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On Waring–Goldbach problem with Piatetski-Shapiro primes ☆

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In this paper, we show that all sufficiently large natural numbers satisfying certain local conditions can be written as the sum of kth powers of Piatetski-Shapiro primes, thereby establishing a variant of Waring–Goldbach problem with primes from a sparse sequence.

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1. Introduction

We define, for a natural number k, and a prime p, $\vartheta = \vartheta(p,k)$ to be the largest natural number such that $p^\vartheta \mid k$, and define $\gamma(p,k)$ by

$$\gamma = \gamma(p,k) = \begin{cases} \vartheta + 2, & \text{if } p = 2 \text{ and } 2 \mid k, \\ \vartheta + 1, & \text{otherwise.} \end{cases}$$

We then put $K(k) = \prod_{(p-1)|k} p^\gamma$.

Let $H_c(k)$ denote the smallest number of variables s such that every sufficiently large integer $n \equiv s \,(\text{mod K(k)})$ can be written in the form

$$n = p_1^k + \cdots + p_s^k, \qquad \text{with } p_1, \ldots, p_s \in \mathcal{A}_c, \tag{1.1}$$

where $p_1, \ldots, p_s$ are *primes* that lie in the set

$$\mathcal{A}_c = \{\lfloor m^c \rfloor : m \in \mathbb{N}\}.$$

This set is named after *I.I. Piatetski-Shapiro*, since he was the first to prove an analog of the Prime Number Theorem (cf. [10]) for primes in $\mathcal{A}_c$ for $c \in (1, 12/11)$.

An asymptotic formula for the number of representations of n as in (1.1) was given by the authors in [1]. In this paper, we intend to give an upper bound for $H_c(k)$ using the recent result of Kumchev and Wooley (cf. [8]). Before we state our result, we shall give some definitions from their paper.

Put $\theta = 1 - 1/k$, $\sigma_k^{-1} = k(k-1)$, $t = \lceil \frac{1}{2} k \log k \rceil$ and $u = \lceil k \log(k^2/2) \rceil - t$. Then, define

$$\begin{aligned} \lambda_i &= \left(\theta + \sigma_{k-1}/k\right)^{i-1} \qquad (1 \leqslant i \leqslant u+1), \\ \lambda_{u+2} &= \frac{k^2 - \theta^{t-3}}{k^2 + k - k\theta^{t-3}} \lambda_{u+1}, \\ \lambda_{u+j} &= \frac{k^2 - k - 1}{k^2 + k - k\theta^{t-3}} \theta^{j-3} \lambda_{u+1} \qquad (3 \leqslant j \leqslant t). \end{aligned}$$

Finally, set $\Lambda = \sum_{i \leqslant u+t} \lambda_i$, $v = \lceil (k - \Lambda)/2\sigma_k \rceil$ and $\lambda = \lambda_{u+t}$. Note that λ is the minimum of the λ_i's.

Theorem 1.1. *For sufficiently large k,*

$$H_c(k) \leqslant (4k-2)\log k - (2\log 2 - 1)k - 1,$$

whenever

$$1 < c < 1 + \frac{\lambda}{152(\Lambda + v) + 13\lambda}.$$

Remark 1.2. It is computed in [8] that

$$\frac{k-\Lambda}{2\sigma_k} = \frac{k}{2} + \log(\sqrt{2}/k) - 3 - \gamma + O(k^{-1/2}),$$

where γ is the fractional part of $u+t$. We can also estimate λ easily to see that $k^2\lambda \in [1.9, 2.1]$. Thus, the length of the interval that c lies in is of order k^{-3}.

2. Preliminaries and notation

2.1. Notation

Throughout the paper, the letters k, m and n are natural numbers with $k \geqslant 4$, and p always denotes a prime number. The notation $x \sim X$ means that $X < x \leqslant 2X$ for any real number X. Furthermore, $c > 1$ is a fixed real number and we put $\delta = 1/c$.

Given a real number x, we write $\mathbf{e}(x) = e^{2\pi i x}$, $\{x\}$ for the fractional part of x, $\lfloor x \rfloor$ for the greatest integer not exceeding x. We write $\mathcal{L} = \log N$.

