PRICING PERPETUAL AMERICAN-TYPE STRANGLE OPTION FOR MERTON'S JUMP DIFFUSION PROCESS

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By

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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A stock price X_t evolves according to jump diffusion process with certain parameters. An asset manager who holds a strangle option on that stock, wants to maximize his/her expected payoff over the infinite time horizon. We derive an optimal exercise rule for asset manager when the underlying stock is dividend paying and non-dividend paying. We conclude that optimal stopping strategy changes according to stock's dividend rate. We also illustrate the solution on numerical examples.

Keywords: Optimal stopping, perpetual, strangle option, Markov jump diffusion processes.

ÖZET

VADESIZ AMERIKAN TIPI STRANGLE OPSIYONUNUN FIYATLANDIRILMASI

Ayşegül Onat
Endüstri Mühendisliği, Yüksek Lisans
Tez Danışmanı: Doç. Dr. Savaş Dayanık
Aralık, 2014

Hissenin fiyatı zamana bağlı olarak belirli parametrelerle ve belirli aralıklarla gerçekleşen zıplamalarla gelişmektedir. Hisse yöneticisi bu hisse üzerine yazılmış süresiz bir strangle opsiyonu yönetmektedir. Hisse yöneticisi kazancını yükseltmek için en uygun durma zamanını seçmek istemektedir. Ara ödemeler yapan ve yapmayan hisse seçenekleri için en iyi durma zamanı ve beklenen kazanç hesaplanmıştır. Durma stratejisinin hissenin ara ödeme yapan ve yapmayan olmas durumuna göre değişkenlik gösterdiği ortaya konmuştur. Çözümler sayısal örneklerle de gösterilmiştir.

Anahtar sözcükler: En iyi durma zamanı, vadesız, strangle opsiyonu, Markov..

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Above all, I would like to express my deepest thanks to my family especially my mother Peluzan Onat and my brother Z. Mert Onat for their love, support and trust at all stages of my life. Lastly, I would like to dedicate this thesis to my father I. Hakkı Onat, his memory lives in our hearts.

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Glossary

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p strike price of put option

c strike price of call option ($p < c$)

X_t stock price process

μ fixed appreciation rate of the underlying stock on which perpetual option is written

δ fixed dividend rate of the underlying stock on which perpetual option is written

λ constant arrival rate op downward jumps

y_0 the fraction that stock price loses every time jumps occurs

Y_t stock price process after diffusions and jumps are separated

$\mathbb{P}$ real world probability measure

$\mathbb{P}^\gamma$ risk neutral probability measure after jump frequency is changed to $\lambda\gamma$

γ the fraction of the new arrival rate after probability change over the old arrival rate

$f(.)$ the payoff function of strangle option

$W(.)$ the Wronskian function

$\varphi(.)$ the decreasing solution of the second order ordinary differential equation

$\psi(.)$ the increasing solution of the second order ordinary differential equation

α_0 power of the decreasing solution of the second order ordinary differential equation ($\alpha_0 < 0$)

α_1 power of the increasing solution of the second order ordinary differential equation ($\alpha_1 > 1$)

Chapter 1

Preliminaries

Definition 1.1. *(Sigma algebra) Let Ω be a given set. Then a family of subsets of ω is called σ-algebra $\mathcal{F}$ on ω if it satisfies*

(i) $\emptyset \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$, *where* $A^c = \Omega - A$ *is the complement of* A *in* Ω

(iii) $A_1, A_2, ... \in \mathcal{F} \Longrightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$

Definition 1.2. *(Filtration) A filtration on $(\Omega, \mathcal{F})$ is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t \geq 0}$ of σ-algebras $\mathcal{M}_t \subset \mathcal{F}$ such that*

$$0 \leq s < t \Longrightarrow \mathcal{M}_s \subset \mathcal{M}_t$$

which means that $\{\mathcal{M}_t\}$ is increasing.

Definition 1.3. *(Probability measure) Let $(\Omega, \mathcal{F})$ be a measurable space. A probability measure $\mathcal{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \mapsto [0, 1]$ such that*

(i) $\mathbb{P}(\emptyset) = 0$ *and* $\mathbb{P}(\Omega) = 1$

(ii) If $A_1, A_2, ... \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Definition 1.4. *(Probability space) A probability space is the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ which contains information about elementary outcomes in the sample space Ω, all events are collected in the σ-algebra $\mathcal{F}$, and the probability of all events is described by the probability measure $\mathbb{P}$.*

Definition 1.5. *(Risk neutral probability measure) A risk-neutral measure, (also called an equilibrium measure, or equivalent martingale measure), is a probability measure such that each stock price is exactly equal to the discounted expectation of the stock price at the future time under this measure. This is heavily used in the pricing of financial derivatives due to the fundamental theorem of asset pricing, which implies that in a complete market a derivative's price is the discounted expected value of the future payoff under the unique risk-neutral measure.*

Definition 1.6. *(Stopping time) Let $(I, \leq)$ be an ordered index set and let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. A random variable $\tau : \Omega \mapsto I$ is called a stopping time if*

$$\{\omega : \tau \leq t\} \in \mathcal{F}_t$$

Definition 1.7. *(Strong Markov property) Suppose that $X = (X_t : t \geq 0)$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Then X is said to have the strong Markov property if for each stopping time τ, conditioning on the event $\{\tau < \infty\}$, and for each bounded Borel function $f : \mathbb{R}^n \mapsto \mathbb{R}$ we have*

$$\mathbb{E}[f(X_{\tau+h})|\mathcal{F}_\tau] = \mathbb{E}[f(X_h)|\sigma(X_\tau)]$$

for all $h \geq 0$.

Chapter 2

Introduction

In a volatile market, investors hedge their risks against the uncertainty of asset prices by using classical instruments such as financial options. A put option gives its holder the right to sell one asset unit for a pre-agreed strike price and a call option grants the right to buy; they are used when expecting the asset prices to fall and to rise, respectively. If a trader believes there will be a significant price movement but is unsure of its direction, in general he would build a long position on a strangle option that creates the two sided payoff as the combination of a put payoff with a lower strike price and a call payoff at a higher strike price written on the same underlying asset. Such a long strangle strategy is often traded in the over-the-counter (OTC) market, and is favored by hedge fund managers, particularly in currency and metal markets and CME, SAXO OTC contracts (see [1]). Figure 2.1 shows a typical payoff of a strangle option for an investor holding a long position. Mathematically, the payoff of a strangle option exercised at stock price $x > 0$ is

$$f(x) = (p - x)^+ + (x - c)^+.$$

The strangle option considered in this thesis is perpetual, namely, the option never expires.

This thesis studies the optimal stopping problem of an hedge fund manager who manages perpetual strangle option written on a continuously dividend-paying

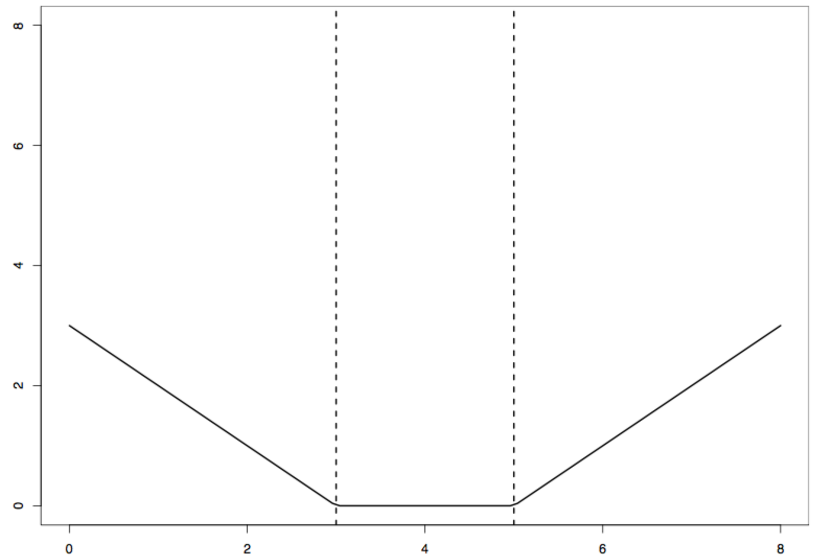

Figure 2.1: Payoff of a strangle option with put option strike price p=3 and call option strike price c=5 ($c > p$)

stock at a fixed rate. At each time point, he has to take a decision between exercising the option or waiting for future observations. He wants to come up with the best optimal stopping strategy in order to maximize his payoff and, in the meantime, he also has to consider downward jumps coming from stock price at some uncertain times which reduce its value by a fixed percentage. Stock price processes with downward jumps have very important economical meaning: In financial market stock prices may be correlated with some other prices. Therefore, any bank crisis or default of a company in a related sector may lead sudden price changes and our model is able to capture of replicating those scenarios.

The no arbitrage pricing theory of mathematical finance requires the problem be setup under a risk-neutral probability measure. The risk-neutral probability measure is not unique and we use one of them. Afterwards, we separate the jump and diffusion parts similar to the ideas of Davis [2] and we introduce a dynamic programming operator. Using this formulation, we solve the optimal stopping problem by means of successive approximations which not only lead to accurate and efficient numerical algorithms but also allow us to establish concretely the form of optimal stopping strategy.

We also study the same optimal stopping problem when underlying stock is not dividend paying and illustrate how asset manager changes his optimal behavior. This case differs from the first one because stock price process appreciates at a higher rate and this encourages holder of the option to wait longer compared to the case of the dividend-paying stock.

The next chapter reviews related studies in the literature. In Chapter 4, we give a mathematical formulation of our problem and define risk neutral probability measure along with the dynamic programming operator. In the first section of Chapter 5, we break the original value function into parts and apply appropriate transformations in order to solve the optimal stopping problem via techniques of Dayanik and Kazatzas [3]. By a back-transformation we finally obtain with the optimal strategy and the optimal stopping time. At the end of Chapter 5, we reconsider the problem for an underlying stock price process paying zero dividend as a special case. Chapter 6 illustrates numeric examples. The computer code used for examples is relegated to the appendix.

Chapter 3

Literature Review

In a recent work [4] related to our study Dayanik and Egami solve optimal stopping problems of an institutional asset manager. The investors entrust their initial funds in the amount of L to the asset manager and receive coupon payments from the asset manager on their initial funds at a fixed rate c (higher than the risk-free interest rate). The asset manager gathers dividend at a fixed rate δ on the market value of the portfolio. At any time, the asset manager has the right to terminate the contract and to walk away with the net terminal value of the portfolio after the payment of the investors initial funds. However, she is not financially responsible for any amount of shortfall. The asset managers problem is to find a stopping rule which maximizes her expected discounted total income which is

$$U(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}_x^{\gamma} \left[e^{-r\tau}(X_\tau - L)^+ + \int_0^\tau e^{-rt}(\delta X_t - cL)dt \right]$$

where $\mathbb{E}^{\gamma}$ is taken under equivalent martingale measure $\mathbb{P}^{\gamma}$ and γ represents market price of jump risk. Our problem mathematically differs in terms of the structure of the reward function.

Chiarella and Ziogas [5] study the pricing of the American type strangle option written on a dividend paying asset. They find the boundaries $a_1(t)$ and

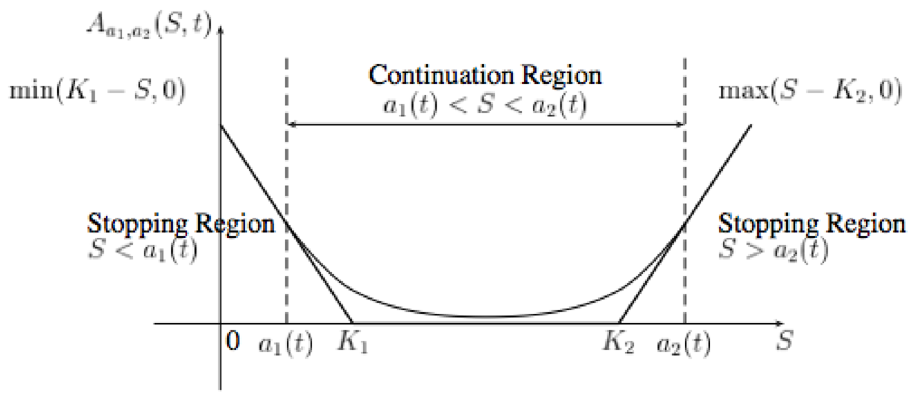

Figure 3.1: Continuation and stopping region of an American strangle option with put and call strike prices are K_1 and K_2, respectively. [5]

$a_2(t)$ depicted in figure 3.1 by applying Fourier transform to Black-Scholes partial differential equation (PDE). Fourier transformation changes Black-Scholes PDE into an ordinary differential equation. However, in their study, stock price process does not contain any jumps. This means that the market has unique risk-neutral probability measure which is highly suitable for no artbitrage pricing theory.

Having jump diffusion stock price process, we need to strip the jumps from the diffusion process as in Dayanik and Egami [4] and define a new process as a sequential diffusions. Dayanik and Karatzas use this approach in order to solve the optimal stopping problems with successive approximations. The idea was inspired by the paper of Davis [2] where he strips jumps from the deterministic trajectories of piecewise-deterministic Markov processes between jump times.

To solve the transformed optimal stopping problems for pure diffusion processes, we use the techniques developed by Dayanik and Karatzas [3] who characterize concave excessive functions for optimal stopping problems of one dimensional diffusion processes. Their study is a generalization of the paper of Dynkin and Yushkevich [6], who solve optimal stopping problems with diffusions restricted to compact subspaces of $\mathbb{R}$. However, our problem definition requires diffusions to be defined on the interval $(0, +\infty)$ and chapter 5.1 of [3] defines the smallest concave majorant when left boundary is absorbing and right boundary is natural. In order to find the smallest excessive function, we use an important proposition of Dayanik and Karatzas which allows us to transform our reward function into a new function whose excessive function is easier to calculate. By back-transformation, the optimal stopping strategy and the optimal stopping time can be found.

Chapter 4

Problem Descripton

Let $(\Omega, \mathcal{F}, P)$ be a probability space hosting Brownian motion $B = \{B_t, t \geq 0\}$ and a homogenous Poisson process $N = \{N_t, t \geq 0\}$ with rate λ, both adapted to filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions.

Let market has a stock whose price process is driven by $X = \{X_t, t \geq 0\}$ with appreciation rate μ and dividend rate δ. At some time points modeled by Poisson process N_t, stock is subject to downward jumps and decreases its value by y_0. The stock price has the dynamics

$$\frac{dX_t}{X_{t-}} = (\mu - \delta)dt + \sigma dB_t - y_0 \left(dN_t - \lambda dt\right),$$

for some constants $\mu > 0, \delta \geq 0, \sigma > 0$ and $y_0 \in (0, 1)$. Therefore, the stock price is modeled by the equation

$$X_t = X_0 \exp\left((\mu - \delta + \lambda y_0)t - \frac{1}{2}\sigma^2 t + \sigma B_t\right)(1 - y_0)^{N_t},$$

for $t \geq 0$,. Hence, stock price process is a geometric Brownian motion subject to downward jumps with constant relative jump sizes.

Imagine a trader holds a perpetual strangle option written on $X = \{X_t, t \geq 0\}$ and at any time $\tau \in (0, \infty)$, the trader has right to exercise the option and gets the payoff

$$f(X_\tau) = (p - X_\tau)^+ + (X_\tau - c)^+.$$

Trader aims to choose $\tau \in (0, \infty)$ so that she will obtain the maximum payoff. To do this, we need to calculate maximum expected discounted payoff

$$V(x) = \sup_{\tau > 0} \mathbb{E}_x^\gamma \left[e^{-r\tau} f(X_\tau) \right]$$

for $x \geq 0$ and over all stopping times τ of X. $\mathbb{E}^\gamma$ is taken under the equivalent martingale measure $\mathbb{P}^\gamma$ for a specified market price of the jump risk γ.

No-arbitrage pricing framework claims that the value of the contract on the asset X is the expectation of the discounted payoff of the contract under some equivalent martingale measure. Since X has jumps, there are more than one equivalent martingale measure. Radon-Nikodym derivative gives class of equivalent martingale measures in the form

$$\left. \frac{d\mathbb{P}^\gamma}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \eta_t$$

where

$$\frac{d\eta_t}{\eta_{t-}} = -\frac{\mu - \delta - r}{\sigma} dB_t + (\gamma - 1)\left(dN_t - \lambda dt \right)$$

which has the solution

$$\eta_t = \exp\left\{ -(\gamma - 1)\lambda t - \frac{\mu - \delta - r}{\sigma} B_t - \frac{1}{2} \frac{(\mu - \delta - r)^2}{\sigma^2} t \right\} \gamma^{N_t},$$

$t \geq 0$. The Girsanov Theorem shows that $B_t^\gamma = \frac{\mu - \delta - r}{\sigma} t + B_t$ is a standard Brownian motion under the probability measure $\mathbb{P}^\gamma$ defined by equation. Here the price process given by

$$\frac{dX_t}{X_{t-}} = (r - \delta)dt + \sigma dB_t^\gamma - y_0 \left(dN_t - \lambda \gamma dt \right)$$

$$X_t = X_0 \exp\left\{ (r - \delta)t + \lambda \gamma y_0 t - \frac{1}{2}\sigma^2 t + \sigma B_t^\gamma \right\} (1 - y_0)^{N_t}$$

where N_t is a poisson process with intensity $\lambda\gamma$ and independent of B_t^γ under the new measure $\mathbb{P}^\gamma$.

