left(-\frac{\ln(\tau)}{d_{\max}} + \frac{d_{\max}}{2}\right)\right)$.

Then the decision of the transmitter is determined by the following:

$$\begin{aligned}
&\pi_0^t(C_{10}^t - C_{00}^t) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtrless}} \pi_0^t(C_{10}^t - C_{00}^t)\mathcal{Q}\left(\zeta\left(\frac{\ln(\tau)}{d_{\max}} + \frac{d_{\max}}{2}\right)\right) \\
&\qquad\qquad + \pi_1^t(C_{01}^t - C_{11}^t)\mathcal{Q}\left(\zeta\left(-\frac{\ln(\tau)}{d_{\max}} + \frac{d_{\max}}{2}\right)\right) \\
&\pi_0^t(C_{10}^t - C_{00}^t)\mathcal{Q}\left(\zeta\left(-\frac{\ln(\tau)}{d_{\max}} - \frac{d_{\max}}{2}\right)\right) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtrless}} \\
&\qquad\qquad \pi_1^t(C_{01}^t - C_{11}^t)\mathcal{Q}\left(\zeta\left(-\frac{\ln(\tau)}{d_{\max}} + \frac{d_{\max}}{2}\right)\right) \\
&\zeta k_0\tau\mathcal{Q}\left(\zeta\left(-\frac{\ln(\tau)}{d_{\max}} - \frac{d_{\max}}{2}\right)\right) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtrless}} \zeta k_1\mathcal{Q}\left(\zeta\left(-\frac{\ln(\tau)}{d_{\max}} + \frac{d_{\max}}{2}\right)\right).
\end{aligned} \tag{4.12}$$

(4.12) can be analyzed based on the values of ζ and τ as follows:

i. $\underline{\zeta = 1 \text{ and } 0 < \tau < 1}$: Since $\ln\tau(k_0 - k_1) \geq 0 \Rightarrow k_0 - k_1 \leq 0$ and $k_0 + k_1 \geq 0$, $k_1 \geq 0$ always. Then, (4.12) becomes

$$\frac{k_0\tau}{k_1}\mathcal{Q}\left(-\frac{\ln(\tau)}{d_{\max}} - \frac{d_{\max}}{2}\right) - \mathcal{Q}\left(-\frac{\ln(\tau)}{d_{\max}} + \frac{d_{\max}}{2}\right) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtrless}} 0\,.$$

ii. $\underline{\zeta = -1 \text{ and } \tau > 1}$: Since $\ln\tau(k_0 - k_1) \geq 0 \Rightarrow k_0 - k_1 \geq 0$ and $k_0 + k_1 \geq 0$, $k_0 \geq 0$ always. Then, (4.12) becomes

$$\frac{k_1}{k_0\tau}\mathcal{Q}\left(\frac{\ln(\tau)}{d_{\max}} - \frac{d_{\max}}{2}\right) - \mathcal{Q}\left(\frac{\ln(\tau)}{d_{\max}} + \frac{d_{\max}}{2}\right) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtrless}} 0\,.$$

(b) $\underline{\zeta(1-\tau)=0 \Leftrightarrow \tau=1}$: Since $k_0+k_1 \geq 0$ and $d^2(k_0+k_1)-2\ln\tau\,(k_0-k_1) \geq 0$, $r^t(\mathcal{S},\delta)$ is a monotone decreasing function of d, which implies $d^*=d_{\max}$ and informative equilibrium.

(c) $\underline{\zeta(1-\tau)<0}$:

i. If $d^*=0$, since $\delta : \zeta \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtreqless}} \zeta\tau$, the receiver always chooses $\mathcal{H}_0$, thus $\mathsf{P}_{00}=\mathsf{P}_{01}=1$ and $\mathsf{P}_{10}=\mathsf{P}_{11}=0$. Then, from (4.7), $r^t(\mathcal{S},\delta) = \pi_0^t C_{00}^t + \pi_1^t C_{11}^t + \pi_1^t(C_{01}^t - C_{11}^t)$.

ii. If $d^*=d_{\max}$, by utilizing (4.7) and (4.10), $r^t(\mathcal{S},\delta) = \pi_0^t C_{00}^t + \pi_1^t C_{11}^t + \pi_0^t(C_{10}^t - C_{00}^t)\mathcal{Q}\left(\zeta\left(\frac{\ln(\tau)}{d_{\max}}+\frac{d_{\max}}{2}\right)\right) + \pi_1^t(C_{01}^t - C_{11}^t)\mathcal{Q}\left(\zeta\left(-\frac{\ln(\tau)}{d_{\max}}+\frac{d_{\max}}{2}\right)\right)$.

Then the decision of the transmitter is determined by the following:

$$
\begin{aligned}
&\pi_1^t(C_{01}^t - C_{11}^t) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtreqless}} \pi_0^t(C_{10}^t - C_{00}^t)\mathcal{Q}\left(\zeta\left(\frac{\ln(\tau)}{d_{\max}}+\frac{d_{\max}}{2}\right)\right) \\
&\qquad\qquad + \pi_1^t(C_{01}^t - C_{11}^t)\mathcal{Q}\left(\zeta\left(-\frac{\ln(\tau)}{d_{\max}}+\frac{d_{\max}}{2}\right)\right) \\
&\pi_1^t(C_{01}^t - C_{11}^t)\mathcal{Q}\left(\zeta\left(\frac{\ln(\tau)}{d_{\max}}-\frac{d_{\max}}{2}\right)\right) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtreqless}} \\
&\qquad\qquad \pi_0^t(C_{10}^t - C_{00}^t)\mathcal{Q}\left(\zeta\left(\frac{\ln(\tau)}{d_{\max}}+\frac{d_{\max}}{2}\right)\right) \\
&\zeta k_1\mathcal{Q}\left(\zeta\left(\frac{\ln(\tau)}{d_{\max}}-\frac{d_{\max}}{2}\right)\right) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtreqless}} \zeta k_0\tau\mathcal{Q}\left(\zeta\left(\frac{\ln(\tau)}{d_{\max}}+\frac{d_{\max}}{2}\right)\right).
\end{aligned}
\tag{4.13}
$$

(4.13) can be analyzed based on the values of ζ and τ as follows:

i. $\underline{\zeta=-1 \text{ and } 0<\tau<1}$: Since $\ln\tau(k_0-k_1) \geq 0 \Rightarrow k_0-k_1 \leq 0$ and $k_0+k_1 \geq 0$, $k_1 \geq 0$ always. Then, (4.13) becomes

$$\frac{k_0\tau}{k_1}\mathcal{Q}\left(-\frac{\ln(\tau)}{d_{\max}}-\frac{d_{\max}}{2}\right) - \mathcal{Q}\left(-\frac{\ln(\tau)}{d_{\max}}+\frac{d_{\max}}{2}\right) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtreqless}} 0\,.$$

ii. $\underline{\zeta=1 \text{ and } \tau>1}$: Since $\ln\tau(k_0-k_1) \geq 0 \Rightarrow k_0-k_1 \geq 0$ and $k_0+k_1 \geq 0$, $k_0 \geq 0$ always. Then, (4.13) becomes

$$\frac{k_1}{k_0\tau}\mathcal{Q}\left(\frac{\ln(\tau)}{d_{\max}}-\frac{d_{\max}}{2}\right) - \mathcal{Q}\left(\frac{\ln(\tau)}{d_{\max}}+\frac{d_{\max}}{2}\right) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtreqless}} 0\,.$$

Thus, the comparison of the transmitter Bayes risks for $d^* = 0$ and $d^* = d_{\max}$ reduces to the following rule:

$$\left(\frac{k_1}{k_0\tau}\right)^{\mathrm{sgn}(\ln(\tau))} \mathcal{Q}\left(\frac{|\ln(\tau)|}{d_{\max}} - \frac{d_{\max}}{2}\right) - \mathcal{Q}\left(\frac{|\ln(\tau)|}{d_{\max}} + \frac{d_{\max}}{2}\right) \underset{d^*=0}{\overset{d^*=d_{\max}}{\gtrless}} 0\,. \tag{4.14}$$

The most interesting case is Case-3 in which $\ln\tau\ (k_0 - k_1) < 0, k_0 + k_1 < 0$, and $d_{\max}^2 \geq \left|\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right|$, since in all other cases, the transmitter chooses either the minimum or the maximum distance between the signal levels. Further, for classical hypothesis-testing in the team setup, the optimal distance corresponds to the maximum separation [72]. However, in Case-3, there is an optimal distance $d^* = \sqrt{\left|\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right|} < d_{\max}$ that makes the Bayes risk of the transmitter minimum as it can be seen in Figure 4.2.

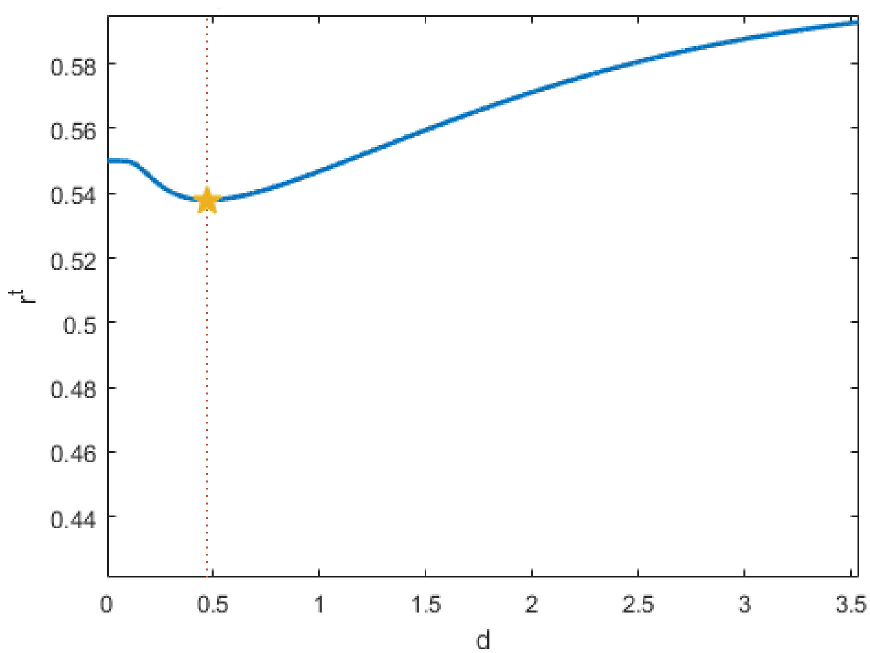

Figure 4.2: The Bayes risk of the transmitter versus d when $C^r_{01} = 0.4, C^r_{10} = 0.9, C^r_{00} = 0, C^r_{11} = 0, C^t_{01} = 0.4, C^t_{10} = 0.4, C^t_{00} = 0.6, C^t_{11} = 0.6, P_0 = 1, P_1 = 1, \sigma = 0.1, \pi^t_0 = 0.25,$ *and* $\pi^r_0 = 0.25$. *The optimal* $d^* = \sqrt{\left|\frac{2\ln\tau(k_0-k_1)}{k_0+k_1}\right|} = 0.4704 < d_{max}$ *and its corresponding Bayes risk are indicated by the star.*

4.7.3 Proof of Theorem 4.4.1

Let the transmitter choose any signals $\mathcal{S} = \{S_0, S_1\}$. Assuming nonzero priors $\pi_0^t, \pi_0^r, \pi_1^t$ and π_1^r, the optimal decision for the receiver is given by (4.9). By applying the same extreme case analysis as in the proof of Theorem 4.2.1, the equilibrium is non-informative if $\tau \leq 0$ or $\tau = \infty$ (see Table 4.4); thus, $0 < \tau < \infty$ can be assumed.

Now assume that the receiver applies a single-threshold rule; i.e., $\delta : \left\{ ay \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta \right.$ where $a \in \mathbb{R} - \{0\}$, and $\eta \in \mathbb{R}$. Since

$$aY \sim \begin{cases} \mathcal{N}\Big(aS_0, a^2\sigma^2\Big) & \text{under } \mathcal{H}_0 \\ \mathcal{N}\Big(aS_1, a^2\sigma^2\Big) & \text{under } \mathcal{H}_1 \end{cases},$$

the conditional probabilities are

$$\begin{aligned} \mathsf{P}_{10} &= \mathsf{Pr}(y \in \Gamma_1 | \mathcal{H}_0) = \mathsf{Pr}(\delta(y) = 1 | \mathcal{H}_0) = 1 - \mathsf{Pr}(\delta(y) = 0 | \mathcal{H}_0) \\ &= 1 - \mathsf{P}_{00} = \mathcal{Q}\left(\frac{\eta - aS_0}{|a|\sigma}\right), \end{aligned} \tag{4.15}$$

$$\begin{aligned} \mathsf{P}_{01} &= \mathsf{Pr}(\delta(y) = 0 | \mathcal{H}_1) = 1 - \mathsf{Pr}(\delta(y) = 1 | \mathcal{H}_1) \\ &= 1 - \mathsf{P}_{11} = 1 - \mathcal{Q}\left(\frac{\eta - aS_1}{|a|\sigma}\right) = \mathcal{Q}\left(-\frac{\eta - aS_1}{|a|\sigma}\right). \end{aligned} \tag{4.16}$$

Then, by inserting (4.15) and (4.16) in (4.7), the Bayes risk of the transmitter becomes

$$\begin{aligned} r^t(\mathcal{S}, \delta) = \pi_0^t C_{00}^t + \pi_1^t C_{11}^t + \pi_0^t (C_{10}^t - C_{00}^t) \mathcal{Q}\left(\frac{\eta - aS_0}{|a|\sigma}\right) \\ + \pi_1^t (C_{01}^t - C_{11}^t) \mathcal{Q}\left(-\frac{\eta - aS_1}{|a|\sigma}\right). \end{aligned}$$

Since the power constraints are $|S_0|^2 \leq P_0$ and $|S_1|^2 \leq P_1$, the signals S_0 and S_1 can be regarded as independent, and the optimum signals $\mathcal{S} = \{S_0, S_1\}$ can be found by analyzing the derivative of the Bayes risk of the transmitter with respect to the signals:

$$\frac{\partial\, r^t(\mathcal{S}, \delta)}{\partial\, S_0} = \pi_0^t (C_{10}^t - C_{00}^t) \frac{-1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\eta - aS_0}{|a|\sigma}\right)^2\right\} \frac{-a}{|a|\sigma}$$

$$= \frac{\operatorname{sgn}(a)}{\sqrt{2\pi}\sigma}\pi_0^t(C_{10}^t - C_{00}^t)\exp\left\{-\frac{1}{2}\left(\frac{\eta - aS_0}{|a|\sigma}\right)^2\right\},$$

$$\frac{\partial\, r^t(\mathcal{S},\delta)}{\partial\, S_1} = \pi_1^t(C_{01}^t - C_{11}^t)\frac{-1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(-\frac{\eta - aS_1}{|a|\sigma}\right)^2\right\}\frac{a}{|a|\sigma}$$

$$= \frac{\operatorname{sgn}(a)}{\sqrt{2\pi}\sigma}\pi_1^t(C_{11}^t - C_{01}^t)\exp\left\{-\frac{1}{2}\left(\frac{\eta - aS_1}{|a|\sigma}\right)^2\right\}.$$

Then, for $i \in \{0,1\}$, the following cases hold:

(i) $\underline{C_{1i}^t = C_{0i}^t} \Rightarrow S_i$ has no effect on the Bayes risk of the transmitter.

(ii) $\underline{C_{1i}^t < C_{0i}^t \text{ or } C_{1i}^t > C_{0i}^t} \Rightarrow r^t(\mathcal{S},\delta)$ is a decreasing (increasing) function of S_i if $a(C_{1i}^t - C_{0i}^t)$ is negative (positive); thus the transmitter chooses the optimal signal levels as $S_0 = -\operatorname{sgn}(a)\operatorname{sgn}(C_{10}^t - C_{00}^t)\sqrt{P_0}$ and $S_1 = \operatorname{sgn}(a)\operatorname{sgn}(C_{01}^t - C_{11}^t)\sqrt{P_1}$.

By using the expressions above, the cases can be listed as follows:

1. $\underline{\tau \leq 0 \text{ or } \tau = \infty} \Rightarrow$ The equilibrium is non-informative.

2. $\underline{C_{10}^t = C_{00}^t \text{ (and/or } C_{01}^t = C_{11}^t)} \Rightarrow S_0$ (and/or S_1) has no effect on the Bayes risk of the transmitter; thus it can arbitrarily be chosen by the transmitter. In this case, if the transmitter chooses $S_0 = S_1$; i.e., he does not send anything useful to the receiver, and the receiver applies the decision rule $\delta : \zeta \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\tau$; i.e., he only considers the prior information (totally discards the information sent by the transmitter). Therefore, there exists a non-informative equilibrium.

3. Notice that, since $0 < \tau < \infty$ is assumed, $\zeta = \operatorname{sgn}(C_{01}^r - C_{11}^r) = \operatorname{sgn}(C_{10}^r - C_{00}^r)$ is obtained. Now, assume that the decision rule of the receiver is $\delta : \left\{ ay \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta\right.$. Then, the transmitter selects $S_0 = -\operatorname{sgn}(a)\operatorname{sgn}(C_{10}^t - C_{00}^t)\sqrt{P_0}$ and $S_1 = \operatorname{sgn}(a)\operatorname{sgn}(C_{01}^t - C_{11}^t)\sqrt{P_1}$ as optimal signals, and the decision rule

becomes (4.9). By combining the best responses of the transmitter and the receiver,

$$
\begin{aligned}
a &= \zeta(S_1 - S_0) = \zeta \mathrm{sgn}(a)\left(\mathrm{sgn}(C^t_{01} - C^t_{11})\sqrt{P_1} + \mathrm{sgn}(C^t_{10} - C^t_{00})\sqrt{P_0}\right)\\
\Rightarrow & \mathrm{sgn}(a) = \zeta \mathrm{sgn}(a)\left(\mathrm{sgn}(C^t_{01} - C^t_{11})\sqrt{P_1} + \mathrm{sgn}(C^t_{10} - C^t_{00})\sqrt{P_0}\right)\\
\Rightarrow & \underbrace{\frac{\mathrm{sgn}(C^t_{01} - C^t_{11})}{\mathrm{sgn}(C^r_{01} - C^r_{11})}}_{=\mathrm{sgn}(\xi_1)}\sqrt{P_1} + \underbrace{\frac{\mathrm{sgn}(C^t_{10} - C^t_{00})}{\mathrm{sgn}(C^r_{10} - C^r_{00})}}_{=\mathrm{sgn}(\xi_0)}\sqrt{P_0} > 0
\end{aligned}
\tag{4.17}
$$

is obtained. Here, unless (4.17) is satisfied, the best responses of the transmitter and the receiver cannot match each other. Then, there are four possible cases:

(a) $\underline{\xi_0 < 0 \text{ and } \xi_1 < 0}$ $\Rightarrow$ (4.17) cannot be satisfied; thus, the best responses of the transmitter and the receiver do not match each other, which results in the absence of a Nash equilibrium.

(b) $\underline{\xi_0 < 0 \text{ and } \xi_1 > 0}$ $\Rightarrow$ (4.17) is satisfied only when $\sqrt{P_1} > \sqrt{P_0}$. If $\sqrt{P_1} < \sqrt{P_0}$, (4.17) cannot be satisfied and the best responses of the transmitter and the receiver do not match each other, which results in the absence of a Nash equilibrium. If $\sqrt{P_1} = \sqrt{P_0}$ (which implies $S_0 = S_1$), then the receiver applies $\delta : \left\{ \zeta \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\tau \right.$ as in Case-2, and the receiver chooses either always $\mathcal{H}_0$ or always $\mathcal{H}_1$. Hence, there exists a non-informative equilibrium; i.e., the transmitter sends dummy signals, and the receiver makes a decision without considering the transmitted signals.

(c) $\underline{\xi_0 > 0 \text{ and } \xi_1 < 0}$ $\Rightarrow$ (4.17) is satisfied only when $\sqrt{P_0} > \sqrt{P_1}$. If $\sqrt{P_0} < \sqrt{P_1}$, (4.17) cannot be satisfied and the best responses of the transmitter and the receiver do not match each other, which results in the absence of a Nash equilibrium. If $\sqrt{P_1} = \sqrt{P_0}$ (which implies $S_0 = S_1$), then the receiver applies $\delta : \left\{ \zeta \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\tau \right.$ as in Case-2, and the equilibrium is non-informative.

(d) $\underline{\xi_0 > 0 \text{ and } \xi_1 > 0}$ $\Rightarrow$ (4.17) is always satisfied; thus, the consistency is established, and there exists an informative equilibrium.