For any function f, we put

$$\Delta f(x) = f\left(-(x+1)^{\delta}\right) - f(-x^{\delta}), \qquad (x > 0).$$

We recall that for functions F and real nonnegative G the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leqslant \alpha G$ holds for some constant $\alpha > 0$. If $F \geqslant 0$ also, then $F \gg G$ is equivalent to $G \ll F$. We also write $F \asymp G$ to indicate that $F \ll G$ and $G \ll F$. In what follows, any implied constants in the symbols $\ll$ and O may depend on the parameters c, ε, k, s, but are absolute otherwise. We shall frequently use ε with a slight abuse of notation to mean a small positive number, possibly a different one each time.

2.2. Preliminaries

Lemma 2.1 *(Vaaler [3, Appendix]). Put $\psi(x) = x - \lfloor x \rfloor - 1/2$. Then, there exists a trigonometric polynomial*

$$\psi^*(x) = \sum_{1 \leqslant |h| \leqslant H} a_h e(hx), \qquad (a_h \ll |h|^{-1})$$

such that for any real x,

$$|\psi(x) - \psi^*(x)| \leqslant \sum_{|h| < H} b_h e(hx), \qquad (b_h \ll H^{-1}).$$

Lemma 2.2 *(Vaughan's Identity [4, Prop. 13.4]). Let $u, v \geqslant 1$ be real numbers. If $n > v$ then,*

$$\Lambda(n) = \sum_{\substack{rs=n \\ r\leqslant u}} \mu(r)\log s - \sum_{\substack{rs=n \\ r>v \\ s>u}} \Lambda(r) \sum_{\substack{d|s \\ d\leqslant u}} \mu(d) - \sum_{\substack{rst=n \\ r\leqslant u \\ s\leqslant v}} \mu(r)\Lambda(s),$$

where Λ is von Mangoldt's function.

Definition 2.3. We put

$$g_k(\alpha, x) = \sum_{p\sim x} \mathbf{e}(\alpha p^k), \qquad f_k(\alpha, x) = \sum_{\substack{p\sim x \\ p\in\mathcal{A}_c}} cp^{1-\delta}\mathbf{e}(\alpha p^k).$$

Lemma 2.4. *Assume that $c > 1$, and there are coprime integers a and q with $0 \leqslant a \leqslant q \leqslant P$ such that $|q\alpha - a| \leqslant Px^{-k}$. Then,*

$$f_k(\alpha, x) = g_k(\alpha, x) + O(x^{(44-14\delta)/31} P^{7/31})$$

for sufficiently large x.

Proof. We shall assume below that $c \in (1, 14/13)$ and $P \leqslant x^{(18-17\delta)/7}$, since otherwise the given error is worse than the trivial estimate.

The function $\lfloor -n^\delta \rfloor - \lfloor -(n+1)^\delta \rfloor$ serves as the characteristic function of the set $\mathcal{A}_c$, and it can be rewritten as

$$\lfloor -n^\delta \rfloor - \lfloor -(n+1)^\delta \rfloor = \delta n^{\delta-1} + \Delta\psi(n) + O(n^{\delta-2}).$$

Thus,

$$f_k(\alpha, x) = g_k(\alpha, x) + \sum_{p\sim x} cp^{1-\delta}\mathbf{e}(\alpha p^k)\Delta\psi(p) + O\big(1/\log x\big).$$

By Lemma 2.1

$$\sum_{p\sim x} cp^{1-\delta}\mathbf{e}(\alpha p^k)\Delta(\psi - \psi^*)(p) \ll H^{-1}x^{2-\delta} + H^{-1} \sum_{1\leqslant h<H} \Big| \sum_{n\sim x} n^{1-\delta}\mathbf{e}(hn^\delta) \Big|.$$

Partial integration yields

$$\sum_{n\sim x} n^{1-\delta}\mathbf{e}(hn^\delta) \ll x^{1-\delta} \sup_{t\sim x} \Big| \sum_{x<n\leqslant t} \mathbf{e}(hn^\delta) \Big|.$$

