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The finite/infinite horizon ruin problem with multi-threshold premiums: a Markov fluid queue approach

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Abstract We present a new numerical method to obtain the finite- and infinite-horizon ruin probabilities for a general continuous-time risk problem. We assume the claim arrivals are modeled by the versatile Markovian arrival process, the claim sizes are PH-distributed, and the premium rate is allowed to depend on the instantaneous risk reserve in a piecewise-constant manner driven by a number of thresholds, i.e., multi-threshold premiums. We introduce a novel sample path technique by which the ruin problems are shown to reduce to the steady-state solution of a certain multi-regime Markov fluid queue. We propose to use the already existing numerically efficient and stable numerical algorithms for such Markov fluid queues. Numerical results are presented to validate the effectiveness of the proposed method regarding the computation of the finite- and infinite-horizon ruin probabilities for risk models including those with relatively large number of thresholds.

Keywords Finite/infinite horizon ruin probabilities · Markov fluid queues · Erlangization

Mathematics Subject Classification 60K25 · 91B30

1 Introduction

We consider a risk reserve (or surplus) process in continuous-time $\{U(t), t \geq 0\}$ with the number of claims up to time t denoted by $\{N(t), t \geq 0\}$. The initial reserve at time $t = 0$ is denoted by u. We assume the claim arrivals are modeled by a Markovian arrival process (MAP) [or called Neuts' versatile point process in Neuts (1977)] with m states and characterized with the matrix pair (D_0, D_1); see Neuts (1989), Lucantoni et al. (1990), and Latouche

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and Ramaswami (2002) for more details on the versatile MAP process. Although MAP refers to a Markov additive process in some contexts such as in Asmussen and Albrecher (2010), a concept dating back to Cinlar (1972), it is used to represent a Markovian arrival process in the current paper. The matrices D_0 and D_1 are $m \times m$, D_0 has negative diagonal elements and non-negative off-diagonal elements, D_1 is non-negative, and $D = D_0 + D_1$ is an irreducible infinitesimal generator. The matrices D_0 and D_1 govern the transitions among the states of the MAP without and with arrivals, respectively. The initial probability vector of the MAP at time $t = 0$ is denoted by the row vector $v = \{v_i\}$, $i = 1, \ldots, m$, where v_i denotes the probability that the MAP starts operation in state i at time $t = 0$. The claims occur at time instants t_1, t_2, ..., with corresponding claim payoffs (or claim sizes) denoted by $S_1, S_2, \ldots$, respectively. We assume the claim sizes are distributed according to a PH (Phase Type) distribution with s phases characterized with the pair (α, T); see Neuts (1981) for PH-distributions and their properties. In this characterization, the $1 \times s$ row vector α represents the initial probability vector of the distribution and the $s \times s$ sub-stochastic matrix T governs the transitions among the s transient phases. Furthermore, we assume that a premium is collected at a rate of $p(x)$ at time t which depends on the instantaneous reserve $U(t) = x$ in a piecewise-constant manner:

$$p(x) = p_k > 0, \quad T^{(k-1)} \leq x < T^{(k)}, \quad 1 \leq k \leq K, \tag{1}$$

where the $K - 1$ values $T^{(k)}$, $1 \leq k < K$ are the premium thresholds satisfying $0 = T^{(0)} < T^{(1)} < \cdots < T^{(K-1)} < T^{(K)} = \infty$. Finally, the risk reserve satisfies the following equation given in Petersen (1989):

$$U(t) = u + \int_0^t p(U(y))dy - \sum_{i=1}^{N(t)} S_i. \tag{2}$$

The time of ruin denoted by $\tau(u, v)$ is given by

$$\tau(u, v) = \inf\{t > 0 : U(t) < 0\}, \tag{3}$$

and the finite-horizon ruin probability before the so-called horizon value H denoted by $\psi(u, v, H)$ is then given by

$$\psi(u, v, H) = \Pr(\tau(u, v) \leq H). \tag{4}$$

The infinite-horizon ruin probability is then expressed as the following limit:

$$\psi(u, v) = \lim_{H \to \infty} \psi(u, v, H). \tag{5}$$

Risk models when the insurer is allowed to borrow money at a debit interest rate when the surplus hits zero, see for example a recent work by Cai and Yang (2014), are not considered in this paper. Risk models described by reserve-dependent premiums according to (1) are referred to as multi-threshold (with $K - 1$ thresholds) or multi-regime (with K regimes). Also note that multi-regime dividend strategies can also be addressed using the same framework where the net rate of change of the surplus process, namely premium income rate minus the dividend payout rate, turns out to be a piecewise-constant function of the current reserve.

Computational risk problems of various types are studied in depth by Asmussen and Albrecher (2010) and Asmussen and Rolski (1991) using duality results between risk problems and queues. The finite-horizon ruin probability with Poisson claim arrivals, PH-type claim sizes, and constant premiums is studied in Asmussen et al. (2002) using Erlangization in which an l-stage Erlang distribution is used to approximate the deterministic horizon value H. The Erlangization idea is extended by Stanford et al. (2005) to the Sparre–Andersen risk model in which the claim arrival process is allowed to be of more general renewal-type. There is however much fewer research on Markovian risk models or those with MAP claim arrivals; see Asmussen (1989) and Cheung et al. (2011) and the references therein. Recently, a connection has been established between risk problems and fluid queues in several studies; see Badescu et al. (2005), Ahn et al. (2007), Badescu and Landriault (2009). This connection allows computational procedures developed for Markov fluid queues (MFQ) to be reused in the risk context.

