DECENTRALIZED BLOCKING ZEROS IN THE CONTROL OF LARGE SCALE SYSTEMS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

KONUR. A. ÜNYELİOĞLU

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I certify that I have read this dissertation and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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Assoc.Prof.Dr. A. Bülent Özgüler (Supervisor)

ABSTRACT

DECENTRALIZED BLOCKING ZEROS IN THE CONTROL OF LARGE SCALE SYSTEMS

KONUR A. ÜNYELİOĞLU

Ph. D. in Electrical and Electronics Engineering

Supervisor: Assoc.Prof.Dr. A. Bülent Özgüler

July 1992

In this thesis, a number of synthesis problems for linear, time-invariant, finite-dimensional systems are addressed. It is shown that the new concept of *decentralized blocking zeros* is as fundamental to controller synthesis problems for large scale systems as the concept of decentralized fixed modes.

The main problems considered are (i) decentralized stabilization problem, (ii) decentralized strong stabilization problem, and (iii) decentralized concurrent stabilization problem.

The decentralized stabilization problem is a fairly well-understood controller synthesis problem for which many synthesis methods exist. Here, we give a new synthesis procedure via a proper stable fractional approach and focus our attention on the *generic solvability* and *characterization of all solutions.*

The decentralized strong stabilization problem is the problem of stabilizing a system using stable local controllers. In this problem, the set of decentralized blocking zeros play an essential role and it turns out that the problem has a solution in case the poles and the real nonnegative decentralized blocking zeros have parity interlacing property. In the more general problem of decentralized stabilization problem with minimum number of unstable controller poles, it is shown that this minimum number is determined by the number of odd distributions of plant poles among the real nonnegative decentralized blocking zeros.

The decentralized concurrent stabilization problem is a special type of simultaneous stabilization problem using a decentralized controller. This problem is of interest, since many large scale synthesis problems turn out to be its special cases. A complete solution to decentralized concurrent stabilization problem is obtained, where again the decentralized blocking zeros play a central role. Three problems that have received wide attention in the literature of large scale systems: *stabilization of composite systems using locally stabilizing subsystem controllers*, *stabilization of composite systems via the stabilization of main diagonal transfer matrices*, and *reliable decentralized stabilization problem* are solved by a specialization of our main result on decentralized concurrent stabilization problem.

Keywords: Control system synthesis, linear systems, multivariable control systems, decentralized stability, large scale systems, poles and zeros.

ÖZET

GENİŞ ÇAPLI SİSTEMLERİN DENETİMİNDE AYRIŞIK TOPTAN SIFIRLAR

KONUR A. ÜNYELİOĞLU

Elektrik ve Elektronik Mühendisliği'nde Doktora

Tez Danışmanı: Doç.Dr. A. Bülent Özgüler

Temmuz 1992

Bu tez doğrusal, zamanla değişmeyen, sonlu boyuttaki geniş-çaplı sistemlerle ilgili çeşitli problemlerin çözümlerini içermektedir. Tezin denetim kuramına temel katkısı ***ayrışık toptan sıfırlar*** olarak isimlendirilen yeni bir sistem sıfırları kümesinin tanımlanmasıdır. Bu yeni sıfır kavramının geniş-çaplı sistemlerdeki tasarım problemlerinde ayrışık değişmez özdeğerler kadar temel bir rol üstlendiği gösterilmektedir.

İncelenen ana problemler şunlardır: (i) ayrışık kararlılaştırma problemi, (ii) ayrışık güçlü kararlılaştırma problemi ve (iii) ayrışık birlikte kararlılaştırma problemi.

Ayrışık kararlılaştırma problemi literatürde iyi incelenmiş bir denetleyici tasarımı problemi olup çözümü bilinmektedir. Bu tezde, kararlı uygun oranlar yaklaşımı ile yeni bir tasarım yöntemi önerilmekte ve ***bütün çözümlerin tanımlanması*** ve ***çözümlerin yapısal özellikleri*** konularına ağırlık verilmektedir.

Ayrışık güçlü kararlılaştırma problemi bir sistemi kararlı yerel denetleyicilerle kararlılaştırma problemidir. Bu problemin çözümü, eğer ve ancak gerçel kararsız ayrışık toptan sıfırlar ile kutuplar arasında bir girişim özelliği sağlandığı zaman vardır. En az sayıda kararsız kutuba sahip kararlaştırıcı ayrışık denetleyicilerin tasarımında da sistem kararsız kutuplarının, ayrışık toptan sıfırlar arasındaki tek sayılı dağılımlarının belirleyici olduğu gösterilmektedir.

Ayrışık birlikte kararlılaştırma problemi özel bir aynı anda kararlılaştırma problemi olup çeşitli geniş çaplı tasarım problemleri bu problemin özel bir hali olarak tanımlanabilmektedir. Bu tezde, ayrışık birlikte kararlılaştırma problemi ayrışık güçlü kararlılaştırma problemine dönüştürülerek çözülmektedir. Bu problemin çözümünde ayrışık toptan sıfırlar yine temel bir rol üstlenmektedir. Literatürde geniş ilgi görmüş olan arabağlı sistemlerle ilgili üç temel tasarım problemi, ayrışık birlikte kararlılaştırma problemine dönüştürülerek çözülmektedir.

Anahtar kelimeler: Denetim sistem tasarımı, doğrusal sistemler, çokdeğişkenli sistemler, ayrışık kararlılık, geniş çaplı sistemler, kutuplar ve sıfırlar.

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I was visiting the University of Michigan, Ann Arbor in the Fall 1991, where I had the opportunity of joining several meetings of the Systems and Control Group. I had also the chance of studying in the University of Michigan libraries where I completed a thorough literature survey on decentralized control and related issues. I would like to thank P. Pramod Khargonekar and A. Bülent Özgüler for making such a visit to University of Michigan possible.

I am also thankful to the members of the examining committee, especially to Altuğ İftar and Ömer Morgül, for their constructive comments on an earlier version of this thesis.

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Chapter 1

INTRODUCTION

This thesis is concerned with the Decentralized Stabilization Problem (DSP), Decentralized Strong Stabilization Problem (DSSP) and Decentralized Concurrent Stabilization Problem (DCSP) of linear time-invariant finite dimensional systems and the applications of the concept of decentralized blocking zeros in the solutions of DSSP and DCSP. In this chapter we will give brief definitions of these problems and discuss their motivation. More precise definitions of the problems are given in the subsequent chapters.

Let Z be a plant with N input-output channels (vector inputs and vector outputs). Consider the decentralized feedback configuration below.