Under the probability measure $\mathbb{P}^\gamma$ we should solve

$$V(x) = \sup_{\tau > 0} \mathbb{E}_x^\gamma \left[e^{-r\tau} \left\{ (p - X_\tau)^+ + (X_\tau - c)^+ \right\} \right]. \tag{4.1}$$

which is a discounted optimal stopping problem with reward function $f(x) = (p-x)^+ + (x-c)^+$.

Let $T_1, T_2, ...$ be the arrival times of process N. Observe that $X_{T_{n+1}} = (1-y_0)\, X_{T_{n+1}-}$ and

$$\frac{X_{T_n+t}}{X_{T_n}} = \exp\left\{\left(r-\delta+\lambda\gamma y_0 - \frac{1}{2}\sigma^2\right)t + \sigma\left(B^\gamma_{T_n+t} - B^\gamma_{T_n}\right)\right\}$$

if $0 \le t < T_{n+1} - T_n$.

Define the standard Brownian motion $B^{\gamma,n}_t = B^\gamma_{T_n+t} - B^\gamma_{T_n}$ for every $n \ge 1$, $t \ge 0$ and Poisson process $T^{(n)}_k = T_{n+k} - T_n$ for $k \ge 0$ respectively under $\mathbb{P}^\gamma$ and one dimensional diffusion process

$$Y^{y,n}_t = y\exp\left\{\left(r-\delta+\lambda\gamma y_0 - \frac{1}{2}\sigma^2\right)t + \sigma B^{\gamma,n}_t\right\}$$

which has the dynamics

$$\begin{aligned} Y^{y,n}_0 &= y \\ \frac{dY^{y,n}_t}{Y^{y,n}_t} &= (r-\delta+\lambda\gamma y_0)\,dt + \sigma B^{\gamma,n}_t. \end{aligned}$$

X coincides with $Y^{X_{T_n},n}_t$ on $[T_n, T_{n+1})$ and jumps to $(1-y_0)\,Y^{X_{T_n},n}_{T_{n+1}-T_n}$ at time T_{n+1} for every $n \ge 0$. Namely,

$$X_{T_n+t} = \begin{cases} Y^{X_{T_n},n}_t & \text{if } 0 \le t < T_{n+1} - T_n \\ (1-y_0)Y^{X_{T_n},n}_{T_{n+1}-T_n} & \text{if } t = T_{n+1} - T_n \end{cases}$$

For $n = 0$, we write $Y^{y,0}_t = y\exp\left\{\left(r-\delta+\lambda\gamma y_0 - \frac{1}{2}\sigma^2\right)t + \sigma B^\gamma_t\right\}$ where $0 \le t < T_1$.

Let S_B be the collection of all stopping times of Y^x or equivalently Brownian motion B. Take arbitrary fixed stopping time $\tau \in S_B$ and consider the following optimal strategy:

(i) on $\{\tau < T_1\}$ stop at time τ.

(ii) on $\{\tau \geq T_1\}$ update X at time T_1 to $X_{T_1} = (1 - y_0)\, Y_{T_1}^{x_0}$ and continue optimality thereafter.

The value of this new strategy is

$$\begin{aligned} & \mathbb{E}_x^{\gamma}\left[e^{-r\tau} f\left(X_\tau\right) \mathbf{1}_{\{\tau<T_1\}} + e^{-rT_1} V\left(\left(1-y_0\right) Y_{T_1}^x\right) \mathbf{1}_{\{\tau \geq T_1\}}\right] \\ = \; & \mathbb{E}_x^{\gamma}\left[e^{-(r+\lambda\gamma)\tau} f\left(Y_\tau^{x_0}\right) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} V\left(\left(1-y_0\right) Y_t^x\right) dt\right]. \end{aligned}$$

For every bounded function $w : \mathbb{R}^+ \to \mathbb{R}^+$, we introduce the operator

$$(Jw)(x) = \sup_{\tau \in S_B} \mathbb{E}_x^{\gamma}\left[e^{-(r+\lambda\gamma)\tau} f\left(Y_\tau^x\right) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} w\left(\left(1-y_0\right) Y_t^x\right) dt\right] \tag{4.2}$$

then we expect the value function $V(.)$ of equation 4.1 to be the unique fixed point of the operator J, namely $V(.) = (JV)(.)$ and $V(.)$ is the pointwise limit of the successive approximations

$$\begin{aligned} v_0(x) &= f(x) = (p-x)^+ + (x-c)^+ \\ v_n(x) &= (Jv_{n-1})(x) \end{aligned}$$

for $x \geq 0$, $n \geq 1$.

Assumption 1. *Let $w : \mathbb{R}^+ \to \mathbb{R}$ be a convex function such that $f(x) \leq w(x) \leq x + p$ for every $x \in \mathbb{R}^+$.*

Assumption 2. *$Jw(.)$ is a non-increasing function up to some point x, then it is non-decreasing.*

Remark 4.1. *For any two functions $w_1(.)$ and $w_2(.)$ satisfying Assumption 1, we have the inequality*

$$\|w_1 - w_2\| \leq p + c$$

where $\|w\| = sup_{x \in \mathbb{R}^+} |w(x)|$.

Remark 4.2. *Under Assumption 1, $Jw(x) \leq \frac{\lambda\gamma}{\delta+\lambda\gamma} x + \frac{\lambda\gamma}{r+\lambda\gamma} p$ for every $x \in \mathbb{R}^+$.*

Proof. From equation 4.2 we have

$$\begin{aligned}
(Jw)(x) &= \sup_{\tau\in S_B} \mathbb{E}_x^\gamma \left[e^{-(r+\lambda\gamma)\tau} f\left(Y_\tau^x\right) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} w\left(\left(1-y_0\right)Y_t^x\right) dt \right] \\
&\leq \mathbb{E}_x^\gamma \left[\int_0^\infty \lambda\gamma e^{-(r+\lambda\gamma)t} \left(\left(1-y_0\right)Y_t^x + p\right) dt \right] \\
&\leq (1-y_0)\lambda\gamma \int_0^\infty x e^{-(r+\lambda\gamma)t} e^{(r-\delta+\lambda\gamma y_0)t} \mathbb{E}_x^\gamma \left[e^{\sigma B_t^\gamma - \frac{\sigma^2}{2}t} \right] dt + \frac{\lambda\gamma}{r+\lambda\gamma} p \\
&\leq \frac{\lambda\gamma}{\delta+\lambda\gamma} x + \frac{\lambda\gamma}{r+\lambda\gamma} p \\
&< \infty
\end{aligned}$$

□

Lemma 4.1 (Monotonicity Lemma)**.** *For any two functions $w_1, w_2 : \mathbb{R}^+ \to \mathbb{R}$ if $w_1(.) \leq w_2(.)$ then we have $(Jw_1)(.) \leq (Jw_2)(.)$. If $w(.)$ is convex function, then $(Jw)(.)$ is also a convex function.*

Proof. From inequality $w_1(.) \leq w_2(.)$, we can get

$$\begin{aligned}
& \mathbb{E}_x^\gamma \left[e^{-(r+\lambda\gamma)\tau} f\left(Y_\tau^x\right) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} w_1\left(\left(1-y_0\right)Y_t^x\right) dt \right] \\
\leq\ & \mathbb{E}_x^\gamma \left[e^{-(r+\lambda\gamma)\tau} f\left(Y_\tau^x\right) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} w_2\left(\left(1-y_0\right)Y_t^x\right) dt \right].
\end{aligned}$$

By taking supremum of both sides over $\tau \in S_B$ we prove $(Jw_1)(.) \leq (Jw_2)(.)$. Because J is a linear operator of $w(.)$, convexity is preserved. □

Proposition 4.1. *For any two functions $w_1, w_2 : \mathbb{R}^+ \to \mathbb{R}$ satisfying Assumption 1, we have*

$$\|Jw_1 - Jw_2\| \leq \frac{\lambda\gamma}{r+\lambda\gamma} \|w_1 - w_2\| \leq \frac{\lambda\gamma}{r+\lambda\gamma}(p+c).$$

This means that J acts as a contraction mapping on the bounded functions.

Proof. For every $\epsilon > 0$ and $x > 0$, there is an $\epsilon-$optimal stopping time $\tau(\epsilon, x)$ which may depend on ϵ and x, such that

$$(Jw_1)(x) - \epsilon \leq \mathbb{E}_x^\gamma \left[e^{-(r+\lambda\gamma)\tau(\epsilon,x)} f\left(Y^x_{\tau(\epsilon,x)}\right) + \int_0^{\tau(\epsilon,x)} \lambda\gamma e^{-(r+\lambda\gamma)t} w_1\left(\left(1-y_0\right) Y_t^x\right) dt \right]$$

so we have,

$$\begin{aligned}
(Jw_1)(x) - (Jw_2)(x) \leq \epsilon + \mathbb{E}_x^\gamma \Bigg[& e^{-(r+\lambda\gamma)\tau(\epsilon,x)} f\left(Y^x_{\tau(\epsilon,x)}\right) \\
& + \int_0^{\tau(\epsilon,x)} \lambda\gamma e^{-(r+\lambda\gamma)t} w_1\left(\left(1-y_0\right) Y_t^{x_0}\right) dt \Bigg] \\
& - \sup_{\tau \in S_B} \mathbb{E}_x^\gamma \Bigg[e^{-(r+\lambda\gamma)\tau} f\left(Y^x_\tau\right) \\
& + \int_0^{\tau} \lambda\gamma e^{-(r+\lambda\gamma)t} w_2\left(\left(1-y_0\right) Y_t^{X_0}\right) dt \Bigg] \\
\leq & \epsilon + \mathbb{E}_x^\gamma \Bigg[e^{-(r+\lambda\gamma)\tau(\epsilon,x)} f\left(Y^x_{\tau(\epsilon,x)}\right) \\
& + \int_0^{\tau(\epsilon,x)} \lambda\gamma e^{-(r+\lambda\gamma)t} w_1\left(\left(1-y_0\right) Y_t^x\right) dt \Bigg] \\
& - \mathbb{E}_x^\gamma \Bigg[e^{-(r+\lambda\gamma)\tau(\epsilon,x)} f\left(Y^x_{\tau(\epsilon,x)}\right) \\
& + \int_0^{\tau(\epsilon,x)} \lambda\gamma e^{-(r+\lambda\gamma)t} w_2\left(\left(1-y_0\right) Y_t^x\right) dt \Bigg] \\
\leq & \epsilon + \mathbb{E}_x^\gamma \Bigg[\int_0^{\tau(\epsilon,x)} \lambda\gamma e^{-(r+\lambda\gamma)t} [w_1\left(\left(1-y_0\right) Y_t^x\right) \\
& - w_2\left(\left(1-y_0\right) Y_t^x\right)] dt \Bigg]
\end{aligned}$$

Therefore,

$$\begin{aligned}
(Jw_1)(x) - (Jw_2)(x) &\leq \epsilon + \|w_1 - w_2\| \int_0^\infty \lambda\gamma e^{-(r+\lambda\gamma)} dt \\
&= \epsilon + \|w_1 - w_2\| \frac{\lambda\gamma}{r+\lambda\gamma} \\
&\leq \epsilon + (p+c) \frac{\lambda\gamma}{r+\lambda\gamma}
\end{aligned}$$

Taking supremum of both sides over $x \geq 0$ completes the proof. □

Lemma 4.2. *The sequence $(v_n)_{n\geq 0}$ is monotonically nondecreasing. Therefore the pointwise limit $v_\infty(x) = \lim_{n\to\infty} v_n(x)$, $x \geq 0$, exists. Every $v_n(.)$, $n \geq 0$ and $v_\infty(.)$ are finite and convex functions.*

Proof. (By induction) For $n = 1$, we have

$$\begin{aligned}
v_1(x) &= (Jv_0)(x) \\
&= \sup_{\tau \in S_B} \mathbb{E}_x^\gamma \left[e^{-(r+\lambda\gamma)\tau} f\left(Y_\tau^{X_0}\right) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} v_0\left((1-y_0)\, Y_t^x\right) dt \right] \\
&\geq f(Y^x) \\
&= v_0(x).
\end{aligned}$$

Base holds. Assume $v_n(.) \geq v_{n-1}(.)$ is true. We must show that $v_{n+1}(.) \geq v_n(.)$ holds as well. By taking the operator J of both sides, we get $(Jv_n)(.) \geq (Jv_{n-1})(.) \Rightarrow v_{n+1}(.) \geq v_n(.)$. This implies that the sequence $(v_n)_{n\geq 0}$ is monotonically nondecreasing. We also know from Assumption 1, $v_n(x) < x + p$, $\forall n \geq 0$, $\forall x \geq 0$. Therefore, the limit $v_\infty(x) = \lim_{n\to\infty} v_n(x)$, $x \geq 0$, exists. □

Proposition 4.2. *The limit $v_\infty(.) = \lim_{n\to\infty} v_n(.) = \sup_{n\geq 0} v_n(.)$ is the unique bounded fixed point operator of $(Jv)(.)$ and*

$$0 \leq v_\infty(x) - v_n(x) \leq (p+c)\left(\frac{\lambda\gamma}{r+\lambda\gamma}\right)^n$$

holds for every $x \geq 0$.

Proof. For any $x > 0$ and $n \geq 0$, we have $v_n(x) \nearrow v_\infty(x)$ as $n \to \infty$ and $0 \leq v_n(x) \leq x + p$. Hence, the monotone convergence theorem implies that

$$\begin{aligned}
v_\infty(x) &= \sup_{n\geq 0} v_n(x) \\
&= \sup_{\tau\in S_B} \lim_{n\to\infty} \mathbb{E}_x^\gamma \left[e^{-(r+\lambda\gamma)\tau} f\left(Y_\tau^x\right) + \int\limits_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} v_{n-1}\left(\left(1-y_0\right) Y_t^x\right) dt \right] \\
&= \sup_{\tau\in S_B} \mathbb{E}_x^\gamma \left[e^{-(r+\lambda\gamma)\tau} f\left(Y_\tau^x\right) + \int\limits_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} v_\infty\left(\left(1-y_0\right) Y_t^x\right) dt \right] \\
&= \left(J v_\infty\right)(x).
\end{aligned}$$

Therefore, $v_\infty(.)$ is the bounded fixed point operator of $(Jv)(.)$

$$\|v_\infty - v_n\| = \|Jv_\infty - Jv_{n-1}\| \leq \|v_\infty - v_{n-1}\| \frac{\lambda\gamma}{r+\lambda\gamma} \leq \ldots \leq (p+c)\left(\frac{\lambda\gamma}{r+\lambda\gamma}\right)^n$$

for every $n \geq 1$. $\square$

Chapter 5

The Optimal Exercise Policy for the Strangle Option

In this chapter, we are going to define an optimal exercise policy for the problem

$$(Jw)(x) = \sup_{\tau \in S_B} \mathbb{E}_x^{\gamma} \left[e^{-(r+\lambda\gamma)\tau} f\left(Y_\tau^x\right) + \int_0^{\tau} \lambda\gamma e^{-(r+\lambda\gamma)t} w\left(\left(1-y_0\right) Y_t^x\right) dt \right]$$

using the methodology of Dayanik and Karatzas [3]. Afterwards, we examine the special case when the underlying asset is non-dividend paying.

