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On Dwell Time Minimization for Switched Delay Systems: Time-Scheduled Lyapunov Functions

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Abstract: In the present paper, dwell time stability conditions of the switched delay systems are derived using scheduled Lyapunov-Krasovskii functions. The derivative of the Lyapunov functions are guaranteed to be negative semidefinite using free weighting matrices method. After representing the dwell time in terms of linear matrix inequalities, the upper bound of the dwell time is minimized using a bisection algorithm. Some numerical examples are given to illustrate effectiveness of the proposed method, and its performance is compared with the existing approaches. The yielding values of dwell time via the proposed technique show that the novel approach outperforms the previous ones.

Keywords: Time delay, dwell time optimization, time-scheduled Lyapunov function, switched systems, free weighting matrices

1. INTRODUCTION

One of the main methods in analysis of switched systems is restricting switching signals to a certain set. We can restrict the switching signals to signals with the property that the time interval between any consecutive switching times is not less than a certain value, which is called the *dwell time* to guarantee the asymptotic stability of the switched systems. Readers can refer to Hespanha and Morse (1999), Yuan and Wu (2015), Mitra and Liberzon (2004), Lin and Antsaklis (2009) and references therein for further information on this topic.

There is a dwell time between switchings in many real life examples such as changing road conditions (dry, wet, dirt) of a car on the road, or the different dynamics of a teleoperation robotic system either contacting a tissue or not (Allerhand and Shaked (2011)). Furthermore, fast switching can cause chattering problems in contrast to the control schemes where switching is restricted by a dwell time (Ishii and Francis (2001)). As a result, stability analysis and stabilization of switched systems with dwell time becoming increasingly popular.

Multiple Lyapunov function method is a common technique in dealing with dwell time stability problems. In Geromel and Colaneri (2006), the dwell time is determined by constraints involving exponential terms, i.e. $e^{A\tau_D}Pe^{A\tau_D}$. Another approach is employing time-scheduled Lyapunov functions which form convex sets in system matrices Allerhand and Shaked (2011). In Xiang (2015), it is shown that stability criteria of Allerhand and Shaked (2011) and Geromel and Colaneri (2006) are equal when sufficiently large number of decision variables and linear matrix inequalities (LMIs) are included in stability criterion of Allerhand and Shaked (2011).

There are some recent results on dwell time stability of switched delay systems. In Sun et al. (2006) and Li et al. (2013), average dwell time is represented with constraints involving exponential terms, which can not be minimized by polynomial-time algorithm. In order to make the problem a tractable one, it is solved for a given average dwell time rather than performing the dwell time minimization. In the literature there are limited studies that address the minimization problem on dwell time (Çalışkan et al. (2013), Yan and Özbay (2008), Yan et al. (2014), Koru et al. (2014)). Model transformation methods are used in Çalışkan et al. (2013), Yan and Özbay (2008), Yan et al. (2014) whereas free-weighting matrices method, which does not include any model transformation, is used in Koru et al. (2014). In those works, none of the conditions contain a term that is formed as a product of transcendental function and its domain to avoid non-convex representation of the problem. However, this leads to some conservatism. Typically, the upper bound of the dwell time, leading to stability, is represented as:

$$\tau_D = \tau_{\max} + T^*$$

where $\tau_{\max}$ is maximum time delay among all of the subsystems and T^* is the cost function. Hence, minimum dwell time is at least $\tau_{\max}$ even for the systems sharing a common Lyapunov function which are known to be stable under arbitrary switching Lin and Antsaklis (2009).

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In order to reduce conservatism, the present paper derives the stability conditions for the switched delay system using time scheduled Lyapunov functions. As a result, upper bound of the dwell time is represented without using $\tau_{\max}$ term. The minimization of the dwell time is formulated as a semi-definite programming in terms of LMIs. Upper bounds of the derivatives of the Lyapunov functions are found via free weighting matrices method (see Wu et al. (2010) for more information).

The notation to be used in the paper is standard: $\mathbb{R}$ ($\mathbb{R}^+$, $\mathbb{R}_0^+$) stands for the set of real numbers (positive real numbers, non-negative real numbers), $\mathcal{C}$ is used to denote the set of differentiable functions, $\mathbb{Z}^+$ symbolizes the set of positive integers. The identity matrices are denoted by I. We use $X \succ 0$ ($\succeq$, $\prec$, $\preceq 0$) to denote a positive definite (positive-semidefinite, negative definite, negative-semidefinite) matrix. The asterisk symbol $(*)$ denotes complex conjugate transpose of a matrix and x_t denotes the translation operator acting on the trajectory such as $x_t(\theta) = x_t(t+\theta)$ for some non-zero interval $\theta \in [-\tau, 0]$, and

$$\text{affine}(X, Y, t_0, \delta) = \left(1 - \frac{t-t_0}{\delta}\right) X + \left(\frac{t-t_0}{\delta}\right) Y.$$

