INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **18** (2001) 509–516 www.iop.org/Journals/cq PII: S0264-9381(01)16093-4

Some higher-dimensional vacuum solutions

Metin Gurses¨[1] and Atalay Karasu[2]

[1] Department of Mathematics, Faculty of Sciences, Bilkent University, 06533 Ankara, Turkey
[2] Department of Physics, Faculty of Arts and Sciences, Middle East Technical University, 06531 Ankara, Turkey

Received 4 August 2000

Abstract
We study an even-dimensional manifold with a pseudo-Riemannian metric with arbitrary signature and arbitrary dimensions. We consider the Ricci flat equations and present a procedure to construct solutions to some higher(even-) dimensional Ricci flat field equations from the four-dimensional Ricci flat metrics. When the four-dimensional Ricci flat geometry corresponds to a colliding gravitational vacuum spacetime our approach provides an exact solution to the vacuum Einstein field equations for colliding gravitational plane waves in an (arbitrary) even-dimensional spacetime. We give explicitly higherdimensional Szekeres metrics and study their singularity behaviour.

PACS numbers: 0450, 0420J, 0430

1. Introduction

In the general theory of relativity there exist several solution-generating techniques for the vacuum and electrovacuum Einstein field equations [1,2]. These techniques basically allow us to construct metrics from the known metrics. Recently [3,4], we have given a direct construction of the metrics of the $2N$-dimensional Ricci flat geometries from the twodimensional minimal surfaces in a pseudo-Euclidean 3-geometry. In this work we present a procedure to obtain solutions to some higher-dimensional Ricci flat field equations from some four-dimensional Ricci flat metrics. We show that starting from a Ricci flat metric of a four-dimensional geometry admitting two Killing vector fields it is possible to generate a whole class $2N$-dimensional Ricci flat metrics. Here, in general, both the four-dimensional and $2N$dimensional geometries have arbitrary signatures. Among these there are some geometries which have physical importance in the general theory of relativity and also in the low-energy limit of string theory. For example, if the four-dimensional geometry describes the colliding gravitational plane-wave geometry then the $2N$-dimensional geometry, for all $N > 2$, describes colliding vacuum gravitational plane waves in the higher-dimensional Einstein theory. We give a direct construction of the 2N-dimensional metrics from the four-dimensional Ricci flat metrics. As an explicit example we give a higher-dimensional extension of the Szekeres [5] colliding vacuum gravitational plane-wave metrics.

0264-9381/01/030509+08$30.00

The singularity structure of these higher-dimensional solutions is examined by using the curvature invariant. It is shown that the singularity becomes weaker or stronger depending upontheparametersofthesolution. Hencethesingularitycharacterofthesolutionmaychange with increasing numbers of dimensions.

Let M be a (2N = 2 + 2n)-dimensional manifold with a metric $ds^2 =$

$$g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

$$= g_{ab}(x^c) dx^a\, dx^b + H_{AB}(x^c) dy^A\, dy^B, \quad (1)$$

where $x^\alpha = (x^a, y^A)$, x^a denote the local coordinates on a two-dimensional manifold and y^A denote the local coordinates on a 2n-dimensional manifold and $a,b = 1,2$, $A,B = 1,2,...,2n$. The Christoffel symbols of the metric $g_{\alpha\beta}$ are given by

$$\Gamma^A_{Ba} = \tfrac{1}{2} H^{AD}\, H_{DB,a}, \qquad \Gamma^a_{AB} = -\tfrac{1}{2} g^{ab}\, H_{AB,b}, \qquad \bar{\Gamma}^a_{bc} = \Gamma^a_{bc}, \quad (2)$$

$$\Gamma^A_{BD} = \Gamma^A_{ab} = \Gamma^a_{AB} = 0, \quad (3)$$

where the Γ^a_{bc} are the Christoffel symbols of the two-dimensional metric g_{ab}.

