

Monodromy groups of real Enriques surfaces

Sultan Erdoğan Demir¹

Bilkent University, Department of Mathematics, 06800 Ankara, Turkey

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ABSTRACT

We compute the monodromy groups of real Enriques surfaces of hyperbolic type. The principal tools are the deformation classification of such surfaces and a modified version of Donaldson's trick, relating real Enriques surfaces and real rational surfaces.

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1. Introduction

An *Enriques surface* is a complex analytic surface E with $\pi_1(E) = \mathbb{Z}_2$ and having a $K3$ -surface X as its universal cover. An Enriques surface is called *real* if it is supplied with an anti-holomorphic involution conj , called *complex conjugation*. The *real part* of a real surface E is the fixed point set $E_{\mathbb{R}} = \text{Fix conj}$. A *topological type* of real surfaces is a class of surfaces with homeomorphic real parts. A real Enriques surface E is a smooth 4-manifold, its real part $E_{\mathbb{R}}$ is either empty or a closed 2-manifold with finitely many components, each being either $S = S^2$, or $S_g = \sharp_g(S^1 \times S^1)$, or $V_p = \sharp_p \mathbb{R}P^2$.

Let E be a real Enriques surface and $p : X \rightarrow E$ its universal covering. Denote by $\tau : X \rightarrow X$ the deck translation of p , called the *Enriques involution*. There are exactly two liftings $t^{(1)}, t^{(2)} : X \rightarrow X$ of conj to X , which are both anti-holomorphic involutions. They commute with each other and with τ , and their composition is τ . For both $i = 1, 2$, the real parts $X_{\mathbb{R}}^{(i)} = \text{Fix } t^{(i)}$, and their images $E_{\mathbb{R}}^{(i)} = p(X_{\mathbb{R}}^{(i)})$ (called the *halves* of $E_{\mathbb{R}}$) are disjoint, $E_{\mathbb{R}}^{(i)}$ consists of whole components of $E_{\mathbb{R}}$, and $E_{\mathbb{R}} = E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$. This decomposition is a deformation invariant of pair (E, conj) . We use the notation $E_{\mathbb{R}} = \{\text{half } E_{\mathbb{R}}^{(1)}\} \sqcup \{\text{half } E_{\mathbb{R}}^{(2)}\}$ for the half decomposition. To describe the topological types of the real part the concept of *topological Morse simplification*, i.e., Morse transformation of the topological type which decreases the total Betti number, is used. A topological Morse simplification is either removing a spherical component ($S \rightarrow \emptyset$) or contracting a handle ($S_{g+1} \rightarrow S_g$ or $V_{p+2} \rightarrow V_p$). The complex deformation type of surfaces being fixed (e.g., $K3$ or Enriques), a topological type is called *extremal* if it cannot be obtained from another one (in the same complex deformation type) by a topological Morse simplification.

The classification of real Enriques surfaces up to deformation was given by A. Degtyarev, I. Itenberg and V. Kharlamov in [3], where one can find a complete list of deformation classes, the invariants necessary to distinguish them, and detailed explanations of the invariants. It turns out that the deformation class of a real Enriques surface is determined by the

E-mail address: erdogan@fen.bilkent.edu.tr.

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topology of its complex conjugation involution. Deformation classification can be regarded as the study of the set of connected components (i.e., π_0) of the moduli space. In this paper, we pose the question about its fundamental group (i.e., π_1). More precisely, we study the canonical representation of the fundamental group of a connected component of the moduli space in the group G of permutations of the components of the real part of the corresponding surfaces. In other words, we discuss the *monodromy groups* of real Enriques surfaces, i.e., the subgroups of G realized by ‘auto-deformations’ and/or automorphisms of the surfaces.

The similar question for various families of $K3$ -surfaces has been extensively covered in the literature. Thus, the monodromy groups have been studied for nonsingular plane sextics by Itenberg [10] and for nonsingular surfaces of degree four in $\mathbb{R}P^3$ by Kharlamov [11–13] and Moriceau [16].

A real Enriques surface is said to be of *hyperbolic, parabolic, or elliptic type* if the minimal Euler characteristic of the components of $E_{\mathbb{R}}$ is negative, zero, or positive, respectively. In the deformation classification, hyperbolic and parabolic cases are treated geometrically (based on Donaldson’s trick [7]) whereas the elliptic cases are treated arithmetically (calculations using the global Torelli theorem for $K3$ -surfaces cf. [1]). There also is a crucial difference between the approaches to surfaces of hyperbolic and parabolic types. In the former case, natural *complex* models of complex DPN -pairs are constructed, and a real structure descends to the model by naturality. In the latter case, it is difficult to study complex DPN -pairs systematically and *real* models of *real* DPN -pairs are constructed from the very beginning. We study the surfaces of hyperbolic types in this work. Thus, we deal with an equivariant version of Donaldson’s trick for Enriques surfaces modified by A. Degtyarev and V. Kharlamov [3], which transforms a real Enriques surface to a real rational surface with a nonsingular real anti-bicanonical curve on it. We analyze this construction and adopt it to the study of the monodromy groups. In particular, we discuss the conditions necessary for an additional automorphism of the rational surface to define an automorphism of the resulting real Enriques surface. The principal result of the paper can be roughly stated as follows (for the exact statements see Theorems 5.1, 5.3 and 5.5): *For a real Enriques surface of hyperbolic type, with some exceptions listed explicitly in each statement, any permutation of homeomorphic components of each half of $E_{\mathbb{R}}$ can be realized by deformations and/or automorphisms.*

The exceptions deserve a separate discussion. In most cases, the nonrealizable permutations are prohibited by a purely topological invariant, the so-called Pontrjagin–Viro form (see [2] and remarks following the relevant statements). There are, however, a few surfaces, those with $E_{\mathbb{R}}^{(1)} = V_3 \sqcup \dots$, for which the Pontrjagin–Viro form is not well defined but the spherical components of $E_{\mathbb{R}}^{(1)}$ cannot be permuted. The question whether these permutations are realizable by equivariant auto-homeomorphisms of the surface remains open. Same question for parabolic and elliptic cases is a subject of a future study as it seems to require completely different means.

Organization of the paper is as follows: In Section 2, we recall some rational surfaces, curves on them, and a few results related to their classification up to rigid isotopy. In Section 3, we describe (modified) Donaldson’s trick and the resulting correspondence theorem, the construction backwards, and recall some results concerning specific families of real Enriques surfaces of hyperbolic type. In Section 4, a few necessary conditions for lifting automorphisms are discussed. In Section 5, the main result is stated and proved in three theorems.

2. Some surfaces and curves on them

2.1. DPN -pairs

A nonsingular algebraic surface admitting a nonempty nonsingular anti-bicanonical curve (i.e., curve in the class $|-2K|$), is called a *DPN-surface*. Most DPN -surfaces are rational.

A pair (Y, B) , where Y is a DPN -surface and $B \in |-2K|$ is a nonsingular curve, is called a *DPN-pair*. A DPN -pair (Y, B) is called *unnodal* if Y is unnodal (does not contain a (-2) -curve), *rational* if Y is rational, and *real* if both Y and B are real. The *degree* of a rational DPN -pair (Y, B) is the degree of Y , i.e., K^2 .

If (Y, B) is a rational DPN -pair, the double covering X of Y ramified along B is a $K3$ -surface. A DPN -surface contains finitely many (-4) -curves. A rational DPN -surface Y of degree d that has r (-4) -curves is called a (g, r) -*surface*, where $g = d + r + 1$. In fact $g \geq 1$ and any nonsingular curve $B \in |-2K_Y|$ is one of the following topological types (see [3]):

1. $B \cong S_g \sqcup rS$ if $g > 1$;
2. $B \cong S_1 \sqcup rS$ or rS if $g = 1$ and $r > 0$;
3. $B \cong 2S_1$ or S_1 if $g = 1$ and $r = 0$.

Let Y be a real surface with $H_1(Y) = 0$. An *admissible branch curve* on Y is a nonsingular real curve $B \subset Y$ such that $[B] = 0$ in $H_2(Y)$, the real part $B_{\mathbb{R}}$ is empty and B is not linked with $Y_{\mathbb{R}}$. An *admissible DPN -pair* is a real rational DPN -pair (Y, B) with B an admissible branch curve.

