Computable Delay Margins for Adaptive Systems With State Variables Accessible

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***Abstract*—Robust adaptive control of plants whose state variables are accessible in the presence of an input time delay is established in this paper. It is shown that a standard model reference adaptive controller modified with projection ensures global boundedness of the overall adaptive system for a range of nonzero delays. The upper bound of such delays, that is, the delay margin, is explicitly defined and can be computed *a priori*.**

***Index Terms*—Adaptive control, robust adaptive control, time delay.**

I. Introduction

ADAPTIVE control theory is a mature control discipline that has evolved over the past four decades and rigorously synthesized [1]–[3]. Researchers have made several attempts in extending the robustness properties of adaptive systems to time delays and unmodeled dynamics (see, for example, [4]–[8]) by introducing modifications to the underlying adaptive law. These results are either 1) semi-global, or 2) global where the delay margin can be shown to exist but is not otherwise computable, or the results are restricted to a small class of plants [8]. In contrast to such results, in this paper, we show that an adaptive system comprised of a single input plant whose states are accessible and an adaptive law modified with projection has an explicitly computable delay margin. That is, global boundedness of the overall adaptive system can be achieved for the system depicted in Fig. 1.

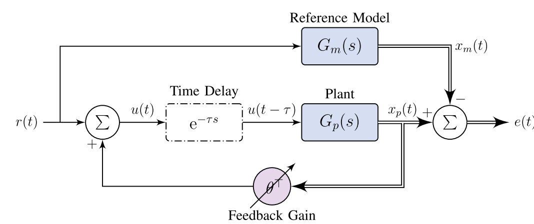

Fig. 1. Adaptive control in the presence of an input time delay.

Several successful adaptive control methods for time delay systems can be found in robust adaptive control literature. One of the first adaptive design methods is given in [9] for systems with input delays and uncertain parameters. A simpler adaptive controller for the same class of systems is proposed in [10]. In [11], by explicitly using future state prediction in the controller derivation and using partial states of the infinite dimensional system in the Lyapunov function, some limiting assumptions on plant dynamics such as the location and multiplicity of the poles are removed. In [12], an adaptive controller is developed for unknown input delays and uncertain parameters and in [13] both state and input delays are addressed. A comprehensive survey on the control of time delay systems, for literature before 2003, can be found in [14]. Extensions of predictor feedback to nonlinear and delay-adaptive systems with actuator dynamics modeled by partial differential equations can be found in [15].

The main contribution of this paper is a proof of robustness of an adaptive controller, for plants whose states are accessible, in the presence of time delays. This adaptive controller uses a conventional control architecture as in [4], an adaptive law that is modified using projection [6]–[8], [16], [17], and is shown to result in globally bounded solutions. Unlike [4] and [5], no normalization is used in the adaptive law. In this paper, unlike [9]–[15], we propose an adaptive controller that is robust to time delays rather than explicitly compensating for the effect of delays. Unlike the standard practice of Lyapunov function-based arguments which suffice for robustness with bounded disturbances, extensive arguments based on first principles are employed in order to prove boundedness. A preliminary version of this result appeared in [18], where the overall approach was first described. Unlike [18], our stability result here is complete, with clear insights provided on the delay margin.

In Section II, we pose the problem and describe the adaptive controller and the projection-based adaptive law. The main result is stated in Section III-E along with a few preliminaries, with its proof in Section IV. A detailed comparison of the main result with earlier work (for example, [5]) is provided in Section V. A numerical example with simulation studies is provided in Section VI to validate the result.

II. Problem Statement

An nth order plant with a scalar input and a parametric uncertainty is given by

$$\dot{x}_p(t) = A_p x_p(t) + b_m u(t-\tau) \tag{1}$$

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where A_p is unknown, b_m is known, and $\tau \geq 0$ is an unknown time delay. A reference model is chosen as

$$\dot{x}_m(t) = A_m x_m(t) + b_m r(t) \tag{2}$$

where A_m is Hurwitz, $x_m(t)$ specifies the desired response, and $r(t)$ is a bounded reference input. We suppose that a standard adaptive control input [4] is chosen as

$$u(t) = \theta^\top(t) x_p(t) + r(t). \tag{3}$$

When no delays are present, it has been shown that the standard adaptive law

$$\dot{\theta}(t) = -\Gamma x_p(t) b_m^\top P e(t) \tag{4}$$

ensures global boundedness and convergence of $x_p(t)$ to $x_m(t)$, where $\Gamma = \Gamma^\top > 0$, P is the solution of the Lyapunov equation $A_m^\top P + P A_m = -qI$, and $e(t) = x_p(t) - x_m(t)$ is the tracking error. The goal, in this paper, is to vary $\theta(t)$ so that the closed-loop adaptive system remains bounded for any initial conditions, in the presence of τ, and for $x_p(t)$ to track $x_m(t)$.

It is well known, from investigations in robust adaptive control over the past thirty years, that the standard adaptive law (4) does not suffice in guaranteeing robustness of adaptive systems to nonparametric perturbations such as external disturbances, unmodeled dynamics, and time delay. Several robustness modifications to the adaptive law were proposed in response (see, for example, [19]). The modification we propose utilizes the projection algorithm and is described in Section II-A.

A. Projection Algorithm

Let Ω_0 and Ω_1 be defined as

$$\begin{aligned} \Omega_0 &= \left\{\Theta \in \mathbb{R}^1 | -\theta'_{\max} \leq \Theta \leq \theta'_{\max}\right\} \\ \Omega_1 &= \left\{\Theta \in \mathbb{R}^1 | -\theta_{\max} \leq \Theta \leq \theta_{\max}\right\} \end{aligned} \tag{5}$$

where $\theta_{\max} > \theta'_{\max}$ are positive constants. We let $\varepsilon = \theta_{\max} - \theta'_{\max}$. A scalar projection algorithm $\mathrm{Proj}(\bullet, \bullet)$, can be defined as

$$\mathrm{Proj}(\Theta, y) = \begin{cases} \dfrac{\theta_{\max}^2 - \Theta^2}{\theta_{\max}^2 - \theta_{\max}'^2} y & \text{if } [\Theta \in \Omega_1 \backslash \Omega_0 \wedge y\Theta > 0] \\ y & \text{otherwise.} \end{cases} \tag{6}$$

The projection algorithm for a scalar Θ is then given by

$$\dot{\Theta} = \mathrm{Proj}(\Theta, y). \tag{7}$$

The following property can now be derived.

Lemma 1: For any time-varying piecewise continuous scalar y, if $\Theta(t_0) \in \Omega_1$ and $\dot{\Theta}$ is updated using the projection algorithm in (5)–(7), then $\Theta(t) \in \Omega_1$ for all $t \geq t_0$.

Remark 1: Lemma 1 implies that the solutions of (7) satisfy

$$|\Theta(t_0)| \leq \theta_{\max} \Rightarrow |\Theta(t)| \leq \theta_{\max}, \qquad \forall t \geq t_0. \tag{8}$$

That is, the projection algorithm in (7) guarantees the boundedness of the parameter $\Theta(t)$ independent of the system dynamics. We refer the reader to [7] and [20] for the proof of Lemma 1. In the subsequent section, we will describe how the projection algorithm in (7) is used to update the parameter $\theta(t)$ in (3).

III. Guaranteed Delay Margins for Adaptive Systems With State Variables Accessible

The following notations are used throughout: For a matrix $A \in \mathbb{R}^{n \times n}$, we define

$$\underline{\lambda}_A \triangleq \min_i |\Re(\lambda_i(A))|$$

$$\overline{\lambda}_A \triangleq \max_i |\Re(\lambda_i(A))|$$

where λ_i is the ith eigenvalue of A and $\Re(\lambda_i)$ denotes its real part. For any vector $x \in \mathbb{R}^{n \times 1}$, we refer to the ith component as x_i for each $i = \{0, \ldots, n-1\}$ and define

$$\overline{x} \triangleq \max_t \|x(t)\|$$

where $\|\cdot\| = \|\cdot\|_2$ represents the Euclidean norm. Similarly, for any scalar $x_i \in \mathbb{R}$, we denote $\overline{x}_i \triangleq \max_t |x_i(t)|$. Lastly, the $n-1$ subvector of x shall be defined as

$$x' \equiv [x_1 \quad x_2 \cdots x_{n-1}]^\top.$$

Before we proceed with the main theorem, we first present the specific adaptive law used to adjust the parameter $\theta(t)$ in (3). The adaptive update law used herein applies projection to a set of *transformed* states. The reason for this stems from the fact that the transformation collapses the analysis of an nth order system into only two key scalars, one each in $e(t)$ and $\theta(t)$, that are central to the proof of global boundedness. This transformation is presented in Section III-A. The adaptive law modified with projection is then introduced in Section III-B in light of the transformation. In Section III-C, a similarity transformation is employed on the reference model and its corresponding properties are discussed. The choice of projection parameters for the adaptive law is discussed in Section III-D. The main result is stated in Section III-E. Before proving the main result, which is done in Section IV, we present a few preliminaries in Section III-F and derive a few properties of the closed-loop adaptive system in Section III-G using the transformation in Section III-A.

A. Nonsingular Transformation

In this section, we will derive the nonsingular transformation matrices C and M that define the transformed error $\mathcal{E}(t)$ and transformed parameter $\vartheta(t)$ as

$$\mathcal{E}(t) \equiv Ce(t), \tag{9}$$

$$\vartheta(t) \equiv M\theta(t). \tag{10}$$

We recall that we will refer to the ith components of the transformed states as $\mathcal{E}_i(t)$ and $\vartheta_i(t)$, respectively, for $i = \{0, 1, \ldots, n-1\}$. The introduction of C and M are needed in order to identify crucial scalar states that capture the dominant effect of the time delay. We now describe the construction of C and M.

First, we begin with the vector

$$c_0 = \frac{Pb_m}{p_{bb}} \tag{11}$$

where P is the solution of the Lyapunov equation $A_m^\top P + PA_m = -Q$ and $p_{bb} \equiv \sqrt{b_m^\top P b_m}$. We note that

$$c_0^\top b_m = \frac{b_m^\top P}{p_{bb}} b_m = p_{bb}. \tag{12}$$

We then construct the $n-1$ vectors c_i for $i=\{1,2,\ldots,n-1\}$, such that

$$c_i^\top P^{-1} c_j = \begin{cases} 0 & i \neq j, \\ 1 & i = j \end{cases} \tag{13}$$

where $j=\{0,1,\ldots,n-1\}$. We therefore note that

$$c_i^\top b_m = c_i^\top P^{-1} c_0 p_{bb} = 0 \text{ for } i = 1,2,\ldots,n-1. \tag{14}$$

Therefore, an invertible matrix C is obtained by defining

$$C = \begin{bmatrix} c_0^\top & c_1^\top & \cdots & c_{n-1}^\top \end{bmatrix}^\top. \tag{15}$$

From (11), (13), and (15), it can be shown that

$$CP^{-1}C^\top = I \tag{16}$$

Lastly, using P and C in (15), we choose M as

$$M = p_{bb} C P^{-1}. \tag{17}$$

B. Modified Adaptive Law with the Projection Algorithm

The adaptive law we propose is of the form[1]

$$\dot{\theta} = M^{-1} w \tag{18}$$

where $w = [w_1 \; w_2 \; \ldots \; w_n]^\top$ and

$$w_i = \mathrm{Proj}\Big(\{M\theta\}_i, -\{M\Gamma x_p b_m^\top P e\}_i\Big) \tag{19}$$

with M in (17), and for the sake of simplicity, let $\Gamma = \gamma P$. The projection operator $\mathrm{Proj}(\bullet,\bullet)$, in (19), produces a scalar output with scalar arguments and is defined in (5)–(7). When projection is not active $(\mathrm{Proj}(\Theta, y) = y)$, the adaptive law given by (18) and (19) reduces to the standard adaptive law (4).

The implications of Lemma 1 on the boundedness of the control parameter θ are obvious. If the adaptive law is chosen as in (11)–(19), it follows from (8) that if $|\{M\theta(t_0)\}_i| \leq \theta_{i,\max}$, then $\{M\theta\}_i$ is bounded $(|\{M\theta(t)\}_i| \leq \theta_{i,\max})$ for all $t \geq t_0$.

C. Properties of the Reference Model

In this section, we define the transformed reference model and its corresponding properties using the transformation matrices given in the previous section. Let the scalars α_{ij} be defined as

$$\alpha_{ij} \equiv c_i^\top A_m P^{-1} c_j, \quad i,j = \{0,\ldots,n-1\} \tag{20}$$

and an $(n \times n)$ matrix

$$\mathcal{A}_m = C A_m P^{-1} C^\top. \tag{21}$$

We partition $\mathcal{A}_m$ as

$$\mathcal{A}_m = \begin{bmatrix} \alpha_{00} & a_1^\top \\ a_0 & \mathcal{A}'_m \end{bmatrix} \tag{22}$$

where $\mathcal{A}'_m$ is an $(n-1)\times(n-1)$ matrix. From (16), it follows that

$$P^{-1}C^\top = C^{-1} \tag{23}$$

which implies that (21) can be rewritten as

$$\mathcal{A}_m = C A_m C^{-1}. \tag{24}$$

[1]For ease of exposition, we suppress the argument "t" in what follows.

It follows immediately from (24) that the eigenvalues of $\mathcal{A}_m$ and those of A_m are identical since $\det(sI - \mathcal{A}_m) = \det(C)\det(sI - A_m)\det(C^{-1})$ and $\det(C) \neq 0$. Since A_m is Hurwitz, this implies that $\mathcal{A}_m$ is also Hurwitz.

In the following lemma, we will show that $\mathcal{A}'_m$ in (22) is Hurwitz.

Lemma 2: $\mathcal{A}'_m$ is Hurwitz.

We refer the reader to Appendix A for the proof of this lemma.