The components of the Riemann tensor are given by

$$R^\alpha_{\beta\gamma\sigma} = \Gamma^\alpha_{\beta\gamma,\sigma} - \Gamma^\alpha_{\beta\sigma,\gamma} + \Gamma^\alpha_{\rho\gamma}\Gamma^\rho_{\beta\sigma} - \Gamma^\alpha_{\rho\sigma}\Gamma^\rho_{\beta\gamma}. \quad (4)$$

The components of the Ricci tensor are

$$\mathcal{R}_{ab} = R^\alpha_{a\alpha b} = R_{ab} + \tfrac{1}{4}\,\mathrm{tr}(\partial_a H^{-1}\partial_b H) - \nabla_a \nabla_b \log\sqrt{\det H}, \quad (5)$$

$$\mathcal{R}_{AB} = -\tfrac{1}{2}(g^{ab} H_{AB,b})_{,a} - \tfrac{1}{2} g^{ab} H_{AB,b}\left[\frac{(\sqrt{\det g})_{,a}}{\sqrt{\det g}} + \frac{(\sqrt{\det H})_{,a}}{\sqrt{\det H}}\right] + \tfrac{1}{2} g^{ab} H_{EA,b} H^{ED} H_{DB,a}, \quad (6)$$

$$\mathcal{R}_{aA} = 0, \quad (7)$$

where R_{ab} is the Ricci tensor of the two-dimensional metric g_{ab}.

2. Ricci flat geometries

The Ricci flat conditions or the vacuum Einstein field equations are given by

$$\partial_a[\sqrt{\det H\, g}\; g^{ab} H^{-1}\partial_b H] = 0, \quad (8)$$

$$R_{ab} + \tfrac{1}{4}\,\mathrm{tr}(\partial_a H^{-1}\partial_b H) - \nabla_a \nabla_b \log\sqrt{\det H} = 0, \quad (9)$$

where H is a 2n×2n matrix of H_{AB} and H^{-1} is its inverse and is the covariant differentiation with respect to the connection Γ^a_{bc} (or with respect to metric g_{ab}). We may rewrite the twodimensional metric as

$$g_{ab} = e^{-M}\eta_{ab}, \tag{10}$$

where η is the metric of flat 2-geometry with arbitrary signature (0 or ±2) and the function M depends on the local coordinates x^a. The corresponding Ricci tensor and the Christoffel symbol are

$$R_{ab} = \frac{1}{2}(\nabla_\eta^2 M)\eta_{ab},$$
$$\Gamma^c_{ba} = \frac{1}{2}[-M_{,b}\delta^c_a - M_{,c}\delta^a_b + M_{,d}\eta^{cd}\eta_{ab}] \tag{11}$$

.

Now let H be a block-diagonal matrix of H_{AB} and each block is a 2 × 2 matrix

$$\begin{pmatrix} \epsilon_1 e^{u_1} h_1 & & & \\ & \ddots & & \bigcirc \\ \bigcirc & & \ddots & \\ & & & \epsilon_n e^{u_i} h_i \end{pmatrix}$$

with det $h_i = 1$ and $_i = \pm 1$ for all $i = 1,2,...,n$. Then

$$\mathrm{tr}(\partial_a H^{-1}\partial_b H) = -2\sum_{i=1}^{n}\partial_a u_i \partial_b u_i + \mathrm{tr}\sum_{i=1}^{n}\partial_a h_i^{-1}\partial_b h_i \tag{12}$$

and

$$\det H = e^{2U}, \qquad \sum_{i=1}^{n} u_i = U. \tag{13}$$

With the above ansatz we can write the higher-dimensional vacuum field equations as

$$\frac{1}{2}\nabla_\eta^2 M\eta_{ab} - U_{,ab} - \frac{1}{2}[M_{,a}U_{,b} + M_{,b}U_{,a} - M_{,d}U^d_{,}\eta_{ab}]$$
$$-\frac{1}{2}\sum_{i=1}^{n}\partial_a u_i \partial_b u_i + \frac{1}{4}\mathrm{tr}\sum_{i=1}^{n}\partial_a h_i^{-1}\partial_b h_i = 0 \tag{14}$$

and

$$\partial_a[\eta^{ab} e^U \partial_b u_i] = 0, \tag{15}$$

$$\partial_a[\eta_{ab} e^U h_i^{-1} \partial_b h_i] = 0, \tag{16}$$

where there is no sum over i (for all i = 1,2,...,n).

