A HIGH RESOLUTION TIME FREQUENCY REPRESENTATION WITH SIGNIFICANTLY REDUCED CROSS–TERMS

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ABSTRACT

A novel algorithm is proposed for efficiently smoothing the slices of the Wigner distribution by exploiting the recently developed relation between the Radon transform of the ambiguity function and the fractional Fourier transformation [1]. The main advantage of the new algorithm is its ability to suppress cross–term interference on chirp–like auto–components without any detrimental effect to the auto–components. For a signal with N samples, the computational complexity of the algorithm is $O(N \log N)$ flops for each smoothed slice of the Wigner distribution.

1. INTRODUCTION

Time–frequency representations have found important application areas in analysis, synthesis and detection of non stationary signals by revealing the signals joint time and frequency content [2, 3]. Much of the research in time–frequency signal processing has been devoted to design of new time–frequency representations. Among the representations developed so far the Wigner distribution [4] has attracted much of the attention because of its nice theoretical properties including the preservation of the marginals and high auto–component concentration [2, 5]. The Wigner distribution of a signal $x(t)$ is given as

$$W_x(t,f) = \int x(t+t'/2)x^*(t-t'/2)e^{-j2\pi t t'}\,\mathrm{d}t' \quad , \qquad (1)$$

where (t, f) denote the time and frequency coordinate. As it becomes clear from this definition, the Wigner distribution is a bilinear representation. Therefore the Wigner distribution of a multi–component signal $x(t) = \sum_{i=1}^{m} x_i(t)$ contains $m(m-1)/2$ cross terms of the form $2\Re\{W_{x_i x_j}(t,f)\}$, $i < j$, in addition to the auto–components $W_{x_i x_i}(t,f)$, where $W_{x_i x_j}(t,f)$ is the cross WD [2, 3] of the signals $x_i(t)$ and $x_j(t)$. The cross–terms usually interfere with the auto–components and decreases the interpreteability of the Wigner distribution. Thus the existence of cross–terms limits the use of the Wigner distribution in some practical applications.

The cross–terms of the Wigner distribution have been extensively analyzed [6, 7]. It has been found that the cross terms lie at mid–time and mid–frequency of the auto–components, they are highly oscillatory and the frequency of oscillations increases with the increasing distance in time and frequency and they might have a peak value as high as twice that of the auto–components. Based on these observations it has been suggested that some sort of smoothing of the Wigner distribution is necessary to suppress the cross–terms at the expense of broadening of the auto–components. In a unified framework, the representation obtained by low–pass filtering the Wigner distribution are studied under the name of Cohen's bilinear class of shift invariant distributions. In this class, the time–frequency representation $TF_x(t,f)$ of a signal $x(t)$ is obtained as [2]

$$TF_x(t,f) = \iint A_x(\nu,\tau)\phi(\nu,\tau)e^{-j2\pi(\nu t+\tau f)}\,\mathrm{d}\nu\,\mathrm{d}\tau \quad , \qquad (2)$$

where $\phi(\nu, \tau)$ is the kernel of the distribution and $A_x(\nu, \tau)$ is the (symmetric) ambiguity function (AF) which is the 2–D inverse Fourier transform of the Wigner distribution:

$$A_x(\nu,\tau) = \int x(t+\tau/2)x^*(t-\tau/2)e^{j2\pi\nu t}\,\mathrm{d}t \quad . \qquad (3)$$

The drawback of this class of distributions is that a fixed kernel can perform well only for a limited class of signals. On the other hand for a large class of signals, there is a trade–off between good cross–term suppression and high auto–component concentration. Therefore to obtain high–quality time–frequency representation, the kernel must be adapted to the characteristics of the input signal to obtain a data–adaptive smoothing. These considerations led to the development of Cohen's class of time–frequency representations with data–dependent kernels [2].

The basis of the recent research on the design of data–dependent kernels is the following observation: In the ambiguity plane, the auto–components lie around the origin and the cross–terms lie away from the origin [6]. Thus by designing a kernel $\phi(\nu, \tau)$ which is apt to the characteristics of the data in the ambiguity plane, higher quality representations (more easily interpretable) are obtained [8, 9]. The disadvantages of this approach are as follows: the obtained methods are computationally expensive, and kernel which is globally optimal does not necessarily produce locally optimal results.

