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A filtration of the modular representation functor

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Abstract

Let $\mathbb{F}$ and $\mathbb{K}$ be algebraically closed fields of characteristics $p > 0$ and 0, respectively. For any finite group G we denote by $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G) = \mathbb{K} \otimes_{\mathbb{Z}} G_0(\mathbb{F}G)$ the modular representation algebra of G over $\mathbb{K}$ where $G_0(\mathbb{F}G)$ is the Grothendieck group of finitely generated $\mathbb{F}G$-modules with respect to exact sequences. The usual operations induction, inflation, restriction, and transport of structure with a group isomorphism between the finitely generated modules of group algebras over $\mathbb{F}$ induce maps between modular representation algebras making $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ an inflation functor. We show that the composition factors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ are precisely the simple inflation functors $S^{\mathrm{i}}_{C,V}$ where C ranges over all nonisomorphic cyclic p'-groups and V ranges over all nonisomorphic simple $\mathbb{K}\,\mathrm{Out}(C)$-modules. Moreover each composition factor has multiplicity 1. We also give a filtration of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$.

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1. Introduction

The purpose of this paper is to describe the structure of the inflation functor $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ mapping a finite group G to $\mathbb{K} \otimes_{\mathbb{Z}} G_0(G)$ where $G_0(G)$ is the Grothendieck group of finite dimensional $\mathbb{F}G$-modules. The cases $\mathbb{C}\mathcal{R}_{\mathbb{C}}$ (as a biset functor) and $k\mathcal{R}_{\mathbb{Q}}$ (as a p-biset functor over a field k of characteristic p) were dealt by Bouc [3, Proposition 27] and Bouc [4]. Another related work is Webb [7] in which he studied inflation and global Mackey functors, and described the structure of cohomology groups as these functors.

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One of our main result Theorem 6.17 states that there is a chain of inflation functors

$$\mathbb{K}\mathcal{R}_{\mathbb{F}} = L_{-1} \supset L_0 \supset L_1 \supset \cdots \supset L_j \supset \cdots$$

such that $\bigcap_j L_j = 0$ and each L_{j-1}/L_j is semisimple with

$$L_{j-1}/L_j \cong \bigoplus_{C,V} S^{\mathfrak{i}}_{C,V}$$

where C ranges over all nonisomorphic cyclic p'-groups with $\ell(C) = j$ and V ranges over all nonisomorphic simple $\mathbb{K}\,\mathrm{Out}(C)$-modules. Here, $\ell(C)$ is the number of prime divisors of the order of C counted with multiplicities. Moreover L_j is the inflation subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ given for any finite group G by

$$L_j(G) = \bigcap_X \mathrm{Ker}\big(\mathbb{K}\mathcal{R}_{\mathbb{F}}\big(\mathrm{Res}^G_X\big) : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(X)\big)$$

where X ranges over all cyclic p'-subgroups of G with $\ell(X) \leqslant j$. The question may be raised as to the finding a similar result for the deflation functor $\mathbb{K}\mathcal{P}_{\mathbb{F}}$, where $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ is the functor mapping a finite group G to $\mathbb{K} \otimes_{\mathbb{Z}} K_0(G)$ and $K_0(G)$ is the Grothendieck group of finite dimensional projective $\mathbb{F}G$-modules. Such a result follows immediately from Theorem 7.1 in which we prove that

$$\mathbb{K}\mathcal{P}_{\mathbb{F}} \cong \mathbb{K}\mathcal{R}^*_{\mathbb{F}}$$

as deflation functors, where $\mathbb{K}\mathcal{R}^*_{\mathbb{F}}$ denotes the dual of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$.

A biset functor, introduced by Bouc [3], is a notion having five kind of operations unifying the similar operations induction, inflation, transport of structure with a group isomorphism, deflation, and restriction which occur in group representation theory. It is defined to be an R-linear (covariant) functor from an R-linear category $\mathfrak{b}$, called the biset category, to the category of (left) R-modules where R is a commutative unital ring.

To realize some representation theoretic algebras as functors one may need to consider functors from some (nonfull) subcategories of the biset category to the category of R-modules because some bisets (morphisms of $\mathfrak{b}$) do not induce maps between these algebras in a natural way. For $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ a similar situation occurs since bisets corresponding to deflations may not induce exact functors between finitely generated module categories of group algebras over the field $\mathbb{F}$ whose characteristic is $p > 0$. For this reason we also consider inflation functors which are defined to be functors from the category $\mathfrak{i}$ to the category of R-modules where $\mathfrak{i}$ is the subcategory of $\mathfrak{b}$ with same objects and with morphisms bisets which are free from right.

The aim of this paper is to study $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ as inflation functor and in particular to find its composition factors together with their multiplicities. Our approach to this problem can be explained briefly as follows.

We first review some of the standard facts on the subject given in Bouc [3]. We then study properties of two specific subfunctors of a given functor M in Section 3 in a slight general form, namely the subfunctors Im^M and Ker^M which are roughly defined to be sum of images and intersection of preimages. Our reason in studying these subfunctors comes from the importance of them in the context of (ordinary) Mackey functors. For a functor M whose Ker^M subfunctor is 0, in Proposition 3.3 we construct a bijective correspondence between the minimal subfunctors of M and the minimal submodules of a coordinate module of M. We next observe that Ker^S subfunctor of any simple inflation functor $S = S^{\mathfrak{i}}_{H,V}$ considered as (global) Mackey functor is 0. This leads us to state Proposition 3.8 saying that any simple inflation functor $S^{\mathfrak{i}}_{H,V}$ has a unique minimal Mackey subfunctor and this subfunctor is isomorphic to $S^{\mathfrak{m}}_{H,V}$. Using the semisimplicity of (global) Mackey functors over fields of characteristic 0, which can be found in Webb [8], we observe in Theorem 3.10 that over fields of characteristic 0, any simple inflation functor $S^{\mathfrak{i}}_{H,V}$ is isomorphic to $S^{\mathfrak{m}}_{H,V}$ as Mackey functors.

These observations imply Proposition 4.5 in which we prove that the multiplicity of a simple inflation functor $S^{\mathfrak{i}}_{H,V}$ in $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ is equal to the multiplicity of the simple Mackey functor $S^{\mathfrak{m}}_{H,V}$ in $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ which is the dimension of the $\mathbb{K}$-space

$$\mathrm{Hom}_{\mathbb{K}\,\mathrm{Out}(H)}\big(V, \mathbb{K}\mathcal{R}_{\mathbb{F}}(H)/I^{\mathfrak{m}}_H \mathbb{K}\mathcal{R}_{\mathbb{F}}(H)\big),$$

where $I^{\mathfrak{m}}_H$ is the ideal of $\mathrm{End}_{\mathfrak{m}}(H)$ spanned by the bisets factorizing through groups of order less than $|H|$, and $\mathrm{End}_{\mathfrak{m}}(H)$ is the $\mathbb{K}$-algebra of (H, H)-bisets which are free from left and right, see Section 2.

We begin to study composition factors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ in Section 5. Using Artin's induction theorem we show in Proposition 5.2 that if $S^{\mathfrak{i}}_{H,V}$ is a composition factor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ then H is a cyclic p'-group. Next we include Lemma 5.4 about the multiplicities of composition factors with minimal subgroups are direct products of two groups of coprime orders. This reduces the problem to computing the multiplicities of composition factors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ of the form $S^{\mathfrak{i}}_{C_{q^n},V}$ where q is a prime different from p, n is a natural number, and C_{q^n} is a cyclic group of order q^n. For this kind of composition factors, by calculating the dimensions of the above Hom spaces we are able to show in Lemma 5.3 that the multiplicities are all equal to 1. We state our final result about this topic as Theorem 5.5.

Our aim in Section 6 is to study subfunctors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ and in particular sections of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ which are semisimple functors. Motivated by the results which we obtained already, we define two subfunctors $K_n \leqslant F_n$ of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ for a natural number n. Given any cyclic p'-group C of order n, we prove in Proposition 6.14 that F_n/K_n is a semisimple inflation functor whose simple summands are the simple inflation functors $S^{\mathfrak{i}}_{C,V}$ where V ranges over all nonisomorphic simple $\mathbb{K}\,\mathrm{Out}(C)$-modules. Finally, using these subfunctors we construct some series of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ whose factors are semisimple inflation functors and cover all composition factors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$, see Theorem 6.15 and its consequences.

Our notations are mostly standard. Let $H \leqslant G \geqslant K$ be finite groups. By the notation $HgK \subseteq G$ we mean that g ranges over a complete set of representatives of double cosets of (H, K) in G. The notation $S \leqslant_* G$ appearing in an index set means that S ranges over all non-G-conjugate subgroups of G. The coefficient rings on which we are working will be explained at the beginnings of each section.

2. Preliminaries

In this section, we simply collect some crucial results on bisets and functors in Bouc [3]. Throughout R is a commutative unital ring. Let G, H, and K be finite groups. A (G, H)-biset is a finite set U having a left G-action and a right H-action such that the two actions commute. Given a (G, H)-biset U and an (H, K)-biset V, the cartesian product $U \times V$ becomes a right H-set with the action $(u, v)h = (uh, h^{-1}v)$. If we let $u \otimes v$ denote the H-orbit of $U \times V$ containing

(u, v), then the set $U \times_H V$ of the H-orbits of $U \times V$ becomes a (G, K)-biset with the actions $g(u \otimes v)k = gu \otimes vk$. Any (G, H)-biset U is a left $G \times H$-set by the action $(g, h)u = guh^{-1}$, and conversely. Terminology for (G, H)-bisets is inherited from terminology for $G \times H$-sets. Thus transitive (G, H)-bisets are isomorphic to bisets of the form $(G \times H)/L$ where L is a subgroup $G \times H$. We write $[U]$ for the isomorphism class of a biset U. Let L be a subgroup of $G \times H$. We define

$$p_1(L) = \{g \in G\colon \exists h \in H,\ (g, h) \in L\}, \quad \text{and} \quad k_1(L) = \{g \in G\colon (g, 1) \in L\},$$
$$p_2(L) = \{h \in H\colon \exists g \in G,\ (g, h) \in L\}, \quad \text{and} \quad k_2(L) = \{h \in H\colon (1, h) \in L\}.$$

Then $k_i(L)$ is a normal subgroup $p_i(L)$, and $k_1(L) \times k_2(L)$ is a normal subgroup of L, and the three quotient groups which we denote by $q(L)$ are isomorphic. If $L \leqslant G \times H$ and $M \leqslant H \times K$ we write

$$L * M = \{(g, k) \in G \times K\colon \exists h \in H,\ (g, h) \in L,\ (h, k) \in M\}.$$

Proposition 2.1. *(See [3].) Let $L \leqslant G \times H$ and $M \leqslant H \times K$. Then*

$$\big((G \times H)/L\big) \times_H \big((H \times K)/M\big) \cong \sum_{p_2(L)hp_1(M) \subseteq H} (G \times K)/\big(L *^{(h,1)}M\big).$$

There are five types of basic bisets so that any transitive biset is isomorphic to a product of them. For $H \leqslant G \trianglerighteq N$ and isomorphism of groups $\psi : G \to G'$, they are

$$\mathrm{Ind}_H^G = (G \times H)/\{(h, h)\colon h \in H\},$$
$$\mathrm{Res}_H^G = (H \times G)/\{(h, h)\colon h \in H\},$$
$$\mathrm{Inf}_{G/N}^G = (G \times G/N)/\{(g, gN)\colon g \in G\},$$
$$\mathrm{Def}_{G/N}^G = (G/N \times G)/\{(gN, g)\colon g \in G\},$$
$$\mathrm{Iso}_G^{G'}(\psi) = (G' \times G)/\{\big(\psi(g), g\big)\colon g \in G\}.$$

For any $L \leqslant G \times H$ we have

$$(G \times H)/L \cong \mathrm{Ind}_{p_1(L)}^G \mathrm{Inf}_{p_1(L)/k_1(L)}^{p_1(L)} \mathrm{Iso}_{p_2(L)/k_2(L)}^{p_1(L)/k_1(L)}(\psi) \mathrm{Def}_{p_2(L)/k_2(L)}^{p_2(L)} \mathrm{Res}_{p_2(L)}^H$$

where $\psi(hk_2(L)) = gk_1(L)$ if and only if $(g, h) \in L$.

Let χ be a family of finite groups closed under taking subgroups, taking isomorphisms and taking quotients. We define the biset category $\mathfrak{b}$ (on χ over R), which is R-linear, as follows:

- The objects are the groups in χ.
- If H and G are in χ then $\mathrm{Hom}_{\mathfrak{b}}(H, G) = RB(G \times H)$ is the Burnside group of (G, H)-bisets, with coefficients in R.
- Composition of morphisms is obtained by R-linearity from the product $(U, V) \mapsto U \times_H V$.

Any R-linear (covariant) functor from the category $\mathfrak{b}$ to the category of left R-modules is called a biset functor (on χ over R). We denote by $\mathfrak{F}_{\mathfrak{b}}$ the category of biset functors, which is an abelian category.

We also want to consider some nonfull subcategories of $\mathfrak{b}$ and R-linear functors from these subcategories to the category of left R-modules. Let $\mathfrak{i}$ be the subcategory of $\mathfrak{b}$ with the same objects and with the morphisms

$$\mathrm{Hom}_{\mathfrak{i}}(H, G) = \bigoplus_{L \leqslant_* G \times H:\ k_2(L)=1} R\big[(G \times H)/L\big].$$

An R-linear functor from $\mathfrak{i}$ to the category of left R-modules is called an inflation functor (on χ over R). We denote by $\mathfrak{F}_{\mathfrak{i}}$ the category of inflation functors.

Let $\mathfrak{m}$ be the subcategory of $\mathfrak{b}$ with the same objects and with the morphisms

$$\mathrm{Hom}_{\mathfrak{m}}(H, G) = \bigoplus_{L \leqslant_* G \times H:\ k_1(L)=1=k_2(L)} R\big[(G \times H)/L\big].$$

An R-linear functor from $\mathfrak{m}$ to the category of left R-modules is called a (global) Mackey functor (on χ over R). We denote by $\mathfrak{F}_{\mathfrak{m}}$ the category of Mackey functors. Mackey functors can also be defined on a family χ of finite groups closed under taking subgroups and taking isomorphism.

These three functor categories have similar theories. For example their simple objects are parameterized in the same manner. From now on in this section, a functor means any of biset, inflation or Mackey.

For any groups X and Y in χ the composition of morphism gives an $(\mathrm{End}(Y), \mathrm{End}(X))$-bimodule structure on $\mathrm{Hom}(X, Y)$, and for a functor M we have an $\mathrm{End}(X)$-module structure on $M(X)$ given by $fm_X = M(f)(m_X)$. For a group X in χ and an $\mathrm{End}(X)$-module V we define a functor $L_{X,V}$ and its subfunctor $J_{X,V}$ as follows:

$$L_{X,V}(Y) = \mathrm{Hom}(X, Y) \otimes_{\mathrm{End}(X)} V,$$
$$L_{X,V}(f): L_{X,V}(Y) \to L_{X,V}(Z), \quad \theta \otimes v \mapsto f\theta \otimes v,$$
$$J_{X,V}(Y) = \bigcap_{f \in \mathrm{Hom}(Y,X)} \mathrm{Ker}\big(L_{X,V}(f)\big).$$

Having defined the functors $L_{X,V}$ we define two important functors between the functor category $\mathfrak{F}$ (i.e., any of $\mathfrak{F}_{\mathfrak{b}}$, $\mathfrak{F}_{\mathfrak{i}}$ or $\mathfrak{F}_{\mathfrak{m}}$) and $\mathrm{End}(X)$-module category,

$$L_{X,-}: \mathrm{End}(X)\text{-Mod} \to \mathfrak{F}, \quad V \mapsto L_{X,V},$$

and if $\varphi: V \to W$ is an $\mathrm{End}(X)$-module homomorphism then $L_{X,-}(\varphi): L_{X,V} \to L_{X,W}$ is the natural transformation whose $Y \in \chi$ component is the map $L_{X,V}(Y) \to L_{X,W}(Y)$, given by $f \otimes v \mapsto f \otimes \varphi(v)$,

$$e_X: \mathfrak{F} \to \mathrm{End}(X)\text{-Mod}, \quad M \mapsto M(X),$$

and if $\pi: M \to N$ is a morphism of functors (i.e., a natural transformation) then $e_X(\pi)$ is the X component $\pi_X: M(X) \to N(X)$ of π.

Proposition 2.2. *(See [3].) Let X be a group in χ. Then:*

(1) *e_X is an exact functor and $L_{X,-}$ is a right exact functor.*
(2) *$(L_{X,-}, e_X)$ is an adjoint pair.*
(3) *If V is a projective $\mathrm{End}(X)$-module then $L_{X,V}$ is a projective functor.*
(4) *If V is an indecomposable $\mathrm{End}(X)$-module then $L_{X,V}$ is an indecomposable functor.*

Let M be a functor. A group H in χ is called a minimal subgroup of M if $M(H) \neq 0$ and $M(K) = 0$ for all $K \in \chi$ with $|K| < |H|$.

Proposition 2.3. *(See [3].) Let X be a group in χ and let V be a simple $\mathrm{End}(X)$-module. Then, $J_{X,V}$ is the unique maximal subfunctor of $L_{X,V}$ and $L_{X,V}/J_{X,V}$ is a simple functor whose evaluation at X is V. However, X may not be a minimal subgroup of this simple functor.*

Proposition 2.4. *(See [3].) For a group G in χ, there is a direct sum decomposition*

$$\mathrm{End}(G) = \mathrm{Ext}(G) \oplus I_G$$

where I_G is a two sided ideal of $\mathrm{End}(G)$ with an R-basis consisting of the elements $[(G \times G)/L]$ of $\mathrm{End}(G)$ with $|q(L)| < |G|$, and $\mathrm{Ext}(G)$ is a unital subalgebra of $\mathrm{End}(G)$ isomorphic to the group algebra $R\,\mathrm{Out}(G)$ of the group of outer automorphisms of G.

A simple functor S with a minimal subgroup H is denoted by $S_{H,V}$ if $S(H) = V$.