3. Four-dimensional geometries

We first consider the four-dimensional case (n = 1). We distinguish the metric functions of the four-dimensional case from the higher-dimensional (n> 1) metric functions by letting

$$M = M, \qquad U = U, \qquad h = h_0. \tag{17}$$

Since there are infinitely many possible solutions of the vacuum four-dimensional Ricci flat equations we shall denote M_i, h_{0i}, $i = 1,2,...,m$ to distinguish this difference. We label all these different solutions by using a subscript $i = 1,2,...,m$. Any two different solutions either have different analytic forms or have the same analytic forms but with different integration constants. We assume that all of these different solutions have the same metric function U. By this choice we lose no generality because it is a matter of choosing a proper coordinate system. The field equations are

$$\frac{1}{2}(\nabla_\eta^2 \mathcal{M}_i)\eta_{ab} - \mathcal{U}_{,ab} - \frac{1}{2}[\mathcal{M}_{i,a}\mathcal{U}_{,b} + \mathcal{M}_{i,b}\mathcal{U}_{,a} - \mathcal{M}_{i,d}\mathcal{U}^d_{,}\eta_{ab}] - \frac{1}{2}\partial_a\mathcal{U}\partial_b\mathcal{U} + \frac{1}{4}\,\mathrm{tr}(\partial_a h_{0i}^{-1}\partial_b h_{0i}) = 0, \qquad (18)$$

and

$$\partial_a[\eta^{ab}e^U\partial_b U] = 0, \qquad (19)$$

$$\partial_a[\eta_{ab}e^U h_{0i}^{-1}\partial_b h_{0i}] = 0. \qquad (20)$$

For each $i = 1,2,...,m$ where m is an arbitrary integer, each triple

$$(M_i, h_{0i}, U)$$

forms a solution to the four-dimensional vacuum field equations and we assume that the function U, for all of these different solutions, is the same.

4. Higher-dimensional Ricci flat geometries

We start with the assumptions that $U = U$ where the function U is defined in (13), $h_i = h_{oi}$ and $m = n$ and using (18) into (14) we obtain

$$\frac{1}{2}\nabla_\eta^2(M - \tilde{M})\eta_{ab} + (n-1)\mathcal{U}_{,ab} - \frac{1}{2}[(M-\tilde{M})_{,a}\mathcal{U}_{,b} + (M-\tilde{M})_{,b}\mathcal{U}_{,a} - (M-\tilde{M})_{,d}\mathcal{U}^d_{,}\eta_{ab}] - \frac{1}{2}\sum_{i=1}^{n}\partial_a u_i \partial_b u_i + \frac{1}{2}n\partial_a\mathcal{U}\,\partial_b\mathcal{U} = 0, \qquad (21)$$

where $\sum_{i=1}^{n}\mathcal{M}_i = \tilde{M}$. Define $M - \tilde{M} = \bar{M}$, the above equation can be written as

$$\frac{1}{2}(\nabla_\eta^2 \bar{M})\eta_{ab} + (n-1)\mathcal{U}_{,ab} - \frac{1}{2}[\bar{M}_{,a}\mathcal{U}_{,b} + \bar{M}_{,b}\mathcal{U}_{,a} - \bar{M}_{,d}\mathcal{U}^d_{,}\eta_{ab}] - \frac{1}{2}\sum_{i=1}^{n}\partial_a u_i\partial_b u_i + \frac{1}{2}n\partial_a\mathcal{U}\partial_b\mathcal{U} = 0. \qquad (22)$$

We assume that U, h_{0i} for $i = 1,2,...,n$ are given functions of x^a. Hence given U we can solve (15) for u_i with $i = 1,2,...,n$, or

$$\nabla^2_\eta u_i + \eta^{ab} U_{,a} u_{i,b} = 0. \qquad (23)$$

Then inserting U, u_i and h_{oi} in (22) we solve the function $\bar{M}$. Then we have the following theorem.