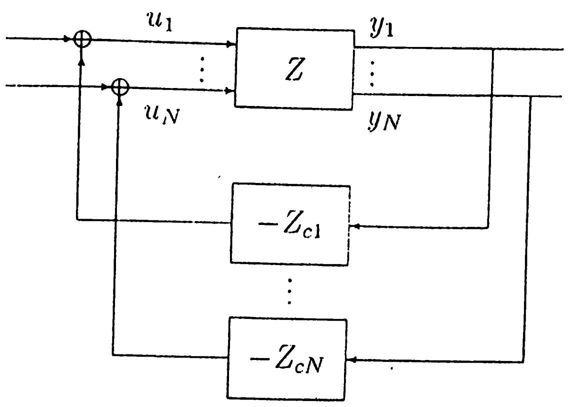

Figure 1.1. Decentralized feedback configuration.

Decentralized Stabilization Problem (DSP). Determine N feedback compensators Z_{c1}, ..., Z_{cN}, such that the pair $(Z, diag\{Z_{c1}, ..., Z_{cN}\})$ is internally stable.

Decentralized Strong Stabilization Problem (DSSP). Solve DSP using a *stable* decentralized controller, i.e., determine N stable feedback compensators $Z_{c1}, ..., Z_{cN}$ such that the pair $(Z, diag\{Z_{c1}, ..., Z_{cN}\})$ is internally stable.

Decentralized Concurrent Stabilization Problem. In addition to the N-channel plant Z, we are also given plants T_1, ..., T_N where the size of T_i is compatible with the size of Z_{ii}, the ith main diagonal subblock of Z, $i = 1, ..., N$. Determine N feedback compensators Z_{c1}, ..., Z_{cN} such that the pairs $(Z, diag\{Z_{c1}, ..., Z_{cN}\})$ and (T_i, Z_{ci}), $i = 1, ..., N$ are all internally stable.

In many feedback control problems, the controller is required to process a constrained feedback information due to some practical reasons which make the centralized (full-feedback) control inefficient or impossible. With this motivation, many researchers have paid attention to investigate the solvability conditions of DSP during the last two decades ([49], [48], [52], [32]). A basic decentralized control example is given below.

Example (1.1)-Steam Generator. *[64] In a steam generator, there are two basic elements: combustor and boiler. Water in the boiler is heated by the combustor and turns into steam. In our simplified model of steam generator, the controlled variables in the plant are the steam pressure in the boiler, water level in the boiler, and the superheated steam temperature. The control variables are the fuel flow into the combustor, water flow into the boiler and the flow of pulverized cooling water into superheated steam. Although each controlled variable depends on each of the control variables, the steam generator is preferably controlled by three local controllers each of which observes only one controlled variable and controls only one control variable, as summarized in the following table.*

	Controlled Variables		Control Variables
y_1 :	steam pressure in the boiler	u_1 :	fuel flow into combustor
y_2 :	water level in the boiler	u_2 :	water flow into boiler
y_3 :	superheated steam temperature	u_3 :	flow of pulverized water into steam

Controller i observes y_i and controls u_i, $i = 1, 2, 3$.

Table 1.1. Local control variables of a steam generator.

In this example, a main reason for controlling the plant using a decentralized compensator is due to the fact that the control variable u_i has a considerably faster effect on the controlled variable y_i compared to other control variables. Moreover, the dependence of y_i on the controller variables else than u_i is significantly weaker than its dependence on u_i.

As can be inferred from the use of a constrained feedback scheme, DSP has more restrictive solvability conditions in comparison with the full-feedback stabilization problem. It has been shown [70] that DSP is solvable if and only if the open loop plant has no unstable *decentralized fixed modes* with respect to the specified decentralized feedback constraint. The fixed modes of a plant are those open loop eigenvalues which remain unchanged in the closed loop for all possible constant decentralized compensators. In [10] the solvability of DSP has been shown to be equivalent to the *completeness* of certain system matrices belonging to *complementary subsystems* in case the open loop plant satisfies a connectivity condition called *strong connectedness*. The construction method of decentralized compensators proposed in [10] is obtained by making the closed loop system stabilizable and detectable from a single channel applying decentralized constant feedback around the other channels. A direct proof of the equivalence of the completeness condition of [10] and the absence of decentralized fixed modes as defined by [70] has been given in [2]. It has later been shown by the fractional representation approach to DSP ([36], [68], [22], [37], [55], [56]) that the strong connectedness assumption can also be removed by applying dynamic compensation to each of the channels instead of constant compensation.

Although the precise conditions for the solution of DSP is well-known, there are still some open problems concerning the synthesis of decentralized stabilizing controllers. Such problems arise especially when the decentralized controller is synthesised for a large-scale system comprising various subsystems where the local controllers are required to satisfy additional properties in addition to the stabilization of the composite (interconnected) system. In this context the following three problems are investigated in the subsequent chapters.

(p1) *Stabilization of composite systems using locally stabilizing subsystem controllers.* Consider a collection of linear time-invariant finite dimensional systems described by

$$\Sigma_i : \begin{array}{rl} \dot{x}_i = & A_i x_i + B_i v_i + u_i \\ y_i = & C_i x_i \end{array} \quad , \quad i \in \{1, ..., N\}$$

where A_i, B_i and C_i are real constant matrices of appropriate dimensions corresponding to states, inputs and outputs, respectively. Assume that these systems are interconnected according to the rule $u_i = \sum_{j=1}^{N} A_{ij} x_j$, $i \in \{1, ..., N\}$ for some constant matrices A_{ij}, $i, j \in \{1, ..., N\}$. The resulting composite system is defined by Σ. The objective is to determine local controllers Σ_{ci}, $i \in \{1, ..., N\}$ such that the pairs (Σ_i, Σ_{ci}), $i \in \{1, ..., N\}$ are stable when the interconnections do not exist. It is also desired that when the interconnections exist the composite system Σ becomes stabilized by the decentralized controller composed of Σ_{ci}, $i \in \{1, ..., N\}$. Such an approach to the stabilization problem of composite systems is a natural one because most of the composite systems are constructed by interconnecting the independently controlled subsystems [63], [49]. Although there is an extensive literature concerning the stabilization of interconnected systems via such a special subsystem feedback, so far a necessary and sufficient solvability condition has not yet been obtained (see the references in Chapter 5). An example for problem (p1) is given below.