4.7.4 Proofs for Section 4.5

4.7.4.1 Proof of Theorem 4.5.1

Let the transmitter choose optimal signals $\mathcal{S} = \{\mathbf{S}_0, \mathbf{S}_1\}$. Then the measurements become $\mathcal{H}_i : \mathbf{Y} \sim \mathcal{N}(\mathbf{S}_i, \Sigma)$ for $i \in \{0,1\}$. As in the scalar case in Theorem 4.2.1, the equilibrium is non-informative for $\tau \leq 0$ or $\tau = \infty$; hence, $0 < \tau < \infty$ can be assumed. Similar to (4.9), the optimal decision rule for the receiver is obtained by utilizing (4.8) as

$$\begin{aligned}
\delta^*_{\mathbf{S}_0,\mathbf{S}_1} &: \left\{ \zeta \frac{p_1(\mathbf{Y})}{p_0(\mathbf{Y})} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta \frac{\pi_0^r (C_{10}^r - C_{00}^r)}{\pi_1^r (C_{01}^r - C_{11}^r)} \triangleq \zeta\tau \right. \\
&: \left\{ \zeta \frac{\frac{1}{\sqrt{(2\pi)^n|\Sigma|}} \exp\left\{-\frac{1}{2}(\mathbf{Y}-\mathbf{S}_1)^T\Sigma^{-1}(\mathbf{Y}-\mathbf{S}_1)\right\}}{\frac{1}{\sqrt{(2\pi)^n|\Sigma|}} \exp\left\{-\frac{1}{2}(\mathbf{Y}-\mathbf{S}_0)^T\Sigma^{-1}(\mathbf{Y}-\mathbf{S}_0)\right\}} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\tau \right. \\
&: \left\{ \zeta \exp\left\{ (\mathbf{S}_1-\mathbf{S}_0)^T \Sigma^{-1} \left(\mathbf{Y} - \frac{\mathbf{S}_1+\mathbf{S}_0}{2} \right) \right\} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\tau \right. \\
&: \left\{ \zeta (\mathbf{S}_1-\mathbf{S}_0)^T \Sigma^{-1} \mathbf{Y} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta \left(\ln(\tau) + \frac{1}{2} (\mathbf{S}_1-\mathbf{S}_0)^T \Sigma^{-1} (\mathbf{S}_1+\mathbf{S}_0) \right) \right. . \qquad (4.18)
\end{aligned}$$

Since $\zeta(\mathbf{S}_1-\mathbf{S}_0)^T\Sigma^{-1}\mathbf{Y} \sim \mathcal{N}\left(\zeta(\mathbf{S}_1-\mathbf{S}_0)^T\Sigma^{-1}\mathbf{S}_i, (\mathbf{S}_1-\mathbf{S}_0)^T\Sigma^{-1}(\mathbf{S}_1-\mathbf{S}_0)\right)$ under hypothesis $\mathcal{H}_i$ for $i \in \{0,1\}$, by defining $d^2 \triangleq (\mathbf{S}_1-\mathbf{S}_0)^T\Sigma^{-1}(\mathbf{S}_1-\mathbf{S}_0)$, the conditional probabilities can be written as follows:

$$\begin{aligned}
\mathsf{P}_{10} &= \mathcal{Q}\left(\zeta \frac{\ln(\tau) + \frac{1}{2}(\mathbf{S}_1-\mathbf{S}_0)^T\Sigma^{-1}(\mathbf{S}_1+\mathbf{S}_0-2\mathbf{S}_0)}{\sqrt{(\mathbf{S}_1-\mathbf{S}_0)^T\Sigma^{-1}(\mathbf{S}_1-\mathbf{S}_0)}} \right) = \mathcal{Q}\left(\zeta \left(\frac{\ln(\tau)}{d} + \frac{d}{2} \right) \right) \\
\mathsf{P}_{01} &= 1 - \mathcal{Q}\left(\zeta \frac{\ln(\tau) + \frac{1}{2}(\mathbf{S}_1-\mathbf{S}_0)^T\Sigma^{-1}(\mathbf{S}_1+\mathbf{S}_0-2\mathbf{S}_1)}{\sqrt{(\mathbf{S}_1-\mathbf{S}_0)^T\Sigma^{-1}(\mathbf{S}_1-\mathbf{S}_0)}} \right) \\
&= 1 - \mathcal{Q}\left(\zeta \left(\frac{\ln(\tau)}{d} - \frac{d}{2} \right) \right) = \mathcal{Q}\left(\zeta \left(-\frac{\ln(\tau)}{d} + \frac{d}{2} \right) \right) . \qquad (4.19)
\end{aligned}$$

Notice that the conditional probabilities are the same in (4.10) and (4.19); therefore, in the vector case, the equilibrium is always informative, and the transmitter always prefers the maximum distance similar to the scalar case. However, selecting optimal vector signals is not as trivial as in the scalar case; see Section 4.7.4.4 for details (which is based on [72, pp. 61–63]).

4.7.4.2 Proof of Theorem 4.5.2

The proof of Theorem 4.3.1 can be applied by modifying the definitions of d and $d_{\max}$ as in the statement. For $d^* = d_{\max}$, the method described in the proof of Theorem 4.5.1 can be applied for the optimal signal selection, whereas, for $d^* = 0$, by choosing $\mathbf{S}_0 = \mathbf{S}_1$, the non-informative equilibrium can be achieved. Further, for Case-3 of Theorem 4.3.1, a similar approach, which utilizes Section 4.7.4.4, can be taken, as explained in Section 4.7.4.5.

4.7.4.3 Proof of Theorem 4.5.3

Let the transmitter choose any signals $\mathcal{S} = \{\mathbf{S}_0, \mathbf{S}_1\}$. Assuming nonzero priors $\pi_0^t, \pi_0^r, \pi_1^t$ and π_1^r, the optimal decision rule for the receiver is given by (4.18). Similar to the team case analysis in Section 4.5.1, the equilibrium is non-informative if $\tau \leq 0$ or $\tau = \infty$; thus, $0 < \tau < \infty$ can be assumed.

Now assume that the receiver applies a single-threshold rule; i.e., $\delta : \left\{ \mathbf{a}^T \mathbf{Y} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta \right.$ where $\mathbf{a} \in \mathbb{R}^n - \{\mathbf{0}\}$, and $\eta \in \mathbb{R}$. Since

$$\mathbf{a}^T \mathbf{Y} \sim \begin{cases} \mathcal{N}\Big(\mathbf{a}^T \mathbf{S}_0, \mathbf{a}^T \Sigma \mathbf{a}\Big) & \text{under } \mathcal{H}_0 \\ \mathcal{N}\Big(\mathbf{a}^T \mathbf{S}_1, \mathbf{a}^T \Sigma \mathbf{a}\Big) & \text{under } \mathcal{H}_1 \end{cases},$$

the conditional probabilities are

$$\mathsf{P}_{10} = 1 - \mathsf{P}_{00} = \mathcal{Q}\left(\frac{\eta - \mathbf{a}^T \mathbf{S}_0}{\sqrt{\mathbf{a}^T \Sigma \mathbf{a}}}\right), \tag{4.20}$$

$$\mathsf{P}_{01} = 1 - \mathsf{P}_{11} = \mathcal{Q}\left(-\frac{\eta - \mathbf{a}^T \mathbf{S}_1}{\sqrt{\mathbf{a}^T \Sigma \mathbf{a}}}\right). \tag{4.21}$$

Then, by utilizing (4.20) and (4.21) in (4.7), the Bayes risk of the transmitter becomes

$$\begin{aligned} r^t(\mathcal{S}, \delta) = \pi_0^t C_{00}^t + \pi_1^t C_{11}^t + \pi_0^t (C_{10}^t - C_{00}^t) \mathcal{Q}\left(\frac{\eta - \mathbf{a}^T \mathbf{S}_0}{\sqrt{\mathbf{a}^T \Sigma \mathbf{a}}}\right) \\ + \pi_1^t (C_{01}^t - C_{11}^t) \mathcal{Q}\left(-\frac{\eta - \mathbf{a}^T \mathbf{S}_1}{\sqrt{\mathbf{a}^T \Sigma \mathbf{a}}}\right). \end{aligned}$$

Since the power constraints are $\|\mathbf{S}_0\|^2 \le P_0$ and $\|\mathbf{S}_1\|^2 \le P_1$, the signals $\mathbf{S}_0$ and $\mathbf{S}_1$ can be regarded as independent. Since $\mathcal{Q}$ function is a monotone decreasing, the following cases hold for $i \in \{0, 1\}$:

(i) $\underline{C^t_{1i} < C^t_{0i}}$ $\Rightarrow$ Then, $r^t(\mathcal{S}, \delta)$ is a decreasing function of $\mathbf{a}^T\mathbf{S}_i$, thus the transmitter always chooses $\mathbf{a}^T\mathbf{S}_i$ as maximum subject to $\|\mathbf{S}_i\|^2 \le P_i$; i.e., $\mathbf{S}_i = \sqrt{P_i}\frac{\mathbf{a}}{\|\mathbf{a}\|}$.

(ii) $\underline{C^t_{1i} = C^t_{0i}}$ $\Rightarrow$ Then $\mathbf{S}_i$ has no effect on the Bayes risk of the transmitter.

(iii) $\underline{C^t_{1i} > C^t_{0i}}$ $\Rightarrow$ Then, $r^t(\mathcal{S}, \delta)$ is an increasing function of $\mathbf{a}^T\mathbf{S}_i$, thus the transmitter always chooses $\mathbf{a}^T\mathbf{S}_i$ as minimum subject to $\|\mathbf{S}_i\|^2 \le P_i$; i.e., $\mathbf{S}_i = -\sqrt{P_i}\frac{\mathbf{a}}{\|\mathbf{a}\|}$.

Thus, the the optimal signals can be characterized as $S_0 = -\mathrm{sgn}(C^t_{10} - C^t_{00})\sqrt{P_0}\frac{\mathbf{a}}{\|\mathbf{a}\|}$ and $S_1 = \mathrm{sgn}(C^t_{01} - C^t_{11})\sqrt{P_1}\frac{\mathbf{a}}{\|\mathbf{a}\|}$.

By using the expressions above, the cases can be listed as follows:

1. $\underline{\tau \le 0 \text{ or } \tau = \infty}$ $\Rightarrow$ The equilibrium is non-informative.

2. $\underline{C^t_{10} = C^t_{00} \text{ (and/or } C^t_{01} = C^t_{11})}$ $\Rightarrow$ $\mathbf{S}_0$ (and/or $\mathbf{S}_1$) has no effect on the Bayes risk of the transmitter, thus it can arbitrarily be chosen by the transmitter. In this case, if the transmitter chooses $\mathbf{S}_0 = \mathbf{S}_1$; i.e., he does not send anything useful to the receiver, and the receiver applies the decision rule $\delta : \zeta \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \zeta\tau$; i.e., he only considers the prior information (totally discards the information sent by the transmitter). Then there exists a non-informative equilibrium.

3. Notice that, since $0 < \tau < \infty$ is assumed, $\zeta = \mathrm{sgn}(C^r_{01} - C^r_{11}) = \mathrm{sgn}(C^r_{10} - C^r_{00})$ is obtained. Now, assume that the decision rule of the receiver is $\delta : \left\{ \mathbf{a}^T\mathbf{Y} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta \right.$. Then, the transmitter selects $S_0 = -\mathrm{sgn}(C^t_{10} - C^t_{00})\sqrt{P_0}\frac{\mathbf{a}}{\|\mathbf{a}\|}$ and $S_1 = \mathrm{sgn}(C^t_{01} - C^t_{11})\sqrt{P_1}\frac{\mathbf{a}}{\|\mathbf{a}\|}$ as optimal signals, and the decision rule

becomes (4.18). By combining the best responses of the transmitter and the receiver,

$$
\begin{aligned}
&\mathbf{a}^T = \zeta(\mathbf{S}_1 - \mathbf{S}_0)^T\Sigma^{-1} = \zeta\frac{\mathbf{a}^T}{\|\mathbf{a}\|}\left(\mathrm{sgn}(C_{01}^t - C_{11}^t)\sqrt{P_1} + \mathrm{sgn}(C_{10}^t - C_{00}^t)\sqrt{P_0}\right)\Sigma^{-1}\\
\Rightarrow &\mathbf{a}^T\mathbf{a} = \frac{\mathbf{a}^T\Sigma^{-1}\mathbf{a}}{\|\mathbf{a}\|}\zeta\left(\mathrm{sgn}(C_{01}^t - C_{11}^t)\sqrt{P_1} + \mathrm{sgn}(C_{10}^t - C_{00}^t)\sqrt{P_0}\right)\\
\Rightarrow &\underbrace{\frac{\mathrm{sgn}(C_{01}^t - C_{11}^t)}{\mathrm{sgn}(C_{01}^r - C_{11}^r)}}_{=\mathrm{sgn}(\xi_1)}\sqrt{P_1} + \underbrace{\frac{\mathrm{sgn}(C_{10}^t - C_{00}^t)}{\mathrm{sgn}(C_{10}^r - C_{00}^r)}}_{=\mathrm{sgn}(\xi_0)}\sqrt{P_0} > 0 \qquad (4.22)
\end{aligned}
$$

is obtained. Notice that the expressions in (4.22) and (4.17) of Theorem 4.4.1 are the same; hence, the Nash equilibrium solution of Theorem 4.4.1 also holds for the vector case.

4.7.4.4 Optimal Signal Selection to Maximize $d^2 = (\mathbf{S}_1 - \mathbf{S}_0)^T\Sigma^{-1}(\mathbf{S}_1 - \mathbf{S}_0)$

The optimal signal selection method here is based on [72, pp. 61–63]. The positive definite covariance matrix of the noise can be decomposed as $\Sigma = \sum_{k=1}^{n}\lambda_k\boldsymbol{\nu}_k\boldsymbol{\nu}_k^T$, where $\lambda_1, \ldots, \lambda_n$ and $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_n$ are the eigenvalues and the corresponding orthonormal eigenvectors of Σ. Let $\lambda_{\min} \triangleq \min\{\lambda_1, \ldots, \lambda_n\}$; then

$$
\begin{aligned}
(\mathbf{S}_1 - \mathbf{S}_0)^T\Sigma^{-1}(\mathbf{S}_1 - \mathbf{S}_0) &= (\mathbf{S}_1 - \mathbf{S}_0)^T\sum_{k=1}^{n}\lambda_k^{-1}\boldsymbol{\nu}_k\boldsymbol{\nu}_k^T(\mathbf{S}_1 - \mathbf{S}_0)\\
&\le \lambda_{\min}^{-1}(\mathbf{S}_1 - \mathbf{S}_0)^T\sum_{k=1}^{n}\boldsymbol{\nu}_k\boldsymbol{\nu}_k^T(\mathbf{S}_1 - \mathbf{S}_0)\\
&= \lambda_{\min}^{-1}(\mathbf{S}_1 - \mathbf{S}_0)^T(\mathbf{S}_1 - \mathbf{S}_0)\\
&= \lambda_{\min}^{-1}\|\mathbf{S}_1 - \mathbf{S}_0\|^2
\end{aligned}
$$

Here, the equality is satisfied if and only if $(\mathbf{S}_1 - \mathbf{S}_0)$ is chosen along an eigenvector corresponding to $\lambda_{\min}$. Since the eigenvector with the largest (smallest) eigenvalue of Σ corresponds to the direction, along which the noise is most (least) powerful, signaling in the least noisy direction results in the highest signal-to-noise power ratio for the system. Thus we have $d^2 = (\mathbf{S}_1 - \mathbf{S}_0)^T\Sigma^{-1}(\mathbf{S}_1 - \mathbf{S}_0) = \frac{\|\mathbf{S}_1 - \mathbf{S}_0\|^2}{\lambda_{\min}}$. In order to maximize d^2, $\mathbf{S}_0$ and $\mathbf{S}_1$ must be chosen in the opposite directions; i.e.,

$\mathbf{S}_1 = -\alpha \mathbf{S}_0$ with $\alpha > 0$. Further, $(\mathbf{S}_1 - \mathbf{S}_0)$ must be chosen along $\boldsymbol{\nu}_{\min}$ (that is, the unit-norm eigenvector corresponding to $\lambda_{\min}$), thus $(\mathbf{S}_1 - \mathbf{S}_0) = -(\alpha+1)\mathbf{S}_0 = c\boldsymbol{\nu}_{\min}$ for $c \in \mathbb{R}$, which implies $\mathbf{S}_0 = -\frac{c}{\alpha+1}\boldsymbol{\nu}_{\min}$ and $\mathbf{S}_1 = \frac{c\alpha}{\alpha+1}\boldsymbol{\nu}_{\min}$. Since $d^2 = \frac{\|\mathbf{S}_1 - \mathbf{S}_0\|^2}{\lambda_{\min}} = \frac{\|c\boldsymbol{\nu}_{\min}\|^2}{\lambda_{\min}} = \frac{c^2}{\lambda_{\min}}$, c^2 must be chosen as maximum as possible to maximize d^2. Accordingly, $\|\mathbf{S}_0\|^2 \leq P_0 \Rightarrow c^2 \leq (\alpha+1)^2 P_0$ and $\|\mathbf{S}_1\|^2 \leq P_1 \Rightarrow c^2 \leq \frac{(\alpha+1)^2}{\alpha^2} P_1$ are obtained. Then,

$$
\begin{aligned}
(\alpha+1)^2 P_0 \leq \frac{(\alpha+1)^2}{\alpha^2} P_1 \Rightarrow \alpha^2 \leq \frac{P_1}{P_0} \text{ and } c^2 = (\alpha+1)^2 P_0 \\
\Rightarrow \alpha = \sqrt{\frac{P_1}{P_0}} \text{ to maximize } c^2, \text{ and } c^2 = (\sqrt{P_0} + \sqrt{P_1})^2\,, \\
\frac{(\alpha+1)^2}{\alpha^2} P_1 \leq (\alpha+1)^2 P_0 \Rightarrow \alpha^2 \geq \frac{P_1}{P_0} \text{ and } c^2 = \left(1 + \frac{1}{\alpha}\right)^2 P_1 \\
\Rightarrow \alpha = \sqrt{\frac{P_1}{P_0}} \text{ to maximize } c^2, \text{ and } c^2 = (\sqrt{P_0} + \sqrt{P_1})^2\,.
\end{aligned}
$$

Thus, $\alpha = \sqrt{\frac{P_1}{P_0}}$ and $c = \mp(\sqrt{P_0} + \sqrt{P_1})$, optimum signals are $\mathbf{S}_0 = \pm\sqrt{P_0}\boldsymbol{\nu}_{\min}$ and $\mathbf{S}_1 = \mp\sqrt{P_1}\boldsymbol{\nu}_{\min}$, and the corresponding $d^2_{\max} = \frac{(\sqrt{P_0}+\sqrt{P_1})^2}{\lambda_{\min}}$.

4.7.4.5 Optimal Signal Selection to Achieve $(d^*)^2 = (\mathbf{S}_1 - \mathbf{S}_0)^T \Sigma^{-1} (\mathbf{S}_1 - \mathbf{S}_0) = \left|\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right|$

Since $d_{\max}$ is achieved when the signals are chosen in the direction of the eigenvector with the smallest eigenvalue of Σ, that is $\boldsymbol{\nu}_{\min}$, it possible to find a signal pair $\{\mathbf{S}_0, \mathbf{S}_1\}$ with distance $d^* = \sqrt{\left|\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right|} < d_{\max}$ in that direction. Accordingly, $\mathbf{S}_0 = (-\sqrt{P_0} + t)\boldsymbol{\nu}_{\min}$ and $\mathbf{S}_1 = (-\sqrt{P_0} + d^* + t)\boldsymbol{\nu}_{\min}$ for $t \in [0, \sqrt{P_1} + \sqrt{P_0} - d^*]$ are possible optimal signal pairs. Similarly, $\mathbf{S}_0 = (\sqrt{P_0} - t)\boldsymbol{\nu}_{\min}$ and $\mathbf{S}_1 = (\sqrt{P_0} - d^* - t)\boldsymbol{\nu}_{\min}$ for $t \in [0, \sqrt{P_1} + \sqrt{P_0} - d^*]$ consist of another set of possible optimal signal pairs. Note that it may be possible to find optimal signal pairs $\{\mathbf{S}_0, \mathbf{S}_1\} \in \mathbb{S}$ that satisfy $(\mathbf{S}_1 - \mathbf{S}_0)^T \Sigma^{-1} (\mathbf{S}_1 - \mathbf{S}_0) = \left|\frac{2\ln\tau(k_0-k_1)}{(k_0+k_1)}\right|$ in any other direction rather than the direction of the eigenvector with the smallest eigenvalue of Σ, that is $\boldsymbol{\nu}_{\min}$; however, finding a single pair that corresponds to an equilibrium would be sufficient.

Chapter 5

On the Number of Equilibria in Static Cheap Talk

In this chapter, Crawford and Sobel's cheap talk formulation [16] is investigated for the exponential and Gaussian sources, after verifying the special case with uniform source considered therein. The upper bounds on the number of the quantization bins (if any) are derived depending on the bias b.

The main contributions of this chapter can be summarized as follows:

(i) Under the uniform source assumption, we verify the upper bound on the number of the quantization bins as a function of the bias b, obtain the total cost at the equilibrium, and show that the equilibrium with more bins is preferable for both the encoder and the decoder.

(ii) Under the exponential source assumption, we obtain an upper bound on the number of bins at the equilibrium for a negative bias; i.e., $b < 0$; on the other hand, when the bias is positive; i.e., $b > 0$, we prove that there is no upper bound on the number of bins, actually, it is possible to have equilibria with infinitely many bins.

(iii) Under the Gaussian source assumption, we show that there always exists an equilibrium with two bins.

(iv) The equilibrium with two bins is investigated for the double-exponential and half-normal sources.

5.1 Preliminary Results

Consider an equilibrium with N bins, and let the k-th bin be the interval $[\overline{m}_{k-1}, \overline{m}_k)$ with $\overline{m}_0 < \overline{m}_1 < \ldots < \overline{m}_N$ for $k = 1, 2, \ldots, N$. By Theorem 2.2.3, at the equilibrium, decoder's best response is characterized by

$$u_k = \mathbb{E}[m | \overline{m}_{k-1} \leq m < \overline{m}_k] \tag{5.1}$$

for the k-th bin. As it can be seen, the optimal decoder action is the centroid for the corresponding bin. From the encoder's point of view, the best response of the encoder is determined by the nearest neighbor condition similar to (2.8) as follows:

$$m_k = \frac{u_k + u_{k+1}}{2} + b\,. \tag{5.2}$$

These best responses in (5.1) and (5.2) characterize the equilibrium; i.e., for a given number of bins, the positions of the bin edges and the centroids can be determined.

Note that, by Theorem 2.2.3 and Theorem 2.2.4, we know that the distance between the optimal decoder actions can be at least $2|b|$, this is the reason why the equilibrium must be quantized. Hence, for bounded sources, it can be easily deduced that the number of bins at the equilibrium must be bounded. At this point, for unbounded sources, the questions to be asked are the following:

- For unbounded sources, either one-sided or two-sided, is there any upper bound on the number of bins at the equilibrium as a function of bias b? As a special case, is it possible to have only a non-informative equilibrium?
- Is it possible to have an equilibrium with infinitely many bins?

At this point, one can ask why the number of bins is important: This is useful since if one can show that there only exists a finite number of bins, and if for every bin there is a finite, or a unique, number of distinct equilibria, then the total number of equlibria will be finite; this will allow for a feasible setting where the decision makers can coordinate their policies. Furthermore, it is a desirable question to see whether in general more number of bins implies more desirable equilibria. If such a monotone behavior holds for a class of sources, then both the encoder and the decoder would prefer to have the equilibrium with the most number of bins.

