ESSAYS ON BARGAINING THEORY

A Ph.D. Dissertation

by
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Ankara
May 2018

To my father and Ertan

ESSAYS ON BARGAINING THEORY

The Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

ELİF ÖZCAN TOK

In Partial Fulfillment of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY IN ECONOMICS

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İHSAN DOĞRAMACI BİLKENT UNIVERSITY
ANKARA

May 2018

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

Assist. Prof. Dr. Emin Karagözoğlu
Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

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ABSTRACT

ESSAYS ON BARGAINING THEORY

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May 2018

Bargaining refers to a situation where two or more agents try to decide over how to divide a surplus generated by the economic transactions among these agents. There are two major approaches to bargaining problems: cooperative and non-cooperative approach. The former one focuses on the axioms that a bargaining outcome should satisfy and it is initiated by Nash (1950). The latter one attempts to specify the bargaining procedure and it is pioneered by Stahl (1972) and Rubinstein (1982). This dissertation consists of five essays. The first three essays employ the non-cooperative bargaining approach; the remaining ones employ the cooperative bargaining approach.

In the first essay, we study an infinite horizon bargaining game on a network, where the network is endogenously formed. Two specifications of the cost structure regarding the link formation is investigated: zero cost and non-zero cost. The equilibrium of the game is obtained for both specifications. Lastly, we focus

on efficiency issue and characterize the efficient networks. In the second essay, an infinite horizon bargaining game between buyers and sellers on a two-sided supply chain network is analyzed where the valuations of the buyers are heterogeneous. We prove that the valuations of the buyers and the network structure have an impact on the equilibrium outcome. In the third essay, we investigate the emergence of reference points in a two-player, infinite horizon, alternating offers bargaining game. The preferences of players preferences exhibit reference-dependence, and their current offers have the potential to influence future reference points of each other. However, this influence is limited in that it expires in a finite number of periods. We describe a subgame perfect equilibrium that involves an immediate agreement. We study the influence of expiration length and reference points on equilibrium strategies and outcomes. In the fourth essay, we study the salience of the reference points in determining the anchors and aspirations in a bargaining problem by introducing two parameters which capture these effects. In the cooperative bargaining literature, the disagreement point or the reference point is employed as an anchor while the ideal (or utopia) point or the tempered aspirations point as an aspiration. In this essay, a bargaining problem with a reference point is studied incorporating these two parameters and hence a family of bargaining solutions is obtained. Consequently, several characterizations for each individual member of this family is proposed. In the fifth essay, we introduce the *iterated egalitarian compromise solution* for two-person bargaining problems. It is defined by using two well-known solutions to bargaining problems, the *egalitarian solution* and the *equal-loss solution*, in an iterative fashion. While neither of these two solutions satisfy *midpoint domination* –an appealing normative property– we

show that the *iterated egalitarian compromise solution* does so. To sum up, this dissertation contributes to the diversified fields and practices of bargaining theory.

Keywords: Alternating Offers, Cooperative Bargaining Theory, Networks, Non-cooperative Bargaining Theory, Reference Dependent Preferences.

ÖZET

PAZARLIK TEORİSİ ÜZERİNE MAKALELER

Özcan Tok, Elif

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Tez Danışmanı: Dr. Öğr. Üyesi Emin Karagözoğlu

Mayıs 2018

Pazarlık, iki veya daha fazla aktörün, kendi aralarındaki ekonomik işlemler sonucu ortaya çıkan değerin nasıl paylaşılacağına ilişkin süreci ifade eder. Pazarlık problemlerinde iki ana yaklaşım mevcuttur: işbirlikçi ve işbirliksiz yaklaşım. Nash (1950) tarafından önerilen işbirlikçi yaklaşım, bir pazarlık sonucunun sağlaması gereken aksiyomlara odaklanmıştır. Stahl (1972) ve Rubinstein (1982) 'nin öncülük ettiği ikinci yaklaşım ise pazarlık sürecini tanımlamaya çalışmaktadır. Bu tez beş makaleden oluşmaktadır. İlk üç makale işbirliksiz pazarlık yaklaşımını, diğerleri ise işbirlikçi pazarlık yaklaşımını kullanmaktadır. İlk makalede içsel olarak oluşturulan ağ üzerinde sonsuz süreli bir pazarlık oyunu çalışılmıştır. Bağlantı kurmaya ilişkin maliyet yapısının iki çeşidi incelenmiştir: sıfır maliyet ve sıfırdan farklı maliyet. Oyunun dengesi her iki tanımlama için de elde edilmiştir. Ayrıca, etkinlik konusuna odaklanılmış ve etkin ağlar karakterize edilmiştir. İkinci makalede, iki taraflı bir tedarik zinciri üzerinde alıcıların değerlemelerinin heterojen olduğu durumlarda alıcı ve satıcılar arasındaki sonsuz süreli pazarlık oyunu analiz edilmiştir. Alıcıların değerlemelerinin ve ağ yapısının denge sonucu üzerinde etkili olduğu gösterilmiştir. Üçüncü makalede, iki oyunculu, sonsuz süreli, sıralı teklifli pazarlık oyununda referans noktalarının ortaya çıkışı incelenmiştir. Oyuncuların tercihleri referansa bağımlılık göstermektedir ve mevcut teklifleri birbirlerinin gelecekteki referans noktalarını etkileme potansiyeline sahip olmaktadır. Ancak, bu etki sonlu sayıda bir dönem içerisinde sona erdiği için sınırlıdır. Gecikmesiz anlaşmayı içeren bir alt-oyun mükemmel dengesi tanımlanmış; sona erme süresinin ve referans noktalarının denge stratejileri ve sonuçları üzerindeki etkisi incelenmiştir. Dördüncü makalede, bir pazarlık problemindeki çapa ve istekleri belirlemede referans noktalarının gücü; bu etkileri yakalayan iki parametrenin tanıtılmasıyla incelenmiştir. İşbirlikçi pazarlık yazınında, çapa olarak anlaşmazlık noktası ya da referans noktası; istek noktası olarak ise ideal nokta (ütopya noktası) kullanılmaktadır. Bu makalede, bu iki parametre dahil edilerek referans noktasına dayalı pazarlık problemi çalışılmış ve böylece pazarlık çözümlerinin bir ailesi elde edilmiştir. Sonuç olarak, bu ailenin her bir üyesi için çeşitli karakterizasyonlar önerilmiştir. Beşinci makalede, iki kişilik pazarlık problemleri için *yinelenen eşitlikçi uzlaşma çözümü* tanıtılmıştır. Bu çözüm, *eşitlikçi* ve *eşit kayıplı* pazarlık çözümlerini tekrarlı bir şekilde kullanarak tanımlanmıştır. Bahsi geçen iki çözüm cazip bir normatif özellik olan *orta nokta baskınlığını* sağlamazken, *yinelenen eşitlikçi uzlaşma çözümü* bu özelliği sağlamaktadır. Özetle, bu tez oyun teorisinin çeşitli alanlarına ve uygulamalarına katkı sağlamaktadır.

Anahtar Kelimeler: Ağ, İşbirlikli Pazarlık Teorisi, İşbirliksiz Pazarlık Teorisi, Referans Bağımlı Tercihler, Sıralı Teklifler.

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CHAPTER 1

INTRODUCTION

In broad terms, bargaining refers to the process involving two or more parties where (*i*) a mutually beneficial agreement is possible, (*ii*) there is a common interest in reaching an agreement but conflict of interests over the terms and conditions of agreement, and (*iii*) agreement requires mutual approval. Many economic, social and political interactions can be described as bargaining situations. Price determination in a market, wage negotiations in labor markets, business relations, international agreements, shopping are some examples of bargaining in a daily life. Hence, better understanding the bargaining process has become a major concern for researchers from several fields and policy makers.

Bargaining situations are commonly described as games and the analysis is based on game-theoretic approach. Traditionally, bargaining theory attempts to address the followings: the outcome of the bargaining game (agreement or disagreement, division of the surplus), the factors affecting the bargaining outcome, the sources of bargaining power, the strategies each player should play, the ways to improve a player's surplus from the bargaining and so on. To achieve these aims, in the literature, there are two main approaches to bargaining problems: cooperative and non-cooperative approach. The first one, cooperative bargaining, deals with identifying the appealing properties that a bargaining solution should satisfy. This strand of literature starts with the seminal works Nash (1950) and Nash (1953). He develops a 2-person bargaining problem and introduces certain axioms determining the solution uniquely. Second approach, non-cooperative bargaining, deals with explicit specification of the bargaining games. It considers the bargaining procedure that is ignored by the cooperative approach. Non-cooperative bargaining theory is pioneered by Nash (1953), Stahl (1972) and Rubinstein (1982). The path breaking paper of this literature, Rubinstein (1982), develops an infinite horizon bargaining game with sequential offers, called as *alternating offers bargaining game*, and shows the uniqueness of the subgame perfect equilibrium. Besides the differences between these two approaches, there essentially exists a close relationship. Binmore (1987) explores the convergence of Rubinstein's solution to Nash' solution as discount factor goes to 1.

This thesis consists of five essays centering on bargaining theory and contributes to both the cooperative and the non-cooperative approaches. In the first essay, we study an infinite horizon bargaining game over a network à la Manea (2011). In our game, the network is not exogenously given. In the first-stage, the network is formed where the link formation is probably costly. Given the network formed in this stage, our second stage game coincides with the one in Manea (2011). We study two alternative cost structures for the first-stage: forming links has (i) zero cost and (ii) non-zero cost. We characterize the subgame perfect Nash equilibrium of this game for each specification. In the equilibria of our game, the bargaining power that is due to an advantageous network position in Manea (2011) disappears since all players have equal opportunities to form links. We also define an appropriate efficiency notion and characterize the set of efficient networks.

The second essay of this thesis also builds upon Manea (2011) with a focus on supply chains. We analyze an infinite horizon bargaining game between buyers and sellers over stationary two-sided supply chain networks. We do not impose any further restrictions on the network structure. We allow both buyers and sellers to make offers. Furthermore, valuations of buyers are heterogeneous. We show that the equilibrium payoffs in the bargaining game we study depend on buyers' valuations and all players' network positions. As such, these two factors are sources of bargaining power.

In the third essay, we analyze an infinite horizon alternating offers bargaining game with reference-dependent preferences. Reference points are initially exogenous but they are adjusted through the bargaining process according to the received offers. Hence, past offers have the potential to affect the current reference points. However, it is assumed that the influence expires in finitely many periods. Further, each player perceives the offer above his reference point as a gain and the offer below his reference point as a loss, i.e., players are both gain-seeking and loss-averse. The equilibria of the game with limited influence and the game with unlimited influence are compared. This comparison reveals that the equilibrium offers are identical while the equilibrium strategies are different.

The fourth essay of this thesis investigates the salience (or the power) of the reference points in determining the anchors and aspirations which are assumed to be two major factors affecting the negotiated settlements in most cooperative bargaining models. The papers in the literature employ the disagreement point or the reference point as an anchor point and employ the ideal (or utopia) point or the tempered aspirations point as an aspiration point. Nevertheless, there is no clear explanation about the choice of a particular salient point over an alternative. In this study, two parameters are introduced into bargaining problems with a reference point. The first parameter represents the influence (or the salience) of the reference point in determining the anchor, while the second parameter represents its influence in shaping agents' aspirations. Utilizing these parameters, a unifying framework for the study of bargaining problems with a reference point have been provided. The two-parameter family of bargaining solutions we obtain encompasses Kalai-Smorodinsky (Kalai and Smorodinsky, 1975), Gupta-Livne (Gupta and Livne, 1988), tempered aspirations (Balakrishnan, Gómez, and Vohra, 2011), and local Kalai-Smorodinsky (Gupta and Livne, 1989) solutions as special cases. We offer multiple characterizations for the individual members of this family.

In the fifth essay, we develop a new solution concept for two-person bargaining problems: *iterated egalitarian compromise solution*. This new solution concept is defined by using two well-known solutions concepts, *egalitarian solution* proposed by Kalai (1977) and *equal loss solution* proposed by Chun (1988), in an iterative fashion. The *egalitarian* and the *equal loss solutions* fail to satisfy midpoint domination which requires that the payoff of each player should be at least the average of his disagreement and his ideal point outcomes. We first show that *iterated egalitarian compromise solution* is well-defined. Afterwards, we prove that *iterated egalitarian compromise solution* satisfies midpoint domination that is violated by the *egalitarian* and *equal loss solutions*.

CHAPTER 2

BARGAINING ON ENDOGENOUSLY FORMED NETWORKS

Bilateral relationships taking place in networks is ubiquitous. Buyer-seller relationships, friendships in school or social media, interactions in job markets, scientific collaborations, information exchange, supply-chains, international trade agreements are just some examples. Theoretical and empirical research in economics on networks in the last three decades consistently argue/show that the network structure in general and the location of an agent in the network in particular can significantly influence the nature of the strategic interaction and corresponding (equilibrium) outcomes (see Calvó-Armengol (2003), Corominas-Bosch (2004), Polanski (2007), Jackson (2008), Manea (2011), Abreu and Manea (2012) and Polanski and Vega-Redondo (2013)). For instance, forming and maintaining a large number of social ties likely increase a person's chances of finding a job. Similarly, an intermediary likely benefits from being well-connected both on the seller-end and the buyer-end of the market. Foreseeing the importance of a key network location, agents strategically form (or avoid) links. In this study, we analyze a bilateral bargaining game à la Manea (2011) over an endogenously formed network. The bargaining game is an extension of the model developed in Rubinstein and Wolinsky (1985) which adopts a variation of Rubinstein (1982) with two population and random matching process.

Manea (2011) considers a model in which players are connected via an exogenously given network. Each pair of players in a link of the network is able to produce one-unit pie. On this network, an infinite horizon bargaining game is played. In particular, at each period, a link is chosen with some probability and one of the two players (in the chosen link) is randomly selected as the proposer. The proposer makes a take-it-or-leave-it offer to concerning the division of the unit pie. His opponent responds the offer by accepting or rejecting. If the responder accepts the offer, then the players in the pair leave the game with agreed shares; and in the next period they are replaced by their exact clones.[1] If the responder rejects the offer, then the players in the pair do not earn any payoffs in this period but they remain in the game. At each period, the same random selection and bargaining procedures are repeated. All player have the same discount factor. Manea (2011) shows that advantageous network positions are translated into bargaining power. More precisely, a player's bargaining power does not depend only on the number of links he has but also his neighbours' positions in the network. Assume that player i has the largest number of links in the network, however all of his neighbours have a monopoly power on their neighbours other than i. Hence, in such a network player i could not get a larger payoff than his neighbours have. This model

[1] This is an important property of the model. The replacement of the agreed pair makes the model stationary, which makes the analysis much more tractable. This modelling assumption is followed by Gale (1987), Manea (2011), Polanski and Lazarova (2015) and Nguyen (2012).

provides very valuable insights about the influence of network position on one's bargaining outcomes. Given that an advantageous network position is crucial for getting more of the pie in the bargaining, a natural question is: What could we expect in an extended game where agents first strategically decide on which links to form and then the bargaining game is played on the network that emerges? In this essay, we tackle with this question.

We construct a two-stage game. In the first stage, the network is formed, whereas in the second stage, a bargaining game is played on the previously formed network. For the first stage, we employ the noncooperative network formation game of Bala and Goyal (2000).[2] More precisely, each player i announces his strategy vector, which contains the list of players with whom he wants to form a link. Link formation is bilateral (and in one model specification, costly). Therefore, for any two players, i and j, for a link to be formed between them, both i and j must list each other. The equilibrium concept we adopt for the network formation game is pairwise Nash equilibrium. Once the network is formed, an infinite horizon bargaining game (very similar to the one in Manea (2011)) is played. A significant difference between the bargaining game in our model from that of Manea (2011) is in the payoffs, which is mainly due to the presence of link formation costs. The cost of each particular link is shared by all players (in all periods of the bargaining game) who occupy that link. Thus, at each period, each player incurs a fraction of the total link cost for each link he has, as long as he remains in the game. So, linking costs are not sunk. Some examples of this setting are the

[2] Kranton and Minehart (2001), Corominas-Bosch (2004), Polanski and Winter (2010) and Condorelli and Galeotti (2012)are other important papers in this literature. For the comprehensive survey of the network formation literature, see Myerson (1991) and Jackson (2005).

business relationships which require certain communication technologies and/or infrastructure to carry on a business and to continue collaboration. Similarly, being a member of a chamber commerce or international organizations such as OECD, WTO or NATO in order to establish relations with other member firms or countries. The equilibrium concept we adopt for the bargaining game is subgame perfect Nash equilibrium. We analyze two different specifications of the cost structure: zero-cost and non-zero cost. For each cost structure, we first find the limit equilibrium payoffs (when the discount factor goes to 1) for all possible networks. This makes it possible to obtain a mapping from the set of possible networks to payoffs. Then, using to these mappings, we obtain the equilibrium outcome of the network formation game.

In case of zero cost, the limit equilibrium payoffs of the bargaining game is the same as those in Manea (2011). He constructs a network decomposition algorithm in order to describe the payoffs in the limit equilibrium. The algorithm picks an oligopoly subnetwork at each step where such subnetwork involves a set of players in which no pair of players have a link and the set of their neighbours. In the equilibrium, the pie is divided among the players in a pair proportional to the shortage ratio within an oligopoly subnetwork. The shortage ratio refers to the relative bargaining power of the players in the link-independent set. Note that the sole source of bargaining power is the position in the network. In the setting with zero cost, the equilibrium outcome of the network formation game is all equitable networks -the networks where the expected equilibrium payoff of each agent is equal to the half of the pie. In case of non-zero cost, the continuation payoffs of players are affected by linking costs since at each period players incur a fraction of these costs. This construction yields two factors that influence one's bargaining power: the position in the network and the linking costs. We modify Manea's network decomposition algorithm of in order to capture the effects of costs. This algorithm picks a unique oligopoly subnetwork at each step. Within an oligopoly subnetwork, payoffs are determined according to not only the shortage ratio but also the advantage/disadvantage provided by the linking costs. The equilibrium set of the network formation game with non-zero link formation costs is all equitable networks. That is, in equilibrium, one unit pie is divided equally in expectation in all pairs of players. In both zero and non-zero costs specifications, we have the same characterization result for the equilibrium network.

An important consequence of letting the network to be endogenously formed is that the differences in limit equilibrium payoffs between players (in two sides of the oligopoly subnetworks) disappear. Intuitively, if players have equal opportunities to choose their bargaining partners, strategic link formation incentives of the players eliminate the differences in the limit equilibrium payoffs among players.

Finally, we study efficiency and check whether the equilibrium networks in our game are efficient or not. In our model, efficiency boils down to maximizing the aggregate utility taking into account link formation costs across all players in the society. We obtain the following characterization result concerning efficient networks: a network is efficient if and only if it is a disjoint union of cycles with odd number of vertices and subgraphs with even number of vertices. Consequently, the endogenously formed networks in our equilibria can be covered by such a union; hence they are efficient.

The rest of the essay is organized as follows. The next section defines the benchmark model and reports the results of endogenous link formation with zero costs and non-zero costs. Section 2 focuses on the efficiency of equilibrium networks. Section 3 concludes.

2.1 Model and Results

The set of players is $N = \{1, 2, \dots, n\}$. For each pair of players $(i, j) \in N \times N$, we use shorthand ij. A network G is the subset of links $\{ij | i \neq j, i, j \in N\}$. If $ij \in G$, i and j are connected. Denote the set of all possible undirected networks as Ω.

2.1.1 Manea (2011)

Since our model shares a lot with Manea (2011), we first introduce the model developed by him. Building upon this benchmark model, we incorporate endogenous network structure with zero and non-zero linking costs. Manea (2011) constructs the following infinite horizon bargaining game over an exogenously given network $G \in \Omega$. Let $(p_{ij})_{ij \in G}$ be the probability distribution over the links in G, which defines the matching probabilities of players. A link $ij \in G$ means that i and j are able to produce one unit pie and they bargain over how to divide the pie. At each period $t = 0, 1, \dots$, a link $ij \in G$ is chosen with probability p_{ij} and one of two players in the chosen link is randomly selected as the proposer.

Say player i is selected. Player i makes an offer to j concerning the division of the unit pie, and player j responds to the offer by accepting or rejecting. If player j accepts the offer, i and j leave the game with their respective agreed shares. In period $t+1$, two new players take i and j's positions. Here, assume that for each player i, there are infinitely many players of type i, i.e., $i = \{i_1, i_2, \ldots, i_\tau, \ldots\}$, where a player's type represents his position in the network. If j rejects the offer, i and j remain in the game. In period $t+1$, the same bargaining procedure is repeated. Link selection probabilities is independent across periods. Players discount the future payoffs and all players have the same discount rate, $\delta \in (0,1)$. The bargaining game with discount rate δ is denoted by Γ^δ. Finally, players have perfect information.

Subgame perfect Nash equilibrium is employed as a solution concept. The equilibrium payoff vector of the game Γ^δ is denoted by $(v_i^{*\delta})_{i\in N}$. The equilibrium agreement network is the subnetwork of G which only involves the links such that agreeing provides the players at the nodes more payoff than proceeding to the next period does. Formally, the equilibrium agreement network of Γ^δ, $G^{*\delta}$, is defined as the subnetwork of G that only consists of the links ij satisfying $\delta(v_i^{*\delta} + v_j^{*\delta}) \leq 1$. The limit equilibrium network, denoted by G^*, is the network that $G^{*\delta}$ converges to as δ goes to 1 and the limit equilibrium payoff vector, v^*, is the payoff vector that $v^{*\delta}$ converges to as δ goes to 1.

Manea (2011) also constructs a network decomposition algorithm by which the equilibrium payoffs are easily calculated. Some additional notation is needed in order to introduce the algorithm. For every network G and a subset of players

M, $L^G(M)$ denotes the set of players who have a link in G (hereafter G-link) with the players in M i.e., $L^G(M) = \{j|ij \in G, i \in M\}$. A set of players is G-independent if there does not exist any G-link between any of two players in the set. A set of players is mutually estranged if it is G^*-independent. The set of nonempty G-independent sets is denoted as $\mathcal{I}(G)$.

Network Decomposition Algorithm, $\mathcal{A}(G)$: For a given network $G \in \Omega$, the algorithm generates the sequence $(r_s, M_s, L_s, N_s, G_s)_{s\in\mathbb{N}}$ as follows where s denotes the step of the algorithm:

Let $N_1 = N$ and $G_1 = G$.

For $s \geq 1$:

If $N_s = \emptyset$, then STOP.

Otherwise, let

$$r_s = \min_{M\subset N_s, M\in\mathcal{I}(G)} \frac{|L^{G_s}(M)|}{|M|}.$$

If $r_s \geq 1$, then STOP.

Else, set M_s as the union of all minimizers M. Let $L_s = L^{G_s}(M_s)$.

Denote $N_{s+1} = N_s \setminus (M_s \cup N_s)$ and G_{s+1} be the induced subnetwork of G by the players in N_{s+1}.

Denote by $\bar{s}$ the step at which the algorithm STOPs.

The algorithm decomposes a given network. At each step, it identifies the mutually G-independent sets that achieve the lowest shortage ratio. (The ratio $|L^G(M)|/|M|$ is called as shortage ratio.) As long as the shortage ratio is less than 1, the algorithm picks the union of these minimizers and the partner set of this union. Call the subnetwork which is induced by the players in the union of these minimizers and its partner set as an oligopoly subnetwork. Then, the picked players and their links are removed from the network. In the next step, the algorithm is repeated with the network induced by the remaining players. The decomposition algorithm stops when all players are removed from the network or there does not exist any oligopoly subnetwork.

The outcome of generated by the algorithm $\mathcal{A}(G)$ determines the payoffs in the limit equilibrium which are given by the following theorem. One of the main results is that any discount factor δ induces the same payoffs in the equilibrium.

Theorem 1 (Manea (2011)). ***(Limit Eq. Payoffs)*** *Let $(r_s, M_s, L_s, N_s, G_s)_{s=1}^{\bar{s}}$ be the sequence defined by the algorithm $\mathcal{A}(G)$ where $\bar{s}$ is the step at which the algorithm terminates and let G^* be the equilibrium network. The limit equilibrium payoffs for Γ^δ as $\delta \to 1$ are given by*

$$\forall s < \bar{s},\ \forall i \in M_s,\ v_i^* = \frac{|L_s|}{|L_s| + |M_s|},$$
$$\forall s < \bar{s},\ \forall j \in L_s,\ v_j^* = 1 - \frac{|L_s|}{|L_s| + |M_s|},$$
$$\forall k \in N_{\bar{s}},\ v_k^* = \frac{1}{2}.$$

Within the oligopoly subnetworks, the unit pie is shared in line with the shortage ratio r_s between two sides of the subnetwork in the limit equilibrium. Players at the same side have identical payoffs. However, the payoffs are differentiated across two sides of an oligopoly subnetwork. Players in the short side obtain more share from the pie than those at the long side obtain. It is noteworthy to emphasize that the limit equilibrium payoffs show that bargaining power is determined not only by the number of links the player has but also by the position of the player in the network.

2.1.2 Endogenous Link Formation

In Manea (2011), the network structure is influential in determining the equilibrium payoffs of players. Inspiring by this result, the following question naturally arises: what would change in the equilibrium payoffs in an extended game where the network structure is endogenous? The idea of endogenous link formation is motivated by the simple observation that in real life players decide their connections individually to maximize their benefits. Accordingly, we develop a two stage model of bargaining over an endogenously formed network. The first stage is devoted to the network formation game. Following Bala and Goyal (2000), we use a simultaneous move game for link formation. We analyze two specifications of the model: zero linking costs and non-zero linking costs. In the second stage, players play an infinite horizon bargaining game concerning the division of a unit pie on the network formed in the first stage.

Network Formation Game: Each player type $i \in N$ announces his strategy $g_i = \{g_{i1}, g_{i2}, \ldots, g_{ii-1}, g_{ii+1}, \ldots, g_{in}\} \in \{0, 1\}^{n-1}$. The interpretation of $g_{ij} = 1$ is that player i wishes to form a link with j and the interpretation of $g_{ij} = 0$ is that player i does not wish to form a link with j. We only consider pure strategies. We assume that a player cannot form a link with himself. Let $\mathcal{G}_i$ be the set of strategies of player i. Link formation is bilateral. In other words, forming a link requires a mutual consent of two players. In one specification of the model, link formation is also costly, in the sense that it needs some time and effort. Players in both nodes of a link incur the linking cost. Denote the total cost that player i incurs for each link he has by TC_i. Linking costs are independent across players. Define a correspondence $\phi : (\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n) \longrightarrow \Omega$ which maps the strategies of players to a network such that $\phi(g) = G$.

We have n positions in the network and each position is reserved to each type of $i \in N$. Hence, we define a sequence $i_0, i_1, \ldots, i_\tau, \ldots$ of players of type i, for each $i \in N$ in order to have stationarity of the game. The network is formed by the first generation before the bargaining stage. Players play the following infinite horizon bargaining game on the network previously formed.

Bargaining Game: Let G be the network formed in the first stage (the outcome of the network formation game). If $ij \in G$, then i and j are able to produce a unit pie and they can bargain over how to divide the pie. The infinite horizon bargaining game is adopted from Manea (2011). Differently, among the links in the network G, a link ij is selected with equal probability, i.e., $p_{ij} = 1/$total number of links in G. Further, in the model with non-zero costs, the total cost of link formation is shared by all players who occupy this link over the periods of bargaining game. Formally, at each period, each player $i \in N$ incurs a cost c_i for each link he has. So, at each period he incurs a total cost $l_i^G c_i$ where l_i^G denotes the number of links that player i has in the network G. Hence, $TC_i = \sum_{t=0}^{\infty} \delta^t c_i$ which is mathematically equivalent to that for all $i \in N$,

$$c_i = (1 - \delta)TC_i.$$

Note that for any player $i \in N$, $TC_i \geq 0$ and player i's total linking cost is less than or equal to the size of the pie, i.e., $l_i^G TC_i \leq 1$. The strategy of player i in the bargaining game is denoted by σ_{i_τ} which consists of offers of player i and responds to the offers he received. For each $i \in N$, if the share of i induced by an offer at some period t of the bargaining game is equal to v_i and if the offer is accepted, then the payoff of i at that period in the network G is defined as

$$u_i(G) = v_i - l_i^G c_i.$$

We will denote the equilibrium share vector of the game Γ^δ as $(v_i^{*\delta})_{i \in N}$. Then, the equilibrium payoff vector of the game Γ^δ is

$$u_i^{*\delta}(G) = v_i^{*\delta} - l_i^G c_i$$

Also, define the equilibrium agreement network of the game Γ^δ, $G^{*\delta}$, as subnetwork of G which only involves the links ij satisfying $\delta(v_i^{*\delta} + v_j^{*\delta}) \leq 1$. G^* is the limit equilibrium agreement network, u^* is the limit equilibrium payoff vector as δ goes to 1.

The network formation game that we employ is simple and easily tractable. For this game, we employ pairwise Nash equilibrium concept, a refinement of Nash equilibrium.[3]

Definition 1. A strategy $g^{PNE} \in \mathcal{G}$ is a pairwise Nash equilibrium of the network formation game if for every player $i \in N$, $u_i^*(\phi(g^{PNE})) \geq u_i^*(\phi(g_i, g_{-i}^{PNE}))$ for every $g_i \in \mathcal{G}_i$ and there does not exist any pair of players (i, j) such that

$$u_i^*(\phi(g^{PNE}) + ij) \geq u_i^*(\phi(g^{PNE})) \; and$$
$$u_j^*(\phi(g^{PNE}) + ij) > u_j^*(\phi(g^{PNE})).$$

A network G^{PNE} is a pairwise Nash network if there exists a pairwise Nash equilibrium g^{PNE} such that $\phi(g^{PNE}) = G^{PNE}$.

Nash equilibrium concept is a weaker notion than pairwise Nash equilibrium, since it only accounts for individual deviations. For instance, the empty network is always a Nash network. However, link formation requires mutual consent in our model. Hence, we also want to consider the bilateral moves by using pairwise Nash equilibrium concept that is immune both to single link deletions and bilateral link creations.

[3]See Calvó-Armengol (2004), Bloch and Jackson (2006) and Calvó-Armengol and İlkılıç (2009) for other studies that use this equilibrium concept.

Endogenous Link Formation with Zero Cost

We first analyze the specification of the model where link formation is costless. Assume that for each $i \in N$, $c_i = 0$. So, the bargaining game reduces to Manea (2011)'s bargaining game with equal matching probabilities. Using his limit equilibrium payoff results, we find the pairwise Nash equilibrium of the network formation game.

Next theorem provides a characterization of networks that are endogenously formed. When we allow players to form their links, they form a network in which an oligopoly subnetwork does not exist. The bargaining game on such a network ends up with an equal division of the pie (1/2) among the players of each pair in the limit equilibrium. Formally, the outcome of the network formation game is an equitable network, which is defined as the network where each player obtains identical payoff in the limit equilibrium.

Theorem 1. *A strategy g^{PNE} is a pairwise Nash equilibrium of the network formation game if and only if for all $i \in N$, $u_i^*(\phi(g^{PNE})) = 1/2$. (The induced network G^{PNE} by the strategy profile g^{PNE} is equitable.)*

Proof of Theorem 1. Suppose that g^{PNE} is a pairwise Nash equilibrium of the network formation game. Assume that there exists a player $i \in N$ such that $u_i^*(\phi(g^{PNE})) \neq 1/2$. Then, $u_i^*(\phi(g^{PNE})) < 1/2$ or $u_i^*(\phi(g^{PNE})) > 1/2$.

Case 1. $u_i^*(\phi(g^{PNE})) < 1/2$.

In this case, for some step $s < \bar{s}$ of the algorithm $\mathcal{A}(G)$, i belongs to M_s and $|L_s|/|M_s| < 1$. Hence, there exists a player $j \in M_s$ such that $L^{G_s}(\{i\}) \cap$

$L^{G_s}(\{j\}) \neq \emptyset$. If we add the link ij to the network, we will have

$$u_i^*(\phi(g^{PNE}) + ij) = \frac{1}{2} > u_i^*(\phi(g^{PNE})) \text{ and}$$
$$u_j^*(\phi(g^{PNE}) + ij) = \frac{1}{2} > u_j^*(\phi(g^{PNE})).$$

So, g^{PNE} is not a pairwise Nash equilibrium of the network formation game, which contradicts with our supposition.

Case 2. $u_i^*(\phi(g^{PNE})) > 1/2$.

In this case, for some step $s < \bar{s}$ of the algorithm $\mathcal{A}(G)$, i belongs to L_s. Then, there exists a player $j \in M_s$ such that $u_j^*(\phi(g^{PNE})) < 1/2$. So, following similar arguments to Case 1 for player j leads to a contradiction with our supposition. Hence, for all $i \in N$, $u_i^*(\phi(g^{PNE})) = 1/2$.

Now, for the other part of the theorem, suppose that for all $i \in N$, $u_i^*(\phi(g)) = 1/2$. Hence, for all mutually estranged sets M, $|L^G(M)|/|M| \geq 1$. Assume that g is not a pairwise Nash equilibrium.

Adding a link to the network does not change the shortage ratio r_s for all s which is minimized in the decomposition algorithm. Hence, there does not exist any pair of players (i, j) such that

$$u_i^*(\phi(g) + ij) \geq u_i^*(\phi(g)) \text{ and}$$
$$u_j^*(\phi(g) + ij) > u_j^*(\phi(g)).$$

Therefore, g violates the first condition of the pairwise Nash equilibrium definition. Hence, there exists a player i and a strategy of him $g'_i \in \mathcal{G}_i$ such that $u^*_i(\phi(g'_i, g_{-i})) > u^*_i(\phi(g))$. Let $\phi(g'_i, g_{-i}) = G'$ and $\phi(g) = G$. So, in G', there exists a mutually estranged set M with $L^{G'}(M) = L$ such that $|L|/|M| < 1$ and $i \in L$. Since any change in the strategy of player i does not affect the links of the players in M with other players, we get $|L| = |L^{G'}(M)| \geq |L^G(M)|$, which contradicts with the fact that for all mutually estranged sets M, $|L^G(M)|/|M| \geq 1$.

It follows that g is a pairwise Nash equilibrium of the network formation game.

□

Endogenous Link Formation with Non-Zero Cost

In this section, we assume that there exists at least one player whose linking cost is different than zero. We start by analyzing the second stage of the game: bargaining stage. Hence, let G be the network formed in the first stage: network formation game. Firstly, we show that in every subgame, the expected payoff of each existing player in the network at that period is uniquely determined.

Theorem 2. *For all $\delta \in (0,1)$, there exists a share vector $(v_i^{*\delta})_{i\in N}$ such that in every subgame perfect equilibrium of Γ^δ, the expected share of existing player i_τ of type i is uniquely given by $v_i^{*\delta}$ for all $i \in N$, $\tau \geq 0$. For every $\delta \in (0,1)$, in any equilibrium of Γ^δ, in any subgame where the link $i_\tau j_{\tau'}$ is selected and i_τ is the proposer, for each $i \in N$ the followings statements hold with probability one:*

(1) if $\delta((v_i^{*\delta} - l_i^G c_i) + (v_j^{*\delta} - l_j^G c_j)) < 1$, then i_τ offers $\delta(v_j^{*\delta} - l_j^G c_j))$ and $j_{\tau'}$

accepts.

(2) if $\delta((v_i^{*\delta} - l_i^G c_i) + (v_j^{*\delta} - l_j^G c_j)) > 1$*, then* i_τ *makes an offer that is rejected by* $j_{\tau'}$*.*

Before moving on to the proof of Theorem 2, we need the following lemma.

Lemma 1. *For all* $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{R}$*,*

$$|\max\{\omega_1, \omega_2\} - \max\{\omega_3, \omega_4\}| \leq \max\{|\omega_1 - \omega_3|, |\omega_2 - \omega_4|\}.$$

Proof of Theorem 2. For each $i \in N$, let $\underline{v}_i^\delta$ and $\bar{v}_i^\delta$ be the infimum and supremum of the expected shares of i_τ in any subgame for all $\tau \geq 0$ in every subgame perfect equilibrium of Γ^δ. For each player $i \in N$, l_i denotes the number of links that i has and l denotes the number of total links in G.

Consider a subgame perfect equilibrium. Suppose that the link ij is chosen and i is selected as the proposer. No player of type j will accept an offer smaller than $\delta(\underline{v}_j^\delta - l_j c_j)$, so i can get a share of at most $1 - (\underline{v}_j^\delta - l_j c_j)$. Moreover, any player of type i accepts any offer larger than $\delta(\bar{v}_i^\delta - l_i c_i)$, since when he rejects the offer, he gets at most $\delta(\bar{v}_i^\delta - l_i c_i)$. So, no player offers him more than $\delta(\bar{v}_i^\delta - l_i c_i)$ in the equilibrium.

Now, suppose that i is not a member of chosen link, i's continuation share from the pie is at most $\delta(\bar{v}_i^\delta - l_i c_i)$. So, for each $\tau \geq 0$, the following is hold:

$$v_{i\tau}^{\delta} \leq (1 - \frac{l_i}{2l})\delta(\bar{v}_i^{\delta} - l_i c_i) + \frac{1}{2l} \sum_{\{j|ij \in G\}} \max\{1 - \delta(\underline{v}_j^{\delta} - l_j c_j), \delta(\bar{v}_i^{\delta} - l_i c_i)\}. \quad (2.1)$$

Since the inequality (2.1) holds for all players of type i, it also holds for $\bar{v}_i^{\delta}$.