Donaldson’s trick (see Section 3.1) establishes a one-to-one correspondence between the set of deformation classes of real Enriques surfaces with distinguished nonempty half (i.e., pairs $(E, E_{\mathbb{R}}^{(1)})$ with $E_{\mathbb{R}}^{(1)} \neq \emptyset$) and the set of deformation classes of admissible DPN -pairs (Y, B) . Inverse Donaldson’s trick (see Section 3.2) establishes a surjective map from the set of deformation classes of unnodal admissible DPN -pairs to the set of deformation classes of real Enriques surfaces with distinguished nonempty half.

2.2. Del Pezzo and geometrically ruled rational surfaces

A Del Pezzo surface Y is a surface such that $K_Y^2 > 0$ and $D \cdot K_Y \leq 0$ for any effective divisor D on Y . An unnodal Del Pezzo surface Y is a surface whose anticanonical divisor is ample, or equivalently, a Del Pezzo surface without (-2) -curves.

We use the notation Σ_a , $a \geq 0$, for the geometrically ruled rational surface (i.e., relatively minimal conic bundle over \mathbb{P}^1) that has a section of square $(-a)$, which is called the exceptional section. The classes of the exceptional section E_0 and of a generic section is denoted by e_0 and e_∞ , respectively, so that $e_0^2 = -a$, $e_\infty^2 = a$, and $e_0 \cdot e_\infty = 0$. The class of the fiber (generatrix) will be denoted by l ; one has $l^2 = 0$ and $l \cdot e_0 = l \cdot e_\infty = 1$. Any irreducible curve in Σ_a with $a \geq 1$, either is E_0 or belongs to $|xl + ye_\infty|$, $x, y \geq 0$. If $a = 0$ then $e_0 = e_\infty$. Thus, if l_1 denotes $e_0 = e_\infty$ and l_2 denotes l then any irreducible curve in Σ_0 belongs to $|xl_1 + yl_2|$, $x, y \geq 0$.

2.3. Rigid isotopies

Recall that an isotopy is a homotopy from one embedding of a manifold M into a manifold N to another embedding such that, at every time, it is an embedding. An isotopy in the class of nonsingular (or, more generally, equisingular, in some appropriate sense) embeddings of analytic varieties is called rigid. Below we are mainly dealing with rigid isotopies of nonsingular curves on rational surfaces. Clearly, such an isotopy is merely a path in the space of nonsingular curves.

An obvious rigid isotopy invariant of a real curve C on a real surface Z is its real scheme, i.e., the topological type of the pair $(Z_{\mathbb{R}}, C_{\mathbb{R}})$.

The deformation classification of real Enriques surfaces and hence the monodromy problem of those leads to a variety of auxiliary classification problems for curves on surfaces and surfaces in projective spaces. Below we give a brief account of the related results and recall the basic definitions and facts about them. Details and further references can be found, e.g., in [3].

2.4. Curves in $\mathbb{P}_{\mathbb{R}}^2$

The real point set $C_{\mathbb{R}}$ of a nonsingular curve C in $\mathbb{P}_{\mathbb{R}}^2$ is a collection of circles A embedded in $\mathbb{P}_{\mathbb{R}}^2$, two- or one-sidedly. In the former case the component is called an oval. Any oval divides $\mathbb{P}_{\mathbb{R}}^2$ into two parts; the interior of the oval, homeomorphic to a disk and the exterior of the oval, homeomorphic to the Möbius band. The relation to be in the interior of defines a partial order on the set of ovals, and the collection A equipped with this partial order determines the real scheme of C . The following notation is used to describe real schemes: If a real scheme has a single component, it is denoted by $\langle J \rangle$, if the component is one-sided, or by $\langle 1 \rangle$, if it is an oval. The empty real scheme is denoted by $\langle 0 \rangle$. If $\langle A \rangle$ stands for a collection of ovals, the collection obtained from it by adding a new oval surrounding all the old ones is denoted by $\langle 1 \langle A \rangle \rangle$. If a real scheme splits into two subschemes $\langle A_1 \rangle$, $\langle A_2 \rangle$ so that no oval of $\langle A_1 \rangle$ (respectively, $\langle A_2 \rangle$) surrounds an oval of $\langle A_2 \rangle$ (respectively, $\langle A_1 \rangle$), it is denoted by $\langle A_1 \sqcup A_2 \rangle$. If a real scheme contains n disjoint copies of $\langle 1 \rangle$ it is denoted by $\langle n \rangle$.

Theorem 2.1. ([14]) A nonsingular real quartic C in \mathbb{P}^2 is determined up to rigid isotopy by its real scheme. There are six rigid isotopy classes, with real schemes $\langle \alpha \rangle$, $\alpha = 0, \dots, 4$ and $\langle 1 \langle 1 \rangle \rangle$.

Lemma 2.2. ([3]) Let C be a nonsingular real quartic with the real scheme $\langle \alpha \rangle$, $\alpha = 2, 3, 4$ in \mathbb{P}^2 . Then any permutation of the ovals of C can be realized by a rigid isotopy.

2.5. Cubic sections on a quadratic cone

Let $U \in |ne_\infty|$ be a nonsingular real curve in Σ_2 with its standard real structure $((\Sigma_2)_{\mathbb{R}} = S_1)$. Each connected component of $U_{\mathbb{R}}$ is either an oval or homologous to $(E_0)_{\mathbb{R}}$. The latter, together with $(E_0)_{\mathbb{R}}$, divide $(\Sigma_2)_{\mathbb{R}}$ into several connected components Z_1, \dots, Z_k . Fixing an orientation of the real part of a real generatrix of Σ_2 determines an order of the components Z_i , and the real scheme of U can be described via $\langle C_1 | \dots | C_k \rangle$, where $|$ stands for a component homologous to $(E_0)_{\mathbb{R}}$ and C_i encodes the arrangement of the ovals in Z_i (similar to the case of plane curves), for each $i \in \{1, 2, \dots, k\}$.

Theorem 2.3. ([3]) A nonsingular real curve $U \in |3e_\infty|$ on Σ_2 is determined up to rigid isotopy by its real scheme. There are 11 rigid isotopy classes, with real schemes $\langle \alpha | 0 \rangle$, $1 \leq \alpha \leq 4$, $\langle 0 | \alpha \rangle$, $1 \leq \alpha \leq 4$, $\langle 0 | 0 \rangle$, $\langle 1 | 1 \rangle$, and $\langle || | \rangle$.

Remark 2.4. By analyzing the proof of Theorem 2.3, one can easily see that the curves with real schemes $\langle \alpha | 0 \rangle$ and $\langle 0 | \alpha \rangle$, $1 \leq \alpha \leq 4$, are isomorphic up to a real automorphism of Σ_2 . Furthermore, a stronger statement holds: any two pairs (U, O) , where the real scheme of U is $\langle \alpha | 0 \rangle$ with $0 \leq \alpha \leq 3$ and O is a distinguished oval of U , are rigidly isotopic. For an alternative proof of Theorem 2.3 and the last assertion, one can use the theory of the trigonal curves, see [4].

2.6. Regular complete intersections of two real quadrics in $\mathbb{P}_{\mathbb{R}}^4$

The following is a special case of the rigid isotopy classification of regular complete intersections of two quadrics in $\mathbb{P}_{\mathbb{R}}^n$, due to S. Lopez de Medrano [15].