Remark 2: $\mathcal{A}_m$, as shown in (21), has a special structure with C chosen using (11), (13), and (15). While, in general, a Hurwitz matrix $\mathcal{X}$ need not have a Hurwitz submatrix $\mathcal{X}'$, because of the special structure of $\mathcal{A}_m$, it is proven in Appendix A that $\mathcal{A}'_m$ is Hurwitz.

D. Choice of Projection Algorithm Parameters

The adaptive update law modified with the projection algorithm in (19) requires $\theta_{i,\max}$ and $\theta'_{i,\max}$ to be specified. The former is defined as $\theta_{i,\max} = \theta'_{i,\max} + \varepsilon_i$, where $\varepsilon_i > 0$. The following discussion addresses the selection of $\theta'_{i,\max}$.

It is assumed that A_m in (2) is chosen such that there exists a $\theta^\star$ satisfying

$$A_p + b_p \theta^{\star\top} = A_m \tag{25}$$

for the plant in (1). In addition to that, the size of admissible parametric variation in A_p is assumed to be known (see, for example, (97) in Section VI). That is, we have *a priori* knowledge on the upper and lower bounds of the elements of $\theta^\star$. Therefore, we define such bounds in the transformed parameter space as

$$\theta^\star_{i,\max} = \max_{\theta^\star} \left|\vartheta^\star_i\right| \tag{26}$$

where $\vartheta^\star = M\theta^\star$ with M in (17) and $\theta^\star$ satisfying (25).

We then choose the parameter bounds $\theta'_{i,\max}$ for $i = \{0,1,\ldots,n-1\}$, such that

$$\theta'_{i,\max} \geq \theta^\star_{i,\max}. \tag{27}$$

It is important to note that (27) implies $\theta^\star_{i,\max} \in \Omega_0$ (5). For $i=0$,

$$\theta'_{0,\max} > \theta^\star_{0,\max} + \alpha_{00} + \frac{(\|P'a_0\| + (\|a_1\| + \phi'_{\max})p_\varphi)^2}{2p_\varphi \underline{\lambda}_{Q'}} \tag{28}$$

must be satisfied in addition to (27), where the constants α_{00}, a_0, and a_1 are defined in (22), P' is the solution of

$$\mathcal{A}'^\top_m P' + P'\mathcal{A}'_m = -Q' \tag{29}$$

with $Q' = Q'^\top > 0$,

$$\phi'_{\max} \equiv \sqrt{\sum_1^{n-1}\left(\theta_{i,\max} + \theta^\star_{i,\max}\right)^2} \tag{30}$$

and p_φ is an arbitrary positive constant. It should be noted that choosing projection bounds that satisfy (28) is always possible by taking sufficiently large $\theta'_{0,\max}$. The derivation of the second inequality constraint (28) on the choice of $\theta'_{0,\max}$ will become clear in Section IV-C. Lastly, we define

$$\Theta_{\max} \equiv \sqrt{\sum_{i=0}^{n-1}\theta^2_{i,\max}} \tag{31}$$

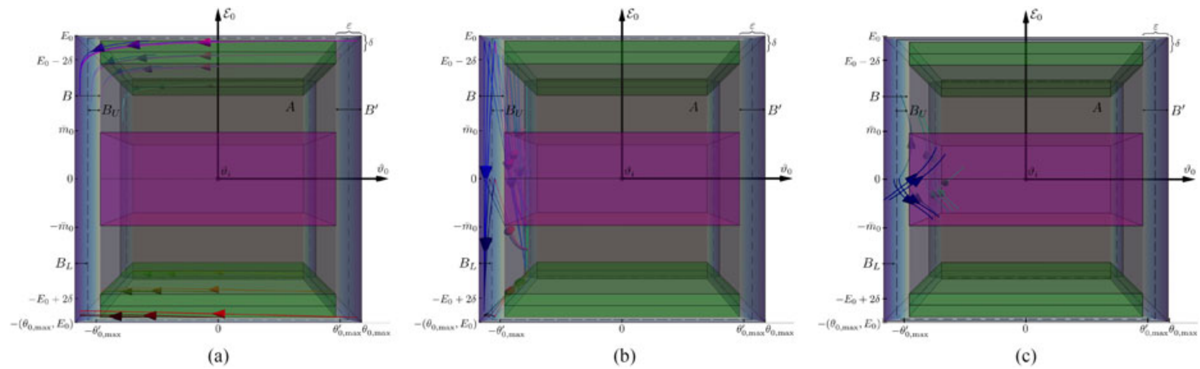

Fig. 2. Phases I–III of the trajectory, z, with boundary regions defined in Definition 1 and Definition 2. (a) Phase I: Entering the boundary. (b) Phase II: In the boundary region, B. (c) Phase III: Exiting the boundary.

and

$$\phi_{\max} \equiv \sqrt{\sum_{i=0}^{n-1} \left(\theta_{i,\max} + \theta^{\star}_{i,\max}\right)^2}. \tag{32}$$

E. Main Result

Theorem 1: There exists a $\tau^\star$ such that the closed-loop adaptive system with the plant in (1), reference model in (2), control law in (3), and adaptive law in (11)–(19) with projection parameters satisfying (27) and (28) has globally bounded solutions for all $\tau \in [0, \tau^\star]$ and any initial conditions $x_p(t) = \chi(t)$, $\theta(t) = \chi_\theta(t)$, $t \in [t_0 - \tau, t_0]$, where $\chi(t) : \mathbb{R} \to \mathbb{R}^n$, $\chi_\theta(t) : \mathbb{R} \to \Omega_1$.

Theorem 1 implies that the overall adaptive system with the projection algorithm in the adaptive law has a nonzero time delay margin $\tau^\star$. The proof of Theorem 1 is given in Section IV and consists of four phases denoted I through IV. The corresponding proof for the scalar case can be found in [21] and uses the same steps outlined in Section IV-A.

The main idea of the proof is as follows: There are two errors, the state error and the parameter error, that completely describe the adaptive system. The latter is guaranteed to be bounded by virtue of the projection algorithm, irrespective of the delay. Global boundedness of the state error, which is the main contribution of this paper, is proven using two major properties of the adaptive system. The first pertains to the behavior of the system trajectories when the parameter is in the boundary of the projection algorithm. The second considers the solutions of the system when the parameter is away from the projection boundary. In the second case, one can guarantee that the parameter will reach the boundary in finite time, which is the first major property. Once inside the projection boundary, the trajectories cannot become unbounded due to the stability of the underlying linear time-varying delay system, which is the second property. Together, these properties are shown to lead to global boundedness for all delays less than a certain bound which is the delay margin.

Before we proceed to the proof, we rewrite the closed-loop adaptive system using the transformation introduced in Section III-A. A few preliminaries are first presented.

F. Preliminaries

Prior to proving Theorem 1, we include a few definitions and specify a condition the trajectory will be shown to satisfy.

Definition 1. We define regions A, B, and B' as follows (see Fig. 2): Let $z(t) = [\mathcal{E}^\top(t)\ \ \vartheta^\top(t)]^\top$

$$A = \left\{z \in \mathbb{R}^{2n} \mid -\theta'_{0,\max} \le \vartheta_0 \le \theta'_{0,\max}\right\}$$
$$B = \left\{z \in \mathbb{R}^{2n} \mid -\theta_{0,\max} \le \vartheta_0 < -\theta'_{0,\max}\right\}$$
$$B' = \left\{z \in \mathbb{R}^{2n} \mid \theta'_{0,\max} < \vartheta_0 \le \theta_{0,\max}\right\}.$$

Definition 2. We further divide the boundary region $\boldsymbol{B}$ into two regions as follows (see Fig. 2):

$$B_L = \left\{z \in \mathbb{R}^{2n} \mid -\theta_{0,\max} \le \vartheta_0 \le -(\theta'_{0,\max} + \varepsilon_0/2)\right\}$$
$$B_U = \left\{z \in \mathbb{R}^{2n} \mid -(\theta'_{0,\max} + \varepsilon_0/2) \le \vartheta_0 < -\theta'_{0,\max}\right\}.$$

We note that $B = B_L \cup B_U$, and that A, B_L, B_U, and B' are all regions in $\mathbb{R}^{2n}$ that lie between two hyperplanes. All of these hyperplanes are specified using only one scalar state variable ϑ_0.

Let positive constants δ and E_0 be defined by

$$\delta \in (0,\, 1] \tag{33}$$

and

$$E_0 = \max\left(\max_{t\in[t_0-\tau,t_0]} |\mathcal{E}_0(t)| + \delta,\ \frac{16}{\delta\gamma}(\Theta^2_{\max} + \gamma)(1 + \overline{m}_0),\ \beta\right) \tag{34}$$

where $\overline{m}_0 \equiv \max_{t\ge t_0} \left|c_0^\top x_m(t)\right|$ and $\beta > 0$ is specified later in Lemma 4. From the definitions of E_0 and δ, it can be shown that $E_0 - 2\delta > \overline{m}_0$. We also define a positive constant E' as

$$E' = \max\left(\sqrt{\frac{\overline{\lambda}_{P'}}{\underline{\lambda}_{P'}}} \max_{t\in[t_0-\tau,t_0]} \|\mathcal{E}(t)\|,\ \sqrt{\frac{\ell^2 r_p}{1-\ell^2 r_p}} E_0\right) \tag{35}$$

where $r_p > 1$ and positive constant ℓ, which is specified later in Proposition 1, satisfies $\sqrt{\frac{\ell^2 r_p}{1-\ell^2 r_p}} < 1$. From the definition of E', it follows that

$$E' < E_0. \tag{36}$$

Using r_p, E_0, and E', we further define

$$E = \sqrt{r_p}\sqrt{E_0^2 + E'^2}. \tag{37}$$

Since $r_p > 1$, it is obvious that

$$E > E_0. \tag{38}$$

Also from the definitions of E' and E, it can be proven that

$$\ell E \leq E'. \tag{39}$$

Condition 1. Given $\tau > 0$, $\pi(t) \in \mathbb{R}^n$ is said to satisfy Condition 1 at time $t_a \geq t_0$ if the following conditions

$$|\pi_0(t)| \leq E \qquad \forall t \in [t_a - \tau, t_a], \tag{40}$$

$$|\pi_0(t_a)| = E_0 - \delta, \tag{41}$$

$$\pi'^\top(t_a - \tau)P'\pi'(t_a - \tau) \leq \underline{\lambda}_{P'} E'^2 \tag{42}$$

are satisfied, where P' is the solution to (29), $E_0 \in \mathbb{R}$ is given in (34), δ in (33), and $E' \in \mathbb{R}$ as in (35) are positive constants with $E_0 - \delta > 0$.

G. Transformed Adaptive System Dynamics

We now return to the overall adaptive system. The closed-loop adaptive system with the plant in (1), reference model in (2), and control law in (3) has error dynamics equivalent to

$$\dot{e} = A_m e + b_m \left((\theta^\top - \theta^{\star\top})(e + x_m) + \eta\right) \tag{43}$$

where η represents the perturbation due to the time delay and is defined as

$$\eta(t) = u(t - \tau) - u(t). \tag{44}$$

The adaptive update law in (18) and (19) can be rewritten as

$$\{M\dot{\theta}\}_i = \text{Proj}\left(\{M\theta\}_i, -\{M\Gamma(e + x_m)b_m^\top Pe\}_i\right). \tag{45}$$

We first note that since $|\chi_{\theta i}(t)| \leq \theta_{i,\max}$, it follows from Lemma 1 that $|\vartheta_i(t)| \leq \theta_{i,\max}\ \forall t \geq t_0$. Theorem 1 is therefore proved if the global boundedness of e is demonstrated. In the following sections, Sections III-G1 and III-G2, the transformed error and parameter dynamics are further discussed.

1) Transformed Error Dynamics: In order to prove global boundedness of e, we will utilize the transformed error $\mathcal{E}$ introduced in (9). It is obvious that the global boundedness of e is demonstrated if the global boundedness of $\mathcal{E}$ is shown. In this section, we will derive the dynamics of $\mathcal{E}$. We note that $c_i^\top$ is the ith row vector of C. It follows from (9) that for $i = \{0, 1, \ldots, n-1\}$

$$\dot{\mathcal{E}}_i = c_i^\top \dot{e}. \tag{46}$$

Using the properties in (14) and (16), we can rewrite P in quadratic form as

$$\sum_{j=0}^{n-1} c_j c_j^\top = P. \tag{47}$$

It then follows from (43) and (47), with some algebraic manipulation, that

$$\dot{\mathcal{E}}_i = c_i^\top A_m I e = c_i^\top A_m P^{-1}\left(\sum_{j=0}^{n-1} c_j c_j^\top\right) e \tag{48}$$

for $i = \{1, 2, \ldots, n-1\}$. Noting the definition of α_{ij} in (20), (48) can be rewritten as $\dot{\mathcal{E}}_i = \sum_{j=1}^{n-1} \alpha_{ij}\mathcal{E}_j + \alpha_{i0}\mathcal{E}_0$. The definition of $\mathcal{A}'_m$ in (21) and a_0 in (22) imply that the subvector $\mathcal{E}'$ of $\mathcal{E}$ given by $\mathcal{E}' \equiv [\mathcal{E}_1\ \mathcal{E}_2\ \ldots\ \mathcal{E}_{n-1}]$ satisfies the error dynamics

$$\dot{\mathcal{E}}' = \mathcal{A}'_m \mathcal{E}' + a_0 \mathcal{E}_0. \tag{49}$$

We now return to (46) and consider the special case when $i = 0$. Using the property in (12) and the definition of α_{ij} in (20), the dynamics of the critical state error $\mathcal{E}_0$ can be obtained from (43) as

$$\begin{aligned}\dot{\mathcal{E}}_0 &= c_0^\top A_m e + p_{bb}\left(\theta^\top - \theta^{\star\top}\right)(e + x_m) + p_{bb}\eta \\ &= \sum_{j=0}^{n-1} \alpha_{0j}\mathcal{E}_j + p_{bb}\left(\theta^\top - \theta^{\star\top}\right)(e + x_m) + p_{bb}\eta. \end{aligned}\tag{50}$$

Defining

$$m_i \equiv c_i^\top x_m \tag{51}$$

and from (47) and (10), the error equation (50) can be rewritten as

$$\begin{aligned}\dot{\mathcal{E}}_0 &= \sum_{j=0}^{n-1} \alpha_{0j}\mathcal{E}_j + \sum_{j=0}^{n-1}\left(\vartheta_j - \vartheta_j^\star\right)\left(\mathcal{E}_j + m_j\right) + p_{bb}\eta \\ &= \left(\alpha_{00} + \vartheta_0 - \vartheta_0^\star\right)\mathcal{E}_0 + \left(\vartheta_0 - \vartheta_0^\star\right)m_0 + p_{bb}\eta \\ &\quad + \left(a_1 + \vartheta' - \vartheta'^\star\right)^\top \mathcal{E}' + \left(\vartheta' - \vartheta'^\star\right)^\top m'. \end{aligned}\tag{52}$$

Since $x_m(t)$ is known to be bounded, boundedness of $m_i(t)$ is straightforward from (51).