In this paper, a novel approach to design a new time–frequency representation is proposed. In contrast to the vast body of previous work, the proposed approach is based

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on the Radon transform of the ambiguity function of the input signal, which is called as the Radon ambiguity function transform (RAFT) [1]. The proposed time-frequency representation cannot be described by either a fixed or signal dependent kernel, therefore it does not belong to Cohen's class. However, by performing windowing on the resultant RAFT's, it eliminates significant part of the cross-terms without reducing the auto-component concentration.

The outline of the paper is as follows. In Section 2 the mathematical details of the new approach are given, in Section 3 some simulation results are presented and finally in Section 4 conclusions are drawn.

2. DIRECTIONAL SMOOTHING OF THE WIGNER DISTRIBUTION

The most important drawback of the approaches based on the low-pass filtering of the WD is that, the low-pass filter is applied in all directions of the Wigner plane. Naturally this leads to broadening of the auto-components, because the auto-components may not have a low-pass characteristic along all orientations. For instance the slice of the WD of a linear chirp has a low-pass characteristic when the slice is along the chirp's major axis, but is has significant high frequency content when the slice is lying along its minor axis as illustrated in Fig. 1. Thus the directional smoothing of the WD by using low-pass filters with data-adaptive cut off frequencies appear to be the natural solution to the problem. By this way the oscillatory cross-terms with significant high-frequency content are suppressed without essentially decreasing the auto-component concentration. At the end what we get is a high resolution time-frequency representation.

In this work, we assume that supports of the auto-components or the regions of the Wigner plane which are suspected to contain auto-components are specified beforehand. What has to be done is to efficiently smooth the slices of these regions with data-adaptive low-pass filters. In the next subsection we develop a procedure to efficiently smooth any arbitrarily chosen slice of the WD.

2.1. Directional smoothing algorithm

Suppose that we want to smooth the non-central slice of the Wigner distribution W_x which passes through the point (t_o, f_o) and makes an angle of ϕ with the time-axis as shown in Fig. 2. It is straightforward to prove that this non-central slice of the Wigner distribution W_x is the same as the central slice of the Wigner distribution of a signal $y(t)$ at the same angle ϕ (see Fig. 2) provided that the latter signal is defined in terms of the original one through the relation

$$y(t) = x(t + t_o)e^{-j2\pi f_o t} \quad . \tag{4}$$

Thus we can formulate the smoothing problem in terms of the WD W_y. By denoting the radial slice of the WD W_y as $SLC\,[W_y](r, \phi) \equiv W_y(r\cos\phi, r\sin\phi)$, and impulse response of the real smoothing filter as $h(t)$, the directional smoothing can be mathematically expressed as

$$s(r, \phi) = h(r) \underset{r}{*} SLC\,[W_y](r, \phi) \quad , \tag{5}$$

where $s(r, \phi)$ is the slice of the smoothed Wigner distribution. By using the projection slice theorem [10], the central slice of the Wigner distribution W_y can be expressed as the Fourier transform of the Radon transform of the ambiguity function A_y:

$$SLC\,[W_y](r, \phi) = \int \mathcal{RDN}\,[A_y](\lambda, \phi) 2^{-j2\pi r\lambda}\, d\lambda \quad , \tag{6}$$

where the Radon transform of the ambiguity function is defined as

$$\mathcal{RDN}\,[A_y](\lambda, \phi) = \int A_y(\lambda\cos\phi - s\sin\phi, \lambda\sin\phi + s\cos\phi)\, ds \; . \tag{7}$$

Thus (5) can be expressed in the (inverse) Fourier transform domain as

$$S(\lambda, \phi) = H(\lambda)\, \mathcal{RDN}\,[A_y](\lambda, \phi) \quad , \tag{8}$$

where $S(\lambda, \phi)$ is the inverse Fourier transform of the slice $s(r, \phi)$ with respect to the radial variable r, and $H(\lambda)$ is the inverse Fourier transform of the smoothing filter $h(t)$. This equation gives the basis of the algorithm for smoothing any slice of the Wigner distribution of a signal $x(t)$:

1. Compute the Radon transform $\mathcal{RDN}\,[A_y](\lambda, \phi)$ of the ambiguity function $A_y(\nu, \tau)$.
2. Design a multiplicative filter $H(\lambda)$ to capture the energy around the origin and suppress the cross-terms away from the origin.
3. Apply the multiplicative filter $H(\lambda)$ to the Radon transform $\mathcal{RDN}\,[A_y](\lambda, \phi)$ to obtain $S(\lambda, \phi)$.
4. Compute the slice $s(r, \phi)$ of the smoothed distribution from $S(\lambda, \phi)$ by using the Fourier transformation.

This procedure can be repeated on different slices where adaptively chosen filters are utilized on each slice depending on the auto-component location in the corresponding $\mathcal{RDN}\,[A_y](\lambda, \phi)$. However to have a practically useful algorithm, we have to obtain the Radon transform of the ambiguity function efficiently. As we prove in Appendix A, the Radon transform of the ambiguity function $A_y(\nu, \tau)$ can be computed as

$$\mathcal{RDN}\,[A_y](\lambda, \phi) = x_{(a-1)}(\frac{\lambda}{2} + d)\, x^*_{(a-1)}(-\frac{\lambda}{2} + d) \quad , \tag{9}$$

where $a = 2\phi/\pi$ and $x_{(a-1)}(t)$ is the $(a-1)^{\text{th}}$ order fractional Fourier transformation [11] of the signal $x(t)$ and in polar format $(d, \phi + \pi/2)$ is the closest point on the non-central slice of the WD to the origin as shown in Fig. 2.

3. SIMULATION

In this section we investigate the performance of the proposed method in removing the cross-terms residing on the auto-components of the Wigner distribution. The synthetic test signal used in this simulation is generated by linearly combining 5 linear frequency modulated chirp signals with Gaussian envelopes. The intelligibility of the Wigner distribution of this multi-component signal is severely degraded by the existence of cross-terms as seen in Fig. 3 (a). In Fig. 3(b), the Wigner distribution is computed on rectangular grids which contain supports of three of the auto-components. By using the new approach, these slices of the auto-components are smoothed by data-adaptive low-pass filtering and the obtained slices are plotted in Fig. 3 (c). In Fig. 3 (d), the difference of the smoothed slices from the actual auto-components is shown to illustrate the high accuracy time-frequency representation provided by the algorithm.

In the next example we investigate the problem, where not only the interference terms but also one of the auto-components are superimposed on another auto-component. As shown in Fig. 4(a), the Wigner distribution of the multi-component signal displays significant cross and auto-term noise on the chirp signal centered at the origin. In Fig. 4(b), the smoothed slices of the WD along this chirp signal are plotted. As it can be seen from this plot, the noise terms are greatly attenuated.

4. CONCLUSIONS

A fast algorithm is developed for smoothing slices of the Wigner distribution to suppress the oscillatory cross-term components yielding a highly accurate representation of the auto-terms of the Wigner distribution. The new algorithm, which is especially tailored for but not limited to chirp-like components, is based on the recently established relationship between the Radon ambiguity function transform and fractional Fourier transform. In contrast to the smoothing algorithms which work by applying a low pass filter globally to the WD, the new algorithm works locally on the slices of the WD. As shown by simulation examples, the proposed algorithm avoids the usual trade-off between cross-term suppression and auto-term broadening by taking into account the characteristics of the cross-terms on the WD slices.