In this chapter, we try to answer these questions for exponential and Gaussian sources, and then based on the answers, we analyze double-exponential and half-normal sources. But first, in order to be familiar to the concepts, we focus on the uniform source between $[0,1]$.

5.2 Uniform Source

The theorems below are valid for $M \sim U\,[0,1]$.

Theorem 5.2.1. *[16] In order to achieve an equilibrium in the case of two levels of quantization, $|b|$ must be limited by $\frac{1}{4}$. Otherwise there cannot be more than one quantization levels, which is completely uninformative equilibrium.*

Theorem 5.2.2. *(Equation (22) of [16]) The relation between the number of bins N and bias b in the scalar case at the equilibrium can be characterized by $|b| < \frac{1}{2N(N-1)}$.*

Theorem 5.2.3. *(Equation (25) of [16]) The total game cost at the equilibrium with any number of bins N is given by the following:*

$$J^e(\gamma^{*,e},\gamma^{*,d})+J^d(\gamma^{*,e},\gamma^{*,d}) = \left(\frac{1}{12N^2}+\frac{b^2(N^2-1)}{3}+b^2\right)+\left(\frac{1}{12N^2}+\frac{b^2(N^2-1)}{3}\right)$$

Theorem 5.2.4. *(Theorem 3 of [16]) The most informative equilibrium is reached with the maximum possible number of bins; i.e., if there are two different equilibria* *with K and N bins for a constant b where $N > K$, the equilibrium with N bins is more informative.*

5.3 Exponential Source

In this section, the source is assumed to be exponential and the number of bins at the equilibria is investigated. Before delving into the technical results on the equilibria, observe the following fact:

Fact 5.3.1. *Let M be an exponentially distributed r.v. with a positive parameter λ: i.e., the probability distribution function of M is $\mathsf{f}(m) = \lambda e^{-\lambda m}$ for $m \geq 0$. The expectation and the variance of the truncated exponential r.v. are $\mathbb{E}[m|a < m < b] = \frac{1}{\lambda} + a - \frac{b-a}{e^{\lambda(b-a)}-1}$ and $\mathrm{Var}\,(m|a < m < b) = \frac{1}{\lambda^2} - \frac{(b-a)^2}{e^{\lambda(b-a)}+e^{-\lambda(b-a)}-2}$, respectively.*

The following result is taken from [37], and it shows the existence of an equilibrium with finitely many bins:

Proposition 5.3.1. *[37] Suppose M is exponentially distributed with parameter λ. Then, for $b < 0$, any Nash equilibrium is deterministic and can have at most $\lfloor -\frac{1}{2b\lambda} + 1 \rfloor$ bins with monotonically increasing bin-lengths.*

The result above does not characterize the equilibrium completely. The following theorem characterizes the equilibrium with two bins, and forms a basis for equilibria with more bins:

Theorem 5.3.1. *When the source has an exponential distribution with a positive parameter λ, there exist only non-informative equilibria if and only if $b \leq -\frac{1}{2\lambda}$. The equilibrium with at least two bins is achievable if and only if $b > -\frac{1}{2\lambda}$.*

Contrarily to the negative bias case, the number of bins at the equilibrium is not bounded when the bias is positive. The following investigates the case in which $b > 0$ and is extended from [37]:

Theorem 5.3.2. *When the source has an exponential distribution with parameter λ, for $b > 0$ and any number of bins N,*

(i) There exists a unique equilibrium,

(ii) Bin-lengths are monotonically increasing.

Further, since the two statements above hold for any $N \in \mathbb{N}$, there exists no upper bound on the number of bins at an equilibrium.

By following a similar approach to that in Theorem 5.3.2, the bounds in Proposition 5.3.1 can be refined as follows:

Corollary 5.3.1. *(i) There exists an equilibrium with at least two bins if and only if $b > -\frac{1}{2\lambda}$.*

(ii) There exists an equilibrium with at least three bins if and only if $b > -\frac{1}{2\lambda}\frac{e-2}{e-1}$.

Proof. (i) In order to have an equilibrium with at least 2 bins, $l_{N-1} > 0$ must be satisfied. From (5.23), if $-(2+2\lambda b) < -1$ is satisfied, then the solution l_{N-1} will be positive. Thus, if $b > -\frac{1}{2\lambda}$, it is possible to obtain an equilibrium with 2 bins; otherwise; i.e., $b \leq -\frac{1}{2\lambda}$, there exists only one bin at the equilibrium.

(ii) In order to have an equilibrium with at least 3 bins, $l_{N-2} > 0$ must be satisfied. From (5.25), if $-\lambda c_{N-2} < -1$ is satisfied, then the solution l_{N-2} will be positive. Then,

$$-\lambda c_{N-2} = -\lambda\left(\frac{2}{\lambda} + 2b - h(l_{N-1})\right) = -\lambda\left(g(l_{N-1}) - h(l_{N-1})\right) = -\lambda l_{N-1} < -1$$
$$\Rightarrow l_{N-1} = \frac{1}{\lambda}W_0\left(-(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}\right) + 2\left(\frac{1}{\lambda} + b\right) > \frac{1}{\lambda}$$
$$\Rightarrow W_0\left(-(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}\right) > -1 - 2\lambda b\,. \tag{5.3}$$

Let $t \triangleq W_0\left(-(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}\right)$, then $t\mathrm{e}^t = -(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}$ and $-1 < t < 0$. Then, from (5.3), since $t\mathrm{e}^t$ is increasing function of t for $t > -1$,

$$t > -1 - 2\lambda b \Rightarrow t\mathrm{e}^t = -(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)} > -(1+2\lambda b)\mathrm{e}^{-(1+2\lambda b)}$$

$$\Rightarrow 2 + 2\lambda b < (1 + 2\lambda b)\mathrm{e} \Rightarrow b > -\frac{1}{2\lambda}\frac{\mathrm{e}-2}{\mathrm{e}-1}. \tag{5.4}$$

Thus, if $b > -\frac{1}{2\lambda}\frac{\mathrm{e}-2}{\mathrm{e}-1}$, it is possible to obtain an equilibrium with three bins; otherwise; i.e., $b \leq -\frac{1}{2\lambda}\frac{\mathrm{e}-2}{\mathrm{e}-1}$, there can exist at most two bins at the equilibrium.

□

Theorem 5.3.2 shows that, when $b > 0$, it possible to have an equilibrium with N bins for any finite N. The following elucidates the existence of equilibria with infinitely many bins:

Theorem 5.3.3. *For the exponential source, assuming a positive bias; i.e. $b > 0$, there exist equilibria with infinitely many bins.*

Heretofore, we show that, at the equilibrium, there is an upper bound on the number of bins when $b < 0$; i.e., there can exist finitely many equilibria with finitely many bins. On the other hand, when $b > 0$, there is no upper bound on the number of bins at the equilibrium, and even it is possible to have equilibria with infinitely many bins. Therefore, at this point, which equilibrium is preferred by the decision makers must be investigated; i.e., which equilibrium is more informative.

Theorem 5.3.4. *The most informative equilibrium is reached with the maximum possible number of bins:*

(i) for $b < 0$, if there are two different equilibria with K and N bins where $N > K$, the equilibrium with N bins is more informative.

(i) for $b > 0$, the equilibrium with infinitely many bins is the most informative one.

5.4 Gaussian Source

Let M be a Gaussian random variable with the mean μ and variance σ^2; i.e., $M \sim \mathcal{N}(\mu, \sigma^2)$. Let the probability distribution function of a standard Gaussian r.v. be $\phi(m) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{m^2}{2}}$ for $m \geq 0$, and let the cumulative distribution function (CDF) of a standard Gaussian r.v. be $\Phi(b) = \int_{-\infty}^{b} \phi(m)\mathrm{d}m$. Then, the expectation of a truncated Gaussian r.v. is the following:

Fact 5.4.1. *The mean of the truncated Gaussian random variable $M \sim \mathcal{N}(\mu, \sigma^2)$ is $\mathbb{E}[m|a < m < b] = \mu - \sigma \frac{\phi(\frac{b-\mu}{\sigma}) - \phi(\frac{a-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}$.*

Proof.

$$
\begin{aligned}
\mathbb{E}[m|a < m < b] &= \int_a^b m \frac{\frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(m-\mu)^2}{2\sigma^2}}}{\int_a^b \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(m-\mu)^2}{2\sigma^2}}} \mathrm{d}m = \frac{\int_a^b m \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(m-\mu)^2}{2\sigma^2}} \mathrm{d}m}{\int_a^b \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(m-\mu)^2}{2\sigma^2}} \mathrm{d}m} \\
&\overset{\substack{s=(m-\mu)/\sigma \\ \mathrm{d}s=\mathrm{d}m/\sigma}}{=} \frac{\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} (\sigma s + \mu) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{s^2}{2}} \mathrm{d}s}{\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{s^2}{2}} \mathrm{d}s} = \mu + \sigma \frac{\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} s \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{s^2}{2}} \mathrm{d}s}{\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{s^2}{2}} \mathrm{d}s} \\
&\overset{\substack{u=s^2/2 \\ \mathrm{d}u=s\mathrm{d}s}}{=} \mu + \sigma \frac{\int_{(a-\mu)^2/(2\sigma^2)}^{(b-\mu)^2/(2\sigma^2)} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-u} \mathrm{d}u}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} = \mu + \sigma \frac{-\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-u} \Big|_{u=(a-\mu)^2/(2\sigma^2)}^{u=(b-\mu)^2/(2\sigma^2)}}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} \\
&= \mu + \sigma \frac{-\phi(\frac{b-\mu}{\sigma}) + \phi(\frac{a-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} \overset{\substack{\alpha \triangleq (a-\mu)/\sigma \\ \beta \triangleq (b-\mu)/\sigma}}{=} \mu - \sigma \frac{\phi(\beta) - \phi(\alpha)}{\Phi(\beta) - \Phi(\alpha)}. \quad (5.5)
\end{aligned}
$$

□

Now, as a preliminary result, consider an equilibrium with two bins as follows:

Theorem 5.4.1. *When the source has a Gaussian distribution as $M \sim \mathcal{N}(\mu, \sigma^2)$, there always exists an equilibrium with two bins regardless of the value of b.*

Since the pdf of a Gaussian r.v. is symmetrical around its mean, and it satisfies the monotonicity properties, the following can be observed due to the similar reasoning in Proposition 5.3.1:

Proposition 5.4.1. *Let there be an equilibrium with N bins for a Gaussian source $M \sim \mathcal{N}(\mu, \sigma^2)$. Then,*

(i) if $b < 0$, bin-lengths are monotonically increasing for $m > \mu$,

(ii) if $b > 0$, bin-lengths are monotonically decreasing for $m < \mu$.

5.5 Other Distributions

Since we have equilibria results for the exponential and Gaussian sources, we will derive the similar results for the double-exponential and half-normal sources.

5.5.1 The Standard Double-Exponential Distribution

Let X be an exponential random variable with a positive parameter λ; i.e., $\mathsf{f}(x) = \lambda \mathrm{e}^{-\lambda x}$. Now consider another random variable M, with pdf which consists of the pdf of X and the pdf of $-X$. Then, M is a Laplacian random variable with location parameter 0 and the scale parameter $\frac{1}{\lambda}$. In other words, M is a standard double-exponential r.v. with pdf $\mathsf{f}(m) = \frac{\lambda}{2}\mathrm{e}^{-\lambda|m|}$ for $m \in (-\infty, \infty)$. Before investigating the equilibrium for the double-exponential source, observe the following fact:

Fact 5.5.1. *The mean of the truncated standard double-exponential distribution with parameter λ is the following:*

$$\mathbb{E}[m|a < m < b] = \begin{cases} \frac{1}{\lambda} + a - \frac{b-a}{\mathrm{e}^{\lambda(b-a)}-1} & 0 < a < b \\ \frac{\frac{1}{\lambda}\left(\mathrm{e}^{\lambda a} - \mathrm{e}^{-\lambda b}\right) - \left(a\mathrm{e}^{\lambda a} + b\mathrm{e}^{-\lambda b}\right)}{2 - \left(\mathrm{e}^{\lambda a} + \mathrm{e}^{-\lambda b}\right)} & a < 0 < b \\ -\frac{1}{\lambda} + b + \frac{b-a}{\mathrm{e}^{\lambda(b-a)}-1} & a < b < 0 \end{cases} \tag{5.6}$$

Now, as a preliminary result, consider an equilibrium with two bins as follows:

Theorem 5.5.1. *When the source has a standard double-exponential distribution, there always exists an equilibrium with two bins for any b, and*

(i) the boundary between the bins lies on the positive side of the real line if $b > 0$,

(ii) the boundary between the bins lies on the negative side of the real line if $b < 0$.

Similar to the Gaussian source case, which also has two-sided infinite support, it is always possible to have an equilibrium with two bins.

5.5.2 Half-Normal Distribution

Let X be a Gaussian random variable with zero mean and variance σ^2; i.e., $X \sim \mathcal{N}(0, \sigma^2)$. Now consider another random variable M, which is a fold of X at zero; i.e., $M = |X|$. Then,

$$\mathsf{Pr}(\mathsf{M} \leq \mathsf{m}) = \mathsf{Pr}(|\mathsf{X}| \leq \mathsf{m}) = \int_{-m}^{m} \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{m^2}{2\sigma^2}} \mathrm{d}m = 2\int_{0}^{m} \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{m^2}{2\sigma^2}} \mathrm{d}m$$
$$\Rightarrow \mathsf{Pr}(\mathsf{M} \leq \mathsf{m}) = \int_{0}^{m} \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \mathrm{e}^{-\frac{m^2}{2\sigma^2}} \mathrm{d}m\,, \tag{5.7}$$

which implies that the pdf of the half-normal distribution is $f(m) = \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \mathrm{e}^{-\frac{m^2}{2\sigma^2}}$ for $m \geq 0$. Now, consider the mean of a truncated half-normal distribution:

Fact 5.5.2. *The mean of the truncated half-normal distribution with parameter σ^2 is $\mathbb{E}[m|a < m < b] = -\sigma \frac{\phi(\frac{b}{\sigma}) - \phi(\frac{a}{\sigma})}{\Phi(\frac{b}{\sigma}) - \Phi(\frac{a}{\sigma})}$.*

Proof.

$$\mathbb{E}[m|a < m < b] = \int_{a}^{b} m \frac{\frac{\sqrt{2}}{\sqrt{\pi}\sigma} \mathrm{e}^{-\frac{m^2}{2\sigma^2}}}{\int_{a}^{b} \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \mathrm{e}^{-\frac{m^2}{2\sigma^2}}} \mathrm{d}m = \frac{\int_{a}^{b} m \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \mathrm{e}^{-\frac{m^2}{2\sigma^2}} \mathrm{d}m}{\int_{a}^{b} \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \mathrm{e}^{-\frac{m^2}{2\sigma^2}} \mathrm{d}m}$$

$$\overset{\substack{s=m/\sigma \\ \mathrm{d}s=\mathrm{d}m/\sigma}}{=} \frac{\int_{a/\sigma}^{b/\sigma} \sigma s \frac{\sqrt{2}}{\sqrt{\pi}} \mathrm{e}^{-\frac{s^2}{2}} \mathrm{d}s}{\int_{a/\sigma}^{b/\sigma} \frac{\sqrt{2}}{\sqrt{\pi}} \mathrm{e}^{-\frac{s^2}{2}} \mathrm{d}s} = \sigma \frac{\int_{a/\sigma}^{b/\sigma} s \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{s^2}{2}} \mathrm{d}s}{\int_{a/\sigma}^{b/\sigma} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{s^2}{2}} \mathrm{d}s}$$

$$\overset{\substack{u=s^2/2 \\ \mathrm{d}u=s\mathrm{d}s}}{=} +\sigma \frac{\int_{a^2/(2\sigma^2)}^{b^2/(2\sigma^2)} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-u} \mathrm{d}u}{\Phi(\frac{b}{\sigma}) - \Phi(\frac{a}{\sigma})} = \sigma \frac{-\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-u} \Big|_{u=a^2/(2\sigma^2)}^{u=b^2/(2\sigma^2)}}{\Phi(\frac{b}{\sigma}) - \Phi(\frac{a}{\sigma})}$$

$$= \sigma \frac{-\phi(\frac{b}{\sigma}) + \phi(\frac{a}{\sigma})}{\Phi(\frac{b}{\sigma}) - \Phi(\frac{a}{\sigma})} \overset{\substack{\alpha \triangleq a/\sigma \\ \beta \triangleq b/\sigma}}{=} -\sigma \frac{\phi(\beta) - \phi(\alpha)}{\Phi(\beta) - \Phi(\alpha)} . \quad (5.8)$$

Note that, for $a = 0$ and $b = \infty$, the mean of the half-normal distribution is obtained as $\mathbb{E}[M] = -\sigma \frac{\phi(\infty) - \phi(0)}{\Phi(\infty) - \Phi(0)} = -\sigma \frac{0 - \frac{1}{\sqrt{2\pi}}}{1 - \frac{1}{2}} = \sigma \sqrt{\frac{2}{\pi}}$. $\square$

Now, as a preliminary result, consider an equilibrium with two bins as follows:

Theorem 5.5.2. *When the source has a half-normal distribution, there always exist an equilibrium with two bins if* $b \geq -\sigma\sqrt{\frac{1}{2\pi}}$*; otherwise, i.e.,* $b < -\sigma\sqrt{\frac{1}{2\pi}}$*, the equilibrium is non-informative; i.e., the equilibrium with one bin.*

Similar to the exponential source case, which also has one-sided infinite support, the existence of an equilibrium with two bins is dependent on b.

5.6 Conclusion

In this chapter, we investigated the upper bounds on the number of bins at the equilibria under different source assumptions. Since cheap talk has always quantized equilibria with at least $2|b|$ separation between the centroids, the bounded sources have always finite number of bins. However, this may not be the case for the sources with one-sided or two-sided infinite support. In this direction, the exponential and Gaussian sources are analyzed, and it is shown that, for an exponential source, there can exist equilibria with infinitely many bins when $b < 0$ whereas the number of bins is bounded when $b < 0$. For a Gaussian source, it is always possible to have an equilibrium with two quantization bins.

5.7 Proofs

5.7.1 Proofs for Section 5.2

5.7.1.1 Proof of Theorem 5.2.1

Suppose that there are only two bins $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$. Recall that, at the equilibrium, from the view of the encoder,

$$m^\alpha \in \mathcal{B}^\alpha \Rightarrow (m^\alpha - u^\alpha - b)^2 < (m^\alpha - u^\beta - b)^2\,,$$

$$m^\beta \in \mathcal{B}^\beta \Rightarrow (m^\beta - u^\beta - b)^2 < (m^\beta - u^\alpha - b)^2\,,$$

where $u^\alpha = \mathbb{E}[m|m \in \mathcal{B}^\alpha]$ and $u^\beta = \mathbb{E}[m|m \in \mathcal{B}^\beta]$ (since the also aims to minimize his cost function $\mathbb{E}[(m-u)^2|X=x]$). Due to the continuity of $(m-u-b)^2$ in m, $\exists\, \overline{m}$ such that

$$u^\alpha + b - \overline{m} = \overline{m} - u^\beta - b \Rightarrow \overline{m} = \frac{u^\alpha + u^\beta}{2} + b\,. \tag{5.9}$$

Thus, two bins can be characterized as $\mathcal{B}^\alpha = [0, \overline{m}]$ and $\mathcal{B}^\beta = [\overline{m}, 1]$. Then,

$$\begin{aligned} u^\alpha &= \mathbb{E}[m|m \in \mathcal{B}^\alpha] = \mathbb{E}[m|m \in [0, \overline{m}]] = \frac{\overline{m}}{2}\,,\\ u^\beta &= \mathbb{E}[m|m \in \mathcal{B}^\beta] = \mathbb{E}[m|m \in [\overline{m}, 1]] = \frac{\overline{m}+1}{2}\,. \end{aligned} \tag{5.10}$$

By utilizing (5.10) in (5.9), we can obtain

$$\overline{m} = \frac{\frac{\overline{m}}{2} + \frac{\overline{m}+1}{2}}{2} + b \Rightarrow \overline{m} = \frac{1}{2} + 2b \in [0,1] \Rightarrow |b| \leq \frac{1}{4} \tag{5.11}$$

Thus, it can be observed that b has limiting boundary in binary equilibrium case.

5.7.1.2 Proof of Theorem 5.2.2

Suppose that there are N bins at the equilibrium and kth bin is $[\overline{m}_{k-1}, \overline{m}_k]$, where $\overline{m}_0 = 0$ and $\overline{m}_N = 1$ for $k = 1, 2, \ldots, N$. At the equilibrium, we have $\overline{m}_k = \frac{u_k + u_{k+1}}{2} + b$ (the nearest neighbor condition, follows from the best response of the encoder) and $u_k = \frac{\overline{m}_{k-1}+\overline{m}_k}{2}$ (the centroid condition, follows from the best response of the decoder). Let the length of the kth bin be defined as $l_k \triangleq \overline{m}_k - \overline{m}_{k-1}$. Then, by combining the nearest neighbor condition and the best response condition, the difference equation for bin-lengths can be found as $2\overline{m}_k = \overline{m}_{k-1} + \overline{m}_{k+1} + 4b \Rightarrow l_{k+1} = l_k - 4b = l_1 - 4kb$. Since the sum of the lengths of the bins is equal to 1, $\sum_{i=1}^{N} l_i = N\overline{m}_1 - 2N(N-1)b = 1 \Rightarrow \overline{m}_1 = \frac{1+2N(N-1)b}{N}$. Since the equilibrium is informative with N bins, all bins must have positive lengths; i.e., $l_1 = \overline{m}_1 - \overline{m}_0 = \frac{1+2N(N-1)b}{N} > 0$ and $l_N = (\overline{m}_1 - \overline{m}_0) - 4(N-1)b = \frac{1-2N(N-1)b}{N} > 0$. By combining these inequalities, we obtain $|b| < \frac{1}{2N(N-1)}$, which is consistent with the results in [16].