For every fixed $w : \mathbb{R}^+ \mapsto \mathbb{R}$ satisfying Assumption 1, we are now ready to solve the optimal stopping problem $(Jw)(.)$. We know that for fixed $x < \infty$, $w(x)$ is bounded from above. See that

$$\begin{aligned}
\mathbb{E}_x^{\gamma} \left[\int_0^{\infty} e^{-(r+\lambda\gamma)t} \left| w\left(\left(1-y_0\right) Y_t^x\right) \right| dt \right] &\leq \mathbb{E}_x^{\gamma} \left[\int_0^{\infty} e^{-(r+\lambda\gamma)t} \left(\left(1-y_0\right) Y_t^x + p\right) dt \right] \\
&\leq \frac{p}{r+\lambda\gamma} + \left(1-y_0\right) \int_0^{\infty} x e^{-(r+\lambda\gamma)t} e^{(r-\delta+\lambda\gamma y_0)t} \mathbb{E}_x^{\gamma} \left[e^{\sigma B_t^{\gamma} - \frac{\sigma^2}{2} t} \right] dt \\
&\leq \frac{x}{\delta+\lambda\gamma} + \frac{p}{r+\lambda\gamma} \\
&< \infty
\end{aligned}$$

for $x \geq 0$. The strong Markov property of $Y_t^{x_0}$ implies that

$$\begin{aligned}(Hw)(x) &= \mathbb{E}_x^\gamma\left[\int_0^\infty e^{-(r+\lambda\gamma)t} w\left(\left(1-y_0\right)Y_t^x\right)dt\right] \\ &= \mathbb{E}_x^\gamma\left[\int_0^\tau e^{-(r+\lambda\gamma)t} w\left(\left(1-y_0\right)Y_t^x\right)dt\right] + \mathbb{E}_x^\gamma\left[e^{-(r+\lambda\gamma)\tau}(Hw)(Y_\tau^x)\right]\end{aligned}$$

for every stopping time $\tau > 0$. The above equality becomes

$$\mathbb{E}_x^\gamma\left[\int_0^\tau e^{-(r+\lambda\gamma)t} w\left(\left(1-y_0\right)Y_t^x\right)\right] = (Hw)(x) - \mathbb{E}_x^\gamma\left[e^{-(r+\lambda\gamma)\tau}(Hw)(Y_t^x)\right]$$

which shows

$$\begin{aligned}&\mathbb{E}_x^\gamma\left[e^{-(r+\lambda\gamma)\tau} f\left(Y_\tau^x\right) + \int_0^\tau \lambda\gamma e^{-(r+\lambda\gamma)t} w\left(\left(1-y_0\right)Y_t^x\right)dt\right] \\ &\qquad = \lambda\gamma(Hw)(x) + \mathbb{E}_x^\gamma\left[e^{-(r+\lambda\gamma)\tau}(f - \lambda\gamma(Hw))(Y_t^x)\right]\end{aligned}$$

for every $\tau > 0$ and $x \geq 0$. Let us define

$$\left(Gw\right)(x) = \sup_{\tau>0} \mathbb{E}_x^\gamma\left[e^{-(r+\lambda\gamma)\tau}(f - \lambda\gamma(Hw))(Y_t^x)\right] \tag{5.1}$$

and let's rewrite value function in equation 4.2 as

$$\left(Jw\right)(x) = \lambda\gamma(Hw)(x) + \left(Gw\right)(x)$$

for $x \geq 0$.

Let $\psi(.)$ and $\varphi(.)$ be increasing and decreasing solutions of $\left(A_0 f\right)(y) - \left(r + \lambda\gamma\right) f(y) = 0$, $y > 0$ with respect to boundary conditions $\psi(0+) = 0$ and $\varphi(+\infty) = 0$ where A_0 is the infinitesimal generator of the diffusion process $Y^x = Y^{x,0}$. We have

$$\frac{\sigma^2 y^2}{2} f''(y) + (r - \delta + \lambda\gamma y_0) y f'(y) - \left(r + \lambda\gamma\right) f(y) = 0$$

which has two linearly independent solutions $\psi(.)$ and $\varphi(.)$ in the form of y^{α_i} for $i = 0, 1$. One can explicitly find α_0 and α_1 from the roots of the characteristic function $g(\alpha) = \alpha(\alpha-1) + \frac{2}{\sigma^2}\left[(r-\delta+\lambda\gamma y_0)\alpha - (r+\lambda\gamma)\right]$ of the above ordinary differential equation. Now we have two solutions $\psi(y) = y^{\alpha_1}$ and $\varphi(y) = y^{\alpha_0}$ for every $y > 0$ and note that

$$\alpha_0 < 0 < 1 < \alpha_1$$

because both $g(0) < 0$ and $g(1) < 0$. Also note that

$$\alpha_0 + \alpha_1 = 1 - \frac{2}{\sigma^2}(r - \delta + \lambda\gamma y_0)$$

$$\alpha_0\alpha_1 = -\frac{2}{\sigma^2}(r + \lambda\gamma).$$

Define the Wronskian

$$W(y) = \psi'(y)\varphi(y) - \psi(y)\varphi'(y) = (\alpha_1 - \alpha_0)y^{\alpha_0+\alpha_1-1}$$

for $y > 0$.

Define the hitting and exit time of the diffusion process Y^x as

$$\begin{aligned}\tau_a &= \inf\{t \geq 0 : Y_t^{x_0} = a\} \\ \tau_{ab} &= \inf\{t \geq 0 : Y_t^{x_0} \notin (a,b)\}\end{aligned}$$

for $0 < a < b < \infty$.

Define the operator

$$(H_{ab}w)(x) = \mathbb{E}_x^\gamma\left[\int_0^{\tau_{ab}} e^{-(r+\lambda\gamma)t} w\left((1-y_0)\,Y_t^x\right) + 1_{\{\tau_{ab}<\infty\}} e^{-(r+\lambda\gamma)\tau_{ab}} f\left(Y_{T_{ab}}^x\right)\right].$$

Lemma 5.1. *For every $x > 0$, we have*

$$\begin{aligned}(Hw)(x) &= \mathbb{E}_x^\gamma\left[\int_0^\infty e^{-(r+\lambda\gamma)t} w\left((1-y_0)\,Y_t^x\right)\right] \\ &= \lim_{a\downarrow 0, b\uparrow\infty} (H_{ab}w)(x) \\ &= \varphi(x)\int_0^x \frac{2\psi(\xi)w\left((1-y_0)\,\xi\right)}{p^2(\xi)W(\xi)}d\xi + \\ &\quad\ \psi(x)\int_x^\infty \frac{2\varphi(\xi)w\left((1-y_0)\,\xi\right)}{p^2(\xi)W(\xi)}d\xi\end{aligned}$$

where $p^2(x) = \sigma^2 x^2$. It is twice continuously differentiable on $\mathbb{R}^+$ and satisfies the ordinary differential equation $\left(A_0 f\right)(x) - (r + \lambda\gamma) f(x) + w\left(\left(1 - y_0\right) x\right) = 0$.

Proof. Proof can be found in Taylor and Karlin [7]. □

We now solve the optimal stopping problem $(Gw)(.)$ in equation 5.1 with the payoff function

$$
\begin{aligned}
(f - \lambda\gamma(Hw))(x) &= (p - x)^+ + (x - c)^+ - \lambda\gamma\Bigg[\varphi(x)\int_0^x \frac{2\psi(\xi) w\left(\left(1 - y_0\right)\xi\right)}{p^2(\xi)W(\xi)} d\xi \\
&+ \psi(x)\int_x^\infty \frac{2\varphi(\xi) w\left(\left(1 - y_0\right)\xi\right)}{p^2(\xi)W(\xi)} d\xi\Bigg] \\
&= (p - x)^+ + (x - c)^+ \\
&- \frac{2\lambda\gamma}{\sigma^2(\alpha_1 - \alpha_0)}\Bigg[x^{\alpha_0}\int_0^x \xi^{-\alpha_0 - 1} w\left(\left(1 - y_0\right)\xi\right) d\xi \\
&+ x^{\alpha_1}\int_x^\infty \xi^{-\alpha_1 - 1} w\left(\left(1 - y_0\right)\xi\right) d\xi\Bigg] \\
&\leq (p - x)^+ + (x - c)^+ \\
&- \frac{2\lambda\gamma}{\sigma^2(\alpha_1 - \alpha_0)}\Bigg[x^{\alpha_0}\int_0^x \xi^{-\alpha_0 - 1}\left(\left(1 - y_0\right)\xi - c\right) d\xi \\
&+ x^{\alpha_1}\int_x^\infty \xi^{-\alpha_1 - 1}\left(\left(1 - y_0\right)\xi - c\right) d\xi\Bigg] \\
&\leq (p - x)^+ + (x - c)^+ - \frac{2\lambda\gamma}{\sigma^2}\left[x\frac{1 - y_0}{(1 - \alpha_0)(\alpha_1 - 1)} - \frac{c}{\alpha_0\alpha_1}\right] \\
&\leq (p - x)^+ + (x - c)^+ - \lambda\gamma\left[\frac{(1 - y_0)x}{\delta + \lambda\gamma(1 - y_0)} + \frac{c}{r + \lambda\gamma}\right]
\end{aligned}
$$

For sufficiently large values of x, we have

$$
(f - \lambda\gamma(Hw))(x) \leq \frac{\delta x}{\delta + \lambda\gamma(1 - y_0)} - \frac{c(r + 2\lambda\gamma)}{r + \lambda\gamma}
$$

and for small enough values of x, we have

$$
(f - \lambda\gamma(Hw))(x) \leq p.
$$

The above inequalities together with boundary conditions $\psi(+\infty) = \varphi(0+) = +\infty$ give the limits

$$l_0 = \limsup_{x\to 0} \frac{(f-\lambda\gamma(Hw))^+(x)}{\varphi(x)} = 0 \quad l_\infty = \limsup_{x\to\infty} \frac{(f-\lambda\gamma(Hw))^+(x)}{\psi(x)} = 0.$$

Therefore, according to Proposition 5.2 of Dayanık [3], value function is finite and optimal stopping strategy exists.
Proposition 5.3 of Dayanık [3] claims that $G(.)$ is the smallest nonnegative majorant of $(f-\lambda\gamma(Hw))(.)$ and by Proposition 5.7 of Dayanık [3],

$$\tau[w] = \inf\{t \geq 0 : Y_t^x \in \Gamma[w]\} \tag{5.2}$$

is an optimal stopping time in the optimal stopping region

$$\Gamma[w] = \{x > 0 : (Gw)(x) = (f-\lambda\gamma(Hw))(x)\} = \{x > 0 : (Jw)(x) = f(x)\}.$$

According to Proposition 5.5 of Dayanık [3], we have function $(Mw)(.)$ which is the smallest nonnegative concave majorant of the function

$$(Lw)(\zeta) = \begin{cases} \frac{f-\lambda\gamma(Hw)}{\varphi} \circ F^{-1}(\zeta) & \text{if } \zeta > 0 \\ 0 & \text{if } \zeta = 0 \end{cases} \tag{5.3}$$

where $F(x) = \frac{\psi(x)}{\varphi(x)}$ and $(Gw)(x) = \varphi(x)(Mw)(F(x))$ for $x \geq 0$. Furthermore, $(Mw)(0) = 0$ and $(Mw)(.)$ is continuous at 0.

In order to explicitly define $(Mw)(.)$, we should observe some important properties of the function $(Lw)(.)$. First, we identify the limiting behavior of $(Lw)(x)$

for large x values. Let us check

$$\lim_{x\uparrow\infty}(Lw)(F^{-1}(x)) = \lim_{x\uparrow\infty}\frac{(p-x)^+ + (x-c)^+ - \lambda\gamma\mathbb{E}_x^\gamma\left[\int\limits_0^\infty e^{-(r+\lambda\gamma)t}w\left(\left(1-y_0\right)Y_t^x\right)dt\right]}{x^{\alpha_0}}$$

$$\geq \lim_{x\uparrow\infty}\frac{x-c-\lambda\gamma\mathbb{E}_x^\gamma\left[\int\limits_0^\infty e^{-(r+\lambda\gamma)t}\left(\left(1-y_0\right)Y_t^x+p\right)dt\right]}{x^{\alpha_0}}$$

$$\geq \lim_{x\uparrow\infty}\frac{x-c-\lambda\gamma(1-y_0)x\int\limits_0^\infty e^{-(r+\lambda\gamma)t}e^{(r-\delta+\lambda\gamma y_0)t}E_x^\gamma\left[e^{\sigma B_t^\gamma-\frac{\sigma^2}{2}t}\right]dt-\lambda\gamma\frac{p}{r+\lambda\gamma}}{x^{\alpha_0}}$$

$$\geq \lim_{x\uparrow\infty}x^{-\alpha_0+1}\left(\frac{1}{\delta+(1-y_0)\lambda\gamma}-\frac{c}{x}-\frac{p}{x(r+\lambda\gamma)}\right)$$

$$= +\infty$$

because of $\alpha_0 < 0$. So we see that $(Lw)(+\infty) = +\infty$.

Let us examine the sign of the first derivative of $(Lw)(x)$

$$\begin{aligned}(Lw)'(x) &= \frac{d}{dx}\left(\frac{f-\lambda\gamma(Hw)}{\varphi}\circ F^{-1}(x)\right)\\ &= \left[\frac{1}{F'}\left(\frac{f-\lambda\gamma(Hw)}{\varphi}\right)'\right]\circ F^{-1}(x)\end{aligned}$$

as x tends to 0, which is

$$\begin{aligned}\lim_{x\downarrow 0}\left[\frac{1}{F'}\left(\frac{f-\lambda\gamma(Hw)}{\varphi}\right)'\right](F^{-1}(x)) &= \lim_{x\downarrow 0}\frac{x^{-\alpha_1}}{\alpha_1-\alpha_0}\bigg[-2\frac{\lambda\gamma x^{\alpha_1}}{\sigma^2}\int_x^\infty \zeta^{-\alpha_1-1}w\left(\left(1-y_0\right)\zeta\right)d\zeta\\ &\quad+(-x-\alpha_0(p-x))1_{\{x<p\}}+(x-\alpha_0(x-c))1_{\{x>c\}}\bigg]\\ &= \lim_{x\downarrow 0}\frac{x^{-\alpha_1}}{\alpha_1-\alpha_0}\bigg[-2\frac{\lambda\gamma x^{\alpha_1}}{\sigma^2}\int_x^\infty \zeta^{-\alpha_1-1}w\left(\left(1-y_0\right)\zeta\right)d\zeta\\ &\quad+(-x(1-\alpha_0)-\alpha_0 p)\bigg]\\ &= +\infty.\end{aligned}$$

because of $\lim_{x\downarrow 0}x^{\alpha_1}\int_x^\infty \zeta^{-\alpha_1-1}w\left(\left(1-y_0\right)\zeta\right)d\zeta = \frac{w(+0)}{\alpha_1} = \frac{p}{\alpha_1}$, $\alpha_1 > 1$ and the positive sign appears due to $-\alpha_0\alpha_1 = \frac{2}{\sigma^2}(r+\lambda\gamma)$.

Proposition 5.1. *The inequality*

$$(Lw)'(F^{-1}(p-)) < (Lw)'(p+) < (Lw)'(c-) < (Lw)'(c+)$$

holds.

Proof. Direct computations give

$$(Lw)'(F^{-1}(p-)) \approx \frac{-2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)} \int_{p-}^{\infty} \zeta^{-\alpha_1-1} w\left((1-y_0)\zeta\right) d\zeta - \frac{(p-)^{1-\alpha_1}}{\alpha_1-\alpha_0}$$
$$(Lw)'(F^{-1}(p+)) \approx \frac{-2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)} \int_{p+}^{\infty} \zeta^{-\alpha_1-1} w\left((1-y_0)\zeta\right) d\zeta$$

which gives $(Lw)'(F^{-1}(p-)) < (Lw)'(F^{-1}(p+))$ since $\frac{-2\lambda\gamma}{\sigma^2} \int_{p-}^{p+} \zeta^{-\alpha_1-1} w\left((1-y_0)\zeta\right) d\zeta \leq 0 < (p-)^{1-\alpha_1}$. Also we have

$$(Lw)'(F^{-1}(c-)) \approx \frac{-2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)} \int_{c-}^{\infty} \zeta^{-\alpha_1-1} w\left((1-y_0)\zeta\right) d\zeta$$
$$(Lw)'(F^{-1}(c+)) \approx \frac{-2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)} \int_{c+}^{\infty} \zeta^{-\alpha_1-1} w\left((1-y_0)\zeta\right) d\zeta + \frac{(c+)^{1-\alpha_1}}{\alpha_1-\alpha_0}$$

which gives $(Lw)'(F^{-1}(c-)) < (Lw)'(F^{-1}(c+))$ since $\frac{-2\lambda\gamma}{\sigma^2} \int_{c-}^{c+} \zeta^{-\alpha_1-1} w\left((1-y_0)\zeta\right) d\zeta \leq 0 < (c+)^{1-\alpha_1}$ and $(Lw)'(F^{-1}(p+)) < (Lw)'(F^{-1}(c-))$ because of $\frac{-2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)} \int_{p+}^{c-} \zeta^{-\alpha_1-1} w\left((1-y_0)\zeta\right) d\zeta \leq 0$. □

Remark 5.1. *(i)*$(Lw)'(F^{-1}(p-)) < 0$

(ii)$(Lw)'(F^{-1}(p+)) < 0$

(iii)$(Lw)'(F^{-1}(c-)) \leq 0$

Proof. Since we have $(Lw)'(F^{-1}(p-)) < (Lw)'(F^{-1}(p+)) < (Lw)'(F^{-1}(c-))$, it is enough to prove that $(Lw)'(F^{-1}(c-)) < 0$ holds. From Assumption 1, we have $0 \leq f(x) \leq w(x) \leq x + p$ then

$$(Lw)'(F^{-1}(c-)) \approx \frac{-2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)} \int_{c-}^{\infty} \zeta^{-\alpha_1-1} w\left((1-y_0)\zeta\right) d\zeta \leq 0$$

since $\frac{-2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)} < 0$. □

We should also analyze the sign of the second derivative of $(Lw)(F(x))$, which is

$$(Lw)''(F^{-1}(x)) = \frac{2\varphi(x)}{p^2(x)W(x)F'(x)}(A_0 - (r+\lambda\gamma))(f - \lambda\gamma(Hw))(x)$$

as Dayanık and Karatzas show. We see that

$$\text{sgn}[(Lw)''(F^{-1}(x))] = \text{sgn}[(A_0 - (r+\lambda\gamma))(f - \lambda\gamma(Hw))(x)]$$

and recall from Lemma 0.3 that $A_0 - (r+\lambda\gamma))(Hw)(x) = -w((1-y_0)x$. So we have that

$$\begin{aligned}A_0 - (r+\lambda\gamma))(f - \lambda\gamma(Hw))(x) =& [(\delta + \lambda\gamma(1-y_0))\, x - (r+\lambda\gamma)p \\ &+ \lambda\gamma w((1-y_0)x)] 1_{\{x<p\}} \\ &+ \lambda\gamma w((1-y_0)x) 1_{\{p\le x\le c\}} \\ &+ [-(\delta + \lambda\gamma(1-y_0))\, x + (r+\lambda\gamma)c \\ &+ \lambda\gamma w((1-y_0)x)] 1_{\{x>c\}}\end{aligned}$$