Therefore,

$$\bar{v}_i^{\delta} \leq (1 - \frac{l_i}{2l})\delta(\bar{v}_i^{\delta} - l_i c_i) + \frac{1}{2l} \sum_{\{j|ij \in G\}} \max\{1 - \delta(\underline{v}_j^{\delta} - l_j c_j), \delta(\bar{v}_i^{\delta} - l_i c_i)\}. \quad (2.2)$$

Consider that i deviates from his equilibrium strategy by offering $\delta(\bar{v}_j^{\delta} - l_j c_j) + \epsilon$ ($\epsilon > 0$) to any player of type j and offering zero to other players. Player j will accept the offer in any subgame perfect equilibrium. Also, player i rejects all offers that he will receive. So, for each $\tau \geq 0$ and for all deviations ($\epsilon > 0$), above cases are captured by the following inequality:

$$v_{i\tau}^{\delta} \geq (1 - \frac{l_i}{2l})\delta(\underline{v}_i^{\delta} - l_i c_i) + \frac{1}{2l} \sum_{\{j|ij \in G\}} \max\{1 - \delta(\bar{v}_j^{\delta} - l_j c_j) - \epsilon, \delta(\underline{v}_i^{\delta} - l_i c_i)\}$$

When the deviation from equilibrium strategy converges to zero ($\epsilon \to 0$),

$$v_{i\tau}^\delta \geq (1-\frac{l_i}{2l})\delta(\underline{v}_i^\delta - l_i c_i) + \frac{1}{2l}\sum_{\{j|ij\in G\}} \max\{1-\delta(\bar{v}_j^\delta - l_j c_j), \delta(\underline{v}_i^\delta - l_i c_i)\}. \quad (2.3)$$

Since the inequality (2.3) holds for all players of type i, it also holds for $\underline{v}_i^\delta$. Therefore,

$$\underline{v}_i^\delta \geq (1-\frac{l_i}{2l})\delta(\underline{v}_i^\delta - l_i c_i) + \frac{1}{2l}\sum_{\{j|ij\in G\}} \max\{1-\delta(\bar{v}_j^\delta - l_j c_j), \delta(\underline{v}_i^\delta - l_i c_i)\}.$$

In order to show the equality of infimum and supremum of the expected shares for each player, we look at the difference between them. Let $D = \max\limits_{k\in N} \bar{v}_k^\delta - \underline{v}_k^\delta$. Take any $i \in \arg\max\limits_{k\in N} \bar{v}_k^\delta - \underline{v}_k^\delta$.

$$\begin{aligned}
D &= \bar{v}_i^\delta - \underline{v}_i^\delta \\
&\leq (1-\frac{l_i}{2l})\delta(\bar{v}_i^\delta - \underline{v}_i^\delta) + \frac{1}{2l}\sum_{\{j|ij\in G\}} [\max\{1-\delta(\underline{v}_j^\delta - l_j c_j), \delta(\bar{v}_i^\delta - l_i c_i)\} \\
&\qquad\qquad\qquad\qquad - \max\{1-\delta(\bar{v}_j^\delta - l_j c_j), \delta(\underline{v}_i^\delta - l_i c_i)\}] \\
&\leq (1-\frac{l_i}{2l})\delta D + \frac{1}{2l}\sum_{\{j|ij\in G\}} \max\{|\delta\bar{v}_j^\delta - \delta\underline{v}_j^\delta|, |\delta\bar{v}_i^\delta - \delta\underline{v}_i^\delta|\} \\
&= (1-\frac{l_i}{2l})\delta D + \frac{1}{2l}\sum_{\{j|ij\in G\}} \delta\max\{\bar{v}_j^\delta - \underline{v}_j^\delta, \bar{v}_i^\delta - \underline{v}_i^\delta\} \\
&= (1-\frac{l_i}{2l})\delta D + \frac{1}{2l}\delta D l_i \\
&= \delta D
\end{aligned}$$

Hence, $D \leq \delta D$. Since $D \geq 0$ and $\delta \in (0,1)$, we have $D = 0$.

For each player, the difference between the infimum and supremum of the expected shares is zero. Hence, for all $k \in N$, $\bar{v}_k^\delta = \underline{v}_k^\delta$. Then, for all $i \in N$, we can write the following equality

$$\bar{v}_i^\delta = (1 - \frac{l_i}{2l})\delta(\bar{v}_i^\delta - l_i c_i) + \frac{1}{2l} \sum_{\{j | ij \in G\}} \max\{1 - \delta(\bar{v}_j^\delta - l_j c_j), \delta(\bar{v}_i^\delta - l_i c_i)\}, \quad (2.4)$$

which means that $\bar{v}_i^\delta = \underline{v}_i^\delta = v_i^{*\delta}$.

To prove the uniqueness of the solution to equation (2.4), we need to define a function $f^\delta : [0,1]^n \longleftrightarrow [0,1]^n$ such that for all $i \in N$,

$$f_i^\delta(v) = (1 - \frac{l_i}{2l})\delta(v_i - l_i c_i) + \frac{1}{2l} \sum_{\{j | ij \in G\}} \max\{1 - \delta(v_j - l_j c_j), \delta v_i\}$$

We argue that the function f^δ has a fixed point by using the contraction mapping theorem. It is enough to prove the following lemma to obtain the uniqueness.

Lemma 2. *f^δ is a contraction mapping with respect to sup norm on $\mathbb{R}^n$.*

Proof of Lemma 2. The proof is relegated to the Appendix. □

This concludes the proof of Theorem 2.

□

It is important to note that the expected share of any player $i \in N$ is given by

$$v_i^\delta = (1-\frac{l_i}{l})\delta(v_i^\delta - l_i c_i) + \frac{l_i}{2l}\delta(v_i^\delta - l_i c_i) + \frac{1}{2l}\sum_{\{j|ij\in G\}} \max\{1-\delta(v_j^\delta - l_j c_j), \delta(v_i^\delta - l_i c_i)\}$$

The probability that any link of i is not selected is equal to $(1 - l_i/l)$ and his expected share is v_i^δ in the next period. First part of the equation covers this case. Second part of the equation covers the case where a link of i is selected but i is not the proposer. The other player makes an offer $\delta(v_i^\delta - l_i c_i)$ or any offer which is rejected by i and so his continuation payoff is also equal to $\delta(v_i^\delta - l_i c_i)$. In the third part, a link of i is selected (say ij) and i is the proposer. i makes an offer $\delta(v_j^\delta - l_j c_j)$ or i makes an offer which will be rejected by player j.

Expected share of any player $i \in N$ is equivalent to

$$v_i^\delta = (1 - \frac{l_i}{2l})\delta(v_i^\delta - l_i c_i) + \frac{1}{2l}\sum_{\{j|ij\in G\}} \max\{1 - \delta(v_j^\delta - l_j c_j), \delta(v_i^\delta - l_i c_i)\}.$$

In any equilibrium, for each δ satisfying $\delta((v_i^{*\delta} - l_i c_i) + (v_j^{*\delta} - l_j c_j)) \neq 1$ for all $ij \in G$, whether the bargaining ends up with agreement or disagreement is captured in Theorem 2. Next lemma completes this analysis by examining the discount factors δ satisfying $\delta((v_i^{*\delta} - l_i c_i) + (v_j^{*\delta} - l_j c_j)) \neq 1$ and shows that the set of such discount factors is finite.

Lemma 3. *The inequality* $\delta((v_i^{*\delta} - l_i c_i) + (v_j^{*\delta} - l_j c_j)) \neq 1$, $\forall ij \in G$ *holds for all* *but a finite set of δ.*

Proof of Lemma 3. The proof is relegated to the Appendix. $\square$

The following result identifies the bound for δ to obtain both the existence of a limit equilibrium network G^* and the existence of limit equilibrium shares as players become more patient.

Theorem 3. *There exists a bound $\underline{\delta}$ and a subnetwork G^* of G such that for all values of $\delta > \underline{\delta}$, $G^{*\delta}$ is equal to G^*. Moreover, the equilibrium share vector at δ converges to v^* as δ goes to* 1.

Proof of Theorem 3. The proof follows from Lemma 3 and the proof of Theorem 2* in Manea (2011), which is stated below. $\square$

Theorem 2* (Manea (2011)): (i) There exists $\underline{\delta} \in (0,1)$ and a subnetwork G^* of G such that the equilibrium agreement network $G^{*\delta}$ of Γ^δ equals G^* for all $\delta > \underline{\delta}$. (ii) The equilibrium payoff vector $v^{*\delta}$ of Γ^δ converges to a payoff vector $v^* \in [0,1]^n$ as δ tend to 1.

By Theorem 2, we know that one of the determinants of the equilibrium shares is the positions of the players in the network. Hence, investigating the structure of the network that is formed will provide cues about the limit equilibrium payoffs of the players. Next theorem identifies the bounds on the limit equilibrium shares of players who get the highest share and the lowest share in a mutually estranged set of a network.

Theorem 4. *For all mutually estranged set M with $L^G(M) = L$, the following inequalities hold:*

$$\min_{i \in M} v_i^* \leq \frac{|L|}{|L| + |M|} + \frac{\sum_{j \in L} l_j^G TC_j}{|L| + |M|} - \frac{\sum_{i \in M} l_i^G TC_i}{|L| + |M|}$$
$$\max_{j \in L} v_j^* \geq \frac{|M|}{|L| + |M|} + \frac{\sum_{i \in M} l_i^G TC_i}{|L| + |M|} - \frac{\sum_{j \in L} l_j^G TC_j}{|L| + |M|}.$$

For the proof of the theorem, we need the following lemma which identifies the division of the pie between players of a pair in the limit equilibrium. In a network G, the produced pie by a link is not wasted. The sum of the limit equilibrium shares of players in the nodes of a link from the pie is equal to 1. In particular, the limit equilibrium network G^* only includes the agreement links.

Lemma 4. *If $ij \in G$, then $v_i^* + v_j^* \geq 1$ and if $ij \in G^*$, then $v_i^* + v_j^* = 1$.*

Proof of Lemma 4. The proof is relegated to the Appendix. □

Proof of Theorem 4. For all δ and for any player i, we can write the equilibrium share as follows

$$v_i^{*\delta} = -\frac{\delta}{1-\delta} l_i^G c_i + \frac{1}{1-\delta} \frac{1}{2l^G} \sum_{\{j | ij \in G\}} \max\{1 - \delta(v_j^{*\delta} - l_j^G c_j) - \delta(v_i^{*\delta} - l_i^G c_i), 0\}.$$

Now, take any mutually estranged set M with $L^G(M) = L$ and take $\delta > \underline{\delta}$. For all i, define the number of links that he has as $l_i = l_i^G$. For all players i in the mutually estranged set M,

$$v_i^{*\delta} = -\frac{\delta}{1-\delta} l_i c_i + \frac{1}{1-\delta}\frac{1}{2l} \sum_{\{j|ij\in G\}} \max\{1 - \delta(v_j^{*\delta} - l_j c_j) - \delta(v_i^{*\delta} - l_i c_i), 0\}. \quad (2.5)$$

If the link ij is not a G^*-link, the players i and j could not reach an agreement on the division of the pie. Hence, in the second part of the equation (6), $\max\{1 - \delta(v_j^{*\delta} - l_j c_j) - \delta(v_i^{*\delta} - l_i c_i), 0\} = 0$.

Since the players in M has G^*-links only with the players in L, we rewrite the equation (2.5) as follows:

$$v_i^{*\delta} = \frac{1}{1-\delta}\frac{1}{2l} \sum_{\{j|ij\in G, j\in L\}} \max\{1 - \delta(v_j^{*\delta} - l_j c_j) - \delta(v_i^{*\delta} - l_i c_i), 0\} - \frac{\delta}{1-\delta} l_i c_i. \quad (2.6)$$

For each $j \in L$,

$$v_j^{*\delta} = \frac{1}{1-\delta}\frac{1}{2l} \sum_{\{k|kj\in G\}} \max\{1 - \delta(v_k^{*\delta} - l_k c_k) - \delta(v_j^{*\delta} - l_j c_j), 0\} - \frac{\delta}{1-\delta} l_j c_j.$$

Applying similar arguments that are used in equation (2.6), we have

$$v_j^{*\delta} \geq \frac{1}{1-\delta}\frac{1}{2l}\sum_{\{i\in M|ij\in G\}} \max\{1-\delta(v_i^{*\delta}-l_ic_i)-\delta(v_j^{*\delta}-l_jc_j),0\} - \frac{\delta}{1-\delta}l_jc_j. \quad (2.7)$$

By taking the summation of (2.6) over all $i \in M$ and taking the summation of (2.7) over each $j \in L$, we get

$$\sum_{i\in M} v_i^{*\delta} = \frac{1}{1-\delta}\frac{1}{2l}\sum_{\{ij\in G|i\in M,j\in L\}} \max\{1-\delta(v_j^{*\delta}-l_jc_j)-\delta(v_i^{*\delta}-l_ic_i),0\} - \sum_{i\in M}\frac{\delta}{1-\delta}l_ic_i. \quad (2.8)$$

and

$$\sum_{j\in L} v_j^{*\delta} \geq \frac{1}{1-\delta}\frac{1}{2l}\sum_{\{ij\in G|i\in M,j\in L\}} \max\{1-\delta(v_j^{*\delta}-l_jc_j)-\delta(v_i^{*\delta}-l_ic_i),0\} - \sum_{j\in L}\frac{\delta}{1-\delta}l_jc_j \quad (2.9)$$

The first part of the summations is the same in both (2.8) and (2.9). Then, we obtain the following simple inequality

$$\sum_{j\in L}(v_j^{*\delta} + \frac{\delta}{1-\delta}l_jc_j) \geq \sum_{i\in M}(v_i^{*\delta} + \frac{\delta}{1-\delta}l_ic_i). \quad (2.10)$$

Since the total linking cost is shared by all players over periods, for each player i, the incurred cost of each link he has is equal to $(1-\delta)TC_i$. Substituting the equality $c_i = (1-\delta)TC_i$ in (2.10), we have

$$\sum_{j \in L}(v_j^{*\delta} + \delta l_j TC_j) \geq \sum_{i \in M}(v_i^{*\delta} + \delta l_i TC_i).$$

When players become more patient, as $\delta \to 1$,

$$\sum_{j \in L}(v_j^* + l_j TC_j) \geq \sum_{i \in M}(v_i^* + l_i TC_i). \tag{2.11}$$

By using the trivial observations that for all $k \in L$, $\max_{j \in L} v_j^* \geq v_k^*$ and for all $l \in M$, $\min_{i \in M} v_i^* \leq v_l^*$, we rewrite the inequality (2.11) as

$$|L| \max_{j \in L} v_j^* + \sum_{j \in L} l_j TC_j \geq |M| \min_{i \in M} v_i^* + \sum_{i \in M} l_i TC_i.$$

Take any player $\underline{i}$ whose limit equilibrium payoff is equal to the minimum limit equilibrium payoff in M, i.e., $\underline{i} \in \{k \in M | v_k^* = \min_{i \in M} v_i^*\}$. Also, take any player $\hat{j}$ who has a G^*-link with $\underline{i}$, i.e., $\hat{j} \in L^{G^*}(\underline{i})$. So,

$$\min_{i \in M} v_i^* = v_{\underline{i}} = 1 - v_{\hat{j}} \geq 1 - \max_{j \in L} v_j^*. \tag{2.12}$$

(the second equality follows from Lemma 4)

Take any player $\bar{j}$ whose limit equilibrium payoff is equal to the maximum limit equilibrium payoff in L, i.e., $\bar{j} \in \{k \in L | v_k^* = \max_{j \in L} v_j^*\}$. Meanwhile, take any player $\hat{i}$ who has a G^*-link with $\bar{j}$, i.e., $\hat{i} \in L^{G^*}(\bar{j})$. So,

$$\max_{j\in L} v_j^* = v_{\bar{j}} = 1 - v_{\hat{i}} \leq 1 - \min_{i\in M} v_i^*. \tag{2.13}$$

(the second equality follows from Lemma 4)

The inequalities (2.12) and (2.13) imply that $\min_{i\in M} v_i^* = 1 - \max_{j\in L} v_j^*$. Then, by using this equality

$$|L|\max_{j\in L} v_j^* + \sum_{j\in L} l_j^G TC_j \geq |M|\min_{i\in M} v_i^* + \sum_{i\in M} l_i^G TC_i \geq |M|(1 - \max_{j\in L} v_j^*) + \sum_{i\in M} l_i^G TC_i.$$

Utilizing above inequality, the followings

$$\min_{i\in M} v_i^* \leq \frac{|L|}{|L|+|M|} + \frac{\sum_{j\in L} l_j^G TC_j}{|L|+|M|} - \frac{\sum_{i\in M} l_i^G TC_i}{|L|+|M|}$$

$$\max_{j\in L} v_j^* \geq \frac{|M|}{|L|+|M|} + \frac{\sum_{i\in M} l_i^G TC_i}{|L|+|M|} - \frac{\sum_{j\in L} l_j^G TC_j}{|L|+|M|}$$

conclude the proof.

□

In the bargaining game, there are two determinants of bargaining outcome: the bargaining power provided by the network structure and the linking costs of the players. The first determinant depends on the position of a player in the network, number of links he has and also the position of his neighbours. Both the network position and linking costs affect the continuation payoff of a player. Theorem 4 that identifies the bounds on limit equilibrium shares of players provides a clue about the impact of both determinants on the limit equilibrium of the game. The former one is captured by the first part of the summation in the bound, $|L|/(|L|+|M|)$. The latter one is captured by the term in the remaining part of the summation. In the oligopoly subnetworks, the players in the set L (short side) have a higher bargaining power compared to the players in M (long side) due to network structure. The impact of linking costs on the bargaining outcome should not make all the players in M better than the players in L. There should exist at least one player in M who is still be less advantageous than the players in L. The advantage/disadvantage obtained from the linking costs should not dominate the advantage/disadvantage obtained from the network structure. This condition is captured by the following assumption.

Assumption 1. For all subsets of players M and M' in a network G such that $|L^G(M)|/|M| < |L^G(M')|/|M'|$ the following holds:

$$\frac{\sum\limits_{j\in L^G(M)} l_j^G TC_j}{|L^G(M)|+|M|} - \frac{\sum\limits_{i\in M} l_i^G TC_i}{|L^G(M)|+|M|} < \frac{\sum\limits_{j\in L^G(M')} l_j^G TC_j}{|L^G(M')|+|M'|} - \frac{\sum\limits_{i\in M'} l_i^G TC_i}{|L^G(M')|+|M'|}.$$

The oligopoly subnetworks that have same shortage ratio are identical in the zero-cost framework. However, in the framework with heterogeneous costs, these subnetworks are differentiated. Hence, we modify the network decomposition algorithm of Manea (2011) by incorporating the cost of link formation.

Network Decomposition Algorithm with Costly Links, $\mathcal{A}^C(G)$: For a given network $G \in \Omega$, the algorithm generates the sequence $(r_s, M_s, L_s, N_s, G_s)_s$ as follows:

Let $N_1 = N$ and $G_1 = G$.

For $s \geq 1$:

If $N_s = \emptyset$, then STOP.

If not, let

$$r_s = \min_{M \subset N_s, M \in \mathcal{I}(G)} \frac{|L^{G_s}(M)|}{|M|}. \tag{2.14}$$

If $r_s < 1$, then define the family

$$\mathcal{N}_s = \left\{ M \subseteq N_s | r_s = \frac{|L^{G_s}(M)|}{|M|} \right\}$$

If $|\mathcal{N}_s| > 1$, then define the component set of the network G_s.

$$\mathcal{C}_s = \{G' \subseteq G_s | \exists M \in \mathcal{N}_s \text{ s.t. } M \in G' \text{ and } G' \text{ is a component of } G_s\}$$

$$\mathcal{M}_s = \{M \subseteq \mathcal{N}_s | \exists G' \in \mathcal{C}_s \text{ s.t. M is the largest mutualy estranged set in } G'\}$$

Otherwise,

if $r_s = 1$ and $|M| = |L^{G_s}(M)| = 1$, then

$$\mathcal{M}_s = \left\{ M \subseteq N_s | r_s = 1 \text{ and } |M| = 1 \right\}.$$

Else, STOP.

Let M_s be union of all mutually estranged sets M in $\mathcal{M}_s$ that minimizes

$$\frac{\sum\limits_{j\in L^{G_s}(M)} l_j^G TC_j}{|L^{G_s}(M)|+|M|} - \frac{\sum\limits_{i\in M} l_i^G TC_i}{|L^{G_s}(M)|+|M|}. \tag{2.15}$$

Denote $L_s = L^{G_s}(M_s)$. Let $N_{s+1} = N_s \backslash (M_s \cup L_s)$ and G_{s+1} be the subnetwork of G induced by the players in N_{s+1}. Denote by $\bar{s}$ the step at which the algorithm ends.

Initially, take the formed network in the network formation game. At each step, the algorithm chooses the sets that minimize the shortage ratio r_s. In case of multiple minimizer sets, it considers the components of the network. Note that a component of a network is defined as the maximal connected subnetwork of the network. Since the players incur a cost for each own link, having a common neighbour will also affect the payoffs. So, in each component, it chooses the maximal set among the minimizer sets. If the number of these maximal sets is more than one, in other words if we have more than one component involving a minimizer set in the active network at step s, compare the advantage or disadvantage provided by linking costs. This is followed by taking the largest set that minimizes this advantage/disadvantage. Remove the players and links that belong to this chosen component from the network. The algorithm terminates when there are no mutually estranged sets of players that make the shortage ratio less than one.

Lemma 5. *The network decomposition algorithm with costly links, $\mathcal{A}^C(G)$, generates a unique sequence $(r_s, M_s, L_s, N_s, G_s)_s$, for all $s = 1, 2, \ldots, \bar{s}$.*

Proof of Lemma 5. The proof is relegated to the Appendix. $\square$

Next lemma gives the monotonicity properties of shortage ratio.

Lemma 6. *Let the sequence $(r_s, M_s, L_s, N_s, G_s)_{s=1,2,\ldots,\bar{s}}$ be defined by the algorithm $\mathcal{A}^C(G)$. For any $s' < s$ $(< \bar{s})$,*

$$\frac{|L_{s'}|}{|L_{s'}| + |M_{s'}|} \leq \frac{|L_s|}{|L_s| + |M_s|}.$$

Proof of Lemma 6. The proof is relegated to the Appendix. $\square$

The outcome of the decomposition algorithm with costly links, $\mathcal{A}^C(G)$, provides the limit equilibrium payoffs which are given by the following theorem.

Theorem 5. *Suppose that in the network formation game, the network G is formed. Let the algorithm $\mathcal{A}^C(G)$ yields the outcome $(r_s, M_s, L_s, N_s, G_s)_{s=1,2,\ldots,\bar{s}}$. Then the limit equilibrium payoffs as $\delta \to 1$ are given by*

$$\begin{aligned}
\forall s < \bar{s},\ \forall i \in M_s,\ u_i^*(G) = v_i^* &= \frac{|L_s|}{|L_s| + |M_s|} + \frac{\sum\limits_{j \in L_s} l_j^G TC_j}{|L_s| + |M_s|} - \frac{\sum\limits_{i \in M_s} l_i^G TC_i}{|L_s| + |M_s|},\\
\forall s < \bar{s},\ \forall j \in L_s,\ u_j^*(G) = v_j^* &= \frac{|M_s|}{|L_s| + |M_s|} + \frac{\sum\limits_{i \in M_s} l_i^G TC_i}{|L_s| + |M_s|} - \frac{\sum\limits_{j \in L_s} l_j^G TC_j}{|L_s| + |M_s|},\\
\forall k \in N_{\bar{s}},\ u_k^*(G) = v_k^* &= \frac{1}{2}.
\end{aligned}$$

Proof of Theorem 5. The proof of the theorem proceeds by induction on s. Suppose that the claim is hold for all $s' < s$. Now, prove it for s.

Case 1. $s < \bar{s}$

Let M_s and L_s be the sets that are generated from the algorithm $\mathcal{A}^C(G)$ at step s. Define the minimum limit equilibrium share as $\underline{x}_s = \min_{i \in N_s} v_i^*$. For notational convenience, let for all i, $l_i = l_i^G$. Also, define

$$\underline{M}_s = \{k \in M_s | v_k^* = \underline{x}_s\} \text{ and } \underline{L}_s = L^{G_s}(\underline{M}_s).$$

$\underline{M}_s$ is the set of players who have the minimum limit equilibrium share in N_s and $\underline{L}_s$ is the partner set of $\underline{M}_s$ in the network G_s.

Claim 1. $\underline{x}_s \leq \frac{|L_s|}{|L_s| + |M_s|} + \frac{\sum_{j \in L_s} l_j TC_j}{|L_s| + |M_s|} - \frac{\sum_{i \in M_s} l_i TC_i}{|L_s| + |M_s|}$

For a contradiction, suppose that

$$\underline{x}_s > \frac{|L_s|}{|L_s| + |M_s|} + \frac{\sum_{j \in L_s} l_j TC_j}{|L_s| + |M_s|} - \frac{\sum_{i \in M_s} l_i TC_i}{|L_s| + |M_s|}.$$

First, we identify the set of players who have G^*-links with players in M_s. Take any $j \in L_{s'}$ where $s' \in \{1, 2, \ldots, s-1\}$. By induction hypothesis,

$$v_j^* = \frac{|M_{s'}|}{|L_{s'}| + |M_{s'}|} + \frac{\sum_{i \in M_{s'}} l_i TC_i}{|L_{s'}| + |M_{s'}|} - \frac{\sum_{j \in L_{s'}} l_j TC_j}{|L_{s'}| + |M_{s'}|}$$

Then, we add up the limit equilibrium payoff of a player in M_s and j to determine whether they have an agreement link. For all players $i \in M_s$,

$$
\begin{aligned}
v_i^* + v_j^* &\geq \underline{x}_s + \frac{|M_{s'}|}{|L_{s'}| + |M_{s'}|} + \frac{\sum\limits_{i \in M_{s'}} l_i TC_i}{|L_{s'}| + |M_{s'}|} - \frac{\sum\limits_{j \in L_{s'}} l_j TC_j}{|L_{s'}| + |M_{s'}|} \\
&> \frac{|L_s|}{|L_s| + |M_s|} + \frac{\sum\limits_{j \in L_s} l_j TC_j}{|L_s| + |M_s|} - \frac{\sum\limits_{i \in M_s} l_i TC_i}{|L_s| + |M_s|} \\
&\quad \frac{|M_{s'}|}{|L_{s'}| + |M_{s'}|} + \frac{\sum\limits_{i \in M_{s'}} l_i TC_i}{|L_{s'}| + |M_{s'}|} - \frac{\sum\limits_{j \in L_{s'}} l_j TC_j}{|L_{s'}| + |M_{s'}|}.
\end{aligned}
$$

Second inequality follows from our supposition. Since $|L_{s'}|/(|L_{s'}| + |M_{s'}|) \leq |L_s|/(|L_s| + |M_s|)$ by Lemma 6, we have

$$
\begin{aligned}
v_i^* + v_j^* > 1 &+ \frac{\sum\limits_{j \in L_s} l_j TC_j}{|L_s| + |M_s|} - \frac{\sum\limits_{i \in M_s} l_i TC_i}{|L_s| + |M_s|} \\
&+ \frac{\sum\limits_{i \in M_{s'}} l_i TC_i}{|L_{s'}| + |M_{s'}|} - \frac{\sum\limits_{j \in L_{s'}} l_j TC_j}{|L_{s'}| + |M_{s'}|}.
\end{aligned}
$$

Utilizing Assumption 1, we get $v_i^*(G) + v_j^*(G) > 1$. So, i does not have a G^*-link with the player j, which means that no player $i \in M_s$ has G^*-links to players $j \in L_1 \cup L_2 \cup \ldots \cup L_{s-1}$. Also, by construction of the decomposition algorithm, M_s is a mutually estranged set implying that M_s is a G^*-independent set. Then, we have $L^{G^*}(M_s) \subseteq L_s$. Theorem 4 implies that

$$
\min_{i \in M_s} v_i^* \leq \frac{|L_s|}{|L_s| + |M_s|} + \frac{\sum\limits_{j \in L_s} l_j TC_j}{|L_s| + |M_s|} - \frac{\sum\limits_{i \in M_s} l_i TC_i}{|L_s| + |M_s|}.
$$

Utilizing the definition of $\underline{x}_s$,

$$\underline{x}_s \leq \frac{|L_s|}{|L_s|+|M_s|} + \frac{\sum\limits_{j \in L_s} l_j TC_j}{|L_s|+|M_s|} - \frac{\sum\limits_{i \in M_s} l_i TC_i}{|L_s|+|M_s|},$$

which contradicts with our supposition.

Claim 2. $\underline{x}_s \geq \frac{|L_s|}{|L_s|+|M_s|} + \frac{\sum\limits_{j \in L_s} l_j TC_j}{|L_s|+|M_s|} - \frac{\sum\limits_{i \in M_s} l_i TC_i}{|L_s|+|M_s|}$

Assumption 1 implies that

$$\frac{|L_s|}{|L_s|+|M_s|} + \frac{\sum\limits_{j \in L_s} l_j TC_j}{|L_s|+|M_s|} - \frac{\sum\limits_{i \in M_s} l_i TC_i}{|L_s|+|M_s|} < \frac{|M_s|}{|L_s|+|M_s|} + \frac{\sum\limits_{i \in M_s} l_i TC_i}{|L_s|+|M_s|} - \frac{\sum\limits_{j \in L_s} l_j TC_j}{|L_s|+|M_s|}$$

Hence, $\underline{x}_s < 1/2$. By Lemma 1 and Claim 1 (that we proved above), for all $j \in \underline{L}_s$, $v^*_j \geq 1 - \underline{x}_s > 1/2$. Thus, $\underline{L}_s$ is a G^*-independent set.

Now, take any $j \in \underline{L}_s$. Since for all players $k \in N_s \setminus \underline{M}_s$, $v^*_k(G) + v^*_j(G) > \underline{x}_s + 1 - \underline{x}_s = 1$, there exists no G^*-link from j to players in $N_s \setminus \underline{M}_s$.

By the construction of the algorithm, there exist no G-link between j and a player in $M_{s'}$ where $s' \in \{1, 2, \ldots, s-1\}$. Further, by Assumption 1 and induction hypothesis, $v^*_k \geq 1/2$ for all $k \in L_{s'}$ where $s' \in \{1, 2, \ldots, s-1\}$, implying that $v^*_k + v^*_j > 1$. Hence, there exists no G^*-link from j to players in $L_1 \cup L_2 \cup \ldots L_{s-1}$.

Therefore, j has G^*-links only with players in $\underline{M}_s$, i.e., $L^{G^*}(\underline{M}_s) \subseteq \underline{L}_s$. Utilizing Theorem 4,

$$\underline{x}_s = \max_{i \in L^{G^*}(\underline{L}_s)} v_i^* \geq \frac{|\underline{L}_s|}{|\underline{M}_s| + |\underline{L}_s|} + \frac{\sum\limits_{j \in \underline{L}_s} l_j TC_j}{|\underline{M}_s| + |\underline{L}_s|} - \frac{\sum\limits_{i \in \underline{M}_s} l_i TC_i}{|\underline{M}_s| + |\underline{L}_s|}.$$

Since $|L_s|/|M_s| < |\underline{L}_s|/|\underline{M}_s|$, we have the following

$$\underline{x}_s \geq \frac{|L_s|}{|L_s| + |M_s|} + \frac{\sum\limits_{j \in L_s} l_j TC_j}{|L_s| + |M_s|} - \frac{\sum\limits_{i \in M_s} l_i TC_i}{|L_s| + |M_s|}.$$

Claim 1 and Claim 2 imply that

$$\underline{x}_s = \frac{|L_s|}{|L_s| + |M_s|} + \frac{\sum\limits_{j \in L_s} l_j TC_j}{|L_s| + |M_s|} - \frac{\sum\limits_{i \in M_s} l_i TC_i}{|L_s| + |M_s|}.$$

Claim 3. $\dfrac{L_s}{M_s} = \dfrac{\underline{L}_s}{\underline{M}_s}$

Since $\underline{L}_s$ is a mutually estranged set with $L^{G_s}(\underline{L}_s) = \underline{M}_s$, utilizing Theorem 4 we obtain

$$\underline{x}_s = \max_{i \in L^{G^*}(\underline{L}_s)} v_i^* \geq \frac{|\underline{L}_s|}{|\underline{M}_s| + |\underline{L}_s|} + \frac{\sum\limits_{j \in \underline{L}_s} l_j TC_j}{|\underline{M}_s| + |\underline{L}_s|} - \frac{\sum\limits_{i \in \underline{M}_s} l_i TC_i}{|\underline{M}_s| + |\underline{L}_s|}.$$

From the construction of the decomposition algorithm which minimizes the shortage ratio, we have $|L_s|/|M_s| \leq |\underline{L}_s|/|\underline{M}_s|$. By Assumption 1, it follows that

$$\frac{\sum_{j\in L_s} l_j TC_j}{|L_s|+|M_s|} - \frac{\sum_{i\in M_s} l_i TC_i}{|L_s|+|M_s|} \leq \frac{\sum_{j\in \underline{L}_s} l_j TC_j}{|\underline{M}_s|+|\underline{L}_s|} - \frac{\sum_{i\in \underline{M}_s} l_i TC_i}{|\underline{M}_s|+|\underline{L}_s|}.$$

These two inequalities imply that

$$\frac{|\underline{L}_s|}{|\underline{L}_s|+|\underline{M}_s|} \leq \frac{|L_s|}{|L_s|+|M_s|}$$

which is equivalent to $|\underline{L}_s|/|\underline{M}_s| \leq |L_s|/|M_s|$. This concludes the proof of Claim 3.

Claim 4. $\underline{M}_s = M_s$

It is clear that $\underline{M}_s \subseteq M_s$. We want to show that $\underline{M}_s \supseteq M_s$. If $|L_s|/|M_s| = 1$ and $|M_s| = 1$, $\underline{M}_s = M_s$ trivially holds. Now, consider the case that $|L_s|/|M_s| < 1$. Suppose for contradiction that $\underline{M}_s \neq M_s$. Then, there exists a player $i \in M_s \setminus \underline{M}_s$. Note that i has no G-links with players in $N_s \setminus L_s$ and players in $M_1 \cup M_2 \cup \ldots \cup M_{s-1}$. Also, by Lemma 4, i has no G^*-links to players in $L_1 \cup L_2 \cup \ldots \cup L_{s-1} \cup \underline{L}_s$. Hence, i has G^*-links only with players in $L_s \setminus \underline{L}_s$. By Theorem 4, it follows that

$$\begin{aligned}
\min_{i\in M_s\setminus \underline{M}_s} v_i^* &\leq \frac{|L_s\setminus \underline{L}_s|}{|M_s\setminus \underline{M}_s|+|L_s\setminus \underline{L}_s|} + \frac{\sum_{j\in L_s\setminus \underline{L}_s} l_j TC_j}{|M_s\setminus \underline{M}_s|+|L_s\setminus \underline{L}_s|} - \frac{\sum_{i\in M_s\setminus \underline{M}_s} l_i TC_i}{|M_s\setminus \underline{M}_s|+|L_s\setminus \underline{L}_s|} \\
&\leq \frac{|\underline{L}_s|}{|\underline{M}_s|+|\underline{L}_s|} + \frac{\sum_{j\in \underline{L}_s} l_j TC_j}{|\underline{M}_s|+|\underline{L}_s|} - \frac{\sum_{i\in \underline{M}_s} l_i TC_i}{|\underline{M}_s|+|\underline{L}_s|},
\end{aligned}$$

which contradicts with $v_i^* > \underline{x}_s$ for all $i \in M_s \setminus \underline{M}_s$.

Hence, $M_s = \underline{M}_s$ and $L_s = \underline{L}_s$. Claim 1 - Claim 4 concludes the proof for any step of the algorithm $s < \bar{s}$.

Case 2. $s = \bar{s}$

The network decomposition algorithm with costly links terminates at $\bar{s}$ when

$$\min_{M \subseteq N_{\bar{s}}, M \in \mathcal{I}(G)} |L^{G_{\bar{s}}}(M)|/|M| > 1 \text{ or } \min_{M \subseteq N_{\bar{s}}, M \in \mathcal{I}(G)} |L^{G_{\bar{s}}}(M)|/|M| = 1 \text{ with } |M_{\bar{s}}| \neq 1.$$

Claim 5. $v_k^*(G) = 1/2$ for all $k \in N_{\bar{s}}$

Define the minimum limit equilibrium share as $\underline{x}_{\bar{s}} = \min_{i \in N_{\bar{s}}} v_i^*$ and the set of players whose limit equilibrium shares are equal to this minimum value, $\underline{M}_{\bar{s}} = \{k \in N_{\bar{s}} | v_k^* = \underline{x}_{\bar{s}}\}$.