5.7.1.3 Proof of Theorem 5.2.3

Suppose again that there are N bins at the equilibrium and kth bin is $B_k = [\overline{m}_{k-1}, \overline{m}_k]$ where $\overline{m}_0 = 0$ and $\overline{m}_N = 1$ for $k = 1, 2, \ldots, N$. Then total costs for encoder and decoder are:

$$J^e(\gamma^{*,e}, \gamma^{*,d}) = \sum_{i=1}^{N} \mathbb{E}[(m - u_i - b)^2 | m \in B_i] \Pr(m \in B_i),$$
$$J^d(\gamma^{*,e}, \gamma^{*,d}) = \sum_{i=1}^{N} \mathbb{E}[(m - u_i)^2 | m \in B_i] \Pr(m \in B_i)].$$

Since the centroids of the bins (optimal decoder actions) are $u_k = \mathbb{E}[m | m \in B_k] = \frac{\overline{m}_{k-1}+\overline{m}_k}{2}$, the costs of the encoder and the decoder can be obtained as follows:

$$\begin{aligned}
J^e(\gamma^{*,e}, \gamma^{*,d}) &= \sum_{i=1}^{N} \mathbb{E}[(m - u_i - b)^2 | m \in B_i] \Pr(m \in B_i) \\
&= \sum_{i=1}^{N} \left(\mathbb{E}[(m - u_i)^2 | m \in B_i] - 2b\mathbb{E}[(m - u_i) | m \in B_i] + b^2\right) \Pr(m \in B_i) \\
&= \sum_{i=1}^{N} \left(\mathbb{E}[(m - u_i)^2 | m \in B_i] - 2b\mathbb{E}[(m - \mathbb{E}[m | m \in B_i]) | m \in B_i] + b^2\right) \\
&\qquad \times \Pr(m \in B_i) \\
&= \sum_{i=1}^{N} \left(\mathbb{E}[(m - u^i)^2 | m \in B_i] + b^2\right) \Pr(m \in B_i) \quad \left(= J^d(\gamma^{*,e}, \gamma^{*,d}) + b^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \left(\int_{\overline{m}_{i-1}}^{\overline{m}_i} (m-u_i)^2 \underbrace{p(\mathrm{d}m)}_{\mathrm{d}m/(\overline{m}_i - \overline{m}_{i-1})} + b^2 \right) \underbrace{\Pr(m \in B_i)}_{\overline{m}_i - \overline{m}_{i-1}} \\
&= \sum_{i=1}^{N} \int_{\overline{m}_{i-1}}^{\overline{m}_i} (m-u_i)^2 \,\mathrm{d}m + b^2 \\
&= \sum_{i=1}^{N} \left[\frac{m^3}{3} - u_i m^2 + (u_i)^2 m \right] \Bigg|_{m=\overline{m}_{i-1}}^{m=\overline{m}_i} + b^2 \\
&= \sum_{i=1}^{N} \left(\frac{(\overline{m}_i)^3 - (\overline{m}_{i-1})^3}{3} - u_i((\overline{m}_i)^2 - (\overline{m}_{i-1})^2) + (u_i)^2(\overline{m}_i - \overline{m}_{i-1}) \right) \\
&\qquad + b^2 \\
&= \sum_{i=1}^{N} \Bigg(\frac{(\overline{m}_i)^3 - (\overline{m}_{i-1})^3}{3} - \frac{\overline{m}_i + \overline{m}_{i-1}}{2}((\overline{m}_i)^2 - (\overline{m}_{i-1})^2) \\
&\qquad + \frac{(\overline{m}_i + \overline{m}_{i-1})^2}{4}(\overline{m}_i - \overline{m}_{i-1}) \Bigg) + b^2 \\
&= \sum_{i=1}^{N} \left(\frac{(\overline{m}_i)^3 - (\overline{m}_{i-1})^3}{3} - \frac{(\overline{m}_i + \overline{m}_{i-1})^2(\overline{m}_i - \overline{m}_{i-1})}{4} \right) + b^2 \\
&= \sum_{i=1}^{N} \frac{(\overline{m}_i - \overline{m}_{i-1})^3}{12} + b^2 \\
&= \sum_{i=1}^{N} \frac{(l_i)^3}{12} + b^2 \\
&= \sum_{i=1}^{N} \frac{(l_1 - 4(i-1)b)^3}{12} + b^2 \qquad (5.12) \\
&= \sum_{i=1}^{N} \frac{(\frac{1+2N(N-1)b}{N} - 4(i-1)b)^3}{12} + b^2 \\
&= \frac{1}{12N^2} + \frac{(N^2-1)b^2}{3} + b^2 \,. \qquad (5.13)
\end{aligned}
$$

5.7.1.4 Proof of Theorem 5.2.4

Suppose there exist two equilibria with K bins and N bins with $N > K$. Then, $|b| < \frac{1}{2N(N-1)} < \frac{1}{2K(K-1)}$ holds by Theorem 5.2.2. Let the decoder cost be $J^{d,K}$

when there are K bins. Then,

$$\begin{aligned}
J^{d,K} - J^{d,N} &= \left(\frac{1}{12K^2} + \frac{b^2(K^2-1)}{3}\right) - \left(\frac{1}{12N^2} + \frac{b^2(N^2-1)}{3}\right) \\
&= (N^2 - K^2)\left(\frac{1}{12K^2N^2} - \frac{b^2}{3}\right) \\
&> (N^2 - K^2)\left(\frac{1}{12K^2N^2} - \frac{1}{12N^2(N-1)^2}\right) \\
&= (N^2 - K^2)\left(\frac{(N-1)^2 - K^2}{12K^2N^2(N-1)^2}\right) \\
&\geq 0\,.
\end{aligned}$$

Thus, as the number of bins increases for a constant b, both the encoder and the decoder cost decrease.

5.7.2 Proofs for Section 5.3

5.7.2.1 Proof of Fact 5.3.1

Consider the following integral:

$$\begin{aligned}
\int \lambda m \mathrm{e}^{-\lambda m}\mathrm{d}m &\overset{\substack{s=\lambda m \\ \mathrm{d}s=\lambda \mathrm{d}m}}{=} \int \frac{1}{\lambda} s\mathrm{e}^{-s}\mathrm{d}s \overset{\substack{u=s,\,\mathrm{d}v=\mathrm{e}^{-s}\mathrm{d}s/\lambda \\ \mathrm{d}u=\mathrm{d}s,\,v=-\mathrm{e}^{-s}/\lambda}}{=} \frac{-s\mathrm{e}^{-s}}{\lambda} - \int \frac{-\mathrm{e}^{-s}}{\lambda}\mathrm{d}s \\
&= \frac{-s\mathrm{e}^{-s}}{\lambda} - \frac{\mathrm{e}^{-s}}{\lambda} \overset{s=\lambda m}{=} -m\mathrm{e}^{-\lambda m} - \frac{\mathrm{e}^{-\lambda m}}{\lambda}\,. \qquad (5.14)
\end{aligned}$$

Then, the expectation of a truncated exponential r.v. will be

$$\begin{aligned}
\mathbb{E}[m|a<m<b] &= \int_a^b m\frac{\lambda \mathrm{e}^{-\lambda m}}{\int_a^b \lambda \mathrm{e}^{-\lambda m}}\mathrm{d}m = \frac{\int_a^b m\lambda \mathrm{e}^{-\lambda m}\mathrm{d}m}{\int_a^b \lambda \mathrm{e}^{-\lambda m}\mathrm{d}m} = \frac{\left(-m\mathrm{e}^{-\lambda m} - \frac{\mathrm{e}^{-\lambda m}}{\lambda}\right)\Big|_a^b}{-\mathrm{e}^{-\lambda m}\Big|_a^b} \\
&= \frac{-b\mathrm{e}^{-\lambda b} - \frac{\mathrm{e}^{-\lambda b}}{\lambda} + a\mathrm{e}^{-\lambda a} + \frac{\mathrm{e}^{-\lambda a}}{\lambda}}{-\mathrm{e}^{-\lambda b} + \mathrm{e}^{-\lambda a}} = \frac{1}{\lambda} + \frac{a\mathrm{e}^{\lambda b} - b\mathrm{e}^{\lambda a}}{\mathrm{e}^{\lambda b} - \mathrm{e}^{\lambda a}} \\
&= \frac{1}{\lambda} + a - \frac{\mathrm{e}^{\lambda a}(b-a)}{\mathrm{e}^{\lambda b} - \mathrm{e}^{\lambda a}} = \frac{1}{\lambda} + a - \frac{b-a}{\mathrm{e}^{\lambda(b-a)} - 1}\,. \qquad (5.15)
\end{aligned}$$

Now consider the following integral:

$$\int \lambda m^2 \mathrm{e}^{-\lambda m} \mathrm{d}m \overset{\substack{u=\lambda m^2,\, \mathrm{d}v=\mathrm{e}^{-\lambda m}\mathrm{d}m \\ \mathrm{d}u=2\lambda m \mathrm{d}m,\, v=-\mathrm{e}^{-\lambda m}/\lambda}}{=} \lambda m^2 \frac{-\mathrm{e}^{-\lambda m}}{\lambda} - \int \frac{-\mathrm{e}^{-\lambda m}}{\lambda} 2\lambda m \mathrm{d}m$$
$$= -m^2 \mathrm{e}^{-\lambda m} + \frac{2}{\lambda} \int \lambda m \mathrm{e}^{-\lambda m} \mathrm{d}m$$
$$\overset{(a)}{=} -m^2 \mathrm{e}^{-\lambda m} - \frac{2m\mathrm{e}^{-\lambda m}}{\lambda} - \frac{2\mathrm{e}^{-\lambda m}}{\lambda^2}, \tag{5.16}$$

where the integral in (5.14) is utilized in (a). Then,

$$\mathrm{Var}\left(m|a<m<b\right) = \mathbb{E}[m^2|a<m<b] - \left(\mathbb{E}[m|a<m<b]\right)^2$$
$$= \int_a^b m^2 \frac{\lambda \mathrm{e}^{-\lambda m}}{\int_a^b \lambda \mathrm{e}^{-\lambda m}} \mathrm{d}m - \left(\frac{-b\mathrm{e}^{-\lambda b} - \frac{\mathrm{e}^{-\lambda b}}{\lambda} + a\mathrm{e}^{-\lambda a} + \frac{\mathrm{e}^{-\lambda a}}{\lambda}}{-\mathrm{e}^{-\lambda b} + \mathrm{e}^{-\lambda a}} \right)^2$$
$$= \frac{-\frac{\mathrm{e}^{-\lambda b}}{\lambda^2}\left(\lambda^2 b^2 + 2\lambda b + 2\right) + \frac{\mathrm{e}^{-\lambda a}}{\lambda^2}\left(\lambda^2 a^2 + 2\lambda a + 2\right)}{-\mathrm{e}^{-\lambda b} + \mathrm{e}^{-\lambda a}}$$
$$- \frac{\left(-\frac{\mathrm{e}^{-\lambda b}}{\lambda}\left(\lambda b + 1\right) + \frac{\mathrm{e}^{-\lambda a}}{\lambda}\left(\lambda a + 1\right)\right)^2}{\left(-\mathrm{e}^{-\lambda b} + \mathrm{e}^{-\lambda a}\right)^2}$$
$$= \frac{\frac{\mathrm{e}^{-2\lambda b}}{\lambda^2}\left(\lambda^2 b^2 + 2\lambda b + 2\right) + \frac{\mathrm{e}^{-2\lambda a}}{\lambda^2}\left(\lambda^2 a^2 + 2\lambda a + 2\right)}{\left(-\mathrm{e}^{-\lambda b} + \mathrm{e}^{-\lambda a}\right)^2}$$
$$- \frac{\frac{\mathrm{e}^{-\lambda(a+b)}}{\lambda^2}\left(\lambda^2 a^2 + \lambda^2 b^2 + 2\lambda a + 2\lambda b + 4\right)}{\left(-\mathrm{e}^{-\lambda b} + \mathrm{e}^{-\lambda a}\right)^2}$$
$$- \frac{\frac{\mathrm{e}^{-2\lambda b}}{\lambda^2}\left(\lambda^2 b^2 + 2\lambda b + 1\right) + \frac{\mathrm{e}^{-2\lambda a}}{\lambda^2}\left(\lambda^2 a^2 + 2\lambda a + 1\right)}{\left(-\mathrm{e}^{-\lambda b} + \mathrm{e}^{-\lambda a}\right)^2}$$
$$+ \frac{\frac{\mathrm{e}^{-\lambda(a+b)}}{\lambda^2}\left(2\lambda^2 ab + 2\lambda a + 2\lambda b + 2\right)}{\left(-\mathrm{e}^{-\lambda b} + \mathrm{e}^{-\lambda a}\right)^2}$$
$$= \frac{\frac{\mathrm{e}^{-2\lambda b}}{\lambda^2} + \frac{\mathrm{e}^{-2\lambda a}}{\lambda^2} - \frac{\mathrm{e}^{-\lambda(a+b)}}{\lambda^2}\left(\lambda^2 a^2 + \lambda^2 b^2 + 2 - 2\lambda^2 ab\right)}{\left(-\mathrm{e}^{-\lambda b} + \mathrm{e}^{-\lambda a}\right)^2}$$
$$= \frac{\frac{\mathrm{e}^{-2\lambda b}}{\lambda^2} + \frac{\mathrm{e}^{-2\lambda a}}{\lambda^2} - \frac{2\mathrm{e}^{-\lambda(a+b)}}{\lambda^2} - \frac{\mathrm{e}^{-\lambda(a+b)}}{\lambda^2}\left(\lambda^2 (b-a)^2\right)}{\mathrm{e}^{-2\lambda b} + \mathrm{e}^{-2\lambda a} - 2\mathrm{e}^{-\lambda(a+b)}}$$
$$= \frac{1}{\lambda^2} - \frac{(b-a)^2}{\mathrm{e}^{-\lambda b + \lambda a} + \mathrm{e}^{-\lambda a + \lambda b} - 2}$$
$$= \frac{1}{\lambda^2} - \frac{(b-a)^2}{\mathrm{e}^{\lambda(b-a)} + \mathrm{e}^{-\lambda(b-a)} - 2}. \tag{5.17}$$

5.7.2.2 Proof of Proposition 5.3.1

Consider an equilibrium with N bins. The k-th bin is defined as the interval $[\overline{m}_{k-1}, \overline{m}_k)$, where $\overline{m}_0 = 0$ and $\overline{m}_N = +\infty$, and l_k denotes the length of the k-th bin; i.e., $l_k = \overline{m}_k - \overline{m}_{k-1}$. Due to the equilibrium definitions, the best responses of the encoder and the decoder must satisfy (5.1) and (5.2). Hence,

$$u_{k+1} - \overline{m}_k = (\overline{m}_k - u_k) - 2b \qquad \text{for } k = 1, 2, \ldots, N-1\,, \tag{5.18}$$

and

$$u_N = E[m|\overline{m}_{N-1} \leq m \leq \overline{m}_N = \infty] = \overline{m}_{N-1} + \frac{1}{\lambda}\,.$$

This leads us to

$$\begin{aligned}
\frac{1}{\lambda} = u_N - \overline{m}_{N-1} &= (\overline{m}_{N-1} - u_{N-1}) - 2b \\
&> (u_{N-1} - \overline{m}_{N-2}) - 2b \\
&= (\overline{m}_{N-2} - u_{N-2}) - 2(2b) \\
&\vdots \\
&> u_1 - \overline{m}_0 - (N-1)(2b) \\
&> -(N-1)(2b)\,.
\end{aligned}$$

Here, the inequalities follow from the fact that the exponential pdf is monotonically decreasing. Hence, for $b < 0$,

$$N < \frac{-1}{2b\lambda} + 1 \Rightarrow N \leq \left\lfloor \frac{-1}{2b\lambda} + 1 \right\rfloor\,.$$

Now, consider bin-lengths as follows:

$$\begin{aligned}
l_k = \overline{m}_k - \overline{m}_{k-1} &= (\overline{m}_k - u_k) + (u_k - \overline{m}_{k-1}) \geq (u_k - \overline{m}_{k-1}) + (u_k - \overline{m}_{k-1}) \\
&= (\overline{m}_{k-1} - u_{k-1} - 2b) + (\overline{m}_{k-1} - u_{k-1} - 2b) \\
&> (\overline{m}_{k-1} - u_{k-1} - 2b) + (u_{k-1} - \overline{m}_{k-2} - 2b) \\
&= (\overline{m}_{k-1} - u_{k-1}) + (u_{k-1} - \overline{m}_{k-2}) - 4b = \overline{m}_{k-1} - \overline{m}_{k-2} - 4b = l_{k-1} - 4b \\
&\Rightarrow l_k > l_{k-1}\,.
\end{aligned} \tag{5.19}$$

Thus, bin-lengths are monotonically increasing; i.e., $l_1 < l_2 < \ldots < l_{N-1} < l_N = \infty$.

5.7.2.3 Proof of Theorem 5.3.1

Let there be an equilibrium with two bins such that the first bin is $[0 = \overline{m}_0, \overline{m}_1)$ and the second bin is $[\overline{m}_1, \overline{m}_2 = \infty)$. The centroids of the bins (the action of the decoder) can be derived from (5.15) as $u_1 = \mathbb{E}[m|0 < m < \overline{m}_1] = \frac{1}{\lambda} - \frac{\overline{m}_1}{\mathrm{e}^{\lambda \overline{m}_1} - 1}$ and $u_2 = \mathbb{E}[m|\overline{m}_1 < m < \infty] = \frac{1}{\lambda} + \overline{m}_1$. Then, by utilizing (5.2),

$$\begin{aligned}\overline{m}_1 &= \frac{u_1 + u_2}{2} + b = \frac{\frac{1}{\lambda} - \frac{\overline{m}_1}{\mathrm{e}^{\lambda \overline{m}_1} - 1} + \frac{1}{\lambda} + \overline{m}_1}{2} + b \Rightarrow \frac{\overline{m}_1}{2} \frac{\mathrm{e}^{\lambda \overline{m}_1}}{\mathrm{e}^{\lambda \overline{m}_1} - 1} = \frac{1}{\lambda} + b \\ &\Rightarrow \mathrm{e}^{\lambda \overline{m}_1} \left(\frac{1}{\lambda} + b - \frac{\overline{m}_1}{2} \right) = \frac{1}{\lambda} + b\,. \end{aligned} \tag{5.20}$$

Note that, in (5.20), $\overline{m}_1 = 0$ is always a solution; however, in order to have an equilibrium with two bins, we need a non-zero solution to (5.20); i.e., $\overline{m}_1 > 0$. For this purpose, the Lambert W-function will be utilized. Although the Lambert W-function is defined for complex variables, we restrict our attention to the real-valued W-function. Then, the W-function is defined as

$$\begin{aligned} W(x\mathrm{e}^x) &= x \text{ for } x \geq 0\,, \\ W_0(x\mathrm{e}^x) &= x \text{ for } -1 \leq x < 0\,, \\ W_{-1}(x\mathrm{e}^x) &= x \text{ for } x \leq -1\,. \end{aligned}$$

As it can be seen, for $x \geq 0$, $W(x\mathrm{e}^x)$ is a well-defined single-valued function. However, for $x < 0$, $W(x\mathrm{e}^x)$ is doubly valued, such as $W(x\mathrm{e}^x) \in (-\frac{1}{\mathrm{e}}, 0)$ and there exists x_1 and x_2 that satisfy $x_1\mathrm{e}^{x_1} = x_2\mathrm{e}^{x_2}$ where $x_1 \in (-1, 0)$ and $x_2 \in (-\infty, -1)$. In order to distinguish these values, the principal branch of the Lambert W-function is defined to represent values greater than -1; e.g., $x_1 = W_0(x_1\mathrm{e}^{x_1}) = W_0(x_2\mathrm{e}^{x_2})$. Similarly, the lower branch of the Lambert W-function represent values smaller than -1; e.g, $x_2 = W_{-1}(x_1\mathrm{e}^{x_1}) = W_{-1}(x_2\mathrm{e}^{x_2})$. Further, for $x = -1$, two branches of the W-function coincide; i.e., $-1 = W_0(-\frac{1}{\mathrm{e}}) = W_{-1}(-\frac{1}{\mathrm{e}})$. Regarding the definition above, by letting $t \triangleq 2\lambda \left(\frac{\overline{m}_1}{2} - \frac{1}{\lambda} - b \right)$, the solution of (5.20) can be found as follows:

$$\mathrm{e}^{t+2+2\lambda b} \left(\frac{-t}{2\lambda} \right) = \frac{1}{\lambda} + b \Rightarrow t\mathrm{e}^t = -(2 + 2\lambda b)\mathrm{e}^{-(2+2\lambda b)} \Rightarrow t = W_0 \left(-(2 + 2\lambda b)\mathrm{e}^{-(2+2\lambda b)} \right)\,. \tag{5.21}$$

Note that, in (5.21), depending on the values of $-(2+2\lambda b)$, the following cases can be considered:

(i) $\underline{-(2+2\lambda b) \geq 0}$: $te^t = -(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)} \Rightarrow t = -(2+2\lambda b) \Rightarrow \overline{m}_1 = 0$, which implies a non-informative equilibrium; i.e., the equilibrium with one bin.

(ii) $\underline{-1 < -(2+2\lambda b) < 0}$: Since $te^t = -(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}$, there are two possible solutions:

  (a) If $t = W_0\left(-(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}\right) = -(2+2\lambda b)$, we have $\overline{m}_1 = 0$ as in the previous case.

  (b) If $t = W_{-1}\left(-(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}\right) \Rightarrow t < -1 \Rightarrow -1 > t = 2\lambda\left(\frac{\overline{m}_1}{2} - \frac{1}{\lambda} - b\right) = \lambda\overline{m}_1 - 2 - 2\lambda b > \lambda\overline{m}_1 - 1 \Rightarrow \lambda\overline{m}_1 < 0$, which is not possible.

(iii) $\underline{-(2+2\lambda b) = -1}$: Since $te^t = -(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}$, there is only one solution, $t = -(2+2\lambda b) = -1 \Rightarrow \overline{m}_1 = 0$, which implies the non-informative equilibrium.

