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CHARACTERISTIC LIE ALGEBRA AND CLASSIFICATION OF SEMIDISCRETE MODELS

I. T. Habibullin*† and A. Pekcan*

We study characteristic Lie algebras of semi-discrete chains and attempt to use this notion to classify Darboux-integrable chains.

Keywords: integrability, discrete equation, Liouville-type equation

1. Introduction

Investigating the class of hyperbolic-type differential equations of the form

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{1}$$

has a very long history. Various approaches have been developed for seeking particular and general solutions of this kind of equation. Several definitions of the integrability of the equation can be found in the literature. According to the one given by Darboux, Eq. (1) is said to be integrable if it reduces to a pair of ordinary (generally, nonlinear) differential equations or, more exactly, if any solution of it satisfies equations of the form [1] (also see [2])

$$F(x, y, u, u_x, u_{xx}, \dots, D_x^m u) = a(x), \qquad G(x, y, u, u_y, u_{yy}, \dots, D_y^n u) = b(y) \tag{2}$$

for appropriately chosen functional parameters $a(x)$ and $b(y)$, where D_x and D_y are differentiation operators with respect to x and y, $u_x = D_x u$, $u_{xx} = D_x u_x$, and so on. The functions F and G are called the y and x integrals of the equation.

Darboux himself proposed an effective criterion for Darboux integrability: Eq. (1) is integrable if and only if the Laplace sequence of the linearized equation terminates at both ends. A rigorous proof of this statement was found only recently [3].

Shabat developed an alternative method for investigating and classifying the Darboux integrable equations based on the notion of a characteristic Lie algebra. We briefly explain this notion. We begin with the basic property of the integrals. Obviously, each y integral satisfies the condition

$$D_y F(x, y, u, u_x, u_{xx}, \dots, D_x^m u) = 0.$$

Differentiating by applying the chain rule, we define a vector field X_1 such that

$$X_1 F = \left(\frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + f \frac{\partial}{\partial u_x} + D_x(f) \frac{\partial}{\partial u_{xx}} + \cdots \right) F = 0. \tag{3}$$

*Department of Mathematics, Faculty of Science, Bilkent University, 06800, Ankara, Turkey, e-mail: asli@fen.bilkent.edu.tr.

†Institute of Mathematics, Ufa Science Center, RAS, Ufa, Russia, e-mail: habibullin_i@mail.rb.ru.

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Hence, the vector field X_1 solves the equation $X_1F = 0$. But the coefficients of the vector field depend on the variable u_y in general while the solution F does not. This severely restricts F; in fact, F must satisfy one more equation, $X_2F = 0$, where $X_2 = \partial/\partial u_y$. But then the commutator of these two operators must also annihilate F. Moreover, for any X from the Lie algebra generated by X_1 and X_2, we obtain $XF = 0$. This Lie algebra is called the characteristic Lie algebra of Eq. (1) in the x direction. The characteristic algebra in the x direction is defined similarly. By virtue of the famous Jacobi theorem, Eq. (1) is Darboux integrable if and only if both of its characteristic algebras are finite-dimensional. The characteristic Lie algebras for the systems of nonlinear hyperbolic equations and their applications were studied in [4].

In this paper, we study semidiscrete chains of the form

$$t_{1x} = f(t, t_1, t_x) \tag{4}$$

from the standpoint of Darboux integrability. Here, the unknown $t = t(n, x)$ is a function of two independent variables: one discrete (n) and one continuous (x). We assume that $\partial f/\partial t_x \neq 0$. A subscript denotes a shift or a derivative, for instance, $t_1 = t(n + 1, x)$ and $t_x = \partial t(n, x)/\partial x$. Below, we let D denote the shift operator and D_x denote the x derivative: $Dh(n, x) = h(n+1, x)$ and $D_xh(n, x) = \partial h(n, x)/\partial x$. We use the subscript for iterated shifts: $D^jh = h_j$.

We now introduce the notions of integrals for semidiscrete chain (4). The x integral is defined similarly to the continuous case. We call a function $F = F(x, n, t, t_1, t_2, \dots)$ depending on a finite number of shifts an x integral of chain (4) if the condition $D_xF = 0$ is satisfied. In accordance with the continuous case, it is natural to call a function $I = I(x, n, t, t_x, t_{xx}, \dots)$ an n integral of chain (4) if it is in the kernel of the difference operator: $(D - 1)I = 0$. In other words, an n integral is invariant under the action of the shift operator $DI = I$ (also see [5]). We can write it in the expanded form

$$I(x, n+1, t_1, f, f_x, f_{xx}, \dots) = I(x, n, t, t_x, t_{xx}, \dots). \tag{5}$$

We note that (5) is a functional equation; the unknown is taken at two different "points." This produces the main difficulty in studying discrete chains. Such problems occur when trying to apply the symmetry approach to discrete equations (see [6]). But the concept of the Lie algebra of characteristic vector fields provides an effective tool for investigating chains.