Equations (49) and (52) represent the transformed tracking error dynamics $\mathcal{E}$. These equations show that the perturbation η due to the time delay τ appears only in the dynamics of $\mathcal{E}_0$ and not in $\mathcal{E}_i$ for all $i = \{1, 2, \ldots, n-1\}$.

In what follows, we will relate the boundedness of $\mathcal{E}'$ to that of $\mathcal{E}_0$ using Lemma 2.

Proposition 1: Suppose

$$|\mathcal{E}_0(t)| \leq W, \quad t \in \mathcal{T}_s = [t_s, t_{ss}] \tag{53}$$

where $t_{ss} > t_s \geq t_0$. Then

$$V'(t) \leq \max\left(V'(t_s), \tfrac{1}{2}\underline{\lambda}_{P'}(\ell W)^2\right) \quad \forall t \in \mathcal{T}_s \tag{54}$$

where the quadratic function $V'(t)$ is defined as

$$V'(t) = \tfrac{1}{2}\mathcal{E}'^\top(t)P'\mathcal{E}'(t) \tag{55}$$

with $P' > 0$ satisfying (29) and positive constant ℓ defined as

$$\ell = \frac{2\overline{\lambda}_{P'}^2\|a_0\|}{\underline{\lambda}_{P'}\underline{\lambda}_{Q'}}. \tag{56}$$

Proof: Since $\mathcal{A}'_m$ is Hurwitz, for any positive definite symmetric matrix Q' there exists $P' = P'^\top > 0$ which satisfies the Lyapunov equation in (29). Considering the Lyapunov-like function in (55), and taking the derivative with respect to time, we obtain

$$\dot{V'} \leq -\tfrac{1}{2}\min_i\left(\Re\left(\lambda_i(Q')\right)\right)\|\mathcal{E}'\|^2 + \|P'a_0\|W\|\mathcal{E}'\|. \tag{57}$$

Noting that

$$\tfrac{1}{2}\underline{\lambda}_{P'}\|\mathcal{E}'(t)\|^2 \leq V'(t) \leq \tfrac{1}{2}\overline{\lambda}_{P'}\|\mathcal{E}'(t)\|^2 \tag{58}$$

(57) can be simplified as $\dot{V}' \leq -k_1 V' + k_2\sqrt{V'}$, where $k_1 = \frac{\underline{\lambda}_{Q'}}{\overline{\lambda}_{P'}}$, $k_2 = \frac{\sqrt{2}\overline{\lambda}_{P'}\|a_0\|W}{\sqrt{\underline{\lambda}_{P'}}}$. Defining $\Delta_1 = \frac{k_1}{2}$ and $\Delta_2 = \frac{k_2^2}{(4\Delta_1)} =$

$\frac{k_2^2}{(2k_1)}$, $\Delta_1 V' + \Delta_2 \geq k_2\sqrt{V'}$ and therefore we obtain

$$\dot{V}' \leq -\frac{k_1}{2}V' + \frac{k_2^2}{2k_1}. \tag{59}$$

Equation (59) implies that $\dot{V}'(t) \leq 0$ if $V'(t) \geq K_1$, where $K_1 = (\frac{k_2}{k_1})^2 = \frac{1}{2}\underline{\lambda}_{P'}(\ell W)^2$. This proves Proposition 1. ■

Corollary 1.1. Suppose (53) is satisfied, where $t_{ss} > t_s \geq t_0$. Then

$$\underline{\lambda}_{P'}\|\mathcal{E}'(t)\|^2 \leq \max\left(\mathcal{E}'^\top(t_s)P'\mathcal{E}'(t_s), \underline{\lambda}_{P'}(\ell W)^2\right) \quad \forall t \in \mathcal{T}_s. \tag{60}$$

Proof: From Proposition 1 and (58), (60) follows. ■

2) Transformed Parameter Error Dynamics: Similar to Section III-G1, we now focus on the transformed parameter error $\vartheta(t)$ in (10). From (45), letting Γ be defined as $\Gamma = \gamma P$, and noting that $\{M\theta\}_i = \vartheta_i$ with M in (17), we obtain

$$\begin{aligned}\dot{\vartheta}_i &= \text{Proj}\left(\vartheta_i, -\gamma p_{bb} c_i^\top (e + x_m) b_m^\top P e\right)\\ &= \gamma p_{bb}\,\text{Proj}\left(\vartheta_i, -(\mathcal{E}_i + m_i) b_m^\top P e\right)\end{aligned}$$

for $i = \{0, \ldots, n-1\}$. We also note that $b_m^\top P e = p_{bb} c_0^\top e = p_{bb}\mathcal{E}_0$ from (9) and (11). Therefore,

$$\dot{\vartheta}_i = \gamma'\,\text{Proj}\left(\vartheta_i, -(\mathcal{E}_i + m_i)\mathcal{E}_0\right), \quad i = \{0, \ldots, n-1\} \tag{61}$$

where $\gamma' = \gamma p_{bb}^2$. We further examine (61) for $i = 0$ in more detail since it was observed in the previous section that $\mathcal{E}_0$ contains η, making the zeroth states of particular interest. From (6), it follows that

$$\dot{\vartheta}_0 = \begin{cases} -\dfrac{\vartheta_{0,\max}^2 - \vartheta_0^2}{\vartheta_{0,\max}^2 - \vartheta_{0,\max}'^2}\gamma'(\mathcal{E}_0 + m_0)\mathcal{E}_0 \\ \qquad \text{if } [z_0 \in (\underline{B} \cup \overline{B}) \wedge -(\mathcal{E}_0 + m_0)\mathcal{E}_0\vartheta_0 > 0] \\ -\gamma'(\mathcal{E}_0 + m_0)\mathcal{E}_0 \qquad \text{otherwise.}\end{cases} \tag{62}$$

It is observed that $\dot{\vartheta}_0 < 0$ when $|\mathcal{E}_0| > \overline{m}_0$ with m_0 in (51). Equation (61) for $i = \{1, 2, \ldots, n-1\}$ and (62) constitute the complete adaptive law.

IV. Proof of the Main Result

From the discussions in Section III-G, it is clear that the overall adaptive system dynamics can be defined with the transformed error $\mathcal{E}$ and the transformer parameter ϑ. The former is given by (49) and (51)–(52), and the latter by (61).

Of the $2n$ states $\mathcal{E}$ and ϑ, two scalar states $\mathcal{E}_0$ and ϑ_0 are shown to be crucial in achieving global boundedness. The reason for this is because η appears explicitly in the dynamics of $\mathcal{E}_0$ only. That is, η does not explicitly appear in the dynamics of $\mathcal{E}_i$ $\forall i \geq 1$. Another interesting observation can be made when considering the parameter dynamics. It follows from (61) that for all $i \geq 1$, $\dot{\vartheta}_i$ depends linearly on $\mathcal{E}_0$. That is, $\dot{\vartheta}_0$ is the only parameter that depends nonlinearly on $\mathcal{E}_0$. The effect of such features is prominently used throughout the proof and will become clear in the following sections.

A. Outline of the Proof

The proof is completed using the following four phases.

(I) The transformed error $\mathcal{E}(t)$ satisfies Condition 1 for some $t = t_a$; this implies that the state z has to enter B at $t_b \in (t_a, t_a + \Delta T_{\text{in},\max})$, where $\Delta T_{\text{in},\max} > 0$ is a finite constant [see Fig. 2(a)].

(II) When the trajectory enters B, the parameter enters the boundary of the projection algorithm; $\mathcal{E}$ is shown to be bounded by making use of the underlying linear time-varying system [see Fig. 2(b)].

(III) There exists $\Delta T_{\text{out},\min}$, such that the trajectory reenters A at $t_c > t_b + \Delta T_{\text{out},\min}$ with $|\mathcal{E}_0(t_c)| < \overline{m}_0$ [see Fig. 2(c)].

(IV) The trajectory has only two options: (A) $|\mathcal{E}_0(t)| < E_0 - \delta$ $\forall t > t_c$ proving Theorem 1, or (B) $\mathcal{E}_0(t)$ satisfies Condition 1 for some $t_d > t_c$. If the latter case holds, we replace t_a by t_d and repeat Phases I through IV.

In the following sections, we prove Phases I–IV in detail. Lemmas and propositions are introduced as needed in order to prove these phases. Proofs of lemmas are provided in the Appendix, unless otherwise noted, while proofs of propositions are retained in the main text.

B. Proof of Phase I: Entering the Boundary B.

We will prove the following proposition in this section.

Proposition 2: Let $\mathcal{E}(t)$ satisfy Condition 1 at $t = t_a$ with δ, E_0, E' given in (33), (34), (35), respectively and $z(t_a) \in A$ where $z = \left[\mathcal{E}^\top\ \vartheta^\top\right]^\top$. Then

(i) $|\mathcal{E}_0(t)| < E_0$, $\forall t \in [t_a, t_a + \Delta T]$

(ii) $\exists t_b' \in [t_a, t_a + \Delta T]$, such that $z(t_b') \in B_L$

where

$$\Delta T = \frac{\delta}{b_0 E + b_1} \tag{63}$$

with

$$\begin{aligned} b_0 &= B + B' \\ b_1 &= \left(\phi_{\max} + 2\frac{\overline{\lambda}_c}{\underline{\lambda}_c}\Theta_{\max}\right)\overline{m} + 2p_{bb}\overline{r} \\ B &= |\alpha_{00}| + |\vartheta_0^\star| + \left(1 + 2\frac{\overline{\lambda}_c}{\underline{\lambda}_c}\right)\Theta_{\max} \\ B' &= \|a_1\| + \|\vartheta'^\star\| + \left(1 + 2\frac{\overline{\lambda}_c}{\underline{\lambda}_c}\right)\Theta_{\max}. \end{aligned} \tag{64}$$

Proof of Proposition 2(i): We note from (52) that

$$\begin{aligned}|\dot{\mathcal{E}}_0(t)| \leq{}& |a_{00} + \vartheta_0(t) - \vartheta_0^\star||\mathcal{E}_0(t)| + |\vartheta_0(t) - \vartheta_0^\star||m_0(t)| \\ &+ p_{bb}|\eta(t)| + \|a_1 + \vartheta'(t) - \vartheta'^\star\|\|\mathcal{E}'(t)\| \\ &+ \|\vartheta'(t) - \vartheta'^\star\|\|m'(t)\|. \end{aligned} \tag{65}$$

From (44) and (3) it can be shown that

$$|\eta(t)| \leq \frac{2}{p_{bb}}\frac{\overline{\lambda}_c}{\underline{\lambda}_c}\Theta_{\max}\left(\max_{[t-\tau,t]}\|\mathcal{E}(t)\| + \overline{m}\right) + 2\overline{r}. \tag{66}$$

From (65) together with (66), it follows after elaborate algebraic manipulations that

$$|\dot{\mathcal{E}}_0(t)| \leq B\hat{\mathcal{E}}_0 + B'\hat{\mathcal{E}}' + b_1 \forall t \in [t_a, t_a + \Delta T] \quad (67)$$

where

$$\hat{\mathcal{E}}_0 = \max_{t\in[t_a-\tau, t_a+\Delta T]} |\mathcal{E}_0(t)|, \quad \hat{\mathcal{E}}' = \max_{t\in[t_a-\tau, t_a+\Delta T]} \|\mathcal{E}'(t)\|. \quad (68)$$

By applying Proposition 1, with $t_a - \tau$ replacing t_s, $t_a + \Delta T$ replacing t_{ss}, and $\hat{\mathcal{E}}_0$ replacing W, we obtain that

$$\mathcal{E}'^{\top}(t)P'\mathcal{E}'(t) \leq \max\left(\mathcal{E}'^{\top}(t_a-\tau)P'\mathcal{E}'(t_a-\tau), \underline{\lambda}_{P'}(\ell\hat{\mathcal{E}}_0)^2\right) \quad (69)$$

$\forall t \in [t_a - \tau, t_a + \Delta T]$. Since $\mathcal{E}(t)$ satisfies Condition 1 (42) at $t = t_a$, the right-hand side can be simplified to obtain $\mathcal{E}'^{\top}(t)P'\mathcal{E}'(t) \leq \max(\underline{\lambda}_{P'}E'^2, \underline{\lambda}_{P'}(\ell\hat{\mathcal{E}}_0)^2)$ for all $t \in [t_a - \tau, t_a + \Delta T]$. Noting the definition of $\hat{\mathcal{E}}'$ in (68), we therefore obtain

$$\hat{\mathcal{E}}' \leq \sqrt{\frac{1}{\underline{\lambda}_{P'}} \max\left(\underline{\lambda}_{P'}E'^2, \underline{\lambda}_{P'}(\ell\hat{\mathcal{E}}_0)^2\right)}.$$

Since $\ell < 1$ and $E' < E_0$ (36), it follows that

$$\hat{\mathcal{E}}' \leq \max\left(E_0, \hat{\mathcal{E}}_0\right). \quad (70)$$

From (70), it can be shown that there are two possible cases: (A) $E_0 \leq \hat{\mathcal{E}}_0$ and (B) $E_0 > \hat{\mathcal{E}}_0$.