Theorem. *If* U, h_{oi} *and* M_i, *for each* $i = 1,2,...,n$, *form a solution to the four-dimensional Ricci flat field equations for the metric*

$$ds^2 = e^{-M_i}\eta_{ab}\,dx^a\,dx^b + e^U(h_{0i})_{ab}\,dy^a\,dy^b, \quad i = 1,2,...,n, \qquad (24)$$

where $M_i = M_i(x^a)$, $U = U(x^a)$ *and* $h_{0i} = h_{0i}(x^a)$, *then the metric of the* $(2n+2)$-

dimensional geometry is defined below

$$ds^2 = e^{-M}\eta_{ab}\,dx^a\,dx^b + \sum_{i=1}^{n} \epsilon_i e^{u_i} (h_{0i})_{ab}\,dy_i^a\,dy_i^b, \qquad (25)$$

solves the Ricci flat equations, where $\epsilon_i = \pm 1$, $M = \bar{M} + \tilde{M}$, $\tilde{M} = \sum_{i=1}^{n} \mathcal{M}_i$, $\bar{M}$ *solves (22) and* u_i *solve (23). Here the local coordinates of the* $(2n+2)$*-dimensional geometry are given by* $x^\alpha = (x^a, y_1^a, y_2^a, \ldots, y_n^a)$.

We shall now consider some examples which will be obtained by the application of the theorem. We shall consider the case which has a physical importance as far as Einstein's theory of general relativity is concerned. We let $\epsilon_i = 1$ for all $i = 1,2,...,n$ and

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad x^1 = u, \quad x^2 = v,$$

then the equations in (22) become

$$\partial_u \bar{M} \partial_u \mathcal{U} = (n-1)\partial_{uu}\mathcal{U} - \tfrac{1}{2}\sum_{i=1}^{n} (\partial_u u_i)^2 + \tfrac{1}{2} n (\partial_u \mathcal{U})^2, \qquad (26)$$

$$\partial_v \bar{M} \partial_v \mathcal{U} = (n-1)\partial_{vv}\mathcal{U} - \tfrac{1}{2}\sum_{i=1}^{n} (\partial_v u_i)^2 + \tfrac{1}{2} n (\partial_v \mathcal{U})^2, \qquad (27)$$

where the (uv) component of (22) is satisfied identically by virtue of equations (26), (27), (23) and (19). The above equations remind us of the construction of the solutions of the Einstein–Maxwell-massless scalar field equations from the metrics of the Einstein–Maxwell spacetimes [8]. Equation (23) becomes

$$2u_{i,uv} + U_{,u}u_{i,v} + U_{,v}u_{i,u} = 0. \qquad (28)$$

Hence for all $n > 1$ to find a solution of higher-dimensional colliding gravitational vacuum plane waves we have to solve the above equations (26)–(28) for $\bar{M}$ and $u_i, i = 1,2,...,n$. We shall now make a further assumption which solves (28) identically. Let $u_i = m_i U$ where m_i $(i = 1,2,...,n)$ are real constants satisfying only the condition

$$\sum_{i=1}^{n} m_i = 1, \qquad (29)$$

otherwise they are arbitrary. Then the solution of (26) and (27) can be found as

$$e^{-\bar{M}} = (f_u g_v)^{-n+1}(f+g)^{(m^2+n-2)/2}. \qquad (30)$$

Here we took

$$e^U = f(u) + g(v), \qquad (31)$$

which is the general solution of (19) where $f(u)$ and $g(v)$ are arbitrary (differentiable) functions of u and v, respectively, and

$$\sum_{i=1}^{n} (m_i)^2 = m^2. \qquad (32)$$

Hence according to our theorem given above this completes the construction of the metric of the corresponding vacuum spacetimes of dimension $2n + 2$. Given any four-dimensional metric of colliding vacuum gravitational plane-wave geometry (see [6] for details) we have their extensions to higher dimensions for arbitrary n without solving any further differential equations. Sometimes to avoid some undesired singularities on the whole $(2n+2)$-dimensional geometry it may be necessary to keep all the integration constants of the original fourdimensional metric variables (M_i, U, h_{0i}). The boundary conditions discussed in [5] and in [6,chapter 7, pp 46–7] of the four-dimensional metrics should be used for the functions M_i to make them continuous across the boundaries $u = 0, v = 0$. Rather we have to use them to make the $(2n + 2)$-dimensional metric function M continuous across these boundaries.