We introduce vector fields as follows. We concentrate on main equation (5). Obviously, its left-hand side contains the variable t_1 while the right-hand side does not. Hence, the total derivative of DI with respect to t_1 must vanish. In other words, the n integral is in the kernel of the operator $Y_1 := D^{-1}\partial D/\partial t_1$. We similarly verify that I is in the kernel of the operator $Y_2 := D^{-2}\partial D^2/\partial t_1$. Indeed, the right-hand side of the equation $D^2I = I$, as follows immediately from (5), is independent of t_1, and the derivative of D^2I with respect to t_1 therefore vanishes. Proceeding thus, we easily prove that for any natural number j, the operator $Y_j = D^{-j}\partial D^j/\partial t_1$ solves the equation $Y_jI = 0$.

So far, we have shifted the argument n forward. We now shift it backward and use main equation (5) written as $D^{-1}I = I$. We rewrite original equation (4) as

$$t_{-1x} = g(t, t_{-1}, t_x), \tag{6}$$

which can be done because of the condition $\partial f/\partial t_x \neq 0$ assumed above. In the expanded form, the equation $D^{-1}I = I$ becomes

$$I(x, n-1, t_{-1}, g, g_x, g_{xx}, \dots) = I(x, n, t, t_x, t_{xx}, \dots). \tag{7}$$

The right-hand side of this equation is independent of t_{-1}, and the total derivative of $D^{-1}I$ with respect to t_{-1} is hence zero, i.e., the operator $Y_{-1} := D\partial D^{-1}/\partial t_{-1}$ solves the equation $Y_{-1}I = 0$. Moreover, the operators $Y_{-j} = D^j\partial D^{-j}/\partial t_{-1}$ also satisfy similar conditions $Y_{-j}I = 0$.

Summarizing the above reasoning, we conclude that the n integral is annihilated by any operator from the Lie algebra $\tilde{L}_n$ generated by the operators [7]

$$\dots, Y_{-2}, Y_{-1}, Y_{-0}, Y_0, Y_1, Y_2, \dots, \tag{8}$$

where $Y_0 = \partial/\partial t_1$ and $Y_{-0} = \partial/\partial t_{-1}$. The algebra $\tilde{L}_n$ consists of the operators from sequence (8), all possible commutators, and linear combinations with coefficients depending on the variables n and x. Obviously, Eq. (4) admits a nontrivial n integral only if the dimension of the algebra $\tilde{L}_n$ is finite. But it is not clear that the finiteness of dimension of $\tilde{L}_n$ suffices for the existence of an n integral. We therefore introduce another Lie algebra called the characteristic Lie algebra of Eq. (4). In addition to the operators $Y_1, Y_2, \dots$, we first define the differential operators $X_j = \partial/\partial_{t_{-j}}$ for $j = 1, 2, \dots$.

The following theorem allows defining the characteristic Lie algebra.

Theorem 1.1. *Equation* (4) *admits a nontrivial* n *integral if and only if the following two conditions hold*:

1. *The linear envelope of the operators* $\{Y_j\}_1^\infty$ *is finite-dimensional* (*its dimension is denoted by* N).

2. *The Lie algebra* L_n *generated by the operators* $Y_1, Y_2, \dots, Y_N, X_1, X_2, \dots, X_N$ *is finite-dimensional. We call* L_n *the characteristic Lie algebra of* (4).

Remark. It is easy to prove that if the dimension of $\{Y_j\}_1^\infty$ is N, then the set $\{Y_j\}_1^N$ constitutes a basis in the linear envelope of $\{Y_j\}_1^\infty$.

2. Characteristic Lie algebra L_n

We study some properties of the characteristic Lie algebra introduced in Theorem 1.1. We begin by proving the remark, which follows immediately from Lemma 2.1.