Using the exponent pair $(1/2, 1/2)$ (cf. [3, eqn. 3.3.4]) we obtain the estimate

$$\sum_{x<n\leqslant t} \mathbf{e}(hn^\delta) \ll h^{1/2}x^{\delta/2} + h^{-1}x^{1-\delta} \qquad (h > 0, t \sim x).$$

Thus, assuming $1 \leqslant H \leqslant x$, we conclude that

$$\sum_{p\sim x} cp^{1-\delta}\mathbf{e}(\alpha p^k)\Delta(\psi-\psi^*)(p) \ll H^{-1}x^{2-\delta}+H^{1/2}x^{1-\delta/2}. \tag{2.1}$$

Next, we observe that the sum

$$\sum_{p\sim x} cp^{1-\delta}\mathbf{e}(\alpha p^k)\Delta\psi^*(p)$$

is

$$\ll \frac{1}{\log x}\sup_{t\sim x}\left|\sum_{x<n\leqslant t} cn^{1-\delta}\mathbf{e}(\alpha n^k)\Lambda(n)\Delta\psi^*(n)\right| + O\big(x^{3/2-\delta}\big),$$

where Λ is von Mangoldt's function. Recalling the definition of ψ^*, noting that $a_h \ll |h|^{-1}$ and that $\phi_h(t)=\mathbf{e}\left(h(t+1)^\delta - ht^\delta\right)-1$ satisfies

$$\phi_h(t)\ll |h|t^{\delta-1}, \qquad \phi_h'(t)\ll |h|t^{\delta-2},$$

we derive by partial integration that

$$\sum_{x<n\leqslant t} cn^{1-\delta}\mathbf{e}(\alpha n^k)\Lambda(n)\Delta\psi^*(n) \ll \sup_{x<z\leqslant t}\sum_{1\leqslant|h|\leqslant H}|F_h(z)|$$

where

$$F_h(z)=\sum_{x<n\leqslant z}\Lambda(n)\mathbf{e}(\alpha n^k+hn^\delta).$$

We have shown so far

$$\sum_{p\sim x} cp^{1-\delta}\mathbf{e}(\alpha p^k)\Delta\psi^*(p) \ll \frac{1}{\log x}\sup_{z\sim x}\sum_{1\leqslant|h|\leqslant H}|F_h(z)| + O\big(x^{3/2-\delta}\big). \tag{2.2}$$

Assume that there exist coprime integers a, q with $0\leqslant a\leqslant q\leqslant P$ such that $|q\alpha-a|\leqslant Px^{-k}$. Then,

$$F_h(z)=q^{-1}\sum_{-q/2<b\leqslant q/2} S(a,b;q)\sum_{x<n\leqslant z}\Lambda(n)\mathbf{e}(G_b(n)), \tag{2.3}$$

where $G_b(t)=\beta t^k+ht^\delta-bt/q$, $\beta=\alpha-a/q$ and

$$S(a,b;q)=\sum_{m=1}^{q}\mathbf{e}\left(\frac{am^k+mb}{q}\right).$$

We shall apply Lemma 2.2 with $u, v \geqslant 1$ and $1 \leqslant uv \leqslant x$ to write

$$\sum_{x<n\leqslant z} \Lambda(n)\mathbf{e}(G_b(n)) = \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where

$$\Sigma_1 = \sum_{r\leqslant u} \mu(r) \sum_{x/r<s\leqslant z/r} \mathbf{e}(G_b(rs)) \log s$$
$$- \sum_{r\leqslant u} \Bigg(\sum_{\substack{r=wt\\ w\leqslant u\\ t\leqslant v}} \mu(w)\Lambda(t) \Bigg) \sum_{x/r<s\leqslant z/r} \mathbf{e}(G_b(rs)),$$

and

$$\Sigma_2 = - \sum_{\substack{x<rs\leqslant z\\ s>v\\ r>u}} \Lambda(s) \sum_{\substack{d|r\\ d\leqslant u}} \mu(d)\mathbf{e}(G_b(rs)),$$
$$\Sigma_3 = - \sum_{\substack{x<rs\leqslant z\\ u<r\leqslant uv}} \Bigg(\sum_{\substack{r=wt\\ w\leqslant u\\ t\leqslant v}} \mu(w)\Lambda(t) \Bigg) \mathbf{e}(G_b(rs)).$$

