About the Wigner distribution of a graded index medium and the fractional Fourier transform operation

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Abstract

Upon propagation through quadratic graded index media, the Wigner distribution of the wavefunction of light rotates uniformly. As a consequence, a definition of fractional Fourier transforms based on rotating the functions Wigner distribution, and another based on propagation through graded index media, are equivalent.

Introduction Recently, two distinct definitions of the fractional Fourier transform were given. In the first one [1, 2, 3] it was defined physically, based on propagation in quadratic graded index (GRIN) media. The ath fractional Fourier transform of a function was defined as follows:

> Let the original function be input from one side of a quadratic GRIN medium, at $z = 0$. Then, the light distribution observed at the plane $z = z_0$ corresponds to the $a = z_0/L$th fractional Fourier transform of the input function, where $L \equiv (\pi/2)\sqrt{n_1/n_2}$ is a characteristic distance. The $a =$ 1st Fourier transform, observed at $z_0 = L$, correponds to the ordinary Fourier transform, by design.

The second definition is based on WDFs [4]. Here the fractional Fourier transform is calculated by finding the WDF of the input image, rotating it by an angle $\alpha = a\pi/2$ and performing the inverse Wigner transform.

Both definitions fulfill two natural postulates: i) The $a =$ 1st Fourier transform corresponds to the ordinary Fourier transform; ii) The fractional operator is additive, i.e. the ath transform of the bth transform is equal to the $a + b$th transform.

We showed that both definitions of the fractional Fourier transform are equivalent. The fact that two distinct definitions turn out to be identical supports the claim as to the naturalness and intrinsicness of the definitions.

The analyses presented here are for 1-dimensional functions, but can be extended to higher dimensions.

The Wigner distribution function (WDF) The WDF of a 1-D signal $f(x)$ can be defined as

$$W(x,\nu) = \int f(x + x'/2) f^*(x - x'/2) \exp(-2\pi i \nu x') dx'. \qquad (1)$$

The WDF is a joint space-frequency representation of a signal, which describes the signal completely. Its properties can be found in Ref. [4]. Here we mention three of them. First is the effect of free space propagation in the z direction on the WDF. It was shown [4] that such propagation causes the WDF to be sheared in the x-direction. The second is the effect of passage through a thin lens. This causes the WDF to be sheared in the ν direction. Finally, how is the WDF of the Fourier transform of a function related to the original function? One observes a $\pi/2$ rotation of the WDF.

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Fractional Fourier transforms - graded index media definition The ath Fourier transform of a function $f(x)$ is denoted as $\mathcal{F}^a[f(x)]$. Our definition should satisfy two basic postulates. $\mathcal{F}^1 f$ should be the usual Fourier transform, and $\mathcal{F}^a[\mathcal{F}^b f] = \mathcal{F}^{a+b} f$. Consistent with these, we suggested [1, 2, 3] defining the fractional Fourier transform as the change of the field due to propagation along a quadratic graded index (GRIN) medium by a length proportional to a. Such a medium has a refractive index profile given by $n^2(x) = n_1^2(1 - (n_2/n_1)x^2)$ where n_1, n_2 are the GRIN medium parameters. $L = (\pi/2)\sqrt{n_1/n_2}$ is the GRIN length that results in the first order Fourier transform. It was shown [1] that the above two postulates are satisfied by this definition.

Fractional Fourier transforms - Wigner distribution function definition Another fractional Fourier transform definition is given in Ref. [4]. Here, the fractional Fourier transform operation is defined as a rotation of its WDF by an angle $\alpha = a\pi/2$. It was shown that the two postulates of the previous section are fulfilled. Since any rotation can be performed as 3 shearing operations (x, ν, x-shearings or ν, x, ν-shearing), it was suggested [4] to use a system consisting of a stretch of free space followed by a lens followed by another stretch of free space to perform the fractional Fourier transform with optical means.

Discussion We have proved that both definitions of the fractional Fourier transform are equivalent. The mathematical details are too complicated to present here and can be found in [5].

In this section we would like to highlight certain consequences of this equivalence. First, this implies that we have two equivalent ways of optically performing the fractional Fourier transform operation.

Another implication of this equivalence is the fact that propagation through quadratic GRIN media results in a rotation of the Wigner distribution function. Till now GRIN media have mostly been utilized as ray optics elements mainly because of the lack of simple interpretations of its effect on the wavefunction of light passing through it. With the Wigner distribution function we have a powerful tool for analyzing and designing systems with GRIN media devices.

Additional insight into the rotation of the Wigner distribution can be gained by examining the ray optics analog of Wigner space, which is a particular kind of phase space. It is found that the collection of points representing a bundle of rays in this phase space rotates just like the Wigner distribution [3, 5].

References

[1] D. Mendlovic and H. M. Ozaktas, "Fractional Fourier transformations and their optical implementation: Part I," JOSA A (accepted).

[2] H. M. Ozaktas and D. Mendlovic. "Fractional Fourier transformations and their optical implementation: Part II," JOSA A (submitted).

[3] D. Mendlovic and H. M. Ozaktas, "Fourier transforms of fractional orders and their optical interpretation," SPIE Proceedind 1983, ICO 16th Meeting, Budapest (1993).

[4] A. W. Lohmann, "Image rotation, Wigner rotation and the fractional Fourier transform," JOSA A (submitted).

[5] D. Mendlovic, H. M. Ozaktas, and A. W. Lohmann, "The effect of propagation in graded index media on the Wigner distribution function and the equivalence of two definitions of the fractional Fourier transform," Appl. Opt. (submitted).