First, we prove that $\underline{x}_{\bar{s}} \geq 1/2$. Suppose for a contradiction that $\underline{x}_{\bar{s}} < 1/2$. By using similar arguments to Claim 2, we can show that any player in $\underline{L}_{\bar{s}}$ has only G^*-links to players in $\underline{M}_{\bar{s}}$ and $\underline{L}_{\bar{s}}$ is G^*-independent. Utilizing Theorem 4,

$$\underline{x}_{\bar{s}} = \max_{i \in L^{G^*}(\underline{L}_{\bar{s}})} v_i^* \geq \frac{|\underline{L}_{\bar{s}}|}{|\underline{M}_{\bar{s}}| + |\underline{L}_{\bar{s}}|} + \frac{\sum_{j \in \underline{L}_{\bar{s}}} l_j TC_j}{|\underline{L}_{\bar{s}}| + |\underline{M}_{\bar{s}}|} - \frac{\sum_{i \in \underline{L}_{\bar{s}}} l_i TC_i}{|\underline{L}_{\bar{s}}| + |\underline{M}_{\bar{s}}|}$$

Since $\underline{x}_{\bar{s}} < 1/2$ from the supposition, we have

$$\frac{|\underline{L}_{\bar{s}}|}{|\underline{M}_{\bar{s}}| + |\underline{L}_{\bar{s}}|} + \frac{\sum_{j \in \underline{L}_{\bar{s}}} l_j TC_j}{|\underline{L}_{\bar{s}}| + |\underline{M}_{\bar{s}}|} - \frac{\sum_{i \in \underline{L}_{\bar{s}}} l_i TC_i}{|\underline{L}_{\bar{s}}| + |\underline{M}_{\bar{s}}|} < \frac{1}{2},$$

which implies that $|\underline{L}_{\bar{s}}|/|\underline{M}_{\bar{s}}| < 1$, a contradiction with

$$\min_{M \subseteq N_{\bar{s}}, M \in \mathcal{I}(G)} |L^{G_{\bar{s}}}(M)|/|M| \geq 1.$$

Therefore, for all $k \in N_{\bar{s}}$,

$$v_k^* \geq \frac{1}{2}. \tag{2.16}$$

Now, we prove that for all $k \in N_{\bar{s}}$, $v_k^* \leq \frac{1}{2}$. Take any $k \in N_{\bar{s}}$. Since $v_k^* \geq \frac{1}{2}$, the construction of the decomposition algorithm implies that player k has no G-links to players in $M_1 \cup M_2 \cup \ldots \cup M_{\bar{s}-1}$ and Lemma 4 implies that player k has no G^*-links to players in $L_1 \cup L_2 \cup \ldots \cup L_{\bar{s}-1}$. Hence, there may exist G^*-links from player k only to players in $N_{\bar{s}}$, yielding that $v_k^* \leq 1/2$. Hence, for all $k \in N_{\bar{s}}$,

$$v_k^* \leq \frac{1}{2}. \tag{2.17}$$

By (2.16) and (2.17), for all $k \in N_{\bar{s}}$, $v_k^* = \frac{1}{2}$.

□

By Theorem 5, we obtain the limit equilibrium payoffs of the bargaining game over a given network G. The example below is a simple but a comprehensive example to understand the process of the algorithm and how the outcome of the algorithm determines the limit equilibrium payoffs.

Example 1. *Consider the network G in Figure 2.1 with the set of players $N = \{1, 2, \ldots, 10\}$ and the total cost vector $TC = (1/10, 0, 1/5, 1/10, 1/10, 0, 0, 1/25,$*

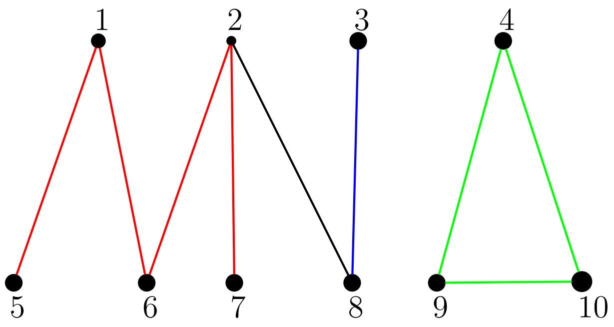


Figure 2.1: Network G

$1/10, 1/10)$. *First, we decompose the network by running the decomposition algorithm* $\mathcal{A}^C(G)$.

s=1: $N_1 = N$ *and* $G_1 = G$

$$r_1 = \min_{M \in N_1, M \in \mathcal{I}(G)} \frac{|L^{G_1}(M)|}{|M|} = \frac{2}{3} < 1$$

The shortage ratio r *is minimized by the sets in the family*

$$\mathcal{N}_1 = \{\{5, 6, 7\}\}.$$

Then,

$$\mathcal{M}_1 = \{\{5, 6, 7\}\}.$$

Since there is only one set in $\mathcal{M}_1$*, no need to check the benefit provided by costs. The mutually estranged set which is chosen in step* 1 *is* $M_1 = \{5, 6, 7\}$*. So,* $L_1 = \{1, 2\}$*. The red subnetwork is decomposed from the initial network.*

s=2: *Consider the network* G_2 *which is induced by the players* $N_2 = \{3, 4, 8, 9, 10\}$.

$$r_2 = \min_{M \in N_2, M \in \mathcal{I}(G)} \frac{|L^{G_2}(M)|}{|M|} = 1$$

Also, we have $|M| = 1$*. Then,*

$$\mathcal{M}_2 = \{\{3\}, \{8\}\}.$$

Since there are more than one set in $\mathcal{M}_2$*, we need to check the costs. Choose the set which minimizes*

$$\frac{\sum\limits_{j \in L} l_j TC_j}{|L| + |M|} - \frac{\sum\limits_{i \in M} l_i TC_i}{|L| + |M|}$$

Since $\dfrac{TC_3 - TC_8}{2} > \dfrac{TC_8 - TC_3}{2}$*, we have* $M_2 = \{8\}$*.*

The blue subnetwork is decomposed from the network G_2*.*

s=3: *Consider the network* G_3 *which is induced by the players* $N_3 = \{4, 9, 10\}$*.*

$$r_3 = \min_{M \in N_3, M \in \mathcal{I}(G)} \frac{|L^{G_3}(M)|}{|M|} = 2 > 1.$$

Then, the algorithm STOPs in this step. So, $\bar{s} = 3$*. The set of remaining players is* $N_3 = \{4, 9, 10\}$ *and the induced network by this set is the green one.*

The decomposition of the network G *is completed. According to the outcome of the algorithm, the payoffs are determined as follows:*

$$\forall s = \{1, 2\},\ \forall i \in M_s,\ u_i^*(G) = \frac{|L_s|}{|L_s| + |M_s|} + \frac{\sum\limits_{j \in L_s} l_j^G TC_j}{|L_s| + |M_s|} - \frac{\sum\limits_{i \in M_s} l_i^G TC_i}{|L_s| + |M_s|}$$

$$\forall s = \{1, 2\},\ \forall j \in L_s,\ u_j^*(G) = \frac{|M_s|}{|L_s| + |M_s|} + \frac{\sum\limits_{i \in M_s} l_i^G TC_i}{|L_s| + |M_s|} - \frac{\sum\limits_{j \in L_s} l_j^G TC_j}{|L_s| + |M_s|}$$

$$\forall k \in N_3,\ u_k^*(G) = \frac{1}{2}.$$

- $M_1 = \{5, 6, 7\}$ *and* $L_1 = \{1, 2\}$

$$u_5^*(G) = u_6^*(G) = u_7^*(G) = \frac{2}{5} + \frac{2TC_1 + 3TC_2 - TC_5 - 2TC_6 - TC_7}{5}$$

$$u_1^*(G) = u_2^*(G) = \frac{3}{5} + \frac{TC_5 + 2TC_6 + TC_7 - 2TC_1 - 3TC_2}{5}$$

- $M_2 = \{8\}$ *and* $L_2 = \{3\}$

$$u_8^*(G) = \frac{1}{2} + \frac{TC_3 - 2TC_8}{2}$$

$$u_3^*(G) = \frac{1}{2} + \frac{2TC_8 - TC_3}{2}$$

- $N_3 = \{4, 9, 10\}$

$$u_4^*(G) = u_9^*(G) = u_{10}^*(G) = \frac{1}{2}$$

Note that the equilibrium agreement network G^* is the network that consists of all the links of G but the link between 2 and 8.

Now, we examine the first stage of the game: network formation game. During this stage, players take into consideration their future limit equilibrium payoffs at the bargaining game. We analyze the strategic decisions of the players and the network structure induced by the bargaining outcome.

Theorem 6. *g^{PNE} is a pairwise Nash equilibrium of network formation game if and only if for all $i \in N$, $u_i^*(\phi(g^{PNE})) = 1/2$. (The induced network G^{PNE} by the strategy profile g^{PNE} is equitable.)*

Proof of Theorem 6. Suppose that g^{PNE} is a pairwise Nash equilibrium of the network formation game. Assume that there exists a player $i \in N$ such that $u_i^*(\phi(g^{PNE})) \neq 1/2$. Then, $u_i^*(\phi(g^{PNE})) < 1/2$ or $u_i^*(\phi(g^{PNE})) > 1/2$.

Case 1. $u_i^*(\phi(g^{PNE})) < 1/2$.

In this case, for some step $s < \bar{s}$ of the algorithm $\mathcal{A}^C(G)$, i belongs to M_s and $|L_s|/|M_s| < 1$. Hence, there exists a player $j \in M_s$ such that $L^{G_s}(\{i\}) \cap L^{G_s}(\{j\}) \neq \emptyset$. If we add the link ij to the network, we will have

$$u_i^*(\phi(g^{PNE}) + ij) = \frac{1}{2} > u_i^*(\phi(g^{PNE})) \text{ and}$$
$$u_j^*(\phi(g^{PNE}) + ij) = \frac{1}{2} > u_j^*(\phi(g^{PNE})).$$

So, g^{PNE} is not a pairwise Nash equilibrium of the network formation game, which contradicts with our supposition.

Case 2. $u_i^*(\phi(g^{PNE})) > 1/2$.

In this case, for some step $s < \bar{s}$ of the algorithm $\mathcal{A}^C(G)$, i belongs to L_s. Then, there exists a player $j \in M_s$ such that $u_j^*(\phi(g^{PNE})) < 1/2$. So, similar arguments to Case 1 follows for player j.

Hence, for all $i \in N$, $u_i^*(\phi(g^{PNE})) = 1/2$.

For the other part of the theorem, suppose that $u_i^*(\phi(g)) = 1/2$. Hence, for all mutually estranged sets M, $|L^G(M)|/|M| > 1$ or $|L^G(M)|/|M| = 1$ with $|M| \neq 1$.

Assume that g is not a pairwise Nash equilibrium.

Adding a link to the network does not change the shortage ratio r_s for all s which is minimized in the decomposition algorithm. Hence, there does not exist any pair of players (i, j) such that

$$u_i^*(\phi(g) + ij) \geq u_i^*(\phi(g)) \text{ and}$$
$$u_j^*(\phi(g) + ij) > u_j^*(\phi(g)).$$

Therefore, g violates the first condition of a pairwise Nash equilibrium. Hence, there exists a player i and a strategy of him $g_i' \in \mathcal{G}_i$ such that $u_i^*(\phi(g_i', g_{-i})) > u_i^*(\phi(g))$. Let $\phi(g_i', g_{-i}) = G'$ and $\phi(g) = G$. So, in G', there exists a mutually estranged set M with $L^{G'}(M) = L$ such that $|L|/|M| < 1$ and $i \in L$ or $|L|/|M| = 1$ with $|M| = 1$ and $i \in L$. Since any change in the strategy of player i does not affect the links of players in M with other players, we get $|L| = |L^{G'}(M)| \geq |L^G(M)|$, which contradicts with the fact that for all mutually estranged sets M, $|L^G(M)|/|M| > 1$ or $|L^G(M)|/|M| = 1$ with $|M| \neq 1$.

It follows that g is a pairwise Nash equilibrium of the network formation game.

□

In this subsection, we prove that with zero linking costs in the network formation stage, the bargaining game reduces the same game in Manea (2011). Hence, the limit equilibrium payoffs are determined in his paper. In this model, the sole determinant of the equilibrium payoffs is the network structure. For the game with non-zero linking cost structure, we obtained the limit equilibirum payoffs for each possible network by constructing a network decomposition algorithm. The algorithm picks the oligopoly subnetworks considering costly links. Here, there are two factors that affecting the limit equilibirium payoffs, the newtork structure and the linking costs. We also investigate the pairwise Nash equilibirum of the network formation game. When players are allowed to strategically form their own links, the differences in limit equilibrium payoffs across players (which exist in Manea (2011)) disappear in both cost specifications. In the equilibrium, two players in the nodes of a link get the same payoff, impyling that the network formed is an equitable network. Network endogeneity leads to equal payoffs between players. Intuitively, having equal opportunities wipes out the heterogeneities in the bargaining outcome.

2.2 Efficiency of Equilibrium Networks

In this section, we are interested in the well-being of the society induced by the networks formed in equilibrium. We, first, define the efficiency notion we employ. The most commonly used welfare measure is defined as the sum of the payoffs of all the players. Formally, let $W : \Omega \longrightarrow R$ be defined as

$$W(G) = \sum_{i \in N} u_i^*(G)$$

for all $G \in \Omega$. Efficiency is basically maximizing the aggregate utility across all players in the society. This definition is referred as strong efficiency in Jackson and Wolinsky (1996).

Definition 2. A network G is efficient if $W(G) \geq W(G')$ for all $G' \in \Omega$.

A network G is efficient if it maximizes $\sum_{i \in N} u_i^*(G)$. For the characterization of the efficient networks, we need the following definitions. A *cycle* is a path from a vertex back to itself with no repeated edges and no repeated vertices except the first and the last vertex. An *odd cycle* is a cycle with odd number of vertices. A *matching* is a graph with even number of vertices. The characterization of the efficient networks from the perspective of the society is provided by the following theorem.

Theorem 7. *A network G is efficient if and only if it is a disjoint union of odd cycles and matchings.*

Proof of Theorem 7. Take any efficient network $G \in \Omega$. Suppose that it is not covered by any disjoint union of odd cycles and matchings. Then, by the decomposition algorithm at some step $s < \bar{s}$ there exists a mutually estranged set M_s such that $|L^{G_s}(M_s)|/|M_s| < 1$. Consider a network G' in which the subnetwork induced by the players $L_s \cup M_s$ is a complete network.

$$
\begin{aligned}
W(G) = \sum_{i\in N} u_i^*(G) &= \sum_{i\in N\setminus(L_s\cup M_s)} u_i^*(G) + \sum_{i\in L_s\cup M_s} u_i^*(G)\\
&= \Big(\sum_{i\in N\setminus(L_s\cup M_s)} u_i^*(G)\Big) + |L_s| + (|M_s| - |L_s|)\Big[\frac{|L_s|}{|L_s|+|M_s|}\\
&\qquad\qquad + \frac{\sum_{j\in L_s} l_j TC_j}{|L_s|+|M_s|} - \frac{\sum_{i\in L_s} l_i TC_i}{|L_s|+|M_s|}\Big]\\
&\leq \Big(\sum_{i\in N\setminus(L_s\cup M_s)} u_i^*(G)\Big) + |L_s| + (|M_s| - |L_s|)\frac{1}{2}\\
&= \Big(\sum_{i\in N\setminus(L_s\cup M_s)} u_i^*(G)\Big) + |M_s| + \frac{|L_s|}{2}\\
&< \Big(\sum_{i\in N\setminus(L_s\cup M_s)} u_i^*(G)\Big) + |M_s| + \frac{|M_s|}{2}\\
&= W(G').
\end{aligned}
$$

This contradicts with the efficiency of G. Hence, G is a disjoint union of odd cycles and matchings.

Now, for the other part of the proof suppose that the network G is a disjoint union of odd cycles and matchings. The social welfare provided by the network G is

$$
W(G) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even}\\ \frac{n}{2} + \frac{1}{2}, & \text{if } n \text{ is odd.} \end{cases}
$$

When we maximize the social welfare over the set of networks Ω, we obtain the the maximum attainable social welfare as

$$\max_{\tilde{G}\in\Omega} W(\tilde{G}) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even} \\ \dfrac{n}{2} + \dfrac{1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Hence, we have $\max_{\tilde{G}\in\Omega} W(\tilde{G}) = W(G)$, which concludes the proof.

□

The corollary below examines whether the endogenously formed network is efficient or not.

Corollary 1. *Any equilibrium network of the network formation game is efficient.*

Proof of Corollary 1. The proof is relegated to the Appendix. □

2.3 Conclusion

We analyze a bargaining model over a network where the network describes the feasible bargaining partners. This study builds upon Manea (2011) by endogenizing the network structure. We construct a two stage game: network formation stage and bargaining stage. In the network formation stage, we allow players to form links where the link formation is bilateral (and could be costly). Next, on the formed network, an infinite horizon bargaining game over how to divide one-unit pie between the players in a pair is played. We investigate two specifications of the cost structure: zero costs and non-zero costs. Firstly, in the model with zero costs, the bargaining game is same as Manea (2011) develops. The network structure is the only determinant of the bargaining outcome. In the limit equilibrium, one-unit pie is divided between the players in a pair in proportion to the relative bargaining power of players provided by their positions in the network. In equilibrium of the network formation game, each player gets half of the pie, i.e., an equitable network is formed. Secondly, we assume that there exists a player whose linking cost is different than zero and the costs are heterogeneous across players. With this specification, the bargaining power implied by the network structure and the linking costs are the two factors affecting the bargaining outcome. Hence, the limit equilibrium payoffs are determined according to both the network structure and the relative advantage/disadvantage provided by linking costs. In the equilibrium of network formation game, the pie is divided equally in all pairs of players (equitable network). Therefore, in both specifications of the cost structure, if the network is endogenously determined by players, the differences among the payoffs of players disappear in equilibrium. Further, with non-zero costs, link formation incentives of the players neutralize the relative advantage/disadvantage induced by cost heterogeneities across players. Briefly, in equilibrium, having equal opportunities sweep away the differences in the bargaining outcome. Finally, we focus on the efficiency issue. We obtain the characterization of the efficient networks. In addition, we prove that endogenously formed network is efficient.

CHAPTER 3

BARGAINING ON SUPPLY CHAIN NETWORKS WITH HETEROGENEOUS VALUATIONS

Bargaining between buyers and sellers plays a key role in determining the terms of trade in supply chains. Hence, bargaining is a well-studied topic in the supply chain literature. We identified three limitations which are mostly common in the literature. First, most of the existing supply chain literature adopt the Stackelberg modelling approach in which only one of the agents always makes an offer and the other one can just accept or reject the offer (i.e, one party is gifted with a significant bargaining power). Second, many papers restrict attention to the supply chain networks with one buyer-one seller (Plambeck and Taylor (2005), Gurnani and Shi (2006) and Feng et al. (2014)), one buyer-multiple sellers (Nagarajan and Bassok (2008)), one seller-multiple buyers (Bernstein and Federgruen (2005)) and two-sellers and two-buyers (Feng and Lu (2013)). Finally, size of the pie subject to bargaining is the same for all bargaining pairs. As a result of these limitations, the effects of bargaining power, network structure and the pie size on equilibrium outcomes could not be investigated in full generality.[1] In this study, we develop a theoretical model of supply chain network where (i) both the sellers and buyers can make an offer in the bargaining, (ii) any number of buyers and sellers is possible, (iii) the size of the pie is allowed to be heterogeneous across links and (iv) the network structure and the size heterogeneity of the pie across links have the potential to affect bargaining outcomes.

Our work is inspired by recent developments in network theory which establish a relationship between the network structure and market/bargaining outcomes. In this setting, the network identifies the trading relationships between the sellers and the buyers. That is, a buyer and a seller can only engage in trade if there is a relationship or a ”link” connecting the two players. In other words, the network structure imposes a restriction on bargaining possibilities. Hence, it is theoretically shown by a large number studies that the network structure has a significant impact on the market outcome (see Calvó-Armengol (2003), Corominas-Bosch (2004), Polanski (2007), Jackson (2008), Manea (2011), Abreu and Manea (2012) and Polanski and Vega-Redondo (2013)). Building on Manea (2011), we study an infinite horizon bargaining game over a two-sided supply chain network with heterogeneous buyers.[2]

In our model, we have sellers producing a homogeneous good and buyers demanding the good. The production cost of the good is same across all sellers. The buyers, on the other hand, value the good differently. Intuitively, they may have

[1] However, in reality, the bargaining power is more evenly distributed in supply chains (see Iyer and Villas-Boas (2003) and Draganska et al. (2010)), there are multiple sellers and buyers, and the size of the pie subject to bargaining is different (e.g, due to heterogeneous valuations.)

[2] Manea (2011) explores the influence of the network structure on the bargaining outcome with homogeneous agents. He shows that the bargaining power of a player does not depend only on the number of his links and his position in the network but also his neighbours’ positions.

different tastes, preferences, habits or business characteristics. The sellers and buyers are connected via an exogenously given two-sided supply chain network. Each pair of players connected by a link generates a surplus equal to the difference between valuation of the buyer for the good and the production cost of the seller. On the network, players play the following infinite horizon bargaining game.[3] At each period, a link is selected with some positive probability and one of two players is randomly selected as a proposer. The proposer makes a take-it-or-leave-it offer concerning the division of the surplus generated by the chosen link. If the offer is accepted, the players in the pair leave the game with the agreed shares. In the next period they are replaced by their identical clones.[4] If the offer is rejected, the players in the pair remain in the game. At each period, the same bargaining procedure is repeated. All players have perfect information and have the same discount factor. The subgame perfect Nash equilibrium is employed as a solution concept.

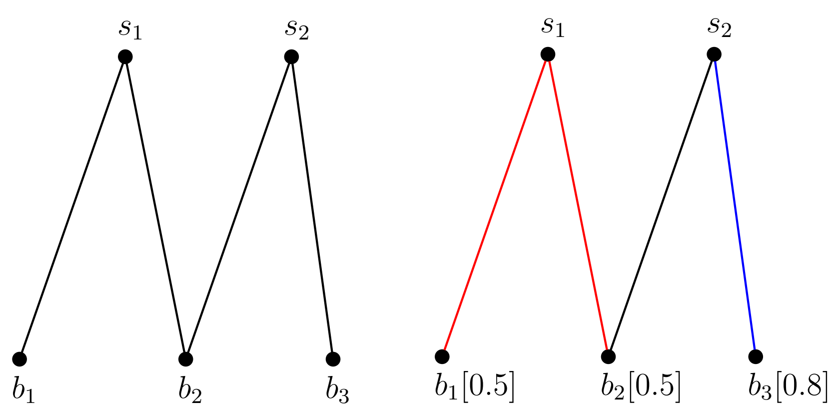
The richness of our model allows us to study the impact of bargaining power provided by the network structure and the valuations of the buyers on the market outcome which is not captured properly by the supply chain literature due to mentioned limitations. In order to investigate these effects on the outcome, we need to identify each player's position in the network and the links in which the trade is feasible. For this purpose, we modify the network decomposition algorithm constructed by Manea (2011) considering the valuation heterogeneity among buyers. This algorithm decomposes a given network into disjoint sub-

[3]Rubinstein (1982) and Rubinstein and Wolinsky (1985) are pioneering papers of the bargaining literature.

[4]The replacement of the players in the agreement pair with their clones makes the model stationary. This modelling assumption is followed by Gale (1987), Manea (2011), Polanski and Lazarova (2015) and Nguyen (2012).

networks. We find the limit equilibrium payoffs in the subnetworks determined by the network decomposition algorithm and prove the uniqueness of the equilibrium. The results show that in equilibrium, the network structure and the valuations of the buyers have an impact on the division of the surplus generated by a pair. Intuitively, the buyers with high valuation and the players who have more links or who have neighbours with less links have a higher bargaining power; and so obtain a larger share from the surplus.

The decomposition algorithm we construct and the algorithm of Manea (2011) yield different outcomes for the same network. For instance, we have a network with two sellers and three buyers as depicted in Figure 3.1. In part a, suppose that the buyers are homogeneous and their valuations are 1. So, the surplus generated by each link is same and equal to 1. On the other hand, in part b, the valuations of b_1, b_2 and b_3 to the good are 0.7, 0.8 and 0.3, respectively. The decomposition outcome for a is $\{\{b_1, b_2, b_3\}, \{s_1, s_2\}\}$ and for b is $\{\{b_1, b_2\}, \{s_1\}\}, \{\{b_3\}, \{s_2\}\}$, which are not equal to each other. As seen from 3.1, a player's position in the network is not the sole source of bargaining power but the valuation heterogeneity among the buyers matters.



(a) Homogeneous Valuations (b) Heterogeneous Valuations

Figure 3.1: Two Sided Supply Chain Network G

The paper that comes closest to ours is Nakkaş and Xu (2014). These authors also study bargaining in a supply chain where (i) the bargaining occur in an alternating order, (ii) there are multiple buyers and sellers and (iii) the pie size does not have to be same across the links. That said there are important differences between our model and theirs. The key differences are the bargaining game and payoffs in the equilibrium. More precisely, in a subnetwork having more sellers than buyers, the equilibrium of Nakkaş and Xu (2014) assigns a seller zero from the surplus and assigns a buyer all the surplus while in the equilibrium of our model, the seller gets a share at least the surplus per player in the subnetwork.

The rest of the essay is organized as follows. Section 1 describes the model and introduces the notation. Section 2 reports the results on the network decomposition outcome and the equilibrium payoffs. Section 3 concludes the study.

3.1 Model

We consider a group of sellers and buyers interconnected by a two-sided supply chain network G. Each seller produces one-unit homogeneous good and each buyer demands the goods. The links in the network represent trading possibilities. $S = \{s_1, s_2, \ldots, s_n\}$ is the set of sellers and $B = \{b_1, b_2, \ldots, b_m\}$ is the set of buyers. Assume that the number of sellers is less than the number of buyers, i.e., $n \leq m$. There doesn't exist any link between any two players within the same group. Denote the utilities of players $s \in S$ and $b \in B$ be u_s and u_b, respectively. Let the buyer b's valuation of the good be v_b and let the production cost of all sellers be c. For simplicity, assume that $c = 0$. Each link in the network generates a surplus equal to the difference of the valuation of the buyer and the production cost of the seller. An infinite horizon bargaining game is played on the supply chain network over the division of this surplus.

We construct the following infinite horizon bargaining game on the network G: At each period $t = 0, 1, \ldots$, a link $(s, b) \in G$ is selected with some probability p_{sb} and one of the players among s and b is randomly selected with equal probability as a proposer, say s. Player s makes an offer to b concerning a division of the surplus generated by the link (s, b) and player b responds to the offer by accepting or rejecting. If the responder b accepts the offer, s and b leave the game with the agreed shares. In period $t + 1$, two new players replace the same positions. If b rejects the offer, s and b remain in the game. In period $t+1$, the same procedure is repeated. The replacement assumption provides the stationarity of the model. Link selection is independent across periods. All players have a common discount rate $\delta \in (0, 1)$. Suppose that all players have perfect information. In this model, we assume that for each player $i \in S \cup B$, there is a continuum of players of type i, i.e., $i = \{i_1, i_2, \ldots, i_\tau, \ldots\}$.

In this game, we employ subgame perfect Nash equilibrium as a solution concept. The equilibrium payoff vector of the game is denoted by $(u_i^{*\delta})_{i \in S \cup B}$. Define the equilibrium agreement network as the subnetwork of G which only involves the links where the agreement gives the players at the nodes of these links more payoff than proceeding to the next period does. More precisely, the equilibrium agreement network at δ, $G^{*\delta}$, is the subnetwork of G that only consists of the links (s, b) satisfying $\delta(u_s^{*\delta} + u_b^{*\delta}) \leq$ *the surplus produced by the link* (s, b). The limit equilibrium network, denoted by G^*, is the network that $G^{*\delta}$ converges to as δ goes to 1 and the limit equilibrium payoff vector, u^*, is the payoff vector that $u^{*\delta}$ converges to as δ goes to 1.

3.2 Results

We analyze the equilibrium of the bargaining game over the network G. The surplus generated by a link (s, b) is equal to $v_b - c$. Note that $c = 0$. Hence, the surplus is equal to v_b. Suppose that in the bargaining game the link (s, b) is selected. The seller s and the buyer b bargain over how to divide the surplus v_b. As an initial step, we show that in every subgame, the expected payoff of each existing player in the network at that period is uniquely determined.

Theorem 8. *For all $\delta \in (0, 1)$, there exists a payoff vector $(u_i^{*\delta})_{i \in S \cup B}$ such that in every subgame perfect equilibrium of the bargaining game on G, the expected payoff of existing player i_τ of type i is uniquely given by $u_i^{*\delta}$ for all $i \in S \cup B$, $\tau \geq 0$. For every $\delta \in (0, 1)$, in any equilibrium, in any subgame where the link $(s_\tau, b_{\tau'})$ is selected and s_τ is the proposer, the followings are hold with probability one:*

> *(1) if $\delta(u_s^{*\delta} + u_b^{*\delta}) < v_b$, then s_τ offers $\delta u_b^{*\delta}$ and $b_{\tau'}$ accepts.*
>
> *(2) if $\delta(u_s^{*\delta} + u_b^{*\delta}) > v_b$, then s_τ makes an offer that is rejected by $b_{\tau'}$*

for each $s \in S$ and $b \in B$.

Before moving on to the proof of Theorem 8, we need the following lemma.

Lemma 7. *For all $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{R}$,*

$$|\max\{\omega_1, \omega_2\} - \max\{\omega_3, \omega_4\}| \leq \max\{|\omega_1 - \omega_3|, |\omega_2 - \omega_4|\}.$$

Proof of Lemma 7. The proof is obvious hence omitted. $\square$

Proof of Theorem 8. Let $\underline{u}_i^\delta$ and $\bar{u}_i^\delta$ be the infimum and supremum of the expected payoffs of i_τ in any subgame for all $\tau \geq 0$ and for each $i \in S \cup B$, in every subgame perfect equilibrium of the game.

Consider a subgame perfect Nash equilibrium of the game. Assume that the link (s, b) is selected and without loss of generality among the players s and b, s is selected as a proposer. Any player of type b does not accept an offer smaller than $\delta \underline{u}_b^\delta$, implying that s can get a payoff of at most $v_b - \delta \underline{u}_b^\delta$. Further, any player of type s accepts any offer greater than $\delta \bar{u}_s^\delta$, since in case of rejection, he gets at most $\delta \bar{u}_s^\delta$. Hence, no player offers him more than $\delta \bar{u}_s^\delta$ in the equilibrium. Now, suppose that any link of s is not selected. In this case, the expected continuation payoff of the player s is at most $\delta \bar{u}_s^\delta$. So, for each player $\tau \geq 0$ of type s, the following is satisfied:

$$u_{s_\tau}^\delta \leq \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right) \delta \bar{u}_s^\delta + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta \underline{u}_b^\delta, \delta \bar{u}_s^\delta\}. \tag{3.1}$$

Since the inequality (3.1) holds for all players of type s, it also holds for $\bar{u}_s^\delta$.

Thus,

$$\bar{u}_s^\delta \leq \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta\bar{u}_s^\delta + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta\underline{u}_b^\delta, \delta\bar{u}_s^\delta\}. \tag{3.2}$$

Consider that player s makes the offer $\delta\bar{u}_b^\delta + \epsilon$ ($\epsilon > 0$) to any player of type b such that $\delta\underline{u}_s^\delta + \delta\bar{u}_b^\delta + \epsilon \leq v_b$ deviating from his equilibrium strategy and offers zero to other players and also rejecting all offers that he receives. Player b accepts the offer in any subgame perfect equilibrium. Hence, for each $\tau \geq 0$ and for all $\epsilon > 0$, we have the following inequality:

$$u_{s_\tau}^\delta \geq \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta\underline{u}_s^\delta + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta\bar{u}_b^\delta - \epsilon, \delta\underline{u}_s^\delta\}.$$

As the deviation converges to zero ($\epsilon \to 0$),

$$u_{s_\tau}^\delta \geq \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta\underline{u}_s^\delta + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta\bar{u}_b^\delta, \delta\underline{u}_s^\delta\}. \tag{3.3}$$

Since the inequality (3.3) holds for all players of type s, it also holds for $\underline{u}_s^\delta$. Therefore,

$$\underline{u}_s^\delta \geq \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta \underline{u}_s^\delta + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta \bar{u}_b^\delta, \delta \underline{u}_s^\delta\}.$$

We take the difference between the infimum and the supremum of the expected payoffs for each seller in S in order to prove the equality of these two. Let $D = \max_{i\in S} \bar{u}_i^\delta - \underline{u}_i^\delta$. Take any $s \in \arg\max_{i\in S} \bar{u}_i^\delta - \underline{u}_i^\delta$.

$$\begin{aligned}
D &= \bar{u}_s^\delta - \underline{u}_s^\delta \\
&\leq \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta(\bar{u}_s^\delta - \underline{u}_s^\delta) + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}[\max\{v_b - \delta \underline{u}_b^\delta, \delta \bar{u}_s^\delta\} \\
&\qquad - \max\{v_b - \delta \bar{u}_b^\delta, \delta \underline{u}_s^\delta\}] \\
&\leq \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta D + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} \max\{|\delta \bar{u}_b^\delta - \delta \underline{u}_b^\delta|, |\delta \bar{u}_s^\delta - \delta \underline{u}_s^\delta|\} \\
&= \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta D + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\delta \max\{\bar{u}_b^\delta - \underline{u}_b^\delta, \bar{u}_s^\delta - \underline{u}_s^\delta\} \\
&= \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta D + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\delta D \\
&= \delta D
\end{aligned}$$

Hence, $D \leq \delta D$. Since $D \geq 0$ and $\delta \in (0,1)$, we get $D = 0$. Hence, for all $s \in S$, $\bar{u}_s^\delta = \underline{u}_s^\delta$. Therefore, for all $s \in S$, we obtain the following equality

$$\bar{u}_s^\delta = \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta \bar{u}_s^\delta + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta \bar{u}_b^\delta, \delta \bar{u}_s^\delta\},$$

which means that $\bar{u}_s^\delta = \underline{u}_s^\delta = u_s^{*\delta}$. The case is similar for any buyer $b \in B$. The statement $\bar{u}_b^\delta = \underline{u}_b^\delta = u_b^{*\delta}$ for all buyers $b \in B$ can be proven similarly.

In order to show the uniqueness, define a function $f^\delta : [0,1]^n \longleftrightarrow [0,1]^n$ such that for all $s \in S$ and $b \in B$,

$$f_s^\delta(u) = \left(1 - \sum_{\{b|sb\in G\}} \frac{p_{sb}}{2}\right)\delta u_s + \sum_{\{b|sb\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta u_b, \delta u_s\}$$
$$f_b^\delta(u) = \left(1 - \sum_{\{s|sb\in G\}} \frac{p_{sb}}{2}\right)\delta u_b + \sum_{\{s|sb\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta u_s, \delta u_b\}$$

We show that the function f^δ has a fixed point by utilizing the contraction mapping theorem. It is enough to prove the following lemma to conclude the proof of the uniqueness.

Lemma 8. *f^δ is a contraction mapping with respect to sup norm on $\mathbb{R}^n$.*

Proof of Lemma 8. See the Appendix. □

This concludes the proof of Theorem 8.

□

The expected payoffs of any players $s \in S$ and $b \in B$ are given by

$$u_s^\delta = \left(1 - \sum_{\{b|sb\in G\}} \frac{p_{sb}}{2}\right)\delta u_s^\delta + \sum_{\{b|sb\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta u_b^\delta, \delta u_s^\delta\}$$
$$u_b^\delta = \left(1 - \sum_{\{s|sb\in G\}} \frac{p_{sb}}{2}\right)\delta u_b^\delta + \sum_{\{s|sb\in G_2\}} \frac{p_{sb}}{2} \max\{v_b - \delta u_s^\delta, \delta u_b^\delta\}$$

When we look into the equation for any seller s, the first part of the equation covers the cases where any link of s is not selected with probability $(1 - \sum_{b|sb\in G} p_{sb}/2)$, leading to the expected payoff of u_s in the next period or a link of s is selected but s is not the proposer. In the latter case, the other player makes an offer δu_s or any offer which is rejected by s and so his expected continuation payoff is also equal to δu_s. In the remaining part, a link of s is selected (say (s, b)) and s is the proposer and he makes an offer δu_b or makes an offer which will be rejected by the buyer b.

The following result indicates the existence of a limit equilibrium network G^* and the existence of limit equilibrium payoffs as δ converges to 1.

Theorem 9. *There exists a bound $\underline{\delta}$ and a subnetwork G^* of G such that for all values of $\delta > \underline{\delta}$, $G^{*\delta}$ is equal to G^*. Moreover, the equilibrium payoff vector at δ converges to u^* as δ goes to* 1.

In any equilibrium, for all δ values such that $\delta(u_s^{*\delta} + u_b^{*\delta}) \neq v_b$ for all $(s, b) \in G$, the solution to bargaining game is determined in Theorem 8. Next lemma extends this finding by demonstrating that the set of such discount factors is finite.

Lemma 9. *The statement for all $(s, b) \in G$, $\delta(u_s^{*\delta} + u_b^{*\delta}) \neq v_b$ holds for all but a* *finite set of* δ.

Proof of Lemma 9. See the Appendix. $\square$

Proof of Theorem 9. The proof follows from Lemma 9 and the proof of Theorem 2* in Manea (2011), which is stated below. $\square$

Theorem 2* (Manea (2011)): (i) There exists $\underline{\delta} \in (0,1)$ and a subnetwork G^* of G such that the equilibrium agreement network $G^{*\delta}$ equals G^* for all $\delta > \underline{\delta}$. (ii) The equilibrium payoff vector $u^{*\delta}$ converges to a payoff vector $u^* \in [0,1]^n$ as δ tend to 1.

In this model, there are two sources of bargaining power: the network structure and the valuation of the good for the buyers. The former depends on the number of links that a player has, his position on the network and also the positions of his bargaining partners. Hence, analyzing the network structure provide cues about the limit equilibrium payoffs of the players. We need some additional notation for this analysis. For every network G and a subset of players $M \subseteq S \cup B$, $L^G(M)$ denotes the set of players who have a link in G with the players in M i.e., $L^G(M) = \{k|(k,l) \in G, l \in M\}$. A set of players is G-independent if there exists no G-link between any of two players in the set. Next theorem identifies the bounds on the limit equilibrium payoffs of players who get the highest share and the lowest share in a subnetwork.

Theorem 10. *For all set of buyers* M *with* $L^{G^*}(M) = L$, *the following inequalities hold:*

$$\max_{s \in L} u_s^* \geq \frac{\sum_{b \in M} v_b}{|L| + |M|}$$
$$\min_{b \in M} u_b^* \leq \frac{\sum_{b \in M} v_b}{|L| + |M|}.$$

Similarly, for all set of sellers M with $L^{G^*}(M) = L$, the following inequalities hold:

$$\max_{b \in L} u_b^* \geq \frac{\sum_{b \in L} v_b}{|L| + |M|}$$
$$\min_{s \in M} u_s^* \leq \frac{\sum_{b \in L} v_b}{|L| + |M|}.$$

Before moving to the proof of the theorem, we need the following lemma that investigates the division of the surplus between players in the nodes of a link in the limit equilibrium. The sum of the limit equilibrium payoffs of players in a pair is equal to the surplus generated by the link, implying that the generated surplus by the link is not wasted. Further, the limit equilibrium network G^* involves only the agreement links.