By partial summation

$$\Sigma_1 \ll \log x \sum_{r\leqslant u} \sup_{z\sim x} \Bigg| \sum_{x/r<s\leqslant z/r} \mathbf{e}(G_b(rs)) \Bigg|.$$

We have

$$\frac{\partial^2 G_b(rs)}{\partial s^2} = k(k-1)\beta r^k s^{k-2} + h\delta(\delta-1)s^{\delta-2}r^{\delta}.$$

Note that since $P \leqslant x^{(18-17\delta)/7}$ and $c < 4/3$, it follows that $P = o(x^{\delta})$. Furthermore, $|\beta| \leqslant Px^{-k}$. Thus, the second term above dominates the first for sufficiently large x when $s \sim x/r$; that is, $f''(s) \asymp r^2 x^{\delta-2}|h|$, where $f(s) = G_b(rs)$. Applying van der Corput's estimate in [4, Cor. 8.13] to $f(s)$ on $(x/r, z/r]$, we conclude that

$$\Sigma_1 \ll \log^2 x \left(u|h|^{1/2}x^{\delta/2} + x^{1-\delta/2}|h|^{-1/2} \right). \tag{2.4}$$

Next, using dyadic division we can write

$$\Sigma_2 \ll x^{\varepsilon} \sum_{R,S} |T(R,S)|$$

where

$$T(R,S)=\sum_{r\sim R}\gamma_r\sum_{\substack{s\sim S\\ x<rs\leqslant z}}\lambda_s\mathbf{e}(G_b(rs))$$

with $R>u$, $S>v$, $RS\asymp x$, and $|\lambda_s|,|\gamma_r|\leqslant 1$. We note that $\Sigma_3\ll\log x\sum_{R,S}|T(R,S)|$, where $T(R,S)$ is a similar bilinear sum with different coefficients and R,S satisfy $u<R\leqslant uv$, $RS\asymp x$. To estimate $T(R,S)$, we apply Weyl–van der Corput inequality (cf. [3, Lemma 2.5]) to get

$$T(R,S)^2\ll\frac{(RS)^2}{L}+\frac{RS^2}{L}\sum_{1\leqslant|\ell|\leqslant L}\max_{S<s,s+\ell\leqslant 2S}|\Gamma(\ell,s,R)|,\tag{2.5}$$

where $1\leqslant L\leqslant S$ is to be chosen optimally, and

$$\Gamma(\ell,s,R)=\sum_{r\in I}\mathbf{e}\left(G_b(r(s+\ell))-G_b(rs)\right).$$

Here, $I\subseteq(R,2R]$ is an interval determined by the conditions $r\sim R$, $x<sr$, $(s+\ell)r\leqslant z$. We have

$$G_b(r(s+\ell))-G_b(rs)=\beta r^k\left((s+\ell)^k-s^k\right)+hr^\delta\left((s+\ell))^\delta-s^\delta\right)-br\ell/q.$$

Thus, we conclude that when $r\sim R$ and for sufficiently large x,

$$\left|\frac{\partial^2\left(G_b(r(s+\ell))-G_b(rs)\right)}{\partial s^2}\right|\asymp x^{\delta-1}|h\ell|R^{-1}.$$

Applying [4, Cor. 8.13] once again, we obtain

$$\Gamma(\ell,s,R)\ll R^{1/2}\big((|h\ell|x^{\delta-1})^{1/2}+(x^{1-\delta}|h\ell|^{-1})^{1/2}\big).$$

Inserting this bound in (2.5) and using [3, Lemma 2.4] to choose $L\in[1,S]$ optimally we obtain

$$\begin{aligned}T(R,S)\ll{}&R^{-1/6}x^{(\delta+5)/6}|h|^{1/6}+x^{1-\delta/4}|h|^{-1/4}\\&+R^{1/2}x^{1/2}+R^{-1/4}x^{(\delta+3)/4}|h|^{1/4}+xR^{-1/4}.\end{aligned}$$