Lemma 2.1. *If the operator* Y_{N+1} *for some integer* N *is a linear combination of the operators with fewer indices,*

$$Y_{N+1} = \alpha_1 Y_1 + \alpha_2 Y_2 + \dots + \alpha_N Y_N, \tag{9}$$

then we have a similar expression for any integer $j > N$,

$$Y_j = \beta_1 Y_1 + \beta_2 Y_2 + \dots + \beta_N Y_N. \tag{10}$$

Proof. Because of the property $Y_{k+1} = D^{-1} Y_k D$, it follows from (9) that

$$Y_{N+2} = D^{-1}(\alpha_1) Y_2 + D^{-1}(\alpha_2) Y_3 + \dots + D^{-1}(\alpha_N)(\alpha_1 Y_1 + \dots + \alpha_N Y_N). \tag{11}$$

We now easily complete the proof of the lemma by induction.

Lemma 2.2. *The commutativity relations*

$$[Y_0, Y_{-0}] = 0, \qquad [Y_0, Y_1] = 0, \qquad [Y_{-0}, Y_{-1}] = 0$$

hold.

Proof. The first of the relations is obvious. To prove the other two, we find a coordinate representation of the operators Y_1 and Y_{-1} acting in the class of locally smooth functions of the variables x, n, $t, t_x, t_{xx}, \dots$. By direct computation,

$$Y_1 I = D^{-1}\frac{d}{dt_1} D I = D^{-1}\frac{d}{dt_1} I(t_1, f, f_x, \dots) =$$
$$= \left\{\frac{\partial}{\partial t} + D^{-1}\left(\frac{\partial f}{\partial t_1}\right)\frac{\partial}{\partial t_x} + D^{-1}\left(\frac{\partial f_x}{\partial t_1}\right)\frac{\partial}{\partial t_{xx}} + \cdots\right\} I(t, t_x, t_{xx}, \dots), \tag{12}$$

we obtain

$$Y_1 = \frac{\partial}{\partial t} + D^{-1}\left(\frac{\partial f}{\partial t_1}\right)\frac{\partial}{\partial t_x} + D^{-1}\left(\frac{\partial f_x}{\partial t_1}\right)\frac{\partial}{\partial t_{xx}} + D^{-1}\left(\frac{\partial f_{xx}}{\partial t_1}\right)\frac{\partial}{\partial t_{xxx}} + \dots. \tag{13}$$

We now note that all the functions f, f_x, $f_{xx}, \dots$ depend on the variables $t_1, t, t_x, t_{xx}, \dots$ and are independent of t_2. Hence, the coefficients of the vector field Y_1 are independent of t_1, and the operators Y_1 and Y_0 therefore commute. Similarly, using the explicit coordinate representation

$$Y_{-1} = \frac{\partial}{\partial t} + D\left(\frac{\partial g}{\partial t_{-1}}\right)\frac{\partial}{\partial t_x} + D\left(\frac{\partial g_x}{\partial t_{-1}}\right)\frac{\partial}{\partial t_{xx}} + D\left(\frac{\partial g_{xx}}{\partial t_{-1}}\right)\frac{\partial}{\partial t_{xxx}} + \dots, \tag{14}$$

we can prove that $[Y_{-0}, Y_{-1}] = 0$.

The following statement proves very useful for studying the characteristic Lie algebra L_n.

Lemma 2.3. *Let the vector field*

$$Y = \alpha(0)\partial_t + \alpha(1)\partial_{t_x} + \alpha(2)\partial_{t_{xx}} + \dots, \tag{15}$$

where $\alpha_x(0) = 0$, solve the equation $[D_x, Y] = 0$. Then $Y = \alpha(0)\partial_t$.

The proof is based on the formula

$$[D_x, Y] = \big(\alpha_x(0) - \alpha(1)\big)\partial_t + \big(\alpha_x(1) - \alpha(2)\big)\partial_{t_x} + \dots. \tag{16}$$

Therefore, if $a_x(0) = 0$, then $a(1) = 0$; but if $a_x(1) = 0$, then $a(2) = 0$; and hence $a(j) = 0$ for all $j > 0$.