Lemma 10. *If $(s, b) \in G$, then $u_s^* + u_b^* \geq v_b$ and if $(s, b) \in G^*$, then $u_s^* + u_b^* = v_b$.*

Proof of Lemma 10. See the Appendix. □

Proof of Theorem 10. For all δ and for any players $s \in S$ and $b \in B$, the equilibrium payoffs are as follows

$$\begin{aligned} u_s^{*\delta} &= \frac{1}{1-\delta} \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta u_b^{*\delta} - \delta u_s^{*\delta}, 0\} \\ u_b^{*\delta} &= \frac{1}{1-\delta} \sum_{\{s|(s,b)\in G\}} \frac{p_{sb}}{2} \max\{v_b - \delta u_s^{*\delta} - \delta u_b^{*\delta}, 0\}. \end{aligned} \tag{3.4}$$

Without loss of generality, take any set of buyer M and let $L^{G^*}(M) = L$. Utilizing Theorem 9, fix $\delta > \underline{\delta}$ for the convergence. If a seller s and a buyer b are not connected in G^*, they could not reach an agreement on the division of the surplus. Thus, in the equations system (3.4), $\max\{v_b - \delta u_s^{*\delta} - \delta u_b^{*\delta}, 0\} = 0$.

Since the buyers in M may have G^*-links only with the sellers in L, the buyer b's expected payoff equation in (3.4) can be written as follows:

$$u_b^{*\delta} = \frac{1}{1-\delta} \sum_{\{s|(s,b)\in G, s\in L\}} \frac{p_{sb}}{2} \max\{v_b - \delta u_s^{*\delta} - \delta u_b^{*\delta}, 0\}. \tag{3.5}$$

For all sellers $s \in L$, applying similar arguments that are used in equation (3.5), we obtain

$$u_s^{*\delta} \geq \frac{1}{1-\delta} \sum_{\{b|(s,b)\in G, b\in M\}} \frac{p_{sb}}{2} \max\{v_b - \delta u_s^{*\delta} - \delta u_b^{*\delta}, 0\}. \tag{3.6}$$

By taking the summation of (3.5) over all $b \in M$ and taking the summation of (3.6) over all $s \in L$, we have

$$\sum_{b\in M} u_b^{*\delta} = \frac{1}{1-\delta} \sum_{\{(s,b)\in G|b\in M, s\in L\}} \frac{p_{sb}}{2} \max\{v_b - \delta u_s^{*\delta} - \delta u_b^{*\delta}, 0\} \tag{3.7}$$

and

$$\sum_{s\in L} u_s^{*\delta} \geq \frac{1}{1-\delta} \sum_{\{(s,b)\in G|b\in M, i\in L\}} \frac{p_{sb}}{2} \max\{v_b - \delta u_s^{*\delta} - \delta u_b^{*\delta}, 0\}. \tag{3.8}$$

The right hand sides of the summations are the same in both (3.7) and (3.8). Then, we have the following inequality

$$\sum_{s\in L} u_s^{*\delta} \geq \sum_{b\in M} u_b^{*\delta}.$$

When players become more patient, as $\delta \to 1$,

$$\sum_{s\in L} u_s^{*} \geq \sum_{b\in M} u_b^{*}.$$

By using the following facts that for all $k \in L$, $\max\limits_{s\in L} u_s^* \geq u_k^*$ and for all $l \in M$, $\min\limits_{b\in M} u_b^* \leq u_l^*$, we obtain

$$|L| \max_{s\in L} u_s^* \geq |M| \min_{b\in M} u_b^*.$$

Utilizing Lemma 10, for all $b \in M$, there exists $s \in L$ such that $u_b^* = v_b - u_s^*$. Hence, for all $b \in M$, $u_b^* \geq v_b - \max_{s\in L} u_s^*$. Therefore we have

$$\begin{aligned}\sum_{b\in M} u_b^* &= \sum_{\{(s,b)|b\in M, s\in L^{G^*}(b)\}} (v_b - u_s^*) \\ &\geq \sum_{\{(s,b)|b\in M, s\in L^{G^*}(b)\}} (v_b - \max_{s\in L} u_s^*) \\ &= \sum_{b\in M} v_b - |M| \max_{s\in L} u_s^*,\end{aligned}$$

which is equivalent to

$$\max_{s\in L} u_s^* \geq \frac{\sum\limits_{b\in M} v_b}{|L|+|M|}.$$

Similarly, utilizing Lemma 10 for the summation of the sellers' payoffs in L, we obtain

$$\sum_{s\in L} u_s^* \leq \sum_{b\in M} v_b - |L| \min_{b\in M} u_b^*,$$

which is equivalent to

$$\min_{b\in M} u_b^* \leq \frac{\sum\limits_{b\in M} v_b}{|L|+|M|}.$$

□

Since the bargaining power depends on the player's position in the network, we need to identify where each player is located in the network. Regarding this aim, we modify the network decomposition algorithm of Manea (2011). We use the outcome generated by this algorithm to identify the limit equilibrium payoffs of the players.

Network Decomposition Algorithm, $\mathcal{A}(G)$: For a given network $G \in \Omega$, the algorithm generates the sequence $(r^t, M^t, L^t, N^t, G^t)_t$ as follows:

Let $N^0 = B \cup S$ and $G^0 = G$.

For $t \geq 0$:

If $N^t = \emptyset$, then STOP.

If not,

$$r^t = \max_{M \subset N^t \cap B} \frac{\sum_{b \in M} v_b}{|L^{G^t}(M)| + |M|}.$$

Set M^t be union of all maximizer sets M. Denote $L^t = L^{G^t}(M^t)$.

If $N^t = M^t \cup L^t$, then STOP.

Otherwise, let $N^{t+1} = N^t \setminus (M^t \cup L^t)$ and G^{t+1} be the subnetwork of G induced by the players in N^{t+1}. Denote the step at which the algorithm ends by $\bar{t}$.

The algorithm initially takes the given network. At each step, it identifies the sets that maximizes the surplus per player in the subnetwork, r^t. It picks the union of these maximizer sets and the partner set of this union and they are removed from the network. If the algorithm picks all the players in the current network, it ends. Otherwise, in the next step, the same procedure is applied to the subnetwork induced by the remaining players. Intuitively, the algorithm decompose a given network into oligopoly subnetworks. The limit equilibrium payoffs can be described utilizing the outcome of the decomposition algorithm, $\mathcal{A}$ and they are given by the following theorem.

Theorem 11. *Let the algorithm $\mathcal{A}(G)$ yields the outcome $(r^t, M^t, L^t, N^t, G^t)_{t=0,1,\ldots,\bar{t}}$. Then the limit equilibrium payoffs as $\delta \to 1$ are given by*

$$\forall t \leq \bar{t},\, \forall s \in L^t,\ u_s^* = \frac{\sum\limits_{b \in M^t} v_b}{|L^t| + |M^t|}$$
$$\forall t \leq \bar{t},\, \forall b \in M^t,\ u_b^* = v_b - \frac{\sum\limits_{b \in M^t} v_b}{|L^t| + |M^t|}$$

Proof of Theorem 11. The proof of the theorem proceeds by induction on t. Suppose that the claim is hold for all $t' < t$. Now, we prove it for t.

Let M^t and L^t be the sets that the algorithm $\mathcal{A}(G)$ generates at step t. Define the maximum limit equilibrium payoff as $\bar{x}^t = \max\limits_{k \in N^t \cap S} u_k^*$. Further, define the sets

$$\bar{L}^t = \{s \in N^t | u_s^* = \bar{x}^t\} \text{ and } \bar{M}^t = \{b \in L^{G^t}(\bar{L}^t) | u_b^* = v_b - \bar{x}^t\}.$$

$\bar{L}^t$ is the set of seller who have the maximum limit equilibrium payoffs in N^t and $\bar{M}^t$ is the set of buyers who have G^*-links with the players in $\bar{L}^t$.

Claim 1. $\bar{x}^t \geq \dfrac{\sum\limits_{k \in M^t} v_k}{|L^t| + |M^t|}$

For a contradiction, suppose that

$$\bar{x}^t < \frac{\sum\limits_{k \in M^t} v_k}{|L^t| + |M^t|}.$$

First of all, we explore the set of players with whom the players in M^t have G^*-links. Take any player $s \in L^{t'}$ where $t' \in \{1, 2, \ldots, t-1\}$. By induction hypothesis,

$$u_s^* = \frac{\sum\limits_{k \in M^{t'}} v_k}{|L^{t'}| + |M^{t'}|}$$

Summing up the limit equilibrium payoffs of s and any buyer $b \in M^t$, we have

$$\begin{aligned}
u_s^* + u_b^* &\geq \frac{\sum\limits_{k \in M^{t'}} v_k}{|L^{t'}| + |M^{t'}|} + v_b - \bar{x}_t \\
&> \frac{\sum\limits_{k \in M^t} v_k}{|L^t| + |M^t|} + v_b - \bar{x}_t \\
&> \bar{x}_t + v_b - \bar{x}_t = v_b.
\end{aligned}$$

Second inequality follows from the construction of the decomposition algorithm and the third one from our supposition. So, no player $b \in M^t$ has a G^*-link with players $s \in L^1 \cup L^2 \cup \ldots \cup L^{t-1}$. Since G is a two-sided network, M^t is a G^*-independent set. Hence, $L^{G^*}(M^t) \subseteq L^t$. Utilizing Theorem 10, we get

$$\max_{s\in L^t} u_s^* \geq \frac{\sum\limits_{k\in M^t} v_k}{|L^t| + |M^t|}.$$

By the definition of $\bar{x}^t$,

$$\bar{x}^t \geq \frac{\sum\limits_{k\in M^t} v_k}{|L^t| + |M^t|},$$

which contradicts with our supposition.

Claim 2. $\bar{x}^t \leq \frac{\sum\limits_{k\in M^t} v_k}{|L^t| + |M^t|}$ and for all $b \in \bar{M}_t$, $u_b^* = v_b - \bar{x}_t$

Take any seller $s \in \bar{L}^t$. For all buyers $b \in B \setminus \bar{M}^t$, $u_b^* + u_s^* > \bar{x}_t + v_b - \bar{x}_t = v_k$. Hence, there exists no G^*-links between s and any buyer $b \in B \setminus \bar{M}^t$. Since we deal with two-sided supply chain networks, it is clear that $\bar{L}^t$ is a G^* independent set. The network decomposition algorithm implies that there exists no G-link between s and any buyer $b \in M^{t'}$, where $t' \in \{1, 2, \ldots, t-1\}$. Therefore, s has G^*-links only with buyers in $\bar{M}^t$, i.e., $L^{G^*}(\bar{L}^t) = \bar{M}^t$. By Theorem 10,

$$\bar{x}_t = \min_{s\in \bar{L}^t} u_s^* \leq \frac{\sum\limits_{k\in L^{G^*}(\bar{L}^t)} v_k}{|L^{G^*}(\bar{L}^t)| + |\bar{L}^t|}.$$

Since M^t maximizes the surplus per player in the subnetwork, we have the following

$$\frac{\sum\limits_{k\in L^{G^*}(\bar{L}^t)} v_k}{|L^{G^*}(\bar{L}^t)| + |\bar{L}^t|} \leq \frac{\sum\limits_{k\in M^t} v_k}{|L^t| + |M^t|},$$

implying that

$$\bar{x}_t \leq \frac{\sum\limits_{b \in M^t} v_k}{|L^t| + |M^t|}.$$

This concludes the proof of our claim. By Claim 1 and Claim 2, we get for all $s \in \bar{L}^t$,

$$\bar{x}^t = u_s^* = \frac{\sum\limits_{k \in M^t} v_k}{|L^t| + |M^t|}$$

and for all $b \in \bar{M}^t$, $u_b^* = v_b - \bar{x}_t$.

Claim 3. $\bar{M}^t = M^t$

Since the network decomposition algorithm picks the union of all maximizer sets M, if $\bar{M}^t$ is the maximizer, then $\bar{M}^t \subseteq M^t$. Hence, $\bar{L}^t = L^{G^t}(\bar{M}^t) \subseteq L^{G^t}(M^t) = L^t$.

For the other side of the equation, suppose for contradiction $M^t \not\subseteq \bar{M}^t$. Then, there exits a player $b \in M^t \backslash \bar{M}^t$. Note that b has no G^*-links with players in $N^t \backslash L^t$. Moreover, by Lemma 10, b has no G-links with players in $L^1 \cup L^2 \cup \ldots \cup L^{t-1} \cup \bar{L}^t$. Hence, the buyer b has G^*-links only with players in $L^t \setminus \bar{L}^t$. Utilizing Theorem 10, we have

$$\max_{s \in L^{G^*}(M^t \backslash \bar{M}^t)} u_s^* \geq \frac{\sum\limits_{k \in M^t \backslash \bar{M}^t} v_k}{|M^t \setminus \bar{M}^t| + |L^t \setminus \bar{L}^t|} = \frac{\sum\limits_{k \in M^t} v_k - \sum\limits_{k \in \bar{M}^t} v_k}{|M^t| + |L^t| - (|\bar{M}^t| + |\bar{L}^t|)}.$$

Note that $\sum_{k \in M^t} v_k/(|L^t| + |M^t|) = \sum_{k \in \bar{M}^t} v_k/(|\bar{L}^t| + |\bar{M}^t|)$. Then, we obtain

$$\max_{s \in L^{G^*}(M^t \setminus \bar{M}^t)} u_s^* \geq \bar{x}_t,$$

which contradicts with $u_s^* < \bar{x}_t$ for all sellers $s \notin \bar{L}^t$.

Hence, $M^t = \bar{M}^t$ and $L^t = \bar{L}^t$. Claim 1 - Claim 3 conclude the proof for any step of the algorithm $t \leq \bar{t}$.

□

By Theorem 11, we obtain the limit equilibrium payoffs of the bargaining game over a given network G. We prove that in the limit equilibrium of the game, the payoffs of the players are determined according to the network structure and the valuations of the buyers for the good.

3.3 Conclusion

In this essay, we examine a bargaining game over a two-sided supply chain network where the sellers producing a homogeneous good and buyers with potentially different valuations for the good bargain over the corresponding surplus. Our model improves upon the existing supply chain literature in multiple dimensions. Consequently, we can investigate the impact of bargaining power due to the network structure and the valuation heterogeneity on the equilibrium market/bargaining outcome. In the current study, the bargaining game is similar to the one developed in Manea (2011). However, in our model, the size of the surplus divided between the players of a pair is not the same for all links which leads to strikingly different equilibrium predictions than that of Manea (2011). More precisely, we show that higher valuation for the good is also a source of bargaining power which can not be captured by Manea (2011).

CHAPTER 4

BARGAINING, REFERENCE POINTS, AND LIMITED INFLUENCE

A plethora of experimental studies in the last two decades almost unequivocally documented the influence of reference points on bargaining behavior and outcomes.[1] One well-known critique of the theories that utilize exogenously given reference points is that almost any sort of behavior can be explained with an appropriate choice of a reference point. This led many researchers to develop theoretical models where the reference point is endogenously derived (preferably from the observables) and/or elicit agents' reference points with experimental methods to check whether the elicited reference points, which were otherwise unobservable, can explain the observed behavior.[2]

[1]Ashenfelter and Bloom (1984), Blount et al. (1996),Kristensen and Gärling (2000), Bohnet and Zeckhauser (2004), Gächter and Riedl (2005, 2006) , Gimpel (2007), Bolton and Karagözoğlu (2016), Bartling and Schmidt (2015), Herweg and Schmidt (2015), Fehr et al. (2015), and Karagözoğlu and Riedl (2015) are only some of these studies, all of which reported that reference points —in the form of reservation prices, informal agreements, existing contracts, historical contractual conditions, expired contracts (fairness) norms— have a significant impact on the the negotiated agreement and on the whole bargaining process

[2]The reader is referred to Benartzi and Thaler (1995), Shalev (2000), Compte and Jehiel (2007), Falk and Knell (2004), Köszegi and Rabin (2006, 2007, 2009), Gimpel (2007), Li (2007), Vartiainen (2007), Abeler et al. (2011), Baucells et al. (2011), Giorgi and Post (2011), Hyndman (2011), Driesen et al. (2012), Sarver (2012), and Roels and Su (2014) among others.

This essay models the emergence of reference points and investigate their influence on bargaining behavior and outcomes in a two-player, infinite horizon, alternating offers bargaining game (Stahl, 1972; Rubinstein, 1982), where players' preferences exhibit reference dependence. We allow past offers in an alternating offers bargaining game to influence players' reference points in later periods (see Shalev, 2002; Driesen et al., 2012 for earlier examples). Players in our model are both gain-seeking and loss-averse. Accordingly, player i ($i = 1, 2$) weights payoffs above his reference point with γ_i and payoffs below his reference point with λ_i, where we assume $\lambda_i \geq \gamma_i$. A novel element we introduce is *limited influence*: an offer made to player i in the current period has the potential to influence his reference points for the next m_i periods in which he responds to player j's offers. In particular, player i's reference point at period $t > 1$ according to which he evaluates player j's current offer, is assumed to be the highest (or the most generous) offer he received in the last m_i periods (in which he received offers). Therefore, in contrast with Driesen et al. (2012), which assumed that player i's reference point at any given period $t > 1$ is the highest offer he received until t, we model those bargaining situations where past offers can have only a *limited influence* on the current reference points. In other words, the influence of past offers *expire* in finitely many periods in our baseline model. Our model is inspired by the *availability* heuristic or *retrievability* bias in decisionmaking (see Kahneman and Tversky, 1974) and the *order effect* (or the *recency effect*) in belief updating and intertemporal decisionmaking (see Hogarth and Einhorn, 1992). There is a strong empirical evidence for these heuristics and biases (see Bartos, 1964; DeBondt and Thaler, 1990; Hogarth and Einhorn, 1992; Grether, 1992; Holt and Smith, 2009;

Malmendier and Nagel, 2016 among many others).

We show that for any exogenously given initial reference point there exists a subgame perfect equilibrium of the game, which induces an immediate agreement. A closer look at this equilibrium reveals that despite the immediate agreement result, *expiration lengths* influence equilibrium behavior. More precisely, the influence of expiration lengths (i.e., m_1 and m_2) are concealed in reference points since m_1 and m_2 determine the corresponding sets of past offers from which the current reference points emerge, and rational players incorporate the information from the continuation game to their actions in the first-period.

We compare the equilibria of the game with limited influence and the game with unlimited influence (i.e., $m_1 = m_2 = \infty$). Our comparison shows that equilibrium outcomes are identical (due to the immediate agreement result) whereas equilibrium strategies are different.

The organization of the essay is as follows: Section 1 introduces the model. Section 2 and its subsections present results from the model with limited influence and the model with unlimited influence, compare the results from the two models, and provide a comparison with Driesen et al. (2012), as well. Finally, Section 3 concludes.

4.1 The Model

We consider an infinite horizon bargaining model in which two players, player 1 and 2, bargain over the division of a pie of a unit size, following an alternating-offers bargaining protocol. More precisely, at odd periods $t = 1, 3, 5, \ldots$, player 1 makes an offer $z = (z_1, z_2)$, where $z_1 + z_2 = 1$ and player 2 decides whether to accept (a) or reject (r) the offer. Similarly, at even periods $t = 2, 4, 6, \ldots$, player 2 makes an offer and player 1 decides whether to accept (a) or reject it (r). If an offer $z = (z_1, z_2)$ is accepted, the game ends with players receiving their corresponding agreed shares. At any period t, if an offer is rejected, then with probability $\delta \in (0, 1)$ the game continues to period $t + 1$ and with probability $1 - \delta$ (i.e, the break-down probability) the game ends. If and when the game ends as a result of a break-down, players do not receive any share from the pie (i.e., the shares of both players are equal to 0). The set of all possible (efficient) offers is denoted by

$$Z = \{(z_1, z_2) \in R^2_+ | z_1 + z_2 = 1\}.$$

For each player $i \in \{1, 2\}$ a strategy $\sigma_i = (\sigma_i^t)_{t=1}^{\infty}$ is a sequence of functions where σ_i^t maps any history up to period t to an offer or a response (i.e., a or r) depending on whose turn it is to make an offer at period t.

We use a framework similar to the ones developed in Shalev (2002) and Driesen et al. (2012) to study the influence of past actions on current decisions through their influence on players' reference points. In particular, player i's current reference point according to which he evaluates player j's current offer is determined by player j's past offers (and the exogenously given initial reference point). However, in contrast with Driesen et al. (2012), the reference point of player i in our model is *not necessarily* the highest offer he received up to the period he has to take an action; instead it is the highest offer he received in the last m_i periods (in which he received offers), where m_i is finite. In other words, the influence of past offers on players' reference points expires in finite periods (after a certain number of periods, a past offer gets simply *too old* to constitute a reference point). For any $i \in \{1, 2\}$ and $t \in \mathbb{N}$, we will use r_i^t to denote the reference point of player i at period t. Thus, after a sequence of offers $(z^s)_{s=1}^{t-1}$ the reference point of the agents at period t are (with the convention $z^0 = (z_1^0, z_2^0) = (0, 0)$):

- If t is odd

$$
\begin{aligned}
r_1^t &= \max\{z_1^s | s \in \{t-1, t-3, \cdots, t-(2m_1-3), t-(2m_1-1)\} \cap \mathbb{Z}_+\} \\
r_2^t &= r_2^{t-1}
\end{aligned}
$$

- If t is even

$$
\begin{aligned}
r_1^t &= r_1^{t-1} \\
r_2^t &= \max\{z_2^s | s \in \{t-1, t-3, \cdots, t-(2m_2-3), t-(2m_2-1)\} \cap \mathbb{Z}_+\}
\end{aligned}
$$

Suppose that at period $t+1$, it is player i's turn to make an offer and he offers z. If player j rejects the offer, his reference point in period $t+2$ will be the maximum of the offers that were made in periods $t-(2m_j-3), t-(2m_j-5), \ldots, t-3, t-1$ and the last offer z_j, since the influence of offer at the period $t-(2m_j-1)$ *expires.*

The following definitions will provide us some convenience in the following discussions. For odd periods t, let

$$\bar{r}_1^t = \max\left\{z_1^s | s \in \{t-1, t-3, \cdots, t-(2m_1-3)\} \cap \mathbb{Z}_+\right\},$$

and for even periods t,

$$\bar{r}_2^t = \left\{z_2^s | s \in \{t-1, t-3, \cdots, t-(2m_2-3)\} \cap \mathbb{Z}_+\right\},$$

Thus $\bar{r}_i^t$ is the maximum of the last $m_i - 1$ offers (or t offers if $m_i - 1 > t$) that the agent has received).

Hence, in the case of a rejection, the reference point of player i in the next period, after he received an offer z, will be $\max\{\bar{r}_i^t, z_i\}$. We assume that player i evaluates the offer he receives at period t, z_i^t, according to his current reference point, r_i^t.

We employ the functional form in Köszegi and Rabin (2006) to incorporate reference-dependent preferences. More precisely, offers below the reference point are regarded as *losses*, whereas the offers above the reference point are regarded as *gains*. Accordingly, at period t, the utility of player i from the realization of z^t is given as follows (assuming that the current reference point is r_i^t):

$$u_i^t(z, r^t) = \begin{cases} z_i^t + \gamma_i(z_i^t - r_i^t) & \text{if } z_i^t \geq r_i^t \\ z_i^t + \lambda_i(z_i^t - r_i^t) & \text{if } z_i^t < r_i^t \end{cases},$$

where $\lambda_i \geq \gamma_i \geq 0$. The first term, z_i^t, is the intrinsic *consumption utility*, which can be considered as the benefit player i obtains from consuming his share of the pie. The second term (i.e., $\lambda_i(z_i^t - r_i^t)$ or $\gamma_i(z_i^t - r_i^t)$) represents *gain-loss utility.*

λ_i is the loss-aversion coefficient, whereas γ_i is the gain-seekingness coefficient. By taking $\lambda_i \geq \gamma_i$ we are assuming that players are more sensitive to losses than to gains, capturing the main idea in Kahneman and Tversky (1979, 1991) . Note that the functional forms used in Shalev (2002) and Driesen et al. (2012) are special cases (i.e., $\gamma_i = 0$) of the functional form that we employ. We denote the game described above by Γ.

4.2 Results

In this section, first we focus on the model with a limited influence (i.e., finite expiration lengths). Later, we analyze a variation of our baseline model, with an unlimited influence (i.e., infinite expiration lengths).

4.2.1 Limited Influence

Throughout this study, equilibrium means subgame perfect equilibrium. Accordingly, in equilibrium, an offer should make the responder indifferent between the current offer and his expected utility from his own offer in the next period.[3] Now, consider an odd period t in which player 1 makes the offer $x \in Z$. Suppose that, if player 2 rejects the offer x in period t, then he will propose $y \in Z$ in period $t+1$.

[3]In the bargaining game Γ, (i) the pie is desirable, (ii) disagreement is the worst outcome, (iii) $u_i^t(x, r) > u_i^{t+1}(x, r)$ for each t, x and r and (iv) u_i is continuous. Moreover, the game is stationary in the sense that player i's preference between his share from the division x at period t and his share from the division y at period $t+1$ is independent of t when t is the period that player i's turn to make an offer. Note that the reference point of player i at period $t+1$ is equal to his reference point at period t, i.e., $r_i^{t+1} = r_i^t$. These properties of Γ allow us to use the expected payoff at period $t+1$ as the continuation payoff of the game (see Osborne and Rubinstein, 1990 pp. 73)

Let $r_2 = r_2^t$ and $\bar{r}_2 = \bar{r}_2^t$. Note that in the case of a rejection, the reference point of player 2 in period $t+1$ will be $r_2^{t+1} = \max\{\bar{r}_2, x_2\}$. For player 2 to be indifferent between accepting the offer x made in period t and rejecting this offer and making the offer y in the next period, which is assumed to be accepted by player 1, we need

$$u_2^t(x, r^t) = \delta u_2^{t+1}(y, r^{t+1}) + (1-\delta)u_2^{t+1}(0, r^{t+1}). \tag{4.1}$$

The left-hand side of the equality is the utility that player 2 gets if he accepts x, whereas the right-hand side is his (expected) continuation utility (i.e., with probability δ the game continues to the next period and player 2 offers y which is assumed to be accepted by player 1 or with probability $1-\delta$ the game ends and player 2 gets zero). Similarly, consider an even period t in which player 2 makes the offer $y \in Z$. Suppose that, if player 1 rejects this offer, then he will propose $x \in Z$ in period $t+1$. Let $r_1 = r_1^t$ and $\bar{r}_1 = \bar{r}_1^t$. Note that in the case of a rejection, the reference point of player 1 in period $t+1$ will be $r_1^{t+1} = \max\{\bar{r}_1, y_1\}$. For player 1 to be indifferent between accepting the offer y made in period t and rejecting this offer and making the offer x in the next period, which is assumed to be accepted by player 2, we need

$$u_1^t(y, r^t) = \delta u_1^{t+1}(x, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}). \tag{4.2}$$

In the rest of the study, we assume that a player does not make an offer that gives him a share less than the offer that he previously rejected, i.e., we assume $x_1 > y_1$.[4]

[4]See Fershtman and Seidmann (1993) for a similar modeling assumption. Note that our

Under this assumption, depending on the reference points, equation (4.2) yields one of the following cases:

1. $r_1 > \bar{r}_1 > y_1$

   a. $x_1 \geq \bar{r}_1 : (1+\gamma_1)\delta x_1 = (1+\lambda_1)y_1 - \lambda_1 r_1 + (1-\delta)\lambda_1\bar{r}_1 + \delta\gamma_1\bar{r}_1$

   b. $x_1 < \bar{r}_1 : (1+\lambda_1)\delta x_1 = (1+\lambda_1)y_1 - \lambda_1 r_1 + \lambda_1\bar{r}_1$

2. $r_1 > y_1 > \bar{r}_1 : (1+\gamma_1)\delta x_1 = (1+\lambda_1+\delta\gamma_1+(1-\delta)\lambda_1)y_1 - \lambda_1 r_1.$

3. $y_1 > r_1 > \bar{r}_1 : (1+\gamma_1)\delta x_1 = (1+\gamma_1+\delta\gamma_1+(1-\delta)\lambda_1)y_1 - \gamma_1 r_1.$

Similarly, for equation (4.1), we have:

I. $r_2 > \bar{r}_2 > x_2$

   a. $y_2 \geq \bar{r}_2 : (1+\gamma_2)\delta y_2 = (1+\lambda_2)x_2 - \lambda_2 r_2 + (1-\delta)\lambda_2\bar{r}_2 + \delta\gamma_2\bar{r}_2$

   b. $y_2 < \bar{r}_2 : (1+\lambda_2)\delta y_2 = (1+\lambda_2)x_2 - \lambda_2 r_2 + \lambda_2\bar{r}_2$

II. $r_2 > x_2 > \bar{r}_2 : (1+\gamma_2)\delta y_2 = (1+\lambda_2+\delta\gamma_2+(1-\delta)\lambda_2)x_2 - \lambda_2 r_2.$

III. $x_2 > r_2 > \bar{r}_2 : (1+\gamma_2)\delta y_2 = (1+\gamma_2+\delta\gamma_2+(1-\delta)\lambda_2)x_2 - \gamma_2 r_2.$

Considering the cases above together, we obtain 16 possible regions for the reference point $(r_1, r_2) \in [0,1]^2$. We will denote these mutually exclusive regions by $R_{1.a-I.a}, R_{1-I.b}, \ldots, R_{3-II}, R_{3-III}$. Let x^ω and y^ω be the offers associated with the corresponding region R_ω, where $\omega \in \{1.a-I.a, 1.a-I.b, 1.a-II, \ldots, 3-III\}$ for player 1 and player 2, respectively. The following theorem describes a subgame perfect equilibrium of the game.

assumption is weaker than their *endogenous commitment* assumption.

Theorem 12. *Take any period $t \geq 1$. Let the reference point be $(r_1^t, r_2^t) \in R_\omega$ and x^ω, y^ω be the offers associated with the corresponding region R_ω, where $\omega \in \{1.a - I.a, 1.a - I.b, 1.a - II, \ldots, 3 - III\}$. For player 1, let σ_1^* be such that if t is odd, player 1 makes the offer x^ω and if t is even, player 1 accepts the offer z if and only if $z_1 \geq y_1^\omega$. For player 2, define the strategy σ_2^* in a similar way. The strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*)$ is a subgame perfect equilibrium of the bargaining game Γ.*

This subgame perfect equilibrium induces an immediate agreement. At any period t, the equilibrium strategy of player i associated with the region R_ω, which directly depends on the reference points of both players, for any $\omega \in \{1.a - I.a, 1.a - I.b, 1.a - II, \ldots, 3 - III\}$, implying indirect dependence on expiration lengths, m_i and m_j.

Before moving to the proof, we first recall the Corollary of Theorem in Hendon et al. (1996), which we will employ in our proof: *One-deviation principle* holds in infinite horizon extensive-form games, which are continuous at infinity.

Definition 3. (Continuity at infinity) A game is continuous at infinity if for any player i and for any $\varepsilon > 0$, there exists a period $\bar{t}$ such that if two strategy profiles σ and σ' satisfy for all $s \leq \bar{t}$, $\sigma^s = \sigma'^s$, then $|U_i(\sigma) - U_i(\sigma')| < \varepsilon$, where $U_i(\sigma)$ is the sum of the discounted utilities accrued at each period in strategy profile σ.

Lemma 11. *The bargaining game Γ is continuous at infinity.*

For the proof of this lemma, see the Appendix.

Proof of Theorem 12. The proof is relegated to the Appendix. □

Equilibrium Outcome: Theorem 12 states that players follow the strategy profile (σ_1^*, σ_2^*) in the equilibrium described:a player makes the offer x^ω or y^ω based on the relevant region R_ω for the reference point, and the agreement is reached immediately. This implies that at $t = 1$, player 1 makes the offer x^ω, where $(r_1^1, r_2^1) \in R_\omega$; and player 2 accepts the offer. For instance, if the initial reference point satisfies $(r_1^1, r_2^1) \in R_{3,III}$, then the equilibrium outcome of the bargaining game $x = (x_1, x_2)$ is given by

$$
\begin{aligned}
x_1 &= \frac{\eta_1(\eta_2 - \gamma_2 r_2^1) - \delta(1+\gamma_2)(\eta_1 - \gamma_1 r_1^1)}{\eta_1\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)} \\
x_2 &= \frac{\eta_1\gamma_2 r_2^1 + \delta(1+\gamma_2)(\eta_1 - \gamma_1 r_1^1) - \delta^2(1+\gamma_1)(1+\gamma_2)}{\eta_1\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}
\end{aligned}
$$

where $\eta_i = (1 + \gamma_i + \delta\gamma_i + (1-\delta)\lambda_i)$.

The division of the pie in the equilibrium depends directly on initial reference points in this case. It implicitly depends on expiration lengths, since they are decisive in the evolution of the reference points. Equilibrium outcomes are given in the Appendix for all regions R_ω, where $\omega \in \{1.a{-}I.a, 1.a{-}I.b, 1.a{-}II, \ldots, 3{-}III\}$.

4.2.2 No Expiration (Unlimited Influence)

In this section, we remove the bounds on the number of periods an offer can influence future reference points. Hence, in any period $t > 1$, the reference points at period t are defined on the basis of the most generous offers players received up to t,

$$\begin{aligned} r_1^t &= \max\{z_1^s | s = 0, 2, 4, 6, \ldots \leq t\} \\ r_2^t &= \max\{z_2^s | s = 0, 1, 3, 5, \ldots \leq t\} \end{aligned}$$

where z^s is the offer made at period s and $z_1^0 = z_2^0 = 0$.

Suppose that at period t it is player i's turn to make an offer; and he offers x. If player j rejects the offer x, his reference point period $t+1$ will be $r_j^{t+1} = \max\{r_j, x_j\}$.

Considering equation (4.2) for the bargaining game with no expiration, we have the following three cases:

1. $r_1 > x_1 > y_1 : \delta x_1 = y_1$.
2. $x_1 \geq r_1 > y_1 : (1+\gamma_1)\delta x_1 = (1+\lambda_1)y_1 + \delta\gamma_1 r_1 - \delta\lambda_1 r_1$.
3. $x_1 > y_1 \geq r_1 : (1+\gamma_1)\delta x_1 = (1+\gamma_1+\delta\gamma_1+(1-\delta)\lambda_1)y_1 - \gamma_1 r_1$.

Similarly, for equation (4.1), we have:

I. $r_2 > x_2 > y_2 : \delta y_2 = x_2$.

II. $x_2 \geq r_2 > y_2 : (1+\gamma_2)\delta y_2 = (1+\lambda_2)x_2 + \delta\gamma_2 r_2 - \delta\lambda_2 r_2$.

III. $x_2 > y_2 \geq r_2 : (1+\gamma_2)y_2 = (1+\gamma_2+\delta\gamma_2+(1-\delta)\lambda_2)y_2 - \gamma_2 r_2$.

It is again clear that $x_1 > y_1$. Let x^ω and y^ω be the associated offers with the regions R_ω where $\omega \in \{1-I, 1-II, \ldots, 3-III\}$ for player 1 and player 2, respectively. The following theorem describes a subgame perfect equilibrium of the corresponding game.

Theorem 13. *Take any period* $t \geq 1$. *Let* $(r_1^t, r_2^t) \in R_\omega$ *where* $\omega \in \{1-I, 1-II, \ldots$

$, 3-III\}$. *For player* 1, *let* σ_1^* *be such that if* t *is odd, player* 1 *makes the offer* x^ω *and if* t *is even, player* 1 *accepts the offer* z *if and only if* $z_1 \geq y_1^\omega$. *For player* 2, *define the strategy* σ_2^* *in a similar way. The strategy profile* $\sigma^* = (\sigma_1^*, \sigma_2^*)$ *is a subgame perfect equilibrium of the bargaining game* Γ.

This game, too, has a subgame perfect equilibrium that induces an immediate acceptance. At any period $t \geq 1$, the equilibrium strategy of player i associated with the region R_ω directly depends on reference points of both players for any $\omega \in \{1-I, 1-II, \ldots, 3-III\}$, namely the highest offer he received and the highest offer he made to player j.

Proof of Theorem 13. The proof of Theorem 13 is similar to the that of Theorem 12 and it is relegated to the Appendix. □

Equilibrium outcome: Suppose that the initial reference point satisfies $(r_1^1, r_2^1) \in R_{3-III}$. Subgame perfect equilibrium outcome of the bargaining game with unlimited influence has the same formulation as that of limited influence.

$$x_1 = \frac{\eta_1(\eta_2 - \gamma_2 r_2^1) - \delta(1+\gamma_2)(\eta_1 - \gamma_1 r_1^1)}{\eta_1\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}$$

$$x_2 = \frac{\eta_1\gamma_2 r_2^1 + \delta(1+\gamma_2)(\eta_1 - \gamma_1 r_1^1) - \delta^2(1+\gamma_1)(1+\gamma_2)}{\eta_1\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}$$

where $\eta_i = (1 + \gamma_i + \delta\gamma_i + (1 - \delta)\lambda_i)$.

The equilibrium outcomes (in different regions) of the bargaining game with unlimited influence are the same with the equilibrium outcomes of the bargaining game with limited influence. This result possibly stems from the equilibrium immediate agreement in combination with the backward focus of the reference point. However, there are still some differences: the strategies are different for two variations of the model since these models differ in the evolution of reference points. Suppose the game is at period $t > m_i$. At this period, in the bargaining game with unlimited influence, the first offer player i received has still an influence on his reference point (and so on his actions) while in the bargaining game with limited influence the impact of the first offer on his reference point expires.