This leads to the estimate

$$\begin{aligned}x^{-\varepsilon}(\Sigma_2+\Sigma_3)\ll{}&xv^{-1/2}+(uv)^{1/2}x^{1/2}+x^{1-\delta/4}|h|^{-1/4}+xu^{-1/4}\\&+u^{-1/6}x^{(\delta+5)/6}|h|^{1/6}+u^{-1/4}x^{(\delta+3)/4}|h|^{1/4}.\end{aligned}\tag{2.6}$$

Combining estimates in (2.4) and (2.6) and choosing $v = (x/u)^{1/2}$ yields the bound

$$x^{-\varepsilon} \sum_{x<n\leqslant z} \Lambda(n)\mathbf{e}(G_b(n)) \ll x^{1-\delta/4}|h|^{-1/4} + u^{1/4}x^{3/4} + u^{-1/4}x + u^{-1/6}x^{(\delta+5)/6}|h|^{1/6} + u^{-1/4}x^{(\delta+3)/4}|h|^{1/4} + ux^{\delta/2}|h|^{1/2}.$$

Inserting this estimate in (2.3) and then applying the bounds (cf. [11, Lemma 4.1 and Theorem 4.2])

$$S(a,b;q) \ll q^{1/2+\varepsilon}\gcd(b,q) \quad \text{for } b \neq 0, \quad S(a,0;q) \ll q^{1-1/k},$$

which hold for $(a,q)=1$, we deduce that

$$F_h(z) \ll x^{\varepsilon}q^{1/2+2\varepsilon}\big(x^{1-\delta/4}|h|^{-1/4} + u^{1/4}x^{3/4} + ux^{\delta/2}|h|^{1/2} + u^{-1/6}x^{(\delta+5)/6}|h|^{1/6} + u^{-1/4}x^{(\delta+3)/4}|h|^{1/4} + u^{-1/4}x\big).$$

Going back to (2.2) and applying [3, Lemma 2.4] to choose $1 \leqslant u \leqslant x$ optimally, we obtain

$$\sum_{p\sim x} cp^{1-\delta}\mathbf{e}(\alpha p^k)\Delta\psi^*(p) \ll x^{3/2-\delta} + x^{\varepsilon}q^{1/2+2\varepsilon}\big(x^{1-\delta/4}H^{3/4} + x^{(\delta+6)/8}H^{9/8} + x^{(3\delta+10)/14}H^{17/14} + x^{7/8}H + x^{\delta/2}H^{3/2} + x^{(3\delta+6)/10}H^{13/10} + x^{(\delta+8)/10}H^{11/10} + x^{3/4}H + x^{(\delta+4)/6}H^{7/6} + x^{(\delta+2)/4}H^{5/4} + x^{(\delta+8)/10}H^{11/10}\big). \tag{2.7}$$

Combining (2.1) and (2.7) we see that $x^{-\varepsilon}(f_k(\alpha,x) - g_k(\alpha,x))$ is

$$\ll H^{-1}x^{2-\delta} + H^{1/2}x^{1-\delta/2} + x^{3/2-\delta} + P^{1/2}\Big(x^{1-\delta/4}H^{3/4} + x^{\delta/2}H^{3/2} + x^{(\delta+6)/8}H^{9/8} + x^{7/8}H + x^{(3\delta+10)/14}H^{17/14} + x^{(3\delta+6)/10}H^{13/10} + x^{(\delta+8)/10}H^{11/10} + x^{(\delta+4)/6}H^{7/6} + x^{(\delta+2)/4}H^{5/4}\Big).$$

Using [3, Lemma 2.4] to choose $H \in [1,x]$ optimally yields the bound

$$x^{(4-2\delta)/3} + x^{(10-4\delta)/7}P^{2/7} + x^{(30-10\delta)/21}P^{5/21} + x^{(24-8\delta)/17}P^{4/17} + x^{(23-8\delta)/16}P^{1/4} + x^{(44-14\delta)/31}P^{7/31} + x^{(32-10\delta)/23}P^{5/23} + x^{(6-2\delta)/5}P^{1/5} + x^{(18-6\delta)/13}P^{3/13} + x^{(12-4\delta)/9}P^{2/9} + P^{1/2}\big(x^{(\delta+8)/10} + x^{7/8} + x^{(3\delta+10)/14}\big).$$