Theorem 2.5. ([15]) *A regular complete intersection Y of two real quadrics in $\mathbb{P}_{\mathbb{R}}^4$ is determined up to rigid isotopy by its real part $Y_{\mathbb{R}}$. There are seven rigid isotopy classes, with $Y_{\mathbb{R}} = V_6, V_4, V_2, S_1, 2S, S$, or \emptyset .*

2.7. Real root schemes

Let $Z = \Sigma_k, k \geq 0$, with the standard real structure. Since we use Σ_2 and Σ_4 in this paper we will consider only the cases $k = 2n$. For $k = 2n + 1$ and further details, see [3]. Consider a real curve $U \in |2e_{\infty} + p|, p \geq 0$, and a real curve $Q = E_0 \cup F$, where E_0 is the exceptional section and $F \in |e_{\infty}|$ is a generic real section of Z . The complement $Z_{\mathbb{R}} \setminus Q_{\mathbb{R}}$ consists of two connected orientable components. Fix one of them and let Z^- denote its closure. Fix an orientation of $F_{\mathbb{R}} \subset \partial Z^-$. Assume that U does not contain any generatrix of Z , is transversal to F and $U_{\mathbb{R}}$ lies entirely in Z^- . Fix an auxiliary real generatrix L of Z transversal to $U \cup E_0$. Consider a real coordinate system (x, y) in the affine part $Z \setminus (E_0 \cup L)$ whose x -axis is F . Choose the positive direction of the y -axis so that the upper half-plane lies in Z^- . In these coordinates U has equation $a(x)y^2 + b(x)y + c(x) = 0$, where a, b , and c are real polynomials of degree $p, p+k$, and $p+2k$, respectively. Let $\Delta = b^2 - 4ac$ and let $\mu(x)$ and $\nu(x)$ denote the multiplicity of a point $x \in F$ in a and Δ , respectively. Consider the sets

$$\begin{aligned} \mathcal{A}_{\mathbb{R}} &= \{x \in F_{\mathbb{R}} \mid \mu(x) \geq 1\}, & \mathcal{A} &= \{x \in F \mid \mu(x) \geq 1\}, \\ \mathcal{D}_{\mathbb{R}} &= \{x \in F_{\mathbb{R}} \mid \Delta(x) \geq 0\}, & \mathcal{D}_r &= \{x \in F \mid \nu(x) \geq r\}, \quad r \geq 1, & \mathcal{D} &= \mathcal{D}_2 \cup \mathcal{D}_{\mathbb{R}}. \end{aligned}$$

The multiplicity functions μ and ν are invariant under complex conjugation. Identify F with the base $B \cong \mathbb{P}^1$ of the ruling of Z . Thus, $B_{\mathbb{R}}$ receives an orientation, \mathcal{A} and \mathcal{D} can be regarded as subsets of B , and, μ and ν are functions defined on B . The root marking of (U, Q) is the triple $(\mathcal{B}, \mathcal{D}, \mathcal{A})$ equipped with the complex conjugation in B and the following structures:

1. the orientation of $B_{\mathbb{R}}$;
2. the multiplicity functions μ and ν .

An isotopy of root markings is an equivariant isotopy of triples $(\mathcal{B}, \mathcal{D}, \mathcal{A})$ followed by a continuous change of the orientation of $B_{\mathbb{R}}, \mu$, and ν restricted to \mathcal{D} . A root scheme is an equivalence class of root markings up to isotopy. The real root marking of (U, Q) is the triple $(\mathcal{B}_{\mathbb{R}}, \mathcal{D}_{\mathbb{R}}, \mathcal{A}_{\mathbb{R}})$ equipped with (1) and (2) above. A real root scheme is an equivalence class of real root markings up to isotopy.

Theorem 2.6. ([3]) *Let $Z = \Sigma_4$ (with the standard real structure), let $U \in |2e_{\infty}|$ be a nonsingular real curve on Z , let $F \in |e_{\infty}|$ be a generic real section transversal to U , and let E_0 be the exceptional section. If $U_{\mathbb{R}}$ belongs to the closure of one of the two components of $Z_{\mathbb{R}} \setminus ((E_0)_{\mathbb{R}} \cup F_{\mathbb{R}})$, then, up to rigid isotopy and automorphism of Z , the pair (U, F) is determined by its real root scheme or, equivalently, by the real scheme of U . The latter consists either of $a = 0, \dots, 4$ ovals (i.e., components bounding disks) or of two components isotopic to $F_{\mathbb{R}}$.*

2.8. Suitable pairs

Let $U \in |2e_{\infty} + 2l|$ be a reduced (does not contain any multiple component) real curve on Σ_2 with the standard real structure. Assume that U is nonsingular outside of E_0 and does not contain E_0 as a component. Then U and E_0 intersect with multiplicity 2. So U either intersects E_0 transversally at two points, or is tangent to it at one point, or has a single singular point of type $A_{r-2}, r \geq 3$, on E_0 ; the grade of U is said to be 1, 2, or r , respectively. A curve U as above is called suitable if either its grade is even or grade is odd and the two branches of U at E_0 are conjugate to each other. A pair (U, F) is called a suitable pair if U is a suitable curve and $F \in |e_{\infty}|$ a nonsingular real section transversal to U such that $U_{\mathbb{R}}$ belongs to the closure of a single connected component of $(\Sigma_2)_{\mathbb{R}} \setminus ((E_0)_{\mathbb{R}} \cup F_{\mathbb{R}})$. The grade of a suitable pair (U, F) is the grade of U . The condition that $U_{\mathbb{R}}$ should belong to the closure of a single connected component of $(\Sigma_2)_{\mathbb{R}} \setminus ((E_0)_{\mathbb{R}} \cup F_{\mathbb{R}})$ guarantees that the real DPN-double (Y, B) (i.e., the resolution of singularities of the double covering of Σ_2 branched over U , where the rational components of B correspond to E_0 and the irrational component of B corresponds to F) of $(\Sigma_2; U, E_0 \cup F)$, where (U, F) is a suitable pair, corresponds to a real Enriques surface by inverse Donaldson's trick.

All the pairs (U, F) satisfying the hypothesis of the following theorem are suitable.

Theorem 2.7. ([3]) *Let $Z = \Sigma_2$ (with the standard real structure), let $U \in |2e_{\infty} + 2l|$ be a reduced real curve on Z , nonsingular outside the exceptional section E_0 and not containing E_0 as a component, and let $F \in |e_{\infty}|$ be a generic real section transversal to U . If $U_{\mathbb{R}}$ belongs to the closure of a single connected component of $Z_{\mathbb{R}} \setminus ((E_0)_{\mathbb{R}} \cup F_{\mathbb{R}})$, then, up to rigid isotopy and automorphism of Z , the pair (U, F) is determined by its real root scheme or, equivalently, by the type of the singular point of U (if any) and the topology of the pair $(Z_{\mathbb{R}}, U_{\mathbb{R}} \cup (E_0)_{\mathbb{R}})$.*

Table 1
Real root schemes of some curves $U \in |2e_\infty + p|$ on Σ_{2k} .

| Real root scheme | $\{E_{\mathbb{R}}^{(1)}\} \sqcup \{E_{\mathbb{R}}^{(2)}\}$ |
|------------------|---|
| | $\{(V_4 \sqcup S) \sqcup (\emptyset)\} \sqcup \{(2S) \sqcup (2S)\}$ |
| | $\{(V_3 \sqcup V_1) \sqcup (\emptyset)\} \sqcup \{(2S) \sqcup (2S)\}$ |
| | $\{(V_3 \sqcup S) \sqcup (\emptyset)\} \sqcup \{(V_1 \sqcup S) \sqcup (2S)\}$ |
| $\times 2$ | $\{(V_3 \sqcup S) \sqcup (V_1)\} \sqcup \{(2S) \sqcup (S)\}$ |
| $\times 3$ | $\{(V_3 \sqcup S) \sqcup (S)\} \sqcup \{(V_1 \sqcup S) \sqcup (S)\}$ |
| $\times 4$ | $\{(V_3 \sqcup V_1 \sqcup S) \sqcup (S)\} \sqcup \{(S) \sqcup (S)\}$ |

Comments: The first column indicates the real root schemes of pairs (U, F) and the second column indicates the quarter decomposition of the real part $E_{\mathbb{R}}$ of the real Enriques surfaces obtained from $(\Sigma_{2k}; U, E_0 \cup F)$. For the first row $p = 0$ and $k = 2$ and for the others $p = 2$ and $k = 1$. In the schemes, \bullet represents a real root of Δ and \circ represents a real root of a (necessary 2-fold), that corresponds to the real intersection point of U and E_0 . The number over a \circ -vertex indicates the multiplicity of the corresponding root in Δ (when greater than 1). The segments $\bullet\text{---}\bullet$ correspond to ovals of $U_{\mathbb{R}}$. Only extremal root schemes are listed; the others are obtained by removing one or several segments $\bullet\text{---}\bullet$.

In Table 1, we list the extremal real root schemes of some pairs (U, F) mentioned in Theorem 2.6 and Theorem 2.7 that are used in the proof of the main result. The complete lists can be found in [3].

Remark 2.8. Each real root marking gives rise to a connected family of pairs (U, Q) such that there is a bijection between the ovals of each curve U and the segments of the real root marking. Recall that these curves are defined by explicit equations. Then both Theorems 2.6 and 2.7 can be refined as follows:

1. Each isotopy of real root markings is followed by a rigid isotopy of curves that is consistent with the bijection between ovals and segments.
2. Any symmetry of a real root marking (not necessarily preserving the orientation of $B_{\mathbb{R}}$) is induced by an automorphism of Σ_{2k} , $k \geq 0$, preserving appropriate pairs (U, F) and consistent with the bijection between ovals and segments.

3. Reduction to DPN-pairs

3.1. Donaldson's trick

The equivariant version of Donaldson's trick employs the hyper-Kähler structure to change the complex structure of the covering $K3$ -surface X so that $t^{(1)}$ is holomorphic, and $t^{(2)}$ and τ are anti-holomorphic. Furthermore, $Y = X/t^{(1)}$ is a real rational surface, where the real structure is the common descent of τ and $t^{(2)}$, and $B \cong \text{Fix}t^{(1)}$ is a nonsingular curve on Y . As a result, the problem about real Enriques surfaces is reduced to the study of certain auxiliary objects, like real plane quartics, space cubics, intersections of two quadrics in $\mathbb{P}_{\mathbb{R}}^4$, etc.

Theorem 3.1. ([5]) *Donaldson's construction establishes a one-to-one correspondence between the set of deformation classes of real Enriques surfaces with distinguished nonempty half (i.e., pairs $(E, E_{\mathbb{R}}^{(1)})$ with $E_{\mathbb{R}}^{(1)} \neq \emptyset$) and the set of deformation classes of pairs (Y, B) , where Y is a real rational surface and $B \subset Y$ is a nonsingular real curve such that*

1. B is anti-bicanonical,
2. the real point set of B is empty, and
3. B is not linked with the real point set $Y_{\mathbb{R}}$ of Y .

One has $E_{\mathbb{R}}^{(2)} = Y_{\mathbb{R}}$ and $E_{\mathbb{R}}^{(1)} = B/t^{(2)}$.

(A real curve $B \subset Y$ with $B_{\mathbb{R}} = \emptyset$ is said to be *not linked with $Y_{\mathbb{R}}$* if for any path $\gamma : [0, 1] \rightarrow Y \setminus B$ with $\gamma(0), \gamma(1) \in Y_{\mathbb{R}}$, the loop $\gamma^{-1} \cdot \text{conj}_Y \gamma$ is $\mathbb{Z}/2$ -homologous to zero in $Y \setminus B$.)

In the above theorem, the first condition on B guarantees that the double covering X of Y branched over B is a $K3$ -surface; and the other two conditions ensure the existence of a fixed point free lift of the real structure on Y to X , see [5]. The statement deals with deformation classes rather than individual surfaces because the construction involves a certain choice (that of an invariant Kähler class).

3.2. Inverse Donaldson's trick

Since we want to construct deformation families of real Enriques surfaces with particular properties, we are using Donaldson's construction backwards. Strictly speaking, Donaldson's trick is not invertible. However, it establishes a bijection

between the sets of deformation classes (see Theorem 3.1); thus, at the level of deformation classes one can speak about ‘inverse Donaldson’s trick’.

Before explaining the construction, recall some properties of $K3$ -surfaces. Let a be a holomorphic involution of a $K3$ -surface X equipped with the complex structure defined by a holomorphic form ω . Then there are three possibilities for the fixed point set $\text{Fix } a$ of a :

1. it may be empty, or
2. it may consist of isolated points, or
3. it may consist of curves.

The following is straightforward:

1. if $\dim_{\mathbb{C}} \text{Fix } a = 0$, then $a^*\omega = \omega$,
2. if $\dim_{\mathbb{C}} \text{Fix } a = \pm 1$, then $a^*\omega = -\omega$.

Let conj be a real structure on X . Then $\text{conj}^*\omega = \lambda\bar{\omega}$ for some $\lambda \in \mathbb{C}^*$. Clearly, w can be chosen (uniquely up to real factor) so that $\text{conj}^*\omega = \bar{\omega}$. We always assume this choice and we denote by $\text{Re } \omega$ and $\text{Im } \omega$ the real part $(\omega + \bar{\omega})/2$ and the imaginary part $(\omega - \bar{\omega})/2$ of ω , respectively.

Let Y be a real rational surface with a nonsingular anti-bicanonical real curve $B \subset Y$ such that $B_{\mathbb{R}} = \emptyset$ and B is not linked with the real point set $Y_{\mathbb{R}}$ of Y . Let X be the (real) double covering $K3$ -surface branched over B , $\tilde{p} : X \rightarrow Y$ the covering projection and $\phi : X \rightarrow X$ the deck translation of \tilde{p} . Then ϕ is a holomorphic involution with nonempty fixed point set. There exist two liftings $c^{(1)}, c^{(2)} : X \rightarrow X$ of the real structure $\text{conj} : Y \rightarrow Y$ to X , which are both anti-holomorphic involutions. They commute with each other and with ϕ , and their composition is ϕ . Because of the requirements on B , at least one of these involutions is fixed point free. Assume that it is $c^{(1)}$.

Pick a holomorphic 2-form μ with the real and imaginary parts $\text{Re } \mu, \text{Im } \mu$, respectively, and a fundamental Kähler form ν . Due to the Calabi–Yau theorem, there exists a unique Kähler–Einstein metric with fundamental class $[\nu]$, see [9]. After normalizing μ so that $\text{Re } \mu^2 = \text{Im } \mu^2 = \nu^2 = 2 \text{Vol } X$, we get three complex structures on X given by the forms:

$$\mu = \text{Re } \mu + i \text{Im } \mu, \quad \tilde{\mu} = \nu + i \text{Re } \mu, \quad \text{and} \quad \text{Im } \mu + i\nu.$$

Let \tilde{X} be the surface X equipped with the complex structure defined by $\tilde{\mu}$. Since $c^{(1)}$ is an anti-holomorphic involution of X , the holomorphic form μ and the fundamental Kähler form ν can be chosen so that $(c^{(1)})^*\mu = -\tilde{\mu}$ and $(c^{(1)})^*\nu = -\nu$. Then $(c^{(1)})^*\tilde{\mu} = -\tilde{\mu}$ and, hence, $c^{(1)}$ is holomorphic on \tilde{X} . Since ϕ is a holomorphic involution of X commuting with $c^{(1)}$, $\phi^*\mu = -\mu$ and ν can be chosen ϕ^* -invariant so that $\phi^*\tilde{\mu} = \tilde{\mu}$, i.e., the involution ϕ is anti-holomorphic on \tilde{X} . Then $E = \tilde{X}/c^{(1)}$ is a real Enriques surface (the real structure being common descent of ϕ and $c^{(2)}$) and the projection $p : \tilde{X} \rightarrow E$ is a real double covering. Hence we have $Y_{\mathbb{R}} = E_{\mathbb{R}}^{(2)}$ and $B/c^{(2)} = E_{\mathbb{R}}^{(1)}$.

3.3. The case of Del Pezzo surfaces

The deformation classification of real Enriques surfaces with a distinguished half $E_{\mathbb{R}}^{(1)} = V_{d+2}$, $d \geq 1$ is reduced to that of real unnodal Del Pezzo surfaces of degree d , $d \geq 1$, with a nonsingular anti-bicanonical curve $B \cong S_g$, $g \geq 2$.

Theorem 3.2. ([3]) *There is a natural surjective map from the set of deformation classes of real unnodal Del Pezzo surfaces Y of degree d , $d \geq 1$, onto the set of deformation classes of real Enriques surfaces with $E_{\mathbb{R}}^{(1)} = V_{d+2}$, $d \geq 1$. Under this correspondence $Y_{\mathbb{R}} = E_{\mathbb{R}}^{(2)}$ and $Y/\text{conj} = E/\text{conj}$.*

Remark 3.3. In fact, the correspondence is bijective.

Proof of Theorem 3.2 reduces, mainly, to showing that a generic deformation of unnodal Del Pezzo surfaces Y_t can be extended to a deformation of pairs (Y_t, B_t) , where $B_t \subset Y_t$ are real anti-bicanonical curves satisfying the hypotheses of Theorem 3.1. This gives a deformation of the covering $K3$ -surfaces. Then it remains to choose a continuous family of invariant Kähler metrics, and inverse Donaldson’s trick applies. Thus, the following stronger result holds.

Theorem 3.4. *A generic deformation of real unnodal Del Pezzo surfaces Y of degree d , $d \geq 1$, defines a deformation of real Enriques surfaces with $E_{\mathbb{R}}^{(1)} = V_{d+2}$, $d \geq 1$, obtained from Y by inverse Donaldson’s trick.*

3.4. The case of $(2, r)$ surfaces

The deformation classification of real Enriques surfaces with disconnected $E_{\mathbb{R}}^{(1)} = V_3 \sqcup \dots$ is reduced to that of real $(2, r)$ -surfaces, $r \geq 1$ with a real nonsingular anti-bicanonical curve $B \cong S_2 \sqcup rS$ and, hence, to the rigid isotopy classification of suitable pairs.

Lemma 3.5. ([3]) *There is a natural surjective map from the set of rigid isotopy classes of suitable pairs of grade r onto the set of deformation classes of real Enriques surfaces with $E_{\mathbb{R}}^{(1)} = V_3 \sqcup \frac{r}{2}S$, if r is even, or $E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1 \sqcup \frac{r-1}{2}S$, if r is odd.*

Proof of the above lemma is based on showing that a generic rigid isotopy of suitable pairs (U_s, F_s) defines a deformation of the DPN-doubles (Y_s, B_s) of $(\Sigma_2; U_s, E_0 \cup F_s)$, so a deformation of the covering K3-surfaces. Then it remains to choose a continuous family of invariant Kähler metrics, to obtain a deformation of the corresponding real Enriques surfaces obtained by inverse Donaldson’s trick which implies the following stronger result.

Theorem 3.6. *A generic rigid isotopy of suitable pairs (U, F) of grade r defines a deformation of the real Enriques surfaces with $E_{\mathbb{R}}^{(1)} = V_3 \sqcup \frac{r}{2}S$, if r is even, or $E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1 \sqcup \frac{r-1}{2}S$, if r is odd.*

4. Lifting involutions

Let Z be a simply connected surface and $\pi : Y \rightarrow Z$ a branched double covering with the branch divisor C . Then any involution $a : Z \rightarrow Z$ preserving C as a divisor admits two lifts to Y , which commute with each other and with the deck translation of the covering. If $\text{Fix } a \neq \emptyset$, then both lifts are also involutions. Any fixed point of a in $Z \setminus C$ has two pullbacks on Y . One of the lifts fixes these two points and the other one permutes them.

In this section we will use the notation of Section 3.2.

Lemma 4.1. *Let Z be a real quadric cone in \mathbb{P}^3 , let $C \subset Z$ be a nonsingular real cubic section disjoint from the vertex, and let $a : Z \rightarrow Z$ be an involution preserving C and such that $\text{Fix } a \cap C \neq \emptyset$. Then a lifts to four distinct involutions on the covering K3-surface X and at least one of the four lifts defines an automorphism of an appropriate real Enriques surface obtained from X by inverse Donaldson’s trick.*

Proof. According to the models of Del Pezzo surfaces [6], the double covering of Z branched at the vertex and over C is a real unnodal Del Pezzo surface Y of degree $d = 1$. The pullback $\tilde{p} \in Y$ of any point $p \in \text{Fix } a \cap C$ is a fixed point of any lift of a to Y . Let $p' \in \text{Fix } a \setminus C$ be in a small neighborhood of p . Then p' has two pullbacks p_1 and p_2 in Y . Let a_1 be the lift of a to Y that permutes p_1 and p_2 . Then \tilde{p} is an isolated fixed point of a_1 . (Note that we do **not** assert that **all** fixed points of a_1 are isolated.) Pick an a_1 -invariant admissible branch curve $B \subset Y$ with $\tilde{p} \notin B$. Denote by X the double covering of Y branched over B and by a_2 , the lift of a_1 to X that fixes the two pullbacks of \tilde{p} . Then the pullbacks of \tilde{p} are isolated fixed points of a_2 . Since X is a K3-surface, $\text{Fix } a_2$ consists of isolated points only, and $(a_2)^*\mu = \mu$. We can choose for ν a generic fundamental Kähler form preserved by $\phi, c^{(1)}, c^{(2)}$, and a_2 . Then we have $(a_2)^*\tilde{\mu} = \tilde{\mu}$, i.e., a_2 is also holomorphic on \tilde{X} . With the projection $p : \tilde{X} \rightarrow E$, a_2 defines an automorphism \tilde{a} of E . \square

Lemma 4.2. *Let Y be a real unnodal Del Pezzo surface of degree $d = 2$ with $Y_{\mathbb{R}} = 2V_1$ and let Γ be the deck translation involution of the double covering $Y \rightarrow \mathbb{P}^2$ whose branch locus is a nonsingular quartic C with $C_{\mathbb{R}} = \emptyset$. Then Γ lifts to two distinct involutions on the covering K3-surface X , and one of the lifts defines an automorphism of an appropriate real Enriques surface obtained from X by inverse Donaldson’s trick.*

Proof. Pick a Γ -invariant admissible branch curve $B \subset Y$ and denote by X the double covering K3-surface of Y branched over B . Due to the adjunction formula, $\text{Fix } \Gamma \cap B \neq \emptyset$. Hence, as in the previous case, we can choose a lift a of Γ to X having isolated fixed points. Since X is a K3-surface, $\text{Fix } a$ consists of isolated points only, and $a^*\mu = \mu$. We can choose for ν a generic fundamental form preserved by $\phi, c^{(1)}, c^{(2)}$, and a . Then we have $a^*\tilde{\mu} = \tilde{\mu}$, i.e., a is also holomorphic on \tilde{X} . With the projection $p : \tilde{X} \rightarrow E$, a defines an automorphism \tilde{a} of E . \square

Lemma 4.3. *Let $Z = \Sigma_4$ (with the standard real structure), and $U \in |2e_{\infty}|$ a nonsingular real curve. Let $a : Z \rightarrow Z$ be an involution preserving U and such that $\text{Fix } a \cap U \neq \emptyset$. Then a lifts to four distinct involutions on the covering K3-surface X and at least one of the four lifts defines an automorphism of an appropriate real Enriques surface obtained from X by inverse Donaldson’s trick.*

Proof. For a nonsingular real curve $F \in |e_{\infty}|$ in Z , if $U_{\mathbb{R}}$ is contained in a connected component of $Z_{\mathbb{R}} \setminus ((E_0)_{\mathbb{R}} \sqcup F_{\mathbb{R}})$ then the DPN-double (Y, B) of $(Z; U, E_0 \cup F)$ is as follows: Y is a real unnodal $(3, 2)$ -surface, and, B is an admissible branch curve with two rational components which are conjugate to each other and $[B] = 0$ in $H_2(X)$ where X is the covering K3-surface of Y branched over B (see [3]). Any point $p \in \text{Fix } a \cap U$ has a unique pullback $\tilde{p} \in Y$ which is a fixed point of both lifts of a to Y . Any point $p' \in \text{Fix } a \setminus U$, in a small neighborhood of p , has two pullbacks p_1 and p_2 in Y . If a_1 is the lift of a to Y that permutes p_1 and p_2 then \tilde{p} is an isolated fixed point of a_1 . Choose $F \in |e_{\infty}|$ and the point $p \in \text{Fix } a \cap U$ in such a way that B is a_1 -invariant and $\tilde{p} \notin B$. Let X be the double covering of Y branched over B and let a_2 be the lift of a_1 to X that fixes the two pullbacks of \tilde{p} . Then the pullbacks of \tilde{p} are isolated fixed points of a_2 . Since X is a K3-surface, $\text{Fix } a_2$ consists of isolated points only, and $(a_2)^*\mu = \mu$. The result follows by making the same choices as in the proof of Lemma 4.1. \square

Lemma 4.4. *Let $Z = \Sigma_2$ (with the standard real structure), let $U \in |2e_\infty + 2l|$ be a suitable curve on Z , and let $a : Z \rightarrow Z$ be an involution preserving U such that $\text{Fix } a \cap U \neq \emptyset$. Then a lifts to four distinct involutions on the covering $K3$ -surface X and at least one of the four lifts defines an automorphism of an appropriate real Enriques surface obtained from X by inverse Donaldson's trick.*

Proof. For a nonsingular real section $F \in |e_\infty|$ in Z , if (U, F) is a suitable pair then the DPN-double of $(Z; U, E_0 \cup F)$ is (Y, B) where Y is a $(2, r)$ -surface and B is an admissible branch curve on Y (see [3]). Thus, for any such curve F , we can make choices of the points $p \in \text{Fix } a \cap U$ and $p' \in \text{Fix } a \setminus U$, and the lift a_1 of a to Y in the same way that we did in the proof of Lemma 4.3 so that $\tilde{p} \in Y$ will be an isolated fixed point of a_1 . Choose $F \in |e_\infty|$ and the point $p \in \text{Fix } a \cap U$ in such a way that B is a_1 -invariant and does not contain \tilde{p} . Then the result follows by making the same choices as in the proof of Lemma 4.3. \square

5. Main results

Theorem 5.1. *With one exception, any permutation of homeomorphic components of the half $E_{\mathbb{R}}^{(2)}$ of a real Enriques surface with a distinguished half $E_{\mathbb{R}}^{(1)} = V_{d+2}$, $d \geq 1$, can be realized by deformations and automorphisms. In the exceptional case $E_{\mathbb{R}} = \{V_3\} \sqcup \{V_1 \sqcup 4S\}$, the realized group is $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset S_4$.*

Remark 5.2. In the exceptional case, the Pontrjagin–Viro form (see [2]) is well defined. It defines a decomposition of $E_{\mathbb{R}}$ into quarters, which is a topological invariant. The decomposition of $E_{\mathbb{R}}^{(2)}$ is $(V_1 \sqcup 2S) \sqcup (2S)$. Obviously, one cannot permute the spheres belonging to different quarters (even topologically), and Theorem 5.1 states that a permutation of the spherical components can be realized if and only if it preserves the quarter decomposition.

Proof of Theorem 5.1. The deformation classification of real Enriques surfaces with a distinguished half $E_{\mathbb{R}}^{(1)} = V_{d+2}$, $d \geq 1$ is reduced to that of real unnodal Del Pezzo surfaces of degree d , $d \geq 1$, with a nonsingular anti-bicanonical curve $B \cong S_g$, $g \geq 2$ (see [6] for the models of Del Pezzo surfaces). It always suffices to construct a particular surface (within each deformation class) that has a desired automorphism or ‘auto-deformation’. We proceed case by case. Among the extremal types listed in [3], we need to consider only the following types (as in the other cases there are no homeomorphic components) and all their derivatives $(E_{\mathbb{R}}^{(1)}, \cdot)$ obtained from the extremal ones by sequences of topological Morse simplifications of $E_{\mathbb{R}}^{(2)}$:

1. $E_{\mathbb{R}}^{(1)} = V_3$; $E_{\mathbb{R}}^{(2)} = V_1 \sqcup 4S$;
2. $E_{\mathbb{R}}^{(1)} = V_4$; $E_{\mathbb{R}}^{(2)} = 2V_1$;
3. $E_{\mathbb{R}}^{(1)} = V_4$; $E_{\mathbb{R}}^{(2)} = 4S$;
4. $E_{\mathbb{R}}^{(1)} = V_6$; $E_{\mathbb{R}}^{(2)} = 2S$.

Case 1: Here we consider the 3 subcases:

$$E_{\mathbb{R}} = \{V_3\} \sqcup \{V_1 \sqcup iS\}, \quad i = 2, 3, 4.$$

The corresponding surface Y obtained by Donaldson's trick (see Section 3.1) is a real unnodal Del Pezzo surface of degree 1 with $Y_{\mathbb{R}} = V_1 \sqcup iS$, $i = 2, 3, 4$. According to the models of Del Pezzo surfaces, the anti-bicanonical system $|-2K|$ maps Y onto an irreducible singular quadric (cone) Z in \mathbb{P}^3 . This map $\varphi : Y \rightarrow Z$ is of degree 2 and its branch locus consists of the vertex V of Z and a nonsingular cubic section C disjoint from V whose real part $C_{\mathbb{R}}$ consists of i ovals and a component noncontractible in $Z_{\mathbb{R}} \setminus \{V\}$. The real part $Y_{\mathbb{R}}$ is the double covering of the domain D consisting of i disks bounded by the ovals of $C_{\mathbb{R}}$ and of the part of $Z_{\mathbb{R}}$ bounded by the noncontractible component of $C_{\mathbb{R}}$ and V . The map φ lifts to a degree 2 map $\tilde{\varphi} : Y \rightarrow Z = \Sigma_2 \subset \mathbb{P}^3 \times \mathbb{P}^1$ (Σ_2 with standard real structure). The branch set of $\tilde{\varphi}$ is the union of E_0 and a real nonsingular curve $C' \in |3e_\infty|$. Rigid isotopy class of C is induced by that of C' . From Theorem 2.3 and Remark 2.4, for each $i = 2, 3$ and 4, there is one rigid isotopy class of C up to isomorphism.

Clearly, a rigid isotopy of C in Z defines a deformation of Y , and an auto-involution of Z , preserving C and having nonempty fixed point set, lifts to an involution on Y . Thus, in view of Theorem 3.4 and Lemma 4.1, it suffices to realize certain permutations of the ovals of a particular curve (in each rigid isotopy class) C by rigid isotopies and/or involutive automorphisms of Z (in the latter case taking care that the fixed point set of the involution intersects C).

For each $i = 2, 3$ and 4, let $C = Z \cap S$, where Z and S in \mathbb{P}^3 are constructed (due to S. Finashin, see [8]) as follows: Let Z be the quadric cone that is the double covering of the plane branched over L_3 and L_4 if $i = 2$, L_1 and L_3 if $i = 3$, and, L_1 and L_2 if $i = 4$ (see Fig. 1). Let S be the cubic surface that is the pullback of the cubic curve, which is symmetric with respect to the line L , and is obtained by a perturbation of the lines P , Q and R (dotted lines, see Fig. 1). For $i = 2$, the symmetry of the cone with respect to the yz -plane permutes the ovals of C . For $i = 3$, it suffices to permute one pair of ovals, see Remark 2.4, and the symmetry of the cone with respect to the yz -plane does permute the opposite ovals of C . For $i = 4$, the symmetries of the cone with respect to the yz -plane and xz -plane permutes the opposite ovals of C . Fixed point set of each symmetry intersects C . Thus, we obtain the groups S_2 , S_3 and $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset S_4$ for $i = 2, 3$ and 4, respectively. For $i = 4$, the fact that other permutations cannot be realized is explained in Remark 5.2.

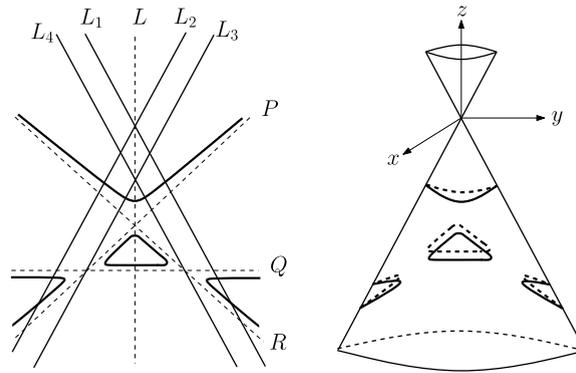


Fig. 1. Elements of the construction of a quadric cone $Z \subset \mathbb{P}^3$ and a symmetric cubic section $C \subset Z$ (left), and an example of the maximal case (right).

In cases 2 and 3 below, the corresponding surface Y obtained by Donaldson's trick is a real unnodal Del Pezzo surface of degree 2. According to the models of Del Pezzo surfaces, the anti-canonical system of Y defines a (real) degree 2 map $Y \rightarrow \mathbb{P}^2$ whose branch locus is a nonsingular quartic C .

Case 2: Here $Y_{\mathbb{R}} = 2V_1$. According to Theorem 2.1, $C_{\mathbb{R}} = \emptyset$ and $Y_{\mathbb{R}}$ is the trivial double covering of $\mathbb{P}_{\mathbb{R}}^2 \cong V_1$. The deck translation involution Γ of the covering permutes the two projective planes. According to Lemma 4.2, Γ defines an automorphism of an appropriate real Enriques surface and the resulting automorphism realizes the permutation of the two copies of V_1 in $E_{\mathbb{R}}^{(2)}$.

Case 3: In this case, $Y_{\mathbb{R}} = iS$, $i = 2, 3, 4$. According to Theorem 2.1, $C_{\mathbb{R}}$ consists of i ovals for each $i = 2, 3, 4$, and $Y_{\mathbb{R}}$ is the double covering of the orientable part of $\mathbb{P}_{\mathbb{R}}^2 \setminus C_{\mathbb{R}}$ branched over $C_{\mathbb{R}}$. According to Lemma 2.2, any permutation of the ovals of C can be realized by a rigid isotopy. The latter defines a deformation of Y and, via inverse Donaldson's trick, a deformation of an appropriate real Enriques surface that realizes the corresponding permutation of the spheres of $E_{\mathbb{R}}^{(2)}$.

Case 4: The corresponding surface Y obtained by Donaldson's trick is a real unnodal Del Pezzo surface of degree 4 with $Y_{\mathbb{R}} = 2S$. According to the models of Del Pezzo surfaces, the anti-canonical system $|-K|$ embeds Y into \mathbb{P}^4 as a surface of degree 4. Unnodal Del Pezzo surfaces of degree 4 are regular intersections of two quadrics in \mathbb{P}^4 . From Theorem 2.5, all such surfaces in \mathbb{P}^4 with real part $2S$ are rigidly isotopic. Thus, we can take for Y the intersection of the 3-sphere $\{x^2 + y^2 + z^2 + t^2 = 2u^2\}$ and the cylinder $\{x^2 + y^2 + z^2 = u^2\}$, where $[x : y : z : t : u]$ are homogeneous coordinates in \mathbb{P}^4 . Rotating the cylinder about the tu -plane through an angle π realizes the desired permutation of the spheres. \square

Theorem 5.3. *With one exception, any permutation of homeomorphic components of both halves of a real Enriques surface with a disconnected half $E_{\mathbb{R}}^{(1)} = V_d \sqcup \dots$, $d \geq 4$, can be realized by deformations and automorphisms. In the exceptional case $E_{\mathbb{R}} = \{V_4 \sqcup S\} \sqcup \{4S\}$, the realized group is $D_8 \subset S_4$.*

Remark 5.4. In the exceptional case, the Pontrjagin–Viro form is well defined. The quarter decomposition of $E_{\mathbb{R}}^{(2)}$ is $(2S) \sqcup (2S)$. A permutation of the spherical components is not realizable if it doesn't preserve the quarter decomposition. Theorem 5.3 states that a permutation of the components can be realized if and only if it preserves the quarter decomposition.

Proof of Theorem 5.3. The problem reduces to a question about appropriate (g, r) -surfaces, $g \geq 3$ and $r \geq 1$ (see [3] for the models of (g, r) -surfaces). We construct a particular surface (within each deformation class) that has a desired automorphism or 'auto-deformation'. Among the extremal types listed in [3], we need to consider only the following types (as in the other cases there are no homeomorphic components) and all their derivatives $(E_{\mathbb{R}}^{(1)}, \cdot)$ obtained from the extremal ones by sequences of topological Morse simplifications of $E_{\mathbb{R}}^{(2)}$:

1. $E_{\mathbb{R}}^{(1)} = V_4 \sqcup 2V_1$; $E_{\mathbb{R}}^{(2)} = \emptyset$;
2. $E_{\mathbb{R}}^{(1)} = V_4 \sqcup S$; $E_{\mathbb{R}}^{(2)} = 4S$.

Case 1: In this case, homeomorphic components are in $E_{\mathbb{R}}^{(1)} = B/t^{(2)}$ so we need to deal with B . By Donaldson's trick, we obtain a DPN-pair (Y, B) , where Y is a real $(3, 2)$ -surface with empty real part and $B \cong S_3 \sqcup 2S$ is an admissible branch curve on Y such that the rational components of B are real. According to the models of $(3, 2)$ -surfaces, Y blows down to $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with the real structure $c_0 \times c_1$, where c_0 is the usual complex conjugation and c_1 is the quaternionic real structure ($\text{Fix } c_1 = \emptyset$) on \mathbb{P}^1 . The image Q of B is the transversal union of smooth components C', C'' and C''' , where $C', C'' \in |l_2|$ are two distinct real generatrices and $C''' \in |4l_1 + 2l_2|$. By Theorem 18.4.1 in [3], there is only one rigid homotopy class of such curves $Q \subset \Sigma_0$ and if Q' is rigidly homotopic to Q then the DPN-resolutions of the pairs (Σ_0, Q') and (Σ_0, Q)

are deformation equivalent in the class of admissible DPN-pairs. By a rigid homotopy of real algebraic curves on Σ_0 , we mean a path Q_s of real curves on Σ_0 such that the members of the path consist of a fixed number of smooth components and have at most type A singular points. Thus, a generic rigid homotopy of Q_s defines a deformation of the DPN-resolutions (Y_s, B_s) of the pairs (Σ_0, Q_s) , thus, a deformation of the covering $K3$ -surfaces. Choosing a continuous family of invariant Kähler metrics leads to a deformation of the corresponding real Enriques surfaces obtained from inverse Donaldson's trick.

It suffices to connect Q with itself by a path that realizes the permutation of C' and C'' , and such that the members Q_s of the path split into sums $C'_s + C''_s + C'''_s$ of distinct real smooth irreducible curves such that $C'_s, C''_s \in |l_2|$ and $C'''_s \in |4l_1 + 2l_2|$. Identify the real part of the base $\mathbb{P}^1_{\mathbb{R}} \cong S^1 = \mathbb{R}/2\pi$. Let $Q = A_0 + A_{\pi} + A$, where A_{α} is the generatrix of Σ_0 over α . Then, the family $\{Q_t\} = \{A_t + A_{\pi+t} + A; t \in [0, \pi]\}$, defines a path that realizes the permutation of the generatrices A_0 and A_{π} .

Case 2: Here we consider the 3 subcases:

$$E_{\mathbb{R}} = \{V_4 \sqcup S\} \sqcup \{iS\}, \quad i = 2, 3, 4.$$

The corresponding DPN-pair resulting from Donaldson's trick is (Y, B) , where Y is a real $(3, 2)$ -surface with $Y_{\mathbb{R}} = iS$, and $B \cong S_3 \sqcup 2S$ is an admissible branch curve such that rational components of B are conjugate and $[B] = 0$ in $H_2(X)$, where X is the covering $K3$ -surface. According to the models of $(3, 2)$ -surfaces, there is a real regular degree 2 map $\phi : Y \rightarrow Z = \Sigma_4$ (with standard real structure, i.e., $Z_{\mathbb{R}} = S_1$) branched over a nonsingular real curve $U \in |2e_{\infty}|$. The irrational component of B is mapped to a real curve $F \in |e_{\infty}|$ and each rational component is mapped isomorphically to the exceptional section E_0 of Z . $U_{\mathbb{R}}$ is contained in a connected component of $Z_{\mathbb{R}} \setminus ((E_0)_{\mathbb{R}} \sqcup F_{\mathbb{R}})$. By Theorem 2.6, up to rigid isotopy and automorphism of Z , the pair (U, F) is determined by its real root scheme. By Theorem 18.4.2 in [3], the real DPN-double of $(Z; U, E_0 \cup F)$ is determined up to deformation in the class of admissible DPN-pairs by the real root scheme of the pair (U, F) . Thus, a generic rigid isotopy of the pairs (U_s, F_s) defines a deformation of the DPN-doubles (Y_s, B_s) of $(Z; U_s, E_0 \cup F_s)$, so a deformation of the covering $K3$ -surfaces. Choosing a continuous family of invariant Kähler metrics gives a deformation of the corresponding real Enriques surfaces obtained from inverse Donaldson's trick. Thus, in view of the above observation and Lemma 4.3, it suffices to realize permutations of certain ovals of U by rigid isotopies of the pair (U, F) and/or involutive automorphisms of Z preserving U . In the latter case the set of fixed points should have nonempty intersection with U .

The real root scheme of (U, F) is a disjoint union of i segments (cf. the first row of Table 1 for $i = 4$), and it has a representative (real root marking) with the desired symmetry group (i.e., S_2, S_3 and D_8 for $i = 2, 3$ and 4 , respectively), generated by rotations and reflections of $B_{\mathbb{R}} \cong S^1$. By Remark 2.8, these symmetries realize permutation of the corresponding ovals. Furthermore, the fixed point set of a reflection symmetry consists of two distinct points on S^1 , which correspond to two distinct real generatrices of Z . Since $U \in |2e_{\infty}|$, U intersects the set of the fixed points of the induced involution and one can apply Lemma 4.3. For $i = 4$, the reason, why other permutations are not allowed is explained in Remark 5.4. \square

Theorem 5.5. For the real Enriques surfaces with disconnected $E_{\mathbb{R}}^{(1)} = V_3 \sqcup \dots$, none of the permutations of the components of the half $E_{\mathbb{R}}^{(1)}$ is realizable by deformations or automorphisms. With the exceptions listed below, any permutation of homeomorphic components of the half $E_{\mathbb{R}}^{(2)}$ can be realized by deformations and automorphisms. The exceptional cases are:

1. surfaces with $E_{\mathbb{R}} = \{V_3 \sqcup V_1\} \sqcup \{4S\}$: the realized group is D_8 ;
2. surfaces with $E_{\mathbb{R}} = \{V_3 \sqcup S\} \sqcup \{V_1 \sqcup 3S\}$: the realized group is S_2 ;
3. surfaces with $E_{\mathbb{R}} = \{V_3 \sqcup V_1 \sqcup S\} \sqcup \{3S\}$: the realized group is S_2 ;
4. surfaces with $E_{\mathbb{R}} = \{V_3 \sqcup 2S\} \sqcup \{V_1 \sqcup 2S\}$: the realized group is trivial.

Remark 5.6. In the exceptional cases, the Pontrjagin–Viro form is well defined. It defines the quarter decompositions as follows:

1. $E_{\mathbb{R}} = \{(V_3 \sqcup V_1) \sqcup (\emptyset)\} \sqcup \{(2S) \sqcup (2S)\}$;
2. $E_{\mathbb{R}} = \{(V_3 \sqcup S) \sqcup (\emptyset)\} \sqcup \{(V_1 \sqcup S) \sqcup (2S)\}$;
3. $E_{\mathbb{R}} = \{(V_3 \sqcup S) \sqcup (V_1)\} \sqcup \{(2S) \sqcup (S)\}$;
4. $E_{\mathbb{R}} = \{(V_3 \sqcup S) \sqcup (S)\} \sqcup \{(V_1 \sqcup S) \sqcup (S)\}$.

One cannot permute homeomorphic components without preserving the quarter decomposition. The above theorem states that a permutation of homeomorphic components of $E_{\mathbb{R}}^{(2)}$ can be realized if and only if it preserves the quarter decomposition.

Proof of Theorem 5.5. For these surfaces, the DPN-pair resulting from Donaldson's trick is (Y, B) , where Y is a real unnodal $(2, r)$ -surface and $B \cong S_2 \sqcup rS$ is an admissible branch curve on Y , $r \geq 1$. According to the models of $(2, r)$ -surfaces ([3]), the anti-bicanonical system of Y defines a surjective degree 2 map $\varphi : Y \rightarrow Z' \subset \mathbb{P}^3$. The branch locus of φ is a cubic section through the vertex. The map φ lifts to a map $\tilde{\varphi} : Y \rightarrow Z = \Sigma_2 \subset \mathbb{P}^3 \times \mathbb{P}^1$ (with standard real structure) and the branch locus of $\tilde{\varphi}$ is a curve $U \in |2e_{\infty} + 2l|$. By $\tilde{\varphi}$, the genus 2 component of B is mapped to a nonsingular real generic section $F \in |e_{\infty}|$ and the rational components of B are mapped to the exceptional section E_0 in Z . The pair (U, F) is a suitable pair (see Section 2.8). The pullback $\tilde{\varphi}^{-1}(E_0)$ consists of the fixed components of $|-2K|$ ((-4) -curves on Y , i.e., the rational



Fig. 2.

components of B) and, possibly, several (-1) -curves. Fig. 2 shows the Dynkin graph of this configuration of curves. It is a linear tree with $2r - 1$ vertices, where the two outermost ones, marked with \tilde{E}_0 , represent the components of the proper transform of E_0 .

We start by proving the first part of the theorem. In view of Theorem 19.1 in [3], we only need to consider the real Enriques surfaces with $E_{\mathbb{R}}^{(1)} = V_3 \sqcup mV_1 \sqcup nS$, $m = 0$ or 1 and $n = 2, 3$ or 4 . By Donaldson's trick we obtain real $(2, r)$ -surfaces with admissible branch curves $B \cong S_2 \sqcup rS$, where $r = m + 2n$ (as $E_{\mathbb{R}}^{(1)} = B/t^{(2)}$). The Dynkin graph of the pullback of the exceptional section E_0 contains $m + 2n$ copies of (-4) -curves that correspond to the spherical components of B . Since the map φ is anti-bicanonical, our model is canonical and both the Dynkin graph and the corresponding Coxeter diagram on the covering $K3$ -surface are rigid. The only map that can realize a permutation of the spherical components of B is the deck translation of the covering φ which changes the order of the curves in the Dynkin graph. But since the spherical components permuted by the deck translation are identified by the map $t^{(2)}$ on the covering $K3$ -surface and $E_{\mathbb{R}}^{(1)} = B/t^{(2)}$, the result follows.

Proof of the second part is based on suitable pairs. Theorem 2.7 states that, up to rigid isotopy and automorphism of Z , a suitable pair (U, F) is determined by its real root scheme. In view of Theorem 3.6 and Lemma 4.4, it is enough to realize the permutations of certain ovals by rigid isotopies of the pair (U, F) and/or involutive automorphisms of Z preserving U , where in the latter case the set of fixed points should intersect U . Proof is very similar to that of Theorem 5.3, case 2. Among the extremal types listed in [3], we need to consider only the following types and all their derivatives $(E_{\mathbb{R}}^{(1)}, \cdot)$ obtained from the extremal ones by sequences of topological Morse simplifications of $E_{\mathbb{R}}^{(2)}$:

1. $E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1$; $E_{\mathbb{R}}^{(2)} = 4S$;
2. $E_{\mathbb{R}}^{(1)} = V_3 \sqcup S$; $E_{\mathbb{R}}^{(2)} = V_1 \sqcup 3S$;
3. $E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1 \sqcup S$; $E_{\mathbb{R}}^{(2)} = 3S$;
4. $E_{\mathbb{R}}^{(1)} = V_3 \sqcup 2S$; $E_{\mathbb{R}}^{(2)} = V_1 \sqcup 2S$;
5. $E_{\mathbb{R}}^{(1)} = V_3 \sqcup V_1 \sqcup 2S$; $E_{\mathbb{R}}^{(2)} = 2S$.

The extremal real root schemes of the pairs (U, F) for these cases are listed in Table 1; the others are obtained by removing several segments not containing a \circ -vertex. As in the proof of Theorem 5.3, one can realize each root scheme by a sufficiently symmetric representative, constructing the desired groups of permutations of the ovals of U , see Remark 2.8. In the case of automorphisms, induced by reflection symmetries, we observe that the fixed point set consists of a pair of generatrices and intersects $U \in |2e_{\infty} + 2|$; hence, Lemma 4.4 applies. Remark 5.6 explains why other permutations are not realizable. \square

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