Theorem 2.5. *(See [3].) In the following an $R\,\mathrm{Out}(H)$-module is considered as an $\mathrm{End}(H)$-module via the natural projection map $\mathrm{End}(H) \to Ext(H) \cong R\,\mathrm{Out}(H)$ given in 2.4.*

(1) *Let H be a group in χ and let V be a simple $R\,\mathrm{Out}(H)$-module. Then H is a minimal subgroup of the simple functor $L_{H,V}/J_{H,V}$. So $L_{H,V}/J_{H,V} = S_{H,V}$.*
(2) *Let S be a simple functor and let H be a minimal subgroup S. Then I_H annihilates $S(H)$, and $S(H)$ is a simple $R\,\mathrm{Out}(H)$-module, and $S \cong S_{H,V}$ where $S(H) = V$.*
(3) *$S_{H,V} \cong S_{K,W}$ if and only if there is a group isomorphism $H \to K$ transporting V to W.*

We use the notations like $S = S^{\mathfrak{b}}_{H,V}$, $L = L^{\mathfrak{i}}_{X,V}$, $I = I^{\mathfrak{m}}_G, \ldots$ to indicate respectively that S is the biset functor, L is the inflation functor, I is the ideal of $\mathrm{End}_{\mathfrak{m}}(G)$ in 2.4. For a functor M we also use the notation M^{χ} to indicate that it is defined on χ. A functor can also be considered as a module of the category algebra of the skeletal category of its domain category (i.e., any of $\mathfrak{b}$, $\mathfrak{i}$, or $\mathfrak{m}$). Identifying the isomorphic groups in χ we can form the category algebra $\Gamma = \bigoplus_{X,Y \in [\chi]} R\,\mathrm{Hom}(X, Y)$ with product being the composition of morphisms whenever they are composable and zero otherwise, where the notation $[\chi]$ denotes the representatives of the isomorphism classes of groups in χ. If M is a functor on χ over R then $\widetilde{M} = \bigoplus_{X \in [\chi]} M(X)$ is a Γ-module with the obvious action, and conversely. In this way one can define functors on a finite family of finite groups χ such that no two groups in χ are isomorphic and if X is in χ then any section of X is isomorphic to a group in χ. Thus in this situation functors may be regarded as modules of finite dimensional algebras, allowing one to apply the theory of modules of finite dimensional algebras. We will follow this approach only when we need to consider composition series, composition factors, etc. of functors. For a more detailed study of this approach see Webb [9] for arbitrary functor categories, and Barker [1] for biset functor categories.

3. Maximal and minimal subfunctors

Our main aim in this section is to show that, over characteristic 0 fields, any simple inflation functor $S^{\mathfrak{i}}_{H,V}$ is isomorphic to $S^{\mathfrak{m}}_{H,V}$ as (global) Mackey functors. We divide this section into two parts. In the first part we include some general results which will be crucial for some later results.

3.1. Some generalities

In this section R is a commutative unital ring, $\mathfrak{A}$ is an (small) R-linear category, and $\mathfrak{F}$ be the category of R-linear (covariant) functors from $\mathfrak{A}$ to the category of left R-modules.

For a functor $M \in \mathfrak{F}$, an object X of $\mathfrak{A}$, and an $\mathrm{End}_{\mathfrak{A}}(X)$-submodule W of $M(X)$, we define two subfunctors $\mathrm{Im}^M_{X,W}$ and $\mathrm{Ker}^M_{X,W}$ of M whose evaluations at any object Y of $\mathfrak{A}$ are given as follows:

$$\mathrm{Im}^M_{X,W}(Y) = \sum_{f \in \mathrm{Hom}_{\mathfrak{A}}(X,Y)} M(f)(W),$$

$$\mathrm{Ker}^M_{X,W}(Y) = \bigcap_{f \in \mathrm{Hom}_{\mathfrak{A}}(Y,X)} M(f)^{-1}(W).$$

We collect some properties of these subfunctors in the following result.

The usage of these subfunctors in (ordinary) Mackey functor categories is well known. And an analogue of 3.5 is proved in Bourizk [6, Lemme 1] for some subfunctors of the Burnside functor considered as biset functors.

Remark 3.1. Let $M \in \mathfrak{F}$ be a functor, X be an object of $\mathfrak{A}$, and N be a subfunctor of M. Suppose that U and W are $\mathrm{End}_{\mathfrak{A}}(X)$-submodules of $N(X)$ and $M(X)$, respectively. Then*:*

(1) $\mathrm{Im}^M_{X,W}$ and $\mathrm{Ker}^M_{X,W}$ are subfunctors of M such that $\mathrm{Im}^M_{X,W}(X) = W$ and $\mathrm{Ker}^M_{X,W}(X) = W$.
(2) If Y is an object of $\mathfrak{A}$, then $\mathrm{Im}^M_{Y,N(Y)} = \mathrm{Im}^N_{Y,N(Y)}$ is a subfunctor of N and N is a subfunctor of $\mathrm{Ker}^M_{Y,N(Y)}$. So $\mathrm{Im}^M_{X,W}$ is the subfunctor of M generated by W.
(3) If W' is an $\mathrm{End}_{\mathfrak{A}}(X)$-submodule of W, then $\mathrm{Im}^M_{X,W'}$ and $\mathrm{Ker}^M_{X,W'}$ are subfunctors of $\mathrm{Im}^M_{X,W}$ and $\mathrm{Ker}^M_{X,W}$, respectively.
(4) If W' is an $\mathrm{End}_{\mathfrak{A}}(X)$-submodule of $M(X)$, then $\mathrm{Im}^M_{X,W'} + \mathrm{Im}^M_{X,W} = \mathrm{Im}^M_{X,W'+W}$ and $\mathrm{Ker}^M_{X,W'} \cap \mathrm{Ker}^M_{X,W} = \mathrm{Ker}^M_{X,W' \cap W}$.
(5) $\mathrm{Ker}^M_{X,U} \cap N = \mathrm{Ker}^N_{X,U}$, and if $I = \mathrm{Ker}^M_{X,0}$ then $\mathrm{Ker}^I_{X,0} = I$.
(6) $(\mathrm{Im}^M_{X,W} + N)/N = \mathrm{Im}^{M/N}_{X,(W+N(X))/N(X)}$ and $\mathrm{Ker}^M_{X,N(X)}/N = \mathrm{Ker}^{M/N}_{X,0}$.

Proof. All parts follow immediately from the definitions of Im and Ker. □

Lemma 3.2. *Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$ such that $M(X)$ is nonzero. Assume that* $\mathrm{Ker}^M_{X,0} = 0$. *Then*:

(1) *If W is a minimal* $\mathrm{End}_{\mathfrak{A}}(X)$*-submodule of $M(X)$, then* $\mathrm{Im}^M_{X,W}$ *is a minimal subfunctor of M.*
(2) *If I is a minimal subfunctor of M, then $I(X)$ is a minimal* $\mathrm{End}_{\mathfrak{A}}(X)$*-submodule of $M(X)$. Moreover $I = \mathrm{Im}^M_{X,I(X)}$.*

Proof. (1) Let W be a minimal $\mathrm{End}_{\mathfrak{A}}(X)$-submodule of $M(X)$. If N is a subfunctor of M such that $N \leqslant \mathrm{Im}^M_{X,W}$, then $N(X)$ is an $\mathrm{End}_{\mathfrak{A}}(X)$-submodule of $\mathrm{Im}^M_{X,W}(X) = W$ implying by the minimality of W that $N(X) = 0$ or $N(X) = W$. Suppose that $N(X) = 0$. Then by 3.1 we have that N is a subfunctor of $\mathrm{Ker}^M_{X,N(X)} = \mathrm{Ker}^M_{X,0} = 0$, implying that $N = 0$. In the case $N(X) = W$, it follows by 3.1 that $\mathrm{Im}^M_{X,W}$ is a subfunctor of N; so $N = \mathrm{Im}^M_{X,W}$. Hence $\mathrm{Im}^M_{X,W}$ is a minimal subfunctor of M.

(2) Let I be a minimal subfunctor of M. As I is a subfunctor of $\mathrm{Ker}^M_{X,I(X)}$ by 3.1, $I(X)$ must be nonzero. If there is a nonzero proper $\mathrm{End}_{\mathfrak{A}}(X)$-submodule W of $I(X)$, then 3.1 implies that $\mathrm{Im}^M_{X,W}$ is a nonzero proper subfunctor of I, contradicting to the minimality of I. Hence $I(X)$ is a minimal $\mathrm{End}_{\mathfrak{A}}(X)$-submodule of $M(X)$. Finally, as $I(X)$ is nonzero it follows by 3.1 that $\mathrm{Im}^M_{X,I(X)} = \mathrm{Im}^I_{X,I(X)}$ is a nonzero subfunctor of I. Now the equality $I = \mathrm{Im}^M_{X,I(X)}$ follows by the minimality of I. □

The previous lemma implies

Proposition 3.3. *Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$ such that $M(X)$ is nonzero. Assume that* $\mathrm{Ker}^M_{X,0} = 0$. *Then the maps $I \to I(X)$,* $\mathrm{Im}^M_{X,W} \leftarrow W$ *define a bijective correspondence between the minimal subfunctors of M and the minimal* $\mathrm{End}_{\mathfrak{A}}(X)$*-submodules of $M(X)$.*

Lemma 3.4. *Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$ such that $M(X)$ is nonzero. Assume that* $\mathrm{Im}^M_{X,M(X)} = M$. *Then*:

(1) *If W is a maximal* $\mathrm{End}_{\mathfrak{A}}(X)$*-submodule of $M(X)$, then* $\mathrm{Ker}^M_{X,W}$ *is a maximal subfunctor of M.*
(2) *If J is a maximal subfunctor of M, then $J(X)$ is a maximal* $\mathrm{End}_{\mathfrak{A}}(X)$*-submodule of $M(X)$. Moreover $J = \mathrm{Ker}^M_{X,J(X)}$.*

Proof. (1) Let W be a maximal $\mathrm{End}_{\mathfrak{A}}(X)$-submodule of $M(X)$. Then by 3.1 $\mathrm{Ker}^M_{X,W}$ is not equal to M. If N is a subfunctor of M containing $\mathrm{Ker}^M_{X,W}$, then the maximality of W implies that $W = N(X)$ or $N(X) = M(X)$. In the case $N(X) = M(X)$, it follows by 3.1 that $M = \mathrm{Im}^M_{X,M(X)}$ is a subfunctor N, implying that $M = N$. Assume now that $N(X) = W$. Then 3.1 gives that N is a subfunctor of $\mathrm{Ker}^M_{X,W}$, and so $N = \mathrm{Ker}^M_{X,W}$. Hence $\mathrm{Ker}^M_{X,W}$ is a maximal subfunctor of M.

(2) Let J be a maximal subfunctor of M. In particular J is not equal to M, implying by the condition $\mathrm{Im}^M_{X,M(X)} = M$ that $J(X)$ is not equal to $M(X)$. If there is an $\mathrm{End}_{\mathfrak{A}}(X)$-submodule W of $M(X)$ containing $J(X)$ then by 3.1 we have $J \leqslant \mathrm{Ker}^M_{X,J(X)} \leqslant \mathrm{Ker}^M_{X,W}$. The maximality of J implies that $\mathrm{Ker}^M_{X,W} = M$ or $\mathrm{Ker}^M_{X,W} = J$. And by evaluating at X we see that $W = M(X)$ or $W = J(X)$. Hence $J(X)$ is a maximal $\mathrm{End}_{\mathfrak{A}}(X)$-submodule of $M(X)$. Finally, by 3.1 we have $J \leqslant \mathrm{Ker}^M_{X,J(X)}$. The equality follows because J is maximal subfunctor of M and $\mathrm{Ker}^M_{X,J(X)}$ is not equal to M. □

The previous lemma implies

Proposition 3.5. *Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$ such that $M(X)$ is nonzero. Assume that $\mathrm{Im}^M_{X,M(X)} = M$. Then the maps $J \to J(X)$, $\mathrm{Ker}^M_{X,W} \leftarrow W$ define a bijective correspondence between the maximal subfunctors of M and the maximal $\mathrm{End}_{\mathfrak{A}}(X)$-submodules of $M(X)$.*

Corollary 3.6. *Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$ such that $M(X)$ is nonzero. Then M is simple if and only if $\mathrm{Im}^M_{X,M(X)} = M$, $\mathrm{Ker}^M_{X,0} = 0$, and $M(X)$ is a simple $\mathrm{End}_{\mathfrak{A}}(X)$-module.*

Proof. Suppose that M is simple. For any nonzero proper $\mathrm{End}_{\mathfrak{A}}(X)$-submodule W of $M(X)$, it follows by 3.1 that $\mathrm{Im}^M_{X,W} \neq 0$ and $\mathrm{Ker}^M_{X,0} \neq M$ are proper subfunctors of M. Since M is simple, $W = M(X)$ and $\mathrm{Ker}^M_{X,0} = 0$. So $M(X)$ is a simple module and $\mathrm{Im}^M_{X,M(X)} = M$. Conversely, if M satisfies the given conditions then it follows by 3.5 that $\mathrm{Ker}^M_{X,0} = 0$ is the unique maximal subfunctor M. So M is simple. □

Using the properties of Im and Ker given in 3.1, we give an obvious generalization of the previous result.

Corollary 3.7. *Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$ such that $N(X)$ is nonzero for all nonzero subfunctors N of M. Then M is semisimple if and only if $\mathrm{Im}^M_{X,M(X)} = M$, $\mathrm{Ker}^M_{X,0} = 0$, and $M(X)$ is a semisimple $\mathrm{End}_{\mathfrak{A}}(X)$-module.*

3.2. Applications

Throughout this section we work over an arbitrary field $\mathbb{L}$. We want to give some applications of the general results obtained in Section 3.1. Especially, we want to reduce the problem of finding multiplicities of simple inflation functors in $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ to the problem of finding multiplicities of simple Mackey functors in $\mathbb{K}\mathcal{R}_{\mathbb{F}}$.

Proposition 3.8. *Any simple inflation functor $S^{\mathfrak{i}}_{H,V}$ has a unique minimal Mackey subfunctor M. Moreover $M \cong S^{\mathfrak{m}}_{H,V}$.*

Proof. Let $S = S^{\mathfrak{i}}_{H,V}$, $L = L^{\mathfrak{i}}_{H,V}$, and $J = J^{\mathfrak{i}}_{H,V}$. We will show that $\mathrm{Ker}^{S,\mathfrak{m}}_{H,0} = 0$. Take any finite group G. For any $T \leqslant H \times G$ with $k_2(T)$=1 and $|q(T)| < |H|$, we see that

$$\big[(H \times G)/T\big]\mathrm{Hom}_{\mathfrak{i}}(H,G) \subseteq I^{\mathfrak{i}}_H$$

and so annihilates $V = L(H)$, see also Bouc [3]. Consequently the image of the map

$$L\big(\big[(H \times G)/T\big]\big) : L(G) \to L(H)$$

is zero. Hence

$$S\big(\big[(H \times G)/T\big]\big)\big(S(G)\big) = \big(L\big(\big[(H \times G)/T\big]\big)\big(L(G)\big) + J(H)\big)/J(H) = 0.$$

As S is a simple inflation functor, $\mathrm{Ker}^{S,\mathfrak{i}}_{H,0} = 0$ by 3.6. As $|q(T)| = |H|$ implies $k_1(T) = 1$, we have

$$\begin{aligned} 0 &= \operatorname{Ker}_{H,0}^{S,\mathfrak{i}} \\ &= \bigcap_{T \leqslant H \times G:\ k_2(L)=1} \operatorname{Ker}\bigl(S\bigl(\bigl[(H \times G)/T\bigr]\bigr)\bigr) \\ &= \bigcap_{T \leqslant H \times G:\ k_2(L)=1=k_1(L)} \operatorname{Ker}\bigl(S\bigl(\bigl[(H \times G)/T\bigr]\bigr)\bigr) \\ &= \operatorname{Ker}_{H,0}^{S,\mathfrak{m}}. \end{aligned}$$

Now 3.3 implies that $M = \operatorname{Im}_{H,V}^{S,\mathfrak{m}}$ is the unique minimal Mackey subfunctor of S, because $S(H)$ is a simple $\operatorname{End}_{\mathfrak{m}}(H)$-module. Finally it is clear that $M \cong S_{H,V}^{\mathfrak{m}}$. □

The next result allows us to give a nice consequence of 3.8.

Theorem 3.9.

(1) (*Bouc*) *Let* $\mathbb{L}$ *be of characteristic* 0. *Then, the biset functor category on* χ *over* $\mathbb{L}$ *is semisimple if and only if every group in* χ *is cyclic.*
(2) (*Thévenaz–Webb*) *Let* $\mathbb{L}$ *be of characteristic* 0. *Then the* (*global*) *Mackey functor category* (*on* χ) *over* $\mathbb{L}$ *is semisimple.*
(3) *The inflation functor category on* χ *over* $\mathbb{L}$ *is semisimple if and only if every group in* χ *is trivial.*

Proof. For the parts (1) and (2), see respectively Barker [1] and Webb [8, Theorem 4.1].

(3) The sufficiency is obvious. Suppose that the inflation functor category is semisimple. So every simple inflation functor, in particular $S_{1,\mathbb{L}}^{\mathfrak{i}}$, is projective. Since $\operatorname{End}_{\mathfrak{i}}(1) \cong \mathbb{L}$ it follows by 2.2 that $L_{1,\mathbb{L}}^{\mathfrak{i}}$ is the projective cover of $S_{1,\mathbb{L}}^{\mathfrak{i}}$. By the definition of the functors $L_{Y,W}$ we see that $L_{1,\mathbb{L}}^{\mathfrak{i}}$ is isomorphic to the Burnside (inflation) functor $B^{\mathfrak{i}}$. Hence $S_{1,\mathbb{L}}^{\mathfrak{i}} \cong B^{\mathfrak{i}}$. Suppose that χ contains a group G with $|G| \neq 1$. Then $\dim_{\mathbb{L}} B^{\mathfrak{i}}(G) \geqslant 2$. So it suffices to show that the dimension of $S_{1,\mathbb{L}}^{\mathfrak{i}}(G)$ is 1 for any finite group G. One way of doing this is to use the arguments in Bouc [3] which show that, for a simple functor S, the dimension of the space $S(G)$ at a finite group G is equal to the rank of a certain matrix. Alternatively, as the referee suggested, we can use an explicit description of the simple functor $S_{1,\mathbb{L}}^{\mathfrak{i}}$. For any finite group G, we let the vector space $M(G)$ be equal to $\mathbb{L}$. If U is a right free (H, G)-biset, then we let the map $M([U]) : \mathbb{L} \to \mathbb{L}$ be equal to multiplication by $|U/G|$, where $|U/G|$ denotes the number of G-orbits on U. Then M becomes an inflation functor, because if V is a right free (K, H)-biset, then $|(V \times_H U)/G| = |V/H||U/G|$. Now one can see easily, for example by using 3.6, that M is the simple inflation functor $S_{1,\mathbb{L}}^{\mathfrak{i}}$. Therefore $G \in \chi$ implies that $G = 1$. □

Theorem 3.10. *Let* $\mathbb{L}$ *be of characteristic* 0. *Then, any simple inflation functor* $S_{H,V}^{\mathfrak{i}}$ *is isomorphic to* $S_{H,V}^{\mathfrak{m}}$ *as Mackey functors.*

Proof. Proposition 3.8 implies that $S_{H,V}^{\mathfrak{i}}$ has a unique minimal Mackey subfunctor isomorphic to $S_{H,V}^{\mathfrak{m}}$. As Mackey functors over $\mathbb{L}$ are semisimple from 3.9, we must have $S_{H,V}^{\mathfrak{i}} \cong n S_{H,V}^{\mathfrak{m}}$ for some natural number n. Evaluation at H shows that $n = 1$. □

Proposition 3.8 gives some information about restriction of a functor to a nonfull subcategory of its domain category. The next result shows that restriction to full subcategories is not interesting. The same result for functors from arbitrary categories (satisfying some finiteness conditions) to the category of left R-modules can be found in Webb [9]. We give its easy justification.