(iv) $\underline{-(2+2\lambda b) < -1}$: Since $te^t = -(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}$, there are two possible solutions:

  (a) If $t = W_{-1}\left(-(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}\right) = -(2+2\lambda b)$, we have $\overline{m}_1 = 0$ as in the first case.

  (b) If $t = W_0\left(-(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}\right)$, we have $-1 < t < 0 \Rightarrow -1 < \lambda\overline{m}_1 - 2 - 2\lambda b < 0 \Rightarrow \frac{1}{\lambda} + 2b < \overline{m}_1 < \frac{2}{\lambda} + 2b$. Thus, if we have $\frac{1}{\lambda} + 2b > 0 \Rightarrow b > -\frac{1}{2\lambda}$, then we have positive $\overline{m}_1$, which implies the existence of an equilibrium with two bins.

Thus, as long as $b \leq -\frac{1}{2\lambda}$, there exists only one bin at the equilibrium; i.e., there can be only non-informative equilibria; and the equilibrium with two bins can be achieved only if $b > -\frac{1}{2\lambda}$. In this case, $\overline{m}_1 = \frac{1}{\lambda}W_0\left(-(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}\right) + 2\left(\frac{1}{\lambda} + b\right)$. Note that, since $-1 < W_0(\cdot) < 0$, the boundary between two bins lies within the interval $\frac{1}{\lambda} + 2b < \overline{m}_1 < \frac{2}{\lambda} + 2b$.

5.7.2.4 Proof of Theorem 5.3.2

The proof consists of three main parts. After characterizing the equilibrium, the monotonicity of bin-lengths and the upper bound on the number of bins are investigated.

Part-I: Equilibrium Solution : Consider an equilibrium with N bins, and let the k-th bin be the interval $[\overline{m}_{k-1}, \overline{m}_k)$ where $\overline{m}_0 = 0$ and $\overline{m}_N = +\infty$, l_k denotes the length of the k-th bin; i.e., $l_k = \overline{m}_k - \overline{m}_{k-1}$, and the centroid of the k-th bin (i.e., the corresponding action of the decoder) is $u_k = \mathbb{E}[m|\overline{m}_{k-1} < m < \overline{m}_k]$. Then, by utilizing (5.2) and (5.15),

$$\begin{aligned}
\overline{m}_{N-1} = \frac{u_{N-1}+u_N}{2} + b &= \frac{\left(\overline{m}_{N-2} + \frac{1}{\lambda} - \frac{l_{N-1}}{\mathrm{e}^{\lambda l_{N-1}}-1}\right) + \left(\overline{m}_{N-1} + \frac{1}{\lambda}\right)}{2} + b \\
\Rightarrow l_{N-1}\frac{\mathrm{e}^{\lambda l_{N-1}}}{\mathrm{e}^{\lambda l_{N-1}}-1} &= \frac{2}{\lambda} + 2b \qquad (5.22) \\
\Rightarrow l_{N-1} &= \frac{1}{\lambda}W_0\left(-(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}\right) + 2\left(\frac{1}{\lambda} + b\right). \qquad (5.23)
\end{aligned}$$

By utilizing (5.15) and (5.18), the length of the k-th bin for $k = 1, 2, \ldots, N-2$ can be found as follows:

$$\begin{aligned}
u_{k+1} - \overline{m}_k = \overline{m}_k - u_k - 2b &= (\overline{m}_k - \overline{m}_{k-1}) - (u_k - \overline{m}_{k-1}) - 2b \\
\Rightarrow \frac{1}{\lambda} - \frac{l_{k+1}}{\mathrm{e}^{\lambda l_{k+1}}-1} &= l_k - \frac{1}{\lambda} + \frac{l_k}{\mathrm{e}^{\lambda l_k}-1} - 2b \\
\Rightarrow l_k\frac{\mathrm{e}^{\lambda l_k}}{\mathrm{e}^{\lambda l_k}-1} &= \frac{2}{\lambda} + 2b - \frac{l_{k+1}}{\mathrm{e}^{\lambda l_{k+1}}-1}. \qquad (5.24)
\end{aligned}$$

If we let $c_k \triangleq \frac{2}{\lambda} + 2b - \frac{l_{k+1}}{\mathrm{e}^{\lambda l_{k+1}}-1}$, the solution to (5.24) is

$$l_k = \frac{1}{\lambda}W_0\left(-\lambda c_k \mathrm{e}^{-\lambda c_k}\right) + c_k. \qquad (5.25)$$

It can be observed from (5.23) and (5.25) that bin-lengths $l_1, l_2, \ldots, l_{N-1}$ have a unique solution, which implies that the bin edges have unique values as $\overline{m}_0 = 0$, $\overline{m}_k = \sum_{i=1}^{k} l_k$ for $k = 1, 2, \ldots, N-1$, and $\overline{m}_N = \infty$.

In order to represent the solutions in a more compact form recursively, define $g(l_k) \triangleq l_k \frac{\mathrm{e}^{\lambda l_k}}{\mathrm{e}^{\lambda l_k}-1}$ and $h(l_k) \triangleq \frac{l_k}{\mathrm{e}^{\lambda l_k}-1}$. Then, the recursions in (5.22) and (5.24) can be written as:

$$g(l_{N-1}) = \frac{2}{\lambda} + 2b\,, \tag{5.26a}$$

$$g(l_k) = \frac{2}{\lambda} + 2b - h(l_{k+1})\,, \qquad \text{for } k = 1, 2, \ldots, N-2\,. \tag{5.26b}$$

Part-II: Monotonically Increasing Bin-Lengths : The proof is based on induction. Before the induction step, in order to utilize (5.26b), we examine the structure of g and h. First note that both functions are continuous and differentiable on $[0, \infty)$. Now, g has the following properties:

- $g(0) = \lim_{s\to 0} \frac{s e^{\lambda s}}{e^{\lambda s}-1} \stackrel{H}{=} \lim_{s\to 0} \frac{1+\lambda s}{\lambda} = \frac{1}{\lambda} > 0$ ($\stackrel{H}{=}$ represents l'Hôspital's rule),
- $\lim_{s\to\infty} g(s) = \lim_{s\to\infty} \frac{s e^{\lambda s}}{e^{\lambda s}-1} \stackrel{H}{=} \lim_{s\to\infty} \frac{1+\lambda s}{\lambda} = \infty$,
- $\frac{d}{ds}(g(s))|_{s=0} = \lim_{s\to 0} \frac{e^{\lambda s}(e^{\lambda s}-\lambda s-1)}{(e^{\lambda s}-1)^2} \stackrel{H}{=} \lim_{s\to 0} \frac{\lambda e^{\lambda s}(2e^{\lambda s}-\lambda s-2)}{2\lambda e^{\lambda s}(e^{\lambda s}-1)} \stackrel{H}{=} \lim_{s\to 0} \frac{2\lambda e^{\lambda s}-\lambda}{2\lambda e^{\lambda s}} = \frac{1}{2} > 0$,
- $\frac{d}{ds}(g(s)) = \frac{e^{\lambda s}(e^{\lambda s}-\lambda s-1)}{(e^{\lambda s}-1)^2} = \frac{e^{\lambda s}\sum_{k=2}^{\infty}\frac{(\lambda s)^k}{k!}}{(e^{\lambda s}-1)^2} > 0, \quad$ for any $s > 0$.

All of the above imply that g is a positive, strictly increasing and unbounded function on $\mathbb{R}_{\geq 0}$.

Similarly, the properties of h can be listed as follows:

- $h(0) = \lim_{s\to 0} \frac{s}{e^{\lambda s}-1} \stackrel{H}{=} \lim_{s\to 0} \frac{1}{\lambda e^{\lambda s}} = \frac{1}{\lambda} > 0$,
- $\lim_{s\to\infty} h(s) = \lim_{s\to\infty} \frac{s}{e^{\lambda s}-1} \stackrel{H}{=} \lim_{s\to\infty} \frac{1}{\lambda e^{\lambda s}} = 0$,
- $\frac{d}{ds}(h(s))|_{s=0} = \lim_{s\to 0} -\frac{e^{\lambda s}(\lambda s-1)+1}{(e^{\lambda s}-1)^2} \stackrel{H}{=} \lim_{s\to 0} \frac{-\lambda^2 s e^{s}}{2(e^{\lambda s}-1)\lambda e^{\lambda s}} \stackrel{H}{=} \lim_{s\to 0} -\frac{\lambda}{2\lambda e^{\lambda s}} = -\frac{1}{2} < 0$,
- $\frac{d}{ds}(h(s)) = -\frac{e^{\lambda s}(\lambda s-1)+1}{(e^{\lambda s}-1)^2} \stackrel{(a)}{<} 0, \quad$ for any $s > 0$,
  where (a) follows from the fact that $\frac{d}{ds}(-e^{\lambda s}(\lambda s-1)-1) = -\lambda^2 s e^{\lambda s} < 0$ for any $s > 0$, and $-e^0(\lambda(0)-1)-1 = 0$.

All of the above imply that h is a positive and strictly decreasing function on $\mathbb{R}_{\geq 0}$.

Further, notice the following properties:

$$
\begin{aligned}
&g(l_k) = h(l_k) + l_k\,,\\
&g(l_k) = l_k \frac{\mathrm{e}^{\lambda l_k}}{\mathrm{e}^{\lambda l_k} - 1} > l_k\,,\\
&h(l_k) = \frac{l_k}{\mathrm{e}^{\lambda l_k} - 1} = \frac{l_k}{\sum_{k=0}^{\infty} \frac{(\lambda l_k)^k}{k!} - 1} = \frac{l_k}{\lambda l_k + \sum_{k=2}^{\infty} \frac{(\lambda l_k)^k}{k!}} < \frac{l_k}{\lambda l_k} = \frac{1}{\lambda}\,.
\end{aligned}
\tag{5.27}
$$

Now consider the length of the $(N-2)$-nd bin. By utilizing the properties in (5.27) on the recursion in (5.26b),

$$
\begin{aligned}
g(l_{N-2}) &= \frac{2}{\lambda} + 2b - h(l_{N-1}) = g(l_{N-1}) - h(l_{N-1}) = l_{N-1}\\
&\Rightarrow l_{N-1} = g(l_{N-2}) = l_{N-2} + h(l_{N-2})\\
&\Rightarrow l_{N-2} < l_{N-1} < l_{N-2} + \frac{1}{\lambda}
\end{aligned}
\tag{5.28}
$$

is obtained. Similarly, for the $(N-3)$-rd bin, the following relations hold:

$$
\begin{aligned}
g(l_{N-3}) &= \frac{2}{\lambda} + 2b - h(l_{N-2}) = g(l_{N-2}) + h(l_{N-1}) - h(l_{N-2}) = l_{N-2} + h(l_{N-1})\\
&g(l_{N-3}) = l_{N-2} + h(l_{N-1}) < l_{N-2} + h(l_{N-2}) = g(l_{N-2}) \Rightarrow l_{N-3} < l_{N-2}\\
&l_{N-2} < l_{N-2} + h(l_{N-1}) = g(l_{N-3}) = l_{N-3} + h(l_{N-3}) < l_{N-3} + \frac{1}{\lambda}\\
&\Rightarrow l_{N-3} < l_{N-2} < l_{N-3} + \frac{1}{\lambda}\,.
\end{aligned}
\tag{5.29}
$$

Now, by following the similar approach, suppose that $l_{N-1} > l_{N-2} > \ldots > l_k$ is obtained. Then, consider the $(k-1)$-st bin:

$$
\begin{aligned}
g(l_{k-1}) &= \frac{2}{\lambda} + 2b - h(l_k) = g(l_k) + h(l_{k+1}) - h(l_k) = l_k + h(l_{k+1})\\
&g(l_{k-1}) = l_k + h(l_{k+1}) < l_k + h(l_k) = g(l_k) \Rightarrow l_{k-1} < l_k\\
&l_k < l_k + h(l_{k+1}) = g(l_{k-1}) = l_{k-1} + h(l_{k-1}) < l_{k-1} + \frac{1}{\lambda}\\
&\Rightarrow l_{k-1} < l_k < l_{k-1} + \frac{1}{\lambda}\,.
\end{aligned}
\tag{5.30}
$$

Thus, bin-lengths form a monotonically increasing sequence.

Part-III: Number of Bins : Consider the length of the $(N-1)$-st bin: Notice that, in (5.23), since $b > 0$, $-(2+2\lambda b) < -2 < -1$, the Lambert W-function returns $t = W_0\left(-(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}\right)$ such that $t\mathrm{e}^t = -(2+2\lambda b)\mathrm{e}^{-(2+2\lambda b)}$ and $-1 < t < 0$, which results in $\frac{2}{\lambda} + 2b > l_{N-1} > \frac{1}{\lambda} + 2b > \frac{1}{\lambda}$; i.e., the $(N-1)$-st bin has a positive length.

For the other bins, since $c_k = \frac{2}{\lambda} + 2b - \frac{l_{k+1}}{\mathrm{e}^{\lambda l_{k+1}}-1} = \frac{2}{\lambda} + 2b - h(l_{k+1}) > \frac{2}{\lambda} + 2b - \frac{1}{\lambda} = \frac{1}{\lambda} + 2b$ for $b > 0$, we have $-\lambda c_k = -1 - 2\lambda b < -1$, which implies that $W_0\left(-\lambda c_k \mathrm{e}^{-\lambda c_k}\right)$ has a solution t such that $t\mathrm{e}^t = -\lambda c_k \mathrm{e}^{-\lambda c_k}$ and $-1 < t < 0$. Hence, from (5.25), $l_k = \frac{1}{\lambda} W_0\left(-\lambda c_k \mathrm{e}^{-\lambda c_k}\right) + c_k > \frac{1}{\lambda}(-1) + \frac{1}{\lambda} + 2b = 2b > 0$ is obtained. This means that, for any given number of bins N, when $b > 0$, it is possible to obtain an equilibrium with positive bin-lengths $l_1, l_2, \ldots, l_{N-2}$.

To summarize the results, for every N, with $l_N = \infty$, there exists a solution $l_1, l_2, \ldots, l_N$ so that

(i) these construct a unique equilibrium,

(ii) each of these are non-zero,

(iii) these form a monotonically increasing sequence.

5.7.2.5 Proof of Theorem 5.3.3

For any equilibrium, consider a bin with a finite length, let's say the k-th bin, and by utilizing (5.26b) and (5.27), we have the following:

$$\begin{aligned}
\frac{2}{\lambda} + 2b =& g(l_k) + h(l_{k+1}) = g(l_k) + g(l_{k+1}) - l_{k+1} > l_k + l_{k+1} - l_{k+1} = l_k \\
&\Rightarrow l_k < \frac{2}{\lambda} + 2b\,, \\
\frac{2}{\lambda} + 2b =& g(l_k) + h(l_{k+1}) = h(l_k) + l_k + h(l_{k+1}) < \frac{1}{\lambda} + l_k + \frac{1}{\lambda} = \frac{2}{\lambda} + l_k \\
&\Rightarrow l_k > 2b\,.
\end{aligned}$$

Thus, all bin-lengths are bounded from above and below: $2b < l_k < \frac{2}{\lambda} + 2b$. Now consider the fixed-point solution of the recursion in (5.26b); i.e., $g(l^*) =$

$\frac{2}{\lambda} + 2b - h(l^*)$. Then, by letting $c \triangleq \frac{2}{\lambda} + 2b$,

$$l^* \frac{\mathrm{e}^{\lambda l^*}}{\mathrm{e}^{\lambda l^*} - 1} = c - \frac{l^*}{\mathrm{e}^{\lambda l^*} - 1} \Rightarrow l^* \frac{\mathrm{e}^{\lambda l^*} + 1}{\mathrm{e}^{\lambda l^*} - 1} = c \Rightarrow (c - l^*)\mathrm{e}^{\lambda l^*} - (c + l^*) = 0\,. \quad (5.31)$$

In order to investigate if (5.31) has a unique solution l^* such that $2b < l^* < \frac{2}{\lambda} + 2b$, let $\Psi(s) \triangleq (c - s)\mathrm{e}^{\lambda s} - (c + s)$ for $s \in \left(2b, \frac{2}{\lambda} + 2b\right)$ and notice the following properties:

- $\Psi(2b) = \frac{2}{\lambda}\mathrm{e}^{2\lambda b} - \left(\frac{2}{\lambda} + 4b\right) = \frac{2}{\lambda}\left(\mathrm{e}^{2\lambda b} - 1 - 2\lambda b\right)$
  $= \frac{2}{\lambda}\left(1 + 2\lambda b + \sum_{k=2}^{\infty} \frac{(2\lambda b)^k}{k!} - 1 - 2\lambda b\right) = \frac{2}{\lambda}\left(\sum_{k=2}^{\infty} \frac{(2\lambda b)^k}{k!}\right) > 0,$
- $\Psi(\frac{2}{\lambda} + 2b) = 0 \times \mathrm{e}^{2+2\lambda b} - \left(\frac{4}{\lambda} + 4b\right) = -\left(\frac{4}{\lambda} + 4b\right) < 0,$
- $\Psi'(s) = \frac{\mathrm{d}}{\mathrm{d}s}(\Psi(s)) = \mathrm{e}^{\lambda s}\left(\lambda(c - s) - 1\right) - 1,$
- $\Psi'(2b) = \mathrm{e}^{2\lambda b}\left(\lambda(\frac{2}{\lambda}) - 1\right) - 1 = \mathrm{e}^{2\lambda b} - 1 > 0,$
- $\Psi'(\frac{2}{\lambda} + 2b) = \mathrm{e}^{2+2\lambda b}\left(\lambda \times 0 - 1\right) - 1 = -\mathrm{e}^{2+2\lambda b} - 1 < 0,$
- $\Psi''(s) = \frac{\mathrm{d}}{\mathrm{d}s}(\Psi') = \lambda \mathrm{e}^{\lambda s}\left(\lambda(c - s) - 2\right)$, since $s \in \left(2b, \frac{2}{\lambda} + 2b\right)$ and $c = \frac{2}{\lambda} + 2b$, we have $0 < c - s < \frac{2}{\lambda} \Rightarrow -2 < \lambda(c - s) - 2 < 0 \Rightarrow \Psi''(s) > 0$.

All of the above implies that $\Psi(s)$ is a concave function of s for $s \in \left(2b, \frac{2}{\lambda} + 2b\right)$, $\Psi(2b) > 0$, $\Psi(s)$ reaches its maximum value on the interval $\left(2b, \frac{2}{\lambda} + 2b\right)$; i.e., when $\Psi'(s^*) = 0$, and $\Psi(\frac{2}{\lambda} + 2b) < 0$; thus, $\Psi(s)$ crosses the s-axis only once, which implies that $\Psi(s) = 0$ has a unique solution on the interval $\left(2b, \frac{2}{\lambda} + 2b\right)$. In other words, the fixed-point solution of the recursion in (5.26b) is unique; i.e., $\Psi(l^*) = 0$.

Hence, if the length of the first bin is l^*; i.e., $l_1 = l^*$, then, all bins must have a length of l^*; i.e., $l_1 = l_2 = l_3 = \ldots = l^*$. Thus, there exist equilibria with infinitely many equi-length bins.

Now, suppose that $l_1 < l^*$. Then, by (5.26b), $h(l_2) = \frac{2}{\lambda} + 2b - g(l_1)$. Since g in an increasing function, $l_1 < l^* \Rightarrow g(l_1) < g(l^*)$. Let $g(l^*) - g(l_1) \triangleq \Delta > 0$, then,

$$g(l^*) + h(l^*) = g(l_1) + h(l_2) = \frac{2}{\lambda} + 2b \Rightarrow \Delta = g(l^*) - g(l_1) = h(l_2) - h(l^*)\,. \quad (5.32)$$

From Theorem 5.3.2, we know that $h(s) = \frac{s}{\mathrm{e}^{\lambda s}-1}$ is a decreasing function with $h'(s) = -\frac{\mathrm{e}^{\lambda s}(\lambda s-1)+1}{(\mathrm{e}^{\lambda s}-1)^2} < 0$ for $s > 0$ and $h'(0) = -\frac{1}{2}$. Slightly changing the notation, let $\widetilde{h}'(s) = h'(\frac{s}{\lambda})$; i.e., $\widetilde{h}'(s) = \frac{\mathrm{e}^s-1-s\mathrm{e}^s}{(\mathrm{e}^s-1)^2}$. Then, $\widetilde{h}''(s) = \frac{\mathrm{d}}{\mathrm{d}s}(\widetilde{h}'(s)) = -\frac{\mathrm{e}^s(\mathrm{e}^s-1)(2\mathrm{e}^s-s\mathrm{e}^s-s-2)}{(\mathrm{e}^s-1)^4}$. Now, let $\varrho(s) \triangleq 2\mathrm{e}^s - s\mathrm{e}^s - s - 2$, and observe the following:

$$
\begin{aligned}
\varrho(s) &= 2\mathrm{e}^s - s\mathrm{e}^s - s - 2 \Rightarrow \varrho(0) = 0\,,\\
\varrho'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}(\varrho(s)) = \mathrm{e}^s - s\mathrm{e}^s - 1 \Rightarrow \varrho'(0) = 0\,,\\
\varrho''(s) &= \frac{\mathrm{d}}{\mathrm{d}s}(\varrho'(s)) = -s\mathrm{e}^s \leq 0 \Rightarrow \varrho''(0) = 0\\
&\quad \Rightarrow \varrho'(s) < 0 \text{ for } s > 0 \Rightarrow \varrho(s) < 0 \text{ for } s > 0 \Rightarrow \widetilde{h}''(s) > 0 \text{ for } s > 0\,.
\end{aligned}
\tag{5.33}
$$

Thus, $\widetilde{h}'(s)$ is an increasing function, which implies that $h'(s)$ is also an increasing function. Since $h'(0) = -\frac{1}{2}$, $h'(s) > -\frac{1}{2}$ for $s > 0$, it follows that $\frac{h(l^*)-h(l_2)}{l^*-l_2} > -\frac{1}{2} \Rightarrow \frac{-\Delta}{l^*-l_2} > -\frac{1}{2} \Rightarrow l^* - l_2 > 2\Delta$. From (5.32),

$$
\begin{aligned}
\Delta + \Delta &= (g(l^*) - g(l_1)) + (h(l_2) - h(l^*)) = g(l^*) - g(l_1) + (g(l_2) - l_2) - (g(l^*) - l^*)\\
&= g(l_2) - g(l_1) + \underbrace{l^* - l_2}_{>2\Delta} \Rightarrow g(l_2) - g(l_1) < 0 \Rightarrow l_2 < l_1\,.
\end{aligned}
\tag{5.34}
$$

Proceeding similarly, $l^* > l_1 > l_2 > \ldots$ can be obtained. Now, notice that, since $h(l_k)$ is a monotone function and $2b < l_k < \frac{2}{\lambda} + 2b$, the recursion in (5.26b) can be satisfied if

$$
g(l_k) = \frac{2}{\lambda} + 2b - h(l_{k+1}) \Rightarrow \frac{2}{\lambda} + 2b - h(2b) < g(l_k) < \frac{2}{\lambda} + 2b - h\left(\frac{2}{\lambda} + 2b\right)\,.
\tag{5.35}
$$

Let $\underline{l}$ and $\bar{l}$ and defined as $g(\underline{l}) = \frac{2}{\lambda} + 2b - h(2b)$ and $g(\bar{l}) = \frac{2}{\lambda} + 2b - h\left(\frac{2}{\lambda} + 2b\right)$, respectively. Thus, if $l_k \notin (\underline{l}, \bar{l})$, then there is no solution to l_{k+1} for the recursion in (5.26b). Since the sequence of bin-lengths is monotonically decreasing, there is a natural number K such that $l_K > \underline{l}$ and $l_{K+1} \leq \underline{l}$, which implies that there is no solution to l_{K+2}. Thus, there cannot be any equilibrium (with finite or infinite bins) if $l_1 < l^*$.