5. Higher-dimensional Szekeres solution

For illustration let us take the Szekeres solutions [5,6] (which contains the Khan–Penrose [7] solution as a special case) as the four-dimensional vacuum solutions. They are given by

$$ds^2 = 2e^{-M_i}dudv + e^{U-V_i}dx^2 + e^{U+V_i}dy^2, \qquad i = 1,2,\ldots,n \tag{33}$$

where

$$V_i = -2k_i \tanh^{-1}\left(\frac{\frac{1}{2}-f}{\frac{1}{2}+g}\right)^{1/2} - 2\ell_i \tanh^{-1}\left(\frac{\frac{1}{2}-g}{\frac{1}{2}+f}\right)^{1/2}, \tag{34}$$

$$\begin{aligned}\mathcal{M}_i = &-\log(c_i f_u h_v) - \tfrac{1}{2}(k_i^2 + \ell_i^2 + 2k_i\ell_i - 1)\log(f+g)\\ &+\tfrac{1}{2}k_i^2\log(\tfrac{1}{2}-f) + \tfrac{1}{2}\ell_i^2\log(\tfrac{1}{2}-g) + \tfrac{1}{2}\ell_i^2\log(\tfrac{1}{2}+f) + \tfrac{1}{2}k_i^2\log(\tfrac{1}{2}+g)\\ &+2k_i\ell_i\log\left(\sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}-g} + \sqrt{\tfrac{1}{2}+f}\sqrt{\tfrac{1}{2}+g}\right),\end{aligned} \tag{35}$$

where k_i, ℓ_i and c_i are constants for all $i = 1,2,\ldots,n$, and

$$f = \tfrac{1}{2} - (e_1 u)^{n_1}, \qquad g = \tfrac{1}{2} - (e_2 v)^{n_2}. \tag{36}$$

Here $e_1, e_2, n_1 \geqslant 2$, and $n_2 \geqslant 2$ are also arbitrary constants. To avoid the discontinuity of the function e^{-M_i} along the boundaries $u = 0$ and $v = 0$ some relations among k_i, ℓ_i and n_1, n_2 are needed. We shall not set these relations, because in our case the continuity of the function e^{-M} is important. For this purpose we give similar relations among these constants. Let us first define

$$k^2 = \sum_{i=1}^{n} k_i^2, \qquad \ell^2 = \sum_{i=1}^{n} \ell_i^2, \qquad s = \sum_{i=1}^{n} k_i\ell_i, \tag{37}$$

and let

$$k^2 = 2\left(1 - \frac{1}{n_1}\right), \qquad \ell^2 = 2\left(1 - \frac{1}{n_2}\right), \tag{38}$$

where n_1 $2, n_2 \geqslant 2$. We observe that the constants k and ℓ are restricted to the range satisfying

$$1 \leqslant k^2 < 2, \qquad 1 \leqslant \ell^2 < 2.$$

It is now easy to calculate M which is continuous across the boundaries $u = 0$ and $v = 0$ (by virtue of the conditions (38)). It reads

$$e^{-M} = \frac{(f+g)^{(k^2+\ell^2+m^2+2s-2)/2}}{(\frac{1}{2}+f)^{k^2/2}(\frac{1}{2}+g)^{\ell^2/2}\left(\sqrt{\frac{1}{2}-f}\sqrt{\frac{1}{2}-g}+\sqrt{\frac{1}{2}+f}\sqrt{\frac{1}{2}+g}\right)^{2s}}. \quad (39)$$