Remark 3.11. Let $\mathfrak{Y} \subseteq \chi$ be families of finite groups satisfying appropriate conditions given in Section 2. Let $S^{\chi}_{H,V}$ be a functor (i.e., any of biset, inflation, or Mackey) on χ. Then its restriction $\downarrow^{\chi}_{\mathfrak{Y}} S^{\chi}_{H,V}$ to the family $\mathfrak{Y}$ is $S^{\mathfrak{Y}}_{H,V}$ if $H \in \mathfrak{Y}$ and 0 otherwise.

Proof. If $\downarrow^{\chi}_{\mathfrak{Y}} S^{\chi}_{H,V}$ is nonzero then there is a $G \in \mathfrak{Y}$ so that $S^{\chi}_{H,V}(G)$ is nonzero, in particular H is isomorphic to a section (to a subgroup in Mackey functor case) of G. Conditions on $\mathfrak{Y}$ imply then that $H \in \mathfrak{Y}$. Let $H \in \mathfrak{Y}$. Since morphism sets are the same for the categories with respective objects elements of χ and of $\mathfrak{Y}$, it is clear that $S^{\chi}_{H,V}$ satisfies the conditions in 3.6 as a functor on $\mathfrak{Y}$ because, being simple, it satisfies them as a functor on χ. Thus $\downarrow^{\chi}_{\mathfrak{Y}} S^{\chi}_{H,V} \cong S^{\mathfrak{Y}}_{H,V}$. □

We close this section by giving further applications of the general results obtained in the first part. However, we will not make use of the following result throughout the paper.

Proposition 3.12.

(1) *Any simple biset functor $S^{\mathfrak{b}}_{H,V}$ has a unique maximal inflation subfunctor M. Moreover $S^{\mathfrak{b}}_{H,V}/M \cong S^{\mathfrak{i}}_{H,V}$.*
(2) (*Referee*) *Let V be a simple $\mathbb{L}\,\mathrm{Out}(H)$-module and H be any finite abelian group. Then the biset functor $L^{\mathfrak{b}}_{H,V}$ has a unique maximal inflation subfunctor M. Moreover $L^{\mathfrak{b}}_{H,V}/M \cong S^{\mathfrak{i}}_{H,V}$.*

Proof. (1) Let $S = S^{\mathfrak{b}}_{H,V}$, $L = L^{\mathfrak{b}}_{H,V}$, and $J = J^{\mathfrak{b}}_{H,V}$. We will show that S is generated by $S(H)$ as an inflation functor. Take any finite group G. By 2.5, $S = L/J$ and the ideal $I^{\mathfrak{b}}_{H}$ annihilates $S(H) = V$. Thus for any $T \leqslant G \times H$ with $|q(T)| < |H|$ we have

$$\big[(G \times H)/T\big] \otimes_{\mathrm{End}_{\mathfrak{b}}(H)} V \subseteq J(G) \quad \text{so that} \quad S\big(\big[(G \times H)/T\big]\big)\big(S(H)\big) = 0,$$

see also Bouc [3]. Since $|q(T)| = |H|$ implies that $k_2(T) = 1$, if $|q(T)| = |H|$ then $[(G \times H)/T] \in \mathrm{Hom}_{\mathfrak{i}}(H, G)$. As S is a simple biset functor, from 3.6 S is generated by $S(H)$ as a biset functor. Hence,

$$S(G) = \sum_{T \leqslant G \times H} S\big(\big[(G \times H)/T\big]\big)\big(S(H)\big) = \sum_{T \leqslant G \times H:\, k_2(L)=1} S\big(\big[(G \times H)/T\big]\big)\big(S(H)\big).$$

Therefore S is generated by $S(H)$ as an inflation functor, that is $S = \mathrm{Im}^{S,\mathfrak{i}}_{H,S(H)}$. Now 3.5 implies that $M = \mathrm{Ker}^{S,\mathfrak{i}}_{H,0}$ is the unique maximal inflation subfunctor of S, because $S(H)$ is a simple $\mathrm{End}_{\mathfrak{i}}(H)$-module. Finally, as $M(H) = 0$ it is clear that S/M is isomorphic to $S^{\mathfrak{i}}_{H,V}$.

(2) Let $L = L^{\mathfrak{b}}_{H,V}$. We will first show that L is generated by $L(H)$ as an inflation functor. For this, we will use a method suggested by the referee which uses the argument of Bouc–Thévenaz

[5, (9.1) Lemma]. Take any finite group G. If $T \leqslant G \times H$, and if $Q = q(T)$, we can factorize $(G \times H)/T$ as

$$(G \times H)/T \cong (G \times Q)/A \times_Q (Q \times H)/B$$

for suitable subgroups $A \leqslant G \times Q$ and $B \leqslant Q \times H$. Since H is an abelian group, any subquotient of H is actually a quotient group of H, see [5, (9.1) Lemma]. In particular, there is a subgroup N of H such that $H/N \cong Q$. So there are subgroups $C \leqslant Q \times H$ and $D \leqslant H \times Q$, such that

$$(Q \times H)/C \times_H (H \times Q)/D$$

is the identity (Q, Q)-biset, where

$$(Q \times H)/C \cong \mathrm{Iso}^{Q}_{H/N} \mathrm{Def}^{H}_{H/N} \quad \text{and} \quad (H \times Q)/D \cong \mathrm{Inf}^{H}_{H/N} \mathrm{Iso}^{H/N}_{Q}.$$

Putting this in the previous factorization gives

$$(G \times H)/T \cong \big((G \times Q)/A \times_Q (Q \times H)/C\big) \times_H \big((H \times Q)/D \times_Q (Q \times H)/B\big),$$

and the (H, H) biset on the right will act by 0 on V, unless $Q \cong H$. In the case $Q \cong H$, it follows that $k_2(T) = 1$ so that $(G \times H)/T$ is a right free (G, H)-biset. This shows that L is generated by $L(H)$ as an inflation functor, because by the very definition of L, it is generated by $L(H)$ as a biset functor.

Now 3.5 implies that $M = \mathrm{Ker}^{L,\mathfrak{i}}_{H,0}$ is the unique maximal inflation subfunctor of L, because $L(H) = V$ is a simple $\mathrm{End}_{\mathfrak{i}}(H)$-module. Moreover, by [5, (9.1) Lemma], $L(X) = 0$ if H is not isomorphic to a section of X. This implies that H is a minimal subgroup of the simple inflation functor L/M, because $M(H) = 0$. Hence L/M must be isomorphic to $S^{\mathfrak{i}}_{H,V}$. $\square$

4. Modules of endomorphisms

In this section we work over a field $\mathbb{L}$, and by a functor we mean any of biset, inflation, or Mackey. We first give some easy results relating functors and modules of endomorphism algebras of objects of the domain categories. Our goal is to obtain that the multiplicity of a simple inflation functor $S^{\mathfrak{i}}_{H,V}$ in $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ is equal to the dimension of the $\mathbb{K}$-space

$$\mathrm{Hom}_{\mathbb{K}\,\mathrm{Out}(H)}\big(V, \mathbb{K}\mathcal{R}_{\mathbb{F}}(H)/I^{\mathfrak{m}}_{H}\mathbb{K}\mathcal{R}_{\mathbb{F}}(H)\big)$$

which follows from part (4) of 4.5.

Remark 4.1. Let G be a finite group, and let S_1 and S_2 be two simple functors with $S_1(G) \neq 0$. Then:

(1) $S_1(G)$ is a simple $\mathrm{End}(G)$-module.
(2) If $W = S_1(G)$ then $S_1 \cong L_{G,W}/J_{G,W}$.
(3) If $S_1(G) \cong S_2(G)$ as $\mathrm{End}(G)$-modules then $S_1 \cong S_2$ as functors.
(4) Let $W = S_1(G)$. Then, I_G annihilates W if and only if $S_1 \cong S_{G,W}$.

Proof. (1) By 3.6.

(2) By 2.2 the pair $(L_{G,-}, e_G)$ is an adjoint pair, implying the existence of an $\mathbb{L}$-space isomorphism between $0 \neq \mathrm{End}_{\mathrm{End}(G)}(W)$ and $\mathrm{Hom}_{\mathfrak{F}}(L_{G,W}, S_1)$. So there is a nonzero functor homomorphism $\pi : L_{G,W} \to S_1$ which is necessarily surjective by the simplicity of S_1. Then the kernel of π is a maximal subfunctor of $L_{G,W}$, and so equal to $J_{G,W}$ because $J_{G,W}$ is the unique maximal subfunctor of $L_{G,W}$ by 2.3. Hence $S_1 \cong L_{G,W}/J_{G,W}$.

(3) If $S_1(G) \cong S_2(G) = W$ then by part (2) both of S_1 and S_2 are isomorphic to $L_{G,W}/J_{G,W}$, implying that $S_1 \cong S_2$.

(4) If I_G annihilates W then W is a simple $\mathbb{L}\,\mathrm{Out}(G)$-module, and part (2) and 2.5 imply that $S_1 \cong L_{G,W}/J_{G,W} \cong S_{G,W}$. If $S_1 \cong S_{G,W}$ then by 2.5 I_G annihilates W. $\square$

The previous result implies

Proposition 4.2. *Let G be a finite group. Then the maps $S_{H,V} \to S_{H,V}(G)$, $L_{G,W}/J_{G,W} \leftarrow W$ define a bijective correspondence between the isomorphism classes of simple functors whose evaluations at G are nonzero and the isomorphism classes of simple* $\mathrm{End}(G)$*-modules.*

If $S_{H,V}$ is a simple functor and E is the $\mathrm{End}(H)$-projective cover of V, then by Bouc [3, Lemme 2] the functor $L_{H,E}$ is the projective cover of $S_{H,V}$. Therefore the following is obvious.

Remark 4.3. (See [3, Lemme 2].) Let $S_{H,V}$ be a simple functor and G be a finite group. If $S_{H,V}(G)$ is nonzero then the $\mathrm{End}(G)$-projective cover $P(S_{H,V}(G))$ of $S_{H,V}(G)$ is isomorphic to $L_{H,P(V)}(G)$ as $\mathrm{End}(G)$-modules, where $P(V)$ is the $\mathrm{End}(H)$-projective cover of V.

In the next section we will need some results about the multiplicities of simple functors as composition factors of a given functor M. Since finitely generated modules of finite dimensional algebras have composition series of finite length whose composition factors are unique up to isomorphism and ordering, to guarantee the same for functors we will assume in the rest of this section that functors are defined on a finite family of χ of finite groups satisfying the conditions given in the last paragraph of Section 2.

We first make an easy remark.

Remark 4.4. Let $\mathbb{L}$ be algebraically closed. Suppose that A is a finite dimensional semisimple $\mathbb{L}$-algebra admitting a direct sum decomposition $A = B \oplus I$ where I is a two sided ideal of A and B is a unital subalgebra of A. Let V be a simple B-module (so we may regard V as an A-module by putting $IV = 0$). Then, for any finitely generated A-module S the multiplicity of V in S as an A-module composition factor is equal to $\dim_{\mathbb{L}} \mathrm{Hom}_B(V, S/IS)$.

Proof. This is obvious, because both of A and B are finite dimensional semisimple $\mathbb{L}$-algebras, and $IV = 0$. $\square$

By the multiplicity of S in M we mean the multiplicity of S in M as a composition factor of M. Part (4) is the only part of the following result that we will use. For completeness we write down all implications.

Proposition 4.5. *Let $\mathbb{L}$ be algebraically closed and let M be a functor such that $M(X)$ is a finite dimensional $\mathbb{L}$-space for all X in χ.*

(1) *Given a simple functor $S_{H,V}$, the following numbers are equal*:
  (a) *The multiplicity of $S_{H,V}$ in M as functors.*
  (b) *The multiplicity of V in $M(H)$ as* $\mathrm{End}(H)$*-modules.*
  (c) $\dim_{\mathbb{L}} \mathrm{Hom}_{\mathrm{End}(H)}(P(V), M(H))$ *where $P(V)$ is the* $\mathrm{End}(H)$*-projective cover of V.*
(2) *Assume that $\mathbb{L}$ is of characteristic* 0. *If H is a cyclic group and M is a biset functor, then for any simple $\mathbb{L}\,\mathrm{Out}(H)$-module V the following numbers are equal*:
  (a) *The multiplicity of $S^{\mathfrak{b}}_{H,V}$ in M as biset functors.*
  (b) *The multiplicity of V in $M(H)/I^{\mathfrak{b}}_H M(H)$ as* $\mathbb{L}\,\mathrm{Out}(H)$*-modules.*
  (c) $\dim_{\mathbb{L}} \mathrm{Hom}_{\mathbb{L}\,\mathrm{Out}(H)}(V, M(H)/I^{\mathfrak{b}}_H M(H))$.
(3) *Assume that $\mathbb{L}$ is of characteristic* 0. *If M is a Mackey functor, then for any simple Mackey functor $S^{\mathfrak{m}}_{H,V}$ the following numbers are equal*:
  (a) *The multiplicity of $S^{\mathfrak{m}}_{H,V}$ in M as Mackey functors.*
  (b) *The multiplicity of V in $M(H)/I^{\mathfrak{m}}_H M(H)$ as* $\mathbb{L}\,\mathrm{Out}(H)$*-modules.*
  (c) $\dim_{\mathbb{L}} \mathrm{Hom}_{\mathbb{L}\,\mathrm{Out}(H)}(V, M(H)/I^{\mathfrak{m}}_H M(H))$.
(4) *Assume that $\mathbb{L}$ is of characteristic* 0. *If M is an inflation functor, then for any simple inflation functor $S^{\mathfrak{i}}_{H,V}$ the following numbers are equal*:
  (a) *The multiplicity of $S^{\mathfrak{m}}_{H,V}$ in M as Mackey functors.*
  (b) *The multiplicity of V in $M(H)/I^{\mathfrak{m}}_H M(H)$ as* $\mathbb{L}\,\mathrm{Out}(H)$*-modules.*
  (c) $\dim_{\mathbb{L}} \mathrm{Hom}_{\mathbb{L}\,\mathrm{Out}(H)}(V, M(H)/I^{\mathfrak{m}}_H M(H))$.
  (d) *The multiplicity of $S^{\mathfrak{i}}_{H,V}$ in M as inflation functors.*

Proof. Let A be a finite dimensional $\mathbb{L}$-algebra and V be a simple A-module and S be a finitely generated A-module. It is well known that the multiplicity of V in S as A-modules is equal to the dimension of $\mathrm{Hom}_A(P(V), S)$ where $P(V)$ is the projective cover of V. Since $\mathbb{L}\,\mathrm{Out}(H)$ is semisimple when $\mathbb{L}$ is of characteristic 0, the numbers in (b) and (c) are equal in all of (1)–(4).

If $P(V)$ is the $\mathrm{End}(H)$-projective cover of V then by 2.2 the functor $L_{H,P(V)}$ is the projective cover of $S_{H,V}$ as functors on χ. So the multiplicity of $S_{H,V}$ in M is equal to the dimension of $\mathrm{Hom}_{\mathfrak{F}}(L_{H,P(V)}, M)$ which is isomorphic to the $\mathbb{L}$-space $\mathrm{Hom}_{\mathrm{End}(H)}(P(V), M(H))$ by the adjointness of the pair $(L_{H,-}, e_H)$ given in 2.2. This shows that the numbers in (a) and (c) of (1) are equal.

Moreover $\mathrm{End}(H) = \mathrm{Ext}(H) \oplus I_H$ and $\mathrm{Ext}(H) \cong \mathbb{L}\,\mathrm{Out}(H)$ by 2.4, so that 4.4 is applicable whenever $\mathrm{End}(H)$ is semisimple. If $\mathrm{End}(H)$ is semisimple then $P(V) = V$ and 4.4 implies that the multiplicity of $S_{H,V}$ in M is equal to the dimension of $\mathrm{Hom}_{\mathbb{L}\,\mathrm{Out}(H)}(V, M(H)/I_H M(H))$. Using the semisimplicity results given in 3.9 we see that the numbers in (a) and (c) are equal in all of (2)–(4).

Up to now we finished the proofs of (1)–(3), and showed the equality of numbers in (a)–(c) of (4).

Given any composition series of M as inflation functors on χ. We see from 3.10 that the same series is also a composition series of M as Mackey functors on χ and any simple inflation functor $S^{\mathfrak{i}}_{H,V}$ is isomorphic to $S^{\mathfrak{m}}_{H,V}$ as Mackey functors, proving the equality of numbers in (a) and (d) of (4). □

5. Composition factors of $\mathbb{K}\boldsymbol{\mathcal{R}}_{\mathbb{F}}$

Throughout this section, $\mathbb{F}$ is an algebraically closed field of characteristic $p > 0$, and $\mathbb{K}$ is an algebraically closed field of characteristic 0.

Let $H \leqslant G$ be finite groups. For $\mathbb{F}H$ and $\mathbb{F}G$-modules W and V, we denote by $\uparrow_H^G W$ and $\downarrow_H^G V$ the $\mathbb{F}G$ and $\mathbb{F}H$-modules $\mathbb{F}G \otimes_{\mathbb{F}H} W$ and $\mathbb{F}G \otimes_{\mathbb{F}G} V$, respectively. We let $\mathrm{Irr}(\mathbb{F}G)$ be a complete set of representatives of the isomorphism classes of simple $\mathbb{F}G$-modules. We write $\mathbb{F}_G$ to indicate the trivial $\mathbb{F}G$-module.

In this section we want to study the composition factors of the modular representation algebra functor $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ as inflation functors over $\mathbb{K}$, where if G is a finite group then $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G) = \mathbb{K} \otimes_{\mathbb{Z}} G_0(\mathbb{F}G)$ and $G_0(\mathbb{F}G)$ is the Grothendieck group of finitely generated $\mathbb{F}G$-modules with respect to exact sequences.

Let G be a finite group. The Grothendieck group $G_0(\mathbb{F}G)$ of the finitely generated $\mathbb{F}G$-modules is defined to be a quotient group A/F where A is the free abelian group freely generated by symbols (V) for each isomorphism classes of finitely generated $\mathbb{F}G$-modules V, and F is the subgroup of A generated by all elements of the form $(V) - (V') - (V'')$ arising from the short exact sequences of $\mathbb{F}G$-modules $0 \to V' \to V \to V'' \to 0$. If we write $[V]$ for the image of $(V) \in A$ in A/F, we have

$$G_0(\mathbb{F}G) = \bigoplus_{V \in \mathrm{Irr}(\mathbb{F}G)} \mathbb{Z}[V] \quad \text{and} \quad \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) = \bigoplus_{V \in \mathrm{Irr}(\mathbb{F}G)} \mathbb{K}[V].$$

Let G and H be finite groups. Any (G,H)-biset S gives an $(\mathbb{F}G, \mathbb{F}H)$-bimodule $\mathbb{F}S$, and so induces a functor $\mathbb{F}S \otimes_{\mathbb{F}H} - : \mathbb{F}H\text{-Mod} \to \mathbb{F}G\text{-Mod}$. For each (G,H)-biset S such that the functor $\mathbb{F}S \otimes_{\mathbb{F}H} -$ is exact (equivalently, the right $\mathbb{F}H$-module $\mathbb{F}S_{\mathbb{F}H}$ is projective), S induces an obvious map

$$\mathbb{K}\mathcal{R}_{\mathbb{F}}\big([S]\big) : \mathbb{K}\mathcal{R}_{\mathbb{F}}(H) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(G), \quad [W] \mapsto [\mathbb{F}S \otimes_{\mathbb{F}H} W].$$

With these maps $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ becomes a functor from the subcategory of the biset category with morphisms from H to G are the $\mathbb{K}$-span of $[S]$ where S is any (G,H)-biset with the property that $\mathbb{F}S_{\mathbb{F}H}$ is projective to the category of $\mathbb{K}$-modules.

We see that for the four type of basic bisets

$$\mathrm{Ind}_H^G, \qquad \mathrm{Res}_H^G, \qquad \mathrm{Inf}_{G/N}^G, \quad \text{and} \quad \mathrm{Iso}_G^{G'},$$

where $H \leqslant G \trianglerighteq N$, and $G' \cong G$, the right modules

$$\mathbb{F}G_{\mathbb{F}H}, \qquad \mathbb{F}G_{\mathbb{F}G}, \qquad \mathbb{F}(G/N)_{\mathbb{F}(G/N)}, \quad \text{and} \quad \mathbb{F}G'_{\mathbb{F}G}$$

are all free (hence projective). While for $\mathrm{Def}_{G/N}^G$, we see that $\mathbb{F}(G/N)_{\mathbb{F}G}$ is projective if and only if p does not divide the order of N.

Therefore $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ has a natural inflation functor structure over $\mathbb{K}$ with the following maps:

$\mathbb{K}\mathcal{R}_{\mathbb{F}}(\mathrm{Ind}_H^G) : \mathbb{K}\mathcal{R}_{\mathbb{F}}(H) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$, $[W] \mapsto [\uparrow_H^G W]$.

$\mathbb{K}\mathcal{R}_{\mathbb{F}}(\mathrm{Res}_H^G) : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(H)$, $[V] \mapsto [\downarrow_H^G V]$.

$\mathbb{K}\mathcal{R}_{\mathbb{F}}(\mathrm{Inf}_{G/N}^G) : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G/N) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$, $[U] \mapsto [\mathrm{Inf}_{G/N}^G U]$, where $\mathrm{Inf}_{G/N}^G U = U$ with the G-action given by $gu = (gN)u$.

$\mathbb{K}\mathcal{R}_{\mathbb{F}}(\mathrm{Iso}_G^{G'}(\varphi)) : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(G')$, $[U] \mapsto [\mathrm{Iso}_G^{G'}(\varphi)U]$, where $\mathrm{Iso}_G^{G'}(\varphi)U = U$ with G'-action given by $g'u = \varphi^{-1}(g')u$.

We finally remind the reader that both of $G_0(\mathbb{F}G)$ and $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$ are commutative algebras with product $[V_1][V_2] = [V_1 \otimes_{\mathbb{F}} V_2]$ and with the unity $[\mathbb{F}_G]$. For simplicity we write ψ instead of $\mathbb{K}\mathcal{R}_{\mathbb{F}}(\psi)$ where ψ is any of Ind, Res, Inf, or Iso.

We begin with an easy consequence of induction theorems.

Lemma 5.1. *Let G be a finite group and M be a Mackey subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$. If $M(H) = \mathbb{K}\mathcal{R}_{\mathbb{F}}(H)$ for all cyclic p'-subgroups H of G then $M(G) = \mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$.*

Proof. By Artin's induction theorem

$$\mathbb{K}\mathcal{R}_{\mathbb{F}}(G) = \sum_{H} \mathrm{Ind}_H^G \mathbb{K}\mathcal{R}_{\mathbb{F}}(H)$$

where H ranges over all cyclic p'-subgroups of G, see Benson [2, Theorem 5.6.1, p. 172]. This proves the result. $\square$

From now on in this section, χ will denote a finite family of finite groups such that no two groups in χ are isomorphic and that if X in χ then any section of X is isomorphic to a group in χ. We will study $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ as an inflation functor on χ and write $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ to stress that. In this situation $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ may be regarded as a module of a finite dimensional $\mathbb{K}$-algebra, see the last paragraph of Section 2. Since the coordinate module $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$ at any finite group G is a finite dimensional $\mathbb{K}$-space, it follows that $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ admits a composition series (of finite length), as inflation functors on χ, whose factors are unique up to isomorphism and ordering.

We now observe that minimal subgroups of the inflation functor composition factors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ are among the cyclic p'-groups in χ.

Proposition 5.2. *If $S^{\mathfrak{i}}_{H,V}$ is a composition factor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ as inflation functors then H is a cyclic p'-group in χ.*

Proof. Suppose that $S^{\mathfrak{i}}_{H,V}$ is a composition factor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ as inflation functors on χ. There are inflation subfunctors $N \leqslant M$ of $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ such that M/N is isomorphic to $S^{\mathfrak{i}}_{H,V}$. Then 3.10 implies that $N \leqslant M$ are Mackey subfunctors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ such that M/N is isomorphic to $S^{\mathfrak{m}}_{H,V}$. By 3.9 the functor $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ is a semisimple Mackey functor on χ over $\mathbb{K}$, because $\mathbb{K}$ is of characteristic 0. Consequently, there must exist a Mackey subfunctor T of $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ such that $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}/T$ is isomorphic to $S^{\mathfrak{m}}_{H,V}$. In particular T is a proper Mackey subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$.

Let $\mathfrak{Y}$ be the family consisting of all cyclic p'-groups in χ. If H is not a cyclic p'-group then $H \notin \mathfrak{Y}$ and 3.11 implies that $\downarrow^{\chi}_{\mathfrak{Y}} (\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}/T) = 0$. Thus

$$\downarrow^{\chi}_{\mathfrak{Y}} T = \downarrow^{\chi}_{\mathfrak{Y}} \mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi},$$

implying that $T(H) = \mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}(H)$ for every group H in $\mathfrak{Y}$. Then by 5.1 we get $T(G) = \mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}(G)$ for every group G in χ, a contradiction because T is a proper Mackey subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$. $\square$

We now calculate the multiplicities in $\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\chi}$ of simple inflation functors whose minimal subgroups are cyclic q-groups where q is a prime different from p.

Lemma 5.3. *Let G be cyclic q-group in χ where q is a prime different from p. For any simple $\mathbb{K}\,\mathrm{Out}(G)$-module V, the multiplicity of the simple inflation functor $S^{\mathfrak{i}}_{G,V}$ in $\mathbb{K}\mathcal{R}^{\chi}_{\mathbb{F}}$ is equal to* 1.

Proof. The dimension of the $\mathbb{K}$-space $\mathrm{Hom}_{\mathbb{K}\,\mathrm{Out}(G)}(V, \mathbb{K}\mathcal{R}_{\mathbb{F}}(G)/I^{\mathfrak{m}}_G\mathbb{K}\mathcal{R}_{\mathbb{F}}(G))$ is the required multiplicity by part (4) of 4.5. We will show that

$$\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)/I^{\mathfrak{m}}_G\mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \cong \mathbb{K}\,\mathrm{Out}(G)$$

as $\mathbb{K}\,\mathrm{Out}(G)$-modules. This shows that the required multiplicity is 1, because $\mathrm{Out}(G)$ is abelian and V is one dimensional.

If $G = 1$ then $V = \mathbb{K}$, $\mathrm{End}_{\mathfrak{i}}(G) \cong \mathbb{K}$, $P(V) = V$, and $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \cong \mathbb{K}$; and in this case part (1) of 4.5 implies that the multiplicity of $S^{\mathfrak{i}}_{1,\mathbb{K}}$ in $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ is 1.

We first set up our notations as follows:

$G = \langle x\rangle$, $H = \langle x^q\rangle$ and $|G| = q^n$ for some natural number $n \geqslant 1$ (the case $n = 0$ was treated above).

For any integer m, we denote by m_q the highest power of q dividing m. That is q^{m_q} divides m but q^{m_q+1} does not divide m.

$\mathrm{Out}(G) = \{\theta_l\colon l = 1, \ldots, q^n, l_q = 0\}$, where $\theta_l : x \mapsto x^l$.

ε is a primitive q^nth root of unity in $\mathbb{F}$ (exists because $q \neq p$).

$\mathrm{Irr}(\mathbb{F}G) = \{W_1, \ldots, W_{q^n}\}$ and $\mathrm{Irr}(\mathbb{F}H) = \{U_1, \ldots, U_{q^{n-1}}\}$ where $W_i = \mathbb{F}w_i$ and $U_j = \mathbb{F}u_j$ with actions $xw_i = \varepsilon^i w_i$ and $x^q u_j = \varepsilon^{qj} u_j$. For any natural number m, by W_m (respectively U_m) we mean the module W_i (respectively U_j) where i (respectively j) is the unique number in $\{1, \ldots, q^n\}$ (respectively in $\{1, \ldots, q^{n-1}\}$) with $m \equiv i \bmod q^n$ (respectively $m \equiv j \bmod q^{n-1}$).

We note that $\theta_l \in \mathrm{Out}(G)$ acts on $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$ as $\theta_l^{-1}[W_i] = [W_{il}]$ because G acts on the $\mathbb{F}G$-module $\mathrm{Iso}^G_G(\theta_l^{-1})W_i = W_i$ by $xw_i = \theta_l(x)w_i = x^l w_i = \varepsilon^{il} w_i$.

For convenience we divide the proof into several parts.

(A) Let $\phi : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \to \mathbb{K}\,\mathrm{Out}(G)$ be the map given by $[W_i] \mapsto \theta_i^{-1}$ if $i_q = 0$, and $[W_i] \mapsto 0$ otherwise. Then ϕ is a $\mathbb{K}\,\mathrm{Out}(G)$-module epimorphism.

Proof of (A). It is clear that ϕ is a surjective $\mathbb{K}$-linear map. Let $\theta_l \in \mathrm{Out}(G)$. As $l_q = 0$, $(il)_q = i_q$. If $i_q \neq 0$, then

$$\phi\big(\theta_l^{-1}[W_i]\big) = \phi\big([W_{il}]\big) = 0 = \theta_l^{-1}0 = \theta_l^{-1}\phi\big([W_i]\big).$$

If $i_q = 0$, then

$$\phi\big(\theta_l^{-1}[W_i]\big) = \phi\big([W_{il}]\big) = \theta_{il}^{-1} = \theta_l^{-1}\theta_i^{-1} = \theta_l^{-1}\phi\big([W_i]\big).$$

Hence ϕ is a $\mathbb{K}\,\mathrm{Out}(G)$-module epimorphism. $\square$

(B) $\mathrm{Ker}\,\phi$ is a permutation $\mathbb{K}\,\mathrm{Out}(G)$-module with permutation basis

$$X = \big\{[W_i]\colon i = 1, \ldots, q^n,\ i_q \neq 0\big\}.$$

If we let $X_t = \{[W_i] \in X\colon i_q = t\}$, then $X_1, \ldots, X_n$ are the $\mathrm{Out}(G)$-orbits on X, and $[W_{q^t}]$ is an element of X_t, whose $\mathrm{Out}(G)$-stabilizer is the subgroup $S_t = \{\theta_l\colon l \equiv 1 \bmod q^{n-t}\}$.

Proof of (B). By the definition of ϕ, it is clear that X is a $\mathbb{K}$-basis of $\operatorname{Ker}\phi$ which is obviously permuted by $\operatorname{Out}(G)$. We note that $\theta_l \in S_t$ if and only if $\theta_l^{-1}[W_{q^t}] = [W_{q^t}]$, equivalently $[W_{q^t l}] = [W_{q^t}]$, i.e., $q^t l \equiv q^t \mod q^n$. Since $l_q = 0$, we see that S_t is the desired subgroup. Let $[W_i] \in X_t$. Then $i_q = t$ and so $i = q^t s$ for some natural number s with $s_q = 0$. Hence $[W_i] = \theta_s^{-1}[W_{q^t}]$, implying that $\operatorname{Out}(G)$ acts on X_t transitively. $\square$

(C) $I_G^{\mathfrak{m}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) = \operatorname{Ind}_H^G \mathbb{K}\mathcal{R}_{\mathbb{F}}(H)$.

Proof of (C). If $[(G \times G)/L] \in I_G^{\mathfrak{m}}$, we then may write

$$(G \times G)/L \cong \operatorname{Ind}_K^G \operatorname{Iso}_K^K \operatorname{Res}_K^G$$

for some proper subgroup $K = p_1(L)$ of G, see Section 2. It is clear that the maps

$$\operatorname{Res}_K^G : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(K) \quad \text{and} \quad \operatorname{Iso}_K^K : \mathbb{K}\mathcal{R}_{\mathbb{F}}(K) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(K)$$

are surjective and bijective, respectively (even for any finite abelian group G and any finite group K). Consequently

$$\big[(G \times G)/L\big] \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) = \operatorname{Ind}_K^G \mathbb{K}\mathcal{R}_{\mathbb{F}}(K).$$

Finally from the relation $\operatorname{Ind}_{K_2}^G \operatorname{Ind}_{K_1}^{K_2} = \operatorname{Ind}_{K_1}^G$, we see that $I_G^{\mathfrak{m}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) = \operatorname{Ind}_H^G \mathbb{K}\mathcal{R}_{\mathbb{F}}(H)$ because H is the unique maximal subgroup of G. $\square$

(D) $I_G^{\mathfrak{m}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$ is a permutation $\mathbb{K}\operatorname{Out}(G)$-module with permutation basis

$$Y = \big\{\operatorname{Ind}_H^G [U_j]\colon\ j = 1, \dots, q^{n-1}\big\}.$$

If we let $Y_t = \{\operatorname{Ind}_H^G [U_j]\colon\ j_q = t-1\}$, then $Y_1, \dots, Y_n$ are the $\operatorname{Out}(G)$-orbits on Y, and $[U_{q^{t-1}}]$ is an element of Y_t, whose $\operatorname{Out}(G)$-stabilizer is the subgroup $T_t = \{\theta_l \in \operatorname{Out}(G)\colon\ l \equiv 1 \mod q^{n-t}\}$.

Proof of (D). It is clear that $\operatorname{Ind}_H^G : \mathbb{K}\mathcal{R}_{\mathbb{F}}(H) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$ is injective. Therefore Y is a $\mathbb{K}$-basis of $I_G^{\mathfrak{m}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$. We note that if $\theta_l \in \operatorname{Out}(G)$ then its restriction $\theta_l|_H$ to H is an element of $\operatorname{Out}(H)$. Since

$$\theta_l^{-1} \operatorname{Ind}_H^G [U_j] = \operatorname{Iso}_G^G\big(\theta_l^{-1}\big) \operatorname{Ind}_H^G [U_j] = \operatorname{Ind}_H^G \operatorname{Iso}_H^H\big(\theta_l^{-1}\big|_H\big)[U_j] = \operatorname{Ind}_H^G [U_{jl}],$$

we see that $\operatorname{Out}(G)$ permutes Y. Now $\theta_l \in T_t$ if and only if $\theta_l^{-1} \operatorname{Ind}_H^G [U_{q^{t-1}}] = \operatorname{Ind}_H^G [U_{q^{t-1}}]$, equivalently $\operatorname{Ind}_H^G [U_{q^{t-1}l}] = \operatorname{Ind}_H^G [U_{q^{t-1}}]$. Then using the injectivity of Ind_H^G, we see that $\theta_l \in T_t$ if and only if $q^{t-1} l \equiv q^{t-1} \mod q^{n-1}$. Since $l_q = 0$, the stabilizer of $[U_{q^{t-1}}]$ is the desired subgroup T_t. Let $\operatorname{Ind}_H^G [U_j] \in Y_t$. Then $j_q = t-1$ and so $j = q^{t-1}s$ for some s with $s_q = 0$. Hence $\operatorname{Ind}_H^G [U_j] = \theta_s^{-1} \operatorname{Ind}_H^G [U_{q^{t-1}}]$, implying that $\operatorname{Out}(G)$ acts on Y_t transitively. $\square$

We have now accumulated all the information necessary to complete the proof. From (B) and (D) the subgroups S_t and T_t are equal for all $t = 1, \ldots, n$ and so we have

$$\operatorname{Ker}\phi \cong \bigoplus_{t=1}^{n} \uparrow_{S_t}^{\mathrm{Out}(G)} \mathbb{K}_{S_t} = \bigoplus_{t=1}^{n} \uparrow_{T_t}^{\mathrm{Out}(G)} \mathbb{K}_{T_t} \cong I_G^{\mathfrak{m}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$$

as $\mathbb{K}\,\mathrm{Out}(G)$-modules. (A) gives that $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)/\operatorname{Ker}\phi \cong \mathbb{K}\,\mathrm{Out}(G)$ as $\mathbb{K}\,\mathrm{Out}(G)$-modules. Then semisimplicity of the $\mathbb{K}\,\mathrm{Out}(G)$-module $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$ implies that

$$\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)/I_G^{\mathfrak{m}}\mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \cong \mathbb{K}\,\mathrm{Out}(G)$$

as $\mathbb{K}\,\mathrm{Out}(G)$-modules, finishing the proof. $\square$

Let A and B be finite dimensional $\mathbb{L}$-algebras where $\mathbb{L}$ is an algebraically closed field. If V is an A-module and W is a B-module, then $V \otimes_{\mathbb{L}} W$ becomes an $A \otimes_{\mathbb{L}} B$-module with the action $(a \otimes b)(v \otimes w) = av \otimes bw$. Moreover $\mathrm{Irr}(A \otimes_{\mathbb{L}} B)$ is the set consisting of all elements $V \otimes_{\mathbb{L}} W$ where $V \in \mathrm{Irr}(A)$ and $W \in \mathrm{Irr}(B)$. If we assume that both of A and B are semisimple, then by the distributivity of $\otimes_{\mathbb{L}}$ over $\oplus$ we easily see that the multiplicity of $V \otimes_{\mathbb{L}} W$ in $M \otimes_{\mathbb{L}} N$ is equal to the product of the multiplicities of V in M and W in N, where $V \in \mathrm{Irr}(A)$, $W \in \mathrm{Irr}(B)$, and M and N are modules for A and B respectively.

We now give an application of the above facts. Let H and K be two groups of coprime orders. Since any subgroup X of $H \times K$ is of the form $X^H \times X^K$ for some $X^H \leqslant H$ and $X^K \leqslant K$, any element

$$\left[\frac{(H \times K) \times (H \times K)}{L}\right] \in \mathrm{End}_{\mathfrak{m}}(H \times K)$$

is of the form

$$\mathrm{Ind}_P^{H \times K} \mathrm{Iso}_Q^P(\varphi) \mathrm{Res}_Q^{H \times K} = \mathrm{Ind}_{P^H \times P^K}^{H \times K} \mathrm{Iso}_{Q^H \times Q^K}^{P^H \times P^K}\left(\varphi^H \times \varphi^K\right) \mathrm{Res}_{Q^H \times Q^K}^{H \times K}$$

where $P = p_1(L)$ and $Q = p_2(L)$ are isomorphic groups, and $\varphi = \varphi^H \times \varphi^K$ with φ^H and φ^K are the respective restrictions of φ to Q^H and Q^K (as $|H|$ and $|K|$ are coprime, $\varphi(Q^H) = P^H$ and $\varphi(Q^K) = P^K$ for any isomorphism $\varphi : Q \to P$). Consequently, the map

$$\left(\mathrm{Ind}_{R_1}^H \mathrm{Iso}_{R_2}^{R_1}(\alpha) \mathrm{Res}_{R_2}^H\right) \otimes_{\mathbb{K}} \left(\mathrm{Ind}_{S_1}^K \mathrm{Iso}_{S_2}^{S_1}(\beta) \mathrm{Res}_{S_2}^K\right) \mapsto \mathrm{Ind}_{R_1 \times S_1}^{H \times K} \mathrm{Iso}_{R_2 \times S_2}^{R_1 \times S_1}(\alpha \times \beta) \mathrm{Res}_{R_2 \times S_2}^{H \times K}$$

gives a $\mathbb{K}$-algebra isomorphism

$$\mathrm{End}_{\mathfrak{m}}(H) \otimes_{\mathbb{K}} \mathrm{End}_{\mathfrak{m}}(K) \to \mathrm{End}_{\mathfrak{m}}(H \times K).$$

Moreover this $\mathbb{K}$-algebra isomorphism transports $\mathbb{K}\mathcal{R}_{\mathbb{F}}(H) \otimes_{\mathbb{K}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(K)$ to $\mathbb{K}\mathcal{R}_{\mathbb{F}}(H \times K)$, because $\mathrm{Irr}(\mathbb{F}(H \times K))$ consists of all elements of the form $V \otimes_{\mathbb{F}} W$ where V and W range in the sets $\mathrm{Irr}(\mathbb{F}H)$ and $\mathrm{Irr}(\mathbb{F}K)$, respectively.

Lemma 5.4. *Let H and K be two groups of coprime orders. Suppose that V and W are simple modules of $\mathbb{K}\,\mathrm{Out}(H)$ and $\mathbb{K}\,\mathrm{Out}(K)$, respectively. Then, the multiplicity of the simple inflation functor $S^{\mathfrak{i}}_{H\times K, V\otimes_{\mathbb{K}} W}$ in $\mathbb{K}\mathcal{R}^{\chi}_{\mathbb{F}}$ is equal to the product of the multiplicities of the simple inflation functors $S^{\mathfrak{i}}_{H,V}$ and $S^{\mathfrak{i}}_{K,W}$ in $\mathbb{K}\mathcal{R}^{\chi}_{\mathbb{F}}$.*

Proof. By part (4) of 4.5, the multiplicity of any simple inflation functor $S^{\mathfrak{i}}_{X,U}$ in $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ is equal to the multiplicity of the simple Mackey functor $S^{\mathfrak{m}}_{X,U}$ in $\mathbb{K}\mathcal{R}_{\mathbb{F}}$, which is then equal to the multiplicity of U in $\mathbb{K}\mathcal{R}_{\mathbb{F}}(X)$ as $\mathrm{End}_{\mathfrak{m}}(X)$-modules by part (1) of 4.5. Since $\mathrm{End}_{\mathfrak{m}}(X)$ is a semisimple $\mathbb{K}$-algebra by 3.9, the result follows by the facts given above with $X = H \times K$ and $U = V \otimes_{\mathbb{K}} W$. □

We now state the main result of this section.

Theorem 5.5. *The composition factors of $\mathbb{K}\mathcal{R}^{\chi}_{\mathbb{F}}$ as inflation functors on χ are precisely the simple inflation functors $S^{\mathfrak{i}}_{C,V}$, where C ranges over cyclic p'-groups in χ and V ranges over elements in* $\mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C))$. *Moreover the multiplicity of each composition factor is* 1.

Proof. Follows by 5.2–5.4. □

6. Subfunctors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$

In this section, by a functor we mean an inflation functor, and we assume the fields $\mathbb{F}$ and $\mathbb{K}$ as in the previous section. We want to find a filtration of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$.

We begin with a simple observation about the evaluations of subfunctors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$.

Remark 6.1. Let M be a subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$. Then the following are equivalent:

(1) $M(P) \neq 0$ for some finite p-group P.
(2) $M(P) \neq 0$ for every finite p-group P.
(3) $[\mathbb{F}_G] \in M(G)$ for every finite group G.
(4) $M(G) \neq 0$ for every finite group G.

Proof. For any finite p-group P, it is clear that $\mathbb{K}\mathcal{R}_{\mathbb{F}}(P) = \mathbb{K}[\mathbb{F}_P]$. Then using the inclusions $\mathrm{Res}^P_1 M(P) \subseteq M(1)$, $\mathrm{Ind}^P_1 M(1) \subseteq M(P)$, and $\mathrm{Inf}^G_{G/G}\,\mathrm{Iso}^{G/G}_1 M(1) \subseteq M(G)$, the result follows. □

For any natural number n and any finite group G, we define a subset $K_n(G)$ of $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$ by:

$$K_n(G) = \bigcap_C \mathrm{Ker}\big(\mathrm{Res}^G_C : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(C)\big)$$

where C ranges over all cyclic subgroups of G of order dividing n.

Lemma 6.2. $K_n = \mathrm{Ker}^{\mathbb{K}\mathcal{R}_{\mathbb{F}},\mathfrak{i}}_{C_n,0}$ *where C_n is any cyclic group of order n. In particular, K_n is a subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$.*

Proof. For any finite group G,

$$\mathrm{Ker}^{\mathbb{K}\mathcal{R}_{\mathbb{F}},\mathrm{i}}_{C_n,0}(G) = \bigcap_{L \leqslant_* C_n \times G\colon\, k_2(L)=1} \mathrm{Ker}\big(\mathbb{K}\mathcal{R}_{\mathbb{F}}\big(\big[(C_n \times G)/L\big]\big) : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(C_n)\big),$$

see Section 3.1. If $L \leqslant C_n \times G$ with $k_2(L) = 1$ then $[(C_n \times G)/L]$ is of the form

$$\mathrm{Ind}^{C_n}_{p_1(L)} \mathrm{Inf}^{p_1(L)}_{p_1(L)/k_1(L)} \mathrm{Iso}^{p_1(L)/k_1(L)}_{p_2(L)} \mathrm{Res}^G_{p_2(L)}.$$

Then from $p_2(L) \cong p_1(L)/k_1(L)$ we see that $p_2(L)$ is a cyclic subgroup of G of order dividing n. Conversely, if C is a cyclic subgroup of G of order dividing n, then C_n has a subgroup p_1 isomorphic to C such that

$$\mathrm{Ind}^{C_n}_{p_1} \mathrm{Iso}^{p_1}_{C} \mathrm{Res}^G_C$$

is of the form $[(C_n \times G)/M]$ with $k_2(M) = 1$. Now we notice that the maps $\mathrm{Ind}^{C_n}_{p_1(L)}$, $\mathrm{Inf}^{p_1(L)}_{p_1(L)/k_1(L)}$, and $\mathrm{Iso}^{p_1(L)/k_1(L)}_{p_2(L)}$ are all injective so that

$$\mathrm{Ker}\big(\mathbb{K}\mathcal{R}_{\mathbb{F}}\big(\big[(C_n \times G)/L\big]\big)\big) = \mathrm{Ker}\,\mathrm{Res}^G_{p_2(L)}.$$

Finally, as $\mathrm{Ker}\,\mathrm{Res}^G_{{}^gC} = \mathrm{Ker}\,\mathrm{Res}^G_C$, for any $g \in G$, we have

$$\mathrm{Ker}^{\mathbb{K}\mathcal{R}_{\mathbb{F}},\mathrm{i}}_{C_n,0}(G) = \bigcap_C \mathrm{Ker}\big(\mathrm{Res}^G_C : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(C)\big)$$

where C ranges over all cyclic subgroups of G of order dividing n. $\square$

Lemma 6.3. *Let n and m be two p'-numbers. If C_m is a cyclic group of order m, then* $\dim_{\mathbb{K}} K_n(C_m) = m - (n, m)$ *where (n, m) is the greatest common divisor of n and m.*

Proof. For $X \leqslant Y \leqslant C_m$, it is clear from the relation $\mathrm{Res}^{C_m}_X = \mathrm{Res}^Y_X \mathrm{Res}^{C_m}_Y$ that $\mathrm{Ker}\,\mathrm{Res}^{C_m}_Y \subseteq \mathrm{Ker}\,\mathrm{Res}^{C_m}_X$. Therefore

$$K_n(C_m) = \mathrm{Ker}\big(\mathrm{Res}^{C_m}_H : \mathbb{K}\mathcal{R}_{\mathbb{F}}(C_m) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(H)\big)$$

where H is the unique maximal subgroup of C_m of order dividing n. Thus $|H| = (n, m)$. Since $\mathrm{Res}^{C_m}_H$ is surjective, $\dim_{\mathbb{K}} K_n(C_m) = \dim_{\mathbb{K}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(C_m) - \dim_{\mathbb{K}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(H)$ which is equal to $m - (n, m)$. $\square$

We now study the subfunctor K_1.

Lemma 6.4. *Let M be a subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ and G be a finite group. Then*:

(1) *$K_1(G)$ is of codimension 1 in $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$.*
(2) *$K_1(G) = 0$ if and only if G is a p-group.*
(3) *$M(1) = 0$ if and only if $M \leqslant K_1$.*
(4) *$M(1) \neq 0$ if and only if $M + K_1 = \mathbb{K}\mathcal{R}_{\mathbb{F}}$.*

Proof. (1) Because Res_1^G is surjective.

(2) Part (1) implies that $K_1(G)=0$ if and only if $\dim_{\mathbb{K}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(G)=1$, which is equivalent to $|\operatorname{Irr}(\mathbb{F}G)|=1$. This proves the result.

(3) If $M(1)=0$ then, for any finite group G, $\mathrm{Res}_1^G M(G) \subseteq M(1)=0$ implying that $M(G) \leqslant \operatorname{Ker}\mathrm{Res}_1^G = K_1(G)$.

(4) Suppose that $M(1) \neq 0$. Take any finite group G. By 6.1, $[\mathbb{F}_G] \in M(G)$. It is clear that $[\mathbb{F}_G]$ is not in $K_1(G)$. So $M(G)+K_1(G) > K_1(G)$. Then by part (1), $M(G)+K_1(G) = \mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$. $\square$

Proposition 6.5. *The functor K_1 is the unique maximal subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ such that $\mathbb{K}\mathcal{R}_{\mathbb{F}}/K_1$ is isomorphic to $S^{\mathfrak{i}}_{1,\mathbb{K}}$.*

Proof. K_1 is a maximal subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ by 6.4. As $K_1(1)=0 \neq \mathbb{K}\mathcal{R}_{\mathbb{F}}(1)$, the simple quotient $\mathbb{K}\mathcal{R}_{\mathbb{F}}/K_1$ must be isomorphic to $S^{\mathfrak{i}}_{1,\mathbb{K}}$.

Suppose that M is a maximal subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ such that $\mathbb{K}\mathcal{R}_{\mathbb{F}}/M \cong S^{\mathfrak{i}}_{1,\mathbb{K}}$. Then $M(1)=0$ implying by 6.4 that $M \leqslant K_1$. So $M=K_1$. $\square$

Corollary 6.6.

(1) *If M is a minimal subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ then $M(1)=0$ so that $M \leqslant K_1$. In particular $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ is not semisimple.*
(2) *Let $N \leqslant M$ be subfunctors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$. Then, $M/N \cong S^{\mathfrak{i}}_{1,\mathbb{K}}$ if and only if $M(1) \neq 0$ and $M \cap K_1 = N$.*
(3) *K_1 intersects every nonzero subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ nontrivially.*

Proof. (1) Assume that $M(1) \neq 0$. Then $M \cong S^{\mathfrak{i}}_{1,\mathbb{K}}$. Let G be any finite group. Since $[\mathbb{F}_1] \in M(1)$, it follows that $[\mathbb{F}G] = [\uparrow_1^G \mathbb{F}_1] \in \mathrm{Ind}_1^G M(1) \subseteq M(G)$. Moreover $[\mathbb{F}_G] \in M(G)$ by 6.1. But 6.5 implies that $\dim_{\mathbb{K}} M(G)=1$. Therefore $[\mathbb{F}G]=[\mathbb{F}_G]$ implying that $G=1$.

(2) Suppose that $M/N \cong S^{\mathfrak{i}}_{1,\mathbb{K}}$. Then $M(1) \neq 0$ and $N(1)=0$. Hence, 6.4 implies that $N \leqslant M \cap K_1 \leqslant M$ and $M+K_1 = \mathbb{K}\mathcal{R}_{\mathbb{F}}$. Consequently, by 6.5 we have

$$S^{\mathfrak{i}}_{1,\mathbb{K}} \cong \mathbb{K}\mathcal{R}_{\mathbb{F}}/K_1 = (M+K_1)/K_1 \cong M/(M \cap K_1).$$

This shows that $N = M \cap K_1$.

Suppose that $M(1) \neq 0$ and $M \cap K_1 = N$. Then by 6.4 and 6.5,

$$M/N \cong M/(M \cap K_1) \cong (M+K_1)/K_1 = \mathbb{K}\mathcal{R}_{\mathbb{F}}/K_1 \cong S^{\mathfrak{i}}_{1,\mathbb{K}}.$$

(3) Let M be a nonzero subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ such that $M \cap K_1 = 0$. Then

$$M \cong M/(M \cap K_1) \cong (M+K_1)/K_1 = \mathbb{K}\mathcal{R}_{\mathbb{F}}/K_1 \cong S^{\mathfrak{i}}_{1,\mathbb{K}}$$

by 6.4 and 6.5. So M is a minimal subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$, and then part (1) shows that $M \leqslant K_1$. Thus $M = M \cap K_1 = 0$. $\square$

We next study the subfunctors K_n for any p'-number n. But we first need a result about the dimensions of simple functors.

Remark 6.7. Let C_n and C_m be cyclic groups of respective orders n and m for some natural numbers n and m. If V is a simple $\mathbb{K}\,\mathrm{Out}(C_n)$-module then $\dim_{\mathbb{K}} S^{\mathrm{i}}_{C_n,V}(C_m)$ is equal to 1 if n divides m and 0 otherwise.

Proof. By Bouc [3], it is easy to see that the required dimension is the rank of a row matrix over $\mathbb{K}$ which contains a nonzero entry if and only if n divides m. Alternatively, one may use the formulas for the evaluations of simple inflation functors (or of simple (global) Mackey functors by 3.10 and 3.9) given in Webb [7] to deduce the result. □

Proposition 6.8. *Let n be a p'-number. Then the composition factors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}/K_n$ are precisely the simple functors $S^{\mathrm{i}}_{C,V}$ where C ranges over all nonisomorphic cyclic groups of order dividing n and V ranges over all nonisomorphic simple $\mathbb{K}\,\mathrm{Out}(C)$-modules. Moreover the multiplicity of each composition factor is* 1.

Proof. For any natural number m we denote by C_m a cyclic group of order m. Using 6.3 we see that if m is a p'-number, then $K_n(C_m) = 0$ if and only if m divides n. Therefore, if m divides n then K_n has no composition factor whose minimal subgroup is C_m. Then 5.5 implies that each element of the set

$$\mathfrak{S} = \left\{S^{\mathrm{i}}_{C_m,V}\colon\ m \in \mathbb{N},\ m \text{ divides } n,\ V \in \mathrm{Irr}\big(\mathbb{K}\,\mathrm{Out}(C_m)\big)\right\}$$

is a composition factor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}/K_n$ with multiplicity equal to 1.

We will show that there is no other composition factor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}/K_n$. Suppose that $S^{\mathrm{i}}_{C_r,W}$ is a composition factor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}/K_n$. By 5.5 we may assume that r is a p'-number so that $\dim_{\mathbb{K}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(C_r) = r$. Then from 6.7 the contribution of the composition factors in $\mathfrak{S}$ to $\dim_{\mathbb{K}}(\mathbb{K}\mathcal{R}_{\mathbb{F}}/K_n)(C_r)$ is equal to

$$d = \sum_m \left|\mathbb{K}\,\mathrm{Out}(C_m)\right|$$

where m ranges over all natural numbers dividing both of n and r. Thus

$$d = \sum_m \phi(m)$$

where m ranges over all natural numbers dividing the greatest common divisor (n,r) of n and r, and ϕ is the Euler's totient function. Now, $\dim_{\mathbb{K}}(\mathbb{K}\mathcal{R}_{\mathbb{F}}/K_n)(C_r) = (n,r)$ by 6.3 and $d = (n,r)$ by Gauss' theorem. Consequently, $S^{\mathrm{i}}_{C_r,W}$ must belong to the set $\mathfrak{S}$. □

The following is an immediate consequences of the previous result. Note that $K_{nm} \leqslant K_n$ for any natural numbers n and m.

Corollary 6.9. *Let n and m be two p'-numbers. Then the composition factors of K_n/K_{nm} are precisely the simple functors $S^{\mathrm{i}}_{C,V}$ where C ranges over all nonisomorphic cyclic groups of order dividing nm but not dividing n, and V ranges over all nonisomorphic simple $\mathbb{K}\,\mathrm{Out}(C)$-modules. Moreover the multiplicity of each composition factor is* 1.

The previous result suggests to define the following subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$. For any natural number n we define

$$F_n = \bigcap_d K_d$$

where d ranges over all natural numbers less than n and dividing n.

It is clear that $F_n \geqslant K_n$ so that it deserves study only when $F_n \neq K_n$.

Remark 6.10. Let M be a subfunctor of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$. Then, $M \leqslant K_n$ if and only if $M(C_n) = 0$ where C_n is a cyclic group of order n.