A similar approach can be taken for $l_1 > l^*$: Since g is an increasing function, $l_1 > l^* \Rightarrow g(l_1) > g(l^*)$. Let $g(l_1) - g(l^*) \triangleq \widetilde{\Delta} > 0 \Rightarrow g(l_1) - g(l^*) = h(l^*) - h(l_2) =$

$\widetilde{\Delta}$. Then, since $h'(s) > -\frac{1}{2}$ for $s > 0$, $\frac{h(l_2)-h(l^*)}{l_2-l^*} > -\frac{1}{2} \Rightarrow \frac{-\Delta}{l_2-l^*} > -\frac{1}{2} \Rightarrow l_2 - l^* > 2\Delta$. From (5.32),

$$\begin{aligned}\widetilde{\Delta} + \widetilde{\Delta} &= (g(l_1) - g(l^*)) + (h(l^*) - h(l_2)) = g(l_1) - g(l^*) + (g(l^*) - l^*) - (g(l_2) - l_2) \\ &= g(l_1) - g(l_2) + \underbrace{l_2 - l^*}_{>2\Delta} \Rightarrow g(l_1) - g(l_2) < 0 \Rightarrow l_1 < l_2\,. \end{aligned} \tag{5.36}$$

Proceeding similarly, $l^* < l_1 < l_2 < \ldots$ can be obtained. Since the sequence of bin-lengths is monotonically increasing, there is a natural number $\widetilde{K}$ such that $l_{\widetilde{K}} < \bar{l}$ and $l_{\widetilde{K}+1} \geq \bar{l}$, which implies that there is no solution to $l_{\widetilde{K}+2}$. Thus, there cannot be any equilibrium with infinite number of bins if $l_1 > l^*$. Notice that, it is possible to have an equilibrium with finite number of bins: for the last bin with a finite length, (5.26a) is utilized. Further, it is shown that, at the equilibria, any finite bin-length must be greater than or equal to l^*; i.e., $2b < l^* \leq l_k < \frac{2}{\lambda} + 2b$ must be satisfied.

5.7.2.6 Proof of Theorem 5.3.4

Suppose there exists an equilibrium with N bins, and the corresponding bin-lengths are $l_1 < l_2 < \ldots < l_N = \infty$ with bin-edges $0 = \overline{m}_0 < \overline{m}_1 < \ldots < \overline{m}_{N-1} < \overline{m}_N = \infty$. Then, the decoder cost is

$$\begin{aligned}J^{d,N} &= \mathbb{E}[(m-u)^2] = \mathbb{E}[(m - \mathbb{E}[m|x])^2] \\ &= \sum_{i=1}^{N} \mathbb{E}\left[(m - \mathbb{E}[m|\overline{m}_{i-1} < m < \overline{m}_i])^2 \,|\overline{m}_{i-1} < m < \overline{m}_i\right] \Pr(\overline{m}_{i-1} < m < \overline{m}_i) \\ &= \sum_{i=1}^{N} \mathrm{Var}\left(m|\overline{m}_{i-1} < m < \overline{m}_i\right) \Pr(\overline{m}_{i-1} < m < \overline{m}_i) \\ &= \sum_{i=1}^{N} \left(\frac{1}{\lambda^2} - \frac{l_i^2}{\mathrm{e}^{\lambda l_i} + \mathrm{e}^{-\lambda l_i} - 2}\right) \left(\mathrm{e}^{-\lambda \overline{m}_{i-1}} \left(1 - \mathrm{e}^{-\lambda l_i}\right)\right)\,. \end{aligned} \tag{5.37}$$

Now, consider an equilibrium with $N+1$ bins with bin-lengths $\widetilde{l}_1 < \widetilde{l}_2 < \ldots < \widetilde{l}_{N+1} = \infty$ and bin-edges $0 = \widetilde{m}_0 < \widetilde{m}_1 < \ldots < \widetilde{m}_N < \widetilde{m}_{N+1} = \infty$. The relation between bin-lengths and bin-edges can be expressed as $l_k = \widetilde{l}_{k+1}$ and $\overline{m}_k = \widetilde{m}_{k+1} - \widetilde{l}_1$, respectively, for $k = 1, 2, \ldots, N$ by Theorem 5.3.2. Then, the decoder cost at the equilibrium with $N+1$ bins can be written as

$$\begin{aligned}
J^{d,N+1} &= \sum_{i=1}^{N+1} \left(\frac{1}{\lambda^2} - \frac{\widetilde{l}_i^2}{\mathrm{e}^{\lambda \widetilde{l}_i} + \mathrm{e}^{-\lambda \widetilde{l}_i} - 2} \right) \left(\mathrm{e}^{-\lambda \widetilde{m}_{i-1}} \left(1 - \mathrm{e}^{-\lambda \widetilde{l}_i} \right) \right) \\
&= \left(\frac{1}{\lambda^2} - \frac{\widetilde{l}_1^2}{\mathrm{e}^{\lambda \widetilde{l}_1} + \mathrm{e}^{-\lambda \widetilde{l}_1} - 2} \right) \left(\mathrm{e}^{-\lambda \widetilde{m}_0} \left(1 - \mathrm{e}^{-\lambda \widetilde{l}_1} \right) \right) \\
&\quad + \sum_{i=2}^{N+1} \left(\frac{1}{\lambda^2} - \frac{\widetilde{l}_i^2}{\mathrm{e}^{\lambda \widetilde{l}_i} + \mathrm{e}^{-\lambda \widetilde{l}_i} - 2} \right) \left(\mathrm{e}^{-\lambda \widetilde{m}_{i-1}} \left(1 - \mathrm{e}^{-\lambda \widetilde{l}_i} \right) \right) \\
&= \left(\frac{1}{\lambda^2} - \frac{\widetilde{l}_1^2}{\mathrm{e}^{\lambda \widetilde{l}_1} + \mathrm{e}^{-\lambda \widetilde{l}_1} - 2} \right) \left(1 - \mathrm{e}^{-\lambda \widetilde{l}_1} \right) \\
&\quad + \sum_{i=2}^{N+1} \left(\frac{1}{\lambda^2} - \frac{l_{i-1}^2}{\mathrm{e}^{\lambda l_{i-1}} + \mathrm{e}^{-\lambda l_{i-1}} - 2} \right) \left(\mathrm{e}^{-\lambda (\overline{m}_{i-2} + \widetilde{l}_1)} \left(1 - \mathrm{e}^{-\lambda l_{i-1}} \right) \right) \\
&= \left(\frac{1}{\lambda^2} - \frac{\widetilde{l}_1^2}{\mathrm{e}^{\lambda \widetilde{l}_1} + \mathrm{e}^{-\lambda \widetilde{l}_1} - 2} \right) \left(1 - \mathrm{e}^{-\lambda \widetilde{l}_1} \right) \\
&\quad + \mathrm{e}^{-\lambda \widetilde{l}_1} \underbrace{\left(\sum_{i=1}^{N} \left(\frac{1}{\lambda^2} - \frac{l_i^2}{\mathrm{e}^{\lambda l_i} + \mathrm{e}^{-\lambda l_i} - 2} \right) \left(\mathrm{e}^{-\lambda \overline{m}_{i-1}} \left(1 - \mathrm{e}^{-\lambda l_i} \right) \right) \right)}_{J^{d,N}} \\
&\overset{(a)}{<} J^{d,N} \left(1 - \mathrm{e}^{-\lambda \widetilde{l}_1} \right) + J^{d,N} \mathrm{e}^{-\lambda \widetilde{l}_1} = J^{d,N} \,. \qquad (5.38)
\end{aligned}$$

Thus, $J^{d,N+1} < J^{d,N}$ is obtained, which implies that the equilibrium with more bins is more informative. Here, (a) follows from the fact below:

$$\begin{aligned}
J^{d,N} &= \sum_{i=1}^{N} \left(\frac{1}{\lambda^2} - \frac{l_i^2}{\mathrm{e}^{\lambda l_i} + \mathrm{e}^{-\lambda l_i} - 2} \right) \left(\mathrm{e}^{-\lambda \overline{m}_{i-1}} \left(1 - \mathrm{e}^{-\lambda l_i} \right) \right) \\
&< \sum_{i=1}^{N} \left(\frac{1}{\lambda^2} - \frac{\widetilde{l}_1^2}{\mathrm{e}^{\lambda \widetilde{l}_1} + \mathrm{e}^{-\lambda \widetilde{l}_1} - 2} \right) \Pr(\overline{m}_{i-1} < m < \overline{m}_i) \\
&= \left(\frac{1}{\lambda^2} - \frac{\widetilde{l}_1^2}{\mathrm{e}^{\lambda \widetilde{l}_1} + \mathrm{e}^{-\lambda \widetilde{l}_1} - 2} \right) \sum_{i=1}^{N} \Pr(\overline{m}_{i-1} < m < \overline{m}_i) \\
&= \left(\frac{1}{\lambda^2} - \frac{\widetilde{l}_1^2}{\mathrm{e}^{\lambda \widetilde{l}_1} + \mathrm{e}^{-\lambda \widetilde{l}_1} - 2} \right) , \qquad (5.39)
\end{aligned}$$

where the inequality holds since $\widetilde{l}_1 < l_1 < l_2 < \ldots < l_N$ and $\varphi(s) \triangleq \frac{s^2}{\mathrm{e}^{\lambda s} + \mathrm{e}^{-\lambda s} - 2}$ is a decreasing function of s, as shown below:

$$\varphi(s) = \frac{s^2}{\mathrm{e}^{\lambda s} + \mathrm{e}^{-\lambda s} - 2} = \frac{s^2 \mathrm{e}^{\lambda s}}{(\mathrm{e}^{\lambda s} - 1)^2} \,,$$

$$\varphi'(s) = \frac{s\mathrm{e}^{\lambda s}(\mathrm{e}^{\lambda s}-1)(2\mathrm{e}^{\lambda s}-\lambda s\mathrm{e}^{\lambda s}-\lambda s-2)}{(\mathrm{e}^{\lambda s}-1)^4} = \frac{s\mathrm{e}^{\lambda s}(\mathrm{e}^{\lambda s}-1)\varrho(\lambda s)}{(\mathrm{e}^{\lambda s}-1)^4} \overset{(a)}{<} 0\,, \quad (5.40)$$

where (a) follows from (5.33).

Now consider the equilibria with infinitely many bins: by Theorem 5.3.3, bin-lengths are $l_1 = l_2 = \ldots = l^*$, where l^* is the fixed-point solution of the recursion in (5.26b); i.e., $g(l^*) = \frac{2}{\lambda} + 2b - h(l^*)$, and bin-edges are $\overline{m}_k = kl^*$. Then, the decoder cost is

$$\begin{aligned}
J^{d,\infty} &= \sum_{i=1}^{\infty}\left(\frac{1}{\lambda^2} - \frac{(l^*)^2}{\mathrm{e}^{\lambda l^*}+\mathrm{e}^{-\lambda l^*}-2}\right)\left(\mathrm{e}^{-\lambda(i-1)l^*}\left(1-\mathrm{e}^{-\lambda l^*}\right)\right) \\
&= \left(\frac{1}{\lambda^2} - \frac{(l^*)^2}{\mathrm{e}^{\lambda l^*}+\mathrm{e}^{-\lambda l^*}-2}\right)\left(1-\mathrm{e}^{-\lambda l^*}\right) \\
&\quad + \sum_{i=2}^{\infty}\left(\frac{1}{\lambda^2} - \frac{(l^*)^2}{\mathrm{e}^{\lambda l^*}+\mathrm{e}^{-\lambda l^*}-2}\right)\left(\mathrm{e}^{-\lambda(i-1)l^*}\left(1-\mathrm{e}^{-\lambda l^*}\right)\right) \\
&= \left(\frac{1}{\lambda^2} - \frac{(l^*)^2}{\mathrm{e}^{\lambda l^*}+\mathrm{e}^{-\lambda l^*}-2}\right)\left(1-\mathrm{e}^{-\lambda l^*}\right) \\
&\quad + \mathrm{e}^{-\lambda l^*}\sum_{i=2}^{\infty}\left(\frac{1}{\lambda^2} - \frac{(l^*)^2}{\mathrm{e}^{\lambda l^*}+\mathrm{e}^{-\lambda l^*}-2}\right)\left(\mathrm{e}^{-\lambda(i-2)l^*}\left(1-\mathrm{e}^{-\lambda l^*}\right)\right) \\
&= \left(\frac{1}{\lambda^2} - \frac{(l^*)^2}{\mathrm{e}^{\lambda l^*}+\mathrm{e}^{-\lambda l^*}-2}\right)\left(1-\mathrm{e}^{-\lambda l^*}\right) \\
&\quad + \mathrm{e}^{-\lambda l^*}\underbrace{\sum_{i=1}^{\infty}\left(\frac{1}{\lambda^2} - \frac{(l^*)^2}{\mathrm{e}^{\lambda l^*}+\mathrm{e}^{-\lambda l^*}-2}\right)\left(\mathrm{e}^{-\lambda(i-1)l^*}\left(1-\mathrm{e}^{-\lambda l^*}\right)\right)}_{J^{d,\infty}} \\
\Rightarrow J^{d,\infty}\left(1-\mathrm{e}^{-\lambda l^*}\right) &= \left(\frac{1}{\lambda^2} - \frac{(l^*)^2}{\mathrm{e}^{\lambda l^*}+\mathrm{e}^{-\lambda l^*}-2}\right)\left(1-\mathrm{e}^{-\lambda l^*}\right) \\
\Rightarrow J^{d,\infty} &= \left(\frac{1}{\lambda^2} - \frac{(l^*)^2}{\mathrm{e}^{\lambda l^*}+\mathrm{e}^{-\lambda l^*}-2}\right)\,. \qquad (5.41)
\end{aligned}$$

Since bin-lengths at the equilibria with finitely many bins are greater than l^* by Theorem 5.3.3, and due to similar reasoning in (5.38) (indeed, by replacing $\widetilde{l}_1$ with l^*), $J^{d,\infty} < J^{d,N}$ can be obtained for any finite N. Actually, $J^{d,N}$ is a monotonically decreasing sequence with limit $\lim_{N\to\infty} J^{d,N} = J^{d,\infty}$. Thus, the lowest equilibrium cost is achieved with infinitely many quantization bins.

5.7.3 Proofs for Section 5.4

5.7.3.1 Proof of Theorem 5.4.1

Consider an equilibrium with two bins such that the first bin is $(-\infty = \overline{m}_0, \overline{m}_1)$ and the second bin is $[\overline{m}_1, \overline{m}_2 = \infty)$. The centroids of the bins (the action of the decoder) can be derived from (5.5) as $u_1 = \mathbb{E}[m | -\infty < m < \overline{m}_1] = \mu - \sigma \frac{\phi(\frac{\overline{m}_1-\mu}{\sigma})}{\Phi(\frac{\overline{m}_1-\mu}{\sigma})}$ and $u_2 = \mathbb{E}[m | \overline{m}_1 \leq m < \infty] = \mu + \sigma \frac{\phi(\frac{\overline{m}_1-\mu}{\sigma})}{1-\Phi(\frac{\overline{m}_1-\mu}{\sigma})}$. Then, by utilizing (5.2),

$$
\begin{aligned}
\overline{m}_1 &= \frac{u_1+u_2}{2} + b = \frac{\mu - \sigma \frac{\phi(\frac{\overline{m}_1-\mu}{\sigma})}{\Phi(\frac{\overline{m}_1-\mu}{\sigma})} + \mu + \sigma \frac{\phi(\frac{\overline{m}_1-\mu}{\sigma})}{1-\Phi(\frac{\overline{m}_1-\mu}{\sigma})}}{2} + b \\
&= \mu + \frac{\sigma}{2}\left(\frac{\phi(\frac{\overline{m}_1-\mu}{\sigma})}{1-\Phi(\frac{\overline{m}_1-\mu}{\sigma})} - \frac{\phi(\frac{\overline{m}_1-\mu}{\sigma})}{\Phi(\frac{\overline{m}_1-\mu}{\sigma})} \right) + b \\
&\overset{c \triangleq \frac{\overline{m}_1-\mu}{\sigma}}{\Rightarrow} \sigma c + \mu = \mu + \frac{\sigma}{2}\left(\frac{\phi(c)}{1-\Phi(c)} - \frac{\phi(c)}{\Phi(c)} \right) + b \\
&\Rightarrow 2c - \frac{\phi(c)}{1-\Phi(c)} + \frac{\phi(c)}{\Phi(c)} = \frac{2b}{\sigma}\,.
\end{aligned} \tag{5.42}
$$

Let $f(c) \triangleq 2c - \frac{\phi(c)}{1-\Phi(c)} + \frac{\phi(c)}{\Phi(c)}$, then, observe the following:

$$
\begin{aligned}
\lim_{c\to-\infty} f(c) &= \lim_{c\to-\infty}\left(2c - \frac{\phi(c)}{1-\Phi(c)} + \frac{\phi(c)}{\Phi(c)}\right) \\
&= \lim_{c\to-\infty}\left(2c + \frac{\phi(c)}{\Phi(c)}\right) - \lim_{c\to-\infty}\left(\frac{\phi(c)}{1-\Phi(c)}\right)^{\nearrow 0} \\
&= \lim_{c\to-\infty}\left(\frac{2c\Phi(c)+\phi(c)}{\Phi(c)}\right) \\
&\overset{H}{=} \lim_{c\to-\infty}\left(\frac{2\Phi(c)+c\phi(c)}{\phi(c)}\right) \quad (\overset{H}{=} \text{ represents l'Hôspital's rule}) \\
&\overset{H}{=} \lim_{c\to-\infty}\left(\frac{3\phi(c)-c^2\phi(c)}{-c\phi(c)}\right) = \lim_{c\to-\infty}\left(\frac{3-c^2}{-c}\right) \overset{H}{=} \lim_{c\to-\infty} 2c \to -\infty\,, \\
\lim_{c\to\infty} f(c) &= \lim_{c\to\infty}\left(2c - \frac{\phi(c)}{1-\Phi(c)} + \frac{\phi(c)}{\Phi(c)}\right) \\
&= \lim_{c\to\infty}\left(2c - \frac{\phi(c)}{1-\Phi(c)}\right) + \lim_{c\to\infty}\left(\frac{\phi(c)}{\Phi(c)}\right)^{\nearrow 0} \\
&= \lim_{c\to\infty}\left(\frac{2c-2c\Phi(c)-\phi(c)}{1-\Phi(c)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{H}{=} \lim_{c\to\infty} \left(\frac{2 - 2\Phi(c) - c\phi(c)}{-\phi(c)} \right) \\
&\overset{H}{=} \lim_{c\to\infty} \left(\frac{-3\phi(c) + c^2\phi(c)}{c\phi(c)} \right) = \lim_{c\to\infty} \left(\frac{-3 + c^2}{c} \right) \overset{H}{=} \lim_{c\to\infty} 2c \to \infty \,, \\
f'(c) &= 2 - \frac{\phi(c)(-c)(1-\Phi(c)) - \phi(c)(-\phi(c))}{(1-\Phi(c))^2} + \frac{\phi(c)(-c)\Phi(c) - \phi(c)\phi(c)}{\Phi(c)^2} \\
&= 2 - \phi(c)^2 \left(\frac{1}{(1-\Phi(c))^2} + \frac{1}{\Phi(c)^2} \right) + c\phi(c) \left(\frac{1}{1-\Phi(c)} - \frac{1}{\Phi(c)} \right) \\
&= 2 - \frac{\phi(c)}{1-\Phi(c)} \left(\frac{\phi(c)}{1-\Phi(c)} - c \right) - \frac{\phi(c)^2}{\Phi(c)^2} - \frac{c\phi(c)}{\Phi(c)} \,. \qquad (5.43)
\end{aligned}
$$

It can be seen that, by using the identities $\phi(c) = \phi(-c)$ and $\Phi(c) = 1 - \Phi(-c)$, $f'(c)$ is an even function of c; i.e., $f(c) = f(-c)$. Thus, it can be assumed that $c \geq 0$. Then, the analysis of (5.43) can be done as follows:

- In [125], the inequality on the upper bound of the Mill's ratio is proved as $\frac{\phi(c)}{1-\Phi(c)} < \frac{\sqrt{c^2+4}+c}{2}$. Then,

$$
\begin{aligned}
\frac{\phi(c)}{1-\Phi(c)} \left(\frac{\phi(c)}{1-\Phi(c)} - c \right) &< \frac{\sqrt{c^2+4}+c}{2} \left(\frac{\sqrt{c^2+4}+c}{2} - c \right) \\
&= \frac{\sqrt{c^2+4}+c}{2} \frac{\sqrt{c^2+4}-c}{2} = 1 \,.
\end{aligned}
$$

- Since $\mathbb{E}[x| -\infty < x < c] = -\frac{\phi(c)}{\Phi(c)}$ for standard normal distribution, $\frac{\phi(c)}{\Phi(c)}$ is a decreasing function of c, and for $c > 0$, $\frac{\phi(c)}{\Phi(c)} < \frac{\phi(0)}{\Phi(0)} = \sqrt{\frac{2}{\pi}}$.