An expanded coordinate form of the operator Y_1 is already given in formula (12). It can be verified that the operator Y_2 is a vector field of the form

$$Y_2 = D^{-1}\big(Y_1(f)\big)\partial_{t_x} + D^{-1}\big(Y_1(f_x)\big)\partial_{t_{xx}} + D^{-1}\big(Y_1(f_{xx})\big)\partial_{t_{xxx}} + \dots. \tag{17}$$

This immediately follows from the equation $Y_2 = D^{-1}Y_1 D$ and coordinate representation (12). We prove similar formulas for an arbitrary j by induction:

$$Y_{j+1} = D^{-1}\big(Y_j(f)\big)\partial_{t_x} + D^{-1}\big(Y_j(f_x)\big)\partial_{t_{xx}} + D^{-1}\big(Y_j(f_{xx})\big)\partial_{t_{xxx}} + \dots. \tag{18}$$

Lemma 2.4. *For the operators D_x, Y_1, and Y_{-1} considered on the space of smooth functions of $t, t_x, t_{xx}, \dots$, the commutativity relations*

$$[D_x, Y_1] = pY_1, \qquad [D_x, Y_{-1}] = qY_{-1} \tag{19}$$

hold, where $p = -D^{-1}(\partial f/\partial t_1)$ and $q = -D(\partial g/\partial t_{-1})$.

Proof. We recall that

$$Y_1 = \frac{\partial}{\partial t} + D^{-1}\left(\frac{\partial f}{\partial t_1}\right)\frac{\partial}{\partial t_x} + D^{-1}\left(\frac{\partial f_x}{\partial t_1}\right)\frac{\partial}{\partial t_{xx}} + \dots. \tag{20}$$

Using (16), we find $[D_x, Y_1]$:

$$[D_x, Y_1] = -D^{-1}(f_{t_1})\partial_t + D^{-1}\big(D_x(f_{t_1}) - f_{xt_1}\big)\partial_{t_x} + \dots. \tag{21}$$

For an arbitrary function H, we have

$$[D_x, \partial_{t_1}]H(t, t_1, t_x, t_{xx}, \dots) = D_x H_{t_1} - \frac{\partial}{\partial_{t_1}} D_x H =$$

$$= (H_{tt_1}t_x + H_{t_1t_1}t_{1x} + \cdots) - \frac{\partial}{\partial_{t_1}}(H_t t_x + H_{t_1}t_{1x} + \cdots) = -H_{t_1}f_{t_1}. \tag{22}$$

Setting $H = f$ and $H = f_x$, we obtain $[D_x, \partial_{t_1}]f = -f_{t_1}f_{t_1}$, $[D_x, \partial_{t_1}]f_x = -f_{xt_1}f_{t_1}$, and so on. We substitute these equations in (21) and find

$$[D_x, Y_1] = -D^{-1}\left(\frac{\partial f}{\partial t_1}\right)\left\{\frac{\partial}{\partial t} + D^{-1}\left(\frac{\partial f}{\partial t_1}\right)\frac{\partial}{\partial t_x} + D^{-1}\left(\frac{\partial f_x}{\partial t_1}\right)\frac{\partial}{\partial t_x} + \cdots\right\} =$$

$$= -D^{-1}\left(\frac{\partial f}{\partial t_1}\right)Y_1. \tag{23}$$

Similarly, we can prove that $[D_x, Y_{-1}] = -D(\partial g/\partial t_{-1})Y_{-1}$.

We now prove Theorem 1.1. We suppose that there exists a nontrivial n integral $F = F(t, t_x, \dots, t_{[N]})$ for Eq. (4) with $t_{[j]} = D_x^j t$ for any natural number j. Then all the vector fields in the Lie algebra M generated by $\{Y_j, X_k\}$ for $j = 1, 2, \dots$ and $k = 1, \dots, N_2$ with an arbitrary N_2 satisfying $N_2 \geq N$ annihilate F. We show that $\dim M < \infty$. We first consider the projection of the algebra M given by the operator P_N:

$$P_N\left(\sum_{i=-N_2}^{-1} x(i)\partial_{t_i} + \sum_{i=0}^{\infty} x(i)\partial_{t_{[i]}}\right) = \sum_{i=-N_2}^{-1} x(i)\partial_{t_i} + \sum_{i=0}^{N} x(i)\partial_{t_{[i]}}. \tag{24}$$

Let $L_n(N)$ be the projection of M. Then the equation $Z_0 F = 0$ is obviously satisfied for any Z_0 in $L_n(N)$. Obviously, $\dim L_n(N) < \infty$. Let the set $\{Z_{01}, Z_{02}, \dots, Z_{0N_1}\}$ form a basis in $L_n(N)$. Any Z_0 in $L_n(N)$ can be represented as a linear combination

$$Z_0 = \alpha_1 Z_{01} + \alpha_2 Z_{02} + \cdots + \alpha_{N_1} Z_{0N_1}. \tag{25}$$

We suppose that the vector fields $Z, Z_1, \dots, Z_{N_1}$ in M are related to the operators $Z_0, Z_{01}, \dots, Z_{0N_1}$ in $L_n(N)$ by the formulas $P_N(Z) = Z_0$, $P_N(Z_1) = Z_{01}, \dots, P_N(Z_{N_1}) = Z_{0N_1}$. We must prove that

$$Z = \alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_{N_1} Z_{N_1}. \tag{26}$$

We use the following lemma in the proof.