Example (1.2)-Interconnected steam generators. *We consider two steam generators G_1, G_2 which supply steam to two independent steam pipelines. Due to operating conditions and consumer demands it is sometimes desired to interconnect the pipelines via an auxiliary network. Let controllers C_1, C_2 control G_1, G_2, respectively, when the interconnection does not exist. It is required* *that when the pipelines are interconnected the same controllers still achieve the prescribed control objectives in the resulting new system.*

(p2) *Stabilization of composite systems via the stabilization of diagonal transfer matrices.* Another approach to the stabilization problem of composite systems via decentralized controllers is based on the extension of Nyquist and Inverse Nyquist Array methods to multi-input/multi-output systems. The starting point of this approach is to assume that the interactions between the subsystems are sufficiently "weak" in some sense so that a set of local controllers which separately stabilize the main diagonal transfer matrices (in case the interconnections are neglected) also guarantees that the closed-loop system remains stable when the interconnections exist. Although several systematic procedures are available in the literature which provide sufficient conditions for the solution of this problem, a necessary and sufficient solvability condition is yet not available [78], [34], [74]. We note that (p1) and (p2) are different problems, because in (p1) the main diagonal transfer matrices in the transfer matrix of the interconnected system Σ are, in general, different than the transfer matrices of subsystems Σ_i, $i = 1, \ldots, N$.

(p3) *Reliable Decentralized Stabilization Problem.* An important design objective for large-scale systems is to ensure reliable performance with respect to the changes in system parameters. These variations can be modelled in several ways. In this thesis we consider the discrete variations of parameters which arise from the interconnection breakdowns or on-off type of variations of open loop system elements. The reliable decentralized stabilization problem is defined as synthesising a decentralized controller which shows a satisfactory performance (stabilization) for the nominal system and for all systems around the nominal system resulting from a prespecified set of discrete variations in the system parameters. We remind that in Example (1.2) above a built-in reliability is ensured in the sense that when the interconnection between the pipelines is removed accidentally the two resulting independent systems (G_1, C_1), (G_2, C_2) still achieve the desired control objectives.

We note that DCSP is a special decentralized simultaneous stabilization problem and all the above problems (p1)-(p3) can be formulated in the DCSP framework. For problem (p1) this fact has already been indicated in [52]. In case of a restricted class of interconnected systems it has recently been shown that the (centralized) strong stabilization problem plays a primary role in the solution of (p1) [32]. The relation between problem (p2) and DSSP has been shown in [35], [57]. A formulation of problem (p3) in terms of DSSP is given in [8], [57]. Relations between problem (p3) and DSSP are also addressed in [65]. We note that DCSP and DSSP are closely related problems in that DCSP is solvable if and only if DSSP is solvable for a subsidiary plant (Chapter 5). This is an extension of the results obtained for the centralized versions of these problems. We refer to [43], [66], [21] and to the references therein for the (centralized) strong and simultaneous stabilization problems.

The contributions of this thesis are the following.

1. A new set of zeros for multivariable systems, the set of *decentralized blocking zeros* is introduced. Decentralized blocking zeros are common blocking zeros of various complementary transfer matrices and the transfer matrices of main diagonal subplants. Miscellaneous interpretations for decentralized blocking zeros are given in terms of system zeros and transmission zeros.

2. We determine the least unstable degree of decentralized stabilizing controllers and give a synthesis procedure for the construction of a least unstable decentralized stabilizing controller. As a particular case, we obtain the solution of DSSP. It is shown that the least unstable degree of decentralized stabilizing controllers is determined by a parity interlacing property among the real unstable poles and real unstable decentralized blocking zeros of the plant. This result is the analogue of the one obtained for centralized feedback systems [66, Theorem. 5.3.1] Several sufficient conditions on the plant zeros which ensure the solvability of DSSP are given. It is also shown that if a strongly connected plant admits a solution to DSP then the unstable poles of the compensator can be distributed among the local controllers nearly arbitrarily.

3. A solution procedure for DCSP is proposed by transforming it to DSSP in a subsidiary plant. Although the subsidiary plant is not unique, an explicit

expression for the set of decentralized blocking zeros of the subsidiary plant is given in terms of the system zeros of original plants $diag\{T_1, ..., T_N\}$ and Z. It is shown that DSSP is generically solvable. It turns out that in a special case which generically holds, a solution to DCSP exists if and only if DSSP is solvable for the difference plant $diag\{T_1, ..., T_N\} - Z$.

The above problems (p1), (p2) and (p3) are solved in a unified framework by transforming them into DCSP. Various sufficient conditions in terms of system zeros are given which ensure the solvability of these problems. It is also shown that each of (p1), (p2), (p3) is generically solvable.

The organization of the thesis is as follows. The next chapter is devoted to technical preliminaries where we first introduce the notation and terminology. Then, several algebraic properties of the rings of proper, stable proper and stable rational functions are briefly reviewed. Characterization of all stabilizing controllers and the graph topology for linear time-invariant finite-dimensional systems are also considered. In Chapter 3 we study the solution of DSP in a stable proper fractional set-up. A new synthesis procedure for decentralized stabilizing controllers and a characterization of all admissible local controllers associated with a fixed channel are given. Genericity properties of decentralized stabilizing controllers are also investigated. The results in Chapter 3 lay the technical background for the subsequent chapters as DSP is a basic part of every other problem considered. Chapter 4 considers decentralized blocking zeros, the synthesis of least unstable decentralized stabilizing controllers, and the solution of DSSP. Chapter 5 is concerned with DCSP. The solutions of problems (p1), (p2), (p3) are also given in Chapter 5 in sections 5.2, 5.3, 5.4 respectively. Chapter 6 includes some concluding remarks and and a discussion of related problems for future investigation.

The results of Chapters 3 and 4 are partially based on [56] and [38], [60], respectively. Section 4 of Chapter 5 considers a generalization of the results in [60], [57].

Chapter 2

NOTATION AND MATHEMATICAL PRELIMINARIES

This chapter includes the notation of the thesis. We also review some mathematical facts used in the following chapters. For a more detailed exposition of the related algebraic and topological concepts the reader is referred to [66].

By $\mathcal{C}$ and $\mathcal{R}$, we denote the fields of complex and real numbers, respectively. We let $\mathcal{C}_e$ be the set of complex numbers including infinity where the subscript 'e' is an abbreviation for 'extended'. The symbol $\mathcal{C}_{+e}$ denotes the closed right half plane including infinity and $\mathcal{R}_{+e}$ denotes the set of real positive numbers including infinity. More precisely, $\mathcal{R}_{+e} = \mathcal{R} \cap \mathcal{C}_{+e}$. The set of proper rational functions with real coefficients is denoted by $\mathbf{P}$. The sets of stable proper rational functions and stable rational functions (with real coefficients) are denoted by $\mathbf{S}$ and $\mathbf{P}_s$, respectively. Note that, $z \in \mathbf{P}$ belongs to $\mathbf{S}$ if and only if its denominator polynomial is stable, i.e., has no $\mathcal{C}_+$ zeros. The set $\mathbf{P}_s$ is precisely the set of rational functions whose denominator polynomials are stable. By definition, $\mathbf{S} \subset \mathbf{P}_s$. Also, $\mathbf{P}$ is a subset of the field of fractions of $\mathbf{S}$. We indicate by $\mathbf{M}(A)$ the set of matrices with entries over the set A. By $\mathbf{R}$ we denote the set of polynomials with real coefficients. The sets $\mathbf{S}$, $\mathbf{P}_s$ and $\mathbf{R}$ are rings. They are also principal ideal domains. We remind that in a principal ideal domain a greatest common divisor of a given finite number of elements always exists.