Case (A): Condition of case (A) and (70) implies that $\hat{\mathcal{E}}' \leq \hat{\mathcal{E}}_0$. This allows us to simplify (67) as

$$|\dot{\mathcal{E}}_0(t)| \leq b_0\hat{\mathcal{E}}_0 + b_1 \forall t \in [t_a, t_a + \Delta T] \quad (71)$$

where $b_0 \equiv B + B'$. Noting that $\forall \Delta t \in [0, \Delta T]$

$$|\mathcal{E}_0(t_a + \Delta t)| \leq |\mathcal{E}_0(t_a)| + \max_{t\in[t_a, t_a+\Delta T]} |\dot{\mathcal{E}}_0(t)|\Delta T. \quad (72)$$

From (71), the definition of ΔT in (63), and (41) in Condition 1 which is satisfied for $t = t_a$, it follows that $|\mathcal{E}_0(t_a + \Delta t)| \leq (E_0 - \delta) + \delta(1 + \frac{b_0(\hat{\mathcal{E}}_0 - E)}{b_0 E + b_1})$. Therefore

$$\max_{t\in[t_a, t_a+\Delta T]} |\mathcal{E}_0(t)| \leq E_0 + b_0\Delta T(\hat{\mathcal{E}}_0 - E). \quad (73)$$

Noting the definition of $\hat{\mathcal{E}}_0$ in (68) and since $\mathcal{E}_0(t)$ satisfies (40), $\hat{\mathcal{E}}_0 = \max\{E, \max_{t\in[t_a, t_a+\Delta T]} |\mathcal{E}_0(t)|\}$ and therefore there are only two possible cases: (A-a) $\hat{\mathcal{E}}_0 = E$ and (A-b) $\hat{\mathcal{E}}_0 > E$.

If (A-a) holds, it immediately implies from (73) that Proposition 2(i) is true. If we suppose case (A-b) holds, it implies $\hat{\mathcal{E}}_0 = \max_{t\in[t_a, t_a+\Delta T]} |\mathcal{E}_0(t)|$ and from (73) it follows that $(1 - b_0\Delta T)\hat{\mathcal{E}}_0 \leq E_0 - b_0\Delta T E$. Noting $E > E_0$ and $1 - b_0\Delta T > 0$, we can therefore obtain $\hat{\mathcal{E}}_0 < \frac{1-b_0\Delta T}{1-b_0\Delta T}E_0 = E_0 < E$. This contradicts the condition of the case and therefore we obtain $\hat{\mathcal{E}}_0 = E$.

Case (B): Condition of case (B) and (70) implies that $\hat{\mathcal{E}}' \leq E_0$. This allows us to simplify (67) as

$$|\dot{\mathcal{E}}_0(t)| \leq b_0 E_0 + b_1, \quad \forall t \in [t_a, t_a + \Delta T].$$

Noting that (72) $\forall \Delta t \in [0, \Delta T]$, we therefore obtain using (63) and (41) that

$$|\mathcal{E}_0(t_a + \Delta t)| \leq (E_0 - \delta) + \delta\frac{b_0 E_0 + b_1}{b_0 E + b_1} < E_0$$

which again implies that Proposition 2(i) is true. ■

Proof of Proposition 2(ii): Equation (67) together with (70) gives

$$|\dot{\mathcal{E}}_0(t)| \leq b_0 \max\left(E_0, \hat{\mathcal{E}}_0\right) + b_1, \quad \forall t \in [t_a, t_a + \Delta T].$$

Thus, since $E \geq \max(E_0, \hat{\mathcal{E}}_0)$ from the proof of Proposition 2(i), $|\mathcal{E}_0(t)| \geq |\mathcal{E}_0(t_a)| - (b_0 E + b_1)\Delta T$ for all $t \in [t_a, t_a + \Delta T]$ which can be simplified, using the fact that $\mathcal{E}_0(t)$ satisfies (41), as $|\mathcal{E}_0(t)| \geq E_0 - 2\delta$ for all $t \in [t_a, t_a + \Delta T]$. From the choices of δ and E_0 in (33) and (34), it can be shown that $E_0 - 2\delta > \overline{m}_0$. Hence,

$$|\mathcal{E}_0(t)| > \overline{m}_0, \quad \forall t \in [t_a, t_a + \Delta T].$$

From (62), this in turn implies that $\dot{\vartheta}_0(t)$ is negative and

$$\begin{aligned} -\dot{\vartheta}_0(t) &\geq \gamma'|\mathcal{E}_0(t)|(|\mathcal{E}_0(t)| - |m_0(t)|) \\ &\geq \gamma'(E_0 - 2\delta)((E_0 - 2\delta) - \overline{m}_0), \quad \forall t \in T_A \end{aligned} \quad (74)$$

where T_A is defined as $T_A : \{t \mid z(t) \in \text{A and } t \in [t_a, t_a + \Delta T]\}$. From (74), it follows that

$$\vartheta_0(t_a) - \vartheta_0(t_a + \Delta t) \geq \gamma'(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0)\Delta t \quad (75)$$

for all $\Delta t \in [0, \Delta T]$ satisfying $[t_a, t_a + \Delta t] \subset T_A$. Hence, defining

$$\Delta T_{\text{in,max}} = \frac{2\theta_{0,\max}}{\gamma'(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0)} \quad (76)$$

and if $\Delta T_{\text{in,max}} \leq \Delta T$, from (75), (8) and the definition of regions A and B, it follows that $z(t)$ enters B at $t_b \in (t_a, t_a + \Delta T_{\text{in,max}})$.

We now show that $z(t)$ enters B_L at $t < t_a + \Delta T'_{\text{in,max}}$ for some $\Delta T'_{\text{in,max}} > \Delta T_{\text{in,max}}$. First, it can be proven that $|\operatorname{Proj}(\theta, y)| > \frac{1}{2}|y| \;\; \forall z \in B_U$. Using similar arguments as above, it can be shown that

$$-\dot{\vartheta}_0(t) > \tfrac{\gamma'}{2}(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0) \quad \forall t \in T_{BU} \quad (77)$$

where T_{BU} is defined as $T_{BU} : \{t \mid z(t) \in B_U \text{ and } t \in [t_a, t_a + \Delta T]\}$. Noting Definition 2, the maximum time that $z(t)$ can spend in B_U can be derived, using (77), to be $\{\varepsilon_0/2\}/\{\frac{\gamma'}{2}(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0)\}$. This implies that $z(t)$ enters region B_L at $t \in (t_a, t_a + \Delta T'_{\text{in,max}})$ where

$$\begin{aligned} \Delta T'_{\text{in,max}} &= \Delta T_{\text{in,max}} + \frac{\varepsilon_0/2}{\gamma'(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0)/2} \\ &= \frac{2\theta_{0,\max} + \varepsilon_0}{\gamma'(E_0 - 2\delta)(E_0 - 2\delta - \overline{m}_0)} \end{aligned}$$

if $\Delta T'_{\text{in,max}} \leq \Delta T$, since then (77) is satisfied for all $t \in (t_b, t_a + \Delta T'_{\text{in,max}}]$. From (34) $E_0 \geq \frac{16}{\delta\gamma'}(\theta^2_{0,\max} + \gamma')(1 + \overline{m}_0)$ and together with (33), it can be shown using algebraic manipulations that $\Delta T'_{\text{in,max}} < \Delta T$ is implied. This proves Proposition 2(ii). ■

C. Proof of Phase II: In the Boundary Region B.

We return to the overall adaptive system. The closed-loop error dynamics in (43) can be rewritten in the transformed space as

$$\dot{\mathcal{E}} = CA_m C^{-1}\mathcal{E} + Cb_m \left(\theta^\top - \theta^{\star\top}\right) C^{-1}(\mathcal{E} + m) + Cb_m \eta. \quad (78)$$

From (9), (10), (21), and noting that $\theta^\top = p_{bb}^{-1}\vartheta^\top C$ from (17) and $\frac{1}{p_{bb}} Cb_m = [1 \;\; 0_{1\times(n-1)}]^\top$ from (12) and (14), (78) can be rewritten as

$$\dot{\mathcal{E}} = M_0\mathcal{E} + M_1\mathcal{E}(t-\tau) + R \quad (79)$$

where the matrices M_0, M_1, and the vector R are defined as

$$\begin{aligned} M_0 &\equiv \mathcal{A}_m - c_I\vartheta^{\star\top} \\ M_1 &\equiv c_I\vartheta^\top(t-\tau) \\ R &\equiv c_I\vartheta^\top(t-\tau)m(t-\tau) - c_I\vartheta^{\star\top}m + p_{bb}c_I\left(r(t-\tau) - r\right) \\ c_I &= [1 \quad 0 \quad \cdots \quad 0]^\top. \end{aligned} \quad (80)$$

Using the error dynamics derived above (79), we continue with the proof of Theorem 1, Phase II.

When the trajectory enters B, the $i = 0$ parameter is in the boundary of the projection algorithm. Let the trajectory stay in B for $t \in (t_b, t_c)$ for some $t_c > t_b$. From the definition of B, it follows that

$$\vartheta_0(t) = -\theta'_{0,\max} - \epsilon(t), \quad \forall t \in (t_b, t_c) \quad (81)$$

where $\epsilon(t) \in (0, \varepsilon_0]$.

We show below that $\mathcal{E}(t)$ is guaranteed to converge to a bounded set if the trajectory remains in B. Before we proceed to this result, we study the properties of $M_0 + M_1$ while in B. Let us define the following set: $\Omega_B = \{(M_0, M_1) \mid z \in B\}$.

Lemma 3: There exists a $\underline{q} > 0$ such that

$$(M_0 + M_1)^\top\mathcal{P} + \mathcal{P}(M_0 + M_1) < -\underline{q}I \quad (82)$$

is satisfied for all $(M_0, M_1) \in \Omega_B$, where $\mathcal{P}$ is a constant matrix defined as

$$\mathcal{P} = \mathcal{I}^\top\mathcal{R}\mathcal{I} \quad (83)$$

with

$$\mathcal{R} = \begin{bmatrix} P' & 0 \\ 0 & p_\varphi \end{bmatrix}, \; \mathcal{I} = \begin{bmatrix} 0_{1\times(n-1)} & 1 \\ I_{(n-1)\times(n-1)} & 0_{(n-1)\times 1} \end{bmatrix} \quad (84)$$

where P' satisfies (29) and p_φ is an arbitrary positive constant.

The choice of the projection parameters satisfying (28) is used to prove this lemma (see Appendix A). Lemma 3 proves a key property, (3), of the time-varying system (79)–(81).

Lemma 4: Consider the uncertain time-varying system (79)–(81) with the selection of the projection parameters satisfying (28). Let the solutions of the system lie in B for $t \in (t_b, t_c)$. Then there exists $\overline{\tau}$ and $\beta > 0$, such that for any $\tau \leq \overline{\tau}$

$$V(\mathcal{E}(t)) \leq \max\left\{V(\mathcal{E}(t_b)), \overline{\lambda}_{\mathcal{P}}\beta^2\right\}, \quad \forall t \in (t_b, t_c) \quad (85)$$

where

$$V(\mathcal{E}) = \mathcal{E}^\top\mathcal{P}\mathcal{E}. \quad (86)$$

Lemma 4 is a vector version of [21, Theorem 2] and its proof is built upon [22, Proposition 6.7] which utilizes Lemma 3, model transformation, and the Razumikhin Theorem. See Appendix A for the proof of Lemma 4.

We conclude this section with the following proposition.

Proposition 3: If $\tau \leq \overline{\tau}$, then $\|\mathcal{E}(t)\| < E$ for all $t \in [t_b, t_c)$.

Proof: From Lemma 4, for all $t \in [t_b, t_c)$

$$\begin{aligned} V(\mathcal{E}(t)) &\leq \max\left\{V(\mathcal{E}(t_b)), \overline{\lambda}_{\mathcal{P}}\beta^2\right\} \\ &\leq \max\left\{\overline{\lambda}_{\mathcal{P}}\left(\mathcal{E}_0(t_b)^2 + \|\mathcal{E}'(t_b)\|^2\right), \overline{\lambda}_{\mathcal{P}}\beta^2\right\}. \end{aligned} \quad (87)$$

We note from Proposition 2 that $|\mathcal{E}_0(t_b)| < E_0$. Also applying Corollary 1.1 (60) with $t_s = t_a - \tau$, $t_{ss} = t_b$, $W = E_0$ and noting that Condition 1 (42) is satisfied at $t = t_a$, it can be shown that $\|\mathcal{E}'(t_b)\| \leq \max(E', \ell E_0)$. Therefore, (87) can be simplified as

$$V(\mathcal{E}(t)) \leq \overline{\lambda}_{\mathcal{P}} \max\left\{\left(E_0^2 + \max(E'^2, \ell^2E_0^2)\right), \beta^2\right\}.$$

Furthermore, from the definition of E_0 (34), $E_0 \geq \beta$. Also from (38) and (39), $E' > \ell E_0$. Therefore, we obtain

$$V(t) \leq \overline{\lambda}_{\mathcal{P}}\left(E_0^2 + E'^2\right), \quad \forall t \in [t_b, t_c). \quad (88)$$

Noting that $\underline{\lambda}_{\mathcal{P}}\|\mathcal{E}(t)\|^2 \leq V(t) \leq \overline{\lambda}_{\mathcal{P}}\|\mathcal{E}(t)\|^2$, (88) implies that

$$\|\mathcal{E}(t)\| \leq \sqrt{\frac{\overline{\lambda}_{\mathcal{P}}\left(E_0^2 + E'^2\right)}{\underline{\lambda}_{\mathcal{P}}}}, \quad \forall t \in [t_b, t_c).$$

By taking $r_p \equiv \frac{\overline{\lambda}_{\mathcal{P}}}{\underline{\lambda}_{\mathcal{P}}}$, it can be concluded that $\|\mathcal{E}(t)\| \leq E$ for all $t \in [t_b, t_c)$.