5. REFERENCES

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A. THE RADON AMBIGUITY FUNCTION TRANSFORMATION

In [1], it has been shown that the Radon transform of the ambiguity function $A_y(\nu, \tau)$, can be computed as

$$\mathcal{RDN}\,[A_y](\lambda, \phi) = y_{(a-1)}(\lambda/2)\, y^*_{(a-1)}(-\lambda/2) \quad , \qquad (10)$$

where $a = 2\phi/\pi$ and $y_{(a-1)}$ is the $(a-1)^{\text{th}}$ order fractional Fourier transformation (FrFT) of the signal $y(t)$. To express the RAFT of $y(t)$ in terms of the input signal $x(t)$, we first obtain the FrFT of $y(t)$ by using the basic properties of the FrFT [12]:

$$y_{(a-1)}(t) = Ce^{\jmath\varphi(t)} x_{(a-1)}(t - t_o \sin\phi + f_o \cos\phi) \quad , \qquad (11)$$

where $\varphi(t) = 2\pi t(f_o \sin\phi + t_o \cos\phi)$ is the linear phase factor and $C = exp(\jmath\pi\cos\phi(f_o^2 \sin\phi + t_o^2 \cos\phi + f_o t_o \sin\phi))$ is a unit magnitude complex constant. Since we have the freedom to choose (t_o, f_o) as any point which lies on the line L_{W_x} shown in Fig. 2 , we use this freedom to simplify the expression for the FrFT of $y(t)$. By choosing $(t_o, f_o) \triangleq (-d\cos\phi, d\sin\phi)$ as the closest point on L_{W_x} to the origin (see Fig. 2) we simplify (11) as

$$y_{(a-1)}(t) = Cx_{(a-1)}(t - d) \quad . \qquad (12)$$

Finally by substituting this relation into (10), we obtain the desired expression for the RAFT of $y(t)$:

$$\mathcal{RDN}\,[A_y](\lambda, \phi) = x_{(a-1)}(\frac{\lambda}{2} + d)\, x^*_{(a-1)}(-\frac{\lambda}{2} + d) \quad . \quad (13)$$

B. THE MODIFIED FAST FRACTIONAL FOURIER TRANSFORM ALGORITHM

To simulate the proposed method, we need a fast algorithm to compute the samples of $x_{(a-1)}(t+d)$. By using the algorithm given in this appendix, the required samples can be computed in $O(N \log N)$ flops by using N uniformly spaced samples of $x(t)$. This algorithm is obtained by modifying the algorithm in [13] to incorporate the delay term d, and removing the condition that the time–bandwidth product of $x(t)$ be integer.

The Fast Fractional Fourier Transform Algorithm

Given $x(n/\Delta_x)$, $-N/2 \le n \le N/2-1$, to compute $x_a(m\Delta_x/(2N)+d)$, $-N \le m \le N-1$. It is assumed that $x(t)$ is scaled before obtaining its samples so that its WD is confined into a circle with diameter $\Delta_x \le \sqrt{N}$ [13].

Steps of the algorithm:

Interpolate the input samples by 2:
$a' := (a+2 \mod 4) - 2$ % $a' \in [-2,\ 2)$
if $|a'| \in [0.5,\ 1.5]$ **then**
  $a'' := a'$
else
  $a'' := (a'+1 \mod 4) - 2$ % $a'' \in (0.5,\ 1.5)$
end if
$\phi := \frac{\pi}{2} a''$
$\alpha := \cot \phi$
$\beta := \csc \phi$
$A_\phi := \frac{\exp(-j\pi \,\mathrm{sgn}(\sin\phi)/4 + j\phi/2)}{|\sin\phi|^{1/2}}$
% Generate the sequence $c_1[m]$, $c_3[m]$ for $-N \le m$
% $\le N-1$ and $c_2[m]$ for $-2N \le m \le 2N-1$:
  $c_1[m] := e^{j\pi[(\frac{\alpha}{4\Delta_x^2} - \frac{\beta}{4N})m^2 - \frac{\beta d}{\Delta_x} m]}$
  $c_2[m] := e^{j\pi \frac{\beta}{4N} m^2}$
  $c_3[m] := e^{j\pi[\alpha(\frac{\Delta_x}{2N} m + d)^2 - \frac{\beta}{4N} m^2]}$
% Compute the following sequences for $-N \le m \le N-1$:
  $g[m] := c_1[m]\, x(m/2\Delta_x)$
  $h_{a''}(m\Delta_x/(2N)) := \frac{A_\phi}{2\Delta_x} c_3[m]\,(c_2 * g)[m]$
%In the last step FFT is used to compute the convolution
% in $O(N \log N)$ flops.
if $|a| \in [0.5,\ 1.5]$ **then**
  $x_a(m\Delta_x/(2N)+d) := h_{a''}(m\Delta_x/(2N))$
else
  % Compute samples of the ordinary FT using FFT.
  $x_a(m\Delta_x/2N + d) := \{\mathcal{F}^1 h_{a''}\}(m\Delta_x/(2N))$
end if