A comparison with Driesen et al. (2012) is, naturally, in place. First of all, naturally, Driesen et al. (2012) does not model expiration length. In our bargaining game with an unlimited influence, reference points evolve as in Driesen et al. (2012), i.e. they are the maxima of the rejected offers. However, reference points that appear in equilibrium strategies are different. Another point that we differ from Driesen et al. (2012) is the players' evaluation of the offers above the reference point. As we mentioned above, the utility function they employ is a special case of ours; for each player i, $\gamma_i = 0$. For low values of reference points, the equilibrium offers in Driesen et al. (2012) are not affected by them since the gain relevant to the reference point does not have any impact on the utility. However, in our model the offers in the equilibrium depend on the reference points, even for low values of those. Finally, Driesen et al. (2012) restrict their attention to the case where the initial reference point is $(0,0)$, whereas we do not impose such a restriction. Arguably, as a consequence of these differences, they can prove the uniqueness of equilibrium, whereas we cannot.

4.3 Concluding Remarks

We studied an infinite horizon, alternating offers bargaining game with endogenous reference points. In our model, (i) players have reference-dependent preferences, (ii) the initial reference point is exogenously given, (iii) but once the bargaining starts the current reference point of player i depends on the most recent m_i offers player j made. To the best of our knowledge, this is the first study that incorporates behavioral phenomena such as the recency effect/retrieveability bias into bargaining model with endogenous reference points. We showed that (i) there exists a subgame perfect equilibrium with an immediate agreement, (ii) expiration lengths influence players' strategies, (iii) but not their payoffs in this equilibrium. Whether there exists other subgame perfect equilibria or not is far from trivial and left as an open question. Future work on the topic may also study similar bargaining games where players' expiration lengths are not known with certainty or incorporate other types of cognitive biases related to recall.

CHAPTER 5

BETWEEN ANCHORS AND ASPIRATIONS: A NEW FAMILY OF BARGAINING SOLUTIONS

What provides a bargaining advantage? Nash (1950) proposed that what an agent would get in the case of a disagreement may constitute a source of bargaining power. On the other hand, Gupta and Livne (1988) argued that reference points in the form of existing or expired contracts, precedents, negotiation text, or norms may also provide bargaining power.[1] Most cooperative bargaining models employ the disagreement point (e.g., Nash (1950); Kalai and Smorodinsky (1975); Kalai (1977); Chun and Thomson (1992); Balakrishnan et al. (2011)) or the reference point (e.g., Brito et al. (1977); Gupta and Livne (1988)) as an *anchor* that influences the negotiated agreements. An important difference between the bargaining advantages provided by these two sources is worth mentioning here: the former can be exercised unilaterally (i.e., an agent would not need the opponent's per-

[1] A plethora of experimental studies provided evidence supporting these arguments. Among others, the reader is referred to Gächter and Riedl (2005, 2006), Bolton and Karagözoğlu (2016), Herweg and Schmidt (2013), Irlenbusch et al. (2017), Anbarcı and Feltovich (2013, 2018), and Bartling and Schmidt (2015).

mission to *disagree*) whereas the latter needs to be —at least tacitly— mutually acknowledged (i.e., a reference point can be employed only if both parties find it sufficiently salient or reasonable). In that sense, the power provided by the disagreement point can be labeled as *hard* power, whereas the power provided by the reference point can be labeled as *soft* power (see Bolton and Karagözoğlu (2016)). At first look, it appears that hard power should dominate soft power, when push comes to shove. But what if there is a well-established norm that implies a salient reference point? In other words, could it be that the effectiveness of hard power depends on the salience of the source of soft power?

The way disagreement and reference points are incorporated into most cooperative bargaining models implies that in addition to their direct influence mentioned above, they may also have an indirect influence on the negotiated agreement through their influence on agents' *aspirations*. Aspirations can be interpreted as agents' expectations on the best case scenario in negotiations. In other words, they provide answers to the question, "What is the most favorable outcome I can get out of this negotiation?". Kalai and Smorodinsky (1975) is among the first to argue that aspirations can influence agreements.[2] In that study, agents' aspirations are directly derived from the disagreement point (and the utility possibility frontier). In accordance with that, the ideal point which is introduced by Kalai and Smorodinsky (1975) is defined as the the maximum attainable utility level by the players in an individually rational agreement. The Kalai-Smorodinsky solution employs the disagreement point (as an anchor point) and the ideal point (as an aspiration point) in proposing a settlement. On the other hand, the solution

[2]Some other studies that have similar arguments are Raiffa (1953) and Rosenthal (1976).

concept introduced by Gupta and Livne (1988) employs the reference point (as an anchor point) and the ideal point (as an aspiration point). Their model describes a bargaining situation where the salience of the reference point "increases the likelihood that no party exercises its threat to break off" (see Gupta and Livne, 1988, p. 1304). Recently, Balakrishnan et al. (2011) introduced a new salient point into bargaining problems: the tempered aspirations point. Such aspirations are derived from the reference point instead of the disagreement point; and their solution —called the tempered aspirations solution— employs the disagreement point (as an anchor point) and the tempered aspirations point (as an aspiration point). Their model describes a bargaining situation where "the salience of the reference point mutes or tempers the negotiators' aspirations" (see Balakrishnan et al., 2011, p. 144).[3]

One can argue that anchor points describe what would happen in the worst case scenario, whereas aspiration points describe what would happen in the best case scenario; and that there are multiple candidates for both types of salient points which may lead to entirely different descriptions. What is common to all three solution concepts mentioned above is that each proposes a settlement as a feasible compromise between the worst case and the best case scenarios. A natural question is: In modeling a simple bargaining situation, why/when should the reference point be preferred over the disagreement point as an anchor point (as in the Kalai-Smorodinsky solution or the tempered aspirations solution), or vice versa (as in the Gupta-Livne solution)? Similarly, why/when should the ideal

[3]The Kalai-Smorodinsky solution proposes the maximum point of the bargaining set on the line segment connecting the ideal point and the disagreement point. The Gupta-Livne (tempered aspirations) solution proposes the maximum point of the bargaining set on the line segment connecting the ideal point (tempered aspirations point) and the reference point (disagreement point).

point be preferred over the tempered aspirations point as an aspiration point (as in the Kalai-Smorodinsky solution or the Gupta-Livne solution), or vice versa (as in the tempered aspirations solution)? As Balakrishnan et al. (2011) correctly pointed out "... the context of the bargain will affect the manner in which the reference point influences the negotiated outcome".

To put it more concretely, consider the following scenarios on wage bargaining: Imagine first that the only piece of information bargaining parties have is the payoffs they would receive if they cannot reach an agreement (Info 1). Now consider a variation where another piece of information is also available, which is the average wage level in the same industry last year (Info 2). In the presence of this new information, it is hard to believe that Info 1 will be as influential as before on the bargaining agreement. Finally, consider yet another variation where instead of Info 2, parties know the average wage level in a different industry five years ago (Info 2′). It is highly likely that Info 2′ will be less influential on the bargaining agreement in the latter variation than Info 2 is in the former variation. Generally speaking, contextual factors (e.g., similarity, temporality, connectivity) likely influence the salience of the reference point and hence its impact on the negotiated outcome (see Ashenfelter and Bloom (1984), Bazerman (1985), among others).

The influence of contextual factors on negotiations (and, in general, economic behavior) is a well-studied topic. In a seminal paper, Sebenius (1992) discussed the need to incorporate such factors into negotiation analysis. Crusius et al. (2012) argued, by referring to some well-known experimental findings, that the context has an important influence on behavioral mechanisms and the effects of economic parameters. Among studies investigating the effect of anchors on behavior, Kahneman (1992) and Wegener et al. (2001) distinguished between extreme anchors and moderate anchors, and they further reported that moderate anchors are more effective in influencing behavior. Along similar lines, it is argued by Yockey and Kruml (2009) and reported by Holm and Runnemark (2014) that the salience of the reference point is influenced by various contextual factors.

In this study, we argue that the salience of the reference point and its influence on agents' aspirations, respectively, provide new insights on the questions we posed above. Accordingly, we incorporate these two factors into bargaining problems with a reference point. In particular, we introduce two parameters, $\alpha \in [0, 1]$ and $\beta \in [0, 1]$, which capture the influence of the reference point on the anchor (i.e., its salience) and its influence on agents' aspirations, respectively. Higher values of α refer to higher influence on the anchor (i.e., greater salience) whereas higher values of β refer to higher influence on agents' aspirations. This gives us a unifying framework for the study of bargaining problems with a reference point.

The two parameters we introduce into bargaining problems with a reference point also allow us to obtain a (two-parameter) family of bargaining solutions. This family encompasses some of the well-known solution concepts as special (corner) cases. For instance, when $\alpha = 0$ and $\beta = 0$ (i.e., the reference point has no influence on the anchor point or the aspiration point), this solution coincides with the Kalai-Smorodinsky solution. When $\alpha = 1$ and $\beta = 0$ (i.e., the reference point completely determines the anchor but has no influence on the aspiration point), this solution coincides with the Gupta-Livne solution. When $\alpha = 0$ and $\beta = 1$ (i.e., the reference point has no influence on the anchor but completely determines the aspiration point), this solution coincides with the tempered aspirations solution. Finally, when $\alpha = 1$ and $\beta = 1$ (i.e., the reference point completely determines both the anchor point and the aspiration point), this solution coincides with what Gupta and Livne (1989) called the local Kalai-Smorodinsky solution. Naturally, in between these four corner cases, there are infinitely many intermediate solution concepts that propose settlements by offering feasible compromises between the worst case and the best case scenarios described by anchors and aspirations, respectively.

An alternative interpretation for α and β can be obtained by resorting to a commonly used argument for bargaining solution concepts in the literature (see Luce and Raiffa (1957); Kıbrıs (2010)). Some scholars argue that bargaining solutions can be understood as the representations of arbitrators' distributive preferences. Under this interpretation, α and β can be considered as the arbitrator's opinion/belief about how effective the reference point (or the disagreement point) should be in reaching a settlement in a given situation. If, on the other hand, one resorts to the argument in Balakrishnan et al. (2011) regarding the *context* of the bargain, then α and β can be thought as summary descriptors of the context.

Next, we present characterization results in bilateral bargaining problems. We first offer multiple characterizations for each (α, β)-solution in our family. In each of these characterizations, the standard axioms *weak Pareto optimality*, *symmetry*, and *invariance under positive affine transformations* are utilized. For the other axioms, we first note that to be able to offer a characterization for each (α, β)-solution, one requires an *axiom family* rather than a single axiom since one would need each member of an axiom family to match a certain member of the solution family. For a given $(a, b) \in [0,1]^2$, in addition to the three standard axioms, each characterization has a (type of) *monotonicity* axiom: *individual* (a, b)*-monotonicity*, *restricted* (a, b)*-monotonicity*, or *b-restricted* (a, b)*-monotonicity*. When combined with the standard axioms, either of individual (a, b)-monotonicity and restricted (a, b)-monotonicity is strong enough to characterize the corresponding solution. On the other hand, the standard axioms and b-restricted (a, b)-monotonicity are not sufficient to characterize the corresponding solution; hence, an accompanying axiom is required to obtain the characterization result. Along these lines, *limited sensitivity to changes in the* (α, β)*-salient point* and *reduction under trivial* (α, β)*-salient points* are separately used in two different characterizations. It is worth noting here that the existing characterizations in Kalai and Smorodinsky (1975), Gupta and Livne (1988), and Balakrishnan et al. (2011) can be obtained as special cases. Furthermore, utilizing a similar argument, we are able to provide three alternative characterizations for the Kalai-Smorodinsky solution, two for the Gupta-Livne solution, and three for the tempered aspirations solution.

The roadmap for the essay is as follows: Section 1 introduces the bargaining problem with a reference point and the (α, β)-family of bargaining solutions. Section 2 presents the inventory of axioms used in the characterization results that follow. Section 3 presents characterizations of the individual members of the (α, β)-family. Finally, Section 4 concludes with some limitations and possible future research.

5.1 The Model

An n-person bargaining problem with a reference point is a triple (S, d, r) where S denotes the set of feasible outcomes, d is the *disagreement point*, and r is the *reference point*. We assume that (i) $S \subset \mathbb{R}^n$ is a non-empty, closed, convex, and comprehensive set; (ii) $\exists p \in \mathbb{R}^n_{++}$, $\exists q \in \mathbb{R}$ such that $\forall x \in S : \sum_i p_i x_i \leq q$; (iii) $d, r \in S$; (iv) $\exists x \in S$ with $x > d$; and (v) $r \geq d$.[4] Let $a(S, x)$ denote the aspiration vector such that for every $i \in \{1, \ldots, n\}$ and every $x \in S$: $a_i(S, x) \equiv \max\{t \in \mathbb{R} \mid (t, x_{-i}) \in S\}$. Accordingly, $a(S, d)$ is the *ideal (or utopia) point* (see Kalai and Smorodinsky, 1975) and $a(S, r)$ is the *tempered aspirations point* (see Balakrishnan et al., 2011).

Let Σ^n be the class of all bargaining problems with a reference point. A solution concept for such problems is a function $F : \Sigma^n \to \mathbb{R}^n$ that associates each $(S, d, r) \in \Sigma^n$ with a unique point of S. Below, we present the definitions of some solution concepts we will use in the remainder of this study.

[4] The convexity assumption means that agents could agree to take a coin toss between two outcomes and that each agent's payoff from the coin toss is the average of his/her payoffs from these outcomes. Closedness of S means that the set of physical agreements is closed and that agents' payoff functions are continuous. Comprehensiveness property stipulates that utility is freely disposable. The assumption $d \in S$ means that agents are able to agree to disagree, the assumption $r \in S$ means that the reference point is feasible, and the assumption $r \geq d$ means that the reference point is individually rational. By assuming that there exists $x \in S$ with $x > d$, we rule out degenerate problems where no agreement can make all agents better-off than the disagreement outcome. Finally, the condition in (ii) implies the boundedness of S from above, which means that the maximum payoff an agent can achieve out of an agreement is finite.

Definition 4. (Kalai-Smorodinsky Solution) For every $(S, d, r) \in \Sigma^n$,

$$KS(S, d, r) = \lambda^* a(S, d) + (1 - \lambda^*)d$$

where $\lambda^* = \max\{\lambda \in [0, 1] \mid \lambda a(S, d) + (1 - \lambda)d \in S\}$.[5]

The Kalai-Smorodinsky solution proposes the maximum point of the bargaining set on the line segment connecting the ideal point, $a(S, d)$, and the disagreement point, d.

Definition 5. (Gupta-Livne Solution) For every $(S, d, r) \in \Sigma^n$,

$$GL(S, d, r) = \lambda^* a(S, d) + (1 - \lambda^*)r$$

where $\lambda^* = \max\{\lambda \in [0, 1] \mid \lambda a(S, d) + (1 - \lambda)r \in S\}$.

The Gupta-Livne solution proposes the maximum point of the bargaining set on the line segment connecting the ideal point, $a(S, d)$, and the reference point, r.

Definition 6. (Tempered Aspirations Solution) For every $(S, d, r) \in \Sigma^n$,

$$TA(S, d, r) = \lambda^* a(S, r) + (1 - \lambda^*)d$$

where $\lambda^* = \max\{\lambda \in [0, 1] \mid \lambda a(S, r) + (1 - \lambda)d \in S\}$.

The tempered aspirations solution proposes the maximum point of the bargaining

[5] Note that the Kalai-Smorodinsky solution does not depend on the reference point and is usually defined on (S, d). Nevertheless, it is mathematically not problematic to define it on (S, d, r).

set on the line segment connecting the tempered aspirations point, $a(S, r)$, and the disagreement point, d.

As we discussed above, we introduce $\alpha \in [0, 1]$ which can be interpreted as the power of the reference point in determining the anchor (or simply the salience of the reference point); and $\beta \in [0, 1]$ which can be interpreted as the influence of the reference point in shaping agents' aspirations/expectations. Accordingly, the (α, β)-solution is defined as follows.

Definition 7. For a given $(\alpha, \beta) \in [0, 1]^2$ and for every $(S, d, r) \in \Sigma^n$,

$$F^{\alpha,\beta}(S, d, r) = \lambda^* a(S, \beta r + (1 - \beta)d) + (1 - \lambda^*)(\alpha r + (1 - \alpha)d)$$

where $\lambda^* = \max\{\lambda \in [0, 1] \mid \lambda a(S, \beta r + (1 - \beta)d) + (1 - \lambda)(\alpha r + (1 - \alpha)d) \in S\}$.

For every $\alpha, \beta \in [0, 1]$, the (α, β)-solution proposes the maximum point of the bargaining set on the line segment connecting $a(S, \beta r+(1-\beta)d)$ and $\alpha r+(1-\alpha)d$ (see Figure 5.1). The collection of all such solutions (for which $0 \leq \alpha, \beta \leq 1$) constitutes the (α, β)-family of bargaining solutions.

As depicted in Figure 5.2, when $(\alpha, \beta) = (0, 0)$, the (α, β)-solution coincides with the Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975); when $(\alpha, \beta) = (0, 1)$, it coincides with the tempered aspirations solution (Balakrishnan et al., 2011); when $(\alpha, \beta) = (1, 0)$, it coincides with the Gupta-Livne solution (Gupta and Livne, 1988); and when $(\alpha, \beta) = (1, 1)$, it coincides with the local Kalai-Smorodinsky solution (Gupta and Livne, 1989). Between these corner cases, the (α, β)-solution family encompasses all other solution concepts of similar sort.

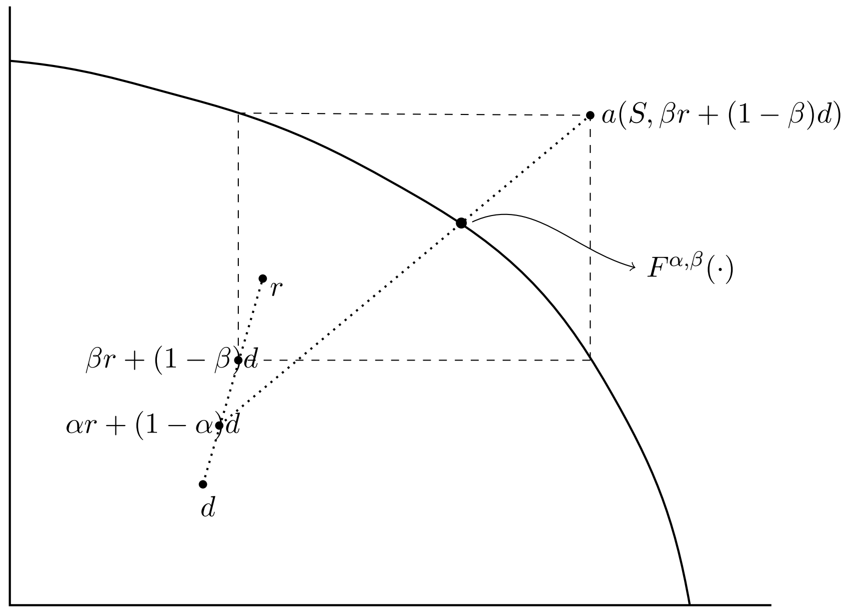
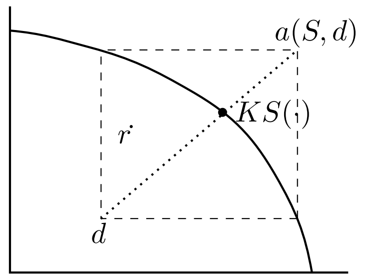


Figure 5.1: The (α, β)-solution

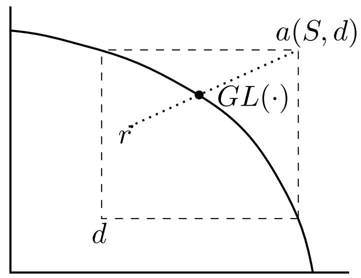
Note that our model allows α and β to be equal, but simply does not only consider this very special case. To the extent that anchors and aspirations are salient points of different nature, this is a natural modelling assumption. Accordingly, we offer a richer description of the bargaining context than the one that restricts attention to cases where α and β are equal. Also, it is worthwhile mentioning that the solution concepts such as Gupta-Livne and the tempered aspirations already implicitly assume that the influences of the reference point on the anchor and aspiration points may be different.

5.2 Inventory of Axioms

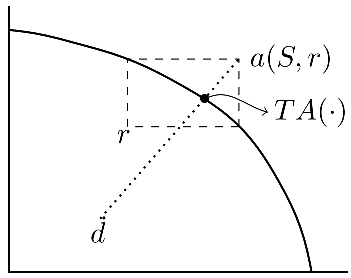
In the following, we present the definitions of the axioms we employ in our characterizations. Since characterization results concern bilateral bargaining problems, these definitions are also given for bilateral bargaining problems.



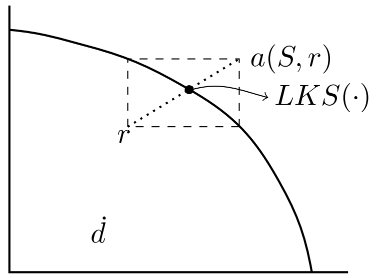
(a) Kalai-Smorodinsky Solution



(b) Gupta-Livne Solution



(c) Tempered Aspirations Solution



(d) Local Kalai-Smorodinsky Solution

Figure 5.2: Four Bargaining Solutions as Members of the (α, β)-Family

First, we define for every $S \subset \mathbb{R}^2$, the set of weakly Pareto optimal outcomes as $WPO(S) = \{x \in S \mid \nexists y \in S \setminus \{x\} : y \gg x\}$ and the set of Pareto optimal outcomes as $PO(S) = \{x \in S \mid \nexists y \in S \setminus \{x\} : y \geq x\}$. The following standard axioms require the solution to be (weakly) Pareto optimal.

Axiom 1. For every $(S, d, r) \in \Sigma^2$, $F(S, d, r) \in WPO(S)$.

Axiom 2. (Pareto Optimality)(PO) For every $(S, d, r) \in \Sigma^2$, $F(S, d, r) \in PO(S)$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (x_2, x_1)$. The following is a primitive fairness axiom, standard in the literature on bargaining problems.

Axiom 3. (Symmetry)(SYM) For every $(S, d, r) \in \Sigma^2$, $F(T(S), T(d), T(r)) = T(F(S, d, r))$.

A bargaining problem $(S, d, r) \in \Sigma^2$ is *symmetric* if $T(S) = S$, $T(d) = d$, and

$T(r) = r$. For such a problem, if a bargaining solution $F : \Sigma^2 \to \mathbb{R}^2$ satisfies SYM, then $F_1(S, d, r) = F_2(S, d, r)$.

We say that $A = (A_1, A_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a positive affine transformation if for each $i \in \{1, 2\}$, the map $A_i(x_1, x_2)$ is of the form $c_i x_i + d_i$ for some positive constant c_i and some constant d_i. Then the following axiom requires the solution to be invariant under positive affine transformations of a given problem.

Axiom 4. (Invariance Under Positive Affine Transformations)(IPAT) For every $(S, d, r) \in \Sigma^2$, $F(A(S), A(d), A(r)) = A(F(S, d, r))$.

In Section 3, we provide characterization results for the individual members of the (α, β)-family. In this regard, we must have *axiom families* rather than single axioms (e.g., Axioms 5-9). This way only one member of the (α, β)-family satisfies only one member of a particular axiom family.

Assume that there is an arbitrator who wants to resolve a conflict in a given bargaining problem with a reference point utilizing a cooperative bargaining solution. Given the *context* of the bargain, the arbitrator has an opinion/belief about the (effective) anchor point and the (effective) aspiration point: $ar + (1 - a)d$ and $a(S, br + (1 - b)d)$, respectively. It is natural to expect that such an arbitrator would care about axioms using these (effective) salient points rather than axioms using disagreement point and/or reference point. Note that Axioms 5-9 are generalizations of the corresponding monotonicity, sensitivity, and relevance axioms in earlier work (see Kalai and Smorodinsky, 1975; Gupta and Livne, 1988; Balakrishnan et al., 2011). Naturally, they have identical interpretations and normative appeal with those axioms in earlier work. For example, Kalai and

Smorodinsky (1975) used the disagreement point in their axiom of *individual monotonicity*, whereas Gupta and Livne (1988) used the reference point in their axioms of *r-restricted S-monotonicity* and *irrelevance of trivial reference points.* In the current work, similar axioms are defined using the (effective) salient points.

The following is a simple monotonicity axiom which is analogous to *individual monotonicity* introduced in Kalai and Smorodinsky (1975).

Axiom 5. (Individual (a, b)-Monotonicity)(IND. (a, b)-MON) Take any (S, d, r), $(S', d', r') \in \Sigma^2$ such that for some $j \in \{1, 2\}$: $a_j(S, br + (1-b)d) = a_j(S', br' + (1-b)d')$ and for $i \neq j$: $a_i(S, x) \leq a_i(S', x)$ for every $x \in S$. If $ar + (1-a)d = ar' + (1-a)d'$, then $F_i(S, d, r) \leq F_i(S', d', r')$.

The following axiom stipulates that if the bargaining set expands in such a way that there is no change in any salient point, then no agent will be worse off.

Axiom 6. (b-Restricted (a, b)-Monotonicity)(b-REST. (a, b)-MON) Take any (S, d, r), $(S', d', r') \in \Sigma^2$ such that $S \subset S'$, $ar + (1-a)d = ar' + (1-a)d'$, and $br + (1-b)d = br' + (1-b)d'$. If $a(S, br + (1-b)d) = a(S', br' + (1-b)d')$, then $F(S, d, r) \leq F(S', d', r')$.

The following axiom indicates that given $(a, b) \in [0, 1]^2$, if the corresponding point from which aspirations are derived is *trivial* (i.e., ineffective), then the situation can be represented by a *reduced* problem in which the reference point and the disagreement point coincide.

Axiom 7. (Reduction under Trivial (a, b)-Salient Points)(RED. T(a, b)-SP) For every $(S, d, r) \in \Sigma^2$, if $a(S, ar + (1-a)d) = a(S, br + (1-b)d)$, then

$F(S, d, r) = F(S, ar + (1-a)d, ar + (1-a)d)$.

Note that Axioms 6 and 7 are analogous to *r-restricted S-monotonicity* and *irrelevance of trivial reference points* introduced in Balakrishnan et al. (2011).

The following axiom requires that if the bargaining problem changes in such a way that the point from which aspirations are derived is the only change, then the solution will not be affected. It is analogous to the *limited sensitivity to changes in the conflict point* axiom introduced in Gupta and Livne (1988).

Axiom 8. (Limited Sensitivity to Changes in the (a, b)-Salient Point)(LSC (a, b)-SP) For every (S, d, r), $(S', d', r') \in \Sigma^2$, if $S = S'$, $ar + (1-a)d = ar' + (1-a)d'$, and $a(S, br + (1-b)d) = a(S', br' + (1-b)d')$, then $F(S, d, r) = F(S', d', r')$.

The following is another monotonicity axiom which is weaker than Axiom 5 and stronger than Axiom 6.

Axiom 9. (Restricted (a, b)-Monotonicity)(REST. (a, b)-MON) For every (S, d, r), $(S', d', r') \in \Sigma^2$, if $S \subset S'$, $ar + (1-a)d = ar' + (1-a)d'$, and $a(S, br + (1-b)d) = a(S', br' + (1-b)d')$, then $F(S, d, r) \leq F(S', d', r')$.

5.3 Characterization Results

In this section, we provide multiple characterizations of the individual members of the (α, β)-family in bilateral bargaining problems. Three of these are closely related to the original characterizations of three of the special cases mentioned in

Section 1 (see Kalai and Smorodinsky, 1975; Gupta and Livne, 1988; Balakrishnan et al., 2011 for these characterizations). Furthermore, stemming from two of these characterizations, we obtain two independent characterizations of the whole family.

Lemma 12. *For a given* $(a, b) \in [0, 1]^2$*, any bargaining solution* $F : \Sigma^2 \to \mathbb{R}^2$ *that satisfies IND.* (a, b)*-MON also satisfies REST.* (a, b)*-MON.*

Proof. Take any $(S, d, r), (S', d', r') \in \Sigma^2$ such that $S \subset S'$, $ar + (1 - a)d = ar' + (1 - a)d'$, and $a(S, br + (1 - b)d) = a(S', br' + (1 - b)d')$. Then the conditions of IND. (a, b)-MON are satisfied for both agents. It follows for every $i \in \{1, 2\}$ that $F_i(S, d, r) \le F_i(S', d', r')$. Hence $F(S, d, r) \le F(S', d', r')$. $\square$

Lemma 13. *For a given* $(a, b) \in [0, 1]^2$*, any bargaining solution* $F : \Sigma^2 \to \mathbb{R}^2$ *that satisfies REST.* (a, b)*-MON also satisfies b-REST.* (a, b)*-MON.*

Proof. Take any $(S, d, r), (S', d', r') \in \Sigma^2$ such that $S \subset S'$, $ar + (1 - a)d = ar' + (1 - a)d'$, $br + (1 - b)d = br' + (1 - b)d'$, and $a(S, br + (1 - b)d) = a(S', br' + (1 - b)d')$. Then the conditions of REST. (a, b)-MON are satisfied. Hence $F(S, d, r) \le F(S', d', r')$. $\square$

Lemma 14. *For a given* $(a, b) \in [0, 1]^2$*, any bargaining solution* $F : \Sigma^2 \to \mathbb{R}^2$ *that satisfies REST.* (a, b)*-MON also satisfies LSC* (a, b)*-SP.*

Proof. Take any $(S, d, r), (S', d', r') \in \Sigma^2$ such that $S = S'$, $ar + (1 - a)d = ar' + (1 - a)d'$, and $a(S, br + (1 - b)d) = a(S, br' + (1 - b)d')$. Then, by REST. (a, b)-MON, it turns out that $F(S, d, r) \le F(S', d', r')$. And considering that $S' = S$, we also have $F(S', d', r') \le F(S, d, r)$. Hence $F(S, d, r) = F(S', d', r')$. $\square$

Lemma 15. *For a given $(a, b) \in [0, 1]^2$, any bargaining solution $F : \Sigma^2 \to \mathbb{R}^2$ that satisfies LSC (a, b)-SP also satisfies RED. T(a, b)-SP.*

Proof. Take any $(S, d, r) \in \Sigma^2$ such that $a(S, ar+(1-a)d) = a(S, br+(1-b)d)$. Set $(S', d', r') = (S, ar+(1-a)d, ar+(1-a)d)$. Note that $ar+(1-a)d = ar'+(1-a)d'$ and $a(S, br+(1-b)d) = a(S, br'+(1-b)d')$. By LSC (a, b)-SP, we have $F(S, d, r) = F(S', d', r')$. Hence $F(S, d, r) = F(S, ar + (1 - a)d, ar + (1 - a)d)$. □

Theorem 14 below presents multiple characterizations of individual members of the (α, β)-family.

Theorem 14. *For a given $(a, b) \in [0, 1]^2$, a bargaining solution $F : \Sigma^2 \to \mathbb{R}^2$ is the (a, b)-solution if and only if F satisfies*

(i) WPO, SYM, IPAT, and IND. (a, b)-MON;

(ii) WPO, SYM, IPAT, and REST. (a, b)-MON;

(iii) WPO, SYM, IPAT, b-REST. (a, b)-MON, and LSC (a, b)-SP.

Moreover, if $a \leq b$, then $F : \Sigma^2 \to \mathbb{R}^2$ is the (a, b)-solution if and only if F satisfies

(iv) WPO, SYM, IPAT, b-REST. (a, b)-MON, and RED. T(a, b)-SP.

Proof of Theorem 14. Fix any $(a, b) \in [0, 1]^2$. It will be enough to prove the "if" part of (i) and the "only if" parts of (iii) and (iv) for $b < a$ and $a \leq b$, respectively. The remaining parts follow by Lemmas 12-15.

The proofs that the (a, b)-solution satisfies WPO, SYM, IPAT, and IND. (a, b)-MON are simple and relegated to the Appendix. As a matter of fact, it is shown in the Appendix that the (a, b)-solution satisfies PO in bilateral bargaining problems.

Conversely, we first focus on the "only if" part of (iii). Assume that $b < a$ and take any solution $F : \Sigma^2 \to \mathbb{R}^2$ satisfying all of the axioms in (iii). Take any $(S, d, r) \in \Sigma^2$. By IPAT, there is no generality lost by assuming that

$$ar + (1-a)d = (0,0) \text{ and } a(S, br + (1-b)d) = (1,1).$$

Notice that $F_1^{a,b}(S, d, r) = F_2^{a,b}(S, d, r)$. Without loss of generality, assume that r is below the 45-degree line. Consider the horizontal lines passing through r, d, and $br+(1-b)d$; and take their intersections with the 45-degree line. Respectively, let these intersections be called $\tilde{r}$, $\tilde{d}$, and $\tilde{b}$. Notice that $a\tilde{r}+(1-a)\tilde{d} = ar+(1-a)d$ and $b\tilde{r} + (1-b)\tilde{d} = \tilde{b}$. Then let S'' be the convex and comprehensive hull of the points $(1, \tilde{b}_2)$, $(\tilde{b}_1, 1)$, and $F^{a,b}(S, d, r)$. By WPO and SYM, we have

$$F(S'', \tilde{d}, \tilde{r}) = F^{a,b}(S, d, r).$$

Also define $S' = \{x \in S \mid x \leq (1,1)\}$. Since $F^{a,b}(S, d, r) \in PO(S')$, $S'' \subset S' \subset S$, and $a(S'', \tilde{b}) = a(S', \tilde{b})$, we utilize b-REST. (a, b)-MON to conclude that

$$F(S'', \tilde{d}, \tilde{r}) = F(S', \tilde{d}, \tilde{r}) = F^{a,b}(S, d, r).$$

By construction, we also have $a(S', br + (1-b)d) = a(S', \tilde{b})$. Then, by LSC

(a,b)-SP,

$$F(S',d,r) = F(S',\tilde{d},\tilde{r}) = F^{a,b}(S,d,r).$$

And by b-REST. (a,b)-MON, we conclude that $F(S,d,r) = F^{a,b}(S,d,r)$.

Finally, assume that $a \leq b$ and take any solution $F : \Sigma^2 \to \mathbb{R}^2$ satisfying all of the axioms in (iv). Take any $(S,d,r) \in \Sigma^2$. By IPAT, there is no generality lost by assuming that

$$ar + (1-a)d = (0,0) \text{ and } a(S, br + (1-b)d) = (1,1).$$

Notice that $F_1^{a,b}(S,d,r) = F_2^{a,b}(S,d,r)$. Without loss of generality, assume that r is below the 45-degree line. Consider the horizontal lines passing through r, d, and $br+(1-b)d$; and take their intersections with the 45-degree line. Respectively, let these intersections be called $\bar{r}$, $\bar{d}$, and $\bar{b}$. Notice that $ar+(1-a)d = a\bar{r}+(1-a)\bar{d}$ and $b\bar{r} + (1-b)\bar{d} = \bar{b}$. Then let S'' be the convex and comprehensive hull of the points $(1,\bar{b}_2)$, $(\bar{b}_1, 1)$, and $F^{a,b}(S,d,r)$. By WPO and SYM, we have

$$F(S'',\bar{d},\bar{r}) = F^{a,b}(S,d,r).$$

Also define $S' = \{x \in S \mid x \leq (1,1)\}$. Since $F^{a,b}(S,d,r) \in PO(S')$, $S'' \subset S' \subset S$, and $a(S'',\bar{b}) = a(S',\bar{d})$, we utilize b-REST. (a,b)-MON to conclude that

$$F(S'',\bar{d},\bar{r}) = F(S',\bar{d},\bar{r}) = F^{a,b}(S,d,r).$$

By construction, we also have $a(S', br+(1-b)d) = a(S', ar+(1-a)d) = a(S',\bar{b})$.

Then, by RED. T(a,b)-SP,

$$F(S',d,r) = F(S', ar+(1-a)d, ar+(1-a)d) = F(S', \bar{d}, \bar{r}) = F^{a,b}(S,d,r).$$

And by b-REST. (a,b)-MON, we conclude that $F(S,d,r) = F^{a,b}(S,d,r)$. $\square$

By Lemmas 12 - 15, we see that the characterizations in Theorem 14 are given in an ascending order of *tightness*. Therefore, if one compares our characterizations using tightness as a measure, one prefers (*iii*) to (*ii*) and (*ii*) to (*i*); and also (*iv*) to all others if $a \leq b$. Yet, we think that each characterization has a value since they allow us to describe and identify the members of the (α,β)-family with different characteristics.

We mentioned that the (α,β)-family encompasses some well-known solutions as special cases. Naturally, this is reflected in the characterizations presented in Theorem 14. In particular, part (*i*) of Theorem 14 encompasses the original characterization of the Kalai-Smorodinsky solution (i.e., if IND. $(0,0)$-MON is considered). In a similar fashion, part (*iii*) encompasses the original characterization of the Gupta-Livne solution[6] and part (*iv*) matches the original characterization of the tempered aspirations solution. On the other hand, to the best of our knowledge, part (*ii*) of Theorem 14 is completely new.

When $(a,b) = (0,0)$, parts (*ii*), (*iii*), and (*iv*) of Theorem 14 become alternative characterizations for the Kalai-Smorodinsky solution; when $(a,b) = (1,0)$, parts (*i*) and (*ii*) become alternative characterizations for the Gupta-Livne solution;

[6] As a matter of fact, there is a minor difference between these characterizations since the one by Gupta and Livne (1988) additionally employs the axiom of *relevant domain*.

and when $(a, b) = (0, 1)$, parts (i), (ii), and (iii) become alternative characterizations for the tempered aspirations solution.