Premiums with continuous reserve dependency have been addressed by Petersen (1989) and Asmussen and Bladt (1996) for Poisson claim arrivals and infinite-horizon ruin probabilities. An efficient simulation-based method is presented by Michaud (1996) to obtain the infinite-horizon ruin probabilities for the case of reserve-dependent premiums. Multi-regime risk models have recently been investigated in a number of studies. Asmussen and Bladt (1996) provide an exact expression for infinite-horizon ruin probabilities for Poisson claim arrivals and general claim sizes for only two regimes. For arbitrary number of regimes, Lin and Sendova (2008) derive a piecewise integro-differential equation for the same problem for the discounted penalty function and the probability of ruin is computed for a three-regime model to illustrate the applicability of the proposed approach. A recursive approach is presented for a similar general problem by Albrecher and Hartinger (2007) and the tractability of the approach is illustrated numerically for a four-regime risk model with exponential claim size distributions. The work of Badescu et al. (2007) studies a multi-regime risk model with MAP claim arrivals and the authors obtain the Laplace-Stieltjes transform (LST) of the time to ruin and obtain the finite-horizon ruin probabilities by LST inversion using the Gaver-Stehfest numerical inversion procedure also detailed by Badescu et al. (2005). A four-regime risk model is presented in this work to illustrate the effectiveness of the approach.

Our work differs from the existing ones in that we unify MAP claim arrivals, PH-type claim sizes, and multi-threshold premiums using the theory of multi-regime MFQs for calculating finite- and infinite-horizon ruin probabilities. The method of Erlangization based on the work of Asmussen et al. (2002) is employed to approximate the deterministic finite horizon by the Erlang distribution. The following lists the contributions of this paper:

- The connection between the multi-threshold risk problem and the arising MFQ is based on a novel sample path technique we introduce in this paper, which is entirely different than the ones used to demonstrate the connection between risk models and fluid queues such as the works of Badescu et al. (2005), Ramaswami (2006), Badescu and Landriault (2009), and Asmussen and Albrecher (2010). In the work of Asmussen and Albrecher (2010, Chapter 3), the duality is established between the surplus process and a storage process constructed by time-reverting the original surplus process. However, the sample path technique that we propose does not use time reversion and we construct a novel embedding of the whole sequence of independent risk processes into a single trajectory of a multi-regime MFQ. With this embedding in place, one can express the finite-horizon ruin probability in terms of the steady-state probability mass accumulations of the associated MFQ at certain levels. This is in contrast with the conventional approach of embedding only one realization of the risk process into again a single realization of certain MFQ, and then study various passage times as opposed to the steady-state behavior; see Badescu et al. (2005) and Ramaswami (2006). Moreover, this novel technique allows us to obtain the ruin probabilities via the calculation of the steady-state solution of the arising fluid queue up to a constant without having to solve the normalization equation which was shown by Yazici and Akar (2013) to be a significant contributor to the overall computation time.

- The numerical algorithm we propose avoids the use of LSTs and is purely matrix-analytical. Moreover, the algorithm relies on the use of well-established matrix algorithms such as ordered Schur decomposition and block LU decomposition described in detail in Golub and van Loan (1996) and also in Yazici and Akar (2013).
- Contrary to only a few regimes used for illustrating the effectiveness of existing multi-regime risk models, the matrix-analytical algorithm we propose allows one to employ arbitrarily large number of regimes stemming from the $O(K)$ computational complexity of the proposed algorithm. With this feature, more granular multi-threshold premium or dividend strategies can be studied. To illustrate this feature, a particular numerical example is provided with 5×10^4 thresholds.

The article is organized as follows. Section 2 addresses single-regime MFQs and their multi-regime extension. In Sect. 3, we reduce the problem of obtaining finite-horizon ruin probabilities to the steady-state solution of a certain multi-regime MFQ using novel sample path arguments and Erlangization. The infinite-horizon counterpart of the same problem is then provided as a particular subcase. Numerical results are presented in Sect. 4. Finally, we conclude.

2 Markov fluid queues

A MFQ is a joint process $\{X(t), Z(t), t \geq 0\}$, where $Z(t)$ represents the state of an N-dimensional continuous-time Markov chain, and $X(t)$ stands for the queue occupancy at time t; see Kulkarni (1997). The net drift into or out of the MFQ at time t is denoted by r_i when $Z(t) = i,\ i = 1, 2, \ldots, N$ and $Z(t)$ is thus called the background (or modulating) process. When the queue capacity is finite, i.e., $X(t) \leq B$ for a non-zero capacity B, the solution of MFQs in terms of the steady-state joint probability density function (pdf) vector of the queue occupancy, i.e., $f(x) = \left[f_1(x) \cdots f_N(x)\right], 0 \leq x \leq B$, is found through the system of differential equations:

$$\frac{d}{dx} f(x) R = f(x) Q, \quad 0 < x < B$$

along with a number of boundary conditions where $R = \mathrm{diag}(r_1, r_2, \ldots, r_N)$ is the diagonal matrix of drifts and Q denotes the infinitesimal generator of the background process in Kulkarni (1997). Numerical methods to obtain $f(x)$ both in infinite and finite queue capacity cases are available; see Kulkarni (1997) and Akar and Sohraby (2004).

A generalization of MFQs is the so-called multi-regime MFQs (MRMFQ) in Kankaya and Akar (2008), also called feedback, level-dependent, multi-layer, and multi-threshold MFQs in the works of Mandjes et al. (2003), Silva and Latouche (2009), Bean and O'Reilly (2008), and Badescu et al. (2007), respectively. In MRMFQs, the finite queue space $(0, B)$ is partitioned into K regimes using $K+1$ thresholds, $0 = T^{(0)} < T^{(1)} < \cdots < T^{(K-1)} < T^{(K)} = B$. The queue is said to be in regime k when $T^{(k-1)} < X(t) < T^{(k)}$. Associated with each regime $k,\ 1 \leq k \leq K$, we have a distinct infinitesimal generator and drift matrix, denoted $Q^{(k)}$ and $R^{(k)}$, respectively. In addition, we define the infinitesimal generator and drift matrices, denoted by $\tilde{Q}^{(k)}$ and $\tilde{R}^{(k)}$, respectively, and the probability mass accumulations, $c^{(k)} = \left[c_1^{(k)} \cdots c_N^{(k)}\right]$, all defined at the regime boundaries $T^{(k)},\ 0 \leq k \leq K$. To be precise,

$$c_i^{(k)} = \lim_{t \to \infty} Pr(Z(t) = i, X(t) = T^{(k)}), \quad 0 \leq k \leq K. \tag{6}$$