Lemma 2.5. *Let $F_1 = D_x F$ and F be an n integral. Then we have $ZF_1 = 0$ for each Z in M.*

Proof. It is easy to verify that F_1 is also an n integral; indeed, $DF_1 = DD_xF = D_xDF = D_xF = F_1$. It was shown above that any Z in M annihilates the n integrals.

We apply the operator $Z - \alpha_1 Z_1 - \alpha_2 Z_2 - \cdots - \alpha_{N_1} Z_{N_1}$ to the function $F_1 = F_1(t, t_x, t_{xx}, \dots, t_{[N+1]})$:

$$(Z - \alpha_1 Z_1 - \alpha_2 Z_2 - \cdots - \alpha_{N_1} Z_{N_1})F_1 = 0. \tag{27}$$

We can write this expression as

$$(Z_0 - \alpha_1 Z_{01} - \alpha_2 Z_{02} - \cdots - \alpha_{N_1} Z_{0N_1})F_1 + \bigl(X(N+1) - \alpha_1 X_1(N+1) - \\ - \alpha_2 X_2(N+1) - \cdots - \alpha_{N_1} X_{N_1}(N+1)\bigr)\frac{\partial}{\partial t_{[N+1]}}F_1 = 0, \tag{28}$$

where $X(N+1), X_1(N+1), \dots, X_{N_1}(N+1)$ are the coefficients of $\partial_{t_{[N+1]}}$ of the vector fields $Z, Z_1, Z_2, \dots, Z_{N_1}$. The first term in (28) vanishes (see linear combination (25)). In the second term, the factor $\partial F_1/\partial t_{[N+1]} = \partial F/\partial t_{[N]}$ is nonzero. We then obtain

$$X(N+1) = \alpha_1 X_1(N+1) + \alpha_2 X_2(N+1) + \cdots + \alpha_{N_1} X_{N_1}(N+1). \tag{29}$$

Equation (29) shows that

$$P_{N+1}(Z) = \alpha_1 P_{N+1}(Z_1) + \alpha_2 P_{N+1}(Z_2) + \cdots + \alpha_{N_1} P_{N+1}(Z_{N_1}). \tag{30}$$

Hence, we can prove formula (26) by induction. Therefore, the Lie algebra M is finite-dimensional. We now construct the characteristic algebra L_n by using M. Because $\dim M < \infty$, the linear envelope of the vector fields $\{Y_j\}_1^\infty$ is finite-dimensional. We choose a basis in this linear space consisting of $Y_1, Y_2, \dots, Y_K$ for $K \le N \le N_2$. Then the algebra generated by $Y_1, Y_2, \dots, Y_K, X_1, X_2, \dots, X_K$ is finite-dimensional because it is a subalgebra of M. This algebra is just the characteristic Lie algebra of Eq. (4).

We suppose that conditions 1 and 2 in Theorem 1.1 are satisfied. Then there exists a finite-dimensional characteristic Lie algebra L_n for Eq. (4). We show that Eq. (4) then admits a nontrivial n integral. Let N_1 be the dimension of L_n and N be the dimension of the linear envelope of the vector fields $\{Y_j\}_{j=1}^\infty$. We take the projection $L_n(N_2)$ of L_n defined by the operator P_{N_2} in (24). Obviously, $L_n(N_2)$ consists of finite sums $Z_0 = \sum_{i=-N}^{-1} x(i)\partial_{t_i} + \sum_{i=0}^{N_2} x(i)\partial_{t_{[i]}}$ where $N = N_1 - N_2$. Let $Z_{01}, \dots, Z_{0N_1}$ form a basis in $L_n(N_2)$. Then we have the $N_1 = N + N_2$ equations $Z_{0j}G = 0$, $j = 1, \dots, N_1$, for a function G depending on $N + N_2 + 1 = N_1 + 1$ independent variables. By the well-known Jacobi theorem, there then exists a function $G = G(t_{-N_2}, t_{-N_2+1}, \dots, t_{-1}, t, t_x, t_{xx}, \dots, t_{[N]})$ that satisfies the equation $ZG = 0$ for any Z in L_n. But it is actually independent of $t_{-N_2}, \dots, t_{-1}$ because $X_1G = 0$, $X_2G = 0, \dots, X_{N_2}G = 0$. Therefore, the function G is $G = G(t, t_x, t_{xx}, \dots, t_{[N]})$.[1]