Finally, Figure 5.3 below summarizes the relations between the axioms used in our characterizations and how they characterize the (a, b)-solution.[7]

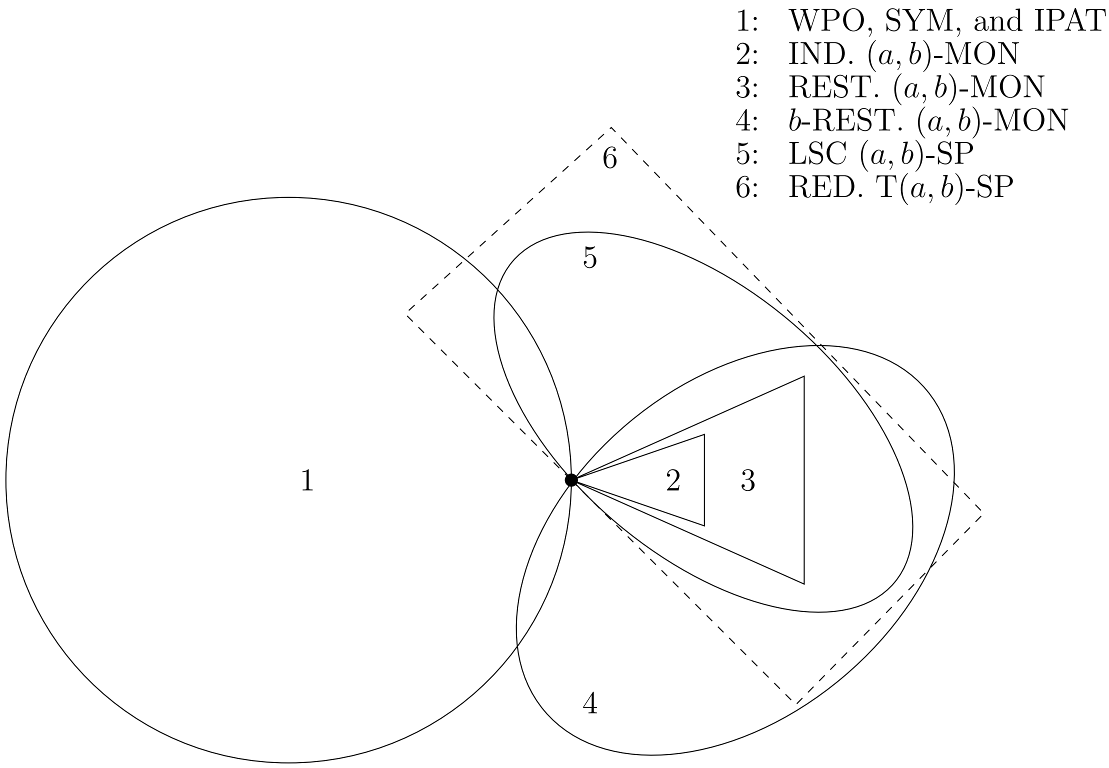


Figure 5.3: The Summary of the Characterization Results

5.4 Conclusion

We introduce two parameters that measure the influences of soft vs hard power on anchor and aspiration formation in bargaining problems with a reference point. These parameters can be thought as descriptors of the specific bargaining con-

[7]Assume that the numbers in the sets denote the name of the sets. The figure utilizes the facts that (i) $2 \subset 3$ by Lemma 12; (ii) $3 \subset 4$ by Lemma 13; and (iii) $3 \subset 5 \subset 6$ by Lemmas 14 - 15.

text which carry information about how influential the reference point (or the disagreement point) is in shaping the (effective) anchor point and the (effective) aspiration point. Alternatively, they can be interpreted as parameters that describe the opinion of an arbitrator about how effectively the reference point should be utilized in reaching a settlement. As a result, we obtain a family of bargaining solutions and present characterizations of individual members of this family in bilateral bargaining problems.

A natural question that may arise at this stage is: What are some caveats in our approach? We would like to note that the solution family we introduce does not cover all possible solutions that employ anchor and aspiration points. For instance, Chun and Thomson (1992) used a *claims point*, which is similar to an aspiration point, and proposed a solution that lies at the intersection of the utility possibility frontier and the line connecting the claims point and the disagreement point. Thus, it behaves similarly to the members of the (α, β)-family. However, this solution does not belong to our family since the claims point is exogenously given and not derived by using a salient point and the utility possibility frontier. Moreover, we follow the assumptions used by Gupta and Livne (1988) in introducing the reference point into the bargaining problem. In particular, the reference point in our model Pareto dominates the disagreement point and lies in the interior of the utility possibility set. Although these restrictions may be satisfied in many instances, there possibly are real-life situations where one or both may fail to be satisfied. We do not address these situations in this study.

Future work may experimentally test the validity of our theory by manipulating the salience of the reference point in simple bargaining games. Both vignettes and incentivized bargaining experiments can be used for this purpose. From a theoretical perspective, axiomatic (see Border and Segal (1997) or strategic (see Van Damme (1986)) selection of the members of the (α, β)-family may be of interest. Moreover, our framework can be used to arrive at alternative characterizations of the aforementioned well-known members of the family by deriving new axiom families from earlier studies (see Livne 1989; Rachmilevitch 2011, 2014, among others). Finally, our model may be used as a natural unifying framework for studying endogenous emergence of reference points in bargaining problems (see Herrero, 1997; Shalev, 2002; Driesen et al., 2011; Bozbay et al., 2012; Birkeland and Tungodden, 2014; Karagözoğlu and Keskin, 2014).

CHAPTER 6

ITERATED EGALITARIAN COMPROMISE SOLUTION TO BARGAINING PROBLEMS AND MIDPOINT DOMINATION

In a seminal paper, Nash (1950) introduced the axiomatic treatment of bargaining problems. Over the last six decades, the axiomatic approach has attracted a considerable attention from researchers studying bargaining (see Kıbrıs, 2010 for an overview). The axiomatic literature on bargaining has been productive in coming up with solution concepts with appealing normative properties. Two prominent solutions of interest for the current study are the *egalitarian solution* (*E,* for short) due to Kalai (1977) and the *equal loss solution* (*EL*, for short) due to Chun (1988). As their names suggest, both solutions apply an egalitarian notion of justice in proposing outcomes to bargaining problems. More precisely, for each bargaining problem, *E* proposes the maximum utility profile that gives each agent an equal gain over his disagreement outcome, whereas *EL* proposes

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the maximum utility profile that gives each agent an equal loss over his ideal outcome.[1]

These two solutions share a common weakness: both solutions fail to satisfy a basic normative requirement that a solution should assign each agent at least half of his ideal point outcome (i.e., the best possible outcome for the agent among the outcomes that are individually rational for both) in all bargaining problems. It can be rephrased as, for any problem, an outcome proposed by a solution should be Pareto superior to the randomized dictatorship outcome. This requirement was introduced by Sobel (1981) and known as *midpoint domination* (*MD*, for short). As Rachmilevitch (2017) points out, midpoint domination has both fairness and efficiency connotations. On one hand, it requires both agents to receive at least half of their ideal point outcomes (fairness) and on the other, it requires the proposed outcome to be Pareto superior to the midpoint (efficiency). Hence, it is an appealing normative property.

In this essay, we, first, introduce a new solution concept for two-person bargaining problems: *iterated egalitarian compromise solution* (*IEC,* for short). For a problem where E and EL propose the same outcome, the outcome proposed by IEC coincides with theirs. For a problem where E and EL propose different outcomes, IEC proposes a compromise in an iterative fashion, by using the proposed outcomes of E and EL at each iteration step. Hence, the name, iterated egalitarian compromise. Second, we show that IEC is well-defined, i.e. for any problem in the domain of two-person bargaining problems we consider, it proposes a unique

[1] As these descriptions may suggest, E and EL are *duals* of each other. For a recent study of this relationship in this journal, see Karagözoğlu and Rachmilevitch (2017).

outcome, defined as the limit of a iterative process. Finally, we show that it satisfies midpoint domination despite the fact that neither of the solutions it is based on does so.

A recent attempt in a similar direction is Rachmilevitch (2017). He proposes and characterizes a midpoint-robust (i.e., satisfying midpoint domination) version of the egalitarian solution.

The essay is organized as follows: in Section 1, we introduce the bargaining problem, define the solutions of interest, and the midpoint domination property. In Section 2, we prove that *IEC* is well-defined and it satisfies midpoint domination. Section 3 concludes with final remarks.

6.1 The Model

A simple two-person *bargaining problem is* denoted by $S \subset \mathbb{R}^2$. It satisfies the following properties: it is (i) non-empty, (ii) closed, and bounded from above, (iii) convex, (iv) comprehensive, (v) $S \cap \mathbb{R}^2_{++} \neq \emptyset$, and (vi) it contains the disagreement outcome, $\mathbf{0} \equiv (0, 0)$. The axiomatic properties of the solutions we will use allow us to normalize the *disagreement outcome* to $(0, 0)$. Since we will do that in what follows, we denote the problem by S instead of (S, d) for short. Intuitively, S represents all the utility vectors that can be achieved by the agents. The non-emptiness is to make the problem non-trivial. The closedness of S means that the set of physical agreements is closed and that the payoff functions of agents are continuous. The boundedness from above means that the maximum utility an agent can achieve out of an agreement is finite. The convexity assumption means that agents could agree to take a coin-toss between two outcomes and that the payoff of each agent from the coin toss is the average of his/her payoffs from these outcomes. Comprehensiveness stipulates that utility is freely disposable down to the disagreement utilities. $S \cap \mathbb{R}^2_{++} \neq \emptyset$ rules out degenerate problems where no agreement can make all agents better off than the disagreement outcome. Finally, $\mathbf{0} \in S$ means that the agents can agree to disagree. We denote the set of all such problems by Σ. For every $S \subset \mathbb{R}^2$, its weak (strong) Pareto optimal set is defined as $WPO(S) \equiv \{y \in S \mid x > y \text{ implies } x \notin S\}(PO(S) \equiv \{y \in S \mid x \gneqq y \text{ implies } x \notin S\})$. Here, we will focus on a subdomain of Σ, denoted by $\widehat{\Sigma}$, whose weak and strong Pareto frontiers coincide (i.e., the bargaining frontier does not have any horizontal or vertical segments). The importance of this assumption will be explained later in the proof of Proposition 1. Finally, a bargaining solution F is a function, which assigns to any bargaining problem S, a unique point in it.

The *egalitarian solution* (Kalai, 1977) equalizes agents' gains over their disagreement outcomes. Accordingly, it assigns to each S the point, $E(S)$ with identical (x, y)–coordinates and $E(S)$ is the maximum possible. This corresponds to selecting the intersection point of the Pareto frontier and the 45–degree line drawn from the disagreement point (in our case, the origin). The *equal loss solution* (Chun, 1988) equalizes agents' losses from their ideal point outcomes. Formally, ideal point, introduced by Kalai and Smorodinsky (1975), is defined as $a_i(S) \equiv \max\{s_i : s \in S\}$, where $a_i(s)$ denotes agent i's ideal point outcome. Accordingly, the *equal loss solution* assigns to each S, the point $EL(S) = a(S) - (l, l)$, where l is the minimum possible. This corresponds to selecting the point at the intersection of the Pareto frontier and the 45-degree line drawn from the ideal point. Note that for all $S \subset \Sigma$, if $a_1(S) > a_2(S)$, then $EL_1(S) > E_1(S)$ and $E_2(S) > EL_2(S)$, and vice-a-versa.

A solution F satisfies *midpoint domination*, if it proposes an outcome $F(S) \geq mp(S) \equiv \frac{1}{2}a(S)$, for all S. Figure 1 shows an example, where both E and EL violate *MD*. Note that the bargaining problem in the example is in $\widehat{\Sigma}$.

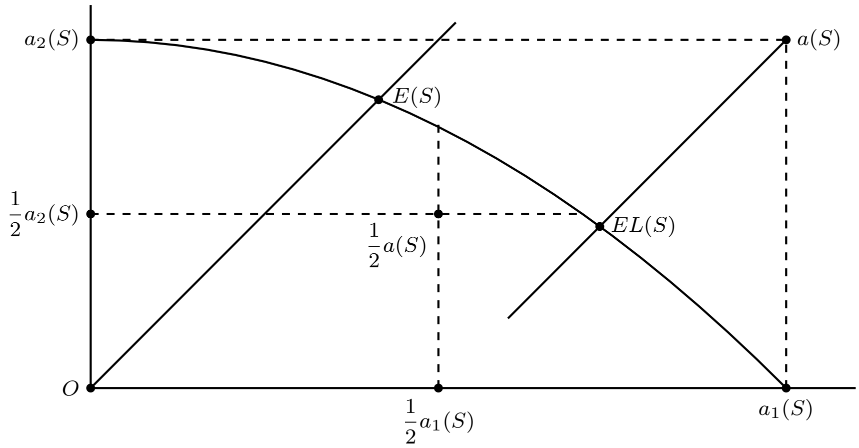

Figure 6.1: E and EL violate *MD*

The *iterated egalitarian compromise solution* (or *IEC*, for short) assigns to each $S \in \widehat{\Sigma}$, the point x, if $E(S) = EL(S) = x$ and assigns the point $y \equiv \cap_{t \in \mathbb{N}} PO(S_t)$, where $S_0 \equiv S$ and the bargaining problem in iteration step t, S_t, for $t \geq 1$ is derived by applying E and EL to S_{t-1} in a way that, the origin (i.e., the disagreement point) of S_t denoted by $o(S_t)$, is $o(S_t) = (\min\{E_1(S_{t-1}), EL_1(S_{t-1})\}, \min\{E_2(S_{t-1}), EL_2(S_{t-1})\})$ and consequently $a(S_t) = (\max\{E_1(S_{t-1}), EL_1(S_{t-1})\}, \max\{E_2(S_{t-1}), EL_2(S_{t-1})\})$.

IEC could be interpreted as a conflict resolution mechanism, which resolves the *conflict* between E and EL in a step-by-step fashion, by using the minimal outcomes in each iteration as starting points and the maximal outcomes as ideals for the bargaining problem in the next step. Figure 2 shows how IEC operates in a problem where E and EL propose different outcomes.

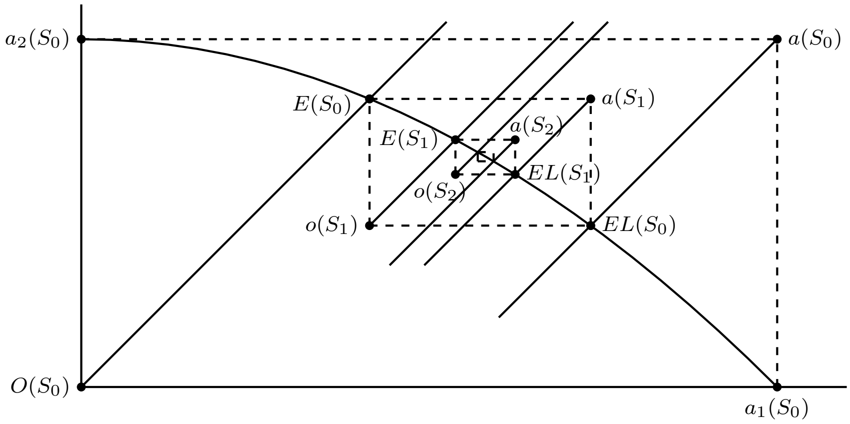

Figure 6.2: Iterated Egalitarian Compromise Solution

6.2 The Result

First, we prove that IEC is well-defined, i.e. for all $S \in \widehat{\Sigma}$ the iterative process embedded in IEC converges to a single point.

Proposition 1. *For all $S \in \widehat{\Sigma}$,* IEC *is well-defined.*

Proof of Proposition 1. First, consider a symmetric bargaining problem, $S \equiv S_0$. In this case, IEC proposes a single outcome, since $E(S_0) = EL(S_0)$. Now, consider an asymmetric problem, $S \equiv S_0 \in \widehat{\Sigma}$. Without loss of generality, suppose that $a_1(S_0) > a_2(S_0)$. For notational convenience, let $a_1(S_t) - o_1(S_t) = \alpha_t$ and $a_2(S_t) - o_2(S_t) = \beta_t$. Since both E and EL operate via upward-sloping 45-degree lines, for each iteration step t, we get $\alpha_{t+1} + \beta_{t+1} = |\alpha_t - \beta_t|$. The sequences (α_t) and (β_t) are decreasing and bounded below ($\alpha_t \geq 0$, $\beta_t \geq 0$). Thus, there exist some $\bar{\alpha}$ and $\bar{\beta}$ such that $\lim_{t\to\infty} \alpha_t = \bar{\alpha} \geq 0$ and $\lim_{t\to\infty} \beta_t = \bar{\beta} \geq 0$. As $t \to \infty$, we have $\bar{\alpha} + \bar{\beta} = |\bar{\alpha} - \bar{\beta}|$, which requires at least one of $\bar{\alpha}$ and $\bar{\beta}$ to be equal to zero. Suppose without loss of generality that $\bar{\alpha} = 0$. Since bargaining frontier has no horizontal or vertical segments, $\bar{\beta} = 0$ as well, which implies that our iteration algorithm converges to a single point (i.e., IEC is single valued). $\square$

Remark 1. For some iteration step t', the relative positions of E and EL on the frontier may change, i.e. $a_2(S_{t'}) > a_1(S_{t'})$. Nevertheless, by the definitions of disagreement point and ideal point, and the way our iteration mechanism operates, $\alpha_{t'} \geq 0$, $\beta_{t'} \geq 0$ and these sequences continue to decrease.

Remark 2. The domain restriction we made (i.e., bargaining frontier has no horizontal/vertical segments) is necessary for the argument in last step of the proof to be valid. If the Pareto frontier had horizontal/vertical segments, iterative process may converge to a set that has more than one element.

The following corollary shows the relationship between $IEC(S_t)$ and $mp(S_t)$ in the limit as $t \to \infty$, and it will be utilized in the proof of Proposition 2.

Corollary 2. *For all* $S \in \widehat{\Sigma}$, $lim_{t\to\infty} IEC(S_t) \geq \lim_{t\to\infty} mp(S_t)$

Proof of Corollary 2. The proof of Proposition 1 clearly implies that any bargaining problem converges to a symmetric bargaining problem in the limit of the iterative process and the IEC solution dominates the midpoint in a symmetric bargaining problem. Hence, the result follows. $\square$

Now, we are ready to state our main result.

Proposition 2. *For all $S \in \widehat{\Sigma}$,* IEC *satisfies* MD.

Proof of Proposition 2. We will prove this statement in two steps. To do that, we partition $\widehat{\Sigma}$ into two subsets: (i) problems with linear bargaining frontiers ($\widehat{\Sigma}_{lin}$), (ii) problems with non-linear bargaining frontiers ($\widehat{\Sigma}_{nlin}$). Below, we will show that *IEC* satisfies *MD* in both subsets.

Claim 1: For all $S \in \widehat{\Sigma}_{lin}$, *IEC* satisfies *MD*.

Proof: If $S \equiv S_0$ is symmetric, i.e., $a_1(S_0) = a_2(S_0)$, then trivially $IEC(S_0) = E(S_0) = EL(S_0) = x$, where $x = mp(S_0)$. Suppose now that $S \equiv S_0$ is asymmetric, i.e., $a_1(S_0) \neq a_2(S_0)$. Furthermore, without loss of generality, assume that $a_1(S_0) > a_2(S_0)$. Then, $E(S_0) \neq EL(S_0)$. The linearity of the Pareto frontier implies that the segments of the frontier cut by E and EL (from two ends) in each iteration step are of equal length. Formally, for all $t > 0$, $o_1(S_t) - o_1(S_{t-1}) = a_1(S_{t-1}) - a_1(S_t)$ and $o_2(S_t) - o_2(S_{t-1}) = a_2(S_{t-1}) - a_2(S_t)$. Therefore, $mp(S_0) = mp(S_t)$, for all $t > 0$. Proposition 1 implies that $IEC(S) = mp(S)$. Hence, the result follows.

Claim 2: For all $S \in \widehat{\Sigma}_{nlin}$, *IEC* satisfies *MD*.

Proof: If $S \equiv S_0$ is symmetric, i.e., $a_1(S_0) = a_2(S_0)$, then trivially $IEC(S_0) = E(S_0) = EL(S_0) = x$, where $x \geq mp(S_0)$. Suppose now that $S \equiv S_0$ is asymmetric, i.e., $a_1(S_0) \neq a_2(S_0)$. Furthermore, without loss of generality, assume that $a_1(S_0) > a_2(S_0)$. Then, $E(S_0) \neq EL(S_0)$. The convexity of S_0 and the non-linearity of the bargaining frontier imply that $o_1(S_t) - o_1(S_{t-1}) \geq a_1(S_{t-1}) - a_1(S_t)$ and $o_2(S_t) - o_2(S_{t-1}) \geq a_2(S_{t-1}) - a_2(S_t)$ for all $t \geq 1$ and these inequalities are strictly hold for some t. But this implies that $mp(S_t) \gg mp(S_{t-1})$ for all $t > 0$ (i.e., each iteration step moves the midpoint in the north-east direction). So, $\lim_{t\to\infty} mp(S_t) \gg mp(S_0)$. From Corollary 1, we know that $y \equiv \lim_{t\to\infty} IEC(S_t) \geq \lim_{t\to\infty} mp(S_t)$. Therefore, $IEC(S) \gg mp(S)$. The same result is valid for the case of $a_1(S_0) < a_2(S_0)$, as well. Hence, the result follows. $\square$

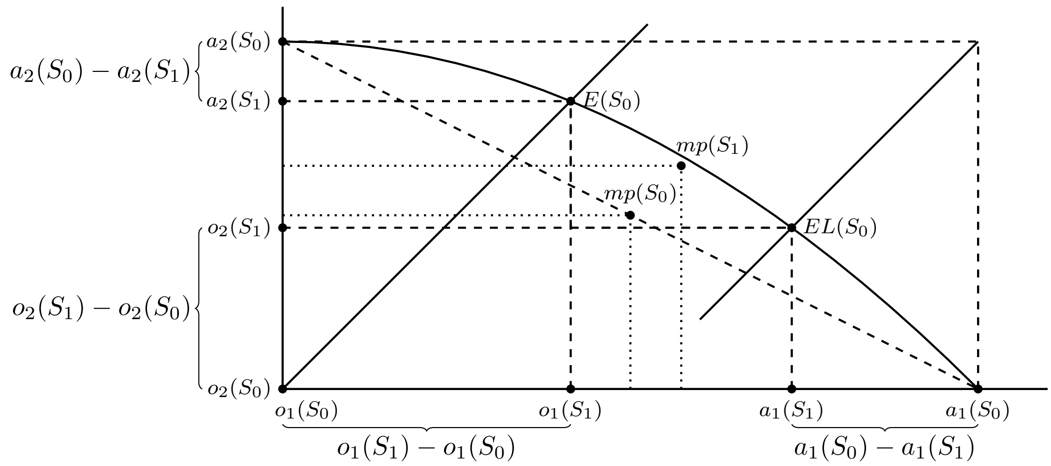

Figure 6.3: Changes in the midpoints (Nonlinear, asymmetric case)

Kalai-Smorodinsky solution (*KS* for short) also satisfies *MD*, and like *E* and *EL*, it utilizes an egalitarian justice norm (*KS* equalizes the ratios of maximal gains across players). As such it can be thought as another alternative, but it rules out inter-personal utility comparisons whereas *IEC*, like *E* and *EL*, is built on the premise that such comparisons are possible. A direct implication of Proposition 2 on the relationship between *IEC* and *KS* is given in the following corollary.

Corollary 3. *For all* $S \in \widehat{\Sigma}_{lin}$, $IEC(S) = KS(S)$.

Proof of Corollary 3. The proof directly follows from the following facts: in a bargaining problem $S \in \widehat{\Sigma}_{lin}$, (i) midpoint is on the $PO(S)$, and thus the only way for a solution to satisfy *MD* is to propose the midpoint, (ii) *KS* proposes the midpoint, and (iii) *IEC* satisfies *MD* in $\widehat{\Sigma}_{lin}$ (from Claim 1 in Proposition 2). $\square$

Note that this statement is not necessarily true for $S \in \widehat{\Sigma}_{nlin}$ (see Figure 4). Furthermore, it is neither valid for *E* nor for *EL*, even in $\widehat{\Sigma}_{lin}$.

It is worth mentioning here that *E* and *EL* are *duals* of each other. Recognizing this fact, one can draw another similarity between *IEC* and *KS*. Recently, Karagözoğlu and Rachmilevitch (2017), in a paper where they provided three new characterizations of *KS*, showed that the outcome proposed by *KS* always lies (i.e., *sandwiched*) in between the outcomes proposed by two other solutions with egalitarian objectives: the equal area solution (*EA* for short) and the dual of the equal area solution (*DEA* for short). Along similar lines, the outcome proposed by *IEC*, by construction, always lies in between the outcomes proposed by *E* and *EL*, again, two egalitarian solutions that are duals of each other. Reader is referred to Anbarcıand Bigelow (1994) for *EA*, Karagözoğlu and Rachmilevitch (2017) for *DEA*, and Lemma 2 in Karagözoğlu and Rachmilevitch (2017) for the above-mentioned "sandwich" result.

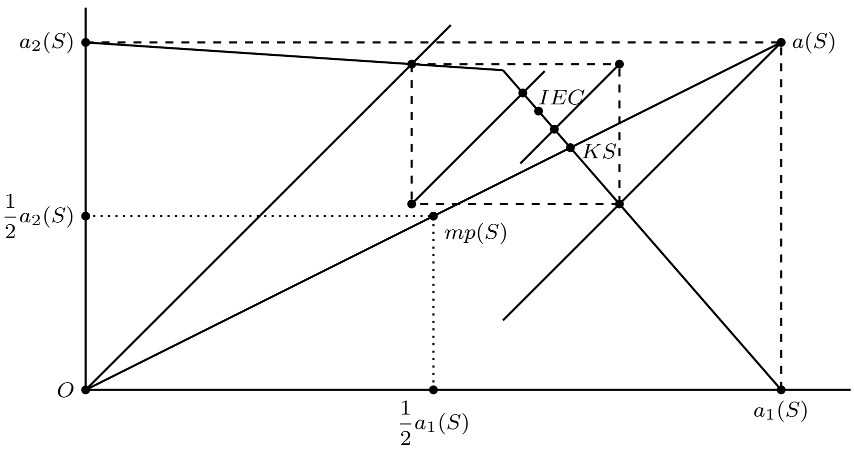

Figure 6.4: The Relation Between *KS* and *IEC*

6.3 Conclusion

We introduced a new solution concept, *IEC*, for two-person bargaining problems, which is based on two well-known egalitarian solution concepts, *E* and *EL*. *IEC* mimics a conflict resolution mechanism and satisfies an appealing normative property, midpoint domination, which is violated by both *E* and *EL*. Thus, *IEC* is a reasonable alternative, especially if one wants to (i) utilize an egalitarian justice norm in problems where *E* and *EL* disagree and (ii) operate in a domain that allows *inter-personal utility comparisons*.

Our results lead to some new questions. Below, we describe three of them.

(1) In addition to *MD*, *IEC* satisfies Pareto optimality, symmetry, and scale invariance Nash (1950), by definition. Furthermore, the proof of Claim 2 in Proposition 2 implies that it satisfies restricted monotonicity Roth (1979), as well. It would be of interest, from a normative perspective, to study which axiomatic properties would characterize *IEC*.

(2) As we argued in Section 2, there are certain similarities between *IEC* and *KS*, in that both are sandwiched between two egalitarian solutions, which are duals of each other: *IEC* is sandwiched by *E* and *EL*, whereas *KS* is sandwiched by *EA* and *DEA*. Further investigation of the relationships between these six solutions with egalitarian objectives would be of interest.

(3) Finally, the iterative process *IEC* utilizes resembles the step-by-step nature of negotiations. Thus, whether a strategic foundation for *IEC* can be provided is an interesting question, in the spirit of the *Nash program* Nash (1953).

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APPENDICES

A Proofs of Chapter 2

Proof of Lemma 2. By the definition of contraction mapping, we need to show that

$$\forall v, w \in [0,1]^n : \|f^\delta(v) - f^\delta(w)\| \leq \delta \|v - w\|,$$

which means for all $i \in N$, $|f_i^\delta(v) - f_i^\delta(w)| \leq \delta \|v - w\|$.

$$\begin{aligned}
&|f_i^\delta(v) - f_i^\delta(w)| \\
&= |(1 - \frac{l_i}{2l})\delta(v_i - w_i) + \frac{1}{2l} \sum_{\{j|ij \in G\}} (\max\{1 - \delta(v_j - l_j c_j), \delta v_i\} - \max\{1 - \delta(w_j - \\
&\qquad l_j c_j), \delta w_i\})| \\
&\leq (1 - \frac{l_i}{2l})\delta|v_i - w_i| + \frac{1}{2l} \sum_{\{j|ij \in G\}} \max\{|1 - \delta v_j - (1 - \delta w_j)|, |\delta v_i - \delta w_i|\} \\
&= (1 - \frac{l_i}{2l})\delta|v_i - w_i| + \frac{1}{2l} \sum_{\{j|ij \in G\}} \delta \max\{|v_j - w_j|, |v_i - w_i|\} \\
&\leq (1 - \frac{l_i}{2l})\delta\|v - w\| + \frac{1}{2l} \sum_{\{j|ij \in G\}} \delta\|v - w\| \\
&= \delta\|v - w\|
\end{aligned}$$

Therefore, the function f^δ is a contraction mapping, implying that it has a fixed point. $\square$

Proof of Lemma 3. $ij \in G^{*\delta}$ means that $ij \in G$ and $\max\{1-\delta(v_j^{*\delta} - l_j c_j), \delta v_i^{*\delta} - l_i c_i\} = 1 - \delta(v_j^{*\delta} - l_j c_j)$. Since $v_i^{*\delta}$ is a fixed point of f^δ, $v^{*\delta}$ solves the linear equation system

$$v_i = (1 - \frac{l_i}{2l})\delta(v_i - l_i c_i) + \frac{1}{2l}\sum_{\{j|ij\in G\}}(1-\delta v_j),\ \forall i = 1,\dots,n.$$

Take any subnetwork H of G and for all $\delta \in (0,1)$, consider the above n x n linear equation system for H

$$v_i = (1 - \frac{l_i^H}{2l^H})\delta(v_i - l_i c_i) + \frac{1}{2l^H}\sum_{\{j|ij\in H\}}(1-\delta(v_j - l_j c_j)). \tag{1}$$

Define the function $h^{\delta,H} : \mathbb{R}^n \to \mathbb{R}^n$ such that for each $i \in N$

$$h_i^{\delta,H}(v) = (1 - \frac{l_i^H}{2l^H})\delta(v_i - l_i c_i) + \frac{1}{2l^H}\sum_{\{j|ij\in H\}}(1-\delta(v_j - l_j c_j)).$$

The function $h^{\delta,H}$ is a contraction mapping. Each equation in the linear system (1) is a linear function of δ. Then, for each $i \in N$, $v_i^{\delta,H}$ is uniquely given by the necessary Cramer's rule.

$$v_i^{\delta,H} = \frac{P_i^H(\delta)}{Q_i^H(\delta)} \tag{2}$$

Since the linear system (1) is non-singular, the denominator of (2), $Q_i^H(\delta)$, is different than zero for all $\delta \in (0,1)$ and for all non-empty subnetworks H of G.

Let $\bar{\Delta}$ be the set of δ for which there exists i, j, H with $\delta((v_i^{\delta,H} - l_i c_i) + (v_j^{\delta,H} -$

$l_jc_j)) = 1$. Take any i, j, H. The equation $\delta((v_i^{\delta,H} - l_ic_i) + (v_j^{\delta,H} - l_jc_j)) = 1$ is equivalent to

$$1 + \delta(l_ic_i + l_jc_j) = \delta\Big(\frac{P_i^H(\delta)}{Q_i^H(\delta)} + \frac{P_j^H(\delta)}{Q_j^H(\delta)}\Big).$$

If the equation above has infinitely many solutions, it holds also for $\delta = 1/3$.

$$1 + \frac{1}{3}(l_ic_i + l_jc_j) = \frac{1}{3}(v_i^{1/3,H} + v_j^{1/3,H}).$$

After some algebraic operations, we have

$$(v_i^{1/3,H} + v_j^{1/3,H}) = 3 + (l_ic_i + l_jc_j),$$

which is a contradiction with $v^{1/3,H} \in [0,1]^n$. Hence, the statement $\delta((v_i^{*\delta} - l_ic_i) + (v_j^{*\delta} - l_jc_j)) = 1$ holds for a finite set of solutions δ.

It follows that for all (i, j, H), the inequality $\delta((v_i^{\delta,H} - l_ic_i) + (v_j^{\delta,H} - l_jc_j)) \neq 1$ holds for all but a finite number of solutions δ. $\square$

Proof of Lemma 4. Take any link $ij \in G$. If $ij \in G \backslash G^*$, then for all $\delta > \underline{\delta}$,

$$\delta((v_i^{*\delta} - l_i^G c_i) + (v_j^{*\delta} - l_j^G c_j)) > 1.$$

Since for all $i \in N$, $c_i \geq 0$, the following holds

$$\delta(v_i^{*\delta} + v_j^{*\delta}) > 1. \tag{3}$$

If $ij \in G^*$, then for all $\delta > \underline{\delta}$,

$$v_i^{*\delta} = \Big(1 - \frac{l_i^G}{2l^G}\Big)\delta(v_i^{*\delta} - l_i^G c_i) + \frac{1}{2l^G}\sum_{\{k|ik\in G^*\}}(1 - \delta(v_k^{*\delta} - l_k^G c_k)).$$

Since for all $k \neq j$ with $ik \in G^*$, $1 - \delta(v_k^{*\delta} - l_k^G c_k) \geq \delta(v_i^{*\delta} - l_i^G c_i)$, we have

$$v_i^{*\delta} \geq \Big(1 - \frac{1}{2l^G}\Big)\delta(v_i^{*\delta} - l_i^G c_i) + \frac{1}{2l^G}(1 - \delta(v_j^{*\delta} - l_j^G c_j)). \tag{4}$$

As $\delta \to 1$, from (3), we have $v_i^* + v_j^* > 1$ for all $ij \in G \backslash G^*$. And by (3) and (4), for all $ij \in G^*$ $v_i^* + v_j^* = 1$. (Note that $\lim_{\delta\to 1} c_i = \lim_{\delta\to 1}(1-\delta)TC_i = 0$.)

□

Proof of Lemma 5. Suppose that the algorithm is at some step $s \in \{1, 2, \ldots, \bar{s}\}$. Let $r_s < 1$. It is possible to have multiple sets that minimize the shortage ratio r_s, $\mathcal{N}_s$ is the family of such sets. If there are more than one set in $\mathcal{N}_s$, the algorithm picks the largest set for each component of G_s. Hence, we have a unique minimizer set for each component. Then, among such maximal minimizer sets in the components, the algorithm chooses the one that minimizes the advantage/disadvantage provided by costs (2.15).

Now, let $r_s = 1$ and the cardinality of the minimizer sets be 1. Then, in case of multiplicity of these sets, the algorithm picks the set that minimizes (16).

Finally, if there exists more than one set that minimizes (2.15), the algorithm unifies them. So, we end up with a unique set M_s. This concludes the proof. □

Proof of Lemma 6. By the definition of algorithm, the shortage ratio is increasing, $\frac{|L_{s'}|}{|M_{s'}|} \leq \frac{|L_s|}{|M_s|}$. It follows that that

$$\frac{|L_{s'}|}{|L_{s'}| + |M_{s'}|} \leq \frac{|L_s|}{|L_s| + |M_s|}.$$

□

Proof of Corollary 1. Let G be the outcome of the network formation game. From Theorem 6, we know that G is equitable, i.e. for all $i \in N$, $u_i^*(G) = 1/2$. The social welfare provided by the network G is

$$W(G) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n}{2} + \frac{1}{2}, & \text{if } n \text{ is odd,} \end{cases}$$

which is equal to the maximum attainable social welfare.

□

B Proofs of Chapter 3

Proof of Lemma 8. By the definition of contraction mapping, we need to prove that

$$\forall u, w \in [0,1]^n : \, \|f^\delta(u) - f^\delta(w)\| \leq \delta \|u - w\|,$$

which means for all $k \in S \cup B$, $|f_k^\delta(u) - f_k^\delta(w)| \leq \delta \|u - w\|$.

Without loss of generality, we prove the above inequality for all $s \in S$. This inequality can be easily shown for any buyer $b \in B$.

$$\begin{aligned}
&\|f_s^\delta(u) - f_s^\delta(w)| \\
&= \left| \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right) \delta(u_s - w_s) + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} (\max\{v_b - \delta u_b, \delta u_s\} \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. - \max\{v_b - \delta w_b, \delta w_s\}) \right| \\
&\leq \left| \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right) \delta(u_s - w_s)| + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} \delta(\max\{|u_b - w_b|, |u_s - w_s|\}) \right| \\
&\leq \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right) \delta\|u - w\| + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2} \delta\|u - w\| \\
&= \delta\|u - w\|,
\end{aligned}$$

implying that the function f^δ is a contraction mapping. Hence, the function has a fixed point. □

Proof of Lemma 9. $(s, b) \in G^{*\delta}$ means that that s and b are connected and $\max\{v_b - \delta u_s^{*\delta}, \delta u_b^{*\delta}\} = v_b - \delta u_s^{*\delta}$. Since $u^{*\delta}$ is a fixed point of f^δ, $u^{*\delta}$ is the solution of the following linear equation system

$$u_s = \left(1 - \sum_{\{b|(s,b)\in G^{*\delta}\}} \frac{p_{sb}}{2}\right)\delta u_s + \sum_{\{b|(s,b)\in G^{*\delta}\}} \frac{p_{sb}}{2}(v_b - \delta u_b), \ \forall s \in S$$
$$u_b = \left(1 - \sum_{\{s|(s,b)\in G^{*\delta}\}} \frac{p_{sb}}{2}\right)\delta u_b + \sum_{\{s|(s,b)\in G^{*\delta}\}} \frac{p_{sb}}{2}(v_b - \delta u_s), \ \forall b \in B.$$

Take any nonempty subnetwork H of G and define a mapping $h^{\delta,H} : \mathbb{R}^n \to \mathbb{R}^n$ such that for each $s \in S$ and $b \in B$,

$$\begin{aligned} h_s^{\delta,H}(u) &= \left(1 - \sum_{\{b|(s,b)\in H\}} \frac{p_{sb}}{2}\right)\delta u_s + \sum_{\{b|(s,b)\in H\}} \frac{p_{sb}}{2}(v_b - \delta u_b), \forall s \in S \\ h_b^{\delta,H}(u) &= \left(1 - \sum_{\{s|(s,b)\in H\}} \frac{p_{sb}}{2}\right)\delta u_b + \sum_{\{s|(s,b)\in H\}} \frac{p_{sb}}{2}(v_b - \delta u_s), \forall b \in B. \end{aligned} \tag{5}$$

$h^{\delta,H}$ is a contraction mapping. All equations in the linear system (5) are a linear functions of δ, implying that for each $k \in S \cup B$, $u_k^{\delta,H}$ is uniquely given by the Cramer's rule.

$$u_k^{\delta,H} = \frac{P_k^H(\delta)}{Q_k^H(\delta)} \tag{6}$$

Since the linear system (6) is non-singular, $Q_k^H(\delta) \neq 0$ for all $\delta \in (0,1)$ and for all non-empty subnetworks H of G.