We also set $\Pi_{i=1}^{n} c_i = (e_1 e_2 n_1 n_2)^{-1}$. Hence the metric of the $(2n + 2)$-dimensional spacetime becomes

$$ds^2 = 2e^{-M}\,du\,dv + \sum_{i=1}^{n}(f+g)^{m_i}(e^{-V_i}\,dx_i^2 + e^{V_i}\,dy_i^2), \quad (40)$$

where $m_i, i = 1,2,...,n$ are constants with the condition given in (29) and V_i are given in (34). Here $x_1 = x, y_1 = y$. When $n = 1$ we have $m_1 = 1$, $m = 1$, $s = k$, which corresponds to the four-dimensional case.

6. Curvature singularities

Next, we calculate the curvature invariant of the metric (1). The components of the Riemannian tensor are

$$\begin{aligned}
R^A_{Bab} &= \Gamma^A_{Bb,a} - \Gamma^A_{Ba,b} + \Gamma^A_{Da}\Gamma^D_{Bb} - \Gamma^A_{Db}\Gamma^D_{Ba} \\
R^A_{abB} &= \Gamma^A_{aB,b} + \Gamma^A_{Db}\Gamma^D_{aB} - \Gamma^A_{cB}\Gamma^c_{ab}, \\
R^a_{bAB} &= \Gamma^a_{DA}\Gamma^D_{bB} - \Gamma^a_{EB}\Gamma^E_{bA}, \\
R^a_{bcd} &= \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}, \\
R^A_{BDE} &= \Gamma^A_{aD}\Gamma^a_{BE} - \Gamma^A_{aE}\Gamma^a_{BD}, \\
R^A_{BDb} &= 0, \qquad R^A_{abc} = 0.
\end{aligned},$$

The curvature invariant is defined by

$$I = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \quad (41)$$

This can be written as

$$I = R_{abcd}R^{abcd} + R_{ABDE}R^{ABDE} + 2R_{ABab}R^{ABab} + 4R_{aAbB}R^{aAbB}, \quad (42)$$

where

$$\begin{aligned}
R^{abcd}R_{abcd} &= R^2, \\
R^{ABCD}R_{ABCD} &= \tfrac{1}{8}g^{ab}g^{cd}[\mathrm{tr}(\partial_c H^{-1}\partial_a H)\,\mathrm{tr}(\partial_d H^{-1}\partial_b H) - \mathrm{tr}(\partial_c H^{-1}\partial_b H\partial_d H^{-1}\partial_a H)] \\
R^{ABab}R_{ABab} &= \tfrac{1}{8}g^{ab}g^{cd}[\mathrm{tr}(\partial_a H^{-1}\partial_d H\partial_c H^{-1}\partial_b H) - \mathrm{tr}(\partial_b H^{-1}\partial_c H\partial_a H^{-1}\partial_d H)], \\
R^{aAbB}R_{aAbB} &= \tfrac{1}{4}g^{ad}g^{eb}\,\mathrm{tr}(H^{-1}\nabla_a\nabla_b H H^{-1}\nabla_d\nabla_e H) \\
&\quad + \tfrac{1}{4}g^{ad}g^{eb}\,\mathrm{tr}(H^{-1}\nabla_a\nabla_b H H^{-1}\partial_d H H^{-1}\partial_e H) \\
&\quad + \tfrac{1}{16}g^{ad}g^{eb}\,\mathrm{tr}(H^{-1}\partial_b H H^{-1}\partial_d H H^{-1}\partial_a H H^{-1}\partial_e H).
\end{aligned},$$