We note one more property of the algebra L_n. Let π be a map that sends each Z in L_n to its conjugate $D^{-1}ZD$. Obviously, the map π acts from the algebra L_n into its central extension $L_n \oplus \{X_{N_1+1}\}$ because we have $D^{-1}Y_jD = Y_{j+1}$ and $D^{-1}X_jD = X_{j+1}$ for the generators of L_n. Obviously, $[X_{N_1+1}, Y_j] = 0$ and $[X_{N_1+1}, X_j] = 0$ for any integer $j \le N_1$. Moreover, $X_{N_1+1}F = 0$ for the function $G = G(t, t_x, \dots, t_{[N]})$ mentioned above, which implies that $ZG_1 = 0$ for $G_1 = DG$ and for any vector field Z in L_n. Indeed, for any Z in L_n, we have a representation of the form $D^{-1}ZD = \widetilde{Z} + \lambda X_{N_1+1}$ where $\widetilde{Z}$ in L_n and λ is a function. Hence,

$$ZG_1 = ZDG = D(D^{-1}ZDG) = D(\widetilde{Z} + \lambda X_{N_1+1})G = 0. \tag{31}$$

[1] Such a function is not unique; any other solution of these equations depending on the same set of variables can be represented as $h(G)$ for some function h.

Therefore, $G_1 = h(G)$ or $DG = h(G)$. In other words, the function $G = G(n)$ satisfies an ordinary first-order difference equation. Its general solution can be written as $G = H(n, c)$, where H is a function of two variables and c is an arbitrary constant. Solving the equation $G = H(n, c)$ for c, we obtain $c = F(G, n)$. The function $F = F(G, n)$ found is just the sought n integral. In fact, $DF(G, n) = Dc = c = F(G, n)$, and hence $DF = F$. This completes the proof of Theorem 1.1.

3. Restricted classification

Different approaches to classifying integrable nonlinear differential (pseudodifferential) equations are known. One of the most popular and powerful is based on higher symmetries. The theoretical aspects of this method were first formulated in the famous paper by Ibragimov and Shabat [8]. Several classes of nonlinear models were tested by this method in [9]. The symmetry approach allowed Yamilov to find all integrable chains of the Volterra type [10]: $u_t(n) = f\big(u(n-1), u(n), u(n+1)\big)$. The consistency approach to classifying integrable discrete equations was studied by Adler, Bobenko, and Suris in [11]. A classification based on the notion of the recursion operator was studied in [12].

In this paper, we attempt to use the notion of the characteristic Lie algebra in the problem of classifying Darboux-integrable discrete equations of form (4). The classification problem is to describe all chains admitting finite-dimensional characteristic Lie algebras in both directions. In fact, the problem of studying the algebra generated by operators (8) seems quite difficult. We therefore start with a very simple case.

Formulation of the problem. We study the problem of finding all Eqs. (4) for which the Lie algebra generated by the operators Y_1 and Y_{-1} is two-dimensional. We set $Y_{1,-1} = [Y_1, Y_{-1}]$ and require that the relation $Y_{1,-1} = \lambda Y_1 + \mu Y_{-1}$ be satisfied. It follows from explicit formulas (13) and (14) that the vector field $Y_{1,-1}$ does not contain a summand with the term $\partial/\partial t$; hence, $\mu = -\lambda$. The commutators of the basic vector fields with the total-derivative operator admit simple expressions (see Lemma 2.4). Evaluating the commutator $[D_x, Y_{1,-1}]$, we have

$$[D_x, Y_{1,-1}] = \big[Y_1, [D_x, Y_{-1}]\big] - \big[Y_{-1}, [D_x, Y_1]\big] = [Y_1, qY_{-1}] - [Y_{-1}, pY_1] =$$
$$= Y_1(q)Y_{-1} + qY_{1,-1} - Y_{-1}(p)Y_1 + pY_{1,-1} = (p+q)Y_{1,-1} + Y_1(q)Y_{-1} - Y_{-1}(p)Y_1.$$

We recall that by the reasoning above, there must exist a coefficient $\lambda = \lambda(n, x)$ such that

$$Y_{1,-1} = \lambda(Y_1 - Y_{-1}). \tag{32}$$

The problem is to find f in the equation $t_{1x} = f(t, t_1, t_x)$ for which constraint (32) holds.