Denote the set of δ satisfying $\delta(u_s^{\delta,H} + u_b^{\delta,H}) = v_b$ for at least one triple (s,b,H) by $\bar{\Delta}$. Take any s,b,H. $\delta(u_s^{\delta,H} + u_b^{\delta,H}) = v_b$ is equivalent to

$$v_b = \delta\Big(\frac{P_s^H(\delta)}{Q_s^H(\delta)} + \frac{P_b^H(\delta)}{Q_b^H(\delta)}\Big).$$

Since the equation above is valid for all $\delta \in (0,1)$, it holds also for $\delta = 1/3$.

Rewriting the equation for this specific δ value, we have

$$3v_b = u_s^{1/3,H} + u_b^{1/3,H},$$

which contradicts with for all $k \in S \cup B$, $u_k^{1/3,H} \leq v_b$. It follows that for all (s, b, H) the statement $\delta(u_s^{1/3,H} + u_b^{1/3,H}) = v_b$ holds for a finite set of solutions δ, that concludes the proof.

□

Proof of Lemma 10. Take any link $(s, b) \in G$. $(s, b) \in G \backslash G^*$ implies that for all $\delta > \underline{\delta}$,

$$\delta(u_s^{*\delta} + u_b^{*\delta}) > v_b. \tag{7}$$

If the link (s, b) is involved in the limit equilibrium network G^*, then for all $\delta > \underline{\delta}$,

$$\begin{aligned} u_s^{*\delta} &= \left(1 - \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta u_s^{*\delta} + \sum_{\{b|(s,b)\in G\}} \frac{p_{sb}}{2}(v_b - \delta u_b^{*\delta}) \\ u_b^{*\delta} &= \left(1 - \sum_{\{s|(s,b)\in G\}} \frac{p_{sb}}{2}\right)\delta u_b^{*\delta} + \sum_{\{s|(s,b)\in G\}} \frac{p_{sb}}{2}(v_b - \delta u_s^{*\delta}). \end{aligned}$$

Since for all $k \neq s \in S$ with $(k, b) \in G^*$, $v_b - \delta u_k^{*\delta} \geq \delta u_b^{*\delta}$ and for all $l \neq b \in B$ with $(s, l) \in G^*$, $v_l - \delta u_l^{*\delta} \geq \delta u_s^{*\delta}$ we have

$$\begin{aligned} u_s^{*\delta} &\geq \left(1 - \frac{p_{sb}}{2}\right)\delta u_s^{*\delta} + \frac{p_{sb}}{2}(v_b - \delta u_b^{*\delta}) \\ u_b^{*\delta} &\geq \left(1 - \frac{p_{sb}}{2}\right)\delta u_b^{*\delta} + \frac{p_{sb}}{2}(v_b - \delta u_s^{*\delta}). \end{aligned} \tag{8}$$

As $\delta \to 1$, by (7), we have $u_s^* + u_b^* \geq v_b$ for all $(s, b) \in G \backslash G^*$. And from (7) and (8), for all $(s, b) \in G^*$, $u_s^* + u_b^* = v_b$.

$\square$

C Proofs of Chapter 4

Definitions of the Regions and the Corresponding Equilibrium Strategies

We define the regions R_ω and find the corresponding equilibrium strategies or proposals , $x^\omega = (x_1^\omega, x_2^\omega)$, where $\omega \in \{1.a - I.a, 1.a - I.b, \dots, 3 - III\}$, in the model with limited influence. Note that $x_2^\omega = 1 - x_1^\omega$ for each ω.Suppose that the game is at period t. Let $(r_1^t, r_2^t) = (r_1, r_2)$ and $(\bar{r}_1^t, \bar{r}_2^t) = (\bar{r}_1, \bar{r}_2)$.

Region 1.a-I.a

$$R_{1.a-I.a} = \Big\{(r_1, r_2)\Big| \frac{(1+\lambda_2)\lambda_1 r_1 + \delta(1+\gamma_1)(\zeta_2\bar{r}_2 - \lambda_2 r_2 + 1 + \lambda_2 - \delta(1+\gamma_2))}{(1+\lambda_1)(1+\lambda_2) - \delta^2(1+\gamma_1)(1+\gamma_2) + (1+\lambda_2)\zeta_1} <$$

$$\bar{r}_1 < \frac{(1+\lambda_1)(\zeta_2\bar{r}_2 - \lambda_2 r_2 + 1 + \lambda_2) + \delta(1+\gamma_2)\lambda_1 r_1 - \delta(1+\gamma_2)(1+\lambda_1)}{(1+\lambda_1)(1+\lambda_2) - \delta^2(1+\gamma_1)(1+\gamma_2) + \delta(1+\gamma_2)\zeta_1}$$

$$\text{and } \frac{\delta(1+\gamma_1)(\zeta_1\bar{r}_1 - \lambda_1 r_1 + 1 + \lambda_1 - \delta(1+\gamma_2)) - (1+\lambda_1)\lambda_2 r_2}{(1+\lambda_1)(1+\lambda_2) - \delta^2(1+\gamma_1)(1+\gamma_2) - (1+\lambda_1)\zeta_2} <$$

$$\bar{r}_2 < \frac{(1+\lambda_2)(\zeta_1\bar{r}_1 - \lambda_1 r_1 + 1 + \lambda_1 - \delta(1+\gamma_1)) + \delta(1+\gamma_1)\lambda_2 r_2}{(1+\lambda_1)(1+\lambda_2) - \delta^2(1+\gamma_1)(1+\gamma_2) + \delta(1+\gamma_1)\zeta_2}\Big\},$$

where $\zeta_i = \delta\gamma_i + (1-\delta)\lambda_i$ for all $i = 1, 2$. The equilibrium strategies are

$$x_1^{1.a-I.a} = \frac{(1+\lambda_1)(\zeta_2\bar{r}_2 - \lambda_2 r_2 + 1 + \lambda_2) - \delta(1+\gamma_1)(\zeta_1\bar{r}_1 - \lambda_1 r_1 + 1 + \lambda_1)}{(1+\lambda_1)(1+\lambda_2) - \delta^2(1+\gamma_1)(1+\gamma_2)}$$

$$y_1^{1.a-I.a} = \frac{\delta(1+\gamma_1)(\zeta_2\bar{r}_2 - \lambda_2 r_2 + 1 + \lambda_2) - (1+\lambda_2)(\zeta_1\bar{r}_1 - \lambda_1 r_1) - \delta^2(1+\gamma_2)}{(1+\lambda_1)(1+\lambda_2) - \delta^2(1+\gamma_1)(1+\gamma_2)}.$$

Region 1.a-I.b

$$R_{1.a-I.b} = \Big\{(r_1, r_2)\Big| \frac{(1+\lambda_2)\lambda_1 r_1 + \delta(1+\gamma_1)(\lambda_2\bar{r}_2 - \lambda_2 r_2 + (1-\delta)(1+\lambda_2))}{(1+\lambda_1)(1+\lambda_2) - (1+\lambda_2)(\delta^2(1+\gamma_1) - \zeta_1)} <$$

$$\bar{r}_1 < \frac{(1+\lambda_1)(\lambda_2\bar{r}_2 - \lambda_2 r_2 + (1-\delta)(1+\lambda_2)) + \delta(1+\lambda_2)\lambda_1 r_1}{(1+\lambda_1)(1+\lambda_2) - \delta(1+\lambda_2)(\delta(1+\gamma_1) - \zeta_1)}$$

$$\text{and } \bar{r}_2 > \frac{(1+\lambda_2)(\zeta_1\bar{r}_1 - \lambda_1 r_1 + 1 + \lambda_1) + \delta(1+\gamma_1)(\lambda_2 r_2 - (1+\lambda_2))}{(1+\lambda_1)(1+\lambda_2) - \delta(1+\gamma_1)(\delta(1+\lambda_2) - \lambda_2)}\Big\},$$

where $\zeta_i = \delta\gamma_i + (1-\delta)\lambda_i$ for all $i = 1, 2$. The equilibrium strategies are

$$x_1^{1.a-I.b} = \frac{(1+\lambda_1)(\lambda_2\bar{r}_2 - \lambda_2 r_2 + (1-\delta)(1+\lambda_2)) - \delta(1+\lambda_2)(\zeta_1\bar{r}_1 - \lambda_1 r_1)}{(1+\lambda_1)(1+\lambda_2) - \delta^2(1+\gamma_1)(1+\lambda_2)}$$

$$y_1^{1.a-I.b} = \frac{\delta(1+\gamma_1)(\lambda_2\bar{r}_2 - \lambda_2 r_2 + (1-\delta)(1+\lambda_2)) - (1+\lambda_2)(\zeta_1\bar{r}_1 - \lambda_1 r_1)}{(1+\lambda_1)(1+\lambda_2) - \delta^2(1+\gamma_1)(1+\lambda_2)}.$$

Region 1.a-II

$$R_{1.a-II} = \Big\{(r_1, r_2)\Big| \frac{\kappa_2\lambda_1 r_1 + \delta(1+\gamma_1)(\lambda_2 - \lambda_2 r_2 + (1-\delta)(1+\lambda_2))}{\kappa_1\kappa_2 - \delta^2(1+\gamma_1)(1+\gamma_2)} <$$

$$\bar{r}_1 < \frac{\delta(1+\gamma_2)\lambda_1 r_1 + (1+\lambda_1)(\lambda_2 - \lambda_2 r_2 + (1-\delta)(1+\lambda_2))}{(1+\lambda_1)\kappa_2 - \delta^2(1+\gamma_1)(1+\gamma_2) + \delta(1+\gamma_2)\zeta_1}$$

$$\text{and } r_2 > \frac{\delta(1+\gamma_2)(\zeta_1\bar{r}_1 - \lambda_1 r_1) + \delta(1+\lambda_1)(1+\gamma_2) - \delta^2(1+\gamma_1)(1+\gamma_2)}{(1+\lambda_1)(\kappa_2 - \lambda_2) - \delta^2(1+\gamma_1)(1+\gamma_2)} > \bar{r}_2\Big\},$$

where $\zeta_i = \delta\gamma_i + (1-\delta)\lambda_i$ and $\kappa_i = 1 + \lambda_i + \delta\gamma_i + (1-\delta)\lambda_i$ for all $i = 1, 2$. The equilibrium strategies are

$$x_1^{1.a-II} = \frac{(1+\lambda_1)(\lambda_2 - \lambda_2 r_2 + (1-\delta)(1+\lambda_2)) - \delta(1+\gamma_2)(\zeta_1\bar{r}_1 - \lambda_1 r_1)}{(1+\lambda_1)(1+\lambda_2) - \delta^2(1+\gamma_1)(1+\lambda_2)}$$

$$y_1^{1.a-II} = \frac{\delta(1+\gamma_1)(\lambda_2 - \lambda_2 r_2 + (1-\delta)(1+\lambda_2)) - \kappa_2(\zeta_1\bar{r}_1 - \lambda_1 r_1)}{(1+\lambda_1)(1+\lambda_2) - \delta^2(1+\gamma_1)(1+\lambda_2)}.$$

Region 1.a-III

$$R_{1.a-III} = \Big\{(r_1, r_2)\Big| \frac{\eta_2\lambda_1 r_1 + \delta(1+\gamma_1)(\gamma_2 - \gamma_2 r_2 + (1-\delta)(1+\lambda_2))}{\kappa_1\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)} <$$

$$\bar{r}_1 < \frac{\delta(1+\gamma_2)\lambda_1 r_1 + (1+\lambda_1)(\gamma_2 - \gamma_2 r_2 + (1-\delta)(1+\lambda_2))}{(1+\lambda_1)\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2) + \delta(1+\gamma_2)\zeta_1}$$

$$\text{and } r_2 < \frac{\delta(1+\gamma_2)(\zeta_1\bar{r}_1 - \lambda_1 r_1) + \delta(1+\lambda_1)(1+\gamma_2) - \delta^2(1+\gamma_1)(1+\gamma_2)}{(1+\lambda_1)(\kappa_2 - \gamma_2) - \delta^2(1+\gamma_1)(1+\gamma_2)}\Big\},$$

where $\zeta_i = \delta\gamma_i + (1-\delta)\lambda_i$, $\kappa_i = 1+\lambda_i+\delta\gamma_i+(1-\delta)\lambda_i$ and $\eta_i = 1+\gamma_i+\delta\gamma_i+(1-\delta)\lambda_i$ for all $i = 1, 2$. The equilibrium strategies are

$$x_1^{1.a-III} = \frac{(1+\lambda_1)(\gamma_2 - \gamma_2 r_2 + (1-\delta)(1+\lambda_2)) - \delta(1+\gamma_2)(\zeta_1\bar{r}_1 - \lambda_1 r_1)}{(1+\lambda_1)\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}$$

$$y_1^{1.a-III} = \frac{\delta(1+\gamma_1)(\gamma_2 - \gamma_2 r_2 + (1-\delta)(1+\lambda_2)) - \eta_2(\zeta_1\bar{r}_1 - \lambda_1 r_1)}{(1+\lambda_1)\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}.$$

Region 1.b-I.b

$$R_{1.b-I.b} = \Big\{(r_1, r_2)\Big|\bar{r}_1 > \frac{\delta(1+\lambda_2)(\lambda_1 r_1 - (1+\lambda_1)) - (1+\lambda_1)(\lambda_2 r_2 - \lambda_2\bar{r}_2 - (1+\lambda_2))}{(1-\delta^2)(1+\lambda_1)(1+\lambda_2) + \delta(1+\lambda_2)\lambda_1}$$

$$\text{and } \bar{r}_2 > \frac{\delta(1+\lambda_1)(\lambda_2 r_2 - (1+\lambda_2)) - (1+\lambda_2)(\lambda_1 r_1 - \lambda_1\bar{r}_1 - (1+\lambda_1))}{(1-\delta^2)(1+\lambda_1)(1+\lambda_2) + \delta(1+\lambda_1)\lambda_2}\Big\},$$

The equilibrium strategies are

$$x_1^{1.b-I.b} = \frac{\delta(1+\lambda_2)(\lambda_1 r_1 - \lambda_1\bar{r}_1 - (1+\lambda_1)) - (1+\lambda_1)(\lambda_2 r_2 - \lambda_2\bar{r}_2 - (1+\lambda_2))}{(1-\delta^2)(1+\lambda_1)(1+\lambda_2)}$$

$$y_1^{1.b-I.b} = \frac{(1+\lambda_2)(\lambda_1 r_1 - \lambda_1\bar{r}_1) - \delta(1+\lambda_1)(\lambda_2 r_2 - \lambda_2\bar{r}_2 - (1-\delta)(1+\lambda_2))}{(1-\delta^2)(1+\lambda_1)(1+\lambda_2)}.$$

Region 1.b-II

$$R_{1.b-II} = \Big\{(r_1, r_2)\Big|\bar{r}_1 > \frac{\delta(1+\gamma_2)\lambda_1 r_1 + (1+\lambda_1)(\lambda_2 - \lambda_2 r_2 + (1-\delta)(1+\lambda_2))}{(1+\lambda_1)\kappa_2 - \delta^2(1+\lambda_1)(1+\gamma_2) + \delta(1+\gamma_2)\lambda_1},$$

$$r_2 > \frac{\delta(1-\delta)(1+\lambda_1)(1+\delta\gamma_2) - \delta^2(1+\lambda_1)(1+\gamma_2) - \delta(1+\gamma_2)(\lambda_1 r_1 - \lambda_1\bar{r}_1)}{(1+\lambda_1)\kappa_2 - \delta^2(1+\lambda_1)(1+\gamma_2) + \delta(1+\lambda_1)\lambda_2} > \bar{r}_2\Big\},$$

where $\kappa_i = 1 + \lambda_i + \delta\gamma_i + (1-\delta)\lambda_i$ for all $i = 1, 2$. The equilibrium strategies are

$$x_1^{1.b-II} = \frac{\delta(1+\gamma_2)(\lambda_1 r_1 - \lambda_1 \bar{r}_1) + (1+\lambda_1)(\lambda_2 - \lambda_2 r_2 + (1-\delta)(1+\lambda_2))}{(1+\lambda_1)\kappa_2 - \delta^2(1+\lambda_1)(1+\gamma_2)}$$

$$y_1^{1.b-II} = \frac{\kappa_2(\lambda_1 r_1 - \lambda_1 \bar{r}_1) + \delta(1+\lambda_1)(\lambda_2 - \lambda_2 r_2 - (1-\delta)(1+\lambda_2))}{(1+\lambda_1)\kappa_2 - \delta^2(1+\lambda_1)(1+\gamma_2)}.$$

Region 1.b-III

$$R_{1.b-III} = \Big\{(r_1, r_2)\Big| \bar{r}_1 > \frac{\delta(1+\gamma_2)\lambda_1 r_1 + (1+\lambda_1)(\gamma_2 - \gamma_2 r_2 + (1-\delta)(1+\lambda_2))}{(1+\lambda_1)\eta_2 - \delta^2(1+\lambda_1)(1+\gamma_2) + \delta(1+\gamma_2)\lambda_1}$$

$$\text{and } r_2 < \frac{(1+\lambda_1)(1+\delta\gamma_2) - \delta^2(1+\lambda_1)(1+\gamma_2) - \delta(1+\gamma_2)(\lambda_1 r_1 - \lambda_1 \bar{r}_1)}{(1+\lambda_1)\eta_2 - \delta^2(1+\lambda_1)(1+\gamma_2) - (1+\lambda_1)\gamma_2}\Big\},$$

where $\eta_i = 1 + \gamma_i + \delta\gamma_i + (1-\delta)\lambda_i$ for all $i = 1, 2$. The equilibrium strategies are

$$x_1^{1.b-III} = \frac{\delta(1+\gamma_2)(\lambda_1 r_1 - \lambda_1 \bar{r}_1) + (1+\lambda_1)(\gamma_2 - \gamma_2 r_2 + (1-\delta)(1+\lambda_2))}{(1+\lambda_1)\eta_2 - \delta^2(1+\lambda_1)(1+\gamma_2)}$$

$$y_1^{1.b-III} = \frac{\eta_2(\lambda_1 r_1 - \lambda_1 \bar{r}_1) + \delta(1+\lambda_1)(\gamma_2 - \gamma_2 r_2 + (1-\delta)(1+\lambda_2))}{(1+\lambda_1)\eta_2 - \delta^2(1+\lambda_1)(1+\gamma_2)}.$$

Region 2-II

$$R_{2-II} = \Big\{(r_1, r_2)\Big| r_1 > \frac{\delta(1+\gamma_1)\kappa_2 - \delta(1+\gamma_1)(\delta(1+\gamma_2) + \lambda_2 r_2)}{(\kappa_1 - \lambda_1)\kappa_2 - \delta^2(1+\gamma_1)(1+\gamma_2)} > \bar{r}_1$$

$$\text{and } r_2 > \frac{\delta(1+\gamma_2)(\kappa_1 - \lambda_1 r_1) - \delta^2(1+\gamma_1)(1+\gamma_2)}{\kappa_1(\kappa_2 - \lambda_2) - \delta^2(1+\gamma_1)(1+\gamma_2)} > \bar{r}_2\Big\},$$

where $\kappa_i = 1 + \lambda_i + \delta\gamma_i + (1-\delta)\lambda_i$ for all $i = 1, 2$. The equilibrium strategies are

$$x_1^{2-II} = \frac{\kappa_1(\kappa_2 - \lambda_2 r_2) - \delta(1+\gamma_2)(\kappa_1 - \lambda_1 r_1)}{\kappa_1\kappa_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}$$

$$y_1^{2-II} = \frac{\kappa_2(\delta(1+\gamma_1) + \lambda_1 r_1) - \delta(1+\gamma_1)(\delta(1+\gamma_2) + \lambda_2 r_2)}{\kappa_1\kappa_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}.$$

Region 2-III

$$R_{2-III} = \Big\{(r_1, r_2)\Big| r_1 > \frac{\delta(1+\gamma_1)\eta_2 - \delta(1+\gamma_1)(\delta(1+\gamma_2)+\gamma_2 r_2)}{(\kappa_1-\lambda_1)\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)} > \bar{r}_1$$

$$\text{and } r_2 < \frac{\delta(1+\gamma_2)(\kappa_1-\lambda_1 r_1) - \delta^2(1+\gamma_1)(1+\gamma_2)}{\kappa_1(\eta_2-\gamma_2) - \delta^2(1+\gamma_1)(1+\gamma_2)}\Big\},$$

where $\kappa_i = 1+\lambda_i+\delta\gamma_i+(1-\delta)\lambda_i$ and $\eta_i = 1+\gamma_i+\delta\gamma_i+(1-\delta)\lambda_i$ for all $i = 1, 2$. The equilibrium strategies are

$$x_1^{2-III} = \frac{\kappa_1(\eta_2-\gamma_2 r_2) - \delta(1+\gamma_2)(\kappa_1-\lambda_1 r_1)}{\kappa_1\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}$$

$$y_1^{2-III} = \frac{\eta_2(\delta(1+\gamma_1)+\lambda_1 r_1) - \delta(1+\gamma_1)(\delta(1+\gamma_2)+\gamma_2 r_2)}{\kappa_1\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}.$$

Region 3-III

$$R_{3-III} = \Big\{(r_1, r_2)\Big| r_1 < \frac{\delta(1+\gamma_1)\eta_2 - \delta(1+\gamma_1)(\delta(1+\gamma_2)+\gamma_2 r_2)}{(\eta_1-\gamma_1)\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}$$

$$\text{and } r_2 < \frac{\delta(1+\gamma_2)(\eta_1-\gamma_1 r_1) - \delta^2(1+\gamma_1)(1+\gamma_2)}{\eta_1(\eta_2-\gamma_2) - \delta^2(1+\gamma_1)(1+\gamma_2)}\Big\},$$

where $\eta_i = 1+\gamma_i+\delta\gamma_i+(1-\delta)\lambda_i$ for all $i = 1, 2$. The equilibrium strategies are

$$x_1^{3-III} = \frac{\eta_1(\eta_2-\gamma_2 r_2) - \delta(1+\gamma_2)(\eta_1-\gamma_1 r_1)}{\eta_1\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}$$

$$y_1^{3-III} = \frac{\eta_2(\delta(1+\gamma_1)+\gamma_1 r_1) - \delta(1+\gamma_1)(\delta(1+\gamma_2)+\gamma_2 r_2)}{\eta_1\eta_2 - \delta^2(1+\gamma_1)(1+\gamma_2)}.$$

We do not describe the regions $\{R_{1.b-I.a}, R_{2,I.a}, R_{2,I.b}, R_{3,I.a}, R_{3,I.b}, R_{3,II}\}$ since they are the symmetric versions of the regions $\{R_{1.a-I.b}, R_{1.a,II}, R_{1.b,II}, R_{1.a,III}, R_{1.b,III}, R_{2,III}\}$, respectively. Hence, the corresponding equilibrium strategies can be found similarly. For the model with unlimited influence, the regions can be defined and the corresponding equilibrium strategies can be easily evaluated by assuming $(r_1, r_2) = (\bar{r}_1, \bar{r}_2)$.

Proofs

Proof of Lemma 11. Let $\varepsilon > 0$ and let σ,σ' be strategy profiles satisfying $\sigma^s = \sigma'^s$ for all $s \leq \bar{t}$ where $\bar{t} > \max_{i=1,2} \log_\delta \left(\frac{\varepsilon}{1+\gamma_i+\lambda_i}\right)$. Let $\bar{\sigma}$ be the strategy profile that maximizes U_i satisfying $(\bar{\sigma}^s)_{s=1}^{\bar{t}} = (\sigma^s)_{s=1}^{\bar{t}}$. Hence, in $\bar{\sigma}$ player i gets the entire pie in period $\bar{t}+1$.

$$\bar{U}_i = \max_{\bar{\sigma} \text{ s.t. } (\bar{\sigma}^s)_{s=1}^{\bar{t}} = (\sigma^s)_{s=1}^{\bar{t}}} U_i(\bar{\sigma}) = \delta^{\bar{t}}(1+\gamma_i(1-r_i^{\bar{t}+1})) + (1-\delta)\sum_{s=1}^{\bar{t}} \delta^{s-1} u_i^s(0)$$

Assume $\underline{\sigma}$ be the strategy profile that minimizes U_i satisfying $(\underline{\sigma}^s)_{s=1}^{\bar{t}} = (\sigma^s)_{s=1}^{\bar{t}}$, that is the strategy profile leading to perpetual disagreement.

$$\underline{U}_i = \min_{\underline{\sigma} \text{ s.t. } (\underline{\sigma}^s)_{s=1}^{\bar{t}} = (\sigma^s)_{s=1}^{\bar{t}}} U_i(\underline{\sigma}) = (1-\delta)\sum_{s=1}^{\infty} \delta^{s-1} u_i^s(0)$$

Note that the largest payoff difference between any two strategy profiles implying same actions in the first $\bar{t}$ periods is $\bar{U}_i - \underline{U}_i$. Utilizing this observation,

$$\begin{aligned}
|U_i(\sigma) - U_i(\sigma')| \leq \bar{U}_i - \underline{U}_i & \\
&= \delta^{\bar{t}}(1+\gamma_i(1-r_i^{\bar{t}+1})) + (1-\delta)\sum_{s=1}^{\bar{t}} \delta^{s-1} u_i^s(0) \\
&- (1-\delta)\sum_{s=1}^{\bar{t}} \delta^{s-1} u_i^s(0) - (1-\delta)\sum_{s=\bar{t}+1}^{\infty} \delta^{s-1} u_i^s(0) \\
&= \delta^{\bar{t}}(1+\gamma_i(1-r_i^{\bar{t}+1})) - (1-\delta)\sum_{s=\bar{t}+1}^{\infty} \delta^{s-1}(-\lambda_i r_i^s) \\
&\leq \delta^{\bar{t}}(1+\gamma_i) + (1-\delta)\sum_{s=\bar{t}+1}^{\infty} \delta^{s-1}\lambda_i \\
&= \delta^{\bar{t}}(1+\gamma_i) + (1-\delta)\delta^{\bar{t}}\lambda_i \frac{1}{1-\delta} \\
&= \delta^{\bar{t}}(1+\gamma_i+\lambda_i) < \varepsilon.
\end{aligned}$$

Hence, the game is continuous at infinity. □

Proof of Theorem 12. We prove that σ^* is a subgame perfect equilibrium. Utilizing Lemma 11, it suffices to check that no player can make a profitable deviation from his strategy σ_i^* in one single period, given that his opponent plays σ_j^*.

Suppose that the game is at period t. Let $(r_1^t, r_2^t) = (r_1, r_2) \in R_\omega$, where $\omega \in \{1.a - I.a, 1.a - I.b, \ldots, 3 - III\}$. $(\bar{r}_1^t, \bar{r}_2^t)$ is denoted by $(\bar{r}_1, \bar{r}_2)$.

The current period t may be either odd or even. First, we investigate the case where t is odd. Hence, player 1 makes an offer $z \in Z$.

Case 1.a. $\omega \in \{1.a - I, 1.a - II, 1.a - III\} : r_1 > \bar{r}_1 > y_1^\omega$ and $x_1^\omega \geq \bar{r}_1$.

1.a.1 We first analyze the case where $x_1^\omega \geq r_1$. We have three possible sub-cases: (i) $z_1 = x_1^\omega$, (ii) $z_1 < x_1^\omega$, and (iii) $z_1 > x_1^\omega$.

(i) If $z_1 = x_1^\omega$, then player 2 accepts the offer by following σ_2^*. So, player 1 gets

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1) = (1 + \gamma_1)x_1^\omega - \gamma_1 r_1.$$

(ii) If $z_1 < x_1^\omega$, then player 2 accepts the offer by following σ_2^* since $z_2 > x_2^\omega$. So, player 1 gets

$$u_1^t(z, r^t) \leq z_1 + \gamma_1(z_1 - r_1) = (1 + \gamma_1)x_1^\omega - \gamma_1 r_1 = u_1^*.$$

(iii) If $z_1 > x_1^\omega$, then player 2 rejects the offer since $z_2 < x_2^\omega$. With probability δ the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, player 1's expected continuation utility

is

$$
\begin{aligned}
& u_1^{t+1}(y^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
= \ & \delta[y_1^\omega + \lambda_1(y_1^\omega - r_1)] - (1-\delta)\lambda_1 r_1 \\
= \ & (1+\lambda_1)\delta y_1^\omega - \delta\lambda_1 r_1 - (1-\delta)\lambda_1 r_1 \\
< \ & y_1^\omega - \lambda_1 \underbrace{(y_1^\omega - r_1)}_{<0} \\
< \ & y_1^\omega - \gamma_1(y_1^\omega - r_1) \\
< \ & x_1^\omega - \gamma_1(x_1^\omega - r_1) = u_1^*.
\end{aligned}
$$

To summarize, making the offer z satisfying $z_1 = x_1^\omega$ gives player 1 the maximum utility given that player 2 follows σ_2^*. Thus, σ_1^* is optimal.

1.a.2 Now, we analyze $x_1^\omega < r_1$ case. Similarly, we look into three possible subcases: (i) $z_1 = x_1^\omega$, (ii) $z_1 < x_1^\omega$, and (iii) $z_1 > x_1^\omega$.

(i) If $z_1 = x_1^\omega$, then player 2 accepts the offer by following σ_2^*. So, player 1 gets

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1) = (1+\lambda_1)x_1^\omega - \lambda_1 r_1.$$

(ii) If $z_1 < x_1^\omega$, then player 2 accepts the offer by following σ_2^* since $z_2 > x_2^\omega$. So, player 1 gets

$$u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1) < (1+\lambda_1)x_1^\omega - \lambda_1 r_1 = u_1^*.$$

(iii) If $z_1 > x_1^\omega$, then player 2 rejects the offer since $z_2 < x_2^\omega$. With probability δ the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, player 1's expected continuation utility

is

$$\begin{aligned} u_1^{t+1}(y^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) &= \delta[y_1^\omega + \lambda_1(y_1^\omega - r_1)] - (1-\delta)\lambda_1 r_1 \\ &= (1+\lambda_1)\delta y_1^\omega - \delta\lambda_1 r_1 - (1-\delta)\lambda_1 r_1 \\ &< (1+\lambda_1)y_1^\omega - \lambda_1 r_1 \\ &< (1+\lambda_1)x_1^\omega - \lambda_1 r_1 = u_1^*. \end{aligned}$$

To summarize, making the offer z satisfying $z_1 = x_1^\omega$ gives player 1 the maximum utility given that player 2 follows σ_2^*. Thus, σ_1^* is optimal.

Case 1.b. $\omega \in \{1.b - I, 1.b - II, 1.b - III\} : r_1 > \bar{r}_1 > y_1^\omega$ and $x_1^\omega < \bar{r}_1$.

In this case, we also have three possible values for the share player 1 gets from z: (i) $z_1 = x_1^\omega$, (ii) $z_1 < x_1^\omega$, and (iii) $z_1 > x_1^\omega$.

(i) If $z_1 = x_1^\omega$, then player 2 accepts the offer by following σ_2^*. So, player 1 gets

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1) = (1+\lambda_1)x_1^\omega - \lambda_1 r_1.$$

(ii) If $z_1 < x_1^\omega$, then player 2 accepts the offer by following σ_2^* since $z_2 > x_2^\omega$. So, player 1 gets

$$u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1) < (1+\lambda_1)x_1^\omega - \lambda_1 r_1 = u_1^*.$$

(iii) If $z_1 > x_1^\omega$, then player 2 rejects the offer since $z_2 < x_2^\omega$. With probability δ the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, player 1's expected continuation utility

is

$$\begin{aligned}
u_1^{t+1}(y^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) &= \delta[y_1^\omega + \lambda_1(y_1^\omega - r_1)] - (1-\delta)\lambda_1 r_1 \\
&= (1+\lambda_1)\delta y_1^\omega - \delta\lambda_1 r_1 - (1-\delta)\lambda_1 r_1 \\
&< (1+\lambda_1)y_1^\omega - \lambda_1 r_1 \\
&< (1+\lambda_1)x_1^\omega - \lambda_1 r_1 = u_1^*.
\end{aligned}$$

To summarize, making the offer z satisfying $z_1 = x_1^\omega$ gives player 1 the maximum utility given that player 2 follows σ_2^*. Thus, σ_1^* is optimal.

Case 2. $\omega \in \{2-I, 2-II, 2-III\} : r_1 > y_1^\omega > \bar{r}_1$

In this region of r_1, we have different results for different sortings of x_1^ω and r_1 at t. We analyze these cases separately.

2.1. We first investigate the case where $x_1^\omega \geq r_1$. Again, we have three distinct sub-cases for z_1.

(i) If $z_1 = x_1^\omega$, then player 2 accepts the offer by following σ_2^*. So, player 1 gets

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1) = (1+\gamma_1)x_1^\omega - \gamma_1 r_1.$$

(ii) If $z_1 < x_1^\omega$, then player 2 accepts the offer by following σ_2^* since $z_2 > x_2^\omega$. So, player 1 gets

$$u_1^t(z, r^t) \leq z_1 + \gamma_1(z_1 - r_1) = (1+\gamma_1)x_1^\omega - \gamma_1 r_1 = u_1^*.$$

(iii) If $z_1 > x_1^\omega$, then player 2 rejects the offer. With probability δ the game continues to the next period and player 2 offers y^ω and with probability

$1-\delta$, the game ends. So, player 1's expected continuation utility is

$$\begin{aligned} u_1^{t+1}(y^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) &= \delta[y_1^\omega + \lambda_1 \underbrace{(y_1^\omega - r_1)}_{<0}] - (1-\delta)\lambda_1 r_1 \\ &< \delta[y_1^\omega + \gamma_1(y_1^\omega - r_1)] - (1-\delta)\gamma_1 r_1 \\ &< (1+\gamma_1)y_1^\omega - \gamma_1 r_1 \\ &< x_1^\omega - \gamma_1(x_1^\omega - r_1) = u_1^*. \end{aligned}$$

To summarize, making the offer z satisfying $z_1 = x_1^\omega$ gives player 1 the maximum utility given that player 2 follows σ_2^*. Thus, σ_1^* is optimal.

2.2. Now, we analyze the second case, $x_1^\omega < r_1$.

(i) If $z_1 = x_1^\omega$, then player 2 accepts the offer by following σ_2^*. So, player 1 gets

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1) = (1+\lambda_1)x_1^\omega - \lambda_1 r_1.$$

(ii) If $z_1 < x_1^\omega$, then player 2 player 2 accepts the offer by following σ_2^* since $z_2 > x_2^\omega$. So, player 1 gets

$$u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1) < (1+\lambda_1)x_1^\omega - \lambda_1 r_1 = u_1^*.$$

(iii) If $z_1 > x_1^\omega$, then player 2 rejects the offer since $z_2 < x_2^\omega$. With probability δ the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, player 1's expected continuation utility

is

$$\begin{aligned}
u_1^{t+1}(y^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) &= \delta[y_1^\omega + \lambda_1(y_1^\omega - r_1)] - (1-\delta)\lambda_1 r_1 \\
&= (1+\lambda_1)\delta y_1^\omega - \delta\lambda_1 r_1 - (1-\delta)\lambda_1 r_1 \\
&< (1+\lambda_1)y_1^\omega - \lambda_1 r_1 \\
&< (1+\lambda_1)x_1^\omega - \lambda_1 r_1 = u_1^*.
\end{aligned}$$

To summarize, making the offer z satisfying $z_1 = x_1^\omega$ gives player 1 the maximum utility given that player 2 follows σ_2^*. Thus, σ_1^* is optimal.

Case 3. $\omega \in \{3-I, 3-II, 3-III\} : y_1^\omega > r_1 > \bar{r}_1$

In this region, the reference point of player 1 is less than his share in equilibrium. We have three possible values for the share he can obtain from the offer z.

(i) If $z_1 = x_1^\omega$, then player 2 accepts the offer by following σ_2^*. So, player 1 gets

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1) = (1+\gamma_1)x_1^\omega - \gamma_1 r_1.$$

(ii) If $z_1 < x_1^\omega$, then player 2 accepts the offer by following σ_2^* since $z_2 > x_2^\omega$. So, player 1 gets

$$u_1^t(z, r^t) \leq z_1 + \gamma_1(z_1 - r_1) < (1+\gamma_1)x_1^\omega - \gamma_1 r_1 = u_1^*.$$

(iii) If $z_1 > x_1^\omega$, then player 2 rejects the offer since $z_2 < x_2^\omega$. With probability δ the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, player 1's expected continuation utility

is

$$
\begin{aligned}
u_1^{t+1}(y^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) &= \delta[y_1^\omega + \gamma_1(y_1^\omega - r_1)] - (1-\delta)\lambda_1 r_1 \\
&< \delta[y_1^\omega + \gamma_1(y_1^\omega - r_1)] - (1-\delta)\gamma_1 r_1 \\
&= (1+\gamma_1)\delta y_1^\omega - \gamma_1 r_1 \\
&< (1+\gamma_1)\delta x_1^\omega - \gamma_1 r_1 = u_1^*.
\end{aligned}
$$

To summarize, making the offer z satisfying $z_1 = x_1^\omega$ gives player 1 the maximum utility given that player 2 follows σ_2^*. Thus, σ_1^* is optimal.