Remark 5.2. *c can be turning point if $(1-y_0)c > p$ holds. In this case $(Lw)'(F^{-1}(c+)) > 0$.*

Proof. We have

$$(Lw)''(F^{-1}(c)) = \lambda\gamma w((1-y_0)c) \ge \lambda\gamma f((1-y_0)c) > \lambda\gamma f(p) = 0$$

and for $w(.) = 0$ we have $(Lw)'(F^{-1}(c+)) = \frac{(c+)^{1-\alpha_1}}{\alpha_1 - \alpha_0} > 0$ shows that c is a turning point. On the other hand, if we have $(1-y_0)c < p$, then

$$(Lw)''(F^{-1}(c)) = \lambda\gamma w((1-y_0)c) \ge \lambda\gamma f((1-y_0)c) > \lambda\gamma(p-(1-y_0)c) > 0.$$

This case $\operatorname{sgn}[(Lw)''(F^{-1}(x))] \neq 0$, therefore it implies that $(Lw)'(F^{-1}(c+)) < 0$. □

Remark 5.3. *The function $(Lw)(F(x))$ is a concave function in some open neighborhood of 0 and $+\infty$.*

Proof. Using Lemma 5.1 we have

$$\begin{aligned}\lim_{x\downarrow 0}(A_0 - (r+\lambda\gamma))(f - \lambda\gamma(Hw))(x) \;&\le\; \lim_{x\downarrow 0}[(\delta + \lambda\gamma(1-y_0))\, x - \\ &\quad (r+\lambda\gamma)p + \lambda\gamma((1-y_0)x + p)] \\ &\le\; -rp \\ &<\; 0\end{aligned}$$

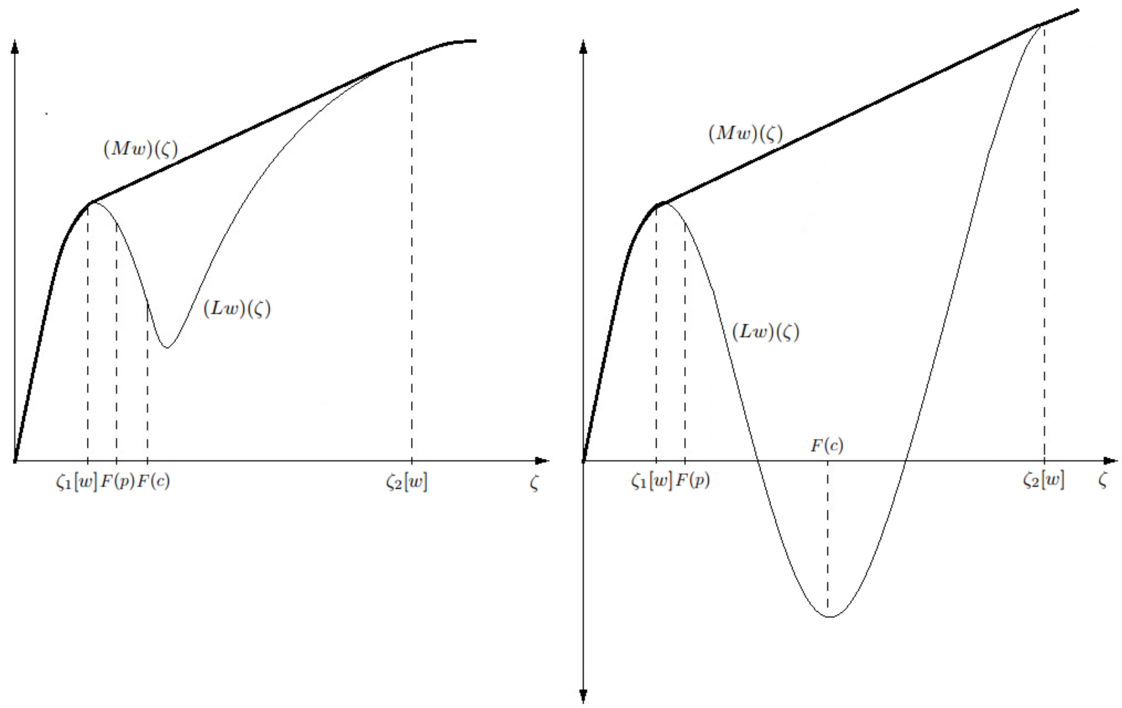

Figure 5.1: Two possible forms of $(Lw)(.)$ and its smallest concave majorant $(Mw)(.)$ when $\delta > 0$

and

$$
\begin{aligned}
\lim_{x\uparrow\infty}(A_0 - (r+\lambda\gamma))(f - \lambda\gamma(Hw))(x) &\leq \lim_{x\uparrow\infty}[-(\delta + \lambda\gamma(1-y_0))\,x + \\
&\quad\ (r+\lambda\gamma)c + \lambda\gamma((1-y_0)x + p)] \\
&\leq \lim_{x\uparrow\infty}[-\delta x + rc + \lambda\gamma(c+p)] \\
&< 0
\end{aligned}
$$

□

The information that we observe so far lead us the following conclusion: there are unique two points $\zeta_1[w]$ and $\zeta_2[w]$ such that $0 < \zeta_1[w] < F^{-1}(p) < F^{-1}(c) < \zeta_2[w] < +\infty$ satisfy

$$(Lw)'(\zeta_1[w]) = (Lw)'(\zeta_2[w]) = \frac{(Lw)(\zeta_2[w]) - (Lw)(\zeta_1[w])}{\zeta_2[w] - \zeta_1[w]}$$

and the smallest nonnegative concave majorant $(Mw)(.)$ of $(Lw)(.)$ coincides with $(Lw)(.)$ on $(0, \zeta_1[w]) \cup (\zeta_2[w], +\infty)$ and straight line which is tangent to $(Lw)(\zeta)$

exactly at $\zeta = \zeta_1[w]$ and $\zeta_2[w]$ on $[\zeta_1[w], \zeta_2[w]]$. More precisely,

$$(Mw)(\zeta) = \begin{cases} (Lw)(\zeta) & \text{if } \zeta \in ((0, \zeta_1[w]) \cup (\zeta_2[w], +\infty)) \\ \frac{\zeta_2[w]-\zeta}{\zeta_2[w]-\zeta_1[w]}(Lw)(\zeta_1[w]) + & \\ \frac{\zeta-\zeta_1[w]}{\zeta_2[w]-\zeta_1[w]}(Lw)(\zeta_2[w]) & \text{if } \zeta \in [\zeta_1[w], \zeta_2[w]]. \end{cases}$$

Let us define $x_1[w] = F^{-1}(\zeta_1[w])$ and $x_2[w] = F^{-1}(\zeta_2[w])$. Then by Proposition 5.5 of Dayanık [3], the value function of the optimal stopping problem in 5.1 equals

$$\begin{aligned} (Gw)(x) &= \varphi(x)(Mw)(F(x)) \\ &= \begin{cases} (f - \lambda\gamma(Hw))(x) & \text{if } x \in ((0, x_1[w]) \cup (x_2[w], +\infty)) \\ \frac{(x_2[w])^{\alpha_1-\alpha_0} - x^{\alpha_1-\alpha_0}}{(x_2[w])^{\alpha_1-\alpha_0} - (x_1[w])^{\alpha_1-\alpha_0}} & \\ (f - \lambda\gamma(Hw))(x_1[w]) & \\ + \frac{x^{\alpha_1-\alpha_0} - (x_1[w])^{\alpha_1-\alpha_0}}{(x_2[w])^{\alpha_1-\alpha_0} - (x_1[w])^{\alpha_1-\alpha_0}} & \\ (f - \lambda\gamma(Hw))(x_2[w]) & \text{if } x \in [x_1[w], x_2[w]]. \end{cases} \end{aligned}$$

Optimal stopping time in equation 5.2 becomes

$$\tau[w] = inf\{t \geq 0 : Y_t^x \in (0, x_1[w]) \cup (x_2[w], +\infty)\}$$

in the optimal stopping region

$$\Gamma[w] = \{x > 0 : (Gw)(x) = (f - \lambda\gamma(Hw))(x)\} = (0, x_1[w]) \cup (x_2[w], +\infty).$$

Proposition 5.2. *The value function $(Gw)(.)$ satisfies*

(i) $(\mathcal{A}_0 - (r + \lambda\gamma))(Gw)(x) = 0$*, for* $x \in (x_1[w], x_2[w])$

(ii) $(Gw)(x) > f(x) - \lambda\gamma(Hw)(x)$*, for* $x \in (x_1[w], x_2[w])$

(ii) $(\mathcal{A}_0 - (r + \lambda\gamma))(Gw)(x) < 0$*, for* $x \in (0, x_1[w]] \cup [x_2[w], +\infty)$

(iv) $(Gw)(x) = f(x) - \lambda\gamma(Hw)(x)$*, for* $x \in (0, x_1[w]] \cup [x_2[w], +\infty)$

Proof. By definition of value function

$$(Gw)\,(x) = \sup_{\tau \in S_B} \mathbb{E}\left[e^{-(r+\lambda\gamma)\tau}(f - \lambda\gamma(Hw))(Y_t^x)\right].$$

For $\tau = 0$, we have

$$(Gw)(x) \geq (f - \lambda\gamma(Hw))(x)$$

and for every small $h > 0$

$$\begin{aligned}
(Gw)(x) &\geq \mathbb{E}[e^{-(r+\lambda\gamma)h}(Gw)(Y_h^x)] \\
&= \mathbb{E}\Big[(1-(r+\lambda\gamma)h+o(h))((Gw)(x)+\int_0^h (Gw)'(Y_t^x)dY_t \\
&\quad +\frac{1}{2}\int_0^h (Gw)''(Y_t^x) < dY_t > +o(h)\Big] \\
&= \mathbb{E}\Big[(1-(r+\lambda\gamma)h+o(h))((Gw)(x) \\
&\quad +\int_0^h x_0(Gw)'(Y_t^x)(r-\delta-\lambda\gamma y_0)dt+\int_0^h x\sigma(Gw)'(Y_t^x)dB_t^\gamma \\
&\quad +\frac{1}{2}\int_0^h (x)^2\sigma^2(Gw)''(Y_t^x)dt+o(h)\Big] \\
&= \mathbb{E}\Big[(1-(r+\lambda\gamma)h+o(h))((Gw)(x)+x(Gw)'(x)(r-\delta-\lambda\gamma y_0)h \\
&\quad +\frac{1}{2}(x)^2\sigma^2(Gw)''(x)h+o(h)\Big]
\end{aligned}$$

there are no remaining stochastic terms, so we can safely remove expectation and ignore the terms whose are in order of h^2. After doing this, we get

$$\begin{aligned}
(Gw)(x) &\geq (Gw)(x)+x_0(Gw)'(x)(r-\delta-\lambda\gamma y_0)h \\
&\quad +\frac{1}{2}(x)^2\sigma^2(Gw)''(x)h-(r+\lambda\gamma)h(Gw)(x)+o(h)
\end{aligned}$$

dividing both sides by h and taking the limits as $h \downarrow 0$ we will have

$$0 \geq x(Gw)'(x)(r-\delta-\lambda\gamma y_0)+\frac{1}{2}(x)^2\sigma^2(Gw)''(x)-(r+\lambda\gamma)(Gw)(x)$$

which equals to

$$0 \geq \mathcal{A}_0(Gw)(x)-(r+\lambda\gamma)(Gw)(x)$$

The solutions of the above term when it equals to zero are $\psi(.)$ and $\varphi(.)$. We have

$$\begin{aligned} 0 &\geq (f-\lambda\gamma(Hw))(x)-(Gw)(x) \\ 0 &\geq \mathcal{A}_0(Gw)(x)-(r+\lambda\gamma)(Gw)(x) \end{aligned}$$

and only one of the above inequalities can be zero. Thus,

$$0=\max\{(f-\lambda\gamma(Hw))(x)-(Gw)(x),(\mathcal{A}_0-(r+\lambda\gamma))(Gw)(x)\}.$$

On the waiting region $(x_1[w],x_2[w])$ we will have

$$\begin{aligned} (\mathcal{A}_0-(r+\lambda\gamma))(Gw)(x) &= 0 \\ (Gw)(x) &> f(x)-\lambda\gamma(Hw)(x) \end{aligned}$$

and on the stopping region $(0,x_1[w]]\cup[x_2[w],+\infty)$ we will have

$$\begin{aligned} (\mathcal{A}_0-(r+\lambda\gamma))(Gw)(x) &< 0 \\ (Gw)(x) &= f(x)-\lambda\gamma(Hw)(x) \end{aligned}$$

This completes the proof. □

Proposition 5.3. *The value function $(Jw)(.)$ satisfies*

(i) $(\mathcal{A}_0-(r+\lambda\gamma))(Jw)(x)+\lambda\gamma w((1-y_0)x)=0$, *for* $x\in(x_1[w],x_2[w])$

(ii) $(Jw)(x)>f(x)$, *for* $x\in(x_1[w],x_2[w])$

(iii) $(\mathcal{A}_0-(r+\lambda\gamma))(Jw)(x)+\lambda\gamma w((1-y_0)x)<0$, *for* $x\in(0,x_1[w]]\cup[x_2[w],+\infty)$

(iv) $(Jw)(x)=f(x)$, *for* $x\in(0,x_1[w]]\cup[x_2[w],+\infty)$

Proof. By Lemma 5.1

$$(\mathcal{A}_0(Hw))(x)-(r+\lambda\gamma)(Hw)(x)=-w((1-y_0)x)$$

and by definition

$$(Jw)(x)=\lambda\gamma(Hw)(x)+(Gw)(x).$$

These equations and Proposition 5.2 complete the proof. □

Theorem 1. *The function $x \mapsto v_\infty(x) = (Jv_\infty)(x)$ satisfies the following variational inequalities*

(i) $(\mathcal{A}_0 - (r + \lambda\gamma))v_\infty(x) + \lambda\gamma v_\infty((1 - y_0)x) = 0$, *for* $x \in (x_1[w], x_2[w])$

(ii) $v_\infty(x) > f(x)$, *for* $x \in (x_1[w], x_2[w])$

(iii) $(\mathcal{A}_0 - (r + \lambda\gamma))v_\infty(x) + \lambda\gamma v_\infty((1 - y_0)x) < 0$, *for* $x \in (0, x_1[w]] \cup [x_2[w], +\infty)$

(iv) $v_\infty(x) = f(x)$, *for* $x \in (0, x_1[w]] \cup [x_2[w], +\infty)$

Proof. Every $v_n(x)$, $n \geq 0$ and $v_\infty(x)$ are convex and bounded for every fixed $x > 0$. Therefore, Proposition 5.3, applied to $w = v_\infty$ completes the proof of theorem. □

Theorem 2. *For every $x > 0$, the expected reward of asset manager is $V(x) = v_\infty(x) = \mathbb{E}_x^\gamma \left[e^{-r\tau[v_\infty]} f\left(X_{\tau[v_\infty]}\right)\right]$ and $\tau[v_\infty]$ is an optimal stopping time for equation 4.2.*

Proof. Define $\tau_{ab} = \inf\{t \leq 0 : X_t \in (0, a] \cup [b, \infty)\}$ for every $0 < a < b < \infty$. Ito's rule gives

$$\begin{aligned} e^{-r(t\wedge\tau\wedge\tau_{ab})} v_\infty(X_{t\wedge\tau\wedge\tau_{ab}}) &= v_\infty(X_0) \\ &+ \int_0^{t\wedge\tau\wedge\tau_{ab}} e^{-rs}(\mathcal{A}_0 - (r + \lambda\gamma))v_\infty(X_s) + \lambda\gamma v_\infty((1 - y_0)X_s)ds \\ &+ \int_0^{t\wedge\tau\wedge\tau_{ab}} e^{-rs}(\mathcal{A}_0 - (r + \lambda\gamma))v_\infty(X_s) + \lambda\gamma v_\infty((1 - y_0)X_s)\sigma X_s dB_s^\gamma \\ &+ \int_0^{t\wedge\tau\wedge\tau_{ab}} e^{-rs}[v_\infty((1 - y_0)X_{s-}) - v_\infty(X_{s-})(dN_s - \lambda\gamma ds)] \end{aligned}$$

for every $t \geq 0$, $\tau \geq 0$ and $0 < a < b < \infty$. We know that $v_\infty(.)$ is continuous and bounded on every compact subintervals of $(0, \infty)$, so stochastic integrals of above equation are martingales and if we take the expectation of both sides we get

$$\begin{aligned}
\mathbb{E}_x^\gamma[e^{-r(t\wedge\tau\wedge\tau_{ab})}v_\infty(X_{t\wedge\tau\wedge\tau_{ab}})] &= v_\infty(x) \\
&+ \mathbb{E}_x^\gamma\bigg[\int_0^{t\wedge\tau\wedge\tau_{ab}} e^{-rs}(\mathcal{A}_0-(r+\lambda\gamma))v_\infty(X_s) \\
&+ \lambda\gamma v_\infty((1-y_0)X_s)ds\bigg]
\end{aligned}$$

From the variational inequalities (i) and (iii) of Theorem 1 if we have

$$(\mathcal{A}_0-(r+\lambda\gamma))v_\infty(x)+\lambda\gamma v_\infty((1-y_0)x) \leq 0$$

then it means

$$\mathbb{E}_x^\gamma[e^{-r(t\wedge\tau\wedge\tau_{ab})}v_\infty(X_{t\wedge\tau\wedge\tau_{ab}})] \leq v_\infty(x) \tag{5.4}$$

for every $t\geq 0$, $\tau\geq 0$ and $0<a<b<\infty$. Because $\lim_{a\downarrow 0,b\uparrow\infty}\tau_{ab}=\infty$ and $f(x)$ is continuous and bounded for every fixed $x>0$, we can take the limits of both sides of equation 5.4 as $t\uparrow\infty$, $a\downarrow 0$, $b\uparrow\infty$ and use the bounded convergence theorem to get

$$\mathbb{E}_x^\gamma[e^{-r\tau}v_\infty(X_\tau)] \leq v_\infty(x)$$

. By taking supremum of both sides we complete the proof of the first inequality

$$\begin{aligned}
\sup_{\tau>0}\mathbb{E}_x^\gamma[e^{-r\tau}v_\infty(X_\tau)] &\leq v_\infty(x) \\
\mathbb{E}_x^\gamma[e^{-r\tau[v_\infty]}v_\infty(X_{\tau[v_\infty]})] &\leq v_\infty(x).
\end{aligned}$$

. We should also prove the reverse inequality and to do this we replace τ and τ_{ab} with $\tau[v_\infty]$. By variational inequality (i) of Theorem 1 we have

$$(\mathcal{A}_0-(r+\lambda\gamma))v_\infty(x)+\lambda\gamma v_\infty((1-y_0)x)=0$$

so we have

$$\mathbb{E}_x^\gamma[e^{-r(t\wedge\tau[v_\infty])}v_\infty(X_{t\wedge\tau[v_\infty]})] = v_\infty(x)$$

for every $t\geq 0$. Because $v_\infty(x)$ is bounded and continuous for every $x>0$ taking limits as $t\uparrow\infty$ and the bounded convergence theorem together with (iv) of Theorem 1 gives

$$\begin{aligned}
\mathbb{E}_x^\gamma[e^{-r\tau[v_\infty]}v_\infty(X_{\tau[v_\infty]})] &= v_\infty(x) \\
V(x)\geq\mathbb{E}_x^\gamma[e^{-r\tau[v_\infty]}f(X_{\tau[v_\infty]})] &= v_\infty(x)
\end{aligned}$$

which completes the proof. □

Special Case: When The Stock is Non-dividend Paying

We consider the underlying asset with $\delta = 0$ and we will see choosing the underlying asset non-dividend paying changes the optimal stopping strategy. The stock price has the dynamics

$$\frac{dX_t}{X_{t-}} = \mu dt + \sigma dB_t - y_0 \left(dN_t - \lambda dt\right).$$

The stock price is modeled by the equation

$$X_t = x \exp\left((\mu + \lambda y_0)t - \frac{1}{2}\sigma^2 t + \sigma B_t\right)(1 - y_0)^{N_t},$$

for $t \geq 0$.

The stock price process X has jumps which gives more than one equivalent martingale measure. Radon-Nikodym derivative gives class of equivalent martingale measures in the form

$$\left.\frac{d\mathbb{P}^\gamma}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \eta_t$$

where

$$\frac{d\eta_t}{\eta_{t-}} = -\frac{\mu - r}{\sigma} dB_t + (\gamma - 1)\left(dN_t - \lambda dt\right)$$

which has the solution

$$\eta_t = \exp\left\{-(\gamma - 1)\lambda t - \frac{\mu - r}{\sigma} B_t - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} t\right\}\gamma^{N_t},$$

$t \geq 0$. The Girsanov Theorem shows that $B_t^\gamma = \frac{\mu - r}{\sigma} t + B_t$ is a standard Brownian motion under the probability measure $\mathbb{P}^\gamma$ defined by equation. Here the price process given by

$$\frac{dX_t}{X_{t-}} = r dt + \sigma dB_t^\gamma - y_0 \left(dN_t - \lambda\gamma dt\right)$$

$$X_t = x \exp\left\{rt + \lambda\gamma y_0 t - \frac{1}{2}\sigma^2 t + \sigma B_t^\gamma\right\}(1 - y_0)^{N_t}$$

where N_t is a poisson process with intensity $\lambda\gamma$ and independent of B_t^γ under the new measure $\mathbb{P}^\gamma$.

Under the probability measure $\mathbb{P}^\gamma$ we should solve

$$V(x) = \sup_{\tau > 0} E_x^\gamma \left[e^{-r\tau} \left\{ (p - X_\tau)^+ + (X_\tau - c)^+ \right\} \right].$$

which is a discounted optimal stopping problem with reward function $f(x) = (p-x)^+ + (x-c)^+$.

Let $T_1, T_2, \ldots$ be the arrival times of process N. Observe that $X_{T_{n+1}} = (1-y_0)\, X_{T_{n+1}-}$ and

$$\frac{X_{T_n+t}}{X_{T_n}} = \exp\left\{ \left(r + \lambda\gamma y_0 - \frac{1}{2}\sigma^2 \right) t + \sigma \left(B^\gamma_{T_n+t} - B^\gamma_{T_n} \right) \right\}$$

if $0 \le t < T_{n+1} - T_n$.

Define the standard Brownian motion $B_t^{\gamma,n} = B^\gamma_{T_n+t} - B^\gamma_{T_n}$ for every $n \ge 1$, $t \ge 0$ and Poisson process $T_k^{(n)} = T_{n+k} - T_n$ for $k \ge 0$ respectively under P^γ and one dimensional diffusion process

$$Y_t^{y,n} = y \exp\left\{ \left(r + \lambda\gamma y_0 - \frac{1}{2}\sigma^2 \right) t + \sigma B_t^{\gamma,n} \right\}$$

which has the dynamics

$$\begin{aligned} Y_0^{y,n} &= y \\ \frac{dY_t^{y,n}}{Y_t^{y,n}} &= (r + \lambda\gamma y_0)\, dt + \sigma B_t^{\gamma,n}. \end{aligned}$$

X coincides with $Y_t^{X_{T_n},n}$ on $[T_n, T_{n+1})$ and jumps to $(1-y_0)\, Y^{X_{T_n},n}_{T_{n+1}-T_n}$ at time T_{n+1} for every $n \ge 0$. Namely,

$$X_{T_n+t} = \begin{cases} Y_t^{X_{T_n},n} & \text{if } 0 \le t < T_{n+1} - T_n \\ (1-y_0) Y^{X_{T_n},n}_{T_{n+1}-T_n} & \text{if } t = T_{n+1} - T_n \end{cases}$$

For an arbitrary but fixed stopping time $\tau \in S_B$ the strategy is

(i) on $\{\tau < T_1\}$ stop at time τ.

(ii) on $\{\tau \ge T_1\}$ update X at time T_1 to $X_{T_1} = (1-y_0)\, Y^x_{T_1}$ and continue optimality thereafter.

The value of this new strategy is

$$\begin{aligned} & \mathbb{E}_x^\gamma\left[e^{-r\tau} f\left(X_\tau\right) \mathbf{1}_{\{\tau<T_1\}}+e^{-r T_1} V\left(\left(1-y_0\right) Y_{T_1}^x\right) \mathbf{1}_{\{\tau \geq T_1\}}\right] \\ = \quad & \mathbb{E}_x^\gamma\left[e^{-(r+\lambda \gamma) \tau} f\left(Y_\tau^x\right)+\int_0^\tau \lambda \gamma e^{-(r+\lambda \gamma) t} V\left(\left(1-y_0\right) Y_t^x\right) d t\right] . \end{aligned}$$

Let $\psi(.)$ and $\varphi(.)$ be increasing and decreasing solutions of $\left(A_0 f\right)(y)-(r+\lambda\gamma) f(y)=0$, $y>0$ with respect to boundary conditions $\psi(0+)=0$ and $\varphi(+\infty)=0$ where A_0 is the infinitesimal generator of the diffusion process $Y^{x_0}=Y^{x_0,0}$. We have

$$\frac{\sigma^2 y^2}{2} f''(y)+(r+\lambda\gamma y_0) y f'(y)-(r+\lambda\gamma) f(y)=0$$

which has two linearly independent solutions $\psi(.)$ and $\varphi(.)$ in the form of y^{α_i} for $i=0,1$. One can explicitly find α_0 and α_1 from the roots of the characteristic function $g(\alpha)=\alpha(\alpha-1)+\frac{2}{\sigma^2}\left[(r+\lambda\gamma y_0)\alpha-(r+\lambda\gamma)\right]$ of the above ordinary differential equation. Now we have two solutions $\psi(y)=y^{\alpha_1}$ and $\varphi(y)=y^{\alpha_0}$ for every $y>0$ and note that

$$\alpha_0<0<1<\alpha_1$$

because both $g(0)<0$ and $g(1)<0$. Also note that

$$\alpha_0+\alpha_1=1-\frac{2}{\sigma^2}(r+\lambda\gamma y_0)$$

$$\alpha_0\alpha_1=-\frac{2}{\sigma^2}(r+\lambda\gamma).$$

Define the Wronskian

$$W(y)=\psi'(y)\varphi(y)-\psi(y)\varphi'(y)=(\alpha_1-\alpha_0) y^{\alpha_0+\alpha_1-1}$$

for $y > 0$.

$$
\begin{aligned}
(f-\lambda\gamma(Hw))(x) &= (p-x)^+ + (x-c)^+ - \lambda\gamma\Bigg[\varphi(x)\int\limits_0^x \frac{2\psi(\xi)w\left((1-y_0)\,\xi\right)}{p^2(\xi)W(\xi)}d\xi \\
&+ \psi(x)\int\limits_x^\infty \frac{2\varphi(\xi)w\left((1-y_0)\,\xi\right)}{p^2(\xi)W(\xi)}d\xi\Bigg] \\
&= (p-x)^+ + (x-c)^+ \\
&- \frac{2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)}\Bigg[x^{\alpha_0}\int\limits_0^x \xi^{-\alpha_0-1}w\left((1-y_0)\,\xi\right)d\xi \\
&+ x^{\alpha_1}\int\limits_x^\infty \xi^{-\alpha_1-1}w\left((1-y_0)\,\xi\right)d\xi\Bigg] \\
&\le (p-x)^+ + (x-c)^+ \\
&- \frac{2\lambda\gamma}{\sigma^2(\alpha_1-\alpha_0)}\Bigg[x^{\alpha_0}\int\limits_0^x \xi^{-\alpha_0-1}\left((1-y_0)\,\xi-c\right)d\xi \\
&+ x^{\alpha_1}\int\limits_x^\infty \xi^{-\alpha_1-1}\left((1-y_0)\,\xi-c\right)d\xi\Bigg] \\
&\le (p-x)^+ + (x-c)^+ - \frac{2\lambda\gamma}{\sigma^2}\left[x\frac{1-y_0}{(1-\alpha_0)(\alpha_1-1)} - \frac{c}{\alpha_0\alpha_1}\right] \\
&\le (p-x)^+ + (x-c)^+ - x - \frac{\lambda\gamma c}{r+\lambda\gamma}
\end{aligned}
$$

For sufficiently enough large values of x, we have

$$
(f-\lambda\gamma(Hw))(x) \le -\frac{c(r+2\lambda\gamma)}{r+\lambda\gamma} < 0
$$

and for small enough values of x, we have

$$
(f-\lambda\gamma(Hw))(x) \le p + \frac{\lambda\gamma c}{r+\lambda\gamma}.
$$

Above inequalities together with the boundary conditions $\psi(+\infty) = \varphi(0+) = +\infty$ give the limits

$$
l_0 = \limsup_{x\to 0}\frac{(f-\lambda\gamma(Hw))^+(x)}{\varphi(x)} = 0 \quad l_\infty = \limsup_{x\to\infty}\frac{(f-\lambda\gamma(Hw))^+(x)}{\psi(x)} = 0
$$

which will guarantee the existence of optimal stopping strategy.

By using proposition 5.5 of Dayanık, we have function $(Mw)(.)$ which is the smallest nonnegative concave majorant of the function

$$(Lw)(\zeta) = \begin{cases} \frac{f-\lambda\gamma(Hw)}{\varphi} \circ F^{-1}(\zeta) & \text{if } \zeta > 0 \\ 0 & \text{if } \zeta = 0 \end{cases}$$

where $F(x) = \frac{\psi(x)}{\varphi(x)}$ and $(Gw)(x) = \varphi(x)(Mw)(F(x))$ for $x \geq 0$. Furthermore, $(Mw)(0) = 0$ and $(Mw)(.)$ is continuous at 0.

In order to explicitly define $(Mw)(.)$, we should observe some important properties of the function $(Lw)(.)$ First, let's identify the limiting behavior of $(Lw)(x)$ for large x values. Let us check

$$\begin{aligned} \lim_{x\uparrow\infty}(Lw)(F^{-1}(x)) &= \lim_{x\uparrow\infty} \frac{(p-x)^+ + (x-c)^+ - \lambda\gamma\mathbb{E}_x^\gamma\left[\int\limits_0^\infty e^{-(r+\lambda\gamma)t} w\left((1-y_0)\, Y_t^x\right) dt\right]}{x^{\alpha_0}} \\ &\leq \lim_{x\uparrow\infty} \frac{-\frac{c(r+2\lambda\gamma)}{r+\lambda\gamma}}{x^{\alpha_0}} \\ &\leq -\infty \end{aligned}$$

because of $\alpha_0 < 0$. So we see that $(Lw)(+\infty) = -\infty$.
Let us examine the sign of the first derivative as x tends to zero and infinity.

$$\begin{aligned} (Lw)'(x) &= \frac{d}{dx}\left(\frac{f-\lambda\gamma(Hw)}{\varphi} \circ F^{-1}(x)\right) \\ &= \left[\frac{1}{F'}\left(\frac{f-\lambda\gamma(Hw)}{\varphi}\right)'\right] \circ F^{-1}(x) \end{aligned}$$

and because $\lim_{x\downarrow 0} x^{\alpha_1} \int_x^\infty \zeta^{-\alpha_1-1} w\left((1-y_0)\,\zeta\right) d\zeta = w(+0)/\alpha_1 = p/\alpha_1$ and $\alpha_1 > 1$, we have

$$\begin{aligned} \lim_{x\downarrow 0}\left[\frac{1}{F'}\left(\frac{f-\lambda\gamma(Hw)}{\varphi}\right)'\right](F^{-1}(x)) =& \frac{x^{-\alpha_1}}{\alpha_1-\alpha_0}\Bigg[-2\frac{\lambda\gamma x^{\alpha_1}}{\sigma^2}\int_x^\infty \zeta^{-\alpha_1-1} w\left((1-y_0)\,\zeta\right) d\zeta \\ &+ (-x-\alpha_0(p-x))1_{\{x<p\}} + (x-\alpha_0(x-c))1_{\{x>c\}}\Bigg] \\ =& +\infty. \end{aligned}$$

For large x values, first derivative becomes

$$\begin{aligned}\lim_{x\downarrow 0}\left[\frac{1}{F'}\left(\frac{f-\lambda\gamma(Hw)}{\varphi}\right)'\right](F^{-1}(x)) &= \frac{x^{-\alpha_1}}{\alpha_1-\alpha_0}\Bigg[-2\frac{\lambda\gamma x^{\alpha_1}}{\sigma^2}\int_x^\infty \zeta^{-\alpha_1-1}w\left((1-y_0)\zeta\right)d\zeta \\ &\quad +(-x-\alpha_0(p-x))1_{\{x<p\}} \\ &\quad +(x-\alpha_0(x-c))1_{\{x>c\}}\Bigg] \\ &= 0\end{aligned}$$

We should also analyze the sign of the second derivative of $(Lw)(F(x))$, which is

$$(Lw)''(F^{-1}(x)) = \frac{2\varphi(x)}{p^2(x)W(x)F'(x)}(A_0-(r+\lambda\gamma))(f-\lambda\gamma(Hw))(x)$$

as Dayanık and Karatzas show. We see that

$$\operatorname{sgn}[(Lw)''(F(x))] = \operatorname{sgn}[(A_0-(r+\lambda\gamma))(f-\lambda\gamma(Hw))(x)]$$

and recall from Lemma 5.1 that $A_0-(r+\lambda\gamma))(Hw)(x) = -w((1-y_0)x$. So we have that

$$\begin{aligned}A_0-(r+\lambda\gamma))(h-\lambda\gamma(Hw))(x) = {} & [\lambda\gamma(1-y_0)x-(r+\lambda\gamma)p \\ & +\lambda\gamma w((1-y_0)x)]1_{\{x<p\}} \\ & +\lambda\gamma w((1-y_0)x)1_{\{p\le x\le c\}} \\ & +[-\lambda\gamma(1-y_0)x+(r+\lambda\gamma)c \\ & +\lambda\gamma w((1-y_0)x)]1_{\{x>c\}}\end{aligned}$$

Remark 5.4. *The function $(Lw)(F(x))$ is a concave function in some open neighborhood of 0 and convex function in some open neighborhood of $+\infty$.*

Proof. Using Lemma 5.1, we get

$$\begin{aligned}\lim_{x\downarrow 0}(A_0-(r+\lambda\gamma))(h-\lambda\gamma(Hw))(x) &\le \lim_{x\downarrow 0}[\lambda\gamma(1-y_0)x-(r+\lambda\gamma)p+ \\ &\quad \lambda\gamma((1-y_0)x+p)] \\ &\le -rp<0\end{aligned}$$

and

$$\begin{aligned}\lim_{x\uparrow\infty}(A_0-(r+\lambda\gamma))(h-\lambda\gamma(Hw))(x) &\ge \lim_{x\uparrow\infty}[-\lambda\gamma(1-y_0)x+(r+\lambda\gamma)c+ \\ &\quad \lambda\gamma((1-y_0)x-c)] \\ &\ge rc>0\end{aligned}$$

□

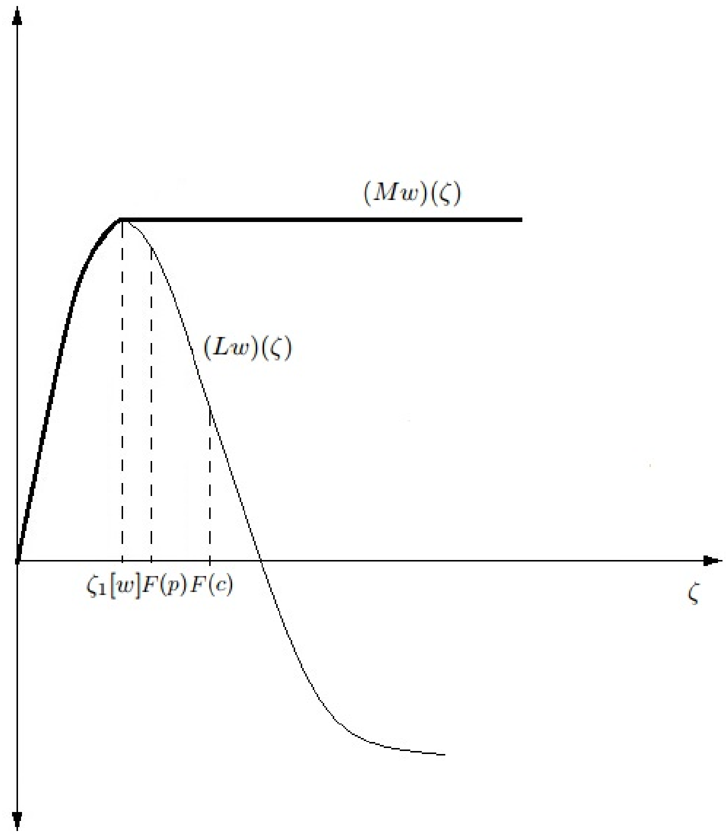

Figure 5.2: Possible form of $(Lw)(.)$ and its smallest concave majorant $(Mw)(.)$ when $\delta = 0$.

The results that are obtained so far concludes that there is a unique number $0 < \zeta_1[w] < F(p) < \infty$ such that

$$(Lw)'(\zeta_1[w]) = 0.$$

The smallest concave majorant $(Mw)(.)$ becomes

$$(Mw)(\zeta) = \begin{cases} (Lw)(\zeta) & \text{if } \zeta \in (0, \zeta_1[w]) \\ (Lw)(\zeta_1[w]) & \text{if } \zeta \in [\zeta_1[w], +\infty). \end{cases}$$

Let us define $x_1[w] = F^{-1}(\zeta_1[w])$. By Proposition 5.5 of Dayanık [3], the value function 5.1 of the optimal stopping problem equals

$$
\begin{aligned}
(Gw)(x) &= \varphi(x)(Mw)(F(x)) \\
&= \begin{cases} (f-\lambda\gamma(Hw))(x) & \text{if } x \in (0, x_1[w]) \\ (f-\lambda\gamma(Hw))(x_1[w]) & \text{if } x \in [x_1[w], +\infty). \end{cases}
\end{aligned}
$$

Optimal stopping time in equation 5.2 becomes

$$\tau[w] = inf\{t \geq 0 : Y_t^x \in (0, x_1[w])\}$$

in the optimal stopping region

$$\Gamma[w] = \{x > 0 : (Gw)(x) = (f - \lambda\gamma(Hw))(x)\} = (0, x_1[w]).$$

Proposition 5.4. *The value function* $(Gw)(.)$ *satisfies*

(i) $(\mathcal{A}_0 - (r + \lambda\gamma))(Gw)(x) = 0$, *for* $x \in (x_1[w], +\infty)$

(ii) $(Gw)(x) > f(x) - \lambda\gamma(Hw)(x)$, *for* $x \in (x_1[w], +\infty)$

(iii) $(\mathcal{A}_0 - (r + \lambda\gamma))(Gw)(x) < 0$, *for* $x \in (0, x_1[w]]$

(iv) $(Gw)(x) = f(x) - \lambda\gamma(Hw)(x)$, *for* $x \in (0, x_1[w]]$

Proof. Proof is similar to the proof of Proposition 5.2 □

Proposition 5.5. *The value function* $(Jw)(.)$ *satisfies*

(i) $(\mathcal{A}_0 - (r + \lambda\gamma))(Jw)(x) + \lambda\gamma w((1 - y_0)x) = 0$, *for* $x \in (x_1[w], +\infty)$

(ii) $(Jw)(x) > f(x)$, *for* $x \in (x_1[w], +\infty)$

(iii) $(\mathcal{A}_0 - (r + \lambda\gamma))(Jw)(x) + \lambda\gamma w((1 - y_0)x) < 0$, *for* $x \in (0, x_1[w]]$

(iv) $(Jw)(x) = f(x)$, *for* $x \in (0, x_1[w]]$

Proof. By Lemma 5.1

$$(\mathcal{A}_0(Hw))(x) - (r + \lambda\gamma)(Hw)(x) = -w((1 - y_0)x)$$

and by definition

$$(Jw)(x) = \lambda\gamma(Hw)(x) + (Gw)(x).$$

These equations and Proposition 5.4 complete the proof. $\square$

Theorem 3. *The function $x \mapsto v_\infty(x) = (Jv_\infty)(x)$ satisfies the following variational inequalities*

(i) $(\mathcal{A}_0 - (r + \lambda\gamma))v_\infty(x) + \lambda\gamma v_\infty((1 - y_0)x) = 0$, *for* $x \in (x_1[w], +\infty)$

(ii) $v_\infty(x) > f(x)$, *for* $x \in (x_1[w], +\infty)$

(iii) $(\mathcal{A}_0 - (r + \lambda\gamma))v_\infty(x) + \lambda\gamma v_\infty((1 - y_0)x) < 0$, *for* $x \in (0, x_1[w]]$

(iv) $v_\infty(x) = f(x)$, *for* $x \in (0, x_1[w]]$

Proof. Every $v_n(x)$, $n \geq 0$ and $v_\infty(x)$ are convex and bounded for every fixed $x > 0$. Therefore, Proposition 5.5, applied to $w = v_\infty$ completes the proof of theorem. $\square$

Theorem 4. *For every $x > 0$, the expected reward of asset manager is $V(x) = v_\infty(x) = \mathbb{E}_x^\gamma\left[e^{-r\tau[v_\infty]} f\left(X_{\tau[v_\infty]}\right)\right]$ and $\tau[v_\infty]$ is an optimal stopping time for equation 4.2.*

Proof. Define $\tau_a = \inf\{t \leq 0 : X_t \in (0, a]\}$ for every $0 < a < \infty$. Ito's rule gives

$$\begin{aligned}
e^{-r(t\wedge\tau\wedge\tau_a)} v_\infty(X_{t\wedge\tau\wedge\tau_a}) &= v_\infty(X_0) \\
&+ \int_0^{t\wedge\tau\wedge\tau_a} e^{-rs}(\mathcal{A}_0 - (r + \lambda\gamma))v_\infty(X_s) + \lambda\gamma v_\infty((1 - y_0)X_s)ds \\
&+ \int_0^{t\wedge\tau\wedge\tau_a} e^{-rs}(\mathcal{A}_0 - (r + \lambda\gamma))v_\infty(X_s) + \lambda\gamma v_\infty((1 - y_0)X_s)\sigma X_s dB_s^\gamma \\
&+ \int_0^{t\wedge\tau\wedge\tau_a} e^{-rs}[v_\infty((1 - y_0)X_{s-}) - v_\infty(X_{s-})](dN_s - \lambda\gamma ds)
\end{aligned}$$

for every $t \geq 0$, $\tau \geq 0$ and $0 < a < \infty$. We know that $v_\infty(.)$ is continuous and bounded on every compact subintervals of $(0, \infty)$, so stochastic integrals of above equation are martingales and if we take the expectation of both sides we get

$$\begin{aligned}\mathbb{E}_x^\gamma[e^{-r(t\wedge\tau\wedge\tau_a)}v_\infty(X_{t\wedge\tau\wedge\tau_a})] &= v_\infty(x)\\ &+ \mathbb{E}_x^\gamma\bigg[\int_0^{t\wedge\tau\wedge\tau_a} e^{-rs}(\mathcal{A}_0-(r+\lambda\gamma))v_\infty(X_s)\\ &+ \lambda\gamma v_\infty((1-y_0)X_s)ds\bigg]\end{aligned}$$

From the variational inequalities (i) and (iii) of Theorem 1 if we have

$$(\mathcal{A}_0-(r+\lambda\gamma))v_\infty(x)+\lambda\gamma v_\infty((1-y_0)x) \leq 0$$

then it means

$$\mathbb{E}_x^\gamma[e^{-r(t\wedge\tau\wedge\tau_a)}v_\infty(X_{t\wedge\tau\wedge\tau_a})] \leq v_\infty(x) \tag{5.5}$$

for every $t \geq 0$, $\tau \geq 0$ and $0 < a < \infty$. Because $\lim_{a\downarrow 0}\tau_a = \infty$ and $f(x)$ is continuous and bounded for every fixed $x > 0$, we can take the limits of both sides of equation 5.5 as $t \uparrow \infty$, $a \downarrow 0$ and use the bounded convergence theorem to get

$$\mathbb{E}_x^\gamma[e^{-r\tau}v_\infty(X_\tau)] \leq v_\infty(x)$$

. By taking supremum of both sides we complete the proof of the first inequality

$$\begin{aligned}\sup_{\tau>0}\mathbb{E}_x^\gamma[e^{-r\tau}v_\infty(X_\tau)] &\leq v_\infty(x)\\ \mathbb{E}_x^\gamma[e^{-r\tau[v_\infty]}v_\infty(X_{\tau[v_\infty]})] &\leq v_\infty(x).\end{aligned}$$

. We should also prove the reverse inequality and to do this we replace τ and τ_a with $\tau[v_\infty]$. By variational inequality (i) of Theorem 1 we have

$$(\mathcal{A}_0-(r+\lambda\gamma))v_\infty(x)+\lambda\gamma v_\infty((1-y_0)x)=0$$

so we have

$$\mathbb{E}_x^\gamma[e^{-r(t\wedge\tau[v_\infty])}v_\infty(X_{t\wedge\tau[v_\infty]})] = v_\infty(x)$$

for every $t \geq 0$. Because $v_\infty(x)$ is bounded and continuous for every $x > 0$ taking limits as $t \uparrow \infty$ and the bounded convergence theorem together with (iv) of Theorem 3 gives

$$\begin{aligned}\mathbb{E}_x^\gamma[e^{-r\tau[v_\infty]}v_\infty(X_{\tau[v_\infty]})] &= v_\infty(x)\\ V(x) \geq \mathbb{E}_x^\gamma[e^{-r\tau[v_\infty]}f(X_{\tau[v_\infty]})] &= v_\infty(x)\end{aligned}$$

which completes the proof. □

Chapter 6

Numerical Illustrations

In this chapter we present several examples to illustrate the structure of the solution. As we already see, the dividend rate plays an essential role in the optimal exercise strategy and the shape of $(Lv)(.)$ function depends on this parameter. We proved that when $\delta > 0$, the behavior of the $(Lv)(.)$ function is concave for large x values and goes to plus infinity as x tends to infinity with decreasing slope. On the other hand, when $\delta = 0$, the function $(Lv)(x)$ is convex for large x values and decreases to $-\infty$ as x tends to $+\infty$ with decreasing slope.

As we implement our solution method to calculate value functions with computer, we use linear approximation technique to achieve computable integrals. After each iteration, $v(.)$ increases monotonically as expected and behaves as a convex function with extremely steep line near 0. This makes integrals impossible to calculate hence we approximate this function near 0 linearly. Even using linear approximation does not change the expected behavior of $(Lv)(.)$ and the smallest concave majorants $(Mv)(.)$. In the implementation of the successive approximations, we decided to stop the iterations as soon as the maximum absolute difference between the last two approximations is less than 0.01.

The following four examples are obtained with different parameters. First figure shows the successive value functions $v(.)$, the second figure shows function

	Figure 1	Figure 2	Figure 3	Figure 4
x	1	1	4	5
p	1	1	1	1
c	2	2	5	5
r	0.1	0.15	0.05	0.2
δ	0.05	0.1	0.0	0.0
σ	0.275	0.275	0.275	0.275
$\lambda\gamma$	0.1	0.2	0.1	0.5
y_0	0.3	0.1	0.3	0.1

Table 6.1: Parameter values used for the illustrations

$(Lw)(.)$ and the third figure shows the smallest concave majorants $(Mw)(.)$ of $(Lw)(.)$ with tangent lines. For illustrations we used the parameter sets provided in the table 6.1.

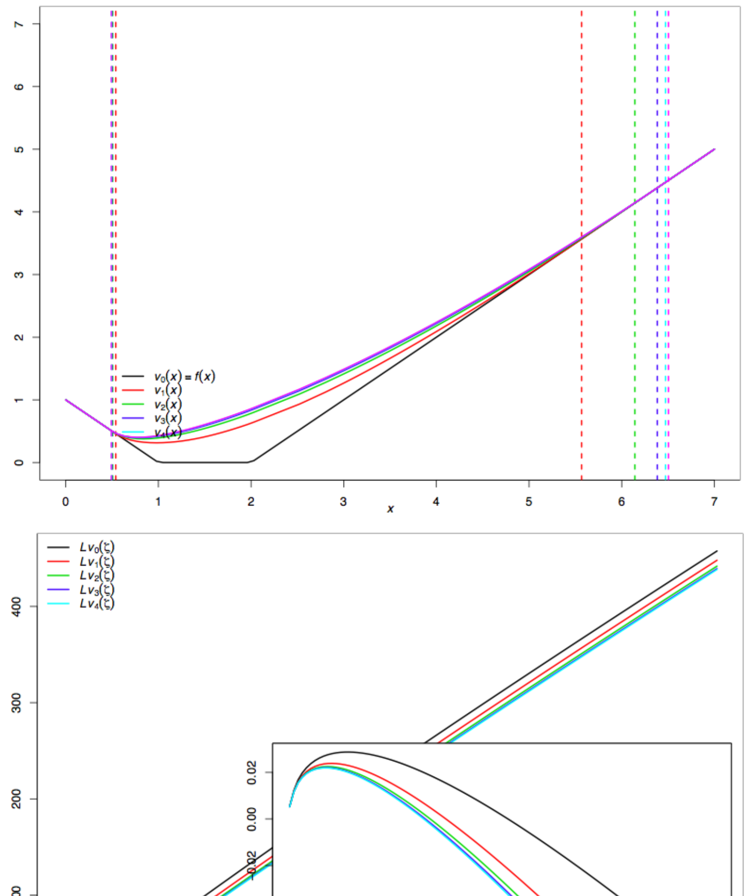

Figure 6.1: Value function iterations, corresponding $(Lv)(.)$ functions and their smallest concave majorants produced with first parameter set. Optimal exercise region is $(0, 0.4925865) \cup (6.504095, \infty)$