For a strictly positive integer N, $\mathbf{N}$ denotes the ordered set $\{1, 2, ..., N\}$. A set $\{i_1, i_2, ..., i_\mu\}$ is called a *proper subset* of $\mathbf{N}$ if $\mathbf{N}-\{i_1, i_2, ..., i_\mu\}$ is nonempty where '$-$' denotes the standard set-difference operation. In case $\{i_1, i_2, ..., i_\mu\}$ is a proper subset of $\mathbf{N}$ we use the following convention: $\mathbf{N} - \{i_1, i_2, ..., i_\mu\} = \{i_{\mu+1}, i_{\mu+2}, ..., i_N\}$. We denote by C_N the set of all proper subsets of $\mathbf{N}$. If a, b are real numbers $min(a,b)$ denotes the minimum of a, b.

The symbols $A := B$, $B =: A$ denote the statement '*A is defined by B*'.

If $c \in \mathcal{C}$ then c^* denotes the complex conjugate of c. For $a \in \mathcal{C}$, $|a|$ denotes the magnitude of a. If $A \in \mathbf{M}(\mathbf{S})$ we denote by $||A||$ the H_∞ norm of A, i.e., $||A|| = \sup_{z\in \mathcal{C}_+} \bar{\sigma}(A(z))$ where $\bar{\sigma}(\,.\,)$ is the largest singular value of its argument. If $A \in \mathbf{M}(\mathcal{C})$ then $||A||$ denotes the spectral matrix norm over $\mathcal{C}$. For a square matrix A, $det(A)$ denotes the determinant of A. For a matrix B, B' denotes the transpose of B. By $diag\{A_1, ..., A_N\}$ we denote the block diagonal matrix having the matrices A_i, $i \in \mathbf{N}$ in its main diagonal blocks. The matrix I_p is the identity matrix with size p. The matrix $0_{p\times r}$ is the zero matrix with p rows and r columns. In case $p = r$, we use 0_p to denote $0_{p\times r}$. Usually the dimension is clear from the context, so the subscripts are dropped.

Let $A = [A_{ij}]$, $i, j \in \mathbf{N}$ be a matrix where A_{ij} denotes the ij'th submatrix of A. Let $\mathbf{r}_1 = \{i_1, ..., i_l\}$, $\mathbf{r}_2 = \{j_1, ..., j_r\}$ be two subsets of $\mathbf{N}$. The matrix $A_{\mathbf{r}_1\mathbf{r}_2}$ is defined as follows.

$$A_{\mathbf{r}_1\mathbf{r}_2} = \begin{bmatrix} A_{i_1j_1} & \cdots & A_{i_1j_r} \\ \vdots & & \vdots \\ A_{i_lj_1} & & A_{i_lj_r} \end{bmatrix}.$$

For any matrix A over $\mathcal{C}$, $\mathbf{P}$ or $\mathbf{P}_s$, $rankA$ denotes the rank of the matrix over the associated field of fractions.

Let S be a set with topology $\mathcal{T}$. We say that a property holds for *almost all* elements of S if the set of elements of S for which that property holds is open and dense in S with respect to $\mathcal{T}$.

2.1 Algebraic Properties

Let $\mathbf{T}$ be a principal ideal domain. The matrices A_i, $i \in \mathbf{N}$ over $\mathbf{T}$ with the same number of rows are said to be *left coprime*, if the matrix $[A_1\ A_2\ ...\ A_N]$ has a right inverse over $\mathbf{T}$. In case A_i, $i \in \mathbf{N}$ are left coprime we say that $(A_1, A_2, ..., A_N)$ is left coprime. Dually, the matrices B_i, $i \in \mathbf{N}$ over $\mathbf{T}$ with the same number of columns are said to be *right coprime*, if the matrix $[B_1'\ B_2'\ ...\ B_N']'$ has a left inverse over $\mathbf{T}$. In case B_i, $i \in \mathbf{N}$ are right coprime we say that $(B_1, B_2, ..., B_N)$ is right coprime.

A square matrix U over $\mathbf{T}$ is called *unimodular* if U is invertible over $\mathbf{T}$. A square matrix $A \in \mathbf{M}(\mathbf{T})$ is called a greatest common left factor of matrices A_i, $i \in \mathbf{N}$, where A_i, $i \in \mathbf{N}$ have the same number of rows if $[A_1\ ...\ A_N] = A[\bar{A}_1\ ...\ \bar{A}_N]$ and $\bar{A}_i$, $i \in \mathbf{N}$ are left coprime. The abbreviation *gclf* stands for "greatest common left factor". Dually, a square matrix $B \in \mathbf{M}(\mathbf{T})$ is called a greatest common right factor of matrices B_i, $i \in \mathbf{N}$, where B_i, $i \in \mathbf{N}$ have the same number of columns if $[B_1'\ ...\ B_N']' = B'[\bar{B}_1'\ ...\ \bar{B}_N']'$ and $\bar{B}_i$, $i \in \mathbf{N}$ are right coprime.

Let $A \in \mathbf{T}^{p \times r}$ where $l = rank\, A \leq min(p, r)$. There exist unimodular matrices U and V over $\mathbf{T}$ of appropriate sizes such that

$$UAV = \left[\begin{array}{cccc|c} \alpha_1 & 0 & & 0 & \\ 0 & \alpha_2 & & 0 & \\ \vdots & & & & 0_{l \times r-l} \\ 0 & 0 & \dots & \alpha_l & \\ \hline & & 0_{p-l \times l} & & 0_{p-l \times r-l} \end{array}\right]$$

where α_i belongs to $\mathbf{T}$, and α_i divides α_{i+1}, $\forall i$. This canonical form for $p \times r$ matrices under unimodular transformations is called the *Smith canonical form* or simply the *Smith form*. The factors α_i's are called the *invariant factors* of A.