We may, in general, discuss the singularity structure of colliding gravitational plane waves in $2n + 2$ dimensions, but the higher-dimensional Szekeres vacuum solutions give a similar feature of this problem. First of all the solutions have delta function curvature singularities across the boundaries $u = 0, v = 0$ or $f = \frac{1}{2}$ and $g = \frac{1}{2}$ when $n_1 = n_2 = 2$. For other values of $n_1 > 2$ and $n_2 > 2$ the curvature has a Heaviside step function discontinuity across these boundaries. In addition to these discontinuities across the boundaries the spacetime has an essential singularity on the surface $f(u)+g(v) = 0$. For this purpose we shall find the form of the curvature invariant I as $f + g \to 0$, which is the singular surface for the four-dimensional case. We find that

$$I \sim (f_u g_v)^2 (f + g)^{-\mu}, \qquad (43)$$

where $\mu = k^2 + ,^2 + m^2 + 2s + 2$. For the four-dimensional case ($n = 1$) let us choose $k = \bar{k}_1, , =$ $,\bar{}_1$, $m_1 = 1$ and $m^2 = 1$. Hence in this case $\mu = \bar{k}_1^2 + \bar{\ell}_1^2 + 2\bar{k}_1\bar{\ell}_1 + 3$. We have

both $1 \leqslant k^2 \qquad \leqslant \ell^2 \qquad \leqslant k^2 \qquad \ell < 2$, $1 < 2$ and $1 \ \bar{}_1 < 2$, $1 \leqslant \bar{}^2_1 < 2$. Hence the constant m plays an important role in the higher-dimensional metrics. On the constants $m_i, i = 1,2,...,n$ we only have the restriction (29). Hence as $f + g \to 0$ we obtain

$$\frac{I_{2n+2}}{I_4} \sim (f + g)^{1-m^2-2s+2\bar{k}_1\bar{\ell}_1}.$$

Here we have made use of conditions (38) for k and , and exactly similar conditions on $\bar{k}_1$ and

$,\bar{}_1$ which imply that $k^2 = \bar{k}_1^2 = 2(1 - 1/n_1)$ and $\ell^2 = \bar{\ell}_1^2 = 2(1 - 1/n_2)$. This means that the singularity structure in the higher-dimensional spacetimes can be made weaker and stronger than the four-dimensional cases by choosing the constants m_i, k_i and $,_i$ properly. We have enough freedom to do this for higher values of n.

7. Conclusion

We have studied some Ricci flat geometries with arbitrary signatures. We proved a theorem saying that to all Ricci flat metrics of four-dimensional pseudo-Riemannian geometries admitting two Killing vector fields there corresponds a class of Ricci flat metrics for some $(2n + 2)$-dimensional pseudo-Riemannian geometries. As an application we presented an explicitconstructionof$(2n+2)$-dimensionalmetricsofcollidinggravitationalwavespacetimes from given four-dimensional metrics. We gave a higher-dimensional generalization of the Szekeres metrics and discussed the singularity structure of the corresponding spacetimes. Further construction of higher-dimensional colliding gravitational plane-wave metrics will be communicated elsewhere. A possible extension of our work to the low-energy limit of string theory is possible for an arbitrary n. Another application of our approach presented here may be to the colliding gravitational plane-wave problem for the Einstein–Maxwell-dilaton field equations [9].

Acknowledgments

This work is partially supported by the Scientific and Technical Research Council of Turkey (TUBITAK) and by the Turkish Academy of Sciences (TUBA).

References

[1] Kramer D, Stephani H, Mac Callum M A H and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press)

[2] Hoenselaers C and Dietz W (ed) 1984 *Solutions of Einstein's Equations: Techniques and Results (Lecture Notes in Physics no 205)* (Berlin: Springer)

[3] Gurses M 2000 Some Ricci flat pseudo Riemannian geometries¨ *Preprint* math. DG/0006040

[4] Gurses M 2000 Sigma models, minimal surfaces and some Ricci flat pseudo Riemannian geometries¨ *Geometry, Integrability and Quantization (Varna, 7–15 July 2000)* ed I M Mladenov and G Naber

[5] Szekeres P 1972 *J. Math. Phys.* **13** 286

[6] Griffiths J B 1991 *Colliding Plane Waves in General Relativity* (Oxford: Clarendon)

[7] Khan K A and Penrose R 1971 *Nature* **229** 185

[8] Eris¸ A and Gurses M 1977¨ *J. Math. Phys.* **18** 1303

[9] Gurses M and Sermutlu E 1995¨ *Phys. Rev.* D **52** 809