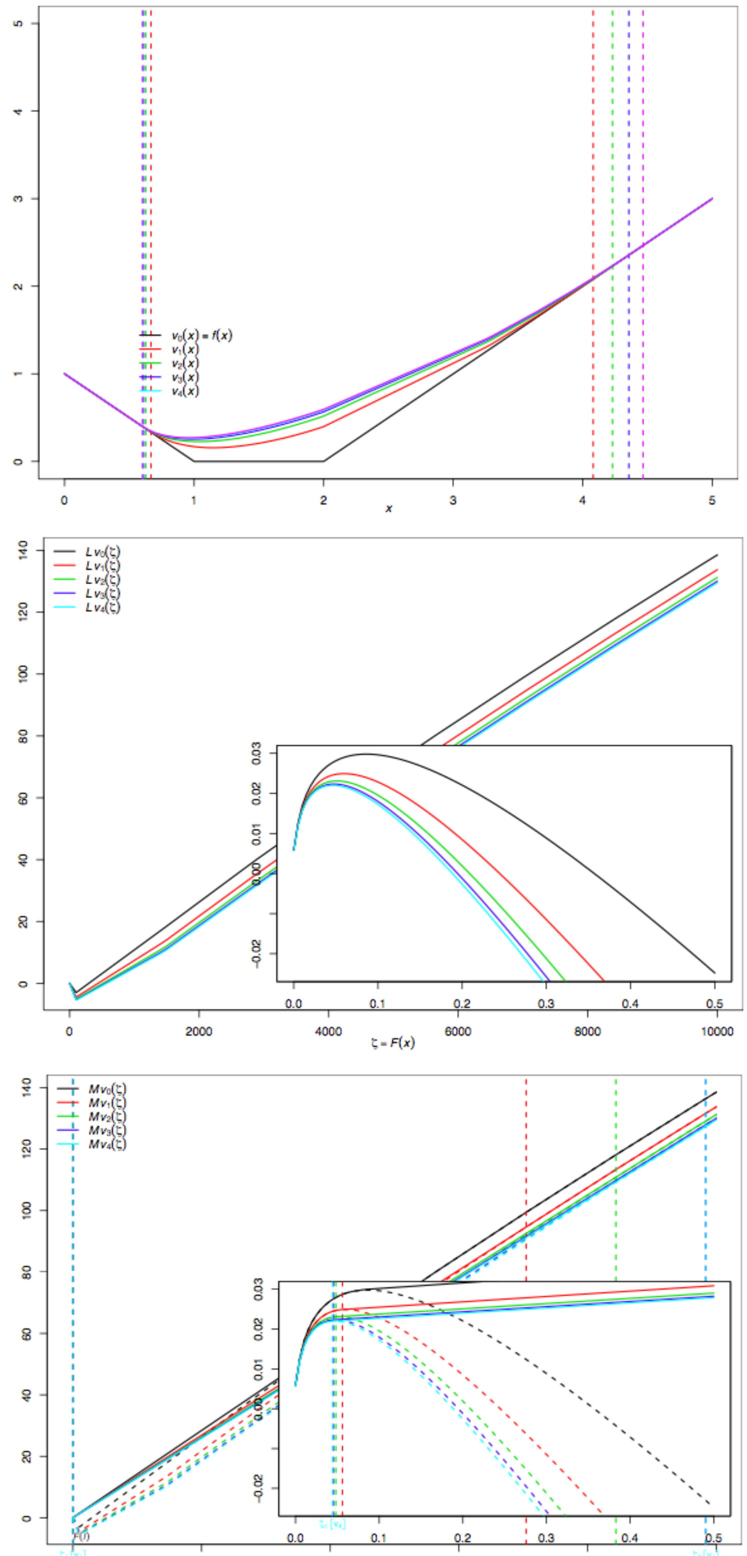

Figure 6.2: Value function iterations, corresponding $(Lv)(.)$ functions and their smallest concave majorants produced with second parameter set. Optimal exercise region is $(0, 0.6015621) \cup (4.46527, \infty)$

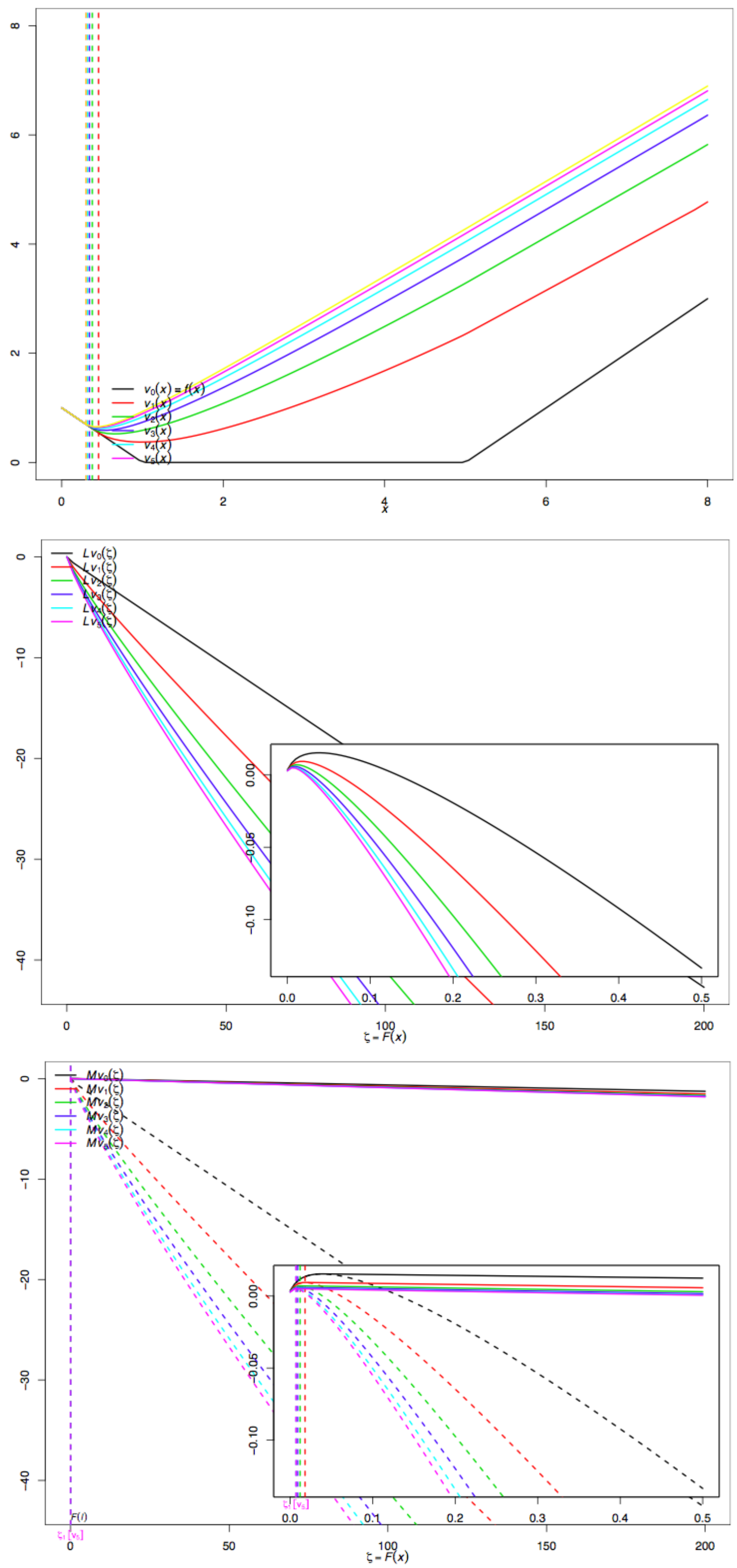

Figure 6.3: Value function iterations, corresponding $(Lv)(.)$ functions and their smallest concave majorants produced with third parameter set. Optimal exercise region is $(0, 0.6015621)$

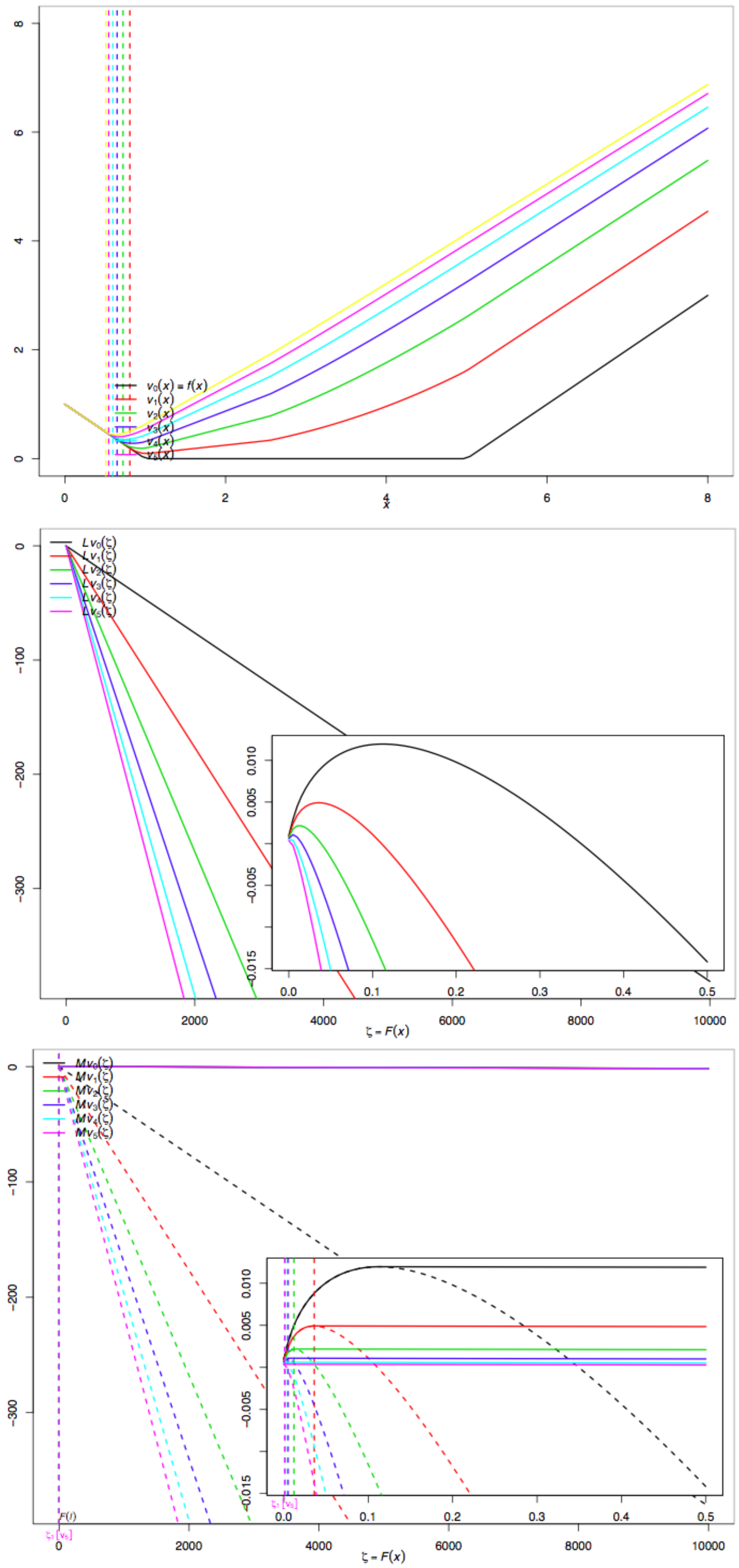

Figure 6.4: Value function iterations, corresponding $(Lv)(.)$ functions and their smallest concave majorants produced with fourth parameter set. Optimal exercise region is $(0, 0.51053)$

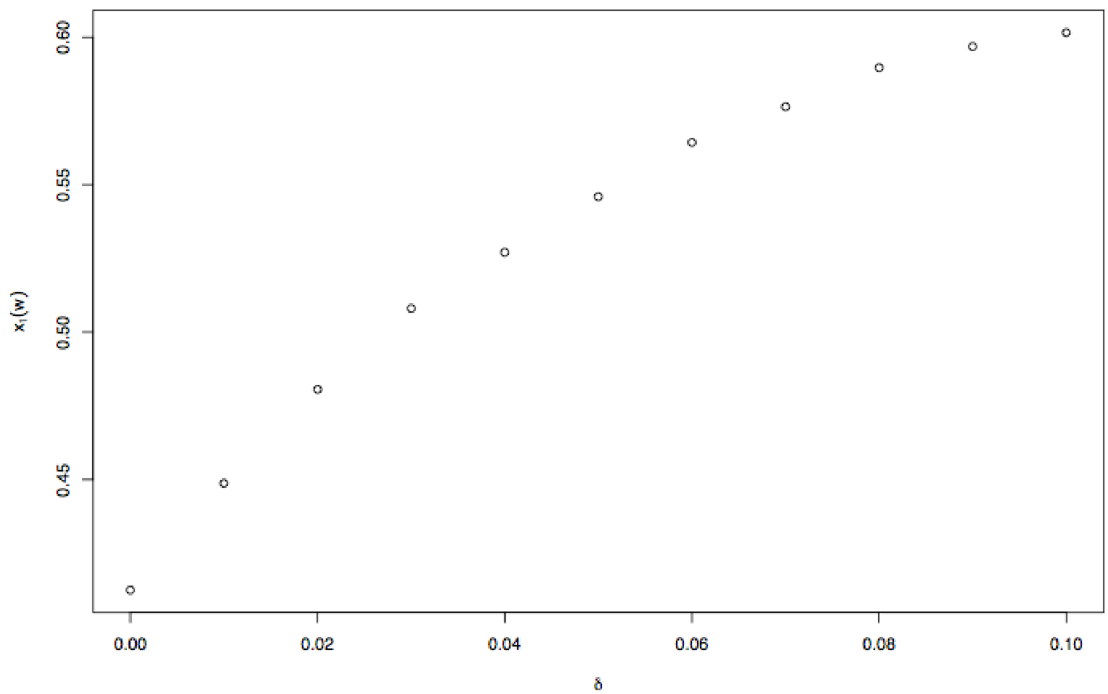

Figure 6.5: Left critical boundary of optimal stopping region as dividend rate δ changes.

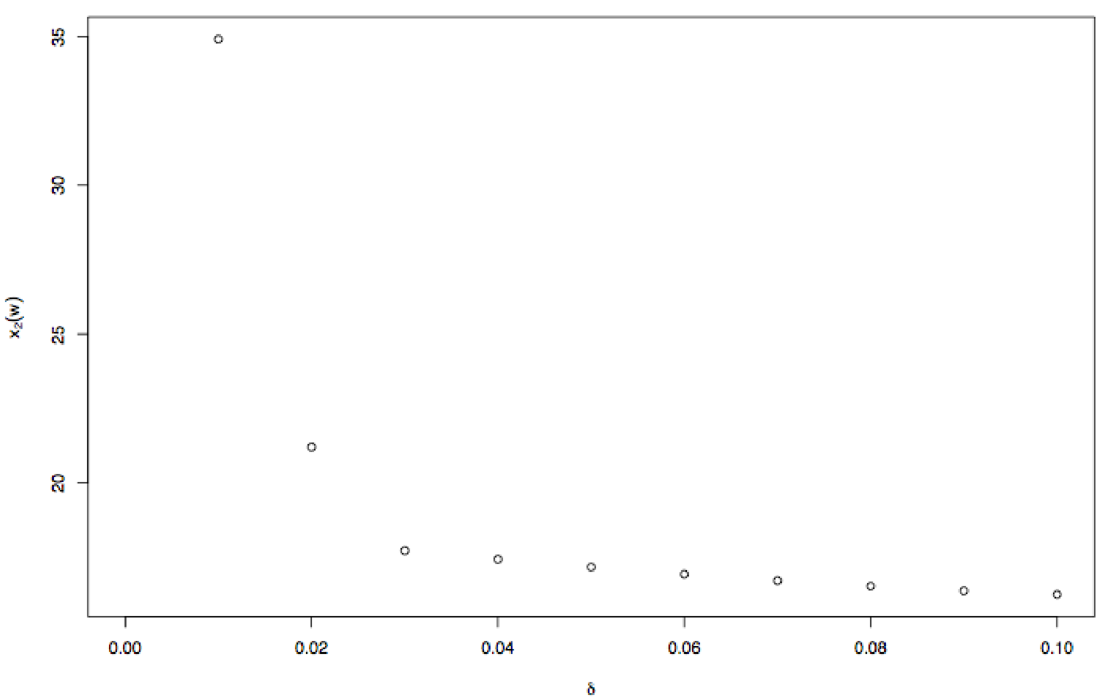

Figure 6.6: Right critical boundary of optimal stopping region as dividend rate δ changes.

Figure 6.5 and figure 6.6 show the changes in $x_1[w]$ and $x_2[w]$ respectively as δ changes. The exponential behavior of $x_1[w]$ and $x_2[w]$ is observed easily. Other parameters used to produce the figure are fixed and they are $x = 5$, $p = 1$, $c = 2$, $r = 0.15$, $\sigma = 0.275$, $\lambda\gamma = 0.2$, $y_0 = 0.1$.

We see that optimal behavior of the hedge fund manager changes with the dividend rate. Specifically, $\tau[w] = \inf\{t \geq 0 : Y_t^x \geq x_2[w]\}$ approaches to $+\infty$ as $\delta \downarrow 0$. This result follows from that $x_2[w]$ increases exponentially as dividend rate δ decreases linearly. Therefore, even though decreasing dividend rate results in higher appreciation rate for stock price process, it will take very large amount of time for stock price process to catch those large values. For the left critical boundary $x_1[w]$, increasing δ also increases $x_1[w]$. These figures show that there are two values $x_1^*[w]$ and $x_2^*[w]$ such that as $\delta \to r$, $x_1[w] \to x_1^*[w]$ and $x_2[w] \to x_2^*[w]$. If we define $\mathcal{R}_\delta$ as the optimal stopping region for a specific δ when all other variables are constant, we have

$$\mathcal{R}_{\delta_1} \subset \mathcal{R}_{\delta_2}$$

for any $0 \leq \delta_1 < \delta_2 < r$. Therefore, optimal stopping regions have nested structure as δ increases.