Let $\mathbf{F}$ be the field of fractions of $\mathbf{T}$ and let $Z \in \mathbf{F}^{p \times r}$ where $l = rank\, Z \leq min(p, r)$. There exist unimodular matrices U and V over $\mathbf{T}$ of appropriate sizes such that

$$UZV = \left[\begin{array}{cccc|c} \frac{\varepsilon_1}{\psi_1} & 0 & & 0 & \\ 0 & \frac{\varepsilon_2}{\psi_2} & \dots & 0 & \\ & & & & 0_{l\times r-l} \\ 0 & 0 & \dots & \frac{\varepsilon_l}{\psi_l} & \\ \hline & 0_{p-l\times l} & & & 0_{p-l\times r-l} \end{array}\right] \tag{2.1}$$

where ε_i, ψ_i belong to T, (ε_i, ψ_i) are coprime, and ε_i divides ε_{i+1}, ψ_{i+1} divides ψ_i, $\forall i$. This canonical form for $p\times r$ matrices in **F** is called the *Smith-McMillan* form.

Let $Z \in \mathbf{F}^{p\times r}$. There exist $D_l \in \mathbf{T}^{p\times p}$, $N_l \in \mathbf{T}^{p\times r}$, $D_r \in \mathbf{T}^{r\times r}$, $N_r \in \mathbf{T}^{p\times r}$, $Q \in \mathbf{T}^{q\times q}$, $P \in \mathbf{T}^{p\times q}$, $R \in \mathbf{T}^{q\times r}$ for some q such that

$$Z = D_l^{-1}N_l = N_rD_r^{-1} = PQ^{-1}R. \tag{2.2}$$

the pairs (D_l, N_l), (Q, R) are left coprime and (D_r, N_r), (Q, P) are right coprime. The fractions in (2.2) are called *left coprime, right coprime and bicoprime fractional representations* of Z, respectively.

Let $Z \in \mathbf{P}^{p\times r}$. The notation $Z = 0$ means that every entry of Z is identically zero (i.e., the zero element of the ring **P**). Note that if Z is nonzero, or equivalently, $Z \neq 0$ then $Z(z) = 0$ only for a finite number of elements z of $\mathcal{C}$. A complex number z_0 is a *blocking zero* of Z if $Z(z_0) = 0$ [16], [17]. If Z is stable, then the unstable blocking zeros are the unstable zeros of the *smallest invariant factor (sif)* of Z over **S**. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two finite collections of numbers in $\mathcal{R}_{+e}$, in which some numbers may occur more than once. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint then we say that the ordered pair $(\mathcal{S}_1, \mathcal{S}_2)$ has *parity interlacing property* if there are an even number of elements from $\mathcal{S}_1$ between each pair of elements from $\mathcal{S}_2$. The terminology is borrowed from [77] in which $\mathcal{S}_1$ and $\mathcal{S}_2$ are, respectively, the poles (with multiplicity) and the blocking zeros of a transfer matrix. Note that, if $\mathcal{S}_1$

is the set of $\mathcal{R}_{+e}$ zeros with multiplicity of $a \in \mathbf{S}$, then $a(z)$ takes the same sign at all elements $z \in \mathcal{S}_2$ if and only if $(\mathcal{S}_1, \mathcal{S}_2)$ has the parity interlacing property.

Let $Z \in \mathbf{P}^{p\times r}$ be given such that

$$Z = P_1 Q_1^{-1} R_1 = P_2 Q_2^{-1} R_2 \tag{2.3}$$

where $Q_1 \in \mathbf{P}_s^{q_1\times q_1}$, $R_1 \in \mathbf{P}_s^{q_1\times r}$, $P_1 \in \mathbf{P}_s^{p\times q_1}$, $Q_2 \in \mathbf{P}_s^{q_2\times q_2}$, $R_2 \in \mathbf{P}_s^{q_2\times r}$, $P_2 \in \mathbf{P}_s^{p\times q_2}$. We say that the representations (P_1, Q_1, R_1), (P_2, Q_2, R_2) are *Fuhrmann equivalent over* $\mathbf{P}_s$ if for some matrices A_1, B_1, A_2, B_2 over $\mathbf{P}_s$ of appropriate dimensions

$$\begin{bmatrix} A_1 & 0 \\ B_1 & I \end{bmatrix} \begin{bmatrix} Q_1 & R_1 \\ -P_1 & 0 \end{bmatrix} = \begin{bmatrix} Q_2 & R_2 \\ -P_2 & 0 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ 0 & I \end{bmatrix}$$

and (Q_1, A_2) is right coprime, (Q_2, A_1) is left coprime [18], [19]. Let a state space realization of Z be given by (C, A, B) where A, B and C are the state, input and output matrices, respectively. Noting that $Z = C(zI - A)^{-1}B$, we use the triple (C, A, B) to denote the representation $(C, zI - A, B)$.

Lemma (2.1). *Let $K = [K_{ij}]$, $K_{ij} \in \mathbf{P}^{p_i\times r_j}$, $i, j \in \mathbf{N}$ be given. Suppose*

$$\left(\begin{bmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{bmatrix}, \hat{A}, \begin{bmatrix} \hat{B}_1 & \dots & \hat{B}_N \end{bmatrix}\right)$$

is a stabilizable and detectable state-space realization of K such that $K_{ij} = \hat{C}_i(zI - \hat{A})^{-1}\hat{B}_j$, $i, j \in \mathbf{N}$. Also let

$$K = \begin{bmatrix} \hat{P}_1 \\ \vdots \\ \hat{P}_N \end{bmatrix} \hat{Q}^{-1} \begin{bmatrix} \hat{R}_1 & \dots & \hat{R}_N \end{bmatrix}$$

be a bicoprime fraction over $\mathbf{S}$ *where $K_{ij} = \hat{P}_i \hat{Q}^{-1} \hat{R}_j$, $i, j \in \mathbf{N}$. Then, for any proper subset $\{i_1, \dots, i_\mu\}$ of* $\mathbf{N}$ *the two systems*

$$\left(\begin{bmatrix} \hat{C}_{i_{\mu+1}} \\ \vdots \\ \hat{C}_{i_N} \end{bmatrix}, \hat{A}, \begin{bmatrix} \hat{B}_{i_1} & \dots & \hat{B}_{i_\mu} \end{bmatrix}\right), \quad \left(\begin{bmatrix} \hat{P}_{i_{\mu+1}} \\ \vdots \\ \hat{P}_{i_N} \end{bmatrix}, \hat{Q}, \begin{bmatrix} \hat{R}_{i_1} & \dots & \hat{R}_{i_\mu} \end{bmatrix}\right)$$

are Fuhrmann equivalent over $\mathbf{P}_s$.