Proof. Definition of K_n implies that $K_n(C_n) = 0$. So if $M \leqslant K_n$ then $M(C_n) = 0$. Conversely, if $M(C_n) = 0$ then it follows by 3.1 that M is a subfunctor of $\operatorname{Ker}_{C_n,0}^{\mathbb{K}\mathcal{R}_{\mathbb{F}},\mathrm{i}}$ and hence by 6.2 a subfunctor of K_n. □

Lemma 6.11. *Let n be a p'-number. Then $F_n \neq K_n$.*

Proof. By 6.10, $F_n = K_n$ if and only if $F_n(C_n) = 0$ where, for any natural number m, we denote by C_m a cyclic group of order m. By the definition of F_n and by the relation $\operatorname{Res}_X^Y \operatorname{Res}_Y^{C_n} = \operatorname{Res}_X^{C_n}$, we may write

$$F_n(C_n) = \bigcap_C \operatorname{Ker}\bigl(\operatorname{Res}_C^{C_n} : \mathbb{K}\mathcal{R}_{\mathbb{F}}(C_n) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(C)\bigr)$$

where C ranges over all maximal subgroups of C_n. Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ be the prime factorization of n, where p_i's are distinct primes and $\alpha_i \geqslant 1$. As maximal subgroups of cyclic groups must have prime index,

$$F_n(C_n) = \bigcap_{s=1}^{r} \operatorname{Ker}\bigl(\operatorname{Res}_{C_{n/p_s}}^{C_n} : \mathbb{K}\mathcal{R}_{\mathbb{F}}(C_n) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(C_{n/p_s})\bigr).$$

By the identification given after 5.3 (i.e., using the isomorphism $\mathbb{K}\mathcal{R}_{\mathbb{F}}(A \times B) \cong \mathbb{K}\mathcal{R}_{\mathbb{F}}(A) \otimes_{\mathbb{K}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(B)$ for two groups A and B of coprime orders), if we put $A_j = C_{p_j^{\alpha_j}}$ for all $j = 1, 2, \dots, r$ we have

$$\operatorname{Res}_{C_{n/p_i}}^{C_n} = \bigotimes_{j=1}^{r}{}_{\mathbb{K}} \operatorname{Res}_{H_j}^{A_j} : \bigotimes_{j=1}^{r}{}_{\mathbb{K}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(A_j) \to \bigotimes_{j=1}^{r}{}_{\mathbb{K}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(H_j)$$

where $H_j = A_j$ if $j \neq i$ and $H_i = C_{p_i^{\alpha_i - 1}}$. Since all the maps $\operatorname{Res}_{H_j}^{A_j}$ except $j = i$ are identities,

$$\operatorname{Ker}\operatorname{Res}_{C_{n/p_i}}^{C_n} = \mathbb{K}\mathcal{R}_{\mathbb{F}}(A_1) \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} \operatorname{Ker}\operatorname{Res}_{H_i}^{A_i} \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} \mathbb{K}\mathcal{R}_{\mathbb{F}}(A_r).$$

Therefore

$$F_n(C_n) = \bigotimes_{s=1}^{r} {}_{\mathbb{K}} \operatorname{Ker}\Bigl(\operatorname{Res}^{C_{p_s^{\alpha_s}}}_{C_{p_s^{\alpha_s-1}}} : \mathbb{K}\mathcal{R}_{\mathbb{F}}(C_{p_s^{\alpha_s}}) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(C_{p_s^{\alpha_s-1}})\Bigr).$$

As the maps $\operatorname{Res}^{C_n}_X$ are surjective for all subgroups X, we see that

$$\dim_{\mathbb{K}} F_n(C_n) = \prod_{s=1}^{r} \bigl(p_s^{\alpha_s} - p_s^{\alpha_s-1}\bigr) = \phi(n)$$

where ϕ is the Euler's function. In particular $F_n(C_n) \neq 0$. $\square$

By the definition of F_n and 6.9, the following is obvious.

Corollary 6.12. *Let C be a cyclic group whose order is a p'-number n. Then the composition factors of F_n/K_n are precisely the simple functors $S^{\mathrm{i}}_{C,V}$ where V ranges over all nonisomorphic simple $\mathbb{K}\operatorname{Out}(C)$-modules. Moreover the multiplicity of each composition factor is* 1.

We will prove that the functors F_n/K_n are semisimple for any p'-number n. We will make use of 3.7 in our proof. For this reason we first give a result stating that the functors F_n/K_n satisfy some of the conditions of 3.7.

Lemma 6.13. *Let n be a p'-number and C_n be a cyclic group of order n. Then*:

(1) $I^{\mathrm{i}}_{C_n}$ *annihilates* $F_n(C_n)$.
(2) $\operatorname{Ker}^{F_n/K_n,\mathrm{i}}_{C_n,0} = 0$.

Proof. (1) Let $[(C_n \times C_n)/L]$ be in $I^{\mathrm{i}}_{C_n}$ and let x be in $F_n(C_n)$. We will show that $[(C_n \times C_n)/L]x = 0$. Then $[(C_n \times C_n)/L]$ is of the form

$$\operatorname{Ind}^{C_n}_{p_1} \operatorname{Inf}^{p_1}_{p_1/k_1} \operatorname{Iso}^{p_1/k_1}_{p_2} \operatorname{Res}^{C_n}_{p_2}$$

where $p_i = p_i(L)$, $k_1 = k_1(L)$, and $|q(L)| < n$. Therefore p_2 is a cyclic subgroup of C_n of order less than n and dividing n so that $\operatorname{Res}^{C_n}_{p_2} x = 0$ by the definition of F_n. Consequently, $I^{\mathrm{i}}_{C_n} F_n(C_n) = 0$.

(2) Using the properties of Ker given in 3.1, we see that

$$\operatorname{Ker}^{F_n/K_n,\mathrm{i}}_{C_n,0} = \operatorname{Ker}^{F_n,\mathrm{i}}_{C_n,0}/K_n = \bigl(\operatorname{Ker}^{\mathbb{K}\mathcal{R}_{\mathbb{F}},\mathrm{i}}_{C_n,0} \cap F_n\bigr)/K_n = (K_n \cap F_n)/K_n = 0$$

where we also use $\operatorname{Ker}^{\mathbb{K}\mathcal{R}_{\mathbb{F}},\mathrm{i}}_{C_n,0} = K_n$ from 6.2. $\square$

Proposition 6.14. *Let C be a cyclic group whose order is a p'-number n. Then F_n/K_n is a semisimple functor such that*

$$F_n/K_n \cong \bigoplus_{V \in \operatorname{Irr}(\mathbb{K}\operatorname{Out}(C))} S^{\mathrm{i}}_{C,V}.$$

Proof. We first show that $F_n(C)$ is a semisimple $\mathrm{End}_{\mathfrak{i}}(C)$-module isomorphic to $\mathbb{K}\,\mathrm{Out}(C)$ as $\mathbb{K}\,\mathrm{Out}(C)$-modules.

By 6.12 we may find a series of functors

$$K_n = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_d = F_n$$

such that each quotient is a simple functor whose minimal subgroup is C and that

$$\mathrm{Irr}\big(\mathbb{K}\,\mathrm{Out}(C)\big) = \big\{(M_{i+1}/M_i)(C)\colon\ i = 0, \ldots, d-1\big\}$$

and $d = |\,\mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C))|$. By evaluating at C we get the series

$$0 = K_n(C) = M_0(C) \subset M_1(C) \subset M_2(C) \subset \cdots \subset M_d(C) = F_n(C)$$

of $\mathrm{End}_{\mathfrak{i}}(C)$-modules such that each quotient is a simple $\mathrm{End}_{\mathfrak{i}}(C)$-module by 4.1. Now by 6.13 the ideal $I_C^{\mathfrak{i}}$ annihilates $F_n(C)$ so that the last series is a composition series of $F_n(C)$ as $\mathbb{K}\,\mathrm{Out}(C)$-modules. Since $\mathbb{K}\,\mathrm{Out}(C)$ is semisimple and since $\mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C)) = \{(M_{i+1}/M_i)(C)\colon\ i = 0, \ldots, d-1\}$, it follows that $F_n(C)$ is a semisimple $\mathrm{End}_{\mathfrak{i}}(C)$-module annihilated by $I_C^{\mathfrak{i}}$ and $F_n(C) \cong \mathbb{K}\,\mathrm{Out}(C)$ as $\mathbb{K}\,\mathrm{Out}(C)$-modules.

We now show that F_n/K_n is generated by $(F_n/K_n)(C)$ as inflation functors. That is $\mathrm{Im}_{C,F_n(C)}^{F_n/K_n,\mathfrak{i}} = F_n/K_n$, see 3.1.

Let the following series

$$0 = X_0 \subset X_1 \subset \cdots \subset X_d = F_n(C)$$

be a composition series of $F_n(C)$ as $\mathbb{K}\,\mathrm{Out}(C)$-modules (and hence as $\mathrm{End}_{\mathfrak{i}}(C)$-modules because $I_C^{\mathfrak{i}}$ annihilates $F_n(C)$). For each i we define a subfunctor $N_i/K_n = \mathrm{Im}_{C,X_i}^{F_n/K_n,\mathfrak{i}}$ of F_n/K_n (note that $K_n(C) = 0$) so that by the properties of Im given in 3.1 we have the following series of functors

$$K_n = N_0 \subset N_1 \subset \cdots \subset N_d \subseteq F_n.$$

If N_d is not equal to F_n then the number of composition factors of F_n/K_n counting with multiplicities must be larger than d which is impossible by 6.12. Thus $N_d = F_n$. This proves that $F_n/K_n = N_d/K_n = \mathrm{Im}_{C,F_n(C)}^{F_n/K_n,\mathfrak{i}}$ as desired.

Up to now we observed that $F_n/K_n = \mathrm{Im}_{C,F_n(C)}^{F_n/K_n,\mathfrak{i}}$ and $\mathrm{Ker}_{C,0}^{F_n/K_n,\mathfrak{i}} = 0$ (by 6.13), and also that $(F_n/K_n)(C) \cong F_n(C)$ is a semisimple $\mathrm{End}_{\mathfrak{i}}(C)$-module. Moreover, any nonzero subfunctor of F_n/K_n must be nonzero at C from 6.12.

Therefore, 3.7 can be applied to deduce that F_n/K_n is semisimple. The rest follows by 6.12. $\square$

If q is a prime different from p then we see that $F_{q^n} = K_{q^{n-1}}$ for any natural number n. And using 6.14 we get a series of functors

$$\mathbb{K}\mathcal{R}_{\mathbb{F}} \supset K_1 \supset K_q \supset K_{q^2} \cdots \supset K_{q^n} \supset \cdots$$

such that the quotients are semisimple and

$$K_{q^{n-1}}/K_{q^n} \cong \bigoplus_{V \in \mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C_{q^n}))} S^{\mathrm{i}}_{C_{q^n},V}$$

where C_{q^n} is a cyclic group of order q^n.

We want to find series of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ as above involving the subfunctors K_n and F_n whose quotients are semisimple and cover all composition factors of $\mathbb{K}\mathcal{R}_{\mathbb{F}}$.

We finish this section by constructing a series of functors

$$\mathbb{K}\mathcal{R}_{\mathbb{F}} \supset K_1 \supset L_1 \supset L_2 \supset \cdots \supset L_j \supset \cdots$$

such that the quotients are semisimple and cover all composition factors whose minimal subgroups are π-groups where π is any set of prime numbers not containing p.

Let $\pi = \{p_1, p_2, \ldots, p_r\}$ be a set of prime numbers not containing p. For any natural number j we define

$$L_0 = K_1, \qquad L_1 = \bigcap_{1 \leqslant i_1 \leqslant r} K_{p_{i_1}}, \qquad L_2 = \bigcap_{1 \leqslant i_1 \leqslant i_2 \leqslant r} K_{p_{i_1} p_{i_2}}, \quad \text{and}$$

$$L_j = \bigcap_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_j \leqslant r} K_{p_{i_1} p_{i_2} \cdots p_{i_j}}.$$

Theorem 6.15. *Let $\pi = \{p_1, p_2, \ldots, p_r\}$ be a set of prime numbers not containing p. Then the series of functors*

$$\mathbb{K}\mathcal{R}_{\mathbb{F}} \supset K_1 \supset L_1 \supset L_2 \supset \cdots \supset L_j \supset \cdots$$

satisfies:

(1) *L_{j-1}/L_j is a semisimple functor for all $j = 1, 2, \ldots$.*

(2)
$$L_{j-1}/L_j \cong \bigoplus_{C} \bigoplus_{V \in \mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C))} S^{\mathrm{i}}_{C,V}$$

where C ranges over all nonisomorphic cyclic groups of order $p_{i_1} p_{i_2} \ldots p_{i_j}$ with $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_j \leqslant r$.

Proof. As $K_d \leqslant K_s$ for any natural numbers d and s such that s divides d, we see by the definition of F_n that $F_n = \bigcap_q K_{n/q}$ for any natural number n where q ranges over all prime divisors of n. This shows that

$$L_{j-1} \subseteq F_{p_{i_1} p_{i_2} \cdots p_{i_j}}$$

for any $j = 1, 2, \ldots$ and $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_j \leqslant r$. (Note that $F_{p_{i_1}} = K_1$.) Consequently, the natural epimorphism

$$F_{p_{i_1} p_{i_2} \cdots p_{i_j}} \to F_{p_{i_1} p_{i_2} \cdots p_{i_j}} / K_{p_{i_1} p_{i_2} \cdots p_{i_j}}$$

induces a monomorphism

$$L_{j-1}/(L_{j-1} \cap K_{p_{i_1}p_{i_2}\cdots p_{i_j}}) \to F_{p_{i_1}p_{i_2}\cdots p_{i_j}}/K_{p_{i_1}p_{i_2}\cdots p_{i_j}}.$$

Then 6.14 implies that

$$L_{j-1}/(L_{j-1} \cap K_{p_{i_1}p_{i_2}\cdots p_{i_j}}) \cong \bigoplus_{V \in \mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C))} e_V S^{\mathfrak{i}}_{C,V}$$

where $e_V \in \{0,1\}$ and C is a cyclic group of order $p_{i_1}p_{i_2}\ldots p_{i_j}$. In particular it is semisimple (if nonzero).

Now the homomorphism

$$L_{j-1} \to \prod_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_j \leqslant r} L_{j-1}/(L_{j-1} \cap K_{p_{i_1}p_{i_2}\cdots p_{i_j}}),$$

which is the product of natural epimorphisms

$$L_{j-1} \to L_{j-1}/(L_{j-1} \cap K_{p_{i_1}p_{i_2}\cdots p_{i_j}}),$$

has kernel equal to L_j. Therefore, if C_m denotes any cyclic group of order m then we have

$$L_{j-1}/L_j \cong \bigoplus_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_j \leqslant r} \ \bigoplus_{V \in \mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C_{p_{i_1}p_{i_2}\cdots p_{i_j}}))} e_V^{i_1 i_2 \ldots i_j} S^{\mathfrak{i}}_{C_{p_{i_1}p_{i_2}\cdots p_{i_j}},V}$$

where $e_V^{i_1 i_2 \ldots i_j} \in \{0,1\}$. In particular it is semisimple (if nonzero).

To show that each $e_V^{i_1 i_2 \ldots i_j}$ is equal to 1, we simply observe that $L_j(C_{p_{i_1}p_{i_2}\cdots p_{i_k}}) = 0$ for any $k \leqslant j$ and $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant r$. This proves by 5.5 that, for any $V \in \mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C_{p_{i_1}p_{i_2}\cdots p_{i_j}}))$, the simple functors $S^{\mathfrak{i}}_{C_{p_{i_1}p_{i_2}\cdots p_{i_j}},V}$ are composition factors of L_{j-1}/L_j with multiplicity 1. Hence, each $e_V^{i_1 i_2 \ldots i_j}$ is equal to 1. □

We have the following immediate consequence. For a group G with $|G| = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$ where q_i are distinct primes and $\alpha_i \geqslant 1$ are integers, we put $\ell(G) = \sum_i \alpha_i$ and $\pi(G) = \{q_1, q_2, \ldots, q_s\}$.

Corollary 6.16. *Let χ be a finite family of groups satisfying the conditions given in the last paragraph of Section 2. Let $\{p_1, \ldots, p_r\}$ be the union of the sets $\pi(C)$ and n be the maximum of the numbers $\ell(C)$ where C ranges over all cyclic p'-groups in χ. Then, the following series*

$$\mathbb{K}\mathcal{R}^{\chi}_{\mathbb{F}} \supset K_1^{\chi} \supset L_1^{\chi} \supset L_2^{\chi} \supset \cdots \supset L_n^{\chi} = 0$$

of $\mathbb{K}\mathcal{R}^{\chi}_{\mathbb{F}}$ as functors on χ satisfies that each $L_{j-1}^{\chi}/L_j^{\chi}$ is semisimple and

$$L^{\chi}_{j-1}/L^{\chi}_{j} \cong \bigoplus_{C} \bigoplus_{V \in \mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C))} S^{\mathfrak{i},\chi}_{C,V}$$

where C ranges over all cyclic p'-groups in χ with $\ell(C) = j$.

Our final result of this section is an immediate consequences of 6.15 obtained by letting π be the set of all prime numbers different from p.

In this case it is clear by the definitions of L_j which depend on the set π that $L_0 = K_1$ and $L_j = \bigcap_n K_n$, where n ranges over all natural p'-numbers whose number of prime divisors counted with multiplicities is j. Then, the definition of K_n implies that

$$L_j(G) = \bigcap_X \mathrm{Ker}\bigl(\mathrm{Res}^G_X : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(X)\bigr)$$

where X ranges over all cyclic p'-subgroups of G satisfying $\ell(X) \leqslant j$.

Theorem 6.17. *There is a chain of functors*

$$\mathbb{K}\mathcal{R}_{\mathbb{F}} = L_{-1} \supset L_0 \supset L_1 \supset \cdots \supset L_j \supset \cdots$$

such that $\bigcap_j L_j = 0$ and each L_{j-1}/L_j is semisimple with

$$L_{j-1}/L_j \cong \bigoplus_{C,V} S^{\mathfrak{i}}_{C,V}$$

where C ranges over all nonisomorphic cyclic p'-groups with $\ell(C) = j$ and V ranges over all nonisomorphic simple $\mathbb{K}\,\mathrm{Out}(C)$-modules.