D. Proof of Phase III: Exiting From the Boundary B.

We have thus far shown that the trajectory will enter the boundary region B at $t_b \in (t_a, t_a + \Delta T_{\text{in},\max})$ where $\Delta T_{\text{in},\max}$ is finite. It was further proven that there exists a finite $t'_b > t_b$, such that $z(t'_b) \in B_L$. For $t > t'_b$, either (i) $z(t) \in B$ for all $t > t'_b$, or (ii) z reenters A at $t = t_c$ for some $t_c > t'_b$.

In the former case, it follows immediately from Proposition 3 with $t_c \to \infty$ that $\|\mathcal{E}(t)\| < E$, proving global boundedness. The latter case is addressed in the following proposition.

Proposition 4: Let $z(t) \in B$ for all $t \in [t'_b, t_c)$ and $z(t_c) \in A$ for some $t_c > t'_b$. Then

$$t_c - t'_b \geq \Delta T_{\text{exit},\min} \quad (89)$$

where

$$\Delta T_{\text{exit},\min} = \frac{2\varepsilon_0}{\gamma'\overline{m}_0^2}, \quad (90)$$

and

$$|\mathcal{E}_0(t_c)| < \overline{m}_0. \quad (91)$$

Proof: From the definition of regions A and B_L in Definition 1 and Definition 2, it follows that

$$\vartheta_0(t'_b) \leq -(\theta'_{0,\max} + \varepsilon_0/2), \quad \vartheta_0(t_c) \geq -\theta'_{0,\max}.$$

In addition, from (62) $\dot{\vartheta}_0(t) \leq \frac{1}{4}\gamma'\overline{m}_0^2 \;\forall\, t$. Hence, $t_c - t'_b \geq \frac{2\varepsilon_0}{\gamma'\overline{m}_0^2}$, completing the proof of (89).

We now prove (91) as follows. The conditions of case (ii) imply

$$\vartheta_0(t_c - \Delta t_c) < -\theta'_{0,\max}, \quad \vartheta_0(t_c) \geq -\theta'_{0,\max}$$

for any $\Delta t_c \in (0, t_c - t_b']$. Letting Δt_c tend to zero from the right-hand side, it follows that $\dot{\vartheta}_0(t_c) > 0$. This in turn implies, from (62), that $|\mathcal{E}_0(t_c)| < |m_0(t)|$, proving (91). ∎

E. Proof of Phase IV: Return to Condition 1.

So far, we have shown the following:

(I) If at $t = t_a$, $\mathcal{E}(t)$ satisfies Condition 1, then $z(t_b) \in B_L$ for $t_b < t_a + \Delta T'_{\text{in,max}}$, with $|\mathcal{E}_0(t)| < E_0 \ \forall t \in [t_a, t_a + \Delta T]$.

(II) $z(t) \in B \ \forall t \in [t_b, t_c)$. If $\tau \leq \overline{\tau}$, then $\|\mathcal{E}(t)\| < E \ \forall t \in [t_b, t_c)$.

(III) Either (a) $t_c = \infty$ or (b) $t_c \geq t_b + \Delta T_{\text{exit,min}}$ where $z(t_c) \in A$ and $|\mathcal{E}_0(t_c)| < \overline{m}_0$.

The following proposition contains the main result of this section.

Proposition 5: Either $\mathcal{E}(t)$ returns to Condition 1 for some $t = t_d$ or the boundedness of $\mathcal{E}(t)$ is immediate.

Proof: In case (a) of Phase III, the boundedness of $\mathcal{E}(t)$ is guaranteed since Phase II implies that $\|\mathcal{E}(t)\| < E \ \forall t \geq t_b$. In Phase III, case (b), noting (91) and that $E_0 - \delta > \overline{m}_0$ from (34), there are only two possibilities:

(A) $|\mathcal{E}_0(t)| < E_0 - \delta$ for all $t \geq t_c$, or

(B) there exists $t_d > t_c$ such that $|\mathcal{E}_0(t_d)| = E_0 - \delta$ and $|\mathcal{E}_0(t)| < E_0 - \delta \ \forall t \in [t_c, t_d)$.

Case (A): In case (A), applying Corollary 1.1 with $t_s = t_c$, $t_{ss} = \infty$, and $W = E_0 - \delta$, it can be shown from (60) that

$$\|\mathcal{E}'(t)\| \leq \max\left(\sqrt{\frac{\overline{\lambda}_{P'}}{\underline{\lambda}_{P'}}}\|\mathcal{E}'(t_c)\|, \ell(E_0 - \delta)\right)$$

for all $t \geq t_c$. This implies that $\mathcal{E}(t)$ and therefore $z(t)$ is bounded.

Case (B): If case (B) holds, then the condition of the case immediately implies that $\mathcal{E}(t)$ satisfies (41) in Condition 1 for $t = t_d$. We note that for all $t \in [t_b, t_c)$, $z(t) \in B$ with $\|\mathcal{E}(t)\| \leq E$. This together with the condition of the case $|\mathcal{E}_0(t)| \leq E_0 - \delta \ \forall t \in [t_c, t_d]$ implies that

$$|\mathcal{E}_0(t)| \leq E, \quad \forall t \in [t_b, t_d]$$

since $|\mathcal{E}_0(t)| \leq \|\mathcal{E}(t)\|$ and $E > E_0$. Hence, if $\tau \leq \Delta T_{\text{exit,min}}$, it follows that $\mathcal{E}_0(t)$ satisfies (40) in Condition 1 for $t = t_d$. Furthermore, since $\mathcal{E}_0(t)$ satisfies (40) in Condition 1 at $t = t_a$, and from Phase I $|\mathcal{E}_0(t)| < E_0 \ \forall t \in [t_a, t_a + \Delta T]$, we obtain

$$|\mathcal{E}_0(t)| < E, \quad \forall t \in [t_a - \tau, t_d].$$

Then, applying Proposition 1 with $t_s = t_a - \tau$, $t_{ss} = t_d - \tau$ and $W = E$, it follows that

$$V'(t_d - \tau) \leq \max\left(V'(t_a - \tau), \frac{1}{2}\underline{\lambda}_{P'}(\ell E)^2\right).$$

Noting that (42) in Condition 1 is satisfied by $\mathcal{E}'(t)$ for $t = t_a$, and using (39), we obtain

$$V'(t_d - \tau) \leq \max\left(\frac{1}{2}\underline{\lambda}_{P'}E'^2, \frac{1}{2}\underline{\lambda}_{P'}E'^2\right) = \frac{1}{2}\underline{\lambda}_{P'}E'^2.$$

Hence, $\|\mathcal{E}'(t)\|$ satisfies Condition 1 (42) for $t = t_d$. This implies that $\mathcal{E}(t)$ satisfies Condition 1 for $t = t_d$. ∎

F. Summary

The above phases imply that starting with $t = t_a$, there are three possibilities:

(i) The trajectory stays in Phase II for all $t \geq t_b$.

(ii) The trajectory stays in Phase IV, case (A) for all $t \geq t_c$.

(iii) The trajectory visits all four phases infinitely often.

The discussions in Sections IV-B–IV-E imply that in all three cases (i)–(iii), $\mathcal{E}(t)$ always remains bounded, proving Theorem 1. In particular, it follows from Proposition 2(i), Lemma 4, and (91) that in all cases, if $\tau \leq \tau^\star$ with $\tau^\star$ defined as

$$\tau^\star = \min\left[\Delta T_{\text{exit,min}}, \overline{\tau}\right] \tag{92}$$

then

$$|\mathcal{E}_0(t)| \leq E, \quad \forall t \geq t_0.$$

Again, applying Proposition 1 with $t_s = t_a - \tau$ and $W = E_0$, we obtain

$$V'(t) \leq \max\left(\frac{1}{2}\underline{\lambda}_{P'}E'^2, \frac{1}{2}\underline{\lambda}_{P'}(\ell E_0)^2\right), \quad \forall t \geq t_a - \tau.$$

Noting (38) and (39), it follows that

$$\|\mathcal{E}'(t)\| \leq E', \quad \forall t \geq t_a - \tau.$$

Hence

$$|z(t)| \leq \sqrt{E^2 + \max\left(E', \max_{[t_0, t_a - \tau]}\|\mathcal{E}'(t)\|\right)^2 + \Theta_{\max}^2}$$

for all $t \geq t_0$, proving global boundedness.

From (C.141), (90), and (92), we obtain that the solutions of the overall adaptive system are bounded for all $\tau \leq \tau^\star$. Hence, the delay margin is given by $\tau^\star$, with

$$\tau^\star < \min\left[\frac{2\varepsilon_0}{\gamma p_{bb}^2 \overline{m}_0^2}, \frac{\underline{q}}{4\Theta_{\max}^2}\sqrt{\frac{\underline{\lambda}_{\mathcal{P}}}{\overline{\lambda}_{\mathcal{P}}^3}}\right] \tag{93}$$

where $\varepsilon_0 \in (0, \theta_{0,\max} - \theta_{0,\max}^\star)$ with $\theta_{0,\max}^\star$ in (27), $\gamma > 0$ an arbitrary and finite constant, p_{bb} defined in Section III-A, $\overline{m}_0 = \max_t \left|c_0^\top x_m\right|$, $\Theta_{\max}$ in (31), $\mathcal{P}$ in (83), and $\underline{q}$ satisfying (82).

Remark 3: The results of Theorem 1 represent an important step in robust adaptive control. From establishing global boundedness in the presence of disturbances and unmodeled dynamics, this paper takes the next step in robust adaptive control and extends it to time delays for a class of adaptive systems. A computable delay margin is demonstrated to exist, thereby providing a theoretical framework for verification of adaptive control systems in flight as well in other applications. The most important point to note is the absence of any Lyapunov function, a fixture in most adaptive control proofs. A first principles approach was used instead in this paper to ensure the global boundedness of the tracking errors, which is a distinctly different type of proof than those employed in robust adaptive control to date. As can be seen in the proof of Theorem 1, the two most crucial pieces of the proof involve the boundary of the projection algorithm in the adaptive law. The first says that the trajectory will hit the boundary in a finite time (Phase I). The second is that once it hits the boundary, it cannot become unbounded while remaining on the boundary. These two were central points that helped establish global boundedness in this challenging problem. Needless to say, more complexities had to be dealt with in the vector case due to the higher dimensions of the errors.

Remark 4: In this paper, for the sake of simplicity, we assumed that b_p is known and let $b_m \equiv b_p$. However, it is expected that the result can be extended straightforwardly for the case $b_p = \lambda b_m$, where $\lambda > 0$ is an unknown parameter.

Remark 5: The matching condition (25) appears limiting but has common and practical use in real-world control problems. For example, in flight control, the structure of the matrix A_p is known and the reference model parameters are chosen so that there exist ideal control parameters that satisfy the matching condition.

Remark 6: The analytic approach presented above can be applied to stability and robustness investigations for a larger class of systems beyond the robust adaptive control problem. Independent of such an applicability, the impact of the presented work lies in that it is the first study that rigorously proves that the standard adaptive law modified with a suitably tuned projection algorithm introduces a computable delay margin, even without any delay compensation method such as predictive feedback.

It should be noted that the computable delay margin $\tau^\star$ in (93) is quite conservative. This is understandable given the complex nonlinear nature of the underlying adaptive system. One of the main reasons for this can be attributed to (82) which is fairly restrictive.

Remark 7: The class of plants addressed in this paper has considered a scalar input. Extensions to the multiple-input case can be carried out in a similar manner. The main property that needs to be established is the dynamics of the transformed error states $\mathcal{E}$ and ϑ which in turn are dictated by C and M in (15) and (17), respectively.

V. Comparison to Earlier Work

In this section, we distinguish the results presented in this paper from earlier work (for example, [5], [23]). As the results in [5] employed both unnormalized and normalized adaptive laws, we provide the comparison by considering these cases separately.

Let us first consider the case of direct model reference adaptive control with unnormalized adaptive laws in [5]. We begin by considering a plant of the form

$$y_p = G_0(s)\,(1 + \Delta_m(s))\,u \tag{94}$$

where $G_0(s)$ represents the nominal plant, and $\Delta_m(s)$ is an unknown multiplicative perturbation. Without loss of generality, we assume a scalar plant and reference model with a modified adaptive law defined as in [5, Section 9.3.2]. It follows that the closed-loop dynamics can be written as

$$y = W(s)(\widetilde{\theta} y + b_m r) \tag{95}$$

where

$$W(s) = \frac{1 + \Delta_m(s)}{s + a_m - \theta^\star \Delta_m(s)}. \tag{96}$$

It is shown in [5] that if $W(s)$ is strictly positive real (SPR) then global boundedness of the overall adaptive system can be concluded. However, if $W(s)$ is not SPR only semi-global stability can be shown. We refer the reader to [5] for details of the proof.

For the problem under consideration, $\Delta_m(s)$ can be addressed either as (i) $\Delta_m(s) = e^{-\tau s} - 1$, or (ii) $\Delta_m(s) \approx \frac{-\tau s}{1+\frac{\tau}{2}s}$ using a first-order Padé approximation of $e^{-\tau s}$.

In both (i) and (ii), $W(s)$ in (96) is not SPR. Therefore, one can use the results in [5] to conclude that the closed-loop adaptive system is semi-globally stable. In contrast, we note that this paper demonstrates global boundedness, which is a stronger result.

We now consider the case of normalized adaptive laws treated in [5], which is addressed in Theorem 9.3.2. This theorem states that all signals of the closed-loop plant are bounded if the overall plant transfer function in (94) is strictly proper and $\Delta_m(s)$ satisfies the following conditions.