Second, we analyze the case where t is even (i.e., player 2's turn to make an offer). Hence, player 2 makes an offer $z \in Z$. If player 1 accepts the offer, he gets

$$
u_1^t(z, r^t) = \begin{cases} z_i + \gamma_i(z_i - r_i) & \text{if } z_i \geq r_i \\ z_i + \lambda_i(z_i - r_i) & \text{if } z_i < r_i \end{cases}
$$

On the other hand, if player 1 rejects the offer, his reference point will be $\max\{\bar{r}_1, z_1\}$. With probability δ, the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. Hence, player 1's expected continuation utility in the case of rejection is

$$
\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}).
$$

Case 1.a. $\omega \in \{1.a-I, 1.a-II, 1.a-III\}$: $r_1 > \bar{r}_1 > y_1^\omega$ and $x_1^\omega \geq \bar{r}_1$

In this case, recall that the following equality holds in equilibrium:

$$
(1+\gamma_1)\delta x_1^\omega = (1+\lambda_1)y_1^\omega - \lambda_1 r_1 + (1-\delta)\lambda_1 \bar{r}_1 + \delta\gamma_1 \bar{r}_1.
$$

We analyze the optimal decisions of player 1 in two sub-cases: (1) $z_1 \geq y_1^\omega$ and

(2) $z_1 < y_1^\omega$.

1.a.1 First, suppose that $z_1 \geq y_1^\omega$. Further, there are two more possibilities: (i) $z_1 \geq r_1$ or (ii) $z_1 < r_1$.

(i) Suppose that $z_1 \geq r_1$. If player 1 accepts the offer, he gets his share z_1 plus the relative gain from the reference point. Since the offer is greater than his reference point, he perceives the difference between the two as a gain and this gain is scaled by γ_1. Accepting yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1).$$

If player 1 rejects the offer, his reference point will be $\max\{\bar{r}_1, z_1\}$. With probability δ, the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, rejection gives player 1

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&\quad \leq \delta[x_1^\omega + \gamma_1(x_1^\omega - \max\{\bar{r}_1, z_1\})] - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&\quad = (1+\gamma_1)\delta x_1^\omega - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
&\quad = (1+\lambda_1)y_1^\omega - \lambda_1 r_1 + (1-\delta)\lambda_1\bar{r}_1 + \delta\gamma_1\bar{r}_1 - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
&\quad \leq (1+\lambda_1)y_1^\omega - \lambda_1 r_1 + (1-\delta)\lambda_1 z_1 + \delta\gamma_1 z_1 - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
&\quad \leq (1+\lambda_1)z_1 - \lambda_1 r_1 \\
&\quad = z_1 + \lambda_1(z_1 - r_1) \\
&\quad \leq z_1 + \gamma_1(z_1 - r_1) = u_1^*.
\end{aligned}$$

In this case, acceptance gives player 1 a higher utility than rejection does. If $z_1 \geq y_1^\omega$, accepting the offer z_1 is optimal.

(ii) Now, suppose that $z_1 < r_1$. In the case of acceptance, player 1 gets his share from the offer z_1. However, since the offer is less than his reference point, player 1 perceives this offer as a loss relative to the reference point. The difference between z_1 and r_1 negatively affects his utility and this effect is scaled by λ_1. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1).$$

If player 1 rejects the offer, he gets

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&\quad \leq \ \delta[x_1^\omega + \gamma_1(x_1^\omega - \max\{\bar{r}_1, z_1\})] - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&\quad = \ (1+\gamma_1)\delta x_1^\omega - \delta\gamma_1 \max\{\bar{r}_1, z_1\} - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&\quad = \ (1+\lambda_1)y_1^\omega - \lambda_1 r_1 + (1-\delta)\lambda_1\bar{r}_1 + \delta\gamma_1\bar{r}_1 - \delta\gamma_1 \max\{\bar{r}_1, z_1\} \\
&\qquad\quad - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&\quad \leq \ (1+\lambda_1)y_1^\omega - \lambda_1 r_1 + (1-\delta)\lambda_1\bar{r}_1 + \delta\gamma_1\bar{r}_1 - \delta\gamma_1\bar{r}_1 - (1-\delta)\lambda_1\bar{r}_1 \\
&\quad = \ z_1 + \lambda_1(z_1 - r_1) = u_1^*.
\end{aligned}$$

In this case, acceptance gives player 1 a higher utility than rejection does. If $z_1 \geq y_1^\omega$, accepting the offer z_1 is optimal.

1.a.2 Second, suppose that $z_1 < y_1^\omega$. Note that the offer is smaller than his reference point. In the case of acceptance, he gets his share from the offer z_1. However, since the offer is less than his reference point, player 1 perceives this offer as a loss relative to the reference point. The difference between z_1 and r_1 negatively affects his utility and this effect is scaled by λ_1. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1).$$

If player 1 rejects the offer, he gets

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&= \delta[x_1^\omega + \gamma_1(x_1^\omega - \max\{\bar{r}_1, z_1\})] - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&= (1+\gamma_1)\delta x_1^\omega - \delta\gamma_1\bar{r}_1 - (1-\delta)\lambda_1\bar{r}_1 \\
&= (1+\lambda_1)y_1^\omega - \lambda_1 r_1 + (1-\delta)\lambda_1\bar{r}_1 + \delta\gamma_1\bar{r}_1 - \delta\gamma_1\bar{r}_1 - (1-\delta)\lambda_1\bar{r}_1 \\
&> z_1 + \lambda_1(z_1 - r_1) = u_1^*.
\end{aligned}$$

In this case, rejecting gives player 1 a higher utility than accepting does. Thus, if $z_1 < y_1^\omega$, then rejecting is optimal.

To summarize, accepting the offer z_1 is optimal if and only if $z_1 \geq y_1^\omega$.

Case 1.b. $\omega \in \{1.b - I, 1.b - II, 1.b - III\}$: $r_1 > \bar{r}_1 > y_1^\omega$ and $x_1^\omega < \bar{r}_1$.

In this case, recall that the following equality holds in equilibrium:

$$(1+\lambda_1)\delta x_1^\omega = (1+\lambda_1)\delta y_1^\omega - \lambda_1 r_1 + \lambda_1\bar{r}_1.$$

We analyze the optimal decisions of player 1 in two sub-cases: (1) $z_1 \geq y_1^\omega$ and (2) $z_1 < y_1^\omega$.

1.b.1 First, suppose that $z_1 \geq y_1^\omega$. Further, there are two more possibilities: (*i*) $z_1 \geq r_1$ or (*ii*) $z_1 < r_1$.

  (i) Let $z_1 \geq r_1$. If player 1 accepts the offer, he gets his share z_1 plus the relative gain from the reference point. Since the offer is greater than his reference point, he perceives the difference between the two as a gain and this gain is scaled by γ_1. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1).$$

If player 1 rejects the offer, his reference point will be $\max\{\bar{r}_1, z_1\}$. With probability δ, the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, rejection gives player 1

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&= \delta[x_1^\omega + \lambda_1(x_1^\omega - \max\{\bar{r}_1, z_1\})] - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&= (1+\lambda_1)\delta x_1^\omega - \delta\lambda_1 z_1 - (1-\delta)\lambda_1 z_1 \\
&= (1+\lambda_1)\delta y_1^\omega - \lambda_1 r_1 + \lambda_1 \bar{r}_1 - \lambda_1 z_1 \\
&\leq (1+\lambda_1)\delta y_1^\omega - \lambda_1 r_1 + \lambda_1 z_1 - \lambda_1 z_1 \\
&= y_1^\omega + \lambda_1(y_1^\omega - r_1) \\
&\leq y_1^\omega + \gamma_1(y_1^\omega - r_1) \\
&\leq z_1 + \gamma_1(z_1 - r_1) = u_1^*.
\end{aligned}$$

In this case, acceptance gives player 1 a higher utility than rejection does. If $z_1 \geq y_1^\omega$, accepting the offer z_1 is optimal.

(ii) Now, let $z_1 < r_1$. In the case of acceptance, player 1 gets his share from the offer z_1. However, since the offer is less than his reference point, player 1 perceives this offer as a loss relative to the reference point. The difference between z_1 and r_1 negatively affects his utility and this effect is scaled by λ_1. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1).$$

If player 1 rejects the offer, he gets

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&= \delta[x_1^\omega + \lambda_1(x_1^\omega - \max\{\bar{r}_1, z_1\})] - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&= (1+\lambda_1)\delta x_1^\omega - \delta\lambda_1 \max\{\bar{r}_1, z_1\} - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&= (1+\lambda_1)\delta y_1^\omega - \lambda_1 r_1 + \lambda_1 \bar{r}_1 - \lambda_1 \max\{\bar{r}_1, z_1\} \\
&\leq (1+\lambda_1)\delta y_1^\omega - \lambda_1 r_1 + \lambda_1 \bar{r}_1 - \lambda_1 \bar{r}_1 \\
&\leq z_1 + \lambda_1(z_1 - r_1) = u_1^*.
\end{aligned}$$

In this case, acceptance gives player 1 a higher utility than rejection does. If $z_1 \geq y_1^\omega$, accepting the offer z_1 is optimal.

1.b.2 Second, suppose that $z_1 < y_1^\omega$. Note that the offer is smaller than his reference point. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1).$$

If player 1 rejects the offer, he gets

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&= \delta[x_1^\omega + \lambda_1(x_1^\omega - \max\{\bar{r}_1, z_1\})] - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&= (1+\lambda_1)\delta x_1^\omega - \lambda_1 \bar{r}_1 \\
&= (1+\lambda_1)\delta y_1^\omega - \lambda_1 r_1 + \lambda_1 \bar{r}_1 - \lambda_1 \bar{r}_1 \\
&> z_1 + \lambda_1(z_1 - r_1) = u_1^*.
\end{aligned}$$

In this case, rejecting gives player 1 a higher utility than accepting does. If $z_1 < y_1^\omega$, then rejecting is optimal.

To summarize, accepting the offer z_1 is optimal if and only if $z_1 \geq y_1^\omega$.

Case 2. $\omega \in \{2-I, 2-II, 2-III\}$: $r_1 > y_1^\omega > \bar{r}_1$

In this case, recall that the following equality holds in equilibrium:

$$(1+\gamma_1)\delta x_1^\omega = (1+\lambda_1+\delta\gamma_1+(1-\delta)\lambda_1)y_1^\omega - \lambda_1 r_1.$$

We analyze the optimal decisions of player 1 in two sub-cases: (1) $z_1 \geq y_1^\omega$ and (2) $z_1 < y_1^\omega$.

2.1. Suppose that $z_1 \geq y_1^\omega$. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = \begin{cases} z_i + \gamma_i(z_i - r_i) & \text{if } z_i \geq r_i \\ z_i + \lambda_i(z_i - r_i) & \text{if } z_i < r_i \end{cases}$$

If player 1 rejects the offer, his reference point will be $\max\{\bar{r}_1, z_1\}$. With probability δ, the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, rejection gives player 1

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&\leq \delta[x_1^\omega + \gamma_1(x_1^\omega - \max\{\bar{r}_1, z_1\})] - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&= (1+\gamma_1)\delta x_1^\omega - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
&= (1+\lambda_1+\delta\gamma_1+(1-\delta)\lambda_1)y_1^\omega - \lambda_1 r_1 - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
&\leq (1+\lambda_1)y_1^\omega + \delta\gamma_1 y_1^\omega + (1-\delta)\lambda_1 y_1^\omega - \lambda_1 r_1 - \delta\gamma_1 y_1^\omega - (1-\delta)\lambda_1 y_1^\omega \\
&\leq y_1^\omega + \lambda_1(y_1^\omega - r_1) \leq u_1^*.
\end{aligned}$$

In this case, acceptance gives player 1 a higher utility than rejection does. If $z_1 \geq y_1^\omega$, accepting the offer z_1 is optimal.

2.2. Now, suppose that $z_1 < y_1^\omega$. In the case of acceptance, he gets his share from the offer z_1. However, since the offer is less than his reference point, player 1 perceives this offer as a loss relative to the reference point. The difference between z_1 and r_1 negatively affects his utility and this effect is scaled by λ_1. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1).$$

If player 1 rejects the offer, he gets

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&= \delta[x_1^\omega + \gamma_1(x_1^\omega - \max\{\bar{r}_1, z_1\})] - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&= (1+\gamma_1)\delta x_1^\omega - \delta\gamma_1 \max\{\bar{r}_1, z_1\} - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&= (1+\lambda_1+\delta\gamma_1+(1-\delta)\lambda_1)y_1^\omega - \lambda_1 r_1 - \delta\gamma_1 \max\{\bar{r}_1, z_1\} \\
&\quad - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&> (1+\lambda_1)y_1^\omega + (\delta\gamma_1 + (1-\delta)\lambda_1)\max\{\bar{r}_1, z_1\} - \lambda_1 r_1 \\
&\quad - (\delta\gamma_1 + (1-\delta)\lambda_1)\max\{\bar{r}_1, z_1\} \\
&= z_1 + \lambda_1(z_1 - r_1) = u_1^*.
\end{aligned}$$

In this case, rejecting gives player 1 a higher utility than accepting does. If $z_1 < y_1^\omega$, then rejecting is optimal.

To summarize, accepting the offer z_1 is optimal if and only if $z_1 \geq y_1^\omega$.

Case 3. $\omega \in \{3-I, 3-II, 3-III\}$: $y_1^\omega > r_1 > \bar{r}_1$

In this case, recall that the following equality holds in equilibrium:

$$(1+\gamma_1)\delta x_1^\omega = (1+\gamma_1+\delta\gamma_1+(1-\delta)\lambda_1)y_1^\omega - \gamma_1 r_1.$$

We analyze the optimal decisions of player 1 in two sub-cases: (1) $z_1 \geq y_1^\omega$ and (2) $z_1 < y_1^\omega$.

3.1. Suppose that $z_1 \geq y_1^\omega$. If player 1 accepts the offer, he gets his share z_1 plus the relative gain from the reference point. Since the offer is greater than his reference point, he perceives the difference between two as a gain and this gain is scaled by γ_1. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1).$$

If player 1 rejects the offer, his reference point will be $\max\{\bar{r}_1, z_1\}$. With probability δ, the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, rejection gives player 1

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
\leq\ & \delta[x_1^\omega + \gamma_1(x_1^\omega - \max\{\bar{r}_1, z_1\})] - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
=\ & (1+\gamma_1)\delta x_1^\omega - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
=\ & (1+\gamma_1+\delta\gamma_1+(1-\delta)\lambda_1)y_1^\omega - \gamma_1 r_1 - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
\leq\ & (1+\gamma_1)y_1^\omega + (\delta\gamma_1+(1-\delta)\lambda_1)z_1 - \gamma_1 r_1 - (\delta\gamma_1+(1-\delta)\lambda_1)z_1 \\
\leq\ & z_1 + \gamma_1(z_1 - r_1) = u_1^*.
\end{aligned}$$

In this case, acceptance gives player 1 a higher utility than rejection does. If $z_1 \geq y_1^\omega$, accepting the offer z_1 is optimal.

3.2. Now, suppose that $z_1 < y_1^\omega$.Accepting the offer yields

$$u_1^t(z, r^t) = \begin{cases} z_i + \gamma_i(z_i - r_i) & \text{if } z_i \geq r_i \\ z_i + \lambda_i(z_i - r_i) & \text{if } z_i < r_i \end{cases}$$

If player 1 rejects the offer, he gets

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&= \delta[x_1^\omega + \gamma_1(x_1^\omega - \max\{\bar{r}_1, z_1\})] - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&= (1+\gamma_1)\delta x_1^\omega - \delta\gamma_1 \max\{\bar{r}_1, z_1\} - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&= (1+\gamma_1+\delta\gamma_1+(1-\delta)\lambda_1)y_1^\omega - \gamma_1 r_1 - \delta\gamma_1 \max\{\bar{r}_1, z_1\} \\
&\qquad - (1-\delta)\lambda_1 \max\{\bar{r}_1, z_1\} \\
&> (1+\gamma_1)z_1 + (\delta\gamma_1 + (1-\delta)\lambda_1)\max\{\bar{r}_1, z_1\} - \gamma_1 r_1 \\
&\qquad - (\delta\gamma_1 + (1-\delta)\lambda_1)\max\{\bar{r}_1, z_1\} \\
&= (1+\gamma_1)z_1 - \gamma_1 r_1 \geq u_1^*.
\end{aligned}$$

In this case, rejecting gives player 1 a higher utility than accepting does. If $z_1 < y_1^\omega$, then rejecting is optimal.

To summarize, accepting the offer z_1 is optimal if and only if $z_1 \geq y_1^\omega$.

Considering all cases, we have the following result: it is optimal to accept the offer z if $z_1 \geq y_1^\omega$ and to reject it otherwise, which means following σ_1^* is optimal. □

Proof of Theorem 13. We prove that σ^* is a subgame perfect equilibrium. Utilizing Lemma 11, it is enough to check that no player can make a profitable deviation from his strategy σ_i^* in one single period, given that his opponent plays σ_j^*. Suppose that the game is at period t. Let $(r_1^t, r_2^t) = (r_1, r_2) \in R_\omega$ where $\omega \in \{1-I, 1-II, \ldots, 3-III\}$.

The current period t may be either odd or even. First, we investigate the case where t is odd. Hence, player 1 makes an offer $z \in Z$.

Case 1. $\omega \in \{1-I, 1-II, 1-III\} : r_1 > x_1^\omega > y_1^\omega$

In this case, the reference point of player 1 is greater than his share in the associated offer with the region R_ω. We have three possible sub-cases: (i) $z_1 = x_1^\omega$, (ii) $z_1 < x_1^\omega$, and (iii) $z_1 > x_1^\omega$.

(i) If $z_1 = x_1^\omega$, then player 2 accepts the offer by following σ_2^*. So, player 1 gets

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1) = (1 + \lambda_1)x_1^\omega - \lambda_1 r_1.$$

(ii) If $z_1 < x_1^\omega$, then player 2 accepts the offer by following strategy σ_2^* since $z_2 > x_2^\omega$. So, player 1 gets

$$u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1) = (1 + \lambda_1)z_1\lambda_1 r_1 < (1 + \lambda_1)x_1^\omega - \lambda_1 r_1) = u_1^*.$$

(iii) If $z_1 > x_1^\omega$, then player 2 rejects the offer since $z_2 < x_2^\omega$. With probability δ the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, player 1's expected continuation utility is

$$\begin{aligned} u_1^{t+1}(y^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) &= \delta[y_1^\omega + \lambda_1(y_1^\omega - r_1)] - (1-\delta)\lambda_1 r_1 \\ &= (1 + \lambda_1)\delta y_1^\omega - \delta\lambda_1 r_1 - (1-\delta)\lambda_1 r_1 \\ &= (1 + \lambda_1)\delta y_1^\omega - \lambda_1 r_1 \\ &< (1 + \lambda_1)\delta x_1^\omega - \lambda_1 r_1 = u_1^*. \end{aligned}$$

To summarize, making the offer z satisfying $z_1 = x_1^\omega$ gives player 1 the maximum utility given that player 2 follows σ_2^*. Thus, σ_1^* is optimal.

Case 2. $\omega \in \{2 - I, 2 - II, 2 - III\} : x_1^\omega \geq r_1 > y_1^\omega$

In this case, the reference point of player 1 belongs to the region which is bounded by his and his opponent's share from the associated offer with the region R_ω.

Again, we have three distinct sub-cases: (i) $z_1 = x_1^\omega$, (ii) $z_1 < x_1^\omega$, and (iii) $z_1 > x_1^\omega$.

(i) If $z_1 = x_1^\omega$, then player 2 accepts the offer by following σ_2^*. So, player 1 gets

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1) = (1+\gamma_1)x_1^\omega - \gamma_1 r_1.$$

(ii) If $z_1 < x_1^\omega$, then player 2 player 2 accepts the offer by following σ_2^* since $z_2 > x_2^\omega$.

If $r_1 > z_1$, then player 1 gets

$$\begin{aligned} u_1^t(z, r^t) &= z_1 + \lambda_1(z_1 - r_1) \leq z_1 + \gamma_1(z_1 - r_1) \\ &= (1+\gamma_1)z_1 - \gamma_1 r_1 < (1+\gamma_1)x_1^\omega - \gamma_1 r_1 = u_1^*. \end{aligned}$$

If $r_1 \leq z_1$, then player 1 gets

$$u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1) < (1+\gamma_1)x_1^\omega - \gamma_1 r_1 = u_1^*.$$

(iii) If $z_1 > x_1^\omega$, then player 2 rejects the offer since $z_2 < x_2^\omega$. With probability δ the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, player 1's expected continuation utility is

$$\begin{aligned} u_1^{t+1}(y^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) &= \delta[y_1^\omega + \lambda_1(y_1^\omega - r_1)] - (1-\delta)\lambda_1 r_1 \\ &< \delta[y_1^\omega + \gamma_1(y_1^\omega - r_1)] - (1-\delta)\gamma_1 r_1 \\ &= (1+\gamma_1)\delta y_1^\omega - \gamma_1 r_1 \\ &< (1+\gamma_1)\delta x_1^\omega - \gamma_1 r_1 = u_1^*. \end{aligned}$$

To summarize, making the offer z satisfying $z_1 = x_1^\omega$ gives player 1 the maximum utility given that player 2 follows σ_2^*. Thus, σ_1^* is optimal.

Case 3. $\omega \in \{3-I, 3-II, 3-III\} : x_1^\omega > y_1^\omega \geq r_1$

In this case, the reference point of the first player is less than both his share offer and his opponent's share from the associated offer with the region R_ω. We have three possible values for his share obtained from the offer z.

(i) If $z_1 = x_1^\omega$, then player 2 accepts the offer by following σ_2^*. So, player 1 gets

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1) = (1+\gamma_1)x_1^\omega - \gamma_1 r_1.$$

(ii) If $z_1 < x_1^\omega$, then player 2 player 2 accepts the offer by following σ_2^* since $z_2 > x_2^\omega$.

If $r_1 > z_1$, then player 1 gets

$$\begin{aligned} u_1^t(z, r^t) &= z_1 + \lambda_1(z_1 - r_1) \leq z_1 + \gamma_1(z_1 - r_1) \\ &= (1+\gamma_1)z_1 - \gamma_1 r_1 < (1+\gamma_1)x_1^\omega - \gamma_1 r_1 = u_1^*. \end{aligned}$$

If $r_1 \leq z_1$, then player 1 gets

$$u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1) < (1+\gamma_1)x_1^\omega - \gamma_1 r_1 = u_1^*.$$

(iii) If $z_1 > x_1^\omega$, then player 2 rejects the offer since $z_2 < x_2^\omega$. With probability δ the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, player 1's expected continuation utility

is

$$\begin{aligned}
u_1^{t+1}(y^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) &= \delta[y_1^\omega + \gamma_1(y_1^\omega - r_1)] - (1-\delta)\lambda_1 r_1 \\
&< \delta[y_1^\omega + \gamma_1(y_1^\omega - r_1)] - (1-\delta)\gamma_1 r_1 \\
&= (1+\gamma_1)\delta y_1^\omega - \gamma_1 r_1 \\
&< (1+\gamma_1)\delta x_1^\omega - \gamma_1 r_1 = u_1^*.
\end{aligned}$$

To summarize, making the offer z satisfying $z_1 = x_1^\omega$ gives player 1 the maximum utility given that player 2 follows σ_2^*. Thus, σ_1^* is optimal.

Now, we analyze the case where t is even (i.e., player 2's turn to make an offer). Hence, player 2 make an offer $z \in Z$. If player 1 accepts the offer, then he gets

$$u_1^t(z, r^t) = \begin{cases} z_i + \gamma_i(z_i - r_i) & \text{if } z_i \geq r_i \\ z_i + \lambda_i(z_i - r_i) & \text{if } z_i < r_i \end{cases}$$

On the other hand, if player 1 rejects the offer, his reference point will be $\max\{r_1, z_1\}$. With probability δ, the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. Hence, player 1's expected continuation utility in the case of rejection is

$$\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}).$$

Case 1. $\omega \in \{1-I, 1-II, 1-III\} : r_1 > x_1^\omega > y_1^\omega$

In this case, recall that the following equality holds in equilibrium:

$$\delta x_1^\omega = y_1^\omega.$$

We analyze the optimal decisions of player 1 in two sub-cases: (1) $z_1 \geq r_1$ and (2) $z_1 < r_1$.

1.1. First, suppose that $z_1 \geq r_1$. If player 1 accepts the offer, he gets his share z_1 plus the relative gain from the reference point. Since the offer is greater than his reference point, he perceives the difference between two as a gain and this gain is scaled by γ_1. Accepting yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1).$$

If player 1 rejects the offer, his reference point will be $\max\{r_1, z_1\}$. With probability δ, the game continues to the next period and player 2 offers y^ω and with probability $1 - \delta$, the game ends. So, rejection gives player 1

$$\begin{aligned}
& u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
= \;& \delta[x_1^\omega + \lambda_1(x_1^\omega - \max\{r_1, z_1\})] - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
< \;& \delta[x_1^\omega + \gamma_1(x_1^\omega - z_1)] - (1-\delta)\gamma_1 z_1 \\
= \;& (1+\gamma_1)\delta x_1^\omega - \gamma_1 z_1 \\
< \;& (1+\gamma_1)y_1^\omega - \gamma_1 r_1 \\
< \;& (1+\gamma_1)z_1 - \gamma_1 r_1 = u_1^*
\end{aligned}$$

In this case, acceptance gives player 1 a higher utility than rejection does. If $z_1 \geq y_1^\omega$, accepting the offer z_1 is optimal.

1.2 Second, suppose that $z_1 < r_1$. In the case of acceptance, player 1 gets his share from the offer z_1. However, since the offer is less than his reference point, player 1 perceives this offer as a loss relative to the reference point. The difference between z_1 and r_1 negatively affects his utility and this effect is scaled by λ_1. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1).$$

If player 1 rejects the offer, he gets

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&= \delta[x_1^\omega + \lambda_1(x_1^\omega - \max\{r_1, z_1\})] - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
&= \delta[x_1^\omega + \lambda_1(x_1^\omega - r_1)] - (1-\delta)\lambda_1 r_1 \\
&= (1+\lambda_1)\delta x_1^\omega - \lambda_1 r_1 \\
&= (1+\lambda_1)y_1^\omega - \lambda_1 r_1.
\end{aligned}$$

Hence, $\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \geq u_1^*$ if and only if $z_1 \geq y_1^\omega$.

In this case, accepting the offer satisfying $z_1 \geq y_1^\omega$ and rejecting it otherwise is optimal.

Case 2. $\omega \in \{2-I, 2-II, 2-III\} : x_1^\omega \geq r_1 > y_1^\omega$

In this case, recall that the following equality holds in equilibrium:

$$(1+\gamma_1)\delta x_1^\omega = (1+\lambda_1)y_1^\omega + \delta\gamma_1 r_1 - \delta\lambda_1 r_1.$$

We analyze the optimal decisions of player 1 in two sub-cases: (1) $z_1 \geq r_1$ and (2) $z_1 < r_1$.

2.1. Suppose that $z_1 \geq r_1$. If player 1 accepts the offer, he gets his share z_1 plus the relative gain from the reference point. Since the offer is greater than his reference point, he perceives the difference between the two as a gain and this gain is scaled by γ_1. Accepting yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1).$$

If player 1 rejects the offer, his reference point will be $\max\{r_1, z_1\}$. With probability δ, the game continues to the next period and player 2 offers y^ω and with probability $1 - \delta$, the game ends. So, rejection gives player 1

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&= \delta[x_1^\omega + \lambda_1(x_1^\omega - \max\{r_1, z_1\})] - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
&< \delta[x_1^\omega + \lambda_1(x_1^\omega - z_1)] - (1-\delta)\lambda_1 z_1 \\
&< \delta[x_1^\omega + \gamma_1(x_1^\omega - z_1)] - (1-\delta)\gamma_1 z_1 \\
&= (1+\gamma_1)\delta x_1^\omega - \gamma_1 z_1 \\
&= (1+\lambda_1)y_1^\omega + \delta\gamma_1 r_1 - \delta\lambda_1 r_1 - \gamma_1 z_1 \\
&< (1+\gamma_1)z_1 - \gamma_1 r_1 = u_1^*.
\end{aligned}$$

In this case, acceptance gives player 1 a higher utility than rejection does. If $z_1 \geq y_1^\omega$, accepting the offer z_1 is optimal.

Now, suppose that $z_1 < r_1$. In the case of acceptance, player 1 gets his share from the offer z_1. However, since the offer is less than his reference point, player 1 perceives this offer as a loss relative to the reference point. The difference between z_1 and r_1 negatively affects his utility and this effect is scaled by λ_1. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1).$$

If player 1 rejects the offer, he gets

$$\begin{aligned}
& \delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta) u_1^{t+1}(0, r^{t+1}) \\
= \; & \delta[x_1^\omega + \gamma_1(x_1^\omega - \max\{r_1, z_1\})] - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
= \; & (1+\gamma_1)\delta x_1^\omega - \delta\gamma_1 \max\{r_1, z_1\} - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
= \; & (1+\lambda_1) y_1^\omega + \delta\gamma_1 r_1 - \delta\lambda_1 r_1 - \delta\gamma_1 \max\{r_1, z_1\} \\
& \quad - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
= \; & (1+\lambda_1) y_1^\omega + \delta\gamma_1 r_1 - \delta\lambda_1 r_1 - \delta\gamma_1 r_1 - (1-\delta)\lambda_1 r_1 \\
= \; & (1+\lambda_1)\delta y_1^\omega - \lambda_1 r_1.
\end{aligned}$$

Thus, $\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta) u_1^{t+1}(0, r^{t+1}) \geq u_1^*$ if and only if $z_1 \geq y_1^\omega$.

In this case, accepting the offer satisfying $z_1 \geq y_1^\omega$ and rejecting it otherwise is optimal.

Case 3. $\omega \in \{3-I, 3-II, 3-III\} : x_1^\omega > y_1^\omega \geq r_1$

Recall that the following equality holds in equilibrium:

$$(1+\gamma_1)\delta x_1^\omega = (1 + \gamma_1 + \delta\gamma_1 + (1-\delta)\lambda_1) y_1^\omega - \gamma_1 r_1.$$

Again, we analyze the optimal decisions of player 1 in two distinct sub-cases: (1) $z_1 \geq r_1$ and (2) $z_1 < r_1$.

3.1. Suppose that $z_1 \geq r_1$. If player 1 accepts the offer, he gets his share z_1 plus the relative gain from the reference point. Since the offer is greater than his reference point, he perceives the difference between two as a gain and this gain is scaled by γ_1. Accepting yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \gamma_1(z_1 - r_1).$$

If player 1 rejects the offer, his reference point will be $\max\{r_1, z_1\}$. With probability δ, the game continues to the next period and player 2 offers y^ω and with probability $1-\delta$, the game ends. So, rejection gives player 1

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&= \delta u_1^{t+1}(x^\omega) - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
&\leq \delta[x_1^\omega + \gamma_1(x_1^\omega - \max\{r_1, z_1\})] - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
&= (1+\gamma_1)\delta x_1^\omega - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
&= (1+\gamma_1)y_1^\omega + \delta\gamma_1 y_1^\omega + (1-\delta)\lambda_1 y_1^\omega - \gamma_1 r_1 - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
&< (1+\gamma_1)y_1^\omega + \delta\gamma_1 z_1 + (1-\delta)\lambda_1 z_1 - \gamma_1 r_1 - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
&= (1+\gamma_1)y_1^\omega - \gamma_1 r_1
\end{aligned}$$

In this case, acceptance gives player 1 a higher utility than rejection does. If $z_1 \geq y_1^\omega$, accepting the offer z_1 is optimal.

3.2. Now, suppose that $z_1 < r_1$. In the case of acceptance, player 1 gets his share from the offer z_1. However, since the offer is less than his reference point, player 1 perceives this offer as a loss relative to the reference point. The difference between z_1 and r_1 negatively affects his utility and this effect is scaled by λ_1. Accepting the offer yields

$$u_1^* = u_1^t(z, r^t) = z_1 + \lambda_1(z_1 - r_1).$$

If player 1 rejects the offer, he gets

$$\begin{aligned}
&\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \\
&= \delta[x_1^\omega + \gamma_1(x_1^\omega - \max\{r_1, z_1\})] - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
&= (1+\gamma_1)\delta x_1^\omega - \delta\gamma_1 \max\{r_1, z_1\} - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
&= (1+\gamma_1)y_1^\omega + \delta\gamma_1 y_1^\omega + (1-\delta)\lambda_1 y_1^\omega - \gamma_1 r_1 - \delta\gamma_1 \max\{r_1, z_1\} \\
&\quad - (1-\delta)\lambda_1 \max\{r_1, z_1\} \\
&> (1+\gamma_1)y_1^\omega + \delta\gamma_1 z_1 + (1-\delta)\lambda_1 z_1 - \gamma_1 r_1 - \delta\gamma_1 z_1 - (1-\delta)\lambda_1 z_1 \\
&= (1+\gamma_1)\delta y_1^\omega - \gamma_1 r_1 > (1+\gamma_1)\delta z_1 - \gamma_1 r_1 = u_1^*.
\end{aligned}$$

Thus, $\delta u_1^{t+1}(x^\omega, r^{t+1}) + (1-\delta)u_1^{t+1}(0, r^{t+1}) \geq u_1^*$ if and only if $z_1 \geq y_1^\omega$.

In this case, accepting the offer satisfying $z_1 \geq y_1^\omega$ and rejecting it otherwise is optimal.

Considering all cases, we have the following result: it is optimal to accept the offer z if $z_1 \geq y_1^\omega$ and to reject it otherwise, which implies that following σ_1^* is optimal. □

D Proofs of Chapter 5

Proof of Theorem 14. Here we prove that for a given $(a, b) \in [0, 1]^2$, the (a, b)-solution satisfies WPO (also PO), SYM, IPAT, and IND. (a, b)-MON.

Take any $(S, d, r) \in \Sigma^2$. For WPO, it is enough to recall that the bargaining set S is convex, closed, and bounded from above. As a matter of fact, since $n = 2$, PO is satisfied as well.

For SYM, note that the *symmetric* transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ maps the straight line passing through $ar + (1-a)d$ and $a(S, br + (1-b)d)$ into the straight line L passing through $T(ar + (1-a)d)$ and $T(a(S, br + (1-b)d))$. Since there cannot be a point in $L \cap T(S)$ which is greater than $T(F^{a,b}(S, d, r))$, it must be that $F^{a,b}(T(S), T(d), T(r)) = T(F^{a,b}(S, d, r))$.

For IPAT, note that an affine transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ (i) preserves the partial ordering of $\mathbb{R}^2$; (ii) maps straight lines into straight lines; (iii) maps $ar + (1-a)d$ into $aA(r) + (1-a)A(d)$; and (iv) maps $a(S, br + (1-b)d)$ into $a(A(S), A(br+(1-b)d))$. These and the definition of $F^{a,b}$ jointly imply the result.

For IND. (a, b)-MON, without loss of generality, consider $i = 1$ and $j = 2$. Take any (S, d, r), $(S', d', r') \in \Sigma^2$ such that $ar + (1-a)d = ar' + (1-a)d'$ and $a_2(S, br + (1-b)d) = a_2(S', br' + (1-b)d')$. Assume that for every $x \in S$: $a_1(S, x) \leq a_1(S', x)$. For notational convenience, we let

$$\begin{aligned}
\bar{a} &= ar + (1-a)d, \\
\bar{b} &= br + (1-b)d, \text{ and} \\
\tilde{b} &= br' + (1-b)d'.
\end{aligned}$$

Let L be the straight line passing through $\bar{a}$ and $a(S, \bar{b})$. By definition of the

(a, b)-solution, we have

$$F^{a,b}(S, d, r) = \max\{\lambda \bar{a} + (1-\lambda)a(S, \bar{b}) \in S \mid \lambda \in [0, 1]\}.$$

That is, $F^{a,b}(S, d, r)$ is the maximal element of $L \cap S$. Let L' be the straight line passing through $\bar{a}$ and $a(S', \tilde{b})$. Consider $L' \cap S$, and set

$$\bar{x} = \max\{\lambda \bar{a} + (1-\lambda)a(S', \tilde{b}) \in S \mid \lambda \in [0, 1]\}.$$

Moreover, since $a_1(S, \bar{b}) \leq a_1(S', \tilde{b})$ and $a_2(S, \bar{b}) = a_2(S', \tilde{b})$, we have

$$F_1^{a,b}(S, d, r) \leq \bar{x}_1.$$

By definition, $F^{a,b}(S', d', r')$ is the maximal element of $L' \cap S'$. That is,

$$F^{a,b}(S', d', r') = \max\{\lambda \bar{a} + (1-\lambda)a(S', \tilde{b}) \in S' \mid \lambda \in [0, 1]\}.$$

By convexity and comprehensiveness, the fact that $a_1(S, \bar{x}) \leq a_1(S', \bar{x})$ implies $\bar{x} \in S'$. It follows that the maximal element of $L' \cap S'$ is not less than $\bar{x}$; i.e., $\bar{x} \leq F^{a,b}(S', d', r')$. Therefore, $F_1^{a,b}(S, d, r) \leq F_1^{a,b}(S', d', r')$. $\square$