Note that an MFQ is an MRMFQ with one single regime. It has been shown in Kankaya and Akar (2008) that the steady-state joint pdf vector $f(x)$ of the K-regime fluid queue satisfies the following differential equation:

$$\frac{d}{dx} f^{(k)}(x) R^{(k)} = f^{(k)}(x) Q^{(k)}, \quad T^{(k-1)} < x < T^{(k)}, \quad 1 \le k \le K, \tag{7}$$

along with the boundary conditions given below:

$$c_m^{(0)} = 0, \quad \forall m \in S_+^{(1)} \tag{8}$$

$$c_m^{(k)} = 0, \quad \forall m \in \left(S_+^{(k)} \cap S_+^{(k+1)}\right) \cup \left(S_-^{(k)} \cap S_-^{(k+1)}\right), \quad 1 \le k < K \tag{9}$$

$$c_m^{(k)} = 0, \quad \forall m \in \left(S_-^{(k)} \cap S_+^{(k+1)}\right) \cap \left(\tilde{S}_+^{(k)} \cup \tilde{S}_-^{(k)}\right), \quad 1 \le k < K \tag{10}$$

$$c_m^{(K)} = 0, \quad \forall m \in S_-^{(K)} \tag{11}$$

$$f^{(1)}(0+) R^{(1)} = c^{(0)} \tilde{Q}^{(0)} \tag{12}$$

$$f^{(k+1)}(T^{(k)}+) R^{(k+1)} - f^{(k)}(T^{(k)}-) R^{(k)} = c^{(k)} \tilde{Q}^{(k)}, \quad 1 \le k < K \tag{13}$$

$$f_m^{(k)}(T^{(k)}-) = 0, \quad \forall m \in S_-^{(k)} \cap \left(\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)}\right), \quad 1 \le k < K \tag{14}$$

$$f_m^{(k+1)}(T^{(k)}+) = 0, \quad \forall m \in \left(\tilde{S}_0^{(k)} \cup \tilde{S}_-^{(k)}\right) \cap S_+^{(k+1)}, \quad 1 \le k < K \tag{15}$$

$$f^{(K)}(B-) R^{(K)} = -c^{(K)} \tilde{Q}^{(K)} \tag{16}$$

$$\left(\sum_{k=1}^{K} \int_{T^{(k-1)}+}^{T^{(k)}-} f^{(k)}(x) dx + \sum_{k=0}^{K} c^{(k)} \right) \mathbf{1}_N = 1, \tag{17}$$

where $f^{(k)}(x) = f(x)$ in regime k and the sets $S_0^{(k)}$, $S_-^{(k)}$ and $S_+^{(k)}$ denote the disjoint subsets of the state space $S = \{1, \ldots, N\} = S_0^{(k)} \cup S_-^{(k)} \cup S_+^{(k)}$ that include the states with zero, negative, and positive drifts, respectively, in regime k. Similarly, $\tilde{S}_0^{(k)}$, $\tilde{S}_-^{(k)}$ and $\tilde{S}_+^{(k)}$ are defined at the regime boundaries $T^{(k)}$. The Eq. (17) is called the normalization equation in Yazici and Akar (2013) and the notation $\mathbf{1}_r$ used in the normalization equation denotes a column vector of ones with size r. In the same spirit, we denote a matrix of zeros of size $r \times w$ by $\mathbf{0}_{r \times w}$ and the identity matrix of size r by I_r. However, we drop the subscripts in the above notation when the sizes of the matrices are clear from the context.

When the MRMFQ in question is infinite, the Eq. (11) is replaced with $c^{(K)} = [0 \ \cdots \ 0]$ as there can be no probability mass at infinity, and (16) is replaced by conditions to ensure stability in the final regime K of infinite length. For details on MRMFQs, we refer the reader to Kankaya and Akar (2008). A numerical method is proposed by Kankaya and Akar (2008) for solving the steady-state joint pdf vector of MRMFQs using the additive decomposition (AD) method which employs the ordered Schur decomposition as its main engine; see also Akar and Sohraby (2004) for details on the AD method in the context of single-regime MFQs. In cases where the number of the thresholds is large, large linear systems of equations stemming from the boundary conditions arise. However, the linear system of equations due to the boundary conditions involves a structured matrix that is block-banded. This enables one to exploit this block-banded structure to avoid the $O(K^3)$ complexity associated with general linear systems. In particular, an algorithm based on block-tridiagonal LU decomposition (Golub and van Loan 1996, pp. 174–175) is described in Yazici and Akar (2013) that is shown to have an $O(K)$ computational complexity.

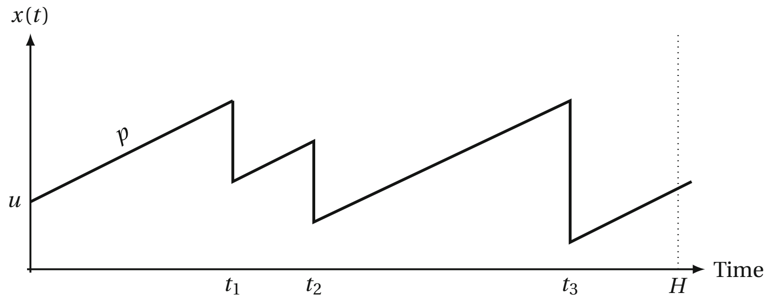

Fig. 1 A sample path of the reserves; claim arrivals occur at times $t_1,\ t_2$ and t_3

In the sequel, we will first reduce the finite-horizon ruin problem to the steady-state solution of a specific MRMFQ. Then, the framework of Kankaya and Akar (2008) along with Yazici and Akar (2013) is to be used to obtain a numerical solution to the MRMFQ.