1) $\Delta_m(s)$ is analytic in $\Re\{s\} \geq \frac{\delta_0}{2}$ for some $\delta_0 > 0$;
2) There exists a strictly proper transfer function $W(s)$ analytic in $\Re\{s\} \geq \frac{\delta_0}{2}$ and such that $W(s)\Delta_m(s)$ is strictly proper;

in addition to the stability bounds given in [5, (9.3.64)]. The stability bounds in (9.3.64) characterize the class of $\Delta_m(s)$ for which global boundedness can be guaranteed. The question therefore is, when $\Delta_m(s) = e^{-\tau s} - 1$, whether $\tau^\star$ can be quantitatively determined for which the bounds in (9.3.64) can be guaranteed. This, however, is an exceedingly difficult task and is not obvious from the deliberations in [5] or [23]. Unlike the above, as will be shown below, a straightforward computation of $\tau^\star$ that satisfies (93) can be provided using the results of this paper. This is the main contribution of this paper. A secondary point is that the unnormalized adaptive law (19) proposed here is significantly less complex than the normalized adaptive law in [5].

VI. Second-Order Example

Let us consider the plant in (1) with

$$A_p = \begin{bmatrix} 0 & 1 \\ -\omega_p^2 & -2\zeta_p\omega_p \end{bmatrix}, \quad b_p = \begin{bmatrix} 0 \\ k_p \end{bmatrix} \tag{97}$$

with $0 < \omega_p \leq \overline{\omega}$ and $|\zeta_p| \leq \overline{\zeta}$ where $\overline{\omega}$ and $\overline{\zeta}$ are known positive constants. Similarly

$$A_m = \begin{bmatrix} 0 & 1 \\ -\omega_m^2 & -2\zeta_m\omega_m \end{bmatrix}, \quad b_m = \begin{bmatrix} 0 \\ k_m \end{bmatrix} \tag{98}$$

with $\zeta_m, \omega_m > 0$ define the reference model in (2). Clearly, from (97) and (98), it follows that the matching condition (25) is satisfied.

To compute $\tau^\star$, we begin with P. For the reference model in (2) and (98) and taking $Q = I_{2\times 2}$, it can be shown that

$$P = \begin{bmatrix} \dfrac{4{\zeta_m}^2 + \omega_m^2 + 1}{4\zeta_m\omega_m} & \dfrac{1}{2\omega_m^2} \\ \dfrac{1}{2\omega_m^2} & \dfrac{\omega_m^2 + 1}{4\zeta_m\omega_m^3} \end{bmatrix} \tag{99}$$

is the solution of the Lyapunov equation $A_m^\top P + P A_m = -Q$.

Second, we proceed to the projection parameters in (27) and (28). These require the $\theta^\star_{i,\max}$, $\mathcal{A}'_m$, and P' which in turn requires $\theta^\star$ and the transformation matrices M and C. For the plant in (97) and reference model in (98), it follows that the unknown parameter θ^* in (25) is given by

$$\theta^\star = \begin{bmatrix} \dfrac{\omega_p^2 - \omega_m^2}{k_p} & -\dfrac{2(\zeta_m\omega_m - \zeta_p\omega_p)}{k_p} \end{bmatrix}^\top. \tag{100}$$

We note that bounds on the elements of (100), however, are known since $0 < \omega_p \le \overline{\omega}$ and $|\zeta_p| \le \overline{\zeta}$ and positive constants $\overline{\zeta}, \overline{\omega}$ are known. Thus, in order to compute $\theta^{\star}_{i,\max}$ in (26) all that remains is the transformation matrix M.

Following the construction of C and M detailed in Section III-A, we obtain

$$C = \begin{bmatrix} \sqrt{\dfrac{\zeta_m}{\omega_m^3 + \omega_m}} & \dfrac{p_{bb}}{k_m} \\ \sqrt{\dfrac{d_m}{4\zeta_m\left(\omega_m^3+\omega_m\right)}} & 0 \end{bmatrix} \tag{101}$$

and

$$M = \begin{bmatrix} 0 & k_m \\ \dfrac{\left(\omega_m^2+1\right)k_m}{\omega_m\sqrt{d_m}} & -\dfrac{2\zeta_m k_m}{\sqrt{d_m}} \end{bmatrix} \tag{102}$$

where

$$p_{bb} = \frac{1}{2}\sqrt{\frac{\left(\omega_m^2+1\right)k_m^2}{\zeta_m\omega_m^3}} \tag{103}$$

and $d_m = \omega_m^4 + \left(4{\zeta_m}^2 + 2\right)\omega_m^2 + 1$. Combining (100) and (102) with all admissible values of ζ_p and ω_p, the bounds on the elements of the uncertain parameter in the transformed parameter space $\theta^{\star}_{i,\max}$ can be determined from (26) for all $i = \{0, 1, \ldots, n-1\}$. This in turn implies that the projection bound $\theta'_{i,\max}$ can be determined from (27) for $i = \{1, 2, \ldots, n-1\}$. Lastly, to choose $\theta'_{0,\max}$, the condition in (28) must also be evaluated. This leads to the computation of $\mathcal{A}_m$ and P' as follows.

For the reference model in (2) and (98), it can be shown that $\mathcal{A}_m$ in (21) and (22) is such that

$$\alpha_{00} = -\frac{2\zeta_m\omega_m^3}{\omega_m^2+1} \quad a_1 = -\frac{\omega_m\left(\omega_m^4 + \left(2-4{\zeta_m}^2\right)\omega_m^2 + 1\right)}{\left(\omega_m^2+1\right)\sqrt{d_m}}$$
$$a_0 = \frac{\omega_m\sqrt{d_m}}{\omega_m^2+1} \quad \mathcal{A}'_m = -\frac{2\zeta_m\omega_m}{\omega_m^2+1} \tag{104}$$

from C in (101). We observe that since $\zeta_m, \omega_m > 0$, it can directly be shown that $\det(\mathcal{A}_m) > 0$ and $\mathrm{Trace}(\mathcal{A}_m) < 0$ which implies that $\mathcal{A}_m$ is Hurwitz. Additionally, it is obvious from (104) that $\mathcal{A}'_m < 0$, validating Lemma 2. Hence, for any $Q' > 0$, it follows that the solution of (29) simplifies to $P' = -\frac{Q'}{2\mathcal{A}'_m}$.

Thus, combining $\theta^{\star}_{0,\max}$ with $P' = -\frac{Q'}{2\mathcal{A}'_m}$ and $\alpha_{00},\ a_0,\ a_1$, and $\mathcal{A}'_m$ in (104), the projection bound $\theta'_{0,\max}$ can be determined from the inequalities in (27) and (28). From the definition of $\theta'_{i,\max} = \theta_{i,\max} - \varepsilon_i$, where $\varepsilon_i > 0$ is an arbitrary finite constant, we have determined all of the projection algorithm parameters needed to define the complete adaptive update law in (61) and $\Theta_{\max}$ in (31).

The third quantity we determine is $\mathcal{P}$. With $\mathcal{A}'_m$ in (104) and $Q' > 0$, we obtain

$$\mathcal{P} = \begin{bmatrix} p_\varphi & 0 \\ 0 & \dfrac{Q'\left(\omega^2+1\right)}{4\zeta\omega} \end{bmatrix} \tag{105}$$

from (83) and (84) where $p_\varphi > 0$ is an arbitrary constant.

With the above three computations, we have thus far determined p_{bb} (103), $\Theta_{\max}$, $\mathcal{P}$ (105), and $\overline{m}_0$ since $\overline{m}_0 = -c_0^\top A_m^{-1} b_m \overline{r}$ and C is defined in (15) and (101). The positive constants γ and $\varepsilon_0 \in (0, \theta_{0,\max} - \theta^{\star}_{0,\max})$ are design parameters that can be chosen arbitrarily. Therefore, $\underline{q}$, which needs to satisfy (82) of Lemma 3, is the only quantity that remains to be computed.

To compute $\underline{q}$, we begin with $\mathcal{Q}$ defined as $\mathcal{Q} \triangleq (M_0 + M_1)^\top\mathcal{P} + \mathcal{P}(M_0 + M_1)$, where M_0 and M_1 are defined in (80) and $\mathcal{P}$ in (105). It follows from (82) that $\underline{q}$ satisfies $\underline{\lambda}_{\mathcal{Q}} > \underline{q}$. That is, one needs to find a $\underline{q}$ such that

$$\max_i \lambda_i\left(\mathcal{Q}(\zeta_p, \omega_p, -\theta_{0,\max} + \epsilon(t), \vartheta_1(t))\right) < -\underline{q} \tag{106}$$

for all admissible ζ_p and ω_p with $\epsilon(t) \in [0, \varepsilon_0]$, where $0 < \varepsilon_0 < \theta_{0,\max} - \theta^{\star}_{0,\max}$. The existence of the solution to (106) is guaranteed by the choice of projection parameters satisfying (28) and is proved in Lemma 3 (See Appendix A for details). The reason for this is because M_1 is the only term in $\mathcal{Q}$ dependent on the projection parameters. This is shown below.

From $\mathcal{A}'_m$ in (104), M in (102) and (27) with $\theta^\star$ in (100), (80) yields Equation (107) as shown at the bottom of this page and

$$M_1 = \begin{bmatrix} -\theta_{0,\max} + \epsilon(t-\tau) & \vartheta_1(t-\tau) \\ 0 & 0 \end{bmatrix}.$$

Suppose we choose $Q' = -2\mathcal{A}'_m$ and $p_\varphi = 1$. It follows then that $\mathcal{P} = I_{2\times 2}$ from (105). Thus, $\mathcal{Q}$ simplifies to $\mathcal{Q} = (M_0 + M_1)^\top + (M_0 + M_1)$ and is given by

$$\mathcal{Q} = \begin{bmatrix} 2\left(M_{0_{11}} - \theta_{0,\max} + \epsilon(t-\tau)\right) & M_{0_{12}} + M_{0_{21}} + \vartheta_1(t-\tau) \\ M_{0_{12}} + M_{0_{21}} + \vartheta_1(t-\tau) & 2M_{0_{22}} \end{bmatrix} \tag{108}$$

where $M_{0_{jk}}$ denotes the elements of M_0 in (107). It is now clear that there exists a $\underline{q}$ that satisfies (106) since it can easily be shown that $\mathrm{Trace}(\mathcal{Q}) < 0$ and $\det(\mathcal{Q}) > 0$ are implicitly satisfied with $\theta'_{0,\max}$ in (28) and $0 < \varepsilon_0 < \theta_{0,\max} - \theta^{\star}_{0,\max}$. It is important to note that the ease in which the stability condition in (28) is derived is largely due to the fact that no cross-coupling between ϑ_0 and ϑ_1 is observed in any of the elements of $\mathcal{Q}$. Furthermore, any numerical procedure can be used to find the solution $\underline{q}$ of (106).

$$M_0 = \begin{bmatrix} \dfrac{2\zeta_m\omega_m}{\omega_m^2+1} - 2\zeta_p\omega_p & \dfrac{-4\zeta_m^2\omega_m}{(\omega_m^2+1)\sqrt{d_m}} + \dfrac{4\zeta_m\zeta_p\omega_p - \left(\omega_m^2+1\right)\dfrac{\omega_p^2}{\omega_m}}{\sqrt{d_m}} \\ \dfrac{\omega_m\sqrt{d_m}}{\omega_m^2+1} & -\dfrac{2\zeta_m\omega_m}{\omega_m^2+1} \end{bmatrix} \tag{107}$$

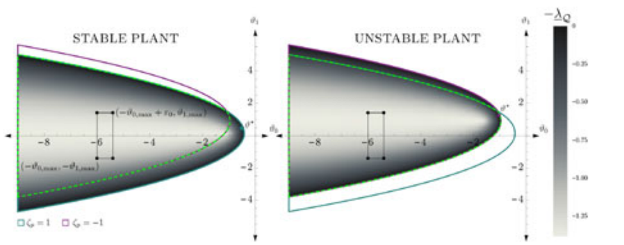

Fig. 3. Density plot of $\max_i \Re(\lambda_i((M_0 + M_1)^\top \mathcal{P} + \mathcal{P}(M_0 + M_1)))$ in (82) for the plant and reference model given in Section VI-A. This plot numerically illustrates the value of $-\underline{q}$ as a function of $-\theta_{0,\max} + \epsilon(t)$ and ϑ_1, illustrated by the boxed region, for both stable and unstable plant with $\omega_p = \overline{\omega}$.

A. Numerical Example

We choose the plant and reference model as in (97) and (98) with

$$(|\zeta_p|, \overline{\omega}) = (1, 0.133),\ (\zeta_m, \omega_m) = (1, 0.4) \tag{109}$$

and $k_m = k_p = \omega_m^2$. That is, we consider both the case when $\zeta_p = 1$ (stable plant) and $\zeta_p = -1$ (unstable plant). The control input

$$u = p_{bb}^{-1} \vartheta^\top C x_p + r \tag{110}$$

and adaptive update law

$$\dot{\vartheta}_i = \gamma' \,\mathrm{Proj}\,(\vartheta_i, -(\mathcal{E}_i + m_i)\mathcal{E}_0) \tag{111}$$

for each $i = \{0, 1\}$, presented in Section III-G2, are implemented. We recall that $\mathcal{E}$, ϑ, and m are the transformed state error, parameter, and reference state, as introduced in (9), (10), and (51), respectively, with transformation matrices C and M in (101) and (102).[2]

To find $\underline{q}$, a numerical scheme was applied to (106) with $\mathcal{Q}$ in (108), the results of which are shown in Fig. 3. As can be seen from the rectangular regions in this figure, $\underline{q}$ depends both on the projection bounds and on the plant parameters ζ_p and ω_p. We now have incurred all necessary components of $\tau^\star$.