2. PRELIMINARIES

Consider a class of switched delay system given by

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)} x(t) + \bar{A}_{\sigma(t)} x(t - r(t)), \quad t \geq 0 \\ x(\theta) &= \varphi(\theta), \qquad \forall\, \theta \in [-h, 0] \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the pseudo-state and $\sigma(t)$ is the piecewise switching signal such that $\sigma(t) : \mathbb{R}^+ \to \mathcal{P}$, $\mathcal{P} := \{1, 2, ..., m\}$ is an index set, $m \in \mathbb{Z}^+$ is the number of subsystems and initial condition belongs to Banach space of continuous functions such that $\varphi(\cdot) \in \mathcal{C}$. Time delay, $r(t)$, is a time-varying differentiable function that satisfies

$$0 \leq r(t) \leq h, \quad (2)$$

$$|\dot{r}(t)| \leq d < 1, \quad (3)$$

where h and d are positive constants. We introduce the notation

$$\Sigma_i := \left(A_i, \bar{A}_i\right) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n \times n}$$

to describe the i^{th} candidate subsystem of (1).

Lemma 1. (Theorem 3.2.1 in Wu et al. (2010)). Consider the non-switched linear subsystem Σ_i for an $i \in \mathcal{P}$ of the switched system (1) with a varying delay, $r(t)$. Given scalar $h > 0$ and $d > 0$ for which both (2) and (3) holds, the i^{th} subsystem is asymptotically stable if there exist symmetric matrices $P_i \succ 0$, $Q_i \succeq 0$, $Z_i \succ 0$, and

$$X_i := \begin{bmatrix} X_{11i} & X_{12i} \\ * & X_{22i} \end{bmatrix} \succeq 0, \quad (4)$$

and any appropriately dimensioned matrices N_{1i} and N_{2i} such that the following LMI's hold:

$$\phi_i := \begin{bmatrix} \phi_{11i} & \phi_{12i} & hA_i^T Z_i \\ * & \phi_{22i} & h\bar{A}_i^T Z_i \\ * & * & -hZ_i \end{bmatrix} \prec 0, \quad (5)$$

$$\psi_i := \begin{bmatrix} X_{11i} & X_{12i} & N_{1i} \\ * & X_{22i} & N_{2i} \\ * & * & Z_i \end{bmatrix} \succeq 0,$$

where

$$\begin{aligned} \phi_{11i} &= P_i A_i + A_i^T P_i + N_{1i} + N_{1i}^T + Q_i + hX_{11i}, \\ \phi_{12i} &= P_i \bar{A}_i - N_{1i} + N_{2i}^T + hX_{12i}, \\ \phi_{22i} &= -N_{2i} - N_{2i}^T - (1-d)Q_i + hX_{22i}. \end{aligned}$$

3. MAIN RESULTS

In this section, for the sake of brevity, the results are presented without proofs. Complete details will be given in Koru et al. (2016), which is under preparation.

In the following lemma, inspired by Allerhand and Shaked (2011), we introduce a Lyapunov function for non-switching time delay systems. The Lyapunov parameters are not constant, but piecewise linear in a time interval. We derive the LMI conditions to guarantee the Lyapunov function is decreasing for that time interval.

Lemma 2. Consider any non-switched linear subsystem Σ_i of (1) for an $i \in \mathcal{P}$. For some time interval $t \in [t_0, t_f]$, let us define $\delta = t_f - t_0$ and the Lyapunov function as

$$\begin{aligned} V_i(t, x_t) &= x^T(t) P_i(t) x(t) + \int_{t-r(t)}^{t} x^T(s) Q_i(t) x(s) ds \\ &+ \int_{t-r(t)}^{t} (s - t + h)\, x^T(s) \dot{Q}_i(t) x(s) ds \\ &+ \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}(s) Z_i(t) \dot{x}(s) ds d\theta \\ &+ \int_{-h}^{0}\int_{t+\theta}^{t} (s - t - \theta)\, \dot{x}(s) \dot{Z}_i(t) \dot{x}(s) ds d\theta \end{aligned} \quad (6)$$

where

$$\begin{aligned} P_i(t) &= \text{affine}(P_{i,1}, P_{i,2}, t_0, \delta), \\ Q_i(t) &= \text{affine}(Q_{i,1}, Q_{i,2}, t_0, \delta), \\ Z_i(t) &= \text{affine}(Z_{i,1}, Z_{i,2}, t_0, \delta). \end{aligned}$$