The result follows by noting that for $P \leqslant x^{(18-17\delta)/7}$ and $c \in (1, 4/3)$, which we assumed above, the term $x^{(44-14\delta)/31}P^{7/31}$ dominates the other terms. $\square$

3. Proof of Theorem 1.1

Given $n \in \mathbb{N}$, we put $N = \frac{1}{2}n^{1/k}$, and define the integral

$$\mathcal{I}_{k,c}(n) = \int_0^1 f_k(\alpha, N)\mathcal{F}(\alpha)^2 \mathbf{e}(-\alpha n)d\alpha,$$

where

$$\mathcal{F}(\alpha) = f_k(\alpha, N)^v \prod_{j=1}^{u+t} f_k(\alpha, N^{\lambda_j}).$$

Definition 3.1 *(Major and minor arcs).* For $1 \leqslant P \leqslant N^{k/2}$, we define the set of major arcs $\mathfrak{M} = \mathfrak{M}(P)$ as the union of the intervals

$$\mathfrak{M}(a, q; P) = \{\alpha \in [0, 1) : |q\alpha - a| \leqslant PN^{-k}\}$$

with $0 \leqslant a \leqslant q \leqslant P$ and $(a, q) = 1$. We define the corresponding set of minor arcs by putting $\mathfrak{m} = \mathfrak{m}(P) = [0, 1) \setminus \mathfrak{M}$.

Lemma 3.2. *For $s = 2(u + t + v) + 1$, $P \leqslant N^{(14\delta-13)\lambda/38-s\varepsilon}$, and any $A > 0$,*

$$\int_{\mathfrak{M}} f_k(\alpha, N)\mathcal{F}(\alpha)^2 \mathbf{e}(-\alpha n)d\alpha = \mathfrak{S}_{s,k}(n)J_{k,s}(n) + O_A\left(XN^{-k}\mathcal{L}^{-s-A}\right),$$

where $X = N^{2\Lambda+2v+1}$, $\mathfrak{S}_{s,k}(n)$ is the singular series

$$\mathfrak{S}_{s,k}(n) = \sum_{q\geqslant 1} \sum_{\substack{1\leqslant a\leqslant q \\ (a,q)=1}} \Bigg(\varphi(q)^{-1} \sum_{\substack{1\leqslant x\leqslant q \\ (x,q)=1}} e\big(ax^k/q\big)\Bigg)^s e(-na/q),$$

and $J_{k,s}(n)$ is the singular integral

$$J_{k,s}(n) = \int_{-\infty}^{\infty} V(\beta; N) \prod_{i=1}^{u+t+v} V(\beta; N^{\lambda_i})^2 \mathbf{e}(-\beta n)d\beta,$$

in which

$$V(\beta; Z) = \int_Z^{2Z} \frac{\mathbf{e}(\beta\gamma^k)}{\log \gamma} d\gamma.$$

Remark 3.3. As noted in [7, eqn. (4.3)], for all sufficiently large integers n with $n \equiv s \pmod{\mathrm{K}(\mathrm{k})}$,

$$\mathfrak{S}_{s,k}(n)J_{k,s}(n) \asymp X\mathcal{L}^{-s},$$

and the conditions $s \geqslant 3k+1$ and $n \equiv s \pmod{\mathrm{K}(\mathrm{k})}$ ensure that the singular series is positive.

Proof. Let

$$\mathcal{G}(\alpha) = g_k(\alpha, N)^v \prod_{j=1}^{u+t} g_k(\alpha, N^{\lambda_j}).$$