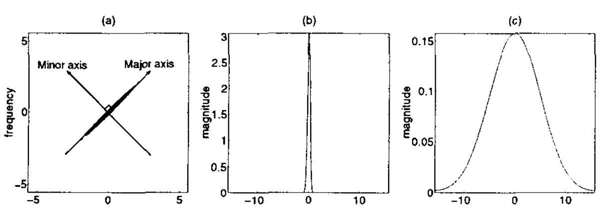

Figure 1: A chirp signal has a low–pass characteristic along its major axis (b), and it has considerable bandwidth in along its minor axis (c).

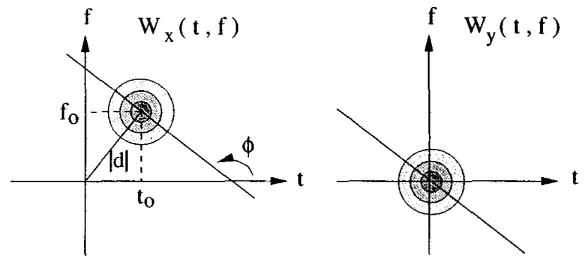

Figure 2: The non–central (left) and central (right) slices of the Wigner distribution $W_x(t,f)$ and $W_y(t,f)$.

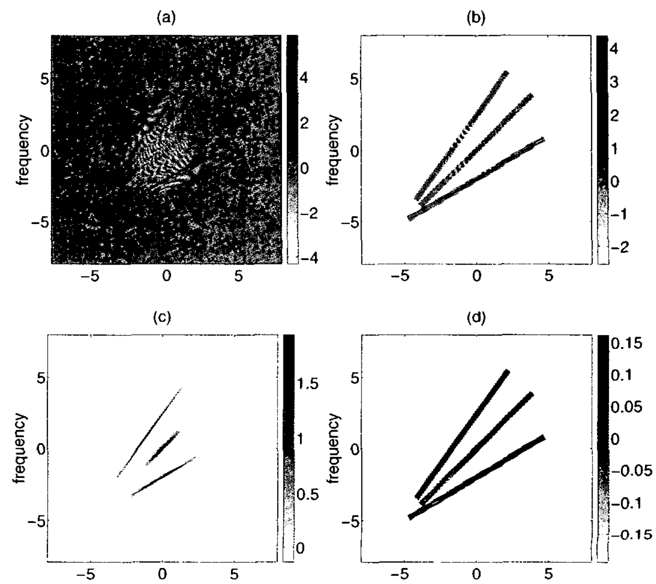

Figure 3: The Wigner distribution (a), slices of the Wigner distribution (b), slices of the Wigner distribution smoothed with data–adaptive directional filtering (c), the difference of the smoothed slices from the auto–components only.

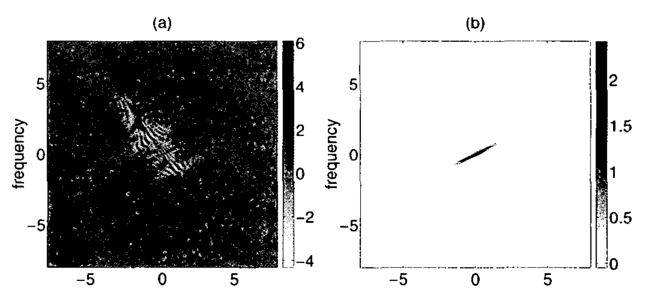

Figure 4: The Wigner distribution (a), smoothed slices of the Wigner distribution along one of the auto–components (b).