We now revisit the delay margin expression in (93). We let the adaptation gain γ be $\gamma = k_\gamma \theta_{0,\max}$, where $k_\gamma > 0$ and $\varepsilon_0 = k_{\varepsilon_0} \theta_{0,\max}$. With our choice of Q' and p_φ from $\mathcal{P}$ in (105), the delay margin simplifies to

$$\tau^\star < \min\left[\left(\frac{2}{p_{bb}^2 \overline{m}_0^2}\right)\frac{k_{\varepsilon_0}}{k_\gamma}, \frac{\underline{q}}{4(\theta_{0,\max}^2 + \theta_{1,\max}^2)}\right] \tag{112}$$

since $\overline{\lambda}_{\mathcal{P}} = \underline{\lambda}_{\mathcal{P}} = 1$. We now compute the delay margin in what follows.

It can be shown that (27) and (28) are satisfied for $\theta_{\max} = [6\ 1.4]^\top$ and $\varepsilon = 0.1\theta_{\max}$ since $\theta_{\max}^\star = [1.07\ 1.22]^\top$ and

$$\mathcal{A}_m' = \begin{bmatrix} -0.110 & -0.173 \\ 0.486 & -0.690 \end{bmatrix}.$$

Additionally, from Fig. 3, it follows that $\underline{q} > 0.727$ in (82). From (103) with (109), we obtain $p_{bb} = 0.341$. Lastly, from $c_0^\top$ (15), (101), and the reference model (98), (109), the definition in (51) implies $\overline{m}_0 = -c_0^\top A_m^{-1} b_m \overline{r} = (1.46)\overline{r}$. We let $k_\gamma = 1/6$. Hence, it follows directly from (112) that $\tau^\star < \min\left[\frac{4.8}{\overline{r}^2},\ 0.00478\right]$s. Therefore, for $\overline{r} < \sqrt{\frac{4.8}{0.00478}}$, we obtain

$$\tau^\star = 4.78 \text{ ms} \tag{113}$$

as the delay margin of the adaptive system.

It is important to review the qualitative implications of tuning the design parameters ($\theta_{i,\max}$, ε_0, k_γ) on the delay margin $\tau^\star$. In (112), the design tradeoff between the size of the parameter bounds and the delay margin can be seen quite readily. The bracketed term in (112) contains two elements. The first term is primarily dependent on the magnitude of the reference input ($\overline{m}_0$), whereas the second term depends largely on the parameter bounds and the corresponding lower bound of the measure of closed-loop LTV stability ($\underline{q}$), while ϑ_0 is in its lower projection boundary (Phase II, Lemma 3). With that being said, we will refer to the former term as $\tau_r^\star$ and the latter as $\tau_\Theta^\star$. We discuss the design tradeoffs in more detail in what follows.

The objective is to find the solution to the optimization problem, $\max_{\theta_{\max}} \tau^\star$. We begin by investigating the design tradeoffs for $\tau_r^\star$ as introduced above. Since $\tau_r^\star = \mathcal{O}\left(\frac{\varepsilon_0}{\gamma}\right)$, it is obvious that increasing ε_0 and decreasing γ are optimizing. That is, choosing ε_0 and γ in such a way results in the largest $\tau_r^\star$. The latter is not surprising since it is well known in the adaptive control community that a high gain on the adaption rate can lead to undesirable closed-loop phenomena. As for the former, increasing ε_0 implies increasing $\theta_{0,\max}$ since $\varepsilon_0 < \theta_{0,\max} - \theta_{0,\max}^\star$. In doing so, $\tau_\Theta^\star$ is inversely effected. The reason for this is twofold. First, it can be observed from Fig. 3 that for any ϑ_0, $\underline{q}$ is maximized for sufficiently small $\theta_{1,\max}$. Second, it can be shown that $\lim_{\vartheta_0 \to -\infty} \underline{\lambda}_{\mathcal{Q}} = 1.38$ for any ϑ_1. Therefore, $\tau_\Theta^\star = \mathcal{O}\left(\|\theta_{\max}\|^{-1}\right)$. Hence, maximizing $\tau_r^\star$ by choosing $\theta_{0,\max}$ sufficiently large, inadvertently minimizes $\tau_\Theta^\star$. Similarly, the solution to the zero-input optimization problem $\max_{\theta_{\max}} \tau_\Theta^\star$ minimizes $\tau_r^\star$. In this case, however, we can counteract such phenomena since $\tau_r^\star$ includes an additional degree of freedom, γ.

The chosen parameter bounds $\theta_{\max} = [6\ 1.4]^\top$ for the particular numerical example presented earlier in this section, in context with the discussion above, are near optimal in the sense that they are approximately the solution to the zero-input optimization problem, $\max\limits_{\theta_x \max} \tau_\Theta^\star$ for all possible values of ζ_p and ω_p.

It is important to note that our discussion here is a result of a design process that yields one particularly clear vantage point. In other words, choosing $Q = I_{2\times 2}$, $Q' = -2\mathcal{A}_m'$ and $p_\varphi = 1$ provides $\tau^\star$ in (112) and invokes the design tradeoff clarity above. Determining the optimal delay margin, however, requires the solution of a complete nonlinear constrained optimization problem.

B. Simulation Studies

In this section, we carry out simulation studies of the adaptive system defined by the plant in (97) in the presence of an input time delay satisfying (113), with the reference model in (98), the controller in (110) and the adaptive law in (111) with $\theta_{\max} = (6, 1.4)$, $\varepsilon_i = 0.1\theta_{i,\max}$ and $\gamma = 1$. With these choices in addition to $\overline{r} < 31$, the adaptive controller in (111) and (110) guarantees globally bounded solutions for any initial conditions $x_p(0)$ and $\theta(0)$ with $\|\theta(0)\| \leq \theta_{\max}$ for any $\tau < \tau^\star$ in (113).

[2]We note that (110) is (3) rewritten with $\theta^\top = \frac{\vartheta^\top C}{p_{bb}}$.

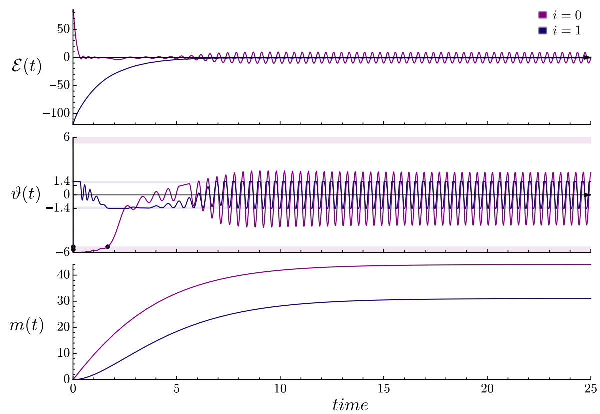

Fig. 4. Simulation of plant in Section VI in the presence of an input delay with $\tau = 0.0047$ s, adaptive law in (111), and $r(t)$ in (114). The points (•) represent t_b, t'_b, and t_c which correspond to Phases I, II, and III as outlined in Section IV-A.

The resulting transformed error $\mathcal{E}$ and transformed parameter, ϑ are illustrated in Fig. 4 for the reference input

$$r(t) = 30, \quad t \geq -\tau \tag{114}$$

time delay $\tau = 4.7$ ms, and initial conditions $\mathcal{E}(\chi) = [82 - 120]^\top$, $\theta(\chi) = [\theta_{0,\max}\, \theta_{1,\max}]^\top$, $\zeta_p(\chi) = 1$, and $\omega_p(\chi) = 0.133$ for all $\chi \in [-\tau, 0]$. It was also observed that the error became unbounded when the projection bound was removed and when the input delay exceeded $\tau = 245$ ms (see Appendix B). In comparison, the analytically computed delay margin was a couple of orders of magnitude smaller.

The numerical simulations show that the behavior of the adaptive system, in terms of which of the four phases reported in Section IV-A occur, is directly dependent on the nature of the reference input and the initial conditions.

VII. Summary

In this paper, robust adaptive control of a class of plants in the presence of an input time delay is investigated. It is shown through analytic methods and validated by simulation results that a projection algorithm in the standard adaptive control law achieves global boundedness of the overall adaptive system for delays less than the computable delay margin. The delay margin bound and the projection bounds are explicitly calculated and demonstrated using a general second-order system with parametric uncertainty.

Appendix A
Proof of Lemmas

A. Proof of Lemma 2

Proof: From (24) and (23)

$$A_m^\top P + PA_m = C^\top \left(\mathcal{A}_m + \mathcal{A}_m^\top\right) C = PC^{-1}\left(\mathcal{A}_m + \mathcal{A}_m^\top\right) C.$$

Noting $P > 0$ and $A_m^\top P + PA_m = -Q$, where $Q \triangleq qI_{n\times n}$, we obtain $C^{-1}\left(\mathcal{A}_m + \mathcal{A}_m^\top\right) C = -P^{-1}Q$. Since $-P^{-1}Q = -qP^{-1} < 0$, $C^{-1}(\mathcal{A}_m^\top + \mathcal{A}_m)C < 0$. Hence, with (22)

$$(\mathcal{A}_m^\top + \mathcal{A}_m) = \begin{bmatrix} 2\alpha_{00} & a_0^\top + a_1^\top \\ a_0 + a_1 & \mathcal{A}_m'^\top + \mathcal{A}_m' \end{bmatrix} < 0. \tag{A.115}$$

Since $\mathcal{A}_m'^\top + \mathcal{A}_m'$ is the $(n-1)\times(n-1)$ trailing principal submatrix of (A.115), it can be shown using a suitable permutation similarity of $\mathcal{A}_m^\top + \mathcal{A}_m$ and Sylvester's Minorant Criterion that $\mathcal{A}_m'^\top + \mathcal{A}_m'$ is negative definite which further implies $\mathcal{A}_m'$ is Hurwitz, proving Lemma 2. ■

B Proof of Lemma 3

Proof: From (80), it is shown that

$$M_0 + M_1 = \mathcal{A}_m + \begin{bmatrix} \vartheta^\top(t-\tau) - \vartheta^{\star\top} \\ 0 \end{bmatrix}. \tag{A.116}$$

From (83), (A.116), and (22), we obtain that

$$\mathcal{M}(t) \equiv \mathcal{I}(M_0 + M_1)\mathcal{I}^\top = \begin{bmatrix} \mathcal{A}_m' & a_0 \\ a_1^\top + \varphi'^\top(t-\tau) & \alpha_{00} + \varphi_0(t-\tau) \end{bmatrix} \tag{A.117}$$

where $\varphi_0 \in \mathbb{R}$, $\varphi' \in \mathbb{R}^{n-1}$ and are given by $[\varphi_0(t)\ \ \varphi'^\top(t)]^\top = \vartheta(t) - \vartheta^\star$.

Defining a symmetric matrix function $\mathcal{S}(\bullet)$ as

$$\mathcal{S}(\mathcal{M}) = -\left(\mathcal{R}\mathcal{M} + \mathcal{M}^\top\mathcal{R}\right) = \begin{bmatrix} Q' & -q_d(\varphi') \\ -q_d^\top(\varphi') & -2p_\varphi(\alpha_{00} + \varphi_0(t-\tau)) \end{bmatrix} \tag{A.118}$$

where $q_d(\varphi') \equiv P'a_0 + (a_1 + \varphi'(t-\tau))p_\varphi$, we can show that $\mathcal{S}(\mathcal{M})$ is positive definite for all $\mathcal{M}(t)$ if $z(t) \in B$ as follows.

From (29), we have that $Q' > 0$. Therefore, all k leading principal minors of $\mathcal{S}$ are positive for $k = 1, 2, \ldots, n-1$. Also, noting from (B.118) that

$$\det(\mathcal{S}) = \det(Q')\left(-2p_\varphi(\alpha_{00} + \varphi_0(t-\tau)) - q_d^\top(\varphi')Q'^{-1}q_d(\varphi')\right) \tag{A.119}$$

and the design of the projection algorithm (28) which implies

$$\varphi_0(t-\tau) < -\alpha_{00} - \frac{1}{2p_\varphi}q_d^\top(\varphi')Q'^{-1}q_d(\varphi') \quad \text{if } z \in B$$

we obtain $\det\{\mathcal{S}\} > 0$ if $z \in B$. Since all the leading principal minors of $\mathcal{S}$ are positive, we obtain that $\mathcal{S}$ is positive definite while $z \in B$.

Noting the definition of $\mathcal{M}$ in (B.117) and from the fact that $\mathcal{I}^\top = \mathcal{I}^{-1}$, we obtain

$$-\mathcal{I}^\top\mathcal{S}\mathcal{I} = \mathcal{I}^\top\mathcal{R}\mathcal{M}\mathcal{I} + \mathcal{I}^\top\mathcal{M}^\top\mathcal{R}\mathcal{I} = (\mathcal{I}^\top\mathcal{R}\mathcal{I})(M_0 + M_1) + (M_0 + M_1)^\top(\mathcal{I}^\top\mathcal{R}\mathcal{I}). \tag{A.120}$$

Equation (B.120) serves as a Lyapunov equation for $M_0 + M_1$, since it can be rewritten into the form of

$$\mathcal{Q} = \mathcal{P}(M_0 + M_1) + (M_0 + M_1)^\top\mathcal{P} \tag{A.121}$$

with $\mathcal{P} \equiv \mathcal{I}^\top\mathcal{R}\mathcal{I}$ and $\mathcal{Q} \equiv -\mathcal{I}^\top\mathcal{S}\mathcal{I}$. From the definition, $\mathcal{P}$ is symmetric and positive definite since $\mathcal{R}$ is a symmetric positive definite matrix. In the same manner, it can be shown that $-\mathcal{Q}$ is symmetric and positive definite for all $(M_0, M_1) \in \Omega_B$ since the symmetric matrix function $\mathcal{S}$ is positive definite while $z \in B$. This proves Lemma 3.