- Let $g(c) \triangleq \frac{c\phi(c)}{\Phi(c)}$, then $g'(c) = \phi(c) \left(\frac{(1-c^2)\Phi(c) - c\phi(c)}{(\Phi(c))^2} \right)$. If we let $h(c) \triangleq (1 - c^2)\Phi(c) - c\phi(c)$, then $h'(c) = -2c\Phi(c) + (1-c^2)\phi(c) - \phi(c) - c\phi(c)(-c) = -2c\Phi(c) < 0$ for $c > 0$. Thus, $g''(c) < 0$, which implies that $g(c)$ is a concave function of c, and takes its maximum value at $g(c^*)$ which satisfies $g'(c^*) = h(c^*) = 0$. By solving numerically, we obtain $c^* \simeq 0.9557$ and $g(c^*) \simeq 0.2908$.

By utilizing the results above, (5.43) becomes

$$
f'(c) > 2 - 1 - \frac{2}{\pi} - 0.2908 \simeq 0.0726 > 0 \,. \qquad (5.44)
$$

Thus, $f(c)$ is a monotone increasing function and it takes values between $(-\infty, \infty)$; thus, (5.42) has always a unique solution to $f(c) = \frac{2b}{\sigma}$. This assures that, there always exists an equilibrium with two bins regardless of the value of b. Further, since $f(0) = 2 \times 0 - \frac{\phi(0)}{1-\Phi(0)} + \frac{\phi(0)}{\Phi(0)} = 0$, the signs of b and c must be the same; i.e., if $b < 0$, the boundary between two bins is smaller than the mean ($\overline{m}_1 < \mu$); if $b > 0$, the boundary between two bins is greater than the mean ($\overline{m}_1 > \mu$).

5.7.3.2 Proof of Proposition 5.4.1

Consider an equilibrium with N bins for a Gaussian source $M \sim \mathcal{N}(\mu, \sigma^2)$: the k-th bin is $[\overline{m}_{k-1}, \overline{m}_k)$, and the centroid of the k-th bin (i..e, the corresponding action of the decoder) is $u_k = \mathbb{E}[m | \overline{m}_{k-1} \leq m < \overline{m}_k]$ so that $-\infty = \overline{m}_0 < u_1 < \overline{m}_1 < u_2 < \overline{m}_2 < \ldots < \overline{m}_{N-2} < u_{N-1} < \overline{m}_{N-1} < u_N < \overline{m}_N = \infty$. Further, assume that μ is in the t-th bin; i.e., $\overline{m}_{t-1} \leq \mu < \overline{m}_t$. Due to the nearest neighbor condition (the best response of the encoder) we have $u_{k+1} - \overline{m}_k = (\overline{m}_k - u_k) - 2b$; and due to the centroid condition (the best response of the decoder), we have $u_k = \mathbb{E}[m | \overline{m}_{k-1} \leq m < \overline{m}_k] = \mu - \sigma \frac{\phi(\frac{\overline{m}_k - \mu}{\sigma}) - \phi(\frac{\overline{m}_{k-1} - \mu}{\sigma})}{\Phi(\frac{\overline{m}_k - \mu}{\sigma}) - \Phi(\frac{\overline{m}_{k-1} - \mu}{\sigma})}$. Now consider the right-half of the Gaussian source; i.e., $m > \mu$, and thus $\overline{m}_{N-1} > \mu$, and for the N-th bin, the following holds:

$$
\begin{aligned}
u_N - \overline{m}_{N-1} &= \mathbb{E}[m | \overline{m}_{N-1} \leq m < \infty] - \overline{m}_{N-1} = \mu + \sigma \frac{\phi(\frac{\overline{m}_{N-1} - \mu}{\sigma})}{1 - \Phi(\frac{\overline{m}_{N-1} - \mu}{\sigma})} - \overline{m}_{N-1} \\
&\overset{(a)}{<} \mu + \sigma \frac{\sqrt{\left(\frac{\overline{m}_{N-1} - \mu}{\sigma}\right)^2 + 4} + \frac{\overline{m}_{N-1} - \mu}{\sigma}}{2} - \overline{m}_{N-1} \\
&= \frac{\sigma}{2} \left(\sqrt{\left(\frac{\overline{m}_{N-1} - \mu}{\sigma}\right)^2 + 4} - \frac{\overline{m}_{N-1} - \mu}{\sigma} \right) \\
&< \frac{\sigma}{2} \left(\sqrt{\left(\frac{\overline{m}_{N-1} - \mu}{\sigma}\right)^2 + 4\left(\frac{\overline{m}_{N-1} - \mu}{\sigma}\right) + 4} - \frac{\overline{m}_{N-1} - \mu}{\sigma} \right) \\
&= \frac{\sigma}{2} \left(\frac{\overline{m}_{N-1} - \mu}{\sigma} + 2 - \frac{\overline{m}_{N-1} - \mu}{\sigma} \right) = \sigma \,. \qquad (5.45)
\end{aligned}
$$

Here, (a) follows from the inequality on the upper bound of the Mill's ratio [125]. Now, observe the following:

$$
\begin{aligned}
\sigma > u_N - \overline{m}_{N-1} &= (\overline{m}_{N-1} - u_{N-1}) - 2b \\
&> (u_{N-1} - \overline{m}_{N-2}) - 2b \\
&= (\overline{m}_{N-2} - u_{N-2}) - 2(2b) \\
&\vdots \\
&> u_{t+1} - \overline{m}_t - (N-t-1)(2b) \\
&= \overline{m}_t - u_t - (N-t)(2b) \\
&> -(N-t)(2b)\,,
\end{aligned}
$$

where the inequalities follow from the fact that the Gaussian pdf of M is monotonically decreasing for $m > \mu$. Hence, for $b < 0$, $N - t < -\frac{\sigma}{2b}$, which implies that the number of bins on the right-half is bounded by $\lfloor -\frac{\sigma}{2b} \rfloor$. Further, when $b < 0$, the following relation holds for bin-lengths:

$$
\begin{aligned}
l_k &= \overline{m}_k - \overline{m}_{k-1} = (\overline{m}_k - u_k) + (u_k - \overline{m}_{k-1}) > (u_k - \overline{m}_{k-1}) + (u_k - \overline{m}_{k-1}) \\
&= (\overline{m}_{k-1} - u_{k-1} - 2b) + (\overline{m}_{k-1} - u_{k-1} - 2b) \\
&> (\overline{m}_{k-1} - u_{k-1} - 2b) + (u_{k-1} - \overline{m}_{k-2} - 2b) \\
&= (\overline{m}_{k-1} - u_{k-1}) + (u_{k-1} - \overline{m}_{k-2}) - 4b = \overline{m}_{k-1} - \overline{m}_{k-2} - 4b = l_{k-1} - 4b \\
&\Rightarrow l_k > l_{k-1}\,. \qquad (5.46)
\end{aligned}
$$

Thus, bin-lengths are monotonically increasing on the right-half when $b < 0$.

Similarly, consider the left-half of the Gaussian source; i.e., $m < \mu$, and thus $\mu > \overline{m}_1$, and for the first bin, the following holds:

$$
\begin{aligned}
\overline{m}_1 - u_1 = \overline{m}_1 - \mathbb{E}[m| -\infty < m < \overline{m}_1] &= \overline{m}_1 - \mu + \sigma \frac{\phi(\frac{\overline{m}_1 - \mu}{\sigma})}{\Phi(\frac{\overline{m}_1 - \mu}{\sigma})} \\
&\overset{(a)}{=} \sigma \frac{\phi(\frac{\mu - \overline{m}_1}{\sigma})}{1 - \Phi(\frac{\mu - \overline{m}_1}{\sigma})} - \sigma \frac{\mu - \overline{m}_1}{\sigma} \\
&\overset{(b)}{<} \sigma \left(\frac{\sqrt{\left(\frac{\mu - \overline{m}_1}{\sigma}\right)^2 + 4} + \frac{\mu - \overline{m}_1}{\sigma}}{2} - \frac{\mu - \overline{m}_1}{\sigma} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sigma}{2}\left(\sqrt{\left(\frac{\mu-\overline{m}_1}{\sigma}\right)^2+4}-\frac{\mu-\overline{m}_1}{\sigma}\right) \\
&< \frac{\sigma}{2}\left(\sqrt{\left(\frac{\mu-\overline{m}_1}{\sigma}\right)^2+4\left(\frac{\mu-\overline{m}_1}{\sigma}\right)+4}-\frac{\mu-\overline{m}_1}{\sigma}\right) \\
&= \frac{\sigma}{2}\left(\frac{\mu-\overline{m}_1}{\sigma}+2-\frac{\mu-\overline{m}_1}{\sigma}\right)=\sigma\,. \qquad (5.47)
\end{aligned}
$$

Here, (a) holds since $\phi(x) = \phi(-x)$ and $\Phi(x) = 1 - \Phi(-x)$, (b) follows from the inequality on the upper bound of the Mill's ratio [125]. Now, observe the following:

$$
\begin{aligned}
\sigma > \overline{m}_1 - u_1 &= u_2 - \overline{m}_1 + 2b \\
&> \overline{m}_2 - u_2 + 2b \\
&= u_3 - \overline{m}_2 + 2(2b) \\
&\vdots \\
&> \overline{m}_{t-1} - u_{t-1} + (t-2)(2b) \\
&= u_t - \overline{m}_{t-1} + (t-1)(2b) \\
&> (t-1)(2b)\,,
\end{aligned}
$$

where the inequalities follow from the fact that the Gaussian pdf of M is monotonically increasing for $m < \mu$. Hence, for $b > 0$, $t - 1 < \frac{\sigma}{2b}$, which implies that the number of bins on the left-half is bounded by $\left\lfloor \frac{\sigma}{2b} \right\rfloor$. Further, when $b > 0$, the following relation holds for bin-lengths:

$$
\begin{aligned}
l_k = \overline{m}_k - \overline{m}_{k-1} &= (\overline{m}_k - u_k) + (u_k - \overline{m}_{k-1}) > (\overline{m}_k - u_k) + (\overline{m}_k - u_k) \\
&= (u_{k+1} - \overline{m}_k + 2b) + (u_{k+1} - \overline{m}_k + 2b) \\
&> (\overline{m}_{k+1} - u_{k+1} + 2b) + (u_{k+1} - \overline{m}_k + 2b) \\
&= (\overline{m}_{k+1} - u_{k+1}) + (u_{k+1} - \overline{m}_k) + 4b = \overline{m}_{k+1} - \overline{m}_k + 4b = l_{k+1} + 4b \\
&\Rightarrow l_k < l_{k+1}\,. \qquad (5.48)
\end{aligned}
$$

Thus, bin-lengths are monotonically decreasing on the left-half when $b > 0$.

5.7.4 Proofs for Section 5.5

5.7.4.1 Proof of Fact 5.5.1

Due to the absolute value in the pdf of M, there may be three different cases as will be investigated below:

(i) $\underline{0 < a < b}$: This case is equivalent to the standard exponential case, thus we have $\mathbb{E}[m|a < m < b] = \frac{1}{\lambda} + a - \frac{b-a}{\mathrm{e}^{\lambda(b-a)}-1}$, as shown below:

$$\begin{aligned}
\mathbb{E}[m|a < m < b] &= \int_a^b m \frac{\frac{\lambda}{2}\mathrm{e}^{-\lambda|m|}}{\int_a^b \frac{\lambda}{2}\mathrm{e}^{-\lambda|m|}} \mathrm{d}m = \frac{\int_a^b m\lambda \mathrm{e}^{-\lambda m}\mathrm{d}m}{\int_a^b \lambda \mathrm{e}^{-\lambda m}\mathrm{d}m} \\
&= \frac{\left(-m\mathrm{e}^{-\lambda m} - \frac{\mathrm{e}^{-\lambda m}}{\lambda}\right)\Big|_a^b}{-\mathrm{e}^{-\lambda m}\Big|_a^b} = \frac{-b\mathrm{e}^{-\lambda b} - \frac{\mathrm{e}^{-\lambda b}}{\lambda} + a\mathrm{e}^{-\lambda a} + \frac{\mathrm{e}^{-\lambda a}}{\lambda}}{-\mathrm{e}^{-\lambda b} + \mathrm{e}^{-\lambda a}} \\
&= \frac{1}{\lambda} + \frac{a\mathrm{e}^{\lambda b} - b\mathrm{e}^{\lambda a}}{\mathrm{e}^{\lambda b} - \mathrm{e}^{\lambda a}} = \frac{1}{\lambda} + a - \frac{\mathrm{e}^{\lambda a}(b-a)}{\mathrm{e}^{\lambda b} - \mathrm{e}^{\lambda a}} \\
&= \frac{1}{\lambda} + a - \frac{b-a}{\mathrm{e}^{\lambda(b-a)} - 1}.
\end{aligned}$$

(ii) $\underline{a < 0 < b}$: In this case, the mean of a truncated standard double-exponential distribution can be obtained as follows:

$$\begin{aligned}
\mathbb{E}[m|a < m < b] &= \int_a^b m \frac{\frac{\lambda}{2}\mathrm{e}^{-\lambda m}}{\int_a^b \frac{\lambda}{2}\mathrm{e}^{-\lambda|m|}} \mathrm{d}m = \frac{\int_a^0 m\lambda \mathrm{e}^{\lambda m}\mathrm{d}m + \int_0^b m\lambda \mathrm{e}^{-\lambda m}\mathrm{d}m}{\int_a^0 \lambda \mathrm{e}^{\lambda m}\mathrm{d}m + \int_0^b \lambda \mathrm{e}^{-\lambda m}\mathrm{d}m} \\
&= \frac{\left(m\mathrm{e}^{\lambda m} - \frac{\mathrm{e}^{\lambda m}}{\lambda}\right)\Big|_a^0 + \left(-m\mathrm{e}^{-\lambda m} - \frac{\mathrm{e}^{-\lambda m}}{\lambda}\right)\Big|_0^b}{\left(\mathrm{e}^{\lambda m}\right)\Big|_a^0 + \left(-\mathrm{e}^{-\lambda m}\right)\Big|_0^b} \\
&= \frac{\left(-\frac{1}{\lambda} - a\mathrm{e}^{\lambda a} + \frac{\mathrm{e}^{\lambda a}}{\lambda}\right) + \left(-b\mathrm{e}^{-\lambda b} - \frac{\mathrm{e}^{-\lambda b}}{\lambda} + \frac{1}{\lambda}\right)}{\left(1 - \mathrm{e}^{\lambda a}\right) + \left(-\mathrm{e}^{-\lambda b} + 1\right)} \\
&= \frac{\frac{1}{\lambda}\left(\mathrm{e}^{\lambda a} - \mathrm{e}^{-\lambda b}\right) - \left(a\mathrm{e}^{\lambda a} + b\mathrm{e}^{-\lambda b}\right)}{2 - \left(\mathrm{e}^{\lambda a} + \mathrm{e}^{-\lambda b}\right)}.
\end{aligned}$$

(iii) $\underline{a < b < 0}$: In this case, the mean of a truncated standard double-exponential distribution can be obtained as follows:

$$\mathbb{E}[m|a < m < b] = \int_a^b m \frac{\frac{\lambda}{2}\mathrm{e}^{-\lambda|m|}}{\int_a^b \frac{\lambda}{2}\mathrm{e}^{-\lambda|m|}} \mathrm{d}m = \frac{\int_a^b m\lambda \mathrm{e}^{\lambda m}\mathrm{d}m}{\int_a^b \lambda \mathrm{e}^{\lambda m}\mathrm{d}m} = \frac{\left(m\mathrm{e}^{\lambda m} - \frac{\mathrm{e}^{\lambda m}}{\lambda}\right)\Big|_a^b}{\mathrm{e}^{\lambda m}\Big|_a^b}$$

$$= \frac{b\mathrm{e}^{\lambda b} - \frac{\mathrm{e}^{\lambda b}}{\lambda} - a\mathrm{e}^{\lambda a} + \frac{\mathrm{e}^{\lambda a}}{\lambda}}{\mathrm{e}^{\lambda b} - \mathrm{e}^{\lambda a}} = -\frac{1}{\lambda} + \frac{b\mathrm{e}^{\lambda b} - a\mathrm{e}^{\lambda a}}{\mathrm{e}^{\lambda b} - \mathrm{e}^{\lambda a}}$$

$$= -\frac{1}{\lambda} + b + \frac{\mathrm{e}^{\lambda a}(b-a)}{\mathrm{e}^{\lambda b} - \mathrm{e}^{\lambda a}} = -\frac{1}{\lambda} + b + \frac{b-a}{\mathrm{e}^{\lambda(b-a)} - 1}.$$

5.7.4.2 Proof of Theorem 5.5.1

Consider an equilibrium with two bins such that the first bin is $[0 = \overline{m}_0, \overline{m}_1)$ and the second bin is $[\overline{m}_1, \overline{m}_2 = \infty)$. Then, based on the value of the boundary between two bins $\overline{m}_1$, there are two possible cases:

(i) $\underline{\overline{m}_1 > 0}$: The centroids of the bins (the action of the decoder) can be derived from (5.6) as $u_1 = \mathbb{E}[m| -\infty < m < \overline{m}_1] = \frac{\frac{1}{\lambda}\left(0-\mathrm{e}^{-\lambda\overline{m}_1}\right) - \left(0+\overline{m}_1\mathrm{e}^{-\lambda\overline{m}_1}\right)}{2-\left(0+\mathrm{e}^{-\lambda\overline{m}_1}\right)} = -\frac{\overline{m}_1+\frac{1}{\lambda}}{2\mathrm{e}^{\lambda\overline{m}_1}-1}$ and $u_2 = \mathbb{E}[m|\overline{m}_1 \leq m < \infty] = \overline{m}_1 + \frac{1}{\lambda}$. Then, by utilizing (5.2),

$$\overline{m}_1 = \frac{u_1+u_2}{2} + b = \frac{-\frac{\overline{m}_1+\frac{1}{\lambda}}{2\mathrm{e}^{\lambda\overline{m}_1}-1} + \overline{m}_1 + \frac{1}{\lambda}}{2} + b \Rightarrow \overline{m}_1 + \frac{\overline{m}_1+\frac{1}{\lambda}}{2\mathrm{e}^{\lambda\overline{m}_1}-1} = \frac{1}{\lambda} + 2b$$

$$\Rightarrow \frac{2\overline{m}_1\mathrm{e}^{\lambda\overline{m}_1} + \frac{1}{\lambda}}{2\mathrm{e}^{\lambda\overline{m}_1}-1} = \frac{1}{\lambda} + 2b \Rightarrow \mathrm{e}^{\lambda\overline{m}_1}\left(\frac{2}{\lambda} + 4b - 2\overline{m}_1\right) = \frac{2}{\lambda} + 2b$$

$$\overset{t=\lambda\overline{m}_1-1-2\lambda b}{\Rightarrow} \mathrm{e}^{t+1+2\lambda b}\frac{-2t}{\lambda} = \frac{2}{\lambda} + 2b \Rightarrow t\mathrm{e}^t = -(1+\lambda b)\,\mathrm{e}^{-(1+2\lambda b)}. \tag{5.49}$$

In order to have a real solution for t, since $t\mathrm{e}^t \geq -\frac{1}{\mathrm{e}}\ \forall t \in \mathbb{R}$, $-(1+\lambda b)\,\mathrm{e}^{-(1+2\lambda b)} \geq -\frac{1}{\mathrm{e}} \Rightarrow (1+\lambda b)\mathrm{e}^{-2\lambda b} \leq 1 \Rightarrow \mathrm{e}^{2\lambda b} - \lambda b - 1 \geq 0$ must be satisfied. Let $f(x) \triangleq \mathrm{e}^{2x} - x - 1$, then we are looking for x values which satisfy $f(x) \geq 0$. Since $f'(x) = 2\mathrm{e}^{2x} - 1$ and $f''(x) = 4\mathrm{e}^{2x} > 0$, $f(x)$ is a convex function of x, and it takes its minimum value when $f'(x) = 2\mathrm{e}^{2x} - 1 = 0 \Rightarrow x = -\frac{\ln 2}{2}$, and $f(-\frac{\ln 2}{2}) < 0$. Therefore,

$f(x)$ takes negative values between two solutions of $f(x) = 0$, and since $-1 < W_0(\cdot) < 0$, the solutions can be found as follows:

$$f(x) = 0 \Rightarrow e^{2x} - x - 1 = 0 \Rightarrow (x+1)e^{-2x} = 1 \overset{u=2x+2}{\Rightarrow} \frac{u}{2}e^{-u+2} = 1$$
$$\Rightarrow -ue^{-u} = -2e^{-2} \Rightarrow u = -W(-2e^{-2}) \Rightarrow x = -\frac{W(-2e^{-2})}{2} - 1$$
$$\Rightarrow x_1 = 0 \text{ and } -1 < x_2 = -\frac{W_0(-2e^{-2})}{2} - 1 < -\frac{1}{2}.$$

Hence, for $-\frac{W_0(-2e^{-2})}{2} - 1 < x < 0$, $f(x)$ takes negative values; in other words, λb cannot take values between $-\frac{W_0(-2e^{-2})}{2} - 1$ and 0 (in other words, it is not possible to have $-\frac{W_0(-2e^{-2})}{2} - 1 < \lambda b < 0$) in order to have an equilibrium with two bins in which the boundary between the bins lies in the positive side of the real line; i.e., $\overline{m}_1 > 0$.