Assume there exists symmetric matrices $P_{i,2}$, $P_{i,1} \succ 0$, $Q_{i,2} \succeq Q_{i,1} \succ 0$, $Z_{i,2} \succeq Z_{i,1} \succ 0$,

$$X_{i,1} = \begin{bmatrix} X_{11i,1} & X_{12i,1} \\ * & X_{22i,1} \end{bmatrix} \succeq 0,$$

$$X_{i,2} = \begin{bmatrix} X_{11i,2} & X_{12i,2} \\ * & X_{22i,2} \end{bmatrix} \succeq 0,$$

and any appropriately dimensioned matrices $N_{1i,1}$, $N_{1i,2}$, $N_{2i,1}$ and $N_{2i,2}$ such that the following LMIs hold:

$${}^1\phi_{i,1} := \begin{bmatrix} {}^1\phi_{11i,1} & {}^1\phi_{12i,1} & -A_i^T\,{}^1\phi_{33i,1} \\ * & {}^1\phi_{22i,1} & -\bar{A}_i^T\,{}^1\phi_{33i,1} \\ * & * & {}^1\phi_{33i,1} \end{bmatrix} \prec 0, \quad (7)$$

$${}^2\phi_{i,1} := \begin{bmatrix} {}^2\phi_{11i,1} & {}^2\phi_{12i,1} & -A_i^T\,{}^2\phi_{33i,1} \\ * & {}^2\phi_{22i,1} & -\bar{A}_i^T\,{}^2\phi_{33i,1} \\ * & * & {}^2\phi_{33i,1} \end{bmatrix} \prec 0, \quad (8)$$

$$\psi_{i,1} := \begin{bmatrix} X_{11i,1} & X_{12i,1} & N_{1i,1} \\ * & X_{22i,1} & N_{2i,1} \\ * & * & Z_{i,1} \end{bmatrix} \succeq 0, \quad (9)$$

$$\psi_{i,2} := \begin{bmatrix} X_{11i,2} & X_{12i,2} & N_{1i,2} \\ * & X_{22i,2} & N_{2i,2} \\ * & * & Z_{i,2} \end{bmatrix} \succeq 0, \quad (10)$$

where

$$^1\phi_{11i,1} = \frac{1}{\delta}(P_{i,2} - P_{i,1}) + P_{i,1}A_i + A_i^T P_{i,1} + N_{1i,1} + N_{1i,1}^T + \frac{h}{\delta}(Q_{i,2} - Q_{i,1}) + Q_{i,1} + hX_{11i,1},$$

$$^2\phi_{11i,1} = \frac{1}{\delta}(P_{i,2} - P_{i,1}) + P_{i,2}A_i + A_i^T P_{i,2} + N_{1i,2} + N_{1i,2}^T + \frac{h}{\delta}(Q_{i,2} - Q_{i,1}) + Q_{i,2} + hX_{11i,2},$$

$$^1\phi_{12i,1} = P_{i,1}\bar{A}_i - N_{1i,1} + N_{2i,1}^T + hX_{12i,1},$$

$$^2\phi_{12i,1} = P_{i,2}\bar{A}_i - N_{1i,2} + N_{2i,2}^T + hX_{12i,2},$$

$$^1\phi_{22i,1} = -N_{2i} - N_{2i}^T - (1-d)Q_{i,1} + hX_{22i,1},$$

$$^2\phi_{22i,1} = -N_{2i} - N_{2i}^T - (1-d)Q_{i,2} + hX_{22i,2},$$

$$^1\phi_{33i,1} = -hZ_{i,1} - \frac{h^2}{2\delta}(Z_{i,2} - Z_{i,1}),$$

$$^2\phi_{33i,1} = -hZ_{i,2} - \frac{h^2}{2\delta}(Z_{i,2} - Z_{i,1}).$$

Then the Lyapunov function in (6) is decreasing in the time interval $t \in [t_0, t_f]$.

In Lemma 2, we get the conditions of time-scheduled Lyapunov-Krasovskii functionals for a segment from $P_{i,1}$ to $P_{i,2}$. In the following theorem, we extend the idea of Lemma 2 to K many segments from P_{i0} up to P_{iK}. By using this extension, the conditions of stability for a given dwell time, which is represented as the total time from P_{i0} to P_{iK}, is given. Before presenting the theorem, let us introduce the new variables

$$\phi_{i,K} := \phi_i \text{ evaluated at } P_i = P_{i,K}, \quad Q_i = Q_{i,K}, \quad Z_i = Z_{i,K}, \quad N_{1i} = N_{1i,K}, \quad N_{2i} = N_{2i,K}, X_i = X_{i,K},$$