Proof. First note that the two representations

$$\left(\begin{bmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{bmatrix}, \hat{A}, \begin{bmatrix} \hat{B}_1 & & \hat{B}_N \end{bmatrix}\right), \left(\begin{bmatrix} \hat{P}_1 \\ \vdots \\ \hat{P}_N \end{bmatrix}, \hat{Q}, \begin{bmatrix} \hat{R}_1 & & \hat{R}_N \end{bmatrix}\right)$$

are Fuhrmann equivalent over $\mathbf{P}_s$ [27]. Fix any proper subset $\{i_1, ..., i_\mu\}$ of $\mathbf{N}$. Let $\hat{B}_I := [\hat{B}_{i_1} \ ... \ \hat{B}_{i_\mu}]$, $\hat{R}_I := [\hat{R}_{i_1} \ ... \ \hat{R}_{i_\mu}]$, $\hat{C}_J := [\hat{C}'_{i_{\mu+1}} \ ... \ \hat{C}'_{i_N}]'$, $\hat{P}_J := [\hat{P}'_{i_{\mu+1}} \ ... \ \hat{P}'_{i_N}]'$, $\hat{B}_J := [\hat{B}_{i_{\mu+1}} \ ... \ \hat{B}_{i_N}]$, $\hat{R}_J := [\hat{R}_{i_{\mu+1}} \ ... \ \hat{R}_{i_N}]$, $\hat{C}_I := [\hat{C}'_{i_1} \ ... \ \hat{C}'_{i_\mu}]'$, $\hat{P}_I := [\hat{P}'_{i_1} \ ... \ \hat{P}'_{i_\mu}]'$. There exist matrices K_1, K_2, L_1, L_2, M_1, M_2 over $\mathbf{P}_s$ such that

$$\begin{bmatrix} K_1 & 0 & 0 \\ L_1 & I & 0 \\ L_2 & 0 & I \end{bmatrix}\begin{bmatrix} zI - \hat{A} & \hat{B}_I & \hat{B}_J \\ -\hat{C}_I & 0 & 0 \\ -\hat{C}_J & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Q} & \hat{R}_I & \hat{R}_J \\ -\hat{P}_I & 0 & 0 \\ -\hat{P}_J & 0 & 0 \end{bmatrix}\begin{bmatrix} K_2 & M_1 & M_2 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

where $(\hat{Q}, K_1)$ is left and $(zI - \hat{A}, K_2)$ are right coprime pairs over $\mathbf{P}_s$. This implies

$$\begin{bmatrix} K_1 & 0 \\ L_2 & I \end{bmatrix}\begin{bmatrix} zI - \hat{A} & \hat{B}_I \\ -\hat{C}_J & 0 \end{bmatrix} = \begin{bmatrix} \hat{Q} & \hat{R}_I \\ -\hat{P}_J & 0 \end{bmatrix}\begin{bmatrix} K_2 & M_1 \\ 0 & I \end{bmatrix}$$

which completes the proof.□

Lemma (2.2). *Let* $(\hat{P}_1, \hat{Q}_1, \hat{R}_1)$ *and* $(\hat{P}_2, \hat{Q}_2, \hat{R}_2)$ *be two Fuhrmann equivalent representations over* $\mathbf{P}_s$. *Then,*

$$rank\begin{bmatrix} \hat{Q}_1 & \hat{R}_1 \\ -\hat{P}_1 & 0 \end{bmatrix}(z) + size(\hat{Q}_2) = rank\begin{bmatrix} \hat{Q}_2 & \hat{R}_2 \\ -\hat{P}_2 & 0 \end{bmatrix}(z) + size(\hat{Q}_1), \ \forall z \in \mathcal{C}_+.$$

Proof. The proof easily follows from the definition of Fuhrmann equivalence.□

Let $Z = C(zI - A)^{-1}B$, where (C, A, B) is a state-space representation of Z. We say that z is an *invariant zero* of system (C, A, B) if it is a zero of some invariant factor of the system matrix

$$\begin{bmatrix} zI - A & B \\ -C & 0 \end{bmatrix}$$

over $\mathbf{R}$. Similarly, let $Z = P_1 Q_1^{-1} R_1$ be some fractional representation of Z over $\mathbf{S}$. We say that z is an invariant zero of system (P_1, Q_1, R_1) if it is a zero of some invariant factor of the system matrix

$$\begin{bmatrix} Q_1 & R_1 \\ -P_1 & 0 \end{bmatrix}$$

over $\mathbf{S}$. Let the representations of Z in (2.3) be Fuhrmann equivalent and satisfy that Q_1, R_1, P_1 are matrices over $\mathbf{S}$ and $Q_2 = zI - A$, $R_2 = B$, $P_2 = C$. Any C_+ invariant zero of (C, A, B) is also an invariant zero of (P_1, Q_1, R_1), and conversely. More precisely, it follows from Lemma (2.2) that $z \in C_+$ is a zero of the $\bar{l}$th invariant factor of

$$\begin{bmatrix} zI - A & B \\ -C & 0 \end{bmatrix}$$

if and only if it is a zero of the $(\bar{l} + size(A) - size(Q_1))$th invariant factor of

$$\begin{bmatrix} Q_1 & R_1 \\ -P_1 & 0 \end{bmatrix}.$$

Let $Z \in \mathbf{P}^{p \times r}$. Consider the Smith-McMillan form of Z over $\mathbf{S}$ as given by (2.1). A complex number $z \in C_{+e}$ which is a zero of any of ε_i, $i = 1, ..., l$, where $l := rank\, Z$ is called a *transmission zero* of Z. For a detailed study of invariant zeros and transmission zeros we refer to [44].

As a final result of this section we consider an interpolation result concerning the ring $\mathbf{S}$.

Lemma (2.3) *Let some distinct real numbers $r_1, \ldots, r_p$ and distinct complex numbers $c_1, \ldots, c_l$ be given such that $c_i \neq c_j^*$, $i, j = 1, \ldots, l$. Also let some real numbers $t_1, \ldots, t_p$ and complex numbers $k_1, \ldots, k_l$ be given. There exists $x \in \mathbf{S}$ such that $x(r_i) = t_i$, $i = 1, \ldots, p$, $x(c_i) = k_i$, $i = 1, \ldots, l$.*

Proof. Although the proof is based on standard interpolation theory, it is repeated here for convenience. Define

$$z_i = \begin{cases} r_i, & i = 1, \ldots, p \\ c_i, & i = p+1, \ldots, p+l \\ c_i^*, & i = p+l+1, \ldots, p+2l \end{cases}$$

and

$$s_i = \begin{cases} t_i, & i = 1, ..., p \\ k_i, & i = p+1, ..., p+l \\ k_i^*, & i = p+l+1, ..., p+2l. \end{cases}$$

We let

$$x(z) = \frac{1}{(z+1)^{2l+p-1}} \sum_{i=1}^{2l+p} s_i(z_i+1)^{2l+p-1} \prod_{j=1, j\neq i}^{2l+p} (z - z_j)/(z_i - z_j)$$

It can be verified that $x \in \mathbf{S}$ and satisfies the desired requirements.□