For $\alpha \in \mathfrak{M}(a,q;P)$ with coprime a, q satisfying $0 \leqslant a \leqslant q \leqslant P$, we have $|q\alpha - a| \leqslant PN^{-k} \leqslant PN^{-\lambda_i k}$. Thus, Lemma 2.4 yields

$$f_k(\alpha, N^{\lambda_i}) = g_k(\alpha, N^{\lambda_i}) + O(N^{\lambda_i(44-14\delta)/31}P^{7/31}) \tag{3.1}$$

for sufficiently large N. Furthermore, by [5, Theorem 2] we have

$$g_k(\alpha, N^{\lambda_i}) \ll P^{1/2}N^{11\lambda_i/20+\varepsilon} + \frac{q^\varepsilon N^{\lambda_i}(\log N)^C}{(q + N^{k\lambda_i}|q\alpha - a|)^{1/2}}$$

for some absolute constant $C > 0$. Here, the second term dominates for each i with $1 \leqslant i \leqslant u+t$, provided that

$$P \leqslant N^{9\lambda/20}, \tag{3.2}$$

in which case, we have

$$g_k(\alpha, N^{\lambda_i}) \ll q^{-1/2}N^{\lambda_i+\varepsilon}.$$

We rewrite $\mathcal{I}_{k,c}(n)$ in the form

$$\int_0^1 \prod_{i=1}^{s} f_k(\alpha, x_i)e(-\alpha n)d\alpha, \tag{3.3}$$

where each x_i stands for $N^{\lambda_{j_i}}$ for some appropriate index $1 \leqslant j_i \leqslant u+t$. Note in this notation, $\prod_{i=1}^{s} x_i = X$. It follows from (3.1) and (3.3) that

$$\int_{\mathfrak{M}} \Big(f_k(\alpha, N)\mathcal{F}(\alpha)^2 - g_k(\alpha, N)\mathcal{G}(\alpha)^2 \Big) \mathbf{e}(-\alpha n)d\alpha \tag{3.4}$$

is

$$\ll \sum_{1\leqslant q\leqslant P}\sum_{\substack{0\leqslant a\leqslant q\\(a,q)=1}} Pq^{-1}N^{-k}\Bigg(X^{(44-14\delta)/31}P^{7s/31}$$
$$+\sum_{\ell=1}^{s-1}\sum_{1\leqslant i_1<\ldots<i_\ell\leqslant s}(x_{i_1}\cdots x_{i_\ell})^{(13-14\delta)/31}P^{7\ell/31}q^{-(s-1-\ell)/2}X^{1+\varepsilon}\Bigg)$$
$$\ll PN^{-k}\Bigg(X^{(44-14\delta)/31}P^{1+7s/31}$$
$$+X^{1+\varepsilon}\sum_{1\leqslant q\leqslant P}q^{-(s-1)/2}\sum_{\ell=1}^{s-1}\left(N^{\lambda(13-14\delta)/31}P^{7/31}q^{1/2}\right)^{\ell}\Bigg).$$

In the last sum over ℓ above, we need $N^{\lambda(13-14\delta)/31}P^{7/31}q^{1/2}<1$, since otherwise the estimate above is worse than the trivial estimate. Thus, with this assumption and the fact that $s>5$, the above estimate is

$$\ll PN^{-k}\left(X^{(44-14\delta)/31}P^{1+7s/31}+X^{1+\varepsilon}N^{\lambda(13-14\delta)/31}P^{7/31}\right).$$

It follows that (3.4) is $\ll_A XN^{-k}\mathcal{L}^{-s-A}$ if we further impose the condition that

$$P\leqslant N^{(14\delta-13)\lambda/38-s\varepsilon}. \tag{3.5}$$

Note that if we assume (3.5), then (3.2) also holds. Finally, it follows from [9, Theorem 3] that for any positive A,

$$\int_{\mathfrak{M}}\prod_{i=1}^{s}g_k(\alpha,x_i)\mathbf{e}(-n\alpha)d\alpha=\mathfrak{S}_{s,k}(n)J_{k,s}(n)+O_A\left(XN^{-k}\mathcal{L}^{-s-A}\right).$$

Therefore, the result follows. □

Next, we deal with minor arc contribution. We choose $P=N^{(14\delta-13)\lambda/38-s\varepsilon}$. Given $\alpha\in\mathfrak{m}$, use Diophantine approximation to find coprime integers a,q with $1\leqslant a\leqslant q\leqslant N^k/P$ such that $|q\alpha-a|\leqslant PN^{-k}$. Since $\alpha\in\mathfrak{m}$, $q>P$. Using [6, Theorem 1.2] with ($\theta=1$, $k\geqslant 4$ and $\rho=\rho(k)$) together with [8, Lemma 2.1] we derive that for any $\varepsilon>0$,

$$\begin{aligned}g_k(\alpha,N)&\ll N^{1-\sigma_k/3+\varepsilon}+N^{1+\varepsilon}P^{-1/2}\\&\ll N^{1+s\varepsilon}\left(N^{-\sigma_k/3}+N^{-(14\delta-13)\lambda/76}\right).\end{aligned}$$