3 Finite-horizon ruin problem as a multi-regime Markov fluid queue

3.1 Problem definition

The finite-horizon ruin problem can be described as follows. An insurer starts business with a certain amount of initial reserve denoted by $U(0) = u$, and collects premiums from its clients with rate $p(x)$ at time t if the risk reserve at time t is $U(t) = x$ according to (1). Recall that claim arrivals are of MAP-type characterized with the matrix pair (D_0, D_1) with initial probability vector v. When a claim arrives, the insurer pays out the claim size which is PH-distributed with parameter pair (α, T). We also define $T^0 = -T\mathbf{1}$. We are interested in finding the finite-horizon ruin probability $\psi(u, v, H)$ defined in (4). A sample path of the risk reserve process for the special case of a fixed premium rate $p(x) = p$ is given in Fig. 1. In this particular example, claim arrivals occur at times $t_1,\ t_2$, and t_3, which lead to downward jumps in the reserve by an amount dictated by the PH-type claim size distribution. In order to model this problem as an MRMFQ, we first need to eliminate the jumps in the sample path. To this end, we employ the transformation first described in Dzial et al. (2005) and used in many others including Yazici and Akar (2013). In this method, downward (upward) jumps are replaced by linear decreases (increases) at fixed but arbitrary rates, but the states used in the background process for these linear displacements are eventually censored out to obtain the performance measures of interest.

3.2 MRMFQ formulation

We first describe the state-space of the background process of the MRMFQ of interest. Barring the instants of claim payoffs, the system wanders in the MAP states of the claim arrival process. Taking into account the l-stage Erlangization for the horizon, we will need l replicas of each MAP state for each of the Erlang levels (or stages); called a *composite* state which corresponds to a pair $(i, j),\ i = 1, \ldots, m,\ j = 1, \ldots, l$ totaling ml composite states. Moreover, once a claim arrives, we need to have a linear decrease in the risk reserve corresponding to the claim size. For this purpose, for each composite state, we need a replica of the PH-type phases representing the claim size, leading to a *merged* state which corresponds

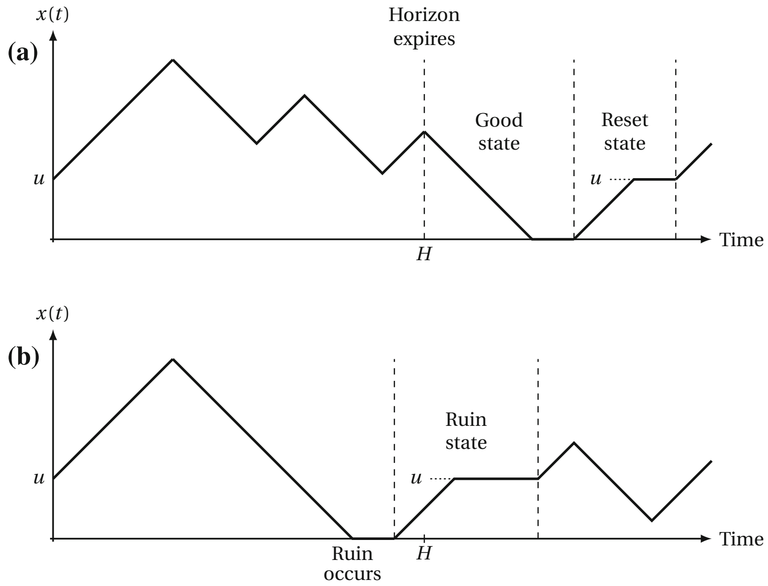

Fig. 2 Sample paths for **a** no ruin (Erlang levels representing the horizon expire before the reserve value hits 0) and **b** ruin (the reserve value hits 0 first)

to a 3-tuple $(i, j, k),\ i = 1, \ldots, m,\ j = 1, \ldots, l,\ k = 1, \ldots, s$ totaling mls merged states. Next, we introduce three auxiliary states, namely the *Ruin*, *Good* and *Reset* states, which are crucial for our MRMFQ model. These will enable us to express the finite-horizon ruin probability in terms of the steady-state solution to the MRMFQ to be described. What we achieve with these three states is quite similar to what one would do to obtain the finite-horizon ruin probability via simulation: One would initialize the risk reserve to u, start a timer that will expire at time H (which we replace with an l-stage Erlang distribution with mean H) and record whether ruin occurs before the timer expires or not. Upon ruin or the expiry of the timer, the parameters are reset and a new iteration is started. For this purpose, with the MRMFQ formulation, in case of ruin before the horizon, the background process transits into Ruin state, which subsequently resets the reserve value to u. On the other hand, if the horizon is reached (i.e., the background process transits out of the final stage of the Erlang distribution) before ruin, the background process transits into the Good state. This state drives the reserve value back to zero, and then a transition into the Reset state occurs. The Reset state, similar to the Ruin state, resets the reserve value to u. Then, from either of these states, the background process transits into one of the composite states according to the initial probability vector v. Obviously, this composite state can only belong to the first Erlang stage.

Two sample paths for each case is given in Fig. 2. In Fig. 2a, the background process exits a composite state belonging to the final Erlang stage while the reserve value is still positive; therefore no ruin is observed in this situation. Therefore, the background process transits to the Good state, followed by the Reset state. In Fig. 2b however, the reserve value hits zero before the last Erlang stage is completed; therefore, the background process transits to the Ruin state. In either case, the reserve value is pulled to the initial value of u, and the cycle starts over from a composite state in the first Erlang stage. It is important to note that the drifts in both the Ruin and Reset states are $+1$ and equal.[1] Therefore, in the steady-state, the probability of ruin and no-ruin before H is proportional to the probability of the MRMFQ being in the Ruin and Reset states, respectively.