Chapter 7

Conclusion

Strangle options are widely used against the significant price movements when the holder of the option is unsure of the direction of the movement. Holding a long position on strangle option is a classical way of building a volatility strategy. In this thesis, we develop an optimal stopping strategy for an hedge fund manager who is holding a long position on a perpetual strangle option. During the solution we used the methodology of Dayanik and Karatzas [3] which decomposes the initial value problem into appropriate processes and aims to find the smallest concave majorant functions to find the boundaries of the continuation and stopping regions.

Dividend rate has a key role in developing the optimal stopping strategy as we see that as the dividend rate approaches to the risk free interest rate, we find bigger optimal stopping region which gives higher chance to exercise the option.

Perpetuality of the strangle option is also an important factor in finding the exercise time. It is known that American call option on a non dividend paying stock should never be exercised early. When American call is perpetual i.e when the maturity time $T \uparrow \infty$, it will not be exercised ever. (see [8]) For perpetual strangles, things are different because these contracts contain both a put side and a call side. Holders of strangles have another reason to exercise early due to the put side in addition to dividend rate. The optionality to exercise the call side is forfeited if the lower exercise boundary is hit first.

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Appendix A

Parameters and Code

A.1 Parameters and Functions

We present the R code used for obtaining the graphics in Chapter 6 when $\delta > 0$ and $\delta = 0$. The parameters used in this code are

`x0`: Initial endowment
`r`: Risk-free interest rate
`sigma`: Volatility of portfolio rate of return
`delta`: Dividend rate
`p`: Strike price of put option
`c`: Strike price of call option
`lg`: Lambda times gamma, the frequency of jumps after probability measure change
`y0`: The fraction of value that portfolio losses at each jump times

The functions used in this code are

`phi.fun(x)`: Computes x^{α_0}

`psi.fun(x)`: Computes x^{α_1}

`F.fun(x)`: Computes $\frac{x^{\alpha_1}}{x^{\alpha_0}}$

`invF.fun(y)`: Computes the inverse function $F^{-1}(y) = y^{\frac{1}{\alpha_1 - \alpha_0}}$

`f.fun(x)`: Computes the payoff of the strangle option $(p-x)^+ + (x-c)^+$

`H.op(w)`: This function computes $(Hw)(x) = \frac{2}{\sigma^2(\alpha_1-\alpha_0)}\left[x^{\alpha_0}\int_0^x \zeta^{-1-\alpha_0} w((1-y_0)\zeta)d\zeta + x^{\alpha_1}\int_x^\infty \zeta^{-1-\alpha_1} w((1-y_0)\zeta)d\zeta\right]$

`L.op(w)`: This function computes $(Lw)(x) = \frac{(f-\lambda\gamma(Hw))}{\phi}(F^{-1}(x))$

A.2 Code

```
rm(list=ls())
setwd("/Users/aysegulonat/Desktop/ThesisTemplate/code")
library(fdrtool)

writepdf=c(TRUE,FALSE)[1]

##Parameters
x0 = 1 #initial endowment
r = 0.15 #risk-free interest rate
sigma = 0.275 # volatility of portfolio rate of return
delta = 0.1 # dividend rate
p = 1 # strike price of put option
c = 2 # strike price of call option
lg = 0.2 # lambda times gamma
y0 = 0.1 # percentage loss upon jump

a=sigma^2
b=(r-delta+lg*y0)*2-sigma^2
cc=-2*(r+lg)

alpha0= (-b-sqrt(b^2-4*a*cc))/(2*a)
alpha1= (-b+sqrt(b^2-4*a*cc))/(2*a)

phi.fun = function (x) x^alpha0
psi.fun = function (x) x^alpha1
F.fun = function(x){x^(alpha1-alpha0)}
invF.fun = function(y) y^(1/(alpha1-alpha0))
f.fun = function (x) pmax(p-x,0)+pmax(x-c,0)
tolerance = 1/100
max.iter = 5

## Place grids on x- and zeta-axes
ub.x = 10*x0
ub.zeta = F.fun(ub.x)
number.of.grid.points.before.F.of.p = 1000
number.of.grid.points.between.F.of.p.and.c = 1000
number.of.grid.points.after.F.of.c = 1000
grid.on.zeta =  unique(c(
```

```
  seq(from=0,to=F.fun(p),length.out=
number.of.grid.points.before.F.of.p),
  seq(from=F.fun(p),,to=F.fun(c),length.out=
number.of.grid.points.between.F.of.p.and.c),
  seq(from=F.fun(c), to=ub.zeta,length.out=
number.of.grid.points.after.F.of.c)))
grid.on.zeta = tail(grid.on.zeta,-1)
grid.on.x = invF.fun(grid.on.zeta)

## H operator defined
H.op = function(w) {
  function (x) {
    f = function (zeta,alpha) (zeta^{-1-alpha})*w((1-y0)*zeta)
    res = c()
    for (i in(1:length(x))) {
      if (x[i]==0) {
        res = c(res, p/(r+lg))
      } else {
        res=c(res,
          (2/((sigma^2)*(alpha1-alpha0)))*
          ((x[i]^alpha0)*integrate(f=f,lower=0,upper=x[i],
      alpha=alpha0,subdivisions=2000)$value+
           (x[i]^alpha1)*integrate(f=f,lower=x[i],upper=Inf,
      alpha=alpha1,subdivisions=2000)$value
           )
          )
      }
    }
    return(res)
  }
}

L.op = function(w) {
  function (zeta) { (f.fun(invF.fun(zeta))-lg*H.op(w)(invF.fun(zeta)))
 /phi.fun(invF.fun(zeta))}
}

filename = sprintf("delta1-2",delta)
save.image(paste(filename,".RData",sep=""))
```

```
library(grid)
library(gridBase)
if (writepdf)
pdf(paste(filename,".pdf",sep=""),paper="a4r",width=0,height=0)
upp=5
par(mfrow=c(1,1),mar=c(3,3,0,0),cex=1.05)
legend.text = c(expression(italic(v[0](x)%==%h(x))))
plot(f.fun,xlim=c(0,upp),ylim=c(0,upp),ylab="",xlab="",
      lwd=2)
title(xlab=expression(italic(x)),line=1.5)
old.w.on.grid = f.fun(grid.on.x)

list.of.obstacles = list()
list.of.concave.majorants = list()

stop.iteration = FALSE
i = 1
print(i)
L.fun.on.grid =
(f.fun(grid.on.x)-lg*H.op(f.fun)(grid.on.x))/phi.fun(grid.on.x)

   list.of.obstacles = c(list.of.obstacles,
      list(list(
                    fun=approxfun(grid.on.zeta,L.fun.on.grid,rule=2:2)
                    ##fun=splinefun(grid.on.zeta,L.fun.on.grid,
         method="natural")
                    )))

   res.lcm = gcmlcm(grid.on.zeta,
      L.fun.on.grid,type="lcm")

   M.x = res.lcm$x.knots
   M.y = res.lcm$y.knots
   lcm.fun = approxfun(x=M.x,y=M.y,rule=2:2)
   zeta1 = max(res.lcm$x.knots[res.lcm$x.knots < F.fun(x0)])
   zeta2 = min(res.lcm$x.knots[res.lcm$x.knots > F.fun(x0)])
print(invF.fun(zeta1))
print(invF.fun(zeta2))
   list.of.concave.majorants = c(list.of.concave.majorants,
      list(list(fun=lcm.fun,
```

```
                 boundaries=c(zeta1,zeta2))))

t = function(x) pmax(p-x,0)

z = approxfun(grid.on.x,
     lg*H.op(f.fun)(grid.on.x)+phi.fun(grid.on.x)*lcm.fun(F.fun(grid.
    on.x)),
     rule=2:2)

w=function(x) pmax(f.fun(x),z(x))

plot(w,xlim=c(0,upp),ylim=c(0,upp),ylab="",xlab="",
      lwd=2, col=i+1, add=TRUE)
abline(v=invF.fun(c(zeta1,zeta2)),lty="dashed",col=i+1,
       lwd=2)
  list.of.approximations = list(
  list(fun=w,
        boundaries=c(NA,NA)))
  list.of.approximations[[i]]$boundaries = invF.fun(c(zeta1,zeta2))
  list.of.approximations = c(list.of.approximations,
    list(list(fun=w,
                 boundaries=c(NA,NA))))

while (!stop.iteration) {

  i = i + 1
  print(i)

  L.fun.on.grid = (f.fun(grid.on.x)-lg*H.op(w)(grid.on.x))/
  phi.fun(grid.on.x)

  list.of.obstacles = c(list.of.obstacles,
    list(list(
                 fun=approxfun(grid.on.zeta,L.fun.on.grid,rule=2:2)
                 )))
  res.lcm = gcmlcm(grid.on.zeta,
    L.fun.on.grid,type="lcm")

  M.x = res.lcm$x.knots
  M.y = res.lcm$y.knots
```

```
  lcm.fun = approxfun(x=M.x,y=M.y,rule=2:2)
  zeta1 = max(res.lcm$x.knots[res.lcm$x.knots < F.fun(x0)])
  zeta2 = min(res.lcm$x.knots[res.lcm$x.knots > F.fun(x0)])
print(invF.fun(zeta1))
print(invF.fun(zeta2))

  list.of.concave.majorants = c(list.of.concave.majorants,
    list(list(fun=lcm.fun,
              boundaries=c(zeta1,zeta2))))

  z = approxfun(grid.on.x,  lg*H.op(w)(grid.on.x)+
  phi.fun(grid.on.x)*lcm.fun(F.fun(grid.on.x)),
         rule=2:2)

    w=function(x)   pmax(f.fun(x),z(x))

  plot(w,xlim=c(0,upp),ylim=c(0,upp),ylab="",
  xlab="",col=i+1, lwd=2, add=TRUE)
         abline(v=invF.fun(c(zeta1,zeta2)),lty="dashed",
  col=i+1, lwd=2)

      legend.text = c(legend.text,substitute(italic(v[s](x)),list(s=i-1)))

   list.of.approximations[[i]]$boundaries = invF.fun(c(zeta1,zeta2))
   list.of.approximations = c(list.of.approximations,
    list(list(fun=w,
              boundaries=c(NA,NA))))

  new.w.on.grid = w(grid.on.x)
  max.diff = max(abs(new.w.on.grid-old.w.on.grid))
  if ((max.diff < tolerance) | (i > max.iter)) {
    stop.iteration = TRUE
  }
  else
    old.w.on.grid = new.w.on.grid
}

legend(x=0.5,y=1.6,legend=legend.text,
       col=c(1:length(list.of.approximations)),
```

```
         lty="solid", lwd=2,
         ##lty=c(1:length(list.of.approximations)),
         bty="n")

##############################################

upper=10000
legend.text = c(expression(italic(Lv[0](zeta))))

plot(list.of.obstacles[[1]]$fun,xlim=c(0, upper),ylab="",xlab="",
      lwd=2)
title(xlab=expression(italic(zeta==F(x))),line=2)
if (length(list.of.obstacles)>1) {
  for (i in (2:length(list.of.obstacles))) {
    plot(list.of.obstacles[[i]]$fun,xlim=c(0, upper),ylab="",xlab
    ="",
          col=i, lwd=2,
          ##lty=i,
          add=TRUE)

      legend.text = c(legend.text,
                      substitute(italic(Lv[s](zeta)),list(s=i-1)))

  }
}

legend("topleft",legend=legend.text,
         col=c(1:length(list.of.obstacles)),
         lty="solid", lwd=2,
         ##lty=c(1:length(list.of.obstacles)),
         bty="n")
#### start drawing inset

vp <- baseViewports()
pushViewport(vp$inner,vp$figure,vp$plot)

## push viewport that will contain the inset
pushViewport(viewport(x=1/3,y=0.06,width=1.95/3,
height=.5,just=c("left","bottom")))

```

```
grid.rect(gp=gpar(fill="white"))

## now either define viewport to contain the whole inset figure
##par(fig=gridFIG(),new=T)  ## or gridPLT()
## ...or just the plotting are (coordinate system)
par(plt=gridPLT(),new=T)

## draw frame around selected area (for illustration only)
##grid.rect(gp=gpar(lwd=3,col="red"))

## plot inset figure
##plot(,xaxs="i",yaxs="i",xlab="",ylab="",cex.axis=0.7,
##      xaxt="n")

plot(list.of.obstacles[[1]]$fun,xlim=c(0,0.5),ylab="",xlab="",
     lwd=2)
if (length(list.of.obstacles)>1) {
  for (i in (2:length(list.of.obstacles))) {
    plot(list.of.obstacles[[i]]$fun,xlim=c(0,0.5),ylab="",xlab="",
         col=i, lwd=2, cex.axis = 0.7,add=TRUE)
         ##lty=i,
  }
}

## pop all viewports from stack
popViewport(1)
par(plt=gridPLT())
popViewport(3)

##################################################

legend.text = c(expression(italic(Mv[0](zeta))))

plot(list.of.obstacles[[1]]$fun,xlim=c(0,upper),ylab="",xlab="",
     lty="dashed",lwd=2)
plot(list.of.concave.majorants[[1]]$fun,xlim=c(0,upper),
     ylab="",xlab="",
     lwd=2,add=TRUE)
title(xlab=expression(italic(zeta==F(x))),line=2)
if (length(list.of.obstacles)>1) {
```

```
  for (i in (2:length(list.of.obstacles))) {
    plot(list.of.obstacles[[i]]$fun,xlim=c(0,upper),ylab="",xlab="",
         col=i, lwd=2,lty="dashed",
        ##lty=i,
         add=TRUE)
    plot(list.of.concave.majorants[[i]]$fun,xlim=c(0,upper),ylab="",
  xlab="",
         col=i, lwd=2,
        ##lty=i,
         add=TRUE)
    abline(v=list.of.concave.majorants[[i]]$boundaries,lty="dashed",
  col=i,
         lwd=2)
##     legend.text = c(legend.text,sprintf("suc. app. %d",i-1))

    legend.text = c(legend.text,substitute(italic(Mv[s](zeta)),
  list(s=i-1)))

  }
}

  mtext(text=substitute(zeta[1]~group("[",v[s],"]"),list(s=i-1)),
        at=list.of.concave.majorants[[i]]$boundaries[1],
  side=1,line=0.3,cex=.8,
          col=i)
  mtext(text=substitute(zeta[2]~group("[",v[s],"]"),list(s=i-1)),
        at=list.of.concave.majorants[[i]]$boundaries[2],
  side=1,line=0.3,cex=.8,
          col=i)

mtext(text=expression(italic(F(l))),at=F.fun(0),side=1,
  line=-1.0,adj=0,cex=.8)

legend("topleft",legend=legend.text,
       col=c(1:length(list.of.obstacles)),
       lty="solid", lwd=2,
      ##lty=c(1:length(list.of.obstacles)),
       bty="n")
#### start drawing inset

```

```r
vp <- baseViewports()
pushViewport(vp$inner, vp$figure, vp$plot)

## push viewport that will contain the inset
pushViewport(viewport(x=1/3,y=0.06,
                      width=1.95/3,height=.5,
                      just=c("left","bottom")
                      ))

grid.rect(gp=gpar(fill="white"))

## now either define viewport to contain the whole inset figure
##par(fig=gridFIG(),new=T)  ## or gridPLT()
## ...or just the plotting are (coordinate system)
par(plt=gridPLT(),new=T)

## draw frame around selected area (for illustration only)
##grid.rect(gp=gpar(lwd=3,col="red"))

## plot inset figure
##plot(,xaxs="i",yaxs="i",xlab="",ylab="",cex.axis=0.7,
##      xaxt="n")

plot(list.of.obstacles[[1]]$fun,xlim=c(0,0.5),ylab="",xlab="",
     lty="dashed",lwd=2)
plot(list.of.concave.majorants[[1]]$fun,xlim=c(0,0.5),ylab="",
     xlab="",
     lwd=2,add=TRUE)

if (length(list.of.obstacles)>1) {
  for (i in (2:length(list.of.obstacles))) {
    plot(list.of.obstacles[[i]]$fun,xlim=c(0,0.5),ylab="",xlab="",
         col=i, lwd=2,lty="dashed",
         ##lty=i,
         add=TRUE)
    plot(list.of.concave.majorants[[i]]$fun,
  xlim=c(0,0.5),ylab="",xlab="",
         col=i, lwd=2,add=TRUE)
         ##lty=i)
    abline(v=list.of.concave.majorants[[i]]$boundaries,
```

```
  lty="dashed",col=i,
         lwd=2)
  }
}

  mtext(text=substitute(zeta[1]~group("[",v[s],"]"),list(s=i-1)),
        at=list.of.concave.majorants[[i]]$boundaries[1],
        side=1,line=0.1,cex=.8,
        col=i)

## pop all viewports from stack
popViewport(1)
par(plt=gridPLT())
popViewport(3)
################################################

par(mfrow=c(1,1))
if (writepdf) dev.off()
```