Journal of Algebra 357 (2012) 34–60

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Bockstein closed 2-group extensions and cohomology of quadratic maps

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ARTICLE INFO

Article history:
Received 29 December 2010
Available online 2 March 2012
Communicated by Luchezar L. Avramov

MSC:
primary 20J06
secondary 17B56

Keywords:
Group cohomology
Group extensions
Quadratic maps
Steenrod operations

ABSTRACT

A central extension of the form $E: 0 \to V \to G \to W \to 0$, where V and W are elementary abelian 2-groups, is called Bockstein closed if the components $q_i \in H^*(W, \mathbb{F}_2)$ of the extension class of E generate an ideal which is closed under the Bockstein operator. In this paper, we study the cohomology ring of G when E is a Bockstein closed 2-power exact extension. The mod-2 cohomology ring of G has a simple form and it is easy to calculate. The main result of the paper is the calculation of the Bocksteins of the generators of the mod-2 cohomology ring using an Eilenberg–Moore spectral sequence. We also find an interpretation of the second page of the Bockstein spectral sequence in terms of a new cohomology theory that we define for Bockstein closed quadratic maps $Q: W \to V$ associated to the extensions E of the above form.

1. Introduction

Let G be a p-group which fits into a central extension of the form

$$E: 0 \to V \to G \to W \to 0$$

where V, W are $\mathbb{F}_p$-vector spaces of dimensions n and m, respectively. E is called Bockstein closed if the components $q_i \in H^*(W, \mathbb{F}_p)$ of the extension class of E generate an ideal which is closed under the Bockstein operator. We say E is p-power exact if the following three conditions are satisfied: (i) $m = n$, (ii) V is the Frattini subgroup of G, and (iii) the p-rank of G is equal to n. Associated to G there is a p-th power map $(\ )^p : W \to V$. When p is odd, the p-th power map is a homomorphism

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2012.01.029

and if E is also p-power exact, then it is an isomorphism. Using this isomorphism, one can define a bracket $[\,,\,]: W \times W \to W$ on W which turns out to be a Lie bracket if and only if the associated extension is Bockstein closed. This was studied by Browder and Pakianathan [3] who also used this fact to give a complete description of the Bockstein cohomology of G in terms of the Lie algebra cohomology of the associated Lie algebra. This theory was later used by Pakianathan [5] to give a counterexample to a conjecture of Adem [1] on exponents in the integral cohomology of p-groups for odd primes p.

In the case where $p = 2$, the 2-power map $(\ )^2 : W \to V$ is not a homomorphism, so the results of Browder–Pakianathan do not generalize to 2-groups in a natural way. In this case, the 2-power map is a quadratic map $Q : W \to V$ where the associated bilinear map $B : W \times W \to V$ is induced by taking commutators in G. The 2-power exact condition is equivalent to the conditions: (i) $m = n$, (ii) the elements $\{Q(w) \mid w \in W\}$ generate V, and (iii) if $Q(w) = 0$ for some $w \in W$, then $w = 0$. We studied the quadratic maps associated to Bockstein closed extensions in an earlier paper, and showed that an extension E is Bockstein closed if and only if there is a bilinear map $P : V \times W \to V$ such that

$$P\big(Q(w), w'\big) = B\big(w, w'\big) + P\big(B\big(w, w'\big), w\big) \tag{1}$$

holds for all $w, w' \in W$ (see Theorem 1.1 in [7]). In some sense this is the Jacobi identity for the $p = 2$ case. If there is a quadratic map $Q : W \to W$ which satisfies this identity with $P = B$, then the vector space W becomes a 2-restricted Lie algebra with 2-power map defined by $w^{[2]} = Q(w) + w$ for all $w \in W$. But in general there are no direct connections between Bockstein closed quadratic maps and mod-2 Lie algebras.

In this paper, we study the cohomology of Bockstein closed 2-power exact extensions. We calculate the mod-2 cohomology ring and give a description of the Bockstein spectral sequence. As in the case when p is odd, the Bockstein spectral sequence can be described in terms of a cohomology theory based on our algebraic data. In this case, the right cohomology theory is the cohomology $H^*(Q, U)$ of a Bockstein closed quadratic map $Q : W \to V$. We define this cohomology using an explicit cochain complex associated to the quadratic map. The definition is given in such a way that the low dimensional cohomology has interpretation in terms of extensions of Bockstein closed quadratic maps. For example, $H^0(Q, U)$ gives the Q-invariants of U and $H^1(Q, U) \cong \mathrm{Hom}_{\mathbf{Quad}}(Q, U)$ if U is a trivial Q-module. Also, $H^2(Q, U)$ is isomorphic to the group of extensions of Q with abelian kernel U (see Proposition 4.4). The definition of $H^*(Q, U)$ is given in Section 4. To keep the theory more general, in the definition of $H^*(Q, U)$ we do not assume that the quadratic map Q is 2-power exact.

In Section 5, we calculate the mod-2 cohomology ring of a Bockstein closed 2-power exact group G using the Lyndon–Hochschild–Serre spectral sequence associated to the extension E. This calculation is relatively easy and it is probably known to experts in the field (see, for example, [4] or [9]). The mod-2 cohomology ring of G has a very nice expression given by

$$H^*(G, \mathbb{F}_2) \cong A^*(Q) \otimes \mathbb{F}_2[s_1, \ldots, s_n]$$

where s_i's are some 2-dimensional generators and the algebra $A^*(Q)$ is given by

$$A^*(Q) = \mathbb{F}_2[x_1, \ldots, x_n]/(q_1, \ldots, q_n)$$

where $\{x_1, \ldots, x_n\}$ forms a basis for $H^1(W)$ and q_i's are components of the extension class $q \in H^2(W, V)$ with respect to a basis for V. The action of the Bockstein operator on this cohomology ring gives valuable information about the question of whether the extension can be uniformly lifted to other extensions. Also finding Bocksteins of generators of the mod-2 cohomology algebra is the starting point for calculating the integral cohomology of G. We prove the following:

Theorem 1.1. *Let $E : 0 \to V \to G \to W \to 0$ be a Bockstein closed 2-power exact extension with extension class q and let $\beta(q) = Lq$. Then the mod-2 cohomology of G is in the above form and $\beta(s) = Ls + \eta$ where s denotes a column matrix with entries in s_i's and η is a column matrix with entries in $H^3(W, \mathbb{F}_2)$.*

The proof of this theorem is given in Section 7 using the Eilenberg–Moore spectral sequence associated to the extension. The key property of the EM-spectral sequence is that it behaves well under the Steenrod operations. The Steenrod algebra structure of the EM-spectral sequence was studied by L. Smith [10,11] and D. Rector [8] independently in a sequence of papers. Here we use only a special case of these results. More precisely, we use the fact that the first two vertical lines in the EM-spectral sequence are closed under the action of the Steenrod algebra. This is stated as Corollary 4.4 in [10].

The column matrix η of the formula $\beta(s) = Ls + \eta$ defines a cohomology class $[\eta] \in H^3(Q, L)$ where L is the Q-module associated to the matrix L. Recall that in the work of Browder and Pakianathan [3], there is a cohomology class lying in the Lie algebra cohomology $H^3(\mathcal{L}, \mathrm{ad})$ which is defined in a similar way and it is an obstruction class for lifting G uniformly twice. We obtain a similar theorem for uniform double lifting of 2-group extensions.

Theorem 1.2. *Let $E : 0 \to V \to G \to W \to 0$ be a Bockstein closed 2-power exact extension with extension class q. Let Q be the associated quadratic map and L denote the Q-module defined by L in the equation $\beta(q) = Lq$. Then, the extension E has a uniform double lifting if and only if $[\eta] = 0$ in $H^3(Q, L)$.*

Another result we have is a description of the second page of the Bockstein spectral sequence in terms of the cohomology of Bockstein closed quadratic maps for the case where the extension has a uniform double lifting.

Theorem 1.3. *Let E, G, Q, and L be as in Theorem 1.2. Assume that E has a uniform double lifting. Then, the second page of the Bockstein spectral sequence for G is given by*

$$B_2^*(G) = \bigoplus_{i=0}^{\infty} H^{*-2i}\big(Q, \mathrm{Sym}^i(L)\big)$$

where $\mathrm{Sym}^i(L)$ denotes the symmetric i-th power of L.

In the p odd case, the B-cohomology has been calculated in cases by comparing it to $H^*(\mathfrak{g}, U(\mathfrak{g})^*)$ where $U(\mathfrak{g})^*$ is the dual of the universal enveloping algebra of $\mathfrak{g}$ equipped with the dual adjoint action, where $\mathfrak{g}$ is an associated complex Lie algebra. This fundamental object has played a role in string topology (homology of free loop spaces) and is analogous to the (classical) ring of modular forms, identified by Eichler–Shimura as $H^*(SL_2(\mathbb{Z}), \mathrm{Poly}(V))$ where V is the complex 2-dimensional canonical representation of $SL_2(\mathbb{Z})$ (see [6] for more details). In string topology contexts, this is referred to as the Hodge decomposition and so the above can be thought of as a Hodge decomposition for the quadratic form Q. It describes the distribution of higher torsion in the integral cohomology of the associated group G.

As in the case of Lie algebras, it is possible to give a suitable definition of a universal enveloping algebra $U(Q)$ for a quadratic map Q so that the representations of Q and representations of the universal algebra $U(Q)$ can be identified in a natural way. However it is not clear to us how to find an isomorphism between the cohomology of the universal algebra $U(Q)$ and the cohomology of the quadratic map Q. There is also the issue of finding analogies of the theorems on universal algebras of Lie algebras such as the Poincaré–Birkhoff–Witt theorem. We leave these as open problems.

The paper is organized as follows: In Section 2, we introduce the category of quadratic maps and show that it is naturally equivalent to the category of extensions of certain type. Then in Section 3, we give the definition of a Bockstein closed quadratic map. The definition of cohomology of Bockstein closed quadratic maps is given in Section 4. Sections 5, 6, and 7 are devoted to the mod-2 cohomology calculations using LHS- and EM-spectral sequences and the calculation of Bocksteins of the generators. In particular, Theorem 1.1 is proven in Section 7. In Section 8, we discuss the obstructions for uniform lifting and in Section 9, we explain the E_2-page of the Bockstein spectral sequence in terms of the cohomology of Bockstein closed quadratic maps.

2. Category of quadratic maps

Let E denote a central extension of the form

$$E : 0 \to V \to G \to W \to 0$$

where V and W are elementary abelian 2-groups. Associated to E, there is a cohomology class $q \in H^2(W, V)$. Also associated to E there is a quadratic map $Q : W \to V$ defined by $Q(w) = (\hat{w})^2$, where $\hat{w}$ denotes an element in G that lifts $w \in W$. Similarly, the commutator induces a symmetric bilinear map $B : W \times W \to V$ defined by $B(x, y) = [\hat{x}, \hat{y}]$ for $x, y \in W$ where $[g, h] = g^{-1}h^{-1}gh$ for $g, h \in G$. It is easy to see that B is the bilinear form associated to Q.

We have shown in [7] that the extension class q and the quadratic form Q are closely related to each other. In particular, we showed that we can take $q = [f]$ where f is a bilinear factor set $f : W \times W \to V$ satisfying the identity $f(w, w) = Q(w)$ for all $w \in W$ (see [7, Lemma 2.3]). We can write this correspondence more explicitly by choosing a basis $\{w_1, \dots, w_m\}$ for W. Then,

$$f(w_i, w_j) = \begin{cases} B(w_i, w_j) & \text{if } i < j, \\ Q(w_i) & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

This gives a very specific expression for q. Let $\{v_1, \dots, v_n\}$ be a basis for V, and let q_k be the k-th component of q with respect to this basis. Then,

$$q_k = \sum_i Q_k(w_i)x_i^2 + \sum_{i<j} B_k(w_i, w_j)x_i x_j$$

where $\{x_1, \dots, x_m\}$ is the dual basis of $\{w_1, \dots, w_m\}$ and Q_k and B_k denote the k-th components of Q and B. This allows one to prove the following:

Proposition 2.1. *(See Corollary 2.4 in [7].) Given a quadratic map $Q : W \to V$, there is a unique (up to equivalence) central extension*

$$E(Q) : 0 \to V \to G(Q) \to W \to 0$$

with a bilinear factor set $f : W \times W \to V$ satisfying $f(w, w) = Q(w)$ for all $w \in W$.

This gives a bijective correspondence between quadratic maps $Q : W \to V$ and the central extensions of the form $E : 0 \to V \to G \to W \to 0$. We will now define the category of quadratic maps and the category of group extensions of the above type and then prove that the correspondence described above indeed gives a natural equivalence between these categories.

2.1. Equivalence of categories

The category of quadratic maps **Quad** is defined as the category whose objects are quadratic maps $Q : W \to V$ where W and V are vector spaces over $\mathbb{F}_2$. For quadratic maps Q_1, Q_2, a morphism $f : Q_1 \to Q_2$ is defined as a pair of linear transformations $f = (f_W, f_V)$ such that the following diagram commutes

$$\begin{tikzcd} W_1 \arrow[r, "f_W"] \arrow[d, "Q_1"] & W_2 \arrow[d, "Q_2"] \\ V_1 \arrow[r, "f_V"] & V_2. \end{tikzcd} \tag{2}$$

The composition of morphisms $f=(f_W, f_V)$ and $g=(g_W, g_V)$ is defined by coordinate-wise compositions. The identity morphism is the pair $(\mathrm{id}_W, \mathrm{id}_V)$. Two quadratic maps Q_1 and Q_2 are isomorphic if there are morphisms $f: Q_1 \to Q_2$ and $g: Q_2 \to Q_1$ such that $f \circ g = \mathrm{id}_{Q_2}$ and $g \circ f = \mathrm{id}_{Q_1}$.

The category **Ext** is defined as the category whose objects are the equivalence classes of extensions of type

$$E: 0 \to V \to G \to W \to 0$$

where V and W are vector spaces over $\mathbb{F}_2$, and the morphisms are given by a commuting diagram as follows:

$$\begin{tikzcd} E_1 : 0 \arrow[r] \arrow[d, "f"] & V_1 \arrow[r] \arrow[d, "f_V"] & G_1 \arrow[r] \arrow[d, "f_G"] & W_1 \arrow[r] \arrow[d, "f_W"] & 0 \\ E_2 : 0 \arrow[r] & V_2 \arrow[r] & G_2 \arrow[r] & W_2 \arrow[r] & 0. \end{tikzcd} \tag{3}$$

Note that two extensions are considered equivalent if there is a diagram as above with $f_W = \mathrm{id}_W$ and $f_V = \mathrm{id}_V$. All such morphisms are taken to be equal to identity morphism in our category. More generally, two morphisms $f, g: E_1 \to E_2$ will be considered equal in **Ext** if $f_V = g_V$ and $f_W = g_W$.

From the discussion at the beginning of this section, it is clear that the assignments $\Phi: Q \to E(Q)$ and $\Psi: E \to Q_E$ give a bijective correspondence between the objects of **Quad** and **Ext**. We just need to extend this correspondence to a correspondence between morphisms. Given a morphism $f: E_1 \to E_2$, we take $\Psi(f)$ to be the pair $(f_V, f_W): Q_1 \to Q_2$. By commutativity of the diagram (3), it is easy to see that $Q_2(f_W(w)) = f_V(Q_1(w))$ holds for all $w \in W_1$. To define the image of a morphism $f: Q_1 \to Q_2$ under Φ, we need to define a group homomorphism $f_G: G(Q_1) \to G(Q_2)$ which makes the diagram given in (3) commute. Note that once f_G is defined, we can define the morphism $\Phi(f): E_1 \to E_2$ as a sequence of maps (f_V, f_G, f_W) as in diagram (3). It is clear that the composition $\Psi \circ \Phi$ is equal to the identity transformation. The composition $\Phi \circ \Psi$ is also equal to the identity in **Ext** although it may not be equal to identity on the middle map f_G. This follows from the fact that two morphisms $f, g: E_1 \to E_2$ between two extensions are equal in **Ext** if $f_V = g_V$ and $f_W = g_W$.

To define a group homomorphism $f_G: G(Q_1) \to G(Q_2)$ which makes the diagram given in (3) commute, first recall that for $i = 1, 2$, we can take $G(Q_i)$ as the set $V_i \times W_i$ with multiplication given by $(v, w)(v', w') = (v + v' + f_i(w, w'), w + w')$ where $f_i: W_i \times W_i \to V_i$ is a bilinear factor set satisfying $f_i(w, w) = Q_i(w)$ for every $w \in W_i$ (see [7, Lemma 2.3]). Note that the choice of the factor set is not unique and if f_i and f_i' are two factor sets for $E(Q_i)$ satisfying $f_i(w, w) = f_i'(w, w) = Q_i(w)$ for all $w \in W_i$, then $f_i + f_i' = \delta(t)$ is a boundary in the bar resolution. When we apply this to the extension associated to the quadratic form $Q_2 f_W = f_V Q_1: W_1 \to V_2$, we see that there is a function $t: W_1 \to V_2$ such that

$$(\delta t)\bigl(w, w'\bigr) = t\bigl(w'\bigr) + t\bigl(w + w'\bigr) + t(w) = f_2\bigl(f_W(w), f_W\bigl(w'\bigr)\bigr) + f_V\bigl(f_1\bigl(w, w'\bigr)\bigr) \tag{4}$$

for all $w, w' \in W_1$. We define $f_G: G(Q_1) \to G(Q_2)$ by $f_G(v, w) = (f_V(v) + t(w), f_W(w))$ for all $v \in V_1$ and $w \in W_1$. To check that f_G is a group homomorphism, we need to show that

$$f_G\bigl((v, w)\bigl(v', w'\bigr)\bigr) = f_G(v, w) f_G\bigl(v', w'\bigr)$$

holds for all $v, v' \in V_1$ and $w, w' \in W_1$. Writing this out in detail, one sees that this equation is equivalent to Eq. (4), hence it holds. So, we obtain a group homomorphism $f_G: G(Q_1) \to G(Q_2)$ as desired. We conclude the following:

Proposition 2.2. *The categories* **Quad** *and* **Ext** *are equivalent.*

An immediate consequence of this equivalence is the following:

Corollary 2.3. *Let Q_1 and Q_2 be quadratic maps with extension classes $q_1 \in H^2(W_1, V_1)$ and $q_2 \in H^2(W_2, V_2)$ respectively. If $f : Q_1 \to Q_2$ is a morphism of quadratic maps, then*

$$(f_W)^*(q_2) = (f_V)_*(q_1)$$

in $H^2(W_1, V_2)$.

Proof. Let $Q' : W_1 \to V_2$ be the quadratic map defined by $Q' = f_V Q_1 = Q_2 f_W$. Then, we have

$$\begin{tikzcd}
W_1 \arrow[r, "="] \arrow[d, "Q_1"] & W_1 \arrow[r, "f_W"] \arrow[d, "Q'"] & W_2 \arrow[d, "Q_2"] \\
V_1 \arrow[r, "f_V"] & V_2 \arrow[r, "="] & V_2.
\end{tikzcd}$$

So, we find a factorization of f in **Quad** as $Q_1 \xrightarrow{f_1} Q' \xrightarrow{f_2} Q_2$. By Proposition 2.2, we obtain a factorization of the corresponding morphism in **Ext**, this gives the following commuting diagram:

$$\begin{tikzcd}
E_1 : 0 \arrow[r] & V_1 \arrow[r] \arrow[d, "f_V"] & G(Q_1) \arrow[r] \arrow[d, "(f_1)_G"] & W_1 \arrow[r] \arrow[d, equal] & 0 \\
E' : 0 \arrow[r] & V_2 \arrow[r] \arrow[d, equal] & \widetilde{G} \arrow[r] \arrow[d, "(f_2)_G"] & W_1 \arrow[r] \arrow[d, "f_W"] & 0 \\
E_2 : 0 \arrow[r] & V_2 \arrow[r] & G(Q_2) \arrow[r] & W_2 \arrow[r] & 0.
\end{tikzcd}$$

Hence, we have $(f_W)^*(q_2) = (f_V)_*(q_1)$ as desired. □

2.2. Extensions and representations of quadratic maps

We now introduce certain categorical notions for maps between quadratic maps such as kernel and cokernel of a map and then give the definition of extensions of quadratic maps.

The kernel of a morphism $f : Q_1 \to Q_2$ is defined as the quadratic map $Q_1|_{\ker f_W} : \ker f_W \to \ker f_V$. We denote this quadratic map as $\ker f$. If $\ker f$ is the zero quadratic map, i.e., the quadratic map from a zero vector space to zero vector space, then we say f is injective. Similarly, we define the image of a quadratic map $f : Q_1 \to Q_2$ as the quadratic map $Q_2|_{\mathrm{Im}\, f_W} : \mathrm{Im}\, f_W \to \mathrm{Im}\, f_V$. We denote this quadratic map by $\mathrm{Im}\, f$ and say f is surjective if $\mathrm{Im}\, f = Q_2$. Given an injective map $f : Q_1 \to Q_2$, we say f is a *normal embedding* if

$$B_2\big(f_W(w_1), w_2\big) \in \mathrm{Im}\, f_V$$

for all $w_1 \in W_1$ and $w_2 \in W_2$. Given a normal embedding $f : Q_1 \to Q_2$, we can define the cokernel of f as the quadratic map $\operatorname{coker} f : \operatorname{coker} f_W \to \operatorname{coker} f_V$ by the formula

$$(\operatorname{coker} f)(w_2 + \mathrm{Im}\, f_W) = Q_2(w_2) + \mathrm{Im}\, f_V.$$

We are now ready to define an extension of two quadratic maps.

Definition 2.4. We say a sequence of quadratic maps of the form

$$\mathcal{E} : 0 \to Q_1 \xrightarrow{f} Q_2 \xrightarrow{g} Q_3 \to 0 \tag{5}$$

is an extension of quadratic maps if f is injective, g is surjective, and $\operatorname{Im} f = \ker g$.

Note that in an extension $\mathcal{E}$ as above, the first map $f : Q_1 \to Q_2$ is a normal embedding because we have

$$B_2(\operatorname{Im} f_W, W_2) = B_2(\ker g_W, W_2) \subseteq \ker g_V = \operatorname{Im} f_V.$$

We say the extension $\mathcal{E}$ is a split extension if there is a morphism of quadratic maps $s : Q_3 \to Q_2$ such that $g \circ s = \mathrm{id}_{Q_3}$. In this case we write $Q_3 \cong Q_1 \rtimes Q_2$.

Later in this paper we consider the extensions where Q_1 is just the identity map $\mathrm{id}_U : U \to U$ of a vector space U. In this case, we denote the extension by

$$\mathcal{E} : 0 \to U \xrightarrow{i} \widetilde{Q} \xrightarrow{\pi} Q \to 0,$$

and say $\mathcal{E}$ is an extension of Q with an abelian kernel U. In Section 4, we define obstructions for splitting such extensions and also give a classification theorem for such extensions in a subcategory of **Quad** where all the quadratic maps are assumed to be Bockstein closed.

Given an extension of quadratic map Q with an abelian kernel U, there is an action of Q on U induced from the bilinear form $\widetilde{B}$ associated to $\widetilde{Q}$. This action is defined as a homomorphism

$$\rho_W : W \to \operatorname{Hom}(U, U)$$

which satisfies $i_V(\rho_W(w)(u)) = \widetilde{B}(i_W(u), \overline{w})$ where $\overline{w}$ is vector in $\widetilde{W}$ such that $\pi_W(\overline{w}) = w$.

In the definition of the representation of a quadratic map, we need the following family of quadratic maps: Let U be a vector space. We define

$$Q_{\mathfrak{gl}(U)} : \operatorname{End}(U) \to \operatorname{End}(U)$$

to be the quadratic map such that $Q_{\mathfrak{gl}(U)}(A) = A^2 + A$ for all $A \in \operatorname{End}(U)$. (See Example 2.5 in [7].)

Definition 2.5. A *representation* of a quadratic form Q is defined as a morphism

$$\rho : Q \to Q_{\mathfrak{gl}(U)}$$

in the category of quadratic maps. In other words, a representation is a pair of maps $\rho = (\rho_W, \rho_V)$ such that the following diagram commutes

$$\begin{tikzcd} W \arrow[r, "\rho_W"] \arrow[d, "Q"] & \operatorname{End}(U) \arrow[d, "Q_{\mathfrak{gl}(U)}"] \\ V \arrow[r, "\rho_V"] & \operatorname{End}(U). \end{tikzcd}$$

If U is a k-dimensional vector space, then we say ρ is a k-dimensional representation of Q. Given a representation as above, we sometimes say U is a Q-module to express the fact that there is an action of Q on $\mathrm{id}_U : U \to U$ via the representation ρ.

3. Bockstein closed quadratic maps

Let $E(Q)$ be a central extension of the form $0 \to V \to G(Q) \to W \to 0$ associated to a quadratic map $Q : W \to V$, where V and W are $\mathbb{F}_2$-vector spaces. Let $q \in H^2(W, V)$ denote the extension class of E. Choosing a basis $\{v_1, \dots, v_n\}$ for V, we can write q as a tuple

$$q = (q_1, \dots, q_n)$$

where $q_i \in H^2(W, \mathbb{F}_2)$ for all i. The elements $\{q_i\}$ generate an ideal $I(Q)$ in the cohomology algebra $H^*(W, \mathbb{F}_2)$. It is easy to see that the ideal $I(Q)$ is independent of the basis chosen for V, and hence is completely determined by Q.

Definition 3.1. We say $Q : W \to V$ is *Bockstein closed* if $I(Q)$ is invariant under the Bockstein operator on $H^*(W; \mathbb{F}_2)$. A central extension $E(Q) : 0 \to V \to G(Q) \to W \to 0$ is called Bockstein closed if the associated quadratic map Q is Bockstein closed.

The following was proven in [7] as Proposition 3.3.

Proposition 3.2. *Let $Q : W \to V$ be a quadratic map, and let $q \in H^2(W, V)$ be the corresponding extension class. Then, Q is Bockstein closed if and only if there is a cohomology class $L \in H^1(W, \mathrm{End}(V))$ such that $\beta(q) = Lq$.*

We often choose a basis for W and V ($\dim W = m$ and $\dim V = n$) and express the formula $\beta(q) = Lq$ as a matrix equation. From now on, let us assume W and V have some fixed basis and let $\{x_1, \dots, x_m\}$ be the dual basis for W. Then, each component q_k is a quadratic polynomial in variables x_i and L is an $n \times n$ matrix with entries given by linear polynomials in x_i's. If we express q as a column matrix whose i-th entry is q_i, then $\beta(q) = Lq$ makes sense as a matrix formula where Lq denotes the matrix multiplication. In general, we can have different matrices, say L_1 and L_2, such that $\beta(q) = L_1 q = L_2 q$. It is known that L is unique when E is 2-power exact. (See Proposition 8.1 in [7].)

Example 3.3. Let G be the kernel of the mod 2 reduction map $GL_n(\mathbb{Z}/8) \to GL_n(\mathbb{Z}/2)$. It is easy to see that G fits into a central short exact sequence

$$0 \to \mathfrak{gl}_n(\mathbb{F}_2) \to G \to \mathfrak{gl}_n(\mathbb{F}_2) \to 0$$

with associated quadratic map $Q_{\mathfrak{gl}_n}$ where $\mathfrak{gl}_n(\mathbb{F}_2)$ is the vector space of $n \times n$ matrices with entries in $\mathbb{F}_2$ and the quadratic map $Q_{\mathfrak{gl}_n} : \mathfrak{gl}_n(\mathbb{F}_2) \to \mathfrak{gl}_n(\mathbb{F}_2)$ is defined by $Q_{\mathfrak{gl}_n}(\mathbb{A}) = \mathbb{A}^2 + \mathbb{A}$ (see [7, Examples 2.5 and 2.6] for more details). We showed in [7, Corollary 3.9] that this extension and its restrictions to suitable subspaces such as $\mathfrak{sl}_n(\mathbb{F}_2)$ or $\mathfrak{u}_n(\mathbb{F}_2)$ are Bockstein closed. Here $\mathfrak{sl}_n(\mathbb{F}_2)$ denotes the subspace of $\mathfrak{gl}_n(\mathbb{F}_2)$ formed by matrices of trace zero and $\mathfrak{u}_n(\mathbb{F}_2)$ denotes the subspace of strictly upper triangular matrices. Note that the extension for $\mathfrak{u}_n(\mathbb{F}_2)$ is also a 2-power exact extension (see [7, Example 9.6]).

We now consider the question of when an extension of two Bockstein closed quadratic maps is also Bockstein closed. The equations that we find in the process of answering this question will give us the motivation for the definition of the cohomology of Bockstein closed quadratic maps.

Let

$$\mathcal{E} : 0 \to U \xrightarrow{i} \widetilde{Q} \xrightarrow{\pi} Q \to 0$$

be an extension of the quadratic map Q with abelian kernel U. We can express this as a diagram of quadratic maps as follows:

$$\begin{array}{ccccc} U & \xrightarrow{i_W} & U \oplus W & \xrightarrow{\pi_W} & W \\ \downarrow{\scriptstyle \mathrm{id}_U} & & \downarrow{\scriptstyle \widetilde{Q}} & & \downarrow{\scriptstyle Q} \\ U & \xrightarrow{i_V} & U \oplus V & \xrightarrow{\pi_V} & V. \end{array}$$

Note that we have $\widetilde{Q}(u,0) = (u,0)$ and $\pi_V \widetilde{Q}(0,w) = Q(w)$ for all $u \in U$ and $w \in W$. Also, there is an action of Q on U given by the linear map $\rho_W : W \to \mathrm{End}(U)$ defined by the equation

$$i_V\big(\rho_W(w)u\big) = \widetilde{B}\big((u,0),(0,w)\big).$$

Hence, we can write

$$\widetilde{Q}(u,w) = \big(u + \rho_W(w)u + f(w), Q(w)\big)$$

where $f : W \to U$ is a quadratic map called factor set. We will study the conditions on f and ρ_W which make $\widetilde{Q}$ a Bockstein closed quadratic map.

Let $k = \dim U$. Choose a basis for U, and let $\{z_1, \ldots, z_k\}$ be the associated dual basis for U^*. Then, we can express the extension class $\tilde{q}$ of $\widetilde{Q}$ as a column matrix

$$\tilde{q} = \begin{bmatrix} q \\ \beta(z) + Rz + f \end{bmatrix}$$

where f and q denote the column matrices for the quadratic maps $f : W \to U$ and $Q : W \to V$ respectively. Here z is the column matrix with i-th entry equal to z_i and R is a $k \times k$ matrix with entries in x_i's which is associated to $\rho_W : W \to \mathrm{End}(U)$. Applying the Bockstein operator, we get

$$\beta(\tilde{q}) = \begin{bmatrix} \beta(q) \\ \beta(R)z + R\beta(z) + \beta(f) \end{bmatrix}.$$

Note that $\widetilde{Q}$ is Bockstein closed if we can find $\widetilde{L}$ such that $\beta(\tilde{q}) = \widetilde{L}\tilde{q}$. Since $\beta(q) = Lq$ for some L, we can take $\widetilde{L}$ as

$$\widetilde{L} = \begin{bmatrix} L & 0 \\ L_{2,1} & L_{2,2} \end{bmatrix}.$$

Note that assuming the top part of the matrix $\widetilde{L}$ is in a special form does not affect the generality of the lower part. So, under the assumption that Q is Bockstein closed, the quadratic map $\widetilde{Q}$ is Bockstein closed if and only if there exist $L_{2,1}$ and $L_{2,2}$ satisfying

$$\beta(R)z + R\beta(z) + \beta(f) = L_{2,1}q + L_{2,2}\big(\beta(z) + Rz + f\big). \tag{6}$$

We have

$$L_{2,2} = \sum_{i=1}^{k} L_{2,2}^{(i)} z_i + \sum_{j=1}^{m} L_{2,2}^{(j)} x_j$$

where $L_{2,2}^{(i)}$ and $L_{2,2}^{(j)}$ are scalar matrices, so we can write $L_{2,2} = L_{2,2}^z + L_{2,2}^x$ where $L_{2,2}^z$ is the first sum and $L_{2,2}^x$ is the second sum in the above formula.

Eq. (6) gives $L^z_{2,2}\beta(z) = 0$ which implies $L^z_{2,2} = 0$. Writing $L_{2,2} = L^x_{2,2}$ in (6), we easily see that we must have $L_{2,2} = R$. Putting this into (6), we get

$$\bigl[\beta(R) + R^2\bigr]z + \bigl[\beta(f) + Rf\bigr] = L_{2,1}q. \tag{7}$$

As we did for $L_{2,2}$, we can write $L_{2,1}$ also as a sum $L_{2,1} = L^z_{2,1} + L^x_{2,1}$ where the entries of $L^z_{2,1}$ are linear polynomials in z_i's and the entries of $L^x_{2,1}$ are linear polynomials in x_i's. So, Eq. (7) gives two equations:

$$\begin{aligned} \bigl[\beta(R) + R^2\bigr]z &= L^z_{2,1}q, \\ \beta(f) + Rf &= L^x_{2,1}q. \end{aligned} \tag{8}$$

From now on, let us write $Z = L^z_{2,1}$. Note that Z is a $k \times n$ matrix ($k = \dim U$ and $n = \dim V$) with entries in the dual space U^*, so it can be thought of as a linear operator $Z : U \to \mathrm{Hom}(V, U)$. Viewing this as a bilinear map $U \times V \to U$, and then using an adjoint trick, we obtain a linear map $\rho_V : V \to \mathrm{Hom}(U, U) = \mathrm{End}(U)$. As a matrix, let us denote ρ_V by T. The relation between Z and T can be explained as follows: If $\{u_1, \dots, u_k\}$ are basis elements for U dual to the basis elements $\{z_1, \dots, z_k\}$ of U^* and if $\{v_1, \dots, v_k\}$ is the basis for V dual to the basis elements $\{t_1, \dots, t_k\}$ of V^*, then $Z(u_i)(v_j) = T(v_j)(u_i)$ for all i, j. So, if $Z = \sum_{i=1}^k Z(i)z_i$ and $T = \sum_{j=1}^n T(j)t_j$, then we have $Z(i)e_j = T(j)e_i$ where e_i and e_j are i-th and j-th unit column matrices. This implies, in particular, that

$$Zq = T(q)z$$

where $T(q)$ is the matrix obtained from T by replacing t_i's with q_i's. So, the first equation in (8) can be interpreted as follows:

Lemma 3.4. *Let $\rho_W : W \to \mathrm{End}(U)$ and $\rho_V : V \to \mathrm{End}(U)$ be two linear maps with corresponding matrices R and T. Let Z denote the matrix for the adjoint of ρ_V in $\mathrm{Hom}(U, \mathrm{Hom}(V, U))$. Then, the equation*

$$\bigl[\beta(R) + R^2\bigr]z = Zq$$

holds if and only if $\rho = (\rho_W, \rho_V) : \mathrm{Q} \to \mathrm{Q}_{\mathfrak{gl}(U)}$ is a representation.

Proof. Note that the diagram

$$\begin{tikzcd} W \arrow[r, "\rho_W"] \arrow[d, "\mathrm{Q}"] & \mathrm{End}(U) \arrow[d, "\mathrm{Q}_{\mathfrak{gl}(U)}"] \\ V \arrow[r, "\rho_V"] & \mathrm{End}(U) \end{tikzcd}$$

commutes if and only if

$$\beta(R) + R^2 = T(q)$$

where $T(q)$ is the $k \times k$ matrix obtained from T by replacing t_i's with q_i's. We showed above that $Zq = T(q)z$, so $\beta(R) + R^2 = T(q)$ holds if and only if

$$\bigl[\beta(R) + R^2\bigr]z = T(q)z = Zq.$$

This completes the proof. $\square$

As a consequence of the above lemma, we can conclude that if the action of W on $\mathrm{End}(U)$ comes from a representation $\rho : Q \to Q_{\mathfrak{gl}(U)}$, then the only obstruction for a quadratic map $\widetilde{Q}$ to be Bockstein closed is the second equation

$$\beta(f) + Rf = L^x_{2,1} q$$

given in (8). Note that both R and $L^x_{2,1}$ are matrices with entries in x_i's, so this equation can be interpreted as saying that $\beta(f) + Rf = 0$ in $A^*(Q) = \mathbb{F}_2[x_1, \ldots, x_m]/(q_1, \ldots, q_n)$. In the next section, we define the cohomology of a Bockstein closed quadratic map using this interpretation.

4. Cohomology of Bockstein closed quadratic maps

Let $Q : W \to V$ be a Bockstein closed quadratic map where W and V are $\mathbb{F}_2$-vector spaces of dimensions m and n, respectively. Let U be a k-dimensional Q-module with associated representation $\rho : Q \to Q_{\mathfrak{gl}(U)}$. We will define the cohomology of Q with coefficients in U as the cohomology of a cochain complex $C^*(Q, U)$. We now describe this cochain complex.

Let $A(Q)^*$ denote the $\mathbb{F}_2$-algebra

$$A^*(Q) = \mathbb{F}_2[x_1, \ldots, x_m]/(q_1, \ldots, q_n)$$

as before. The algebra $A^*(Q)$ is a graded algebra where the grading comes from the usual grading of the polynomial algebra. We define p-cochains of Q with coefficients in U as

$$C^p(Q, U) = A(Q)^p \otimes U.$$

We describe the differentials using a matrix formula. Choosing a basis for U, we can express a p-cochain f as a $k \times 1$ column matrix with entries $f_i \in A^p(Q)$. Let $R \in H^1(W, \mathrm{End}(U))$ be the cohomology class associated to ρ_W. We can express R as a $k \times k$ matrix with entries in x_i's. We define the boundary maps

$$\delta : C^p(Q, U) \to C^{p+1}(Q, U)$$

by

$$\delta(f) = \beta(f) + Rf.$$

Note that

$$\delta^2(f) = \beta(R)f + R\beta(f) + R\beta(f) + R^2 f = \left[\beta(R) + R^2\right] f = 0$$

in $A^*(Q)$ because $\beta(R) + R^2 = T(q)$ by the argument given in the proof of Lemma 3.4. So, $C^*(Q, U)$ with the above boundary maps is a cochain complex.

Note that although the definition of δ only uses R, i.e., ρ_W, the existence of ρ_V is needed to ensure that $\delta^2 = 0$. Thus both maps in the structure of U as a Q-module play a role in establishing δ as a differential. Also note that we need the quadratic map Q to be Bockstein closed for the well-definedness of the differential δ.

Definition 4.1. The cohomology of a Bockstein closed quadratic form Q with coefficients in a Q-module U is defined as

$$H^*(Q, U) := H^*\big(C^*(Q, U), \delta\big)$$

where $C^*(Q, U) = A^*(Q) \otimes U$ and the boundary maps δ are given by $\delta(f) = \beta(f) + Rf$.

Let U be the 1-dimensional trivial Q-module, i.e., $\rho = (\rho_W, \rho_V) = (0,0)$. In this case we write $U = \mathbb{F}_2$. Then, $H^*(Q, \mathbb{F}_2)$ is just the cohomology of the complex $A(Q)^*$ and the boundary map δ is equal to the Bockstein operator. In this case, the cohomology group $H^*(Q, \mathbb{F}_2)$ also has a ring structure coming from the usual multiplication of polynomials. Note that given two cocycles $f, g \in A^*(Q)$, we have $\beta(fg) = \beta(f)g + f\beta(g) = 0$ modulo $I(Q)$. So, we define the product of two cohomology classes $[f], [g] \in H^*(Q, \mathbb{F}_2)$ by

$$[f][g] = [fg]$$

where fg denotes the usual multiplication of polynomials.

Given a morphism $\varphi : Q_1 \to Q_2$, we have $(\varphi_W)^*(q_2) = (\varphi_V)_*(q_1)$ by Corollary 2.3. This shows that $(\varphi_W)^* : H^*(W_2, \mathbb{F}_2) \to H^*(W_1, \mathbb{F}_2)$ takes the entries of q_2 into the ideal $I(Q_1)$. Thus, φ_W induces an algebra map $\varphi^* : A^*(Q_2) \to A^*(Q_1)$ which gives a chain map $C^*(Q_2, U_2) \to C^*(Q_1, U_1)$ where U_2 is representation of Q_2 and U_1 is a representation of Q_1 induced by φ. So, φ induces a homomorphism

$$\varphi^* : H^*(Q_2, U_2) \to H^*(Q_1, U_1).$$

If $U_1 = U_2 = \mathbb{F}_2$, the induced map is also an algebra map.

In the rest of the section, we discuss the interpretations of low dimensional cohomology, $H^i(Q, U)$ for $i = 0, 1, 2$, in terms of extension theory. First we calculate $H^0(Q, U)$. Note that $C^0(Q, U) = U$ and given $u \in C^0(Q, U)$, we have $\delta(u)(w) = \rho_W(w)u$. So, $H^0(Q, U) = U^Q$ where

$$U^Q = \{u \mid \rho_W(w)(u) = 0 \text{ for all } w \in W\}.$$

Note that this is analogous to Lie algebra invariants

$$U^{\mathfrak{g}} = \{u \mid x \cdot u = 0 \text{ for all } x \in \mathfrak{g}\}.$$

We refer to the elements of U^Q as Q-invariants of U.

Now, we consider $H^1(Q, U)$. Note that $C^1(Q, U) \cong \operatorname{Hom}(W, U)$. Let $d_W : W \to U$ be a 1-cochain. By our definition of differentials, d_W is a derivation if and only if $\delta(d_W) = \beta(d_W) + Rd_W = 0$ in $A(Q)^2 \otimes U$. The last equation can be interpreted as follows: There is a linear map $d_V : V \to U$ such that

$$\big(1 + \rho_W(w)\big)d_W(w) + d_V\big(Q(w)\big) = 0.$$

A trivial derivation will be a derivation $d_W : W \to U$ of the form $d_W(w) = \rho_W(w)u$ for some $u \in U$. Note that when U is a trivial module, $d_W : W \to U$ is a derivation if and only if there is a linear map $d_V : V \to U$ such that the following diagram commutes

$$\begin{tikzcd} W \arrow[r, "d_W"] \arrow[d, "Q"] & U \arrow[d, "\mathrm{id}"] \\ V \arrow[r, "d_V"] & U. \end{tikzcd}$$

So, when U is a trivial Q-module, we have

$$H^1(Q, U) \cong \operatorname{Hom}_{\mathbf{Quad}}(Q, U).$$

If $U = \mathbb{F}_2$, then

$$H^1(Q, \mathbb{F}_2) = \ker\{\beta : H^1(W, \mathbb{F}_2) \to A^2(Q)\}.$$

So, we have the following:

Proposition 4.2. *Let $Z(Q)$ be the vector space generated by k-invariants $q_1, \dots, q_n$ and let $Z(Q)^\beta = \{z \in Z(Q) \mid \beta(z) = 0\}$. Then, $H^1(Q, \mathbb{F}_2) \cong Z(Q)^\beta$.*

We refer to the elements of $Z(Q)^\beta$ as the Bockstein invariants of Q. For an arbitrary Q-module U, we have the following:

Proposition 4.3. *There is a one-to-one correspondence between $H^1(Q, U)$ and the splittings of the split extension $0 \to U \to U \rtimes Q \to Q \to 0$.*

Proof. Observe that $s : Q \to U \rtimes Q$ is a morphism in **Quad** if and only if $\widetilde{Q}(s_W(w)) = s_V(Q(w))$. Since $\pi s = \mathrm{id}$, we can write $s_W(w) = (d_W(w), w)$ and $s_V(v) = (d_V(v), v)$. So, s is a morphism in **Quad** if and only if $d_W(w) + \rho(w)d_W(w) = d_V(Q(w))$, i.e., d_W is a derivation. It is easy to see that trivial derivation corresponds to a splitting which is trivial up to an automorphism of $U \rtimes Q$. □

We now consider extensions of a Bockstein closed quadratic map Q with an abelian kernel U, and show that $H^2(Q, U)$ classifies such extensions up to an equivalence. If there is a diagram of quadratic maps of the following form

$$\begin{tikzcd}
\mathcal{E}_1 : 0 \arrow[r] & U \arrow[r] \arrow[d, equal] & Q_1 \arrow[r] \arrow[d, "\varphi"] & Q \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{E}_2 : 0 \arrow[r] & U \arrow[r] & Q_2 \arrow[r] & Q \arrow[r] & 0,
\end{tikzcd} \tag{9}$$

then we say $\mathcal{E}_1$ is equivalent to $\mathcal{E}_2$. Let $\mathrm{Ext}(Q, U)$ denote the set of equivalence classes of extensions of the form $0 \to U \to \widetilde{Q} \to Q \to 0$ with abelian kernel U where $\widetilde{Q}$ and Q are Bockstein closed. We can define the summation of two extensions as it is done in group extension theory. So, $\mathrm{Ext}(Q, U)$ is an abelian group. We prove the following:

Proposition 4.4. $H^2(Q, U) \cong \mathrm{Ext}(Q, U)$.

Proof. Note that we already have a fixed decomposition for the domain and the range of $\widetilde{Q}$, so we will write our proof using these fixed decompositions. We skip some of the details which are done exactly as in the case of group extensions.

First we show there is a 1–1 correspondence between 2-cocycles and Bockstein closed extensions. Recall that a 2-cocycle is a quadratic map $f : W \to U$ such that $\beta(f) + Rf = 0$ in $A(Q)^*$. Consider the extension $\mathcal{E} : 0 \to U \to \widetilde{Q} \to Q \to 0$ where

$$\widetilde{Q}(u, w) = \big(u + \rho_W(w)u + f(w), Q(w)\big).$$

We have seen earlier that $\widetilde{Q}$ is Bockstein closed if and only if $\beta(f) + Rf = 0$ in $A(Q)^*$. So, $\mathcal{E}$ is an extension of Bockstein closed quadratic maps if and only if f is a cocycle.

Now, assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are two equivalent extensions. Let $\varphi : Q_1 \to Q_2$ be a morphism which makes the diagram (9) commute. Then, we can write $\varphi_W(u, w) = (u + a(w), w)$ and $\varphi_V(u, v) = (u + b(v), v)$. Let f_1 and f_2 be the cocycles corresponding to extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. Then, the identity $Q_2(\varphi_W(u, w)) = \varphi_V(Q_1(u, w))$ gives

$$\big(u + a(w) + \rho(w)\big(u + a(w)\big) + f_2(w), Q(w)\big) = \big(u + \rho(w)u + f_1(w) + b\big(Q(w)\big), Q(w)\big).$$

So, we have

$$f_2(w) + f_1(w) = \big(1 + \rho(w)\big)a(w) + b\big(Q(w)\big). \tag{10}$$

Thus $f_1 + f_2 = \delta(a)$ in $A(Q)^*$. Conversely, if $f_1 + f_2 = \delta(a)$ in $A(Q)^*$, then there is a $b : V \to U$ such that Eq. (10) holds, so we can define the morphism $\varphi : Q_1 \to Q_2$ as above so that the diagram (9) commutes. $\square$

5. LHS-spectral sequence for 2-power exact extensions

In this section we study the Lyndon–Hochschild–Serre (LHS) spectral sequence associated to a Bockstein closed 2-power exact extension. We first recall the definition of a 2-power exact extension.

Definition 5.1. A central extension of the form

$$E(Q) : 0 \to V \to G(Q) \to W \to 0$$

with corresponding quadratic map $Q : W \to V$ is called 2-*power exact* if the following conditions hold:

(i) $\dim(V) = \dim(W)$,
(ii) the extension is a Frattini extension, i.e., image of Q generates V, and
(iii) the extension is effective, i.e., $Q(w) = 0$ if and only if $w = 0$.

In this section, we calculate the mod-2 cohomology of $G(Q)$ using a LHS-spectral sequence when $E(Q)$ is a Bockstein closed 2-power exact extension. The mod-2 cohomology ring structure of 2-power exact groups has a simple form and it is not very difficult to obtain once certain algebraic lemmas are established. Similar calculations were given by Rusin [9, Lemma 8] and Minh and Symonds [4].

We first prove an important structure theorem concerning the k-invariants of Bockstein closed 2-power exact extensions.

Proposition 5.2. *Let $E(Q) : 0 \to V \to G(Q) \to W \to 0$ be a Bockstein closed, 2-power exact extension with $\dim(W) = n$. Then, the k-invariants $q_1, \dots, q_n$, with respect to some basis of V, form a regular sequence in $H^*(W, \mathbb{F}_2) = \mathbb{F}_2[x_1, \dots, x_n]$ and $A^*(Q) = \mathbb{F}_2[x_1, \dots, x_n]/(q_1, \dots, q_n)$ is a finite dimensional $\mathbb{F}_2$-vector space.*

Proof. We have shown in [7, Proposition 7.8] that the k-invariants $q_1, \dots, q_m$ form a regular sequence in $H^*(W; \mathbb{F}_2)$. This sequence is regular in any order. To show the second statement, let K denote the algebraic closure of $\mathbb{F}_2$. Since the dimension of the variety associated to $I(Q) = (q_1, \dots, q_m)$ is zero, the (projective) Nullstellensatz shows that $A^*(Q)$ is a nilpotent algebra (elements u in positive degree have $u^k = 0$ for some k, depending on u). However since $A^*(Q)$ is a finitely generated and commutative algebra, this shows that $A^*(Q)$ is finite dimensional as a vector space over K. $\square$

Recall that a regular sequence in a polynomial algebra is always algebraically independent (see [12, Proposition 6.2.1]). So, if $E(Q)$ is a Bockstein closed 2-power extension, then the subalgebra generated by the k-invariants $q_1, \dots, q_n$ is a polynomial algebra. We denote this subalgebra by $\mathbb{F}_2[q_1, \dots, q_n]$. We have the following:

Proposition 5.3. *Let $E(Q) : 0 \to V \to G(Q) \to W \to 0$ be a Bockstein closed, 2-power exact extension with $\dim(W) = n$. Then, $H^*(W; \mathbb{F}_2)$ is free as an $\mathbb{F}_2[q_1, \dots, q_n]$-module. As $\mathbb{F}_2[q_1, \dots, q_n]$-modules $H^*(W; \mathbb{F}_2) \cong \mathbb{F}_2[q_1, \dots, q_n] \otimes A^*(Q)$ where $A^*(Q)$ is given the trivial module structure.*

Proof. Let $P = \mathbb{F}_2[q_1, \dots, q_n]$ be the subalgebra generated by $q_1, \dots, q_n$ in $H^*(W; \mathbb{F}_2)$. Since $A^*(Q)$ is finite dimensional, $H^*(W; \mathbb{F}_2)$ is finitely generated over P. For example, if we take $\hat{A}^*(Q)$ an $\mathbb{F}_2$-vector subspace of $H^*(W; \mathbb{F}_2)$ mapping $\mathbb{F}_2$-isomorphically to $A^*(Q)$ under the projection $H^*(W; \mathbb{F}_2) \to A^*(Q)$, then a homogeneous $\mathbb{F}_2$-basis of $\hat{A}^*(Q)$ gives a set of generators of $H^*(W; \mathbb{F}_2)$ as a P-module.

To prove this, one inducts on the degree of $\alpha \in H^*(W;\mathbb{F}_2)$. Let $\{a_i \mid i \in I\}$ be an $\mathbb{F}_2$-basis of $\hat{A}^*(Q)$. We want to show α is in the P-span of the a_i. For degree of α equal to one or two, this is immediate since the degree of q_i is 2 for all i. In general, by subtracting a suitable linear combination of a_i's from α, we get an element β in the ideal $(q_1, \dots, q_n)$. To show α is in the P-span of the a_i, it is enough to show β is. However $\beta = \sum_{i=1}^n \beta_i q_i$ where the degree of the β_i is 2 less than the degree of β. By induction, each β_i and hence β is in the P-span of the a_i and so we are done.

Since $H^*(W;\mathbb{F}_2)$ is a polynomial algebra, it is trivially Cohen–Macaulay. Since $H^*(W;\mathbb{F}_2)$ is a finitely generated P-module, the fact that $H^*(W;\mathbb{F}_2)$ is a free P-module follows from the fact that P is a polynomial algebra (see Theorem 5.4.10 of [2] for example). Thus $H^*(W;\mathbb{F}_2)$ is a free P-module.

Finally if $\{b_j \mid j \in J\}$ is a basis for the free P-module $H^*(W;\mathbb{F}_2)$, then every element $\alpha \in H^*(W;\mathbb{F}_2)$ can be written uniquely in the form $\sum_{j\in J} \alpha_j b_j$ where $\alpha_j \in \mathbb{F}_2[q_1, \dots, q_n]$. Projecting to $A^*(Q)$, this shows every element of $A^*(Q)$ can be written uniquely as a span of the corresponding images of the b_j. In other words, the $\{b_j \mid j \in J\}$ projects to a basis of $A^*(Q)$. Thus one can define a map of $\mathbb{F}_2[q_1, \dots, q_n]$-modules

$$\mathbb{F}_2[q_1, \dots, q_n] \otimes A^*(Q) \to H^*(W;\mathbb{F}_2)$$

which is an isomorphism. $\square$

Now, consider the LHS-spectral sequence in mod-2 coefficients

$$E_2^{p,q} = H^p\big(W, H^q(V, \mathbb{F}_2)\big) \Rightarrow H^{p+q}\big(G(Q), \mathbb{F}_2\big)$$

associated to the extension $E(Q)$. Let $\{x_1, \dots, x_n\}$ denote a basis for the dual of W and let $\{t_1, \dots, t_n\}$ be a basis for the dual of V.

Lemma 5.4. *For each $1 \leqslant k \leqslant m$, we have $d_2(t_k) = q_k$.*

Proof. From standard theory, the central extension $E(Q)$ corresponds to a principal BV-bundle $BV \to BG(Q) \to BW$ with classifying map $f : BW \to BBV$ where $BBV = K(V, 2)$. Also $H^2(BBV, V) \cong \mathrm{Hom}(V, V)$ and $q = f^*(\mathrm{id})$ where $\mathrm{id} \in \mathrm{Hom}(V, V)$ is the identity map.

In the LHS-spectral sequence for the fibration $BV \to EBV \to K(V, 2)$, we have $d_2 : H^1(BV, V) \to H^2(BBV, V)$ is an isomorphism (since EBV is contractible). In fact, it identifies $H^1(BV, V) = H^2(BBV, V) = \mathrm{Hom}(V, V)$, so $d_2(\mathrm{id}) = \mathrm{id}$. This identity pulls back to our fibration $BV \to BG(Q) \to BW$ as $d_2(\mathrm{id}) = f^*(\mathrm{id}) = q$. From this the lemma easily follows after taking a basis for V. $\square$

Now we are ready for some computations.

Theorem 5.5. *Let $E(Q) : 0 \to V \to G(Q) \to W \to 0$ be a Bockstein closed, 2-power exact extension with $\dim(W) = n$. Then*

$$H^*\big(G(Q); \mathbb{F}_2\big) \cong \mathbb{F}_2[s_1, \dots, s_n] \otimes A^*(Q)$$

as graded algebras, where $deg(s_i) = 2$ for $i = 1, \dots, n$.

Proof. Consider the LHS-spectral sequence for the (central) extension $E(Q)$. The E_2-page has the form

$$E_2^{*,*} = H^*(W, \mathbb{F}_2) \otimes H^*(V, \mathbb{F}_2) = \mathbb{F}_2[x_1, \dots, x_n] \otimes \mathbb{F}_2[t_1, \dots, t_n].$$

We have previously seen that $d_2(t_k) = q_k$. Here we have chosen basis for W, V and their duals as done previously. Note that $d_2(t_k^2) = 0$ and $d_3(t_k^2) = \beta(q_k)$ by a standard theorem of Serre. Let

$$\bigwedge^*(t_1,\ldots,t_n)$$

be the $\mathbb{F}_2$-subspace of $H^*(V,\mathbb{F}_2)$ generated by the monomials $t_1^{\epsilon_1}\ldots t_n^{\epsilon_n}$ where $\epsilon_i = 0, 1$ for each $i = 1,\ldots,n$. Then,

$$H^*(V,\mathbb{F}_2) \cong \mathbb{F}_2\big[t_1^2,\ldots,t_k^2\big] \otimes \bigwedge^*(t_1,\ldots,t_n)$$

as $\mathbb{F}_2$-vector spaces. (Not as algebras!) Using Proposition 5.3, we also write

$$H^*(W,\mathbb{F}_2) \cong \mathbb{F}_2[q_1,\ldots,q_n] \otimes A^*(Q)$$

as vector spaces. Thus as a differential graded complex, $(E_2^{*,*}, d_2)$ splits as a tensor product

$$E_2^{*,*} \cong \mathbb{F}_2\big[t_1^2,\ldots,t_n^2\big] \otimes A^*(Q) \otimes \Big(\bigwedge^*(t_1,\ldots,t_n) \otimes \mathbb{F}_2[q_1,\ldots,q_n], d_2\Big)$$

where the differential on the first two tensor summands is trivial. By Künneth's theorem,

$$E_3^{*,*} \cong \mathbb{F}_2\big[t_1^2,\ldots,t_n^2\big] \otimes A^*(Q) \otimes H^*\Big(\bigwedge^*(t_1,\ldots,t_n) \otimes \mathbb{F}_2[q_1,\ldots,q_n], d_2\Big).$$

Again by Künneth's theorem, the final term is just $H^*(pt,\mathbb{F}_2)$ since it breaks up as the tensor of

$$H^*\Big(\bigwedge^*(t_j) \otimes \mathbb{F}_2[q_j], d_2(t_j) = q_j\Big)$$

which is the cohomology of a point. Thus

$$E_3^{*,*} = \mathbb{F}_2\big[t_1^2,\ldots,t_n^2\big] \otimes A^*(Q)$$

with $d_3(t_j^2) = \beta(q_j)$. Since Q is Bockstein closed, $\beta(q_j) = 0$ in $A^*(Q)$ and so $d_3(t_j^2) = 0$. Thus $E_3^{*,*} = E_4^{*,*}$. By dimensional considerations, there can be no further differentials in the spectral sequence and so we see $E_\infty^{*,*} = E_3^{*,*}$.

This says, in particular, that there are elements $s_i \in H^*(G,\mathbb{F}_2)$ such that

$$\mathrm{res}_V^G(s_i) = t_i^2.$$

Since the t_i's are algebraically independent in $H^*(V,\mathbb{F}_2)$, we conclude that the s_i's are algebraically independent in $H^*(G,\mathbb{F}_2)$. Thus when we define an $\mathbb{F}_2$-vector space homomorphism

$$\mathbb{F}_2[s_1,\ldots,s_n] \otimes A^*(Q) \to H^*(G,\mathbb{F}_2)$$

using the inclusion map on the first factor and the inflation map on the second, we will have a well-defined map of algebras. (Note that the inflation map is always an algebra map.) Finally by the structure of $E_\infty^{*,*}$, it is clear that this map is onto, and since our algebras have finite type, this means that it is an isomorphism of algebras as desired. $\square$

Remark 5.6. Note that although we assumed that the extension $E(Q)$ is 2-power exact in the above calculation, we only use the fact that the k-invariants $q_1,\ldots,q_n$ form a regular sequence. If $E(Q): 0 \to V \to G(Q) \to W \to 0$ is a Bockstein closed extension with $\dim V = n$, $\dim W = m$ such that the k-invariants $q_1,\ldots,q_n$ form a regular sequence, then the k-invariants will be algebraically independent in $\mathbb{F}_2[x_1,\ldots,x_m]$, in particular, they will be linearly independent. This shows that in this case, $E(Q)$ is a Frattini extension, i.e., $\{Q(w) \mid w \in W\}$ generates V. Also, we must have $n \leqslant m$ for dimension reasons. The assumption that Q is Bockstein closed will imply that the variety of $I(Q)$ is $\mathbb{F}_2$-rational, so W must have an $(m-n)$-dimensional subspace W' such that Q restricted to W' is zero. But it may happen that this W' has nontrivial commutators with the rest of the elements in W. So, we can not conclude that $G(Q)$ splits as $G(Q) \cong G' \times \mathbb{Z}/2$. Hence, these groups are still interesting and the calculation above shows that the mod-2 cohomology of these groups is also in the form $H^*(G, \mathbb{F}_2) \cong A^*(Q) \otimes \mathbb{F}_2[s_1, \dots, s_n]$.

The group extensions associated to the strictly upper triangular matrices $\mathfrak{u}_n(\mathbb{F}_2)$ are Bockstein closed 2-power exact extensions (see Example 3.3). So, we have a complete calculation for the mod-2 cohomology of these groups. To calculate the Bockstein's of the generators of mod-2 cohomology of a 2-power exact extension, we need to consider also the Eilenberg–Moore spectral sequence associated to the extension.

6. Eilenberg–Moore spectral sequence

In this section we study the Eilenberg–Moore spectral sequence of a Bockstein closed 2-power exact extension

$$E(Q): 0 \to V \to G(Q) \to W \to 0$$

associated to a quadratic map $Q: W \to V$. Although we have already found the algebra structure of the mod-2 cohomology of $G(Q)$ as

$$H^*\big(G(Q); \mathbb{F}_2\big) \cong \mathbb{F}_2[s_1, \dots, s_n] \otimes A^*(Q)$$

using the LHS-spectral sequence, we find the EM-spectral sequence more useful in studying the Steenrod algebra structure of $H^*(G(Q); \mathbb{F}_2)$. In fact, the theorems laid out in Larry Smith's paper [10] directly compute the behavior of the EM-spectral sequence in our case and give us the Steenrod structure we desire. In order to give a fuller picture of the underlying topology, we summarize some essentials of the EM-spectral sequence here.

Consider a pullback square of spaces

$$\begin{tikzcd} X \times_B Y \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

where $p: Y \to B$ is a fibration and B is simply-connected. The space $X \times_B Y$ is given as

$$X \times_B Y = \big\{(x, y) \in X \times Y \bigm| f(x) = p(y)\big\}$$

and can be viewed as an amalgamation of X and Y over B. Fix k a field, and let $C^*(X)$ denote the dga (differential graded algebra) given by the cochain complex of X with coefficients in k. One can show that the natural map $\alpha: C^*(X) \otimes_{C^*(B)} C^*(Y) \to C^*(X \times_B Y)$ yields an isomorphism

$$\mathrm{Tor}_{C^*(B)}\big(C^*(X), C^*(Y)\big) \cong H^*(X \times_B Y)$$

of algebras. (Though one should be careful in interpreting the algebra structure of the Tor term.) For more details on this isomorphism see [10, Theorem 3.2].

One then can show algebraically, through various filtrations of resolutions computing the Tor term above, that there is a spectral sequence starting at

$$E_2^{*,*} = \mathrm{Tor}_{H^*(B)}\big(H^*(X), H^*(Y)\big)$$

converging to $\mathrm{Tor}_{C^*(B)}(C^*(X), C^*(Y)) \cong H^*(X \times_B Y)$.

Explicitly, this can be constructed as a second quadrant spectral sequence in the (p,q)-plane using the **bar resolution**. The picture of the E_1-page using the bar resolution is as follows: on the $p = -n$ line, one has the algebra

$$H^*(X) \otimes \bar{H}^*(B) \otimes \cdots \otimes \bar{H}^*(B) \otimes H^*(Y)$$

where all tensor products are over k, $\bar{H}^*$ denotes the positive degree elements of H^*, and n factors of $\bar{H}^*(B)$ are used in the above tensor product. We place the graded complex above on the $p = -n$ line in such a way that the elements of total degree q are placed at the $(-n,q)$ lattice point in the (p,q)-plane. Let $[a|b_1|\cdots|b_n|c]$ be short hand for $a \otimes b_1 \otimes \cdots \otimes b_n \otimes c$. The differential d_1 is horizontal moving one step to the right and is given explicitly in characteristic 2 (we will only use this case and we do this also to avoid stating the signs!) by

$$\begin{aligned} d_1\big([a|b_1|\cdots|b_n|c]\big) &= \big[af^*(b_1)|b_2|\cdots|b_n|c\big] + \sum_{i=1}^{n-1}[a|b_1|\cdots|b_ib_{i+1}|\cdots|b_n|c] \\ &\quad + \big[a|b_1|\cdots|b_{n-1}|p^*(b_n)c\big] \end{aligned}$$

for all $a \in H^*(X)$, $b_i \in \bar{H}^*(B)$, $c \in H^*(Y)$.

The power of the EM-spectral sequence is the availability of this geometric resolution to represent its E_1 term. For example the $p = -1$ line can be interpreted as a portion of $H^*(X \times B \times Y)$. If A is the kernel of d_1 on this line, then A is the set of elements of the form $[a|b|c]$ satisfying

$$\big[af^*(b)|c\big] = \big[a|p^*(b)c\big],$$

and the elements of A are permanent cycles in the EM-spectral sequence. As shown in [10], there is a Steenrod module structure on the $p = -1$ and $p = 0$ lines via the natural identification of them inside $H^*(X \times B \times Y)$ and $H^*(X \times Y)$ respectively. Furthermore, this Steenrod module structure persists through all pages of the spectral sequence. Finally the associated filtration on $H^*(X \times_B Y)$ is a filtration of Steenrod modules and components of the associated graded module agree with the Steenrod module structure on the E_∞-page on the $p = 0$ and $p = -1$ lines of the EM-spectral sequence with the bar resolution. In fact various authors have shown that the whole EM-spectral sequence has a natural structure of a module over the Steenrod algebra (using the bar resolution model) with "vertical" and "diagonal" Steenrod operations (corresponding to the fact that the EM-spectral sequence is not an unstable module over $\mathcal{A}_2$). We won't need these here – we will only need the structure on the $p = 0$ and $p = -1$ lines which was already laid out in [10] very naturally.

The bar resolution mentioned above is a nice natural resolution that can be used to compute $E_2^{*,*} = \mathrm{Tor}_{H^*(B)}(H^*(X), H^*(Y))$ in the EM-spectral sequence and has the benefits of being "natural" and "geometric" and hence can be used to study natural operations on the spectral sequence. However typically this resolution is too big to carry out computations directly. In the case where $H^*(B)$ is a polynomial algebra, a smaller Koszul resolution is available to compute this tor-term and the E_2-page is more tractable. Finally if the ideal generated by $f^*(H^*(B))$ in $H^*(X)$ is nice enough, then a change of rings isomorphism can also be used to simplify the calculations immensely. For details see [10]. We will summarize the main result below after a necessary definition.

Definition 6.1. Let k be a field and Λ be a graded commutative algebra over k. An ideal I in Λ is called a *Borel ideal* if there is a regular sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$ (either finite or infinite) that generates the ideal I. Recall that a regular sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$ is one such that x_1 is not a zero divisor in Λ and x_{i+1} is not a zero divisor of $\Lambda/(x_1, \ldots, x_i)$ for all $i \geqslant 1$.

Theorem 6.2 *(Collapse theorem). Let*

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B \end{array}$$

be a pullback square of spaces, with B simply-connected, p a Serre fibration. Furthermore, assume that

(i) $p^* : H^*(B) \to H^*(Y)$ *is onto,*
(ii) $\ker(p^*)$ *is a Borel ideal of* $H^*(B)$*, and*
(iii) $\mathrm{Im}(f^*)$ *generates a Borel ideal J of* $H^*(X)$.

Then, the associated Eilenberg–Moore spectral sequence collapses at the E_2-page and we have

$$E_2^{*,*} = E_\infty^{*,*} \cong \big(H^*(X)/J\big) \otimes \Lambda^*(u_1, \ldots) \cong H^*(X \times_B Y)$$

where $\Lambda^*(u_1, \ldots)$ is an exterior algebra on generators all of which lie on the $p = -1$ line of the spectral sequence.

Proof. See Theorem 3.1 in [10]. □

Remark 6.3. Note that in the above theorem, the final isomorphism is an isomorphism of k-vector spaces in general due to lifting issues when lifting the algebra structure over the associated filtrations. We will come back to this in our specific example later.

In our case we are interested in studying the cohomology of groups $G(Q)$ given by central extensions

$$E(Q) : 0 \to V \to G(Q) \to W \to 0$$

where V and W are elementary abelian. We have seen that the category of such extensions is naturally equivalent to the category of quadratic maps from W to V. However it is basic in group cohomology that the equivalence classes of extensions of this form are uniquely specified by a cohomology class in $H^2(W, V)$. By the representability of cohomology, $H^2(W, V)$ is itself isomorphic to $[BW, BBV]$ where $[-,-]$ denotes homotopy classes of maps, BG denotes the classifying space of a monoid G, and $BBV = K(V, 2)$.

Under this correspondence we will denote the homotopy class representing a specific quadratic form $Q : W \to V$ by $BQ : BW \to BBV$. Thus we have a pullback square

$$\begin{array}{ccc} BG(Q) & \longrightarrow & EBV \\ \downarrow & & \downarrow{\scriptstyle p} \\ BW & \xrightarrow{BQ} & BBV \end{array}$$

for which we have an EM-spectral sequence

$$E_2^{*,*} = \mathrm{Tor}_{H^*(BBV)}\big(H^*(BW), \mathbb{F}_2\big) \Rightarrow H^*\big(G(Q), \mathbb{F}_2\big).$$

In fact the same pullback square above exhibits $BG(Q)$ as the homotopy fiber of the map BQ. We record these observations:

Proposition 6.4. *Let V, W be elementary abelian 2-groups. Then there is a bijective correspondence between equivalence classes of quadratic forms $Q : W \to V$ and $[BW, BBV]$. If $BQ : BW \to BBV$ is the homotopy class of maps associated to a quadratic form $Q : W \to V$ then $BG(Q) = HF(BQ)$ where HF stands for homotopy fiber.*

Similarly, there is a natural equivalence between n-ary forms $W \to V$ and homotopy classes of maps $[BW, B^nV]$ where $B^nV = K(V, n)$. However in this case the homotopy fiber is not an Eilenberg–MacLane space but has a two stage Postnikov tower.

It is well known that if $\dim(V) = 1$ then the characteristic element $\kappa \in H^n(K(V, n))$ is a generator of mod-2 cohomology $H^*(K(V, n))$ as a free module over the Steenrod algebra $\mathcal{A}_2$. As an algebra $H^*(K(V, n))$ is a polynomial algebra on all "permissible Steenrod operation sequences" on κ of excess less than n. The $\dim(V) > 1$ case follows similarly from Künneth's theorem. Using these facts, one sees that conditions (i) and (ii) in Theorem 6.2 hold trivially in this case. Thus it remains to consider condition (iii).

When Q is Bockstein closed, it is clear that the ideal J generated by $\mathrm{Im}(BQ)^*$ is exactly the ideal $I(Q)$ generated by the components of Q. This is because the Bockstein is Sq^1 and all other Steenrod squares on these components are determined as they have degree 2 and automatically lie in $I(Q)$. Thus in the case of a Bockstein closed quadratic form, J is a Borel ideal if and only if $I(Q)$ is. This yields the following theorem immediately.

Theorem 6.5 *(EM-SS collapse for $G(Q)$). Let $E(Q) : 0 \to V \to G(Q) \to W \to 0$ be a Bockstein closed extension associated to the quadratic map $Q : W \to V$. Assume that $I(Q)$ is a Borel ideal, i.e., $\{q_1, \dots, q_n\}$ is a regular sequence in $H^*(BW)$. This holds for example when Q is Bockstein closed and 2-power exact. Then, the EM-spectral sequence collapses at the E_2-page and*

$$E_2^{*,*} = E_\infty^{*,*} = A^*(Q) \otimes \Lambda^*(u_{ij}).$$

This lifts to give $H^*(G(Q), \mathbb{F}_2) \cong A^*(Q) \otimes \Lambda^*(\hat{u}_{ij})$ as $\mathbb{F}_2$-vector spaces (not as algebras). Here $A^*(Q) = H^*(W)/I(Q)$ and the u_{ij} are exterior Koszul generators corresponding to the polynomial subalgebra $\mathbb{F}_2[\kappa_{ij} \mid i = 1, \dots, n,\ j \geqslant 0]$ of $H^*(K(V, 2))$. Specifically, $\kappa_{ij} = Sq^{2^j} \cdots Sq^2 Sq^1 \kappa_i$ were κ_i is the i-th characteristic element in $H^2(K(V, 2))$. Thus u_{ij} lies in bidegree $(-1, 2^{j+1} + 1)$ in the EM-spectral sequence.

Proof. See the discussion above. The only thing that was not explained was the selection of the exterior elements. This comes from the fact that $(BQ)^*(\kappa_i) = q_i$ and a change of rings isomorphism. For details see [10]. □

The reader might have noticed that when we compare the result of the EM-spectral sequence calculation with the result previously obtained with the LHS-spectral sequence, they look different. The LHS-spectral sequence gave under the same conditions that

$$H^*\big(G(Q), \mathbb{F}_2\big) \cong A^*(Q) \otimes \mathbb{F}_2[s_1, \dots, s_n]$$

as algebras. However the EM-spectral sequence above shows that

$$H^*\big(G(Q), \mathbb{F}_2\big) \cong A^*(Q) \otimes \Lambda^*(\hat{u}_{ij})$$

as vector spaces. The point is this last isomorphism is not of algebras and the mentioned exterior algebra is isomorphic to the polynomial algebra $\mathbb{F}_2[s_1, \dots, s_n]$ as $\mathbb{F}_2$-vector spaces. Indeed note that $\hat{u}_{ij}$ lives in degree $(2^{j+1} + 1) - 1 = 2^{j+1}$. The vector space isomorphism comes from mapping $\hat{u}_{ij}$ to $s_i^{2^j}$ and extending to square free powers of the $\hat{u}_{ij}$ as if making an algebra map. To see this is indeed an isomorphism of graded vector spaces just note that a power of s_i say s_i^n can be expressed uniquely as a (square free) product of 2-power powers of s_i using the 2-adic expansion of n. For example $s_3^5 = s_3^4 s_3^1$ and hence would be associated to $\hat{u}_{32}\hat{u}_{30}$ since $2^2 = 4$ and $2^0 = 1$. We record this vector space isomorphism for reference.

Proposition 6.6 *(Exterior-polynomial vector space isomorphism). If*

$$\Lambda^* = \Lambda^*(\hat{u}_{ij} \mid i = 1, \dots, n,\ j \geqslant 0)$$

and $P^* = \mathbb{F}_2[s_1, \dots, s_n]$ *with* $|s_i| = 2$, $|\hat{u}_{ij}| = 2^{j+1}$ *then* $\Lambda^* \cong P^*$ *as graded vector spaces via the isomorphism that takes* $\hat{u}_{ij}$ *to* $s_i^{2^j}$ *extended to the canonical vector space basis of* Λ^* *in the obvious way.*

7. Proof of Theorem 1.1

In this section we prove Theorem 1.1 using the Eilenberg–Moore spectral sequence calculations given in the previous section.

Before the proof, we first discuss the Steenrod algebra structure of the EM-spectral sequence. For this purpose we will want to use the bar resolution and not the smaller Koszul resolutions used previously as it is more "geometric". This bar resolution has an induced Steenrod module structure on the $p = -1$ and $p = 0$ lines that induces the Steenrod module structure on $H^*(G(Q))$. This is proven as Corollary 4.4 in [10]. In fact, we will work out representatives of the u_{i0} in the bar resolution and work out a formula for Sq^1 on them. Just for dimensional reasons, if we throw the u_{i0} into a vector u_0, one has $Sq^1(u_0) = Ru_0 +$ terms in lower filtration where $R \in \mathrm{Hom}(W, \mathrm{End}(V))$, or intuitively, is a matrix with entries in $A^1(Q)$. By results in [10], this lifts to the identity $Sq^1(s) = Rs +$ term in $A^3(Q)$. Thus, if the calculation for $Sq^1(u_0)$ gives $R = L$, then we will have the proof of Theorem 1.1.

Let us first set out finding explicit representatives of the u_{i0} in the E_1 term of the bar resolution. Since we know they survive to $E_\infty^{-1,3}$ we know they will have to be permanent cycles.

In our case, the $p = -1$ line of the EM-spectral sequence (in the bar resolution) is $H^*(BW) \otimes \bar{H}^*(BBV) \otimes k$. Let us write $H^*(BW) = \mathbb{F}_2[x_1, \dots, x_n]$. Then $Sq^1(q_i) = \sum_j L_{ij} q_j$ where $L_{ij} \in H^1(W)$ since our quadratic form is Bockstein closed. Using the usual bar notation, let us define elements $v_i = [1|\kappa_{i0}|1] + \sum_j [L_{ij}|\kappa_j|1] \in E_1^{-1,3}$ where $\kappa_{i0} = Sq^1(\kappa_i)$ and κ_i is the i-th characteristic element of $H^*(BBV)$. Let us compute $d_1(v_i)$. Let $p : EBV \to BBV$ be the path loop fibration then

$$\begin{aligned} d_1(v_i) &= \big(\big[1(BQ)^*(\kappa_{i0})|1\big] + \big[1|p^*(\kappa_{i0})1\big]\big) + \sum_j \big(\big[L_{ij}(BQ)^*(\kappa_j)|1\big] + \big[L_{ij}|p^*(\kappa_j)1\big]\big) \\ &= \big[Sq^1(q_i)|1\big] + \sum_j [L_{ij}q_j|1] \\ &= \left[Sq^1(q_i) + \sum_j L_{ij}q_j|1\right] \\ &= 0. \end{aligned}$$

Since v_i lies on the $p = -1$ line, all further differentials on v_i must vanish for dimensional reasons and so indeed the v_i are permanent cycles. Also note that since $E_1^{-2,3} = 0$ for dimensional reasons, the vector space spanned by the v_i is not the image of d_1. Similarly for dimensional reasons it cannot be the image of any d_r. (This is because one has to use s tensors of $\bar{H}^*(BBV)$ in the line $p = -s$ and hence the line vanishes below $q = 2s$.) Thus the space spanned by the v_i embeds into $E_\infty^{-1,3}$. However comparing with the Koszul resolutions $E_\infty^{-1,3}$ term we see then that the span of the v_i must be the same as the span of the u_{i0} in E_∞. Thus, without loss of generality we may assume (by changing our basis for the span of the s_i) that v_i represents u_{i0}. Thus it remains to compute $Sq^1(v_i)$ to figure out $Sq^1(s_i)$ up to terms of lower filtration.

We now carry out the computation of $Sq^1(v_i)$ using Larry Smith's work showing that Sq^1 on $H^*(BW) \otimes \bar{H}^*(BBV) \otimes \mathbb{F}_2$ naturally induced by viewing that $p = -1$ line as $H^*(BW \times BBV \times pt)$

is compatible with the Sq^1 action on the cohomology of the total space $H^*(BG(Q),\mathbb{F}_2)$. Note $v_i = [1|\kappa_{i0}|1] + \sum_j [L_{ij}|\kappa_j|1]$. Here recall that $[a|b|c]$ can be identified with the cross product $a \times b \times c$ in $H^*(BW \times BBV \times pt)$. Using the Cartan formula for Steenrod squares on cross products, we see that Sq^1 will operate like a derivation and so we get

$$\begin{aligned} Sq^1(v_i) &= \bigl[1|Sq^1(\kappa_{i0})|1\bigr] + \sum_j \bigl(\bigl[Sq^1(L_{ij})|\kappa_j|1\bigr] + \bigl[L_{ij}|Sq^1(\kappa_j)|1\bigr]\bigr) \\ &= \sum_j \bigl(\bigl[L_{ij}^2|\kappa_j|1\bigr] + [L_{ij}|\kappa_{j0}|1]\bigr) \end{aligned}$$

since $Sq^1(\kappa_{i0}) = Sq^1(Sq^1(\kappa_i)) = 0$ and $Sq^1(L_{ij}) = L_{ij}^2$ since the degree of L_{ij} is one. Since the $p = -1$ line is a module over the $p = 0$ line, we can write

$$[L_{ij}|\kappa_{j0}|1] = L_{ij}[1|\kappa_{j0}|1] = L_{ij}\left(v_j + \sum_k [L_{jk}|\kappa_k|1]\right).$$

Plugging this into the above equations and simplifying, one gets

$$Sq^1(v_i) = \sum_j L_{ij} v_j + \left(\sum_j L_{ij}^2[1|\kappa_j|1] + \sum_j \sum_k L_{ij} L_{jk}[1|\kappa_k|1]\right).$$

Let v be the column vector with i-th coordinate v_i, and let L be the matrix with (i,j)-entry L_{ij} and c be the column vector with i-th coordinate $[1|\kappa_i|1]$. Then the above equation becomes

$$Sq^1(v) = Lv + \bigl(Sq^1(L) + L^2\bigr)c$$

where Sq^1 on a matrix just means apply it on each entry. In the next theorem, we will show that the second term $(Sq^1(L) + L^2)c$ in the above equation is zero in $E_2^{*,*} = E_\infty^{*,*}$ by showing that it is a boundary under d_1 and this then shows that $Sq^1(v) = Lv$ in $E_\infty^{*,*}$ which lifts to show $Sq^1(s) = Ls + \eta$ with η a column matrix with entries in $A^3(Q)$.

Theorem 7.1 *(Steenrod structure). Let $Q : W \to V$ be a Bockstein closed quadratic map whose components q_i form a regular sequence in $H^*(W)$ (for example if it is 2-power exact). Let $\dim(V) = n$, then $H^*(G(Q)) \cong k[s_1,\dots,s_n] \otimes A^*(Q)$ as algebras. If we write $\beta(q) = Sq^1(q) = Lq$ where q is the column vector with entries the components of Q (we can write this as Q is Bockstein closed) then we have*

$$\beta(s) = Ls + \eta$$

where η is a column vector with entries in $A^3(Q)$ and s is the column vector with entries the s_i. Since $A^*(Q)$ is the image of $H^*(BW) \to H^*(BG(Q))$, this determines the structure of $H^*(G(Q))$ over the Steenrod algebra completely as all Steenrod operations can be determined from this information and the axioms of the Steenrod algebra.

Proof. In the paragraph before the statement of this theorem, it was shown in the bar resolution that

$$Sq^1(v) = Lv + \bigl(Sq^1(L) + L^2\bigr)c$$

in $E_1^{-1,4}$ of the bar resolution model of the Eilenberg–Moore spectral sequence. Furthermore we showed that v consists of permanent cycles that survive and represent s. Thus to get the main formula of the theorem it is sufficient to show $(Sq^1(L) + L^2)c$ is zero in $E_2^{-1,4} = E_\infty^{-1,4}$ as this will give

$$Sq^1(v) = Lv$$

which will lift to

$$Sq^1(s) = Ls + \eta$$

where η's components have to live in lower filtration, i.e., on $E_\infty^{0,3} = A^3(Q)$. This will prove the theorem as the other statements have been proven in previous sections. Thus it remains to show that $(Sq^1(L) + L^2)c$ is a boundary under d_1.

Note that by the regularity of the sequence $q_1, \dots, q_n$, the equality $[\beta(L) + L^2]q = 0$ gives that

$$\beta(L) + L^2 = \sum_i T_i q_i$$

for some scalar matrices $T_1, \dots, T_n$ (see Proposition 8.2 in [7]). Note that $[B(L) + L^2]q = 0$ gives that for all k,

$$\sum_i \sum_j T_i(k, j) q_i q_j = 0.$$

Hence, $T_i(k,i) = 0$ for every k and i, and $T_i(k, j) = T_j(k,i)$ for every $1 \leqslant i, j \leqslant n$. Here we are using the fact that the set

$$\{q_i q_j \mid 1 \leqslant i < j \leqslant n\}$$

is linearly independent in $H^*(W)$ as the q_i are algebraically independent since they form a regular sequence. Let us call this common entry

$$a_k(i, j) = T_i(k, j) = T_j(k,i).$$

The a_k are then (skew) symmetric scalar matrices with zeros down the diagonal as $a_k(i,i) = T_i(k,i) = 0$. Since we are in characteristic 2, we can view these as skew symmetric matrices which will be useful in what follows.

Let

$$\alpha_k = \sum_{i,j} a_k(i, j)[1|\kappa_i|\kappa_j|1]$$

in $E_1^{-2,4}$. Then,

$$d_1(\alpha_k) = \sum_{i,j} a_k(i, j)\big([q_i|\kappa_j|1] + [1|\kappa_i\kappa_j|1]\big) = \sum_{i,j} a_k(i, j)[q_i|\kappa_j|1]$$

as the second term in the first sum is zero due to the skew symmetry of the matrices a_k. Thus using that the $p = -1$ line is a module over the $p = 0$ line with an action that is purely in the left slot we get

$$d_1(\alpha_k) = \sum_{ij} a_k(i, j) q_i[1|\kappa_j|1].$$

Using that $\sum_i a_k(i,j)q_i = \sum_i T_i(k,j)q_i$ is the (k,j)-entry of $B(L)+L^2$ we get

$$d_1(\alpha_k) = \sum_j \big(B(L)+L^2\big)_{k,j}[1|\kappa_j|1].$$

Writing c to be the column vector with components $[1|\kappa_j|1]$ as before and α to be the column vector with entries the α_k, then this reads as

$$d_1(\alpha) = \big(B(L)+L^2\big)c$$

which shows that indeed the desired term is a d_1 boundary and hence completes the proof of the theorem. $\square$

8. An obstruction class for uniform double lifting

Let $E(Q): 0 \to V \to G(Q) \to W \to 0$ be a Bockstein closed 2-power exact extension with associated quadratic map $Q: W \to V$. Recall that this means $\dim W = \dim V = n$, and the quadratic map Q is Frattini and effective. Note that in this case k-invariants $q_1,\dots,q_n$ form a regular sequence, and as a consequence there is a unique $L \in \mathrm{Hom}(W, \mathrm{End}(V))$ such that $\beta(q) = Lq$ (see Proposition 8.1 of [7]).

First we make a simple observation about L. Note that applying the Bockstein operator to the equation $\beta(q) = Lq$ gives

$$0 = \beta(L)q + L\beta(q) = \big[\beta(L)+L^2\big]q.$$

Since $q_1,\dots,q_n$ forms a regular sequence, the entries of $\beta(L)+L^2$ must be linear combinations of components of q (see Proposition 8.2 in [7]). This means that linear map $L: W \to \mathrm{End}(V)$ extends to a morphism $\rho_L: Q \to Q_{\mathfrak{gl}(V)}$ in **Quad**. So, L defines a representation of Q. Let us denote this representation also with L.

Now, we consider the cohomology ring of the group $G(Q)$. By Theorem 5.5, we have

$$H^*\big(G(Q);\mathbb{F}_2\big) \cong \mathbb{F}_2[s_1,\dots,s_n] \otimes A^*(Q)$$

as graded algebras, where $\deg(s_i) = 2$ for $i = 1,\dots,n$. In the previous section, we calculated the images of Bockstein operator on s_i's. We obtained that

$$\beta(s) = Ls + \eta$$

where η is a column vector whose entries are in $H^3(W,\mathbb{F}_2)$. Applying the Bockstein operator again, we get

$$0 = \beta(L)s + L\beta(s) + \beta(\eta) = \big[\beta(L)+L^2\big]s + \big[\beta(\eta)+L\eta\big].$$

Since, we already know that $\beta(L)+L^2 = 0$ in $A^*(Q)$, we obtain that

$$\beta(\eta) + L\eta = 0$$

in $A(Q)^*$. This shows that η is a 3-dimensional cocycle in the chain complex $C^*(Q,L)$. So, it defines a 3-dimensional cohomology class $[\eta] \in H^3(Q,L)$. We will now show that this class is an obstruction class for uniformly lifting the extension $E(Q)$ twice.

Let $E': 0 \to V \to \Gamma \to G \to 0$ denote the extension with extension class $s \in H^2(G,V)$ where $G = G(Q)$. Since $\mathrm{res}^G_V s_i = t_i^2$ for all i, the restriction of this extension to $V \subseteq G$ gives the extension

$0 \to V \to M \to V \to 0$ where $M \cong (\mathbb{Z}/4)^n$ as an abelian group. Let $0 \to V \to M \to V \to 0$ be the $\mathbb{Z}/4W$-lattice whose logarithm is $L : W \to \mathrm{End}(V)$. By this we mean that the representation for M is of the form $w \to I + 2L(w)$ mod 4 (see Lemma 4.3 [7] for details). By Theorem 4.6 in [7], E' lifts to an extension

$$E'' : 0 \to M \to \widetilde{\Gamma} \to G \to 0$$

if and only if $\beta(s) = Ls$. So, $\eta = \beta(s) + Ls$ is the obstruction for such a lifting. But, the extension E' is determined by our choices of generators s_i. We can replace each s_i with

$$s_i' = s_i + \xi_i$$

for some $\xi_i \in H^1(W, \mathbb{F}_2)$ and the resulting extension E' will still satisfy the above condition. It may happen that for this new extension E', the obstruction for lifting it to E'' is zero. Let us calculate the new obstruction

$$\eta' = \beta\bigl(s'\bigr) + Ls' = \beta(s + \xi) + L(s + \xi) = \eta + \bigl(\beta(\xi) + L\xi\bigr).$$

Note that the class $\beta(\xi) + L\xi$ is equal to $\delta(\xi)$ where δ is the boundary map of the chain complex $C^*(Q, L)$. So, we proved that in the set of all possible extensions $E' : 0 \to V \to \Gamma \to G \to 0$ whose restriction to $V \subseteq G$ gives the extension $0 \to V \to M \to V \to 0$, there is at least one extension E' that lifts to E'' of the form

$$E'' : 0 \to M \to \widetilde{\Gamma} \to G \to 0$$

if and only if η is a coboundary, i.e., the cohomology class $[\eta] = 0$ in $H^3(Q, L)$. So, the proof of Theorem 1.2 is complete.

9. The Bockstein spectral sequence

Fix a prime p and let $\mathbb{F}_p$ be the field with p elements. The Bockstein spectral sequence is a well-known technique of relating the mod-p cohomology of a space with its integral cohomology. It arises from the Massey exact triple

$$\begin{tikzcd} H^*(X;\mathbb{Z}) \arrow[rr, "\cdot p"] & & H^*(X;\mathbb{Z}) \arrow[dl, "\varphi"] \\ & B_1^* = H(X, \mathbb{F}_p) \arrow[ul, "\hat{\beta}"] & \end{tikzcd}$$

associated to the sequence of coefficients $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_p \to 0$. Here $\hat{\beta}$ is the integral Bockstein operator and φ is the map induced by mod p reduction. The composition $\varphi \circ \hat{\beta}$ yields the differential given by the standard Bockstein β on $H^*(X; \mathbb{F}_p)$. If we denote the cohomology of this differential by B_2^* then the Massey exact triple above yields an exact triple

$$\begin{tikzcd} pH^*(X;\mathbb{Z}) \arrow[rr, "\cdot p"] & & pH^*(X;\mathbb{Z}) \arrow[dl, "\varphi_2"] \\ & B_2^* \arrow[ul, "\hat{\beta}_2"] & \end{tikzcd}$$

and by iteration we get Massey exact triples of the form

$$\begin{tikzcd} p^r H^*(X;\mathbb{Z}) \arrow[rr, "\cdot p"] & & p^r H^*(X;\mathbb{Z}) \arrow[dl, "\varphi_{r+1}"] \\ & B^*_{r+1} \arrow[ul, "\hat{\beta}_{r+1}"] & \end{tikzcd}$$

where $\varphi_{r+1} \circ \hat{\beta}_{r+1} = \beta_{r+1}$ gives a differential on B^*_{r+1} called the $(r+1)$-st higher Bockstein. The collection of the Massey exact triangles above is referred to as the *Bockstein spectral sequence* for X at the prime p.

Furthermore when X is the classifying space of a finite p-group, it is known that $B^*_\infty = H^*(pt, \mathbb{F}_p)$ is the limit of the B^*_r. Note if $s > 0$ and $\exp(H^s(X;\mathbb{Z})) = p^{r+1}$, then the multiplication by p in this last diagram is zero. Hence φ_{r+1} gives an injection $0 \neq p^r H^s(X;\mathbb{Z}) \to B^s_{r+1}$ and hence $B^s_{r+1} \neq 0$. On the other hand it is clear that $B^s_{r+2} = 0$ from the next Massey triangle. Thus it is not hard to see that $\exp(H^s(X;\mathbb{Z})) = p^{r+1}$ if and only if $B^s_{r+1} \neq 0$ but $B^s_{r+2} = 0$. So, by seeing how quickly any level of the Bockstein spectral sequence converges to zero, we can determine the exponent of the integral cohomology of the p-group at that level.

Now we have seen that if $Q : W \to V$ is a quadratic map associated to a Bockstein closed 2-power exact sequence $E(Q) : 0 \to V \to G(Q) \to W \to 0$ which lifts uniformly twice then

$$H^*\big(G(Q);\mathbb{F}\big) \cong \mathbb{F}[s_1, \ldots, s_n] \otimes A^*(Q)$$

where $n = \dim(W)$. As Q is Bockstein closed, we may write $\beta q = Lq$ for $L \in \mathrm{Hom}(W, \mathrm{End}(V))$ as usual and we have seen in this case we can choose s so that $\beta s = Ls$ also. Thus $(B^*_1(G(Q)), \beta_1)$ is completely known from the matrix L given from the Bockstein closed quadratic form Q.

To compute B^*_2, we decompose the polynomial algebra $\mathbb{F}[s_1, \ldots, s_n] = \bigoplus_{i=0}^\infty S^i$ into its homogeneous components S^i. We then get a decomposition of differential algebras:

$$\big(B^*_1\big(G(Q)\big), \beta_1\big) = \bigoplus_{i=0}^{\infty} \big(S^i \otimes A^*(Q), \beta\big)$$

and thus

$$B^*_2\big(G(Q)\big) = \bigoplus_{i=0}^{\infty} H^*\big(S^i \otimes A^*(Q), \beta\big).$$

Note that $B^*_2(G(Q))$ contains the information for the distribution of all the higher torsion (of exponent greater than p) for $H^*(G(Q);\mathbb{Z})$. Note that $S^0 = \mathbb{F}_2$, so the term corresponding to $i = 0$ is just the cohomology of $A^*(Q)$ under β which is $H^*(Q;\mathbb{F}_2)$.

Note that for the summand coming from S^1, we have a differential given by $\delta(f) = \beta f + L(f)$ and so the cohomology of that term corresponds to $H^*(Q;L)$ where L denotes the Q-module associated to the matrix L. However since the elements of S^1 are degree 2 elements, the contribution of $H^*(Q;L)$ will be in degree $*+2$. Since S^i is just the i-fold symmetric product of S^1, the complexes for these terms are derived from those for S^1, and we will denote the corresponding matrix and Q-module by $\mathrm{Sym}^i(L)$. Note also that since the elements of S^i have degree $2i$, their contribution will be shifted in degree by $2i$. By these observations we can conclude the following:

Theorem 9.1. *Let $G(Q)$ be a group associated to a Bockstein closed, 2-power exact quadratic form with $\beta(q) = Lq$. Furthermore suppose $Q : W \to V$ can be lifted uniformly twice. Then*

$$H^*\big(G(Q);\mathbb{F}_2\big) \cong \mathbb{F}_2[s_1, \ldots, s_n] \otimes A^*(Q)$$

where $n = \dim(W)$ *and the higher torsion is given by*

$$B_2^* \cong \bigoplus_{i=0}^{\infty} H^{*-2i}\big(Q, \mathrm{Sym}^i(L)\big).$$

Thus, the higher torsion in $G(Q)$ *is computable from the cohomology of the quadratic form* Q *in coefficients given by various symmetric powers of the defining module* L.

Note that for each $j \geqslant 0$, the j-th term B_2^j in the above formula is given by a finite direct sum since there are only a finite number of nonzero coefficient modules $\mathrm{Sym}^i(L)$ in the sum. Furthermore, note that the cochain complex used to compute $H^*(Q, \mathrm{Sym}^i(L))$ has length always given by the length of the cochain complex $A^*(Q)$ which is finite. Thus the equation above does yield an algorithm to compute B_2^j for every $j \geqslant 0$.

Acknowledgment

We thank the referee for careful reading of the paper and for many helpful comments.

References

[1] A. Adem, Cohomological exponents of $\mathbb{Z}G$-lattices, J. Pure Appl. Algebra 58 (1989) 1–5.
[2] D.J. Benson, Representations and Cohomology II: Cohomology of Groups and Modules, Cambridge Stud. Adv. Math., vol. 31, Cambridge Univ. Press, 1998.
[3] W. Browder, J. Pakianathan, Cohomology of uniformly powerful p-groups, Trans. Amer. Math. Soc. 352 (2000) 2659–2688.
[4] P.A. Minh, P. Symonds, The cohomology of pro-p groups with a powerfully embedded subgroup, J. Pure Appl. Algebra 189 (2004) 221–246.
[5] J. Pakianathan, Exponents and the cohomology of finite groups, Proc. Amer. Math. Soc. 128 (1999) 1893–1897.
[6] J. Pakianathan, N. Rogers, Higher torsion in p-groups, Casimir operators and the classifying spectral sequence of a Lie algebra, preprint, 2010.
[7] J. Pakianathan, E. Yalçın, Quadratic maps and Bockstein closed group extensions, Trans. Amer. Math. Soc. 359 (2007) 6079–6110.
[8] D. Rector, Steenrod operations in the Eilenberg–Moore spectral sequence, Comment. Math. Helv. 45 (1970) 540–552.
[9] D. Rusin, The mod 2 cohomology of metacyclic 2-groups, J. Pure Appl. Algebra 44 (1987) 315–327.
[10] L. Smith, Homological algebra and the Eilenberg–Moore spectral sequence, Trans. Amer. Math. Soc. 129 (1967) 58–93.
[11] L. Smith, On the Künneth Theorem I, Math. Z. 116 (1970) 94–140.
[12] L. Smith, Polynomial Invariants of Finite Groups, A.K. Peters, London, 1995.