In order to completely formulate the MRMFQ, we need to rigorously describe the behavior of the background process at $x = 0$ and $x = u$ in the Ruin, Good and Reset states. When the background process transits into the Ruin state, the reserve value is always zero since ruin has occurred. The drift in the Ruin state is $+1$ causing the reserve to increase. This continues until the reserve becomes u. Meanwhile, the transition rate out of the Ruin state is zero. When the reserve is u, the drift in the Ruin state becomes zero, and the transition rate out of the Ruin state is set to unity. Therefore, the background process stays in the Ruin state at u for an exponential amount of time with unity mean. On the other hand, when the background process transits into the Good state, the reserve value is always positive. The drift in the Good state is -1 and the transition rate out is zero, causing the reserve to be depleted. At $x = 0$, the background process waits for an exponentially distributed amount of time with unity mean, and subsequently transits into the Reset state. The drift in the Reset state is $+1$ and the transition rate out is zero until the reserve becomes u, causing the reserve to be driven to u. After this occurs, the background process stays in the Reset state at u for an exponentially distributed amount of time with unity mean, and then transits into one of the composite states in the first Erlang stage. Lastly, when the background process is in one of the merged states, the transition rate into the Ruin state is 0 as long as the reserve stays positive. When the queue hits zero however at this state, the transition rate into the Ruin state is set to $+1$ and the transition rate into any other state is zero, causing the background process to transit into the Ruin state after an exponentially distributed amount of time with unity mean that is spent at the reserve level 0. Moreover, when the background process is in one of the composite states in the final Erlang stage while the reserve is positive, and a transition out of this state occurs due to the Erlang stage expiring, the background process transits invariably into the Good state.

In the formulation of the MRMFQ, we assume a PH-type representation of the l-stage Erlang distribution characterized with the pair (α_h, T_h) where the initial probability vector $\alpha_h = [1\ 0\ \cdots\ 0]$ and

$$T_h = \begin{bmatrix} -\frac{l}{H} & \frac{l}{H} & & & \\ & -\frac{l}{H} & \frac{l}{H} & & \\ & & & \ddots & \\ & & & & -\frac{l}{H} \end{bmatrix},$$

where l is the number of Erlang stages used to represent the horizon. Clearly, the length of the vector α_h is l. Also we define $T_h^0 = -T_h\mathbf{1}$. We order the MRMFQ states as (1) ruin state, (2) reset state, (3) good state, (4) lexicographically-ordered composite states and (5) lexicographically-ordered merged states, comprising an overall of $ml(s + 1) + 3$ states. In light of the description of the behavior of the MRMFQ in either of the five types of states, one can write the generator matrix $Q^{(k)}$, $1 \le k \le K$, in each regime of the MRMFQ in block form as follows:

[1] What is important is that the two drift values are equal, rather than their values. The same arguments would be valid if the drifts were selected some value different than $+1$, but still equal.

$$Q^{(k)} = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & \mathbf{0}_{1\times ml} & \mathbf{0}_{1\times mls} \\ \hline 0 & 0 & 0 & \mathbf{0}_{1\times ml} & \mathbf{0}_{1\times mls} \\ \hline 0 & 0 & 0 & \mathbf{0}_{1\times ml} & \mathbf{0}_{1\times mls} \\ \hline \mathbf{0}_{ml\times 1} & \mathbf{0}_{ml\times 1} & T_h^0 \otimes \mathbf{1}_m & (I_l \otimes D_0) + (T_h \otimes I_m) & (I_l \otimes D_1) \otimes \alpha \\ \hline \mathbf{0}_{mls\times 1} & \mathbf{0}_{mls\times 1} & \mathbf{0}_{mls\times 1} & I_{ml} \otimes T^0 & I_{ml} \otimes T \end{array}\right]. \quad (18)$$

Here, the thresholds of the MRMFQ are determined by the thresholds of the premium rate function. Without loss of generality, we assume that $u \in \{T^{(0)}, T^{(1)}, \ldots, T^{(K-1)}\}$, as one can always define $p(x)$ using a spurious threshold at u even if u is not a discontinuity point, in which case the value of $p(x)$ in $[T^{(J-1)}, T^{(J)})$ would be equal to the value of $p(x)$ in $[T^{(J)}, T^{(J+1)})$, where $T^{(J)} = u$. The generator matrices at the thresholds satisfy $\tilde{Q}^{(k)} = Q^{(k)},\ k \in \{1, 2, \ldots, J-1, J+1, \ldots, K-1\}$. On the other hand, the matrix $\tilde{Q}^{(J)}$ can be written as:

$$\tilde{Q}^{(J)} = \left[\begin{array}{c|c|c|c|c} -1 & 0 & 0 & \alpha_h \otimes v & \mathbf{0}_{1\times mls} \\ \hline 0 & -1 & 0 & \alpha_h \otimes v & \mathbf{0}_{1\times mls} \\ \hline 0 & 0 & 0 & \mathbf{0}_{1\times ml} & \mathbf{0}_{1\times mls} \\ \hline \mathbf{0}_{ml\times 1} & \mathbf{0}_{ml\times 1} & T_h^0 \otimes \mathbf{1}_m & (I_l \otimes D_0) + (T_h \otimes I_m) & (I_l \otimes D_1) \otimes \alpha \\ \hline \mathbf{0}_{mls\times 1} & \mathbf{0}_{mls\times 1} & \mathbf{0}_{mls\times 1} & I_{ml} \otimes T^0 & I_{ml} \otimes T \end{array}\right]. \quad (19)$$