Note that Theorem 1.2 in [6] can be used with the improved exponent σ_k as is mentioned in the proof of [8, Lemma 2.2] due to the recent developments in Vinogradov's mean value theorem.

It follows from [2, Lemma 2.11] that for $c \in (1,2)$ and any $\varepsilon > 0$,

$$f_k(\alpha, N) - g_k(\alpha, N) \ll N^{1-(\delta(1-A)-A)+\varepsilon},$$

where $A = (\nu - 1)/(2\nu - 1)$ and

$$\nu = \begin{cases} k(k+1)^2, & \text{if } 4 \leqslant k \leqslant 11, \\ \dfrac{2\lfloor 3k/2 \rfloor \left(\lfloor 3k/2 \rfloor^2 - 1\right)}{\lfloor 3k/2 \rfloor - k}, & \text{if } k \geqslant 12. \end{cases}$$

Note that

$$\delta(1-A) - A < 1 - 2A = \frac{1}{2\nu - 1} < \frac{1}{3k(k-1)} = \sigma_k/3.$$

Therefore, combining above estimates, we can write

$$\sup_{\alpha \in \mathfrak{m}} |f_k(\alpha, N)| \ll N^{1+\varepsilon}\left(N^{-\eta_1} + N^{-\eta_2}\right),$$

where $\eta_1 = (14\delta - 13)\lambda/76$, and $\eta_2 = \delta(1-A) - A$. Thus,

$$\begin{aligned} \int_{\mathfrak{m}} \left|f_k(\alpha, N)\mathcal{F}(\alpha)^2\right| d\alpha &\leqslant N^{1+\varepsilon}\left(N^{-\eta_1} + N^{-\eta_2}\right) \int_0^1 |\mathcal{F}(\alpha)|^2 d\alpha \\ &\ll N^{1+\varepsilon}\left(N^{-\eta_1} + N^{-\eta_2}\right)(X/N)^{1-\delta} \int_0^1 |\mathcal{G}(\alpha)|^2 d\alpha, \end{aligned}$$

where the passage from the integrand $|\mathcal{F}|^2$ to $|\mathcal{G}|^2$ is justified by interpreting the integral as a weighted sum over the solution set of a system of Diophantine equations. By [8, Lemma 2.3],

$$\int_0^1 \left|\mathcal{G}(\alpha)\right|^2 d\alpha \ll XN^{-1-k+\varepsilon}.$$

Hence, we conclude

$$\begin{aligned} \int_{\mathfrak{m}} \left|f_k(\alpha, N)\mathcal{F}(\alpha)^2\right| d\alpha &\ll \left(N^{-\eta_1} + N^{-\eta_2}\right) N^{2(1-\delta)(\Lambda+v)+2\varepsilon} XN^{-k} \\ &= o(X\mathcal{L}^{-s}N^{-k}), \end{aligned}$$

provided that

$$c - 1 < \min\left\{\frac{1}{(4\nu - 2)(\Lambda + v) + \nu - 1}, \frac{\lambda}{152(\Lambda + v) + 13\lambda}\right\}. \tag{3.6}$$

It follows from Remark 1.2 that for sufficiently large k, $\Lambda + v > \frac{7k}{6}$. We can also see that $\nu < 27k^2/2$ and $\lambda < 2.1/k^2$. Then,

$$\lambda\Big(4\nu - 2 + \frac{\nu - 14}{\Lambda + v}\Big) < \frac{2.1}{k^2}\left(54k^2 + \frac{81k^2}{7k}\right) < 152.$$

This shows that the second term on the right side in (3.6) is the minimum. Finally, the proof of Theorem 1.1 follows by combining the results in this section and using [8, Theorem 1].

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