We commute each side of Eq. (32) with the operator D_x,

$$[D_x, Y_{1,-1}] = [D_x, \lambda Y_1] - [D_x, \lambda Y_{-1}] =$$
$$= (p+q)\lambda(Y_1 - Y_{-1}) + Y_1(q)Y_{-1} - Y_{-1}(p)Y_1 =$$
$$= D_x(\lambda)Y_1 + \lambda p Y_1 - D_x(\lambda)Y_{-1} - \lambda q Y_{-1},$$

and compare two different expressions for the commutator. This gives the conditions

$$q\lambda - Y_{-1}(p) = D_x(\lambda), \qquad p\lambda - Y_1(q) = D_x(\lambda), \tag{33}$$

which form an overdetermined system for the unknown λ (which must satisfy two equations simultaneously). Solving them for λ and $D_x(\lambda)$, we obtain the equations

$$\lambda = \frac{Y_{-1}(p) - Y_1(q)}{q - p}, \qquad D_x(\lambda) = \frac{qY_1(q) - pY_{-1}(p)}{p - q}, \tag{34}$$

which immediately yield

$$D_x\left(\frac{Y_{-1}(p)-Y_1(q)}{q-p}\right)=\frac{pY_{-1}(p)-qY_1(q)}{q-p}. \tag{35}$$

We first note that this equation contains both f and its inverse g. We eliminate g. We recall that $t_{1x}=f(t,t_1,t_x)$ and $t_x=f(t_{-1},t,t_{-1x})$, where $t_{-1x}=g(t,t_{-1},t_x)$. Differentiating the identity $t_x=f\big(t_{-1},t,g(t,t_{-1},t_x)\big)$ with respect to t_{-1}, we obtain

$$D^{-1}\left(\frac{\partial f}{\partial t}(t,t_1,t_x)\right)+D^{-1}\left(\frac{\partial f}{\partial t_x}(t,t_1,t_x)\right)\frac{\partial g}{\partial t_{-1}}=0, \tag{36}$$

which implies that

$$g_{t_{-1}}=-D^{-1}\left(\frac{f_t}{f_{t_x}}\right), \tag{37}$$

and hence $D(g_{t_{-1}})=-f_t/f_{t_x}$. We write Eq. (35) explicitly. We first evaluate $Y_1(q)$ and $Y_{-1}(p)$, where $p=-D^{-1}(f_{t_1})$ and $q=f_t/f_{t_x}$,

$$\begin{aligned}Y_1(q)&=\big\{\partial_t+D^{-1}(f_{t_1})\partial_{t_x}+D^{-1}(f_{xt_1})\partial_{t_{xx}}+\cdots\big\}\frac{f_t}{f_{t_x}}=\\&=\left(\frac{f_t}{f_{t_x}}\right)_t+D^{-1}(f_{t_1})\left(\frac{f_t}{f_{t_x}}\right)_{t_x},\end{aligned} \tag{38}$$

$$\begin{aligned}Y_{-1}(p)&=-\left\{\partial_t-\frac{f_t}{f_{t_x}}\partial_{t_x}-D\left(\frac{\partial g_x}{\partial t_{-1}}\right)\partial_{t_{xx}}-\cdots\right\}D^{-1}(f_{t_1})=\\&=-\big(D^{-1}(f_{t_1})\big)_t+\frac{f_t}{f_{t_x}}\big(D^{-1}(f_{t_1})\big)_{t_x}.\end{aligned} \tag{39}$$

Substituting these equations in (35), we obtain

$$\begin{aligned}D_x&\left\{\frac{-\big(D^{-1}(f_{t_1})\big)_t+(f_t/f_{t_x})\big(D^{-1}(f_{t_1})\big)_{t_x}-\big((f_t/f_{t_x})_t+D^{-1}(f_{t_1})(f_t/f_{t_x})_{t_x}\big)}{f_t/f_{t_x}+D^{-1}(f_{t_1})}\right\}=\\&=\frac{D^{-1}(f_{t_1})\big(\big(D^{-1}(f_{t_1})\big)_t-(f_t/f_{t_x})\big(D^{-1}(f_{t_1})\big)_{t_x}\big)}{f_t/f_{t_x}+D^{-1}(f_{t_1})}-\\&\quad-\frac{(f_t/f_{t_x})\big((f_t/f_{t_x})_t+D^{-1}(f_{t_1})(f_t/f_{t_x})_{t_x}\big)}{f_t/f_{t_x}+D^{-1}(f_{t_1})}.\end{aligned} \tag{40}$$