We also write $\tilde{Q}^{(0)}$ as follows:

$$\tilde{Q}^{(0)} = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & \mathbf{0}_{1\times ml} & \mathbf{0}_{1\times mls} \\ \hline 0 & 0 & 0 & \mathbf{0}_{1\times ml} & \mathbf{0}_{1\times mls} \\ \hline 0 & 1 & -1 & \mathbf{0}_{1\times ml} & \mathbf{0}_{1\times mls} \\ \hline \mathbf{0}_{ml\times 1} & \mathbf{0}_{ml\times 1} & T_h^0 \otimes \mathbf{1}_m & (I_l \otimes D_0) + (T_h \otimes I_m) & (I_l \otimes D_1) \otimes \alpha \\ \hline \mathbf{1}_{mls} & \mathbf{0}_{mls\times 1} & \mathbf{0}_{mls\times 1} & \mathbf{0}_{mls\times ml} & -I_{mls} \end{array}\right]. \quad (20)$$

The reserve-dependent diagonal drift matrices are given as follows:

$$\begin{aligned} R^{(k)} &= \text{diag}(1, 1, -1, p_k I_{ml}, -I_{mls}),\ \text{for } 1 \le k \le J, \\ R^{(k)} &= \text{diag}(-1, -1, -1, p_k I_{ml}, -I_{mls}),\ \text{for } J+1 \le k \le K, \\ \tilde{R}^{(k)} &= R^{(k)},\ \text{for } k \in \{1, 2, \ldots, J-1, J+1, \ldots, K-1\}, \\ \tilde{R}^{(0)} &= \text{diag}(1, 1, 0, p_1 I_{ml}, \mathbf{0}_{mls\times mls}), \\ \tilde{R}^{(J)} &= \text{diag}(0, 0, -1, p_J I_{ml}, -I_{mls}). \end{aligned} \quad (21)$$

Based on the framework given in Kankaya and Akar (2008) and Yazici and Akar (2013), the steady-state pdf vector in each regime is written in mixed matrix-exponential form as

$$f^{(k)}(x) = a^{(k)} \begin{bmatrix} L_0^{(k)} \\ e^{A_-^{(k)}(x - T^{(k-1)})} L_-^{(k)} \\ e^{-A_+^{(k)}(T^{(k)} - x)} L_+^{(k)} \end{bmatrix}, \quad T^{(k-1)} < x < T^{(k)},$$

where $a^{(k)} = \begin{bmatrix} a_0^{(k)} & a_-^{(k)} & a_+^{(k)} \end{bmatrix}$ is a coefficient vector that needs to be solved for, using the boundary conditions (8)–(17). Here, the matrix defined by

$$Y^{(k)} = \begin{bmatrix} L_0^{(k)} \\ L_-^{(k)} \\ L_+^{(k)} \end{bmatrix}^{-1}$$

is a similarity transformation that puts $A^{(k)} = Q^{(k)} \left(R^{(k)}\right)^{-1}$ into block-diagonal form:

$$\left(Y^{(k)}\right)^{-1} A^{(k)} Y^{(k)} = \begin{bmatrix} 0 & & \\ & A_-^{(k)} & \\ & & A_+^{(k)} \end{bmatrix},$$

where $A_-^{(k)}$ has all its eigenvalues in the open left half plane and $A_+^{(k)}$ has all its eigenvalues in the open right half plane. See Kankaya and Akar (2008) for a numerically efficient and stable algorithm based on ordered Schur decomposition along with a Sylvester equation to calculate $Y^{(k)}$. Then, using (8)–(17), a system of linear equations in $a^{(k)}$, $1 \leq k \leq K$ and $c^{(k)}$, $0 \leq k \leq K$, is formed. Observe that each $a^{(k)}$ appears in equations involving only $a^{(k-1)}$, $a^{(k+1)}$, $c^{(k-1)}$ and $c^{(k)}$; and each $c^{(k)}$ appears in equations involving only $a^{(k)}$ and $a^{(k+1)}$. Therefore, the system of linear equations can be made block-banded by ordering the unknown coefficient vectors as

$$\left[c^{(0)}\ a^{(1)}\ a^{(2)} \cdots a^{(J)}\ c^{(J)}\ a^{(J+1)} \cdots a^{(K-1)}\ a_-^{(K)}\right] \tag{22}$$

since $a_+^{(K)}$, $a_0^{(K)}$ and $c^{(K)}$ would be zero in the infinite queue capacity case; see Kankaya and Akar (2008). The block-banded structure can then be exploited in order to reduce the computation time significantly. An algorithm with linear complexity in terms of the number of regimes K based on block-tridiagonal LU factorization (Golub and van Loan 1996, pp. 174–175) is given in Yazici and Akar (2013) for this purpose.

By inspecting R and $\tilde{R}$ matrices, we see that the boundary conditions (9), (10) and (14) do not introduce equations. Moreover, since $p(x)$ is bounded away from zero for any x, there are no probability masses apart from the ones at zero and u, which is a fact also reflected in (22). The probability masses at $x = 0$, namely $c^{(0)}$, are due to the Good state and the merged states, whereas the probability masses at $x = u$, namely $c^{(J)}$, are due to only the Ruin and Reset states. Denoting the probability masses at u in the Ruin and Reset states by $c_{\text{ru}}^{(u)}$ and $c_{\text{rs}}^{(u)}$ respectively, the finite-horizon ruin probability is given by

$$\psi(u, v, H) = \frac{c_{\text{ru}}^{(u)}}{c_{\text{ru}}^{(u)} + c_{\text{rs}}^{(u)}} = \left(1 + \frac{c_{\text{rs}}^{(u)}}{c_{\text{ru}}^{(u)}}\right)^{-1}. \tag{23}$$

In fact, $\psi(u, v, H)$ can be written in terms of other quantities in this context. For instance, denoting the probability mass at $x = 0$ in the Good state by $c_{\text{gd}}^{(0)}$, it is obvious that $c_{\text{gd}}^{(0)} = c_{\text{rs}}^{(u)}$ and hence we can write

$$\psi(u, v, H) = \frac{c_{\text{ru}}^{(u)}}{c_{\text{ru}}^{(u)} + c_{\text{gd}}^{(0)}} = \left(1 + \frac{c_{\text{gd}}^{(0)}}{c_{\text{ru}}^{(u)}}\right)^{-1}. \tag{24}$$

Another point to observe here is that neither of these formulas actually require the true values of the probability masses provided in the expressions (23) and (24). Instead, their ratios are sufficient to obtain the quantity $\psi(u, v, H)$. Therefore, the normalization step due to the boundary condition (17) can be skipped altogether. This might be significant in certain scenarios where the number of states of the background process as well as the number of regimes are large, since empirical evidence due to Yazici and Akar (2013, Table 3) suggests that normalization accounts for up to about a quarter of the whole computation time.