Now, assume that $\lambda b < -\frac{W_0(-2e^{-2})}{2} - 1 \Rightarrow \lambda b < -\frac{W_0(-2e^{-2})}{2} - 1 < -\frac{1}{2} \Rightarrow 1 + 2\lambda b < 0$. Further, since we must have $\overline{m}_1 > 0 \Rightarrow \frac{t}{\lambda} + \frac{1}{\lambda} + 2b > 0 \Rightarrow t > -(1+2\lambda b)$. Then, by combining the inequalities, we have $t > -(1+2\lambda b) > 0 \Rightarrow -(1+\lambda b)\, e^{-(1+2\lambda b)} = te^t > -(1+2\lambda b)e^{-(1+2\lambda b)} \Rightarrow \lambda b > 0$, which is a contradiction. Thus, it is not possible to have $\lambda b < -\frac{W_0(-2e^{-2})}{2} - 1$.

As it can be observed above, in order to have $\overline{m}_1 > 0$, we must have $\lambda b > 0$. Further, the converse also holds: as long as $\lambda b > 0 \Rightarrow b > 0$, it is always possible to have an equilibrium with two bins where $\overline{m}_1 > 0$.

(ii) $\underline{\overline{m}_1 < 0}$: The similar analysis holds, but the details are provided below for completeness:

The centroids of the bins (the action of the decoder) can be derived from (5.6) as $u_1 = \mathbb{E}[m| -\infty < m < \overline{m}_1] = \overline{m}_1 - \frac{1}{\lambda}$ and $u_2 = \mathbb{E}[m|\overline{m}_1 \le m < \infty] = \frac{\frac{1}{\lambda}\left(e^{\lambda\overline{m}_1} - 0\right) - \left(\overline{m}_1 e^{\lambda\overline{m}_1} + 0\right)}{2 - \left(e^{\lambda\overline{m}_1} + 0\right)} = -\frac{\overline{m}_1 - \frac{1}{\lambda}}{2e^{-\lambda\overline{m}_1} - 1}$. Then, by utilizing (5.2),

$$\overline{m}_1 = \frac{u_1 + u_2}{2} + b = \frac{\overline{m}_1 - \frac{1}{\lambda} - \frac{\overline{m}_1 - \frac{1}{\lambda}}{2e^{-\lambda\overline{m}_1} - 1}}{2} + b \Rightarrow \overline{m}_1 + \frac{\overline{m}_1 - \frac{1}{\lambda}}{2e^{-\lambda\overline{m}_1} - 1} = -\frac{1}{\lambda} + 2b$$
$$\Rightarrow \frac{2\overline{m}_1 e^{-\lambda\overline{m}_1} - \frac{1}{\lambda}}{2e^{-\lambda\overline{m}_1} - 1} = -\frac{1}{\lambda} + 2b \Rightarrow e^{-\lambda\overline{m}_1}\left(-\frac{2}{\lambda} + 4b - 2\overline{m}_1\right) = -\frac{2}{\lambda} + 2b$$
$$\overset{t=\lambda\overline{m}_1+1-2\lambda b}{\Rightarrow} e^{-t+1-2\lambda b}\frac{-2t}{\lambda} = -\frac{2}{\lambda} + 2b \Rightarrow -te^{-t} = (-1+\lambda b)\, e^{-1+2\lambda b}. \tag{5.50}$$

In order to have a real solution for t, since $te^t \geq -\frac{1}{\mathrm{e}}\ \forall t \in \mathbb{R}$, $(-1+\lambda b)\,\mathrm{e}^{-1+2\lambda b} \geq -\frac{1}{\mathrm{e}} \Rightarrow (-1+\lambda b)\mathrm{e}^{2\lambda b} \geq -1 \Rightarrow \mathrm{e}^{-2\lambda b} + \lambda b - 1 \geq 0$ must be satisfied. Let $g(x) \triangleq \mathrm{e}^{-2x} + x - 1$, then we are looking for x values which satisfy $g(x) \geq 0$. Since $g'(x) = -2\mathrm{e}^{-2x} + 1$ and $g''(x) = 4\mathrm{e}^{-2x} > 0$, $g(x)$ is a convex function of x, and it takes its minimum value when $g'(x) = -2\mathrm{e}^{-2x} + 1 = 0 \Rightarrow x = \frac{\ln 2}{2}$, and $g(\frac{\ln 2}{2}) < 0$. Therefore, $g(x)$ takes negative values between two solutions of $g(x) = 0$, and since $-1 < W_0(\cdot) < 0$, the solutions can be found as follows:

$$\begin{aligned} g(x) = 0 \Rightarrow & \mathrm{e}^{-2x} + x - 1 = 0 \Rightarrow (1-x)\mathrm{e}^{2x} = 1 \overset{u=2x-2}{\Rightarrow} -\frac{u}{2}\mathrm{e}^{u+2} = 1 \\ \Rightarrow & u\mathrm{e}^{u} = -2\mathrm{e}^{-2} \Rightarrow u = W(-2\mathrm{e}^{-2}) \Rightarrow x = \frac{W(-2\mathrm{e}^{-2})}{2} + 1 \\ \Rightarrow & x_1 = 0 \text{ and } \frac{1}{2} < x_2 = \frac{W_0(-2\mathrm{e}^{-2})}{2} + 1 < 1\,. \end{aligned}$$

Hence, for $0 < x < \frac{W_0(-2\mathrm{e}^{-2})}{2} + 1$, $g(x)$ takes negative values; in other words, λb cannot take values between 0 and $\frac{W_0(-2\mathrm{e}^{-2})}{2} + 1$ (in other words, it is not possible to have $0 < \lambda b < \frac{W_0(-2\mathrm{e}^{-2})}{2} + -1$) in order to have an equilibrium with two bins in which the boundary between the bins lies in the negative side of the real line; i.e., $\overline{m}_1 < 0$.

Now, assume that $\lambda b > \frac{W_0(-2\mathrm{e}^{-2})}{2} + 1 \Rightarrow \lambda b > \frac{W_0(-2\mathrm{e}^{-2})}{2} + 1 > \frac{1}{2} \Rightarrow -1 + 2\lambda b > 0$. Further, since we must have $\overline{m}_1 < 0 \Rightarrow \frac{t}{\lambda} - \frac{1}{\lambda} + 2b < 0 \Rightarrow t < 1 - 2\lambda b$. Then, by combining the inequalities, we have $-t > -1 + 2\lambda b > 0 \Rightarrow (-1+\lambda b)\,\mathrm{e}^{-1+2\lambda b} = -t\mathrm{e}^{-t} > (-1+2\lambda b)\mathrm{e}^{-1+2\lambda b} \Rightarrow \lambda b < 0$, which is a contradiction. Thus, it is not possible to have $\lambda b > \frac{W_0(-2\mathrm{e}^{-2})}{2} + 1$.

As it can be observed above, in order to have $\overline{m}_1 < 0$, we must have $\lambda b < 0$. Further, the converse also holds: as long as $\lambda b < 0 \Rightarrow b < 0$, it is always possible to have an equilibrium with two bins where $\overline{m}_1 < 0$.

5.7.4.3 Proof of Theorem 5.5.2

Consider an equilibrium with two bins such that the first bin is $[0 = \overline{m}_0, \overline{m}_1)$ and the second bin is $[\overline{m}_1, \overline{m}_2 = \infty)$. The centroids of the bins (the action of the decoder) can be derived from (5.8) as $u_1 = \mathbb{E}[m|0 < m < \overline{m}_1] = -\sigma\frac{\phi(\frac{\overline{m}_1}{\sigma})-\frac{1}{\sqrt{2\pi}}}{\Phi(\frac{\overline{m}_1}{\sigma})-\frac{1}{2}}$ and $u_2 = \mathbb{E}[m|\overline{m}_1 \leq m < \infty] = \sigma\frac{\phi(\frac{\overline{m}_1}{\sigma})}{1-\Phi(\frac{\overline{m}_1}{\sigma})}$. Then, by utilizing (5.2),

$$\begin{aligned}
\overline{m}_1 &= \frac{u_1+u_2}{2}+b = \frac{-\sigma\frac{\phi(\frac{\overline{m}_1}{\sigma})-\frac{1}{\sqrt{2\pi}}}{\Phi(\frac{\overline{m}_1}{\sigma})-\frac{1}{2}}+\sigma\frac{\phi(\frac{\overline{m}_1}{\sigma})}{1-\Phi(\frac{\overline{m}_1}{\sigma})}}{2}+b \\
&= \frac{\sigma}{2}\left(-\frac{\phi(\frac{\overline{m}_1}{\sigma})-\frac{1}{\sqrt{2\pi}}}{\Phi(\frac{\overline{m}_1}{\sigma})-\frac{1}{2}}+\frac{\phi(\frac{\overline{m}_1}{\sigma})}{1-\Phi(\frac{\overline{m}_1}{\sigma})}\right)+b \\
&\overset{c\triangleq\frac{\overline{m}_1}{\sigma}}{\Rightarrow} \sigma c = \frac{\sigma}{2}\left(-\frac{\phi(c)-\frac{1}{\sqrt{2\pi}}}{\Phi(c)-\frac{1}{2}}+\frac{\phi(c)}{1-\Phi(c)}\right)+b \\
&\Rightarrow 2c+\frac{\phi(c)-\frac{1}{\sqrt{2\pi}}}{\Phi(c)-\frac{1}{2}}-\frac{\phi(c)}{1-\Phi(c)} = \frac{2b}{\sigma}\,.
\end{aligned} \tag{5.51}$$

Let $f(c) \triangleq 2c+\frac{\phi(c)-\frac{1}{\sqrt{2\pi}}}{\Phi(c)-\frac{1}{2}}-\frac{\phi(c)}{1-\Phi(c)}$, then, observe the following:

$$\begin{aligned}
\lim_{c\to 0} f(c) &= \lim_{c\to 0}\left(2c+\frac{\phi(c)-\frac{1}{\sqrt{2\pi}}}{\Phi(c)-\frac{1}{2}}-\frac{\phi(c)}{1-\Phi(c)}\right) \\
&= \lim_{c\to 0}\left(\frac{\phi(c)-\frac{1}{\sqrt{2\pi}}}{\Phi(c)-\frac{1}{2}}\right)+\lim_{c\to 0}\left(2c-\frac{\phi(c)}{1-\Phi(c)}\right) \\
&= \lim_{c\to 0}\left(\frac{\phi(c)-\frac{1}{\sqrt{2\pi}}}{\Phi(c)-\frac{1}{2}}\right)-\sqrt{\frac{2}{\pi}} \\
&\overset{H}{=} \lim_{c\to 0}\left(\frac{\phi(c)(-c)}{\phi(c)}\right)-\sqrt{\frac{2}{\pi}} \\
&= \lim_{c\to 0}(-c)-\sqrt{\frac{2}{\pi}} = -\sqrt{\frac{2}{\pi}}\,, \\
\lim_{c\to\infty} f(c) &= \lim_{c\to\infty}\left(2c+\frac{\phi(c)-\frac{1}{\sqrt{2\pi}}}{\Phi(c)-\frac{1}{2}}-\frac{\phi(c)}{1-\Phi(c)}\right) \\
&= \lim_{c\to\infty}\left(2c-\frac{\phi(c)}{1-\Phi(c)}\right)+\lim_{c\to\infty}\left(\frac{\phi(c)-\frac{1}{\sqrt{2\pi}}}{\Phi(c)-\frac{1}{2}}\right) \\
&= \lim_{c\to\infty}\left(\frac{2c-2c\Phi(c)-\phi(c)}{1-\Phi(c)}\right)-\sqrt{\frac{2}{\pi}} \\
&\overset{H}{=} \lim_{c\to\infty}\left(\frac{2-2\Phi(c)-c\phi(c)}{-\phi(c)}\right)-\sqrt{\frac{2}{\pi}} \\
&\overset{H}{=} \lim_{c\to\infty}\left(\frac{-3\phi(c)+c^2\phi(c)}{c\phi(c)}\right)-\sqrt{\frac{2}{\pi}}
\end{aligned}$$

$$= \lim_{c\to\infty}\left(\frac{-3+c^2}{c}\right) - \sqrt{\frac{2}{\pi}} \overset{H}{=} \lim_{c\to\infty} 2c - \sqrt{\frac{2}{\pi}} \to \infty\,,$$

$$\begin{aligned} f'(c) &= 2 + \frac{\phi(c)(-c)(\Phi(c)-\frac{1}{2}) - \left(\phi(c) - \frac{1}{\sqrt{2\pi}}\right)\phi(c)}{\left(\Phi(c)-\frac{1}{2}\right)^2} \\ &\qquad - \frac{\phi(c)(-c)(1-\Phi(c)) - \phi(c)(-\phi(c))}{(1-\Phi(c))^2} \\ &= 2 - \frac{c\phi(c)}{\Phi(c)-\frac{1}{2}} + \frac{\left(\frac{1}{\sqrt{2\pi}} - \phi(c)\right)\phi(c)}{\left(\Phi(c)-\frac{1}{2}\right)^2} - \frac{\phi(c)}{1-\Phi(c)}\left(\frac{\phi(c)}{1-\Phi(c)} - c\right) \\ &= 2 - \frac{c\phi(c)}{\Phi(c)-\frac{1}{2}} + \frac{\left(\frac{1}{\sqrt{2\pi}} - \phi(c)\right)\phi(c)}{\left(\Phi(c)-\frac{1}{2}\right)^2} - \frac{\phi(c)}{1-\Phi(c)}\left(\frac{\phi(c)}{1-\Phi(c)} - c\right). \end{aligned} \tag{5.52}$$

Then, observe the following:

- In [125], the inequality on the upper bound of the Mill's ratio is proved as $\frac{\phi(c)}{1-\Phi(c)} < \frac{\sqrt{c^2+4}+c}{2}$. Then,

$$\begin{aligned} \frac{\phi(c)}{1-\Phi(c)}\left(\frac{\phi(c)}{1-\Phi(c)} - c\right) &< \frac{\sqrt{c^2+4}+c}{2}\left(\frac{\sqrt{c^2+4}+c}{2} - c\right) \\ &= \frac{\sqrt{c^2+4}+c}{2}\,\frac{\sqrt{c^2+4}-c}{2} = 1\,. \end{aligned}$$

- Since $0 \le \phi(c) \le \frac{1}{\sqrt{2\pi}}$, we have $\frac{\left(\frac{1}{\sqrt{2\pi}}-\phi(c)\right)\phi(c)}{\left(\Phi(c)-\frac{1}{2}\right)^2} \ge 0$.

- Let $g(c) \triangleq \frac{c\phi(c)}{\Phi(c)-\frac{1}{2}}$, then $g'(c) = \phi(c)\left(\frac{(1-c^2)(\Phi(c)-\frac{1}{2})-c\phi(c)}{(\Phi(c)-\frac{1}{2})^2}\right)$. If we let $h(c) \triangleq (1-c^2)(\Phi(c)-\frac{1}{2})-c\phi(c)$, then $h'(c) = -2c(\Phi(c)-\frac{1}{2}) + (1-c^2)\phi(c) - \phi(c) - c\phi(c)(-c) = -2c(\Phi(c)-\frac{1}{2}) < 0$ for $c > 0$. Thus, $g''(c) < 0$, which implies that $g(c)$ is a concave function of c, and takes its maximum value at $g(c^*)$ which satisfies $g'(c^*) = h(c^*) = 0$. By solving numerically, we obtain $c^* \simeq 0$ and $g(c^*) \simeq 1$.

By utilizing the results above, (5.52) becomes

$$f'(c) > 2 - 1 + 0 - 1 > 0\,. \tag{5.53}$$

Thus, $f(c)$ is a monotone increasing function and it takes values between $\left(-\sqrt{\frac{2}{\pi}}, \infty\right)$; however, if $\frac{2b}{\sigma} < -\sqrt{\frac{2}{\pi}}$, (5.42) cannot have a solution to $f(c) = \frac{2b}{\sigma}$. Therefore,

(i) if $\frac{2b}{\sigma} \geq -\sqrt{\frac{2}{\pi}}$, then (5.42) has always a unique solution to $f(c) = \frac{2b}{\sigma}$, which assures that, there always exists an equilibrium with two bins.

(ii) if $\frac{2b}{\sigma} < -\sqrt{\frac{2}{\pi}}$, there cannot be an equilibrium with two bins; i.e., the only equilibrium is the non-informative one (the equilibrium with one bin).

Chapter 6

Summary and Conclusion

6.1 Summary

In this dissertation, we considered the cheap talk setup of Crawford and Sobel [16], and introduced a signaling game formulation in the hypothesis testing context.

In Chapter 2, we studied the decentralized quadratic cheap talk and signaling game problems, which refers to a class of two-player games of incomplete information in which an informed decision maker (encoder) transmits information to another decision maker (decoder). For a strategic information transmission problem under quadratic criteria with a non-zero bias term leading to a mismatch in the encoder and the decoder objective functions, Nash and Stackelberg equilibria have been investigated in a number of setups. It has been proven that for any arbitrary scalar source, in the presence of misalignment, the quantized nature of Nash equilibrium policies hold whereas all Stackelberg equilibrium policies are fully informative. Further, it has been shown that the Nash equilibrium policies may be non-discrete and even linear for a multi-dimensional cheap talk problem, unlike the scalar case. The additive noisy channel setup with Gaussian statistics has also been studied, such a case leads to a signaling game due to the communication constraints in the transmission. Conditions for the existence of informative affine Nash equilibrium policies were presented for both the scalar and multi-dimensional setups. Lastly, we proved that the only equilibrium in the Stackelberg noisy setup is the linear equilibrium. Table 2.1 summarizes the results of this chapter.

In Chapter 3, dynamic (multi-stage) signaling games involving an encoder and a decoder who have subjective models on the cost functions or the probabilistic model were considered. Nash (simultaneous-move game) and Stackelberg (leader-follower game) equilibria of multi-stage quadratic cheap talk and signaling game problems were investigated under a perfect Bayesian formulation. We established qualitative (e.g. on full revelation, quantization nature, linearity, informativeness and non-informativeness) and quantitative properties (on linearity or explicit computation) of Nash and Stackelberg equilibria under either subjective/inconsistent cost models or priors. For the multi-stage scalar cheap talk, a zero-delay communication setup was considered for i.i.d. and Markov sources; it was shown that the final stage equilibrium is always quantized and under further conditions the equilibria for all time stages must be quantized under the Nash assumption. In contrast, the Stackelberg equilibria are always fully revealing. In the multi-stage signaling game where the transmission of a Gauss-Markov source over a memoryless Gaussian channel was considered, affine policies constitute an invariant subspace under best response maps for scalar and multi-dimensional sources under Nash equilibria. However, for multi-stage Stackelberg signaling games involving Gauss-Markov sources and memoryless Gaussian channels, we have proved that, for scalar setups, linear policies are optimal and the only equilibrium is the linear one, whereas this is not the case for general multi-dimensional setups. We have obtained an explicit dynamic recursion for optimal linear encoding policies for multi-dimensional sources, and derive conditions under which Stackelberg equilibria are non-informative. For the case where the encoder and the decoder have subjective priors on the source distribution, under identical costs and mutual absolute continuity, we have shown that there exist fully informative Nash and Stackelberg equilibria for the dynamic cheap talk as in the team theoretic setup. In particular, for the cheap talk problem, the equilibrium behavior is robust to a class of perturbations in the priors, but not to the perturbations in the cost models in general. For the signaling game, however, Stackelberg equilibrium policies are robust to perturbations in the cost but not to the priors considered in this chapter. Table 3.1 summarizes the results of this chapter.

In Chapter 4, we considered binary signaling problems in the hypothesis testing context, in which the decision makers (the transmitter and the receiver) have subjective priors and/or misaligned objective functions. Depending on the commitment nature of the transmitter to his policies, we formulated the binary signaling problem as a Bayesian game under either Nash or Stackelberg equilibrium concepts and established equilibrium solutions and their properties. We showed that there can be informative or non-informative equilibria in the binary signaling game under the Stackelberg assumption, but there always exists an equilibrium. However, apart from the informative and non-informative equilibria cases, there may not be a Nash equilibrium when the receiver is restricted to use deterministic policies. We also studied the effects of small perturbations at the point of identical priors and costs and showed that the game equilibrium behavior around the team setup is robust under the Nash assumption, whereas it is not robust under the Stackelberg assumption.

In Chapter 5, we investigated the number of bins at the equilibrium under the cheap talk setup with exponential and Gaussian sources as to whether there are finitely many bins or countably infinite number of bins in any equilibrium. It is shown that, for exponential sources, when the bias is negative, the number of bins at the equilibrium is bounded, whereas there is no bound on the number of bins for the positive bias, indeed, it is possible to have an equilibrium with infinitely many bins. For the Gaussian case, there can always exist an equilibrium with two bins.

6.2 Future Directions

A possible future research direction related to Chapter 2 and Chapter 3 is to investigate the convergence properties for Nash and Stackelberg equilibria; i.e., an iterative convergence rate of the best responses can be analyzed. Another direction is to consider more general cost functions and source distributions in cheap talk and signaling game formulations. Obtaining the structural results depending on the cost functions and different type of sources will give significant perspectives to model and comment on the equilibrium states of many decentralized and networked control systems possibly will be in use in the near future.

Regarding Chapter 4, the binary setup considered here can be extended to the M-ary hypothesis testing setup, and the corresponding signaling game structure can be formed in order to model a game between players with a multiple-bit communication channel. The extension to more general noise distributions is possible: the Nash equilibrium analysis holds identically when the noise distribution leads to a single-threshold test. Finally, in addition to the Bayesian approach considered here, different cost structures and parameters can be introduced by investigating the game under Neyman-Pearson and minimax criteria.

Generalization of Lloyd-Max quantizers can be studied related to Chapter 5 in order to analyze the number of bins (whether finite or infinite) at the equilibrium. The possible approaches can be the best-response analysis and fixed point analysis. Further, for more general sources, besides the number of bins, it is a desirable question to see whether in general more number of bins implies more desirable equilibria.

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