Theorem 3. Consider the system (1) with time varying delay, $r(t)$. Assume that for given scalars h, d and some dwell time $\tau_D > 0$, there exists collection of symmetric matrices $P_{i,k} \succ 0$, $Q_{i,k} \succ 0$, $Z_{i,k} \succ 0$ for $k = 0, \ldots, K$, where K is a prechosen integer, and a sequence

$$\left\{ \delta_k > 0, \quad \sum_{k=1}^{K} \delta_k = \tau_D \right\}$$

such that, for all $i \in \mathcal{P}$ following LMIs hold:

$$\begin{aligned}
&^1\phi_{i,k} \prec 0, \quad ^2\phi_{i,k} \prec 0, && \forall k = 0, \ldots, K-1 \\
&\psi_{i,k} \succeq 0, \quad X_{i,k} \succeq 0, && \forall k = 0, \ldots, K \\
&\phi_{i,K} \prec 0, && \\
&Q_{i,k+1} - Q_{i,k} \succeq 0, && \forall k = 0, \ldots, K-1 \\
&Z_{i,k+1} - Z_{i,k} \succeq 0, && \forall k = 0, \ldots, K-1 \\
&P_{i,K} - P_{j,0} \succeq 0, && \forall j \in \mathcal{P},\ j \neq i \quad (11) \\
&Q_{i,K} - Q_{j,0} - \frac{h}{\delta_k}(Q_{j,1} - Q_{j,0}) \succeq 0, && \forall j \in \mathcal{P},\ j \neq i \quad (12) \\
&Z_{i,K} - Z_{j,0} - \frac{h}{\delta_k}(Z_{j,1} - Z_{j,0}) \succeq 0, && \forall j \in \mathcal{P},\ j \neq i \quad (13)
\end{aligned}$$

Then, the system (1) is globally asymptotically stable for any switching law with dwell time greater than or equal to τ_D.

For given scalars K and τ_D, time intervals can be chosen equidistant as

$$\delta_k = \frac{\tau_D}{K}, \quad \forall k = 1, \ldots, K,$$

and all of the conditions are LMIs. In the virtue of the proposed representation for the stability condition, one can derive the dwell time by using a bisection algorithm. In the numerical examples, we solved the problems by using SeDuMi, see Sturm (1999).

A discussion on the effect of selection of K for the non-delayed case can be found in Xiang (2015). As K increases, the results are less conservative. On the other hand, the increment in K leads to high computational cost.

4. NUMERICAL EXAMPLES

In this section, the examples are taken from published papers for comparison purposes. The examples 1 and 2 can be found in Çalışkan et al. (2013), Yan and Özbay (2008), and Koru et al. (2014). Comparison results are summarized in Table 1.

Example 1. Let Σ_1 be

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \qquad \bar{A}_1 = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix},$$

and let Σ_2 be

$$A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \qquad \bar{A}_2 = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}.$$

Upper bounds for the time delay are $h = 0.6$ and $d = 0$. Resulting dwell time is $\tau_D = 1.06 \times 10^{-5}$. For those parameters, the switched delay system admits common Lyapunov functions, e.g., $P = Q = Z = I$.

Example 2. Let Σ_1 be

$$A_1 = \begin{bmatrix} -1.799 & -0.814 \\ 0.2 & -0.714 \end{bmatrix}, \qquad \bar{A}_1 = \begin{bmatrix} -1 & 0 \\ -0.45 & -1 \end{bmatrix},$$

and let Σ_2 be

$$A_2 = \begin{bmatrix} -1.853 & -0.093 \\ -0.853 & -1.1593 \end{bmatrix}, \qquad \bar{A}_2 = \begin{bmatrix} -1 & 0 \\ 0.05 & -1 \end{bmatrix}.$$

Upper bounds for the time delay are $h = 0.2$ and $d = 0$. Similar to Example 1, resulting dwell time is 7.26×10^{-6} seconds and this switched delay system admits common Lyapunov functions, e.g., $P = Q = Z = I$.

5. CONCLUSIONS

We performed the calculation of minimum dwell time to ensure stability of switched delay systems. Time-scheduled Lyapunov-Krasovskii functionals are found using free weighting matrices method. Represented dwell time is minimized using SDP techniques. Improvements over the results obtained in previously published papers are shown in numerical examples.

Table 1. Comparative Dwell Time Results of Examples 1 and 2

	Ex. 1	Ex. 2
Yan and Özbay (2008)	6.51 s	–
Çalışkan et al. (2013)	3.4 s	0.72 s
Koru et al. (2014)	1.11 s	0.58 s
Present Paper	1.06×10^{-5} s	7.26×10^{-6} s

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