3.3 Extension to the infinite-horizon ruin problem

We would like to point out that by selecting H arbitrarily large, a good approximation to the infinite-horizon ruin probability can also be obtained with the framework laid out in the previous subsection. Moreover, for computational convenience, one can simply set the number of Erlang stages to $l = 1$, which corresponds to modeling the horizon with an exponential distribution with an arbitrarily large mean H, if the interest is in finding infinite-horizon ruin probabilities only. Consequently, the infinite-horizon ruin problem reduces to solving an MRMFQ with only $m(s + 1) + 3$ states.

4 Numerical examples

We provide four numerical examples in this section each emphasizing a different feature of the proposed numerical algorithm.

Example I We start with an example from Wikstad (1971) involving a constant premium rate. Inter-arrival time between claims is PH-type with cumulative distribution function $1 - 0.25e^{-0.4t} - 0.75e^{-2t}$ which gives rise to a MAP representation

$$D_0 = \begin{bmatrix} -0.4 & 0 \\ 0 & -2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.5 & 1.5 \end{bmatrix}. \tag{25}$$

We set the initial vector v to the initial probability vector $\begin{bmatrix} 0.25 & 0.75 \end{bmatrix}$ of the associated PH-type distribution. The claim size distribution is PH-type with

$$\alpha = \begin{bmatrix} 0.0039793 & 0.1078392 & 0.8881815 \end{bmatrix},$$
$$T = \text{diag}(-0.014631, -0.190206, -5.514588).$$

The finite-horizon ruin probabilities for $p \in \{1.1, 2\}$, $H \in \{1, 100, 10^6\}$, $u \in \{0, 1, 10, 100\}$ and $l \in \{1, 3, 5, 7, 9\}$ are given in Table 1. The simulation results presented in Wikstad (1971, Table IIB), indicated by 'S', are given for comparison. The simulation values for $H = 10^6$ are taken as the infinite-horizon ruin probabilities given in Wikstad (1971). We observe that the need for increased number of stages l in the Erlangization process is more emphasized especially for relatively small values of H and u.

Example II In the second example, we consider a scenario in which the premium rate is given by

$$p(x) = p_0 + \delta T^{(k-1)}, \quad T^{(k-1)} \leq x < T^{(k)}, \quad 1 \leq k \leq K,$$

where $T^{(k)} = k \times 10^{-3}$, $0 \leq k \leq K - 1 = 5 \times 10^4$, $T^{(K-1)} = 50$, $T^{(K)} = \infty$. This particular function is picked since it approximates the premium rate $p(x) = p_0 + \delta x$ that models a scenario investigated in Michaud (1996) in which an interest of the reserves is earned by the insurer on top of the premiums collected. Here, δ is the interest rate. The claim arrivals are Poisson with rate 1, and the claim sizes are exponentially distributed with mean 1. The analysis is carried out for $p_0 = 1$ and 1.5, and $u \in \{0, 2, 4, 6, 8, 10\}$. Also, the interest rate is $\delta = 0.05$, and $l = 1$ Erlang level is used as the interest is on the infinite-horizon ruin. The finite-horizon ruin probabilities obtained with our framework with $H = 10^{10}$ along with the performance figures obtained by Michaud (1996) are presented in Table 2.

Example III Next, we consider Example 5.1 from Lin and Sendova (2008) in which claims arrive according to a Poisson process with rate 1 with exponentially distributed sizes with mean 100. The premium rate is given by $p(x) = 100(1 + \theta)$, where

Table 1 Finite-horizon ruin probabilities for several values of p, H, u, and l

l	$p = 1.1$				$p = 2$			
	u				u			
	0	1	10	100	0	1	10	100
$H = 1$								
1	0.2771	0.1023	0.0255	0.0012	0.2066	0.0916	0.0230	0.0012
3	0.3082	0.1118	0.0267	0.0013	0.2266	0.1018	0.0247	0.0013
5	0.3142	0.1142	0.0270	0.0013	0.2302	0.1043	0.0251	0.0013
7	0.3166	0.1152	0.0271	0.0013	0.2316	0.1054	0.0253	0.0013
9	0.3179	0.1158	0.0272	0.0013	0.2324	0.1061	0.0254	0.0013
S	0.3221	0.1180	0.0275	0.0013	0.2352	0.1084	0.0257	0.0013
$H = 100$								
1	0.7270	0.6196	0.3943	0.0802	0.4993	0.3951	0.2082	0.0409
3	0.7652	0.6689	0.4417	0.0862	0.5219	0.4208	0.2315	0.0474
5	0.7711	0.6771	0.4516	0.0876	0.5255	0.4251	0.2364	0.0491
7	0.7735	0.6803	0.4558	0.0882	0.5270	0.4269	0.2385	0.0498
9	0.7748	0.6820	0.4582	0.0886	0.5279	0.4279	0.2397	0.0503
S	0.7789	0.6878	0.4662	0.0898	0.5308	0.4314	0.2439	0.0519
$H = 10^6$								
1	0.9245	0.8931	0.8121	0.5494	0.5582	0.4645	0.2853	0.0754
3	0.9247	0.8933	0.8125	0.5502	0.5582	0.4644	0.2853	0.0754
5	0.9247	0.8933	0.8125	0.5502	0.5582	0.4644	0.2853	0.0754
7	0.9247	0.8933	0.8125	0.5502	0.5582	0.4645	0.2853	0.0754
9	0.9247	0.8933	0.8125	0.5502	0.5582	0.4644	0.2853	0.0754
S	0.9247	0.8933	0.8125	0.5502	0.5582	0.4645	0.2853	0.0754