Equation (40) is rather difficult to study, and we impose one more restriction on f. We suppose that $f=a(t)+b(t_1)+c(t_x)$. We then find the variables p, q, $Y_1(q)$, and $Y_{-1}(p)$ in terms of a, b, and c:

$$\begin{aligned}p&=-D^{-1}(f_{t_1})=-b'(t),\\q&=-D(g_{t_{-1}})=\frac{f_t}{f_{t_x}}=\frac{a'(t)}{c'(t_x)},\\Y_1(q)&=\big(\partial_t+b'(t)\partial_{t_x}\big)\frac{a'(t)}{c(t_x)}=\frac{a''(t)}{c'(t_x)}-\frac{b'(t)a'(t)c''(t_x)}{\big(c'(t_x)\big)^2},\\Y_{-1}(p)&=\left(\partial_t-\frac{a'(t)}{c'(t_x)}\partial_{t_x}\right)\big(-b'(t)\big)=-b''(t).\end{aligned}$$

Substituting these expressions in (35) gives

$$D_xG(t,t_x)=\frac{b'(t)b''(t)-\big(a'(t)/c'(t_x)\big)\big[\big(a''(t)/c'(t_x)\big)-\big(b'(t)a'(t)c''(t_x)/\big(c'(t_x)\big)^2\big)\big]}{a'(t)/c'(t_x)+b'(t)}, \tag{41}$$

where

$$G(t,t_x)=\frac{-b''(t)-\big(a''(t)/c'(t_x)\big)+\big(b'(t)a'(t)c''(t_x)/\big(c'(t_x)\big)^2\big)}{a'(t)/c'(t_x)+b'(t)}. \tag{42}$$

Obviously, the left-hand side of Eq. (41) is of the form $(\partial G/\partial t)t_x+(\partial G/\partial t_x)t_{xx}$ and contains the variable t_{xx}, while the right-hand side does not contain it. This gives the additional constraint $\partial G/\partial t_x=0$.

The investigation of Eq. (41) is tediously long. We therefore give only the answers. The details can be found in [13].

Theorem 3.1. *If Eq. (4) with a particular choice of $f(t,t_1,t_x)=a(t)+b(t_1)+c(t_x)$ has the operators Y_1 and Y_{-1} such that the Lie algebra generated by these two operators is two-dimensional, then $f(t,t_1,t_x)$ has one of the forms*

1. $f(t,t_1,t_x)=c(t_x)+\gamma t_1+\beta$,
2. $f(t,t_1,t_x)=\gamma\log|t_x|+(1/\gamma)\log(e^t-e)+\beta$,
3. $f(t,t_1,t_x)=\gamma(t_x+\beta e^{\alpha t})+\beta e^{\alpha t_1}+\eta$,
4. $f(t,t_1,t_x)=\gamma\big(t_x+\beta\sinh(\alpha t+\lambda)\big)+\beta\cosh t_1+\eta$, *or*
5. $f(t,t_1,t_x)=\gamma {t_x}^2+\beta t_x+\alpha t+\eta$,

where $c(t_x)$ is an arbitrary function and α, β, γ, λ, and η are arbitrary constants.

Moreover, in cases 1, 3, and 4, if the corresponding characteristic Lie algebras are also two-dimensional, then the equations have the forms

a. $t_{1x}=t_x$,
b. $t_{1x}=t_x+e^t+e^{t_1}$, or
c. $t_{1x}=t_x+\beta\sinh t+\beta\cosh t_1$,

and they have the respective n integrals

$$I=t_{xx},$$

$$I=\frac{e^{2t}}{2}+\frac{t_x^2}{2}-t_{xx},$$

$$I=\frac{\beta^2}{2}\cosh^2 t-\beta t_x\cosh t+\frac{t_x^2}{2}+\beta t_x\sinh t-t_{xx}+\frac{\beta^2}{2}n.$$

We note that b also has an x integral of the form

$$F=e^{t_1-t}+e^{2t_1-t_2-t}+e^{t_1-t_2}. \tag{43}$$

Hence, $t_{1x}=t_x+e^t+e^{t_1}$ is a discrete analogue of the Liouville equation.

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