C. Proof of Lemma 4

Proof: Using

$$\mathcal{E}(t-\tau) = \mathcal{E}(t) - \int_{-\tau}^{0} \dot{\mathcal{E}}(t+\zeta)d\zeta \tag{A.122}$$

with $\dot{\mathcal{E}}(t+\zeta)$ replaced by the right-hand side of the system equation (79) with appropriate time shift, and substituting the resulting expression for $\mathcal{E}(t-\tau)$ back in to (79), we obtain the following transformed error dynamics:

$$\begin{aligned}\dot{\mathcal{E}}(t) &= \big(M_0 + M_1(t)\big)\mathcal{E}(t) + R(t) \\ &\quad - M_1(t)\int_{-\tau}^{0}\big(M_0\mathcal{E}(t+\zeta) + M_1(t+\zeta)\mathcal{E}(t+\zeta-\tau) \\ &\quad + R(t+\zeta)\big)d\zeta \end{aligned}\tag{A.123}$$

with M_0 and $M_1(t)$ defined in (80). Equation (C.123) can be rewritten compactly as

$$\dot{\mathcal{E}}(t) = \overline{M}_0\mathcal{E}(t) + \int_{-2\tau}^{0} \overline{M}(t,\zeta)\mathcal{E}(t+\zeta)d\zeta + \overline{R}(t) \tag{A.124}$$

for all $\big(\overline{M}_0(t), \overline{M}(t,\zeta)\big) \in \overline{\Omega}$, where $\overline{M}_0$, $\overline{M}(t,\zeta)$, $\overline{\Omega}$, and $\overline{R}(t)$ are defined as

$$\begin{aligned}\overline{M}_0 &= M_0 + M_1(t) \\ \overline{M}(t,\zeta) &= \begin{cases} -M_1(t)M_0 & -\tau \le \zeta < 0 \\ -M_1(t)M_{1(\zeta+\tau)} & -2\tau \le \zeta < -\tau \end{cases} \\ \overline{\Omega} &= \big\{\big(\overline{M}_0, \overline{M}(t,\zeta)\big) \mid M_1(t) \in \Omega_B \wedge M_{1(\zeta+\tau)} \in \Omega_B\big\}\end{aligned}\tag{A.125}$$

and

$$\begin{aligned} M_{1\zeta}(t) &= M_1(t+\zeta) \\ \overline{R}(t) &\equiv R(t) - M_1(t)\int_{-\tau}^{0} R(t+\zeta)d\zeta. \end{aligned}\tag{A.126}$$

We note that $\overline{R}(t)$ is bounded since $R(t)$ and $M_1(t)$ are bounded. That is, there exists a scalar $R_{\max}$ such that $\|\mathcal{P}\overline{R}(t)\| \le R_{\max}$ $\forall t \ge t_0$. Equation (C.124) represents a system with distributed delays, whose stability can be shown using the Razumikhin method, as shown below.

Define

$$\overline{V}(\mathcal{E}_t) = \max_{\zeta\in[-2\tau,0]} V(\mathcal{E}(t+\zeta)) \tag{A.127}$$

and a set Ω_t

$$\Omega_t \equiv \big\{t \mid t \in (t_b, t_c), V(\mathcal{E}(t)) = \overline{V}(\mathcal{E}_t)\big\}. \tag{A.128}$$

It follows that for all $t \in (t_b, t_c)$, there are two cases, (a) $t \in \Omega_t$, (b) $t \in (t_b, t_c)\backslash\Omega_t$. We provide the proof for each case separately.

Case a: From the definitions in (C.127) and (C.128), it follows that in this case

$$V(\mathcal{E}(t+\zeta)) \le V(\mathcal{E}(t)), \quad -2\tau \le \zeta \le 0. \tag{A.129}$$

Hence, we obtain from (86) and (C.124) that

$$\begin{aligned}\dot{V}(\mathcal{E}) &\le 2\mathcal{E}^\top(t)\mathcal{P}\overline{M}_0(t)\mathcal{E}(t) \\ &+2\int_{-2\tau}^{0}\mathcal{E}^\top(t)\mathcal{P}\overline{M}(t,\zeta)\mathcal{E}(t+\zeta)d\zeta + 2\mathcal{E}^\top(t)\mathcal{P}\overline{R}(t) \\ &+\int_{-2\tau}^{0}\alpha(\zeta)\big[\mathcal{E}^\top(t)\mathcal{P}\mathcal{E}(t) - \mathcal{E}^\top(t+\zeta)\mathcal{P}\mathcal{E}(t+\zeta)\big]d\zeta\end{aligned}\tag{A.130}$$

with any scalar positive function $\alpha(\zeta)$, since the last term then becomes nonnegative due to (C.129). Equation (C.130) can be simplified as

$$\dot{V}(\mathcal{E}) \le \int_{-2\tau}^{0} E_\zeta^\top(t)\Psi(t,\zeta)E_\zeta(t)d\zeta + 2R_{\max}\|\mathcal{E}(t)\|$$

where

$$\begin{aligned}\Psi(t,\zeta) &\equiv \begin{bmatrix} N_p(t,\zeta) & \mathcal{P}\overline{M}(t,\zeta) \\ \big(\mathcal{P}\overline{M}(t,\zeta)\big)^\top & -\alpha(\zeta)\mathcal{P}\end{bmatrix}, \\ N_p(t,\zeta) &= \frac{1}{2\tau}[\mathcal{P}(M_0+M_1) + (M_0+M_1)^\top\mathcal{P}] + \alpha(\zeta)\mathcal{P}\end{aligned}\tag{A.132}$$

and $E_\zeta(t) = [\mathcal{E}^\top(t) \;\; \mathcal{E}^\top(t+\zeta)]^\top$. We take

$$\alpha(\zeta) = \Theta_{\max}\sqrt{\frac{\overline{\lambda}_{\mathcal{P}}}{\underline{\lambda}_{\mathcal{P}}}}\cdot\begin{cases}\|M_{0\zeta}\| & -\tau < \zeta \le 0 \\ \|M_{1\zeta}\| & -2\tau \le \zeta \le -\tau.\end{cases}\tag{A.133}$$

We now state and prove a sublemma.

Sublemma 5. There exists $\varepsilon_v, \overline{\tau}$ such that $\Psi(t,\zeta) \le -\varepsilon_v I$ if $\tau \le \overline{\tau}$.

Proof: From (C.132), (C.133), and Lemma 3 (82), it can be shown that if

$$\tau < \frac{1}{2\Theta_{\max}\|M_{k\zeta}\|\overline{\lambda}_{\mathcal{P}}}\sqrt{\frac{\underline{\lambda}_{\mathcal{P}}}{\overline{\lambda}_{\mathcal{P}}}}\underline{q}, \quad k=0,1 \tag{A.134}$$

then

$$N_p(t,\zeta) < 0, \quad \forall t,\zeta. \tag{A.135}$$

Using (C.135), it can be then shown that for any vectors v_1, $v_2 \in \mathbb{R}^{n\times 1}$

$$\begin{aligned}&\big[\,v_1^\top \; v_2^\top\,\big]\Psi(t,\zeta)\begin{bmatrix}v_1\\v_2\end{bmatrix} \\ &\le -\underline{\lambda}_{N_p(t,\zeta)}\left(\|v_1\| - \frac{\|PM_1M_{k\zeta}\|\|v_2\|}{\underline{\lambda}_{N_p(t,\zeta)}}\right)^2 \\ &+\left(\frac{\|PM_1M_{k\zeta}\|^2}{\underline{\lambda}_{N_p(t,\zeta)}} - \alpha(\zeta)\underline{\lambda}_{\mathcal{P}}\right)\|v_2\|^2\end{aligned}\tag{A.136}$$

and also noting (C.132) and (82)

$$\underline{\lambda}_{N_p(t,\zeta)} \ge \tfrac{1}{2\tau}\underline{q} - \alpha(\zeta)\overline{\lambda}_{\mathcal{P}} \tag{A.137}$$

holds. From the definition of M_k, $k = \{0, 1\}$ given in (80) and noting (25), it can be obtained that

$$\|M_k\| \leq \Theta_{\max}. \tag{A.138}$$

Therefore, noting that $\|\mathcal{P}M_1 M_{k\zeta}\| \leq \overline{\lambda}_{\mathcal{P}}\Theta_{\max}\|M_{k\zeta}\|$, and integrating (C.137) into (C.136), we can further simplify the inequality as

$$\begin{aligned}&\begin{bmatrix} v_1^\top & v_2^\top \end{bmatrix}\Psi(t,\zeta)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\\ &\quad\leq \left(\frac{(\overline{\lambda}_{\mathcal{P}}\Theta_{\max}\|M_{k\zeta}\|)^2}{\frac{1}{2\tau}\underline{q} - \alpha(\zeta)\overline{\lambda}_{\mathcal{P}}} - \alpha(\zeta)\underline{\lambda}_{\mathcal{P}}\right)\|v_2\|^2\end{aligned} \tag{A.139}$$

where $k = 0$ if $-\tau < \zeta \leq 0$ and $k = 1$ if $-2\tau \leq \zeta \leq -\tau$. With α substituted by (C.133), it is shown that the parenthesis in (C.139) becomes negative which in turn implies that $\Psi(t, \zeta) < 0$ for all t, ζ if

$$\tau < \frac{1}{4\Theta_{\max}\|M_{k\zeta}\|\overline{\lambda}_{\mathcal{P}}}\sqrt{\frac{\underline{\lambda}_{\mathcal{P}}}{\overline{\lambda}_{\mathcal{P}}}}\underline{q}, \quad k = 0, 1. \tag{A.140}$$

Noting (C.138) again, it can be shown that (C.134) and (C.140) are satisfied if $\tau < \frac{1}{4\Theta_{\max}^2\overline{\lambda}_{\mathcal{P}}}\sqrt{\frac{\underline{\lambda}_{\mathcal{P}}}{\overline{\lambda}_{\mathcal{P}}}}\underline{q}$. We let

$$\overline{\tau} \equiv \frac{1}{(4+\varsigma)\Theta_{\max}^2\overline{\lambda}_{\mathcal{P}}}\sqrt{\frac{\underline{\lambda}_{\mathcal{P}}}{\overline{\lambda}_{\mathcal{P}}}}\underline{q}, \quad \varsigma > 0. \tag{A.141}$$

Then, defining $\varepsilon_v \equiv \min_{t,\zeta,\tau\in[0,\overline{\tau}]}(-\text{eig}(\Psi(t,\zeta)))$, $\Psi(t,\zeta) \leq -\varepsilon_v I$ is satisfied. This proves Sublemma 5 ■.

(C.131) can therefore be simplified as

$$\dot{V}(\mathcal{E}(t)) \leq -\varepsilon_v\|\mathcal{E}(t)\|^2 + 2R_{\max}\|\mathcal{E}(t)\|. \tag{A.142}$$

From (C.142), $\dot{V}(\mathcal{E}(t)) < 0\ \forall t \in \Omega_t \backslash \{t | \|\mathcal{E}(t)\| > \beta\}$, where

$$\beta = 2R_{\max}/\varepsilon_v. \tag{A.143}$$

Since $\overline{V}(\mathcal{E}_t(t)) = V(\mathcal{E}(t))$ as we defined Ω_t in (C.128), it can be concluded that

$$\dot{\overline{V}}(\mathcal{E}_t(t)) < 0, \quad \forall t \in \Omega_t \backslash \{t | \|\mathcal{E}(t)\| > \beta\}. \tag{A.144}$$

Case (b): From the definitions in (C.127) and (C.128), it follows that for any t in Case (b)

$$\overline{V}(\mathcal{E}_t(t)) > V(\mathcal{E}(t)). \tag{A.145}$$

Suppose there exists a $t = t_s \in (t_b, t_c)\backslash\Omega_t$ such that $\dot{\overline{V}}(\mathcal{E}_t(t_s)) > 0$. Then, it follows that $V(\mathcal{E}(t_s^+)) > \overline{V}(\mathcal{E}_t(t_s))$ from the definition of $\overline{V}(\mathcal{E}_t)$ in (C.127). This contradicts (C.145), and therefore we can conclude that

$$\dot{\overline{V}}(\mathcal{E}_t(t)) \leq 0, \quad \forall t \in (t_b, t_c)\backslash\Omega_t. \tag{A.146}$$

From cases (a) and (b) [(C.144) and (C.146)], with β as in (C.143)

$$\dot{\overline{V}}(\mathcal{E}_t(t)) \leq 0, \quad \forall t \in (t_b, t_c)\backslash\{t | \|\mathcal{E}(t)\| > \beta\}.$$

Therefore,

$$\overline{V}(\mathcal{E}_t(t)) \leq \max\left\{\overline{V}(\mathcal{E}_t(t_b)), \overline{\lambda}_{\mathcal{P}}\beta^2\right\}. \tag{A.147}$$

Since $V(\mathcal{E}(t)) \leq \overline{V}(\mathcal{E}_t(t))$ from the definition given by (C.127), (C.147) implies that $V(\mathcal{E}(t)) \leq \max\left\{V(\mathcal{E}(t_b)), \overline{\lambda}_{\mathcal{P}}\beta^2\right\}$ for all $t \in (t_b, t_c)$, completing the proof.

Appendix B
Simulation Studies

See Fig. 5.

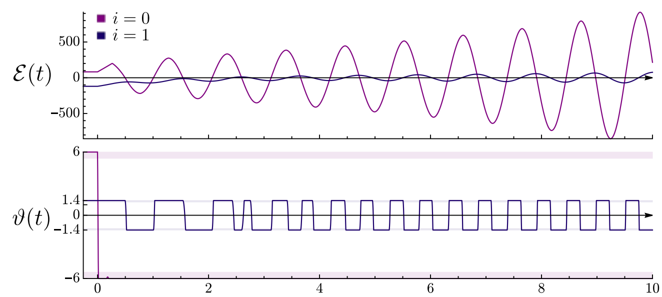

Fig. 5. Simulation of plant in Section VI in the presence of an input delay with $\tau = 257$ ms and $r(t) = 0$ for all time. It is observed that all of the signals grow without bound as $t \to \infty$.

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