Simulation results from Wikstad (1971) are indicated by 'S'. Each of the instances in this table is obtained in less than a second on a laptop PC with Intel Core i7, 2.20 GHz processor and 8 GB RAM

Table 2 Infinite-horizon ruin probabilities with the interest-earning ($\delta = 0.05$) scenario

u	$p_0 = 1$		$p_0 = 1.5$	
	MRMFQ ($H = 10^{10}$)	Sim. Michaud (1996)	MRMFQ ($H = 10^{10}$)	Sim. Michaud (1996)
0	0.841120	0.841108	0.619924	0.619915
2	0.547385	0.547364	0.264767	0.264757
4	0.322436	0.322416	0.106257	0.106251
6	0.173189	0.173175	0.040307	0.040303
8	0.085516	0.085508	0.014528	0.014525
10	0.039127	0.039123	0.004999	0.004997

The simulation results are taken from Michaud (1996, Tables 1 and 3)

$$\theta = \begin{cases} 0.5 & \text{if } x < T^{(1)}, \\ 0.4 & \text{if } T^{(1)} \leq x < T^{(2)}, \\ 0.3 & \text{if } T^{(2)} \leq x. \end{cases}$$

Table 3 Infinite-horizon ruin probabilities for the scenario in Lin and Sendova (2008, Example 5.1) with $H = 10^{10}$ for the MRMFQ solution

u	MRMFQ	Lin and Sendova (2008)
0	0.723784	0.723784
50	0.638975	0.638976
100	0.567187	0.567188
150	0.506419	0.506421
200	0.451776	0.451779
250	0.403694	0.403697
300	0.361987	0.361986
350	0.322538	0.322538
400	0.287389	0.287389
450	0.256070	0.256070
500	0.228165	0.228164

The thresholds satisfy $\exp\left(-T^{(1)}/100\right) = 0.2$, and $\exp\left(-T^{(2)}/100\right) = 0.05$. A closed form expression is derived for the ultimate ruin probability under this scenario in Lin and Sendova (2008). We present the figures for varying values of u obtained via this closed form expression as well as our numerical method using $H = 10^{10}$ with a single Erlang stage in Table 3. We observe that the numerical results match up to four significant digits.

Example IV In the final example, the claim arrivals occur according to a MAP characterized by

$$D_0 = \begin{bmatrix} -2 & 0.3 & 0.3 \\ 0 & -2 & 1 \\ 0.1 & 0.1 & -0.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.4 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0.1 & 0.1 & 0.1 \end{bmatrix},$$

and we use three different initial probability vectors, namely $v = e_i$ where e_i is a row vector of zeros except for a one at the ith position. The claim sizes have the same hyper-exponential distribution as in the first example, and the premium rate is in the same form as in Example II with $T^{(k)} = k \times 10^{-2},\ 0 \le k \le K - 1 = 1.5 \times 10^3$, $T^{(K-1)} = 15$, $p_0 = 1.5$ and $\delta = 0.05$. In Fig. 3, the finite-horizon ruin probabilities for H values starting from 0.1 up to 5 with steps of 0.1 are plotted for $u = 0$, 1, 2 and 3. The number of Erlang levels used in this example is $l = 7$. The results demonstrate a strong dependence on the initial state of the MAP claim arrival process.

5 Conclusions

We present a novel numerical method to obtain the finite-horizon ruin probabilities for a general continuous-time risk problem with claim arrivals of MAP type, PH-distributed claim sizes, and reserve-dependent premium rates, using l-stage Erlangization. The same approach can be used to find infinite-horizon ruin probabilities as well when H is allowed to approach infinity but with using only a single Erlangian stage. The proposed method is validated by numerical examples in comparison with simulation results and closed-form expressions available in the literature for some special cases. It has also been shown that the numerical algorithm is capable of handling cases with a relatively large number of thresholds. The

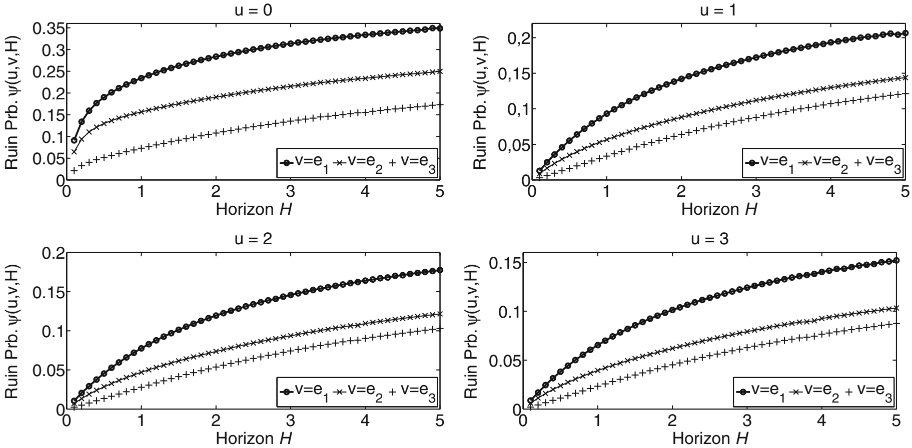

Fig. 3 Finite-horizon ruin probabilities under reserve-dependent premium rate. Claim arrivals occur according to a MAP with three states and the initial probability vector v is set to one of the three possibilities; $e_1 = [1\ 0\ 0]$, $e_2 = [0\ 1\ 0]$, and $e_3 = [0\ 0\ 1]$

numerical examples demonstrate that the proposed method shows promise in approximate computation of ruin probabilities in scenarios with premium rates that are continuously dependent on the instantaneous surplus. However, more work needs to be done in this respect towards proving the convergence of the MRMFQ solution to the exact solution under various discretization schemes.

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