2.2 Graph Topology

Let some left and right coprime fractional representations of a plant $Z_0 \in \mathbf{P}^{p\times r}$ over $\mathbf{S}$ be given as follows:

$$Z_0 = D_l^{-1} N_l = N_r D_r^{-1}.$$

There exists a positive real number $\rho(D_l, N_l)$ such that for any pair of matrices (D, N) over $\mathbf{S}$ where

$$\| \begin{bmatrix} D_l - D & N_l - N \end{bmatrix} \| < \rho(D_l, N_l)$$

it holds that D is nonsingular and (D, N) is left coprime. Let a *basic neighborhood* around Z_0 be defined as

$$B(Z_0, \varepsilon) = \{ Z = D^{-1}N \in \mathbf{P}^{p\times r} | \; \| \begin{bmatrix} D_l - D & N_l - N \end{bmatrix} \| < \varepsilon \}$$

where $0 < \varepsilon < \rho(D_l, N_l)$. Then, the collection of basic neighborhoods $B(Z_0, \varepsilon)$ as Z_0 varies on $\mathbf{P}^{p\times r}$ and ε varies between 0 and $\rho(D_l, N_l)$ is a base for a topology on $\mathbf{P}^{p\times r}$ where a set is open if and only if it is a collection of basic neighborhoods of the above type [66]. This topology is called *graph topology*[1].

Using dual arguments one can define the graph topology using the right coprime representation $Z = N_r D_r^{-1}$ as well. We refer the reader to [66] for details.

[1]This definition of graph topology is slightly different than the one stated in V as we restrict the definition to proper rational matrices.

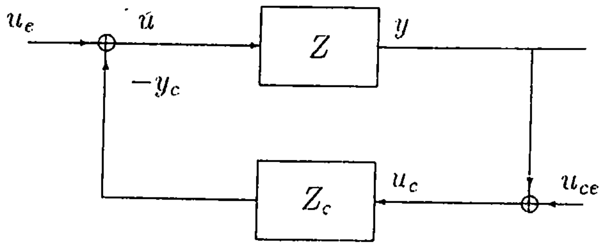

Figure 2.1. The closed loop system.

2.3 Characterization of Stabilizing Controllers

Referring to figure 2.1, let $y = Zu$ and $y_c = Z_c u_c$ be the transfer matrix representations of a plant and compensator respectively, where $Z \in \mathbf{P}^{p\times r}$ and $Z_c \in \mathbf{P}^{r\times p}$. These are interconnected by the laws: $u = u_e - y_c$, $u_c = u_{ce} + y$. We say that *the closed loop system is well defined* if $(I + ZZ_c)$ has an inverse over $\mathbf{P}$, denoted by $(I + ZZ_c)^{-1}$. In this case $[y' \ y_c']' = G[u_e' \ u_{ce}']'$ where

$$G = \begin{bmatrix} Z - ZZ_c(I + ZZ_c)^{-1}Z & -ZZ_c(I + ZZ_c)^{-1} \\ Z_c(I + ZZ_c)^{-1}Z & Z_c(I + ZZ_c)^{-1} \end{bmatrix}. \tag{2.4}$$

It is said that (Z, Z_c) is (internally) *stable* if the closed loop system is well defined and $G \in \mathbf{M}(\mathbf{S})$. The following statements are equivalent by definition: (Z, Z_c) is stable, Z_c stabilizes Z, Z_c is a stabilizing compensator for Z.

If $Z = PQ^{-1}R$ is a bicoprime fractional representation of Z over $\mathbf{S}$ then (Z, Z_c) is a stable pair if and only if

$$\begin{bmatrix} Q & RP_c \\ -P & Q_c \end{bmatrix}$$

is unimodular over $\mathbf{S}$ where $Z_c = P_c Q_c^{-1}$ is a right coprime fractional representation of Z_c over $\mathbf{S}$. In particular, if Z_c is a stable matrix, i.e., if $Z_c \in \mathbf{S}^{r\times p}$ then (Z, Z_c) is stable if and only if $Q + RZ_cP$ is unimodular over $\mathbf{S}$.

Let

$$\bar{Z}_{11} = D_l^{-1}N_l = N_r D_r^{-1} \tag{2.5}$$

be some left and right coprime fractional representations of a plant transfer matrix $\bar{Z}_{11} \in \mathbf{P}^{p\times r}$ over $\mathbf{S}$. Then, there exist matrices T_l, S_l, S_r, T_r over $\mathbf{S}$ such that

$$\begin{bmatrix} T_l & S_l \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -S_r \\ N_r & T_r \end{bmatrix} = I. \tag{2.6}$$

It follows from the standard Youla-Bongiorno-Jabr-Kučera [76], [29] parametrization that a transfer matrix $Z_c \in \mathbf{P}^{r\times p}$ is a stabilizing compensator for $\bar{Z}_{11}$ if and only if

$$\begin{aligned} Z_c &= (S_r + D_r X)(T_r - N_r X)^{-1} \\ &= (T_l - X N_l)^{-1}(S_l + X D_l) \end{aligned} \tag{2.7}$$

for some $X \in \mathbf{S}^{r\times p}$ provided $(T_r - N_r X)$ and $(T_l - X N_l)$ are biproper. This result is now utilized to define a topology over $\mathcal{Z}_c(\bar{Z}_{11})$, the set of all proper rational stabilizing compensators of $\bar{Z}_{11}$. Let $P_c(X) := S_r + D_r X$ and $Q_c(X) := T_r - N_r X$. If $Z_{c0} \in \mathcal{Z}_c(\bar{Z}_{11})$, then for some X_0, $Z_{c0} = P_{c0}(X_0)Q_{c0}^{-1}(X_0)$. Let a real number $\varepsilon > 0$ be sufficiently small to ensure that $Q_c(X)$ is nonsingular for all X satisfying $||X - X_0|| < \varepsilon$. (See [66, Sec. 7.2].) We define a ***basic neighborhood*** around $Z_{c0} = P_c(X_0)Q_c^{-1}(X_0) \in \mathcal{Z}_c(\bar{Z}_{11})$ as

$$\{P_c(X)Q_c^{-1}(X) \in \mathbf{P}^{p\times m} \mid \, ||X - X_0|| < \varepsilon\}.$$

Then, using arguments similar to those in Section 7.2 of [66], it is straightforward to show that the collection of the basic neighborhoods is a base for a topology on $\mathcal{Z}_c(\bar{Z}_{11})$. A similar topology can be defined using the left coprime fractional representation of the compensator. More precisely, let $\tilde{R}_c(X) := S_l + X D_l$, $\tilde{Q}_c(X) := T_l - X N_l$. A basic neighborhood around $Z_{c0} = \tilde{Q}_c^{-1}(X_0)\tilde{R}_c(X_0)$ for some X_0, is defined as

$$\{\tilde{Q}_c^{-1}(X)\tilde{R}_c(X) \in \mathbf{P}^{p\times m} \mid \, ||X - X_0|| < \varepsilon\},$$

where $\varepsilon > 0$ is sufficiently small to ensure that $\tilde{Q}_c(X)$ is nonsingular for all X satisfying $||X - X_0|| < \varepsilon$. Then, the collection of basic neighborhoods in the above form constitutes a base for a topology on $\mathcal{Z}_c(\bar{Z}_{11})$. Note that a property holds for almost all $\mathcal{Z}_c(\bar{Z}_{11})$ with respect to one of the topologies if and only if it holds for almost all $\mathcal{Z}_c(\bar{Z}_{11})$ with respect to the other topology.