Proof. We observed above that the subfunctors L_j can be defined as in Section 1. Thus all the assertions except $\bigcap_j L_j = 0$ follow immediately from 6.15. We will show that $\bigcap_j L_j = 0$. If it is nonzero then its evaluation at some finite group G must be nonzero. Then, considering $\bigcap_j L_j$ as a functor defined on the finite family of groups consisting of representatives of the isomorphism classes of subquotients of G, we may regard $\bigcap_j L_j$ as a nonzero finite dimensional module of a finite dimensional $\mathbb{K}$-algebra. See the last paragraph of Section 2. Therefore, if $\bigcap_j L_j$ is nonzero at some finite group G, then it must have a simple section of the form $S^{\mathfrak{i}}_{C,V}$ where C is a cyclic p'-group of order dividing $|G|$. On the other hand, 6.15 (with π is the set of all primes different from p) implies that $S^{\mathfrak{i}}_{C,V}$ is a summand of L_{j-1}/L_j where $j = \ell(C)$. Since $L_{j-1} \supset L_j \supset \bigcap_j L_j$, it follows that the multiplicity of $S^{\mathfrak{i}}_{C,V}$ in $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ is greater than or equal to 2, which is not the case by 5.5. Hence $\bigcap_j L_j = 0$. $\square$

7. Composition factors of $\mathbb{K}\mathcal{P}_{\mathbb{F}}$

We still assume that the fields $\mathbb{K}$ and $\mathbb{F}$ satisfy the same conditions of Sections 5 and 6. In this section we briefly explain that one can use similar arguments to find the composition factors of the deflation functor $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ whose evaluation at any finite group G is $\mathbb{K} \otimes_{\mathbb{Z}} K_0(G)$, where $K_0(G)$

is the Grothendieck group of finitely generated projective $\mathbb{F}G$-modules. By definition, $K_0(G)$ is generated by expressions $[P]$, one for each isomorphism class (P) of finitely generated projective $\mathbb{F}G$-modules, with relations $[P' \oplus P''] = [P'] + [P'']$. Therefore

$$\mathbb{K}\mathcal{P}_{\mathbb{F}}(G) = \bigoplus_{P} \mathbb{K}[P]$$

where P ranges over a complete set of isomorphism classes of principal indecomposable $\mathbb{F}G$-modules.

Let S be a (G, H)-biset. If the functor ${}_{\mathbb{F}G}\mathbb{F}S \otimes_{\mathbb{F}H} - : \mathbb{F}H\text{-Mod} \to \mathbb{F}G\text{-Mod}$ sends projectives to projectives, then it induces a map

$$\mathbb{K}\mathcal{P}_{\mathbb{F}}(H) \to \mathbb{K}\mathcal{P}_{\mathbb{F}}(G), \quad [P] \mapsto [\mathbb{F}S \otimes_{\mathbb{F}H} P].$$

This is equivalent to the projectivity of ${}_{\mathbb{F}G}\mathbb{F}S$. For the four type of basic bisets

$$\mathrm{Ind}_H^G, \qquad \mathrm{Iso}_{G'}^G, \qquad \mathrm{Def}_{G/N}^G, \quad \text{and} \quad \mathrm{Res}_H^G,$$

we see that the left modules

$${}_{\mathbb{F}G}\mathbb{F}G, \qquad {}_{\mathbb{F}G}\mathbb{F}G', \qquad {}_{\mathbb{F}(G/N)}\mathbb{F}(G/N) \quad \text{and} \quad {}_{\mathbb{F}H}\mathbb{F}G$$

are all free and so projective where $H \leqslant G \trianglerighteq N$ and $G' \cong G$. While for $\mathrm{Inf}_{G/N}^G$, we see that ${}_{\mathbb{F}G}\mathbb{F}(G/N)$ is projective if and only if p does not divide the order of N. Therefore $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ has a natural deflation functor structure over $\mathbb{K}$.

Let $\mathfrak{d}$ be the subcategory of the biset category $\mathfrak{b}$ with the same objects and with the morphisms

$$\mathrm{Hom}_{\mathfrak{d}}(H, G) = \bigoplus_{L \leqslant_* G \times H:\ k_1(L)=1} R\big[(G \times H)/L\big].$$

An R-linear functor from $\mathfrak{d}$ to the category of left R-modules is called a deflation functor.

We now exhibit that there is an isomorphism between the deflation functors $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ and $\mathbb{K}\mathcal{R}_{\mathbb{F}}^*$.

For a (G, H)-biset S we define the opposite S^{op} of S as the (H, G)-biset S with the (H, G)-action given by $h.s.g = g^{-1}sh^{-1}$. It is clear that the opposites of the bisets Ind_H^G, $\mathrm{Inf}_{G/N}^G$ and $\mathrm{Iso}_G^{G'}(\psi)$ are the bisets Res_H^G, $\mathrm{Def}_{G/N}^G$ and $\mathrm{Iso}_{G'}^G(\psi^{-1})$, respectively. See Bouc [3].

Recall that the dual of a biset functor F over a field $\mathbb{L}$ is the biset functor F^* given on objects G and on morphisms $[S] \in \mathrm{Hom}_{\mathfrak{b}}(H, G)$ as follows:

$$F^*(G) = \mathrm{Hom}_{\mathbb{L}}\big(F(G), \mathbb{L}\big),$$
$$F^*\big([S]\big) : \mathrm{Hom}_{\mathbb{L}}\big(F(H), \mathbb{L}\big) \to \mathrm{Hom}_{\mathbb{L}}\big(F(G), \mathbb{L}\big), \quad f \mapsto f \circ F\big(\big[S^{\mathrm{op}}\big]\big).$$

Evidently, dual of an inflation functor is a deflation functor.

While V ranges over $\mathrm{Irr}(\mathbb{F}G)$, the elements $[P(V)]$ and $[V]$ range over respective $\mathbb{K}$-bases of $\mathbb{K}\mathcal{P}_{\mathbb{F}}(G)$ and $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$ where $P(V)$ is the projective cover of V. Therefore the $\mathbb{K}$-linear extensions of the maps, whose images at the above basis elements are given as

$$r_G : \mathbb{K}\mathcal{P}_{\mathbb{F}}(G) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}(G), \quad \big[P(V)\big] \mapsto [V],$$
$$f_G^* : \mathbb{K}\mathcal{R}_{\mathbb{F}}(G) \to \mathbb{K}\mathcal{R}_{\mathbb{F}}^*(G), \quad [V] \mapsto [V]^*,$$

are well-defined $\mathbb{K}$-space isomorphisms, where $[V]^*$ is a dual basis element of $\mathbb{K}\mathcal{R}_{\mathbb{F}}^*(G)$ that corresponds the basis element $[V] \in \mathrm{Irr}_{\mathbb{F}}(G)$ of $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G)$.

Theorem 7.1. *$\mathbb{K}\mathcal{P}_{\mathbb{F}}$ and $\mathbb{K}\mathcal{R}_{\mathbb{F}}^*$ are isomorphic deflation functors.*

Proof. For simplicity we write $\mathcal{R}$ for $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ and $\mathcal{P}$ for $\mathbb{K}\mathcal{P}_{\mathbb{F}}$. Let Ψ be the map from $\mathcal{P}$ to $\mathcal{R}^*$ whose G-component Ψ_G is given by $f_G^* \circ r_G$. By construction, the map Ψ_G is a $\mathbb{K}$-space isomorphism. We will show Ψ is a deflation functor homomorphism by observing that it commutes with Ind, Iso, Def and Res.

We first note that for any simple $\mathbb{F}G$-module V and any $\mathbb{F}G$-module X, one has

$$\Psi_G\big(\big[P(V)\big]\big)\big([X]\big) = \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}G}\big(P(V), X\big)$$

which is the multiplicity of V as a composition factor of X. Moreover, given $\mathbb{F}G$-modules M_1 and M_2 we have

$$P(M_1 \oplus M_2) \cong P(M_1) \oplus P(M_2) \quad \text{and}$$
$$\mathrm{Hom}_{\mathbb{F}G}(M_1 \oplus M_2, X) \cong \mathrm{Hom}_{\mathbb{F}G}(M_1, X) \oplus \mathrm{Hom}_{\mathbb{F}G}(M_2, X).$$

This shows that

$$\Psi_G\big([M]\big)\big([X]\big) = \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}G}(M, X)$$

for any projective $\mathbb{F}G$-module M and any $\mathbb{F}G$-module X. Recall that the functors

$$\downarrow_H^G, \qquad \uparrow_H^G, \qquad {}_{\mathbb{F}G}\mathbb{F}(G/N) \otimes_{\mathbb{F}(G/N)} -, \quad \text{and} \quad {}_{\mathbb{F}(G/N)}\mathbb{F}(G/N) \otimes_{\mathbb{F}G} -$$

between the module categories of group algebras (over $\mathbb{F}$) satisfy that the pairs

$$\big(\uparrow_H^G, \downarrow_H^G\big), \qquad \big(\downarrow_H^G, \uparrow_H^G\big) \quad \text{and} \quad \big({}_{\mathbb{F}(G/N)}\mathbb{F}(G/N) \otimes_{\mathbb{F}G} -, {}_{\mathbb{F}G}\mathbb{F}(G/N) \otimes_{\mathbb{F}(G/N)} -\big)$$

are adjoint pairs. Moreover, all of the functors

$$\downarrow_H^G, \qquad \uparrow_H^G, \quad \text{and} \quad {}_{\mathbb{F}(G/N)}\mathbb{F}(G/N) \otimes_{\mathbb{F}G} -$$

send projectives to projectives.

Now we can see by using the adjointness of the above functors that Ψ commutes with Ind, Iso, Def, and Res.

Let $H \leqslant G$. Given a projective $\mathbb{F}H$-module W and an $\mathbb{F}G$-module X, we have

$$\begin{aligned}\mathcal{R}^*\big(\mathrm{Ind}_H^G\big)\big(\Psi_H\big([W]\big)\big)\big([X]\big) &= \big(\Psi_H\big([W]\big)\circ\mathcal{R}\big(\mathrm{Res}_H^G\big)\big)\big([X]\big)\\ &= \Psi_H\big([W]\big)\big(\big[\downarrow_H^G X\big]\big)\\ &= \dim_{\mathbb{F}}\mathrm{Hom}_{\mathbb{F}H}\big(W, \downarrow_H^G X\big)\\ &= \dim_{\mathbb{F}}\mathrm{Hom}_{\mathbb{F}G}\big(\uparrow_H^G W, X\big)\\ &= \Psi_G\big(\big[\uparrow_H^G W\big]\big)\big([X]\big)\\ &= \Psi_G\big(\mathcal{P}\big(\mathrm{Ind}_H^G\big)\big([W]\big)\big)\big([X]\big).\end{aligned}$$

Therefore $\Psi_G\circ\mathcal{P}(\mathrm{Ind}_H^G)=\mathcal{R}^*(\mathrm{Ind}_H^G)\circ\Psi_H$.

For the above commuting relation we used the adjointness of the pair $(\uparrow_H^G, \downarrow_H^G)$. Similarly, one may show by using the adjointness of the pair $(\downarrow_H^G, \uparrow_H^G)$ that Ψ commutes with Res.

Let $N \trianglelefteq G$. Given a projective $\mathbb{F}G$-module W and an $\mathbb{F}(G/N)$-module X, we have

$$\begin{aligned}\mathcal{R}^*\big(\mathrm{Def}_{G/N}^G\big)\big(\Psi_G\big([W]\big)\big)\big([X]\big) &= \big(\Psi_G\big([W]\big)\circ\mathcal{R}\big(\mathrm{Inf}_{G/N}^G\big)\big)\big([X]\big)\\ &= \Psi_G\big([W]\big)\big(\big[\mathbb{F}(G/N)\otimes_{\mathbb{F}(G/N)} X\big]\big)\\ &= \dim_{\mathbb{F}}\mathrm{Hom}_{\mathbb{F}G}\big(W, \mathbb{F}(G/N)\otimes_{\mathbb{F}(G/N)} X\big)\\ &= \dim_{\mathbb{F}}\mathrm{Hom}_{\mathbb{F}(G/N)}\big(\mathbb{F}(G/N)\otimes_{\mathbb{F}G} W, X\big)\\ &= \Psi_{G/N}\big(\big[\mathbb{F}(G/N)\otimes_{\mathbb{F}G} W\big]\big)\big([X]\big)\\ &= \Psi_{G/N}\big(\mathcal{P}\big(\mathrm{Def}_{G/N}^G\big)\big([W]\big)\big)\big([X]\big).\end{aligned}$$

Therefore $\Psi_{G/N}\circ\mathcal{P}(\mathrm{Def}_{G/N}^G)=\mathcal{R}^*(\mathrm{Def}_{G/N}^G)\circ\Psi_G$.

Any group isomorphism $\psi: G\to G'$ induces an $\mathbb{F}$-algebra isomorphism $\mathbb{F}G\to\mathbb{F}G'$ so that $\mathrm{Iso}_G^{G'}$ transposes the module structure via this isomorphism. Therefore the fact that Ψ commutes with Iso is obvious.

Consequently, $\Psi:\mathcal{P}\to\mathcal{R}^*$ is a deflation functor isomorphism. □

Obviously, the proof of 7.1 implies the isomorphism of the deflation functors K_0 and G_0^* over $\mathbb{Z}$.

We now explain how to find a filtration of $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ by using 7.1 and 6.17. This will follow from some basic facts about dual of a vector space, provided we show that similar results hold also for dual of a biset functor which is the content of the next result.

Let F be a biset functor over a field $\mathbb{L}$ and K be a biset subfunctor of F. We define a subset $K^\perp$ of F^* given on objects G as follows:

$$K^\perp(G)=\big\{f\in F^*(G):\ f\big(K(G)\big)=0\big\}.$$

For any biset functor homomorphism $\varphi: F\to L$ we denote by φ^* the map $L^*\to F^*$ whose G-component $\varphi_G^*: L^*(G)\to F^*(G)$ is given by $f\mapsto f\circ\varphi_G$ for any $f\in L^*(G)$.

Remark 7.2. Let F be a biset functor, K be a biset subfunctor of F, and $\varphi: F\to L$ be a biset functor homomorphism. Then:

(1) $K^{\perp}$ is a biset subfunctor of F^*.
(2) $\varphi^*: L^* \to F^*$ is a biset functor homomorphism.

Proof. (1) For any morphism $[S] \in \mathrm{Hom}_{\mathfrak{b}}(H, G)$, we must show that

$$F^*\big([S]\big)\big(K^{\perp}(H)\big) \subseteq K^{\perp}(G).$$

We first note that $F([S^{\mathrm{op}}])(K(G)) \subseteq K(H)$ because K is a biset subfunctor of F. For this end, we take any element f of $K^{\perp}(H)$ and compute that

$$F^*\big([S]\big)(f)\big(K(G)\big) = f\big(F\big(\big[S^{\mathrm{op}}\big]\big)\big(K(G)\big)\big) \subseteq f\big(K(H)\big) = 0.$$

Thus $F^*([S])(f) \in K^{\perp}(G)$.

(2) We only need to check that φ^* commutes with morphisms of the biset functor category. Thus, for any morphism $[S] \in \mathrm{Hom}_{\mathfrak{b}}(H, G)$, we must show that the following maps

$$\varphi_G^* L^*\big([S]\big): \mathrm{Hom}_{\mathbb{L}}\big(L(H), \mathbb{L}\big) \to \mathrm{Hom}_{\mathbb{L}}\big(F(G), \mathbb{L}\big), \quad f \mapsto f \circ L\big(\big[S^{\mathrm{op}}\big]\big) \mapsto f \circ L\big(\big[S^{\mathrm{op}}\big]\big) \circ \varphi_G,$$
$$F^*\big([S]\big)\varphi_H^*: \mathrm{Hom}_{\mathbb{L}}\big(L(H), \mathbb{L}\big) \to \mathrm{Hom}_{\mathbb{L}}\big(F(G), \mathbb{L}\big), \quad f \mapsto f \circ \varphi_H \mapsto f \circ \varphi_H \circ F\big(\big[S^{\mathrm{op}}\big]\big)$$

are equal. But this is obvious because

$$L\big(\big[S^{\mathrm{op}}\big]\big) \circ \varphi_G = \varphi_H \circ F\big(\big[S^{\mathrm{op}}\big]\big)$$

from the fact that $\varphi: F \to L$ is a biset functor homomorphism. $\square$

We note that the definition of $K^{\perp}$ may sometimes be confusing because it depends on F having K as a subfunctor. In the following both of $K^{\perp}$ and $L^{\perp}$ depend on F so that $L^{\perp} \leqslant K^{\perp}$ when $K \leqslant L \leqslant F$. Note also that for $F \leqslant F \geqslant 0$ we have $0^{\perp} = F^*$ and $F^{\perp} = 0$.

Lemma 7.3. *For any chain $K \leqslant L \leqslant F$ of biset functors, we have*

$$(L/K)^* \cong K^{\perp}/L^{\perp}.$$

Proof. The inclusion map $\iota : L \to F$ of biset functors induces the surjective biset functor homomorphism $\iota^*: F^* \to L^*$ by 7.2 because each component ι_G^* is a surjective $\mathbb{L}$-space map. It is easy to see that $\mathrm{Ker}\,\iota = L^{\perp}$. Consequently,

$$L^* \cong F^*/L^{\perp}, \quad \iota_G^*(x_G) \leftrightarrow x_G + L^{\perp}(G).$$

The natural epimorphism $\pi : L \to L/K$ of biset functors induces the biset functor monomorphism $\pi^*: (L/K)^* \to L^*$ and its image is equal to $K^{\perp}$, by 7.2 and by the similar results in the context of vector spaces over $\mathbb{L}$. Thus we have the following biset functor monomorphism

$$(L/K)^* \to F^*/L^{\perp}$$

whose image is $K^{\perp}/L^{\perp}$. $\square$

It is clear now that from the chain of inflation functors given in 6.17 we obtain the following chain of deflation functors

$$0=\mathbb{K}\mathcal{R}_{\mathbb{F}}^{\perp}\subset L_{-1}^{\perp}\subset L_{0}^{\perp}\subset L_{1}^{\perp}\subset\cdots\subset L_{j}^{\perp}\subset\cdots\subset 0^{\perp}=\mathbb{K}\mathcal{R}_{\mathbb{F}}^{*}\cong\mathbb{K}\mathcal{P}_{\mathbb{F}}.$$

We also see from 7.3 that

$$L_{j}^{\perp}/L_{j-1}^{\perp}\cong(L_{j-1}/L_{j})^{*}\cong\bigoplus_{C,V}\big(S_{C,V}^{\mathfrak{i}}\big)^{*}\cong\bigoplus_{C,V}\big(S_{C,V}^{\mathfrak{d}}\big).$$

Furthermore, we can also find explicit description of evaluations $\Psi^{-1}(L_j^{\perp})(G)$ as a sum of images of Ind by the help of the isomorphism $\Psi:\mathcal{P}\to\mathcal{R}^*$ given in the proof of 7.1. Indeed, for any $\mathbb{K}$-space homomorphisms $f:V\to W$ and $g:V\to W'$ between $\mathbb{K}$-spaces W, W' and V, it is easy to see that $(\operatorname{Ker} f)^{\perp}=\operatorname{Im} f^*$ and hence $(\operatorname{Ker} f\cap\operatorname{Ker} g)^{\perp}=\operatorname{Im} f^*+\operatorname{Im} g^*$ where f^* and g^* are the usual dual maps. And note that for any biset S, the usual dual map $(\mathcal{R}([S]))^*$ of the $\mathbb{K}$-space map $\mathcal{R}([S])$ is equal to $\mathcal{R}^*([S^{\mathrm{op}}])$. Now we can easily calculate that

$$\begin{aligned}
\Psi^{-1}\big(L_j^{\perp}\big)(G)&=\Psi_G^{-1}\big(L_j^{\perp}(G)\big)\\
&=\Psi_G^{-1}\bigg(\Big(\bigcap_X\operatorname{Ker}\big(\mathcal{R}\big(\operatorname{Res}_X^G\big)\big)\Big)^{\perp}\bigg)\\
&=\Psi_G^{-1}\Big(\sum_X\operatorname{Im}\big(\big(\mathcal{R}\big(\operatorname{Res}_X^G\big)\big)^*\big)\Big)\\
&=\Psi_G^{-1}\Big(\sum_X\operatorname{Im}\mathcal{R}^*\big(\operatorname{Ind}_X^G\big)\Big)\\
&=\sum_X\mathcal{P}\big(\operatorname{Ind}_X^G\big)\Psi_X^{-1}\big(\mathcal{R}^*(X)\big)\\
&=\sum_X\mathcal{P}\big(\operatorname{Ind}_X^G\big)\big(\mathcal{P}(X)\big).
\end{aligned}$$

Remark 7.4. Let $\mathbb{L}$ be a field, and $\mathfrak{F}'_{\mathfrak{b}}$ be the category of biset functors over $\mathbb{L}$ whose evaluations at any finite group are finite dimensional over $\mathbb{L}$. Then the duality $F\mapsto F^*$ sending a biset functor to its dual over $\mathbb{L}$ induces a category equivalence between the category $\mathfrak{F}'_{\mathfrak{b}}$ and the opposite category of $\mathfrak{F}'_{\mathfrak{b}}$.

Proof. This is clear from the definition of the dual of a biset functor and by part (2) of 7.2. □

We now explicitly state what we have obtained about the dual of a functor.

Theorem 7.5. *Let $\mathfrak{F}'_{\mathfrak{i}}$ (respectively, $\mathfrak{F}'_{\mathfrak{d}}$) be the category of inflation (respectively, deflation) functors over $\mathbb{K}$ whose evaluations at any finite group are finite dimensional over $\mathbb{K}$. Then, the duality $F\mapsto F^*$ sending an inflation functor to its dual over $\mathbb{K}$ induces an equivalence of categories between $\mathfrak{F}'_{\mathfrak{i}}$ and the opposite category of $\mathfrak{F}'_{\mathfrak{d}}$. This equivalence maps $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ to a deflation functor* *isomorphic to* $\mathbb{K}\mathcal{P}_{\mathbb{F}}$, *and a simple inflation functor of the form* $S^{\mathfrak{i}}_{H,V}$ *to a simple deflation functor of the form* $S^{\mathfrak{d}}_{H,V}$. *Moreover, it reverses filtrations in the sense that if*

$$0 = X_0 \leqslant X_1 \leqslant X_2 \leqslant \cdots \leqslant X_n = F$$

is a chain of inflation functors, then

$$0 = Y_n \leqslant Y_{n-1} \leqslant Y_{n-2} \leqslant \cdots \leqslant Y_0 = F^*$$

is a chain of deflation functors such that

$$Y_{i-1}/Y_i \cong (X_i/X_{i-1})^*$$

for all $i = 1, 2, \dots, n$, *where* $Y_i(G) = \{f \in F^*(G)\colon\ f(X_i(G)) = 0\}$ *for any finite group* G.

Proof. As the dual of an inflation (respectively, deflation) functor is a deflation (respectively, inflation) functor, it follows by 7.4 that the duality induces an equivalence of categories between the desired categories. By 7.1, this equivalence maps $\mathbb{K}\mathcal{R}_{\mathbb{F}}$ to a deflation functor isomorphic to $\mathbb{K}\mathcal{P}_{\mathbb{F}}$. Moreover, as the duality is a category equivalence, it maps a simple inflation functor of the form $S^{\mathfrak{i}}_{H,V}$ to a simple deflation functor, which must be of the form $S^{\mathfrak{d}}_{H,V}$ by the definition of the dual of a functor. The remaining part of the theorem follows easily by 7.2 and 7.3. □

In the rest of this paper, we give a different way of obtaining a filtration of the deflation functor $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ without using the duality. To be more precise, we demonstrate that one can modify easily our earlier results to find a filtration of $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ without using the results 7.1– 7.5.

Proposition 7.6. *Over any field* $\mathbb{L}$ *we have*:

(1) *Any simple biset functor* $S^{\mathfrak{b}}_{H,V}$ *has a unique minimal deflation subfunctor* M. *Moreover* $M \cong S^{\mathfrak{d}}_{H,V}$.
(2) *Any simple deflation functor* $S^{\mathfrak{d}}_{H,V}$ *has a unique maximal Mackey subfunctor* M. *Moreover* $S^{\mathfrak{d}}_{H,V}/M \cong S^{\mathfrak{m}}_{H,V}$.

Proof. (1) This is similar to the proof of 3.8. Because, putting $S = S^{\mathfrak{b}}_{H,V}$ we easily observe that $\mathrm{Ker}^{S,\mathfrak{d}}_{H,0} = 0$.

(2) This part is similar to the proof of part (1) of 3.12. Because we easily see that $S = S^{\mathfrak{d}}_{H,V}$ is generated by $S(H)$ as Mackey functor. □

Now part (2) of the previous result and semisimplicity result 3.9 imply that over characteristic 0 fields, any simple deflation functor $S^{\mathfrak{d}}_{H,V}$ is isomorphic to $S^{\mathfrak{m}}_{H,V}$ as Mackey functors. Consequently we have the following analogy of 4.5.

Proposition 7.7. *Assume that* $\mathbb{L}$ *is an algebraically closed field of characteristic* 0. *If* M *is a deflation functor whose evaluation at any finite group is finite dimensional over* $\mathbb{L}$, *then for any simple deflation functor* $S^{\mathfrak{d}}_{H,V}$ *the following numbers are equal*:

(a) $\dim_{\mathbb{L}} \operatorname{Hom}_{\mathbb{L}\operatorname{Out}(H)}(V, M(H)/I_H^{\mathfrak{m}} M(H))$.
(b) *The multiplicity of* $S_{H,V}^{\mathfrak{d}}$ *in* M *as deflation functors.*

For any finite group G, the Cartan map $c: K_0(G) \to G_0(G)$ becomes an isomorphism of abelian groups if we extend scalars to $\mathbb{K}$, see Benson [2, Corollary 5.3.6, p. 165]. Since c commutes with Ind, the following follows easily by 5.1.

Lemma 7.8. *Let* G *be a finite group and* M *be a Mackey subfunctor of* $\mathbb{K}\mathcal{P}_{\mathbb{F}}$. *If* $M(H) = \mathbb{K}\mathcal{P}_{\mathbb{F}}(H)$ *for all cyclic* p'*-subgroups* H *of* G *then* $M(G) = \mathbb{K}\mathcal{P}_{\mathbb{F}}(G)$.

We let χ be a family of groups satisfying the same conditions of Section 5.

Lemma 7.9. *If* $S_{H,V}^{\mathfrak{d}}$ *is a composition factor of* $\mathbb{K}\mathcal{P}_{\mathbb{F}}^{\chi}$ *as deflation functors then* H *is a cyclic* p'*-group in* χ.

Proof. Using 7.8, it is same as the proof of 5.2. □

It is clear from the proof of 5.4 that 5.4 is still valid for deflation functors and $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ so that it suffices to compute multiplicities in $\mathbb{K}\mathcal{P}_{\mathbb{F}}^{\chi}$ of simple deflation functors whose minimal subgroups are cyclic q-groups where q is a prime different from p. As $\mathbb{K}\mathcal{R}_{\mathbb{F}}(G) = \mathbb{K}\mathcal{P}_{\mathbb{F}}(G)$ for any finite p'-group G, the next result follows by 5.3 and by what we have observed in this section.

Theorem 7.10. *The composition factors of* $\mathbb{K}\mathcal{P}_{\mathbb{F}}^{\chi}$ *as deflation functors on* χ *are precisely the simple deflation functors* $S_{C,V}^{\mathfrak{d}}$, *where* C *ranges over cyclic* p'*-groups in* χ *and* V *ranges over elements in* $\operatorname{Irr}(\mathbb{K}\operatorname{Out}(C))$. *Moreover the multiplicity of each composition factor is* 1.

One may also construct some series of $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ using the ideas of Section 6. From now on, a functor means a deflation functor. We give analogues of some results obtained in Section 6. Since proofs are parallel to the corresponding proofs we gave in Section 6, we omit the justification of some results.

For any p'-number n, we define a subset K_n' of $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ whose evaluations at a finite group G is given as follows:

$$K_n'(G) = \sum_C \operatorname{Ind}_C^G \mathbb{K}\mathcal{P}_{\mathbb{F}}(C)$$

where C ranges over all cyclic subgroups of G of order dividing n. For any natural number m, let C_m be a cyclic group of order m.

Remark 7.11.

(1) If n is a p'-number then $K_n' = \operatorname{Im}_{C_n, \mathbb{K}\mathcal{P}_{\mathbb{F}}(C_n)}^{\mathbb{K}\mathcal{P}_{\mathbb{F}}, \mathfrak{d}}$. In particular, K_n' is the subfunctor of $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ generated by $\mathbb{K}\mathcal{P}_{\mathbb{F}}(C_n)$.
(2) If n and m are p'-numbers then $\dim_{\mathbb{K}} K_n'(C_m) = (n, m)$ where (n, m) is the greatest common divisor of n and m.

For a p'-number n, if d divides n then by the previous result $K_n'(C_d) = \mathbb{K}\mathcal{P}_{\mathbb{F}}(C_d)$. Then by counting dimensions we get the following result similar to 6.8.

Proposition 7.12. *Let n be a p'-number. Then the composition factors of K'_n are precisely the simple functors $S^{\mathfrak{d}}_{C,V}$ where C ranges over all nonisomorphic cyclic groups of order dividing n and V ranges over all nonisomorphic simple $\mathbb{K}\operatorname{Out}(C)$-modules. Moreover the multiplicity of each composition factor is* 1.

Now for any p'-number n we define the following subfunctor F'_n of K'_n.

$$F'_n = \sum_d K'_d$$

where d ranges over all natural numbers less than n and dividing n. We note that, for a subfunctor M of $\mathbb{K}\mathcal{P}_{\mathbb{F}}$, $K'_n \leqslant M$ if and only if $M(C_n) = \mathbb{K}\mathcal{P}_{\mathbb{F}}(C_n)$. Therefore, arguing as in the proof of 6.11, we can show that

$$F'_n(C_n) = \sum_{s=1}^{r} \Big(\mathbb{K}\mathcal{P}_{\mathbb{F}}(C_{p_1^{\alpha_1}}) \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} \operatorname{Ind}_{C_{p_s^{\alpha_s - 1}}}^{C_{p_s^{\alpha_s}}} \mathbb{K}\mathcal{P}_{\mathbb{F}}(C_{p_s^{\alpha_s - 1}}) \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} \mathbb{K}\mathcal{P}_{\mathbb{F}}(C_{p_r^{\alpha_r}}) \Big)$$

where $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ is the prime factorization of n. This shows that F'_n is not equal to K'_n, and so we have the following consequence of 7.12.

Corollary 7.13. *Let C be a cyclic group whose order is a p'-number n. Then the composition factors of K'_n/F'_n are precisely the simple functors $S^{\mathfrak{d}}_{C,V}$ where V ranges over all nonisomorphic simple $\mathbb{K}\operatorname{Out}(C)$-modules. Moreover the multiplicity of each composition factor is* 1.

We next show that K'_n/F'_n is semisimple by using 3.7.

Lemma 7.14. *Let n be a p'-number. Then*:

(1) *$I^{\mathfrak{d}}_{C_n}$ annihilates $K'_n(C_n)/F'_n(C_n)$.*
(2) *K'_n/F'_n is generated by $K'_n(C_n)/F'_n(C_n)$ as deflation functor.*
(3) $\operatorname{Ker}^{K'_n/F'_n, \mathfrak{d}}_{C_n, 0} = 0$.

Proof. (1) Let $[(C_n \times C_n)/L] \in I^{\mathfrak{d}}_{C_n}$. Then it is of the form

$$\operatorname{Ind}^{C_n}_{p_1} \operatorname{Iso}^{p_1}_{p_2/k_2} \operatorname{Def}^{p_2}_{p_2/k_2} \operatorname{Res}^{C_n}_{p_2}$$

where $p_i = p_i(L)$, $k_2 = k_2(L)$, and $|q(L)| < n$. Thus p_1 is a cyclic subgroup of C_n of order less than n and dividing n, implying that $I^{\mathfrak{d}}_{C_n} \mathbb{K}\mathcal{P}_{\mathbb{F}}(C_n) \subseteq F'_n(C_n)$.

(2) Using the properties of Im given in 3.1 we see that

$$\operatorname{Im}^{K'_n/F'_n, \mathfrak{d}}_{C_n, K'_n(C_n)/F'_n(C_n)} = \Big(\operatorname{Im}^{K'_n, \mathfrak{d}}_{C_n, K'_n(C_n)} + F'_n \Big) / F'_n = \big(K'_n + F'_n\big) \big/ F'_n = K'_n / F'_n$$

where we also use $K'_n = \operatorname{Im}^{\mathbb{K}\mathcal{P}_{\mathbb{F}}, \mathfrak{d}}_{C_n, \mathbb{K}\mathcal{P}_{\mathbb{F}}(C_n)}$ from 7.11.

(3) 7.13 implies the existence of a series

$$F'_n(C_n) = X_0 \subset X_1 \subset \cdots \subset X_d = K'_n(C_n)$$

of $\mathrm{End}_{\mathfrak{D}}(C_n)$-modules such that

$$\mathrm{Irr}\big(\mathbb{K}\,\mathrm{Out}(C_n)\big) = \{X_{i+1}/X_i\colon\ i = 0, \ldots, d-1\}$$

and $d = |\,\mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C_n))|$. For each i we define a subfunctor N_i' of K_n' containing F_n' by setting $N_i' = \mathrm{Ker}^{K_n'}_{C_n, X_i}$. By the properties of Ker given in 3.1 we have the following series of functors

$$F_n' \subseteq N_0 \subset N_1 \subset \cdots \subset N_d = K_n'.$$

If N_0 is not equal to F_n' then the number of composition factors of K_n'/F_n' counting with multiplicities must be greater than d which is not the case by 7.13. Consequently, $N_0 = F_n'$. This shows by 3.1 that

$$0 = N_0/F_n' = \mathrm{Ker}^{K_n'}_{C_n, F_n'(C_n)}/F_n' = \mathrm{Ker}^{K_n'/F_n'}_{C_n, 0}. \qquad \Box$$

Proposition 7.15. *Let C be a cyclic group whose order is a p'-number n. Then K_n'/F_n' is a semisimple functor such that*

$$K_n'/F_n' \cong \bigoplus_{V \in \mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C))} S^{\mathfrak{D}}_{C,V}.$$

Proof. Since $I_C^{\mathfrak{D}}$ annihilates $K_n'(C)/F_n'(C)$ by 7.14, it follows from the semisimplicity of $\mathbb{K}\,\mathrm{Out}(C)$ that $K_n'(C)/F_n'(C)$ is a semisimple $\mathrm{End}_{\mathfrak{D}}(C)$-module. Now it is clear from 7.14 that 3.7 implies the desired result. $\Box$

We can now construct a series of functors

$$0 \subset K_1' \subset L_1' \subset L_2' \subset \cdots \subset L_j' \subset \cdots \subset \mathbb{K}\mathcal{P}_{\mathbb{F}}$$

such that the quotients are semisimple and cover all composition factors whose minimal subgroups are π-groups where π is any set of prime numbers not containing p. Let $\pi = \{p_1, p_2, \ldots, p_r\}$ be a set of prime numbers not containing p. For any natural number j we define

$$L_0' = K_1', \qquad L_1' = \sum_{1 \leqslant i_1 \leqslant r} K'_{p_{i_1}}, \qquad L_2' = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant r} K'_{p_{i_1} p_{i_2}}, \quad \text{and}$$

$$L_j' = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_j \leqslant r} K'_{p_{i_1} p_{i_2} \cdots p_{i_j}}.$$

Theorem 7.16. *Let $\pi = \{p_1, p_2, \ldots, p_r\}$ be a set of prime numbers not containing p. Then the series of functors*

$$0 \subset K_1' \subset L_1' \subset L_2' \subset \cdots \subset L_j' \subset \cdots \subset \mathbb{K}\mathcal{P}_{\mathbb{F}}$$

satisfies:

(1) L_j'/L_{j-1}' *is a semisimple functor for all* $j = 1, 2, \ldots$.

(2) $$L'_j/L'_{j-1} \cong \bigoplus_{C} \bigoplus_{V \in \mathrm{Irr}(\mathbb{K}\,\mathrm{Out}(C))} S^{\mathfrak{d}}_{C,V}$$

where C ranges over all nonisomorphic cyclic groups of order $p_{i_1}p_{i_2}\dots p_{i_j}$ with $1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_j \leqslant r$.

Proof. It is clear that $K'_d \leqslant K'_s$ for any p'-numbers d and s such that d divides s. Thus by the definition of F'_n we have $F'_n = \sum_q K'_{n/q}$ where q ranges over all prime divisors of n. This shows that

$$L'_{j-1} = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_j \leqslant r} F'_{p_{i_1}p_{i_2}\dots p_{i_j}}.$$

Therefore, each semisimple quotient $K'_{p_{i_1}p_{i_2}\dots p_{i_j}}/F'_{p_{i_1}p_{i_2}\dots p_{i_j}}$ embeds into L'_j/L'_{j-1}. On the other hand, 7.12 implies that the composition factors of L'_j/L'_{j-1} have multiplicities all equal to 1 and are among the simple functors $S^{\mathfrak{d}}_{C,V}$ where C ranges over all nonisomorphic cyclic groups of order $p_{i_1}p_{i_2}\dots p_{i_j}$ with $1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_j \leqslant r$. Now the result follows by the above embeddings and by 7.13. □

We finally record the following filtration of $\mathbb{K}\mathcal{P}_{\mathbb{F}}$ which is immediate from the previous result. In the following L'_j is the subfunctor given on any finite group G by

$$L'_j(G) = \sum_X \mathrm{Ind}_X^G \mathbb{K}\mathcal{P}_{\mathbb{F}}(X)$$

where X runs over all cyclic p'-subgroups of G with $\ell(X) \leqslant j$.

Corollary 7.17. *There is a chain of functors*

$$0 = L'_{-1} \subset L'_0 \subset L'_1 \subset L'_2 \subset \dots \subset L'_j \subset \dots \subset \mathbb{K}\mathcal{P}_{\mathbb{F}}$$

such that $\sum_j L'_j = \mathbb{K}\mathcal{P}_{\mathbb{F}}$ and each L'_j/L'_{j-1} is a semisimple with

$$L'_j/L'_{j-1} \cong \bigoplus_{C,V} S^{\mathfrak{d}}_{C,V}$$

where C ranges over all nonisomorphic cyclic p'-groups with $\ell(C) = j$ and V ranges over all nonisomorphic simple $\mathbb{K}\,\mathrm{Out}(C)$-modules.

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