Chapter 3

DECENTRALIZED STABILIZATION PROBLEM

This chapter considers the decentralized stabilization problem of linear time-invariant, finite-dimensional systems. The main results of the chapter can be summarized as follows. Theorem (3.1) solves DSP for 2-channel plants whose proof is adapted from [37]. Theorem (3.2) states a solvability condition for DSP of N-channel plants. In fact, that solvability condition is not different than the ones stated in [22], [10], [2]. The main contribution of Theorem (3.2) is the new synthesis procedure for decentralized stabilizing controllers proposed in its constructive proof. As a result of this procedure, the set of all admissible local compensators that can be applied to a specified channel, as an element of some decentralized stabilizing compensator is characterized in I of Theorem (3.3). The characterization is obtained in terms of only two parameters, independent of the number of channels. This yields the characterization of all decentralized stabilizing compensators of a plant. The conditions under which the class of admissible local compensators is generic have been determined in II of Theorem (3.3). These are purely structural conditions and correspond to certain connectivity relations among the subsystems. It has further been shown in III of Theorem (3.3) that, in case these conditions fail to hold, the set of admissible local compensators is precisely the set of internally stabilizing compensators of the corresponding channel. The proof of Theorem (3.2) also yields that the internally stabilizing compensators of a channel is generically admissible for that channel, independent of structural conditions. In Theorem (3.4) the problem of making a multi-channel system stabilizable and detectable from a single channel applying decentralized feedback around the other channels has been shown to be generically solvable for a given set of dynamic local compensators if and only if the plant is strongly connected and is free of unstable decentralized fixed modes.

3.1 Problem Definitions and Preliminaries

A rigorous definition of decentralized stabilization problem is given as follows.

Decentralized Stabilization Problem (DSP). *Let $Z = [Z_{ij}]$, $Z_{ij} \in \mathbf{P}^{p_i \times m_j}$, $i, j = 1, \ldots, N$ be the transfer matrix of a given plant where $p = \sum_{i=1}^{N} p_i$, $r := \sum_{i=1}^{N} r_i$. Determine local compensators $Z_{c1} \in \mathbf{P}^{r_1 \times p_1}$, ..., $Z_{cN} \in \mathbf{P}^{r_N \times p_N}$ such that the pair of plants (Z, Z_c) is stable where $Z_c = diag\{Z_{c1}, \ldots, Z_{cN}\}$.*

Let the plant have the following bicoprime fractional representation over $\mathbf{S}$

$$\begin{bmatrix} Z_{11} & & Z_{1N} \\ \vdots & & \vdots \\ Z_{N1} & & Z_{NN} \end{bmatrix} = \begin{bmatrix} P_1 \\ \\ P_N \end{bmatrix} Q^{-1}[R_1 \ldots R_N] \tag{3.1}$$

where $P_i \in \mathbf{S}^{p_i \times q}$, $R_i \in \mathbf{S}^{q \times r_i}$, and $Q \in \mathbf{S}^{q \times q}$.

The plant (3.1) is said to be *strongly connected* if $Z_{\mathbf{N}-\mathbf{r},\mathbf{r}} \neq 0$ for all $\mathbf{r} \in C_N$ [10]. Strong connectedness is a structural property playing an important role in the characterization of decentralized stabilizing controllers (Theorem (3.3)). Very briefly, if a plant is not strongly connected it can be put into a lower triangular form with a symmetric row and column permutation (for details see [10]). The notion of strong connectedness is also important in case of time-varying controllers. It is known that both in continuous and discrete time systems, strongly connected plants always admit solution to DSP if the decentralized controller is chosen as time varying [4], [28], [61].

From section 2 of the previous chapter it follows that DSP is solvable if and only if there exists P_{ci}, Q_{ci} such that $Z_{ci} := P_{ci}Q_{ci}^{-1}$ is proper and

$$\Upsilon := \begin{bmatrix} Q & R_1 P_{c1} & & R_N P_{cN} \\ -P_1 & Q_{c1} & & 0 \\ & \vdots & & \vdots \\ -P_N & 0 & & Q_{cN} \end{bmatrix} \tag{3.2}$$

is unimodular, in which case $diag\{Z_{c1}, ..., Z_{cN}\}$ solves DSP.

A closely related problem to DSP is the single channel canonicity (more precisely, stabilizability and detectability) problem which is defined as follows.

Single Channel Canonicity Problem (SCCP). *Given the N-channel plant (3.1), determine $N-1$ compensators $Z_{c2}, ..., Z_{cN}$ such that the closed loop system that results by the application of feedback $u_i = -Z_{ci}y_i$, $i = 2, ..., N$ is stabilizable from u_1 and detectable at y_1, i.e. the fractional representation of the closed loop transfer matrix $[P_1\ 0 \dots 0]\ \tilde{\Upsilon}^{-1}\ [R_1'\ 0\ \dots\ 0]'$, where*

$$\tilde{\Upsilon} := \begin{bmatrix} Q & R_2 P_{c2} & & R_N P_{cN} \\ -P_2 & Q_{c2} & & \vdots \\ & \vdots & & \\ -P_N & 0 & & Q_{cN} \end{bmatrix}, \tag{3.3}$$

is bicoprime. By definition, if SCCP is solved by some Z_{ci}, $i = 2, ..., N$ then DSP can be solved by applying a stabilizing compensator to the first channel. Conversely, if DSP is solved by $diag\{Z_{c1}, ..., Z_{cN}\}$ then SCCP can be solved by Z_{ci}, $i = 2, ..., N$. In other words, *DSP is solvable if and only if SCCP is solvable.* This conclusion has been first stated in [37, Theorem 3.2] for 2-channel plants. A similar result is also stated in [10] for strongly connected plants, where Z_{c2}, $...,Z_{cN}$ are restricted to be constant compensators.

In the solution of DSP, the notion of "completeness" of system matrices plays a key role. The following is the definition of completeness over the ring $\mathbf{S}$ [37].

Consider

$$\Pi := \begin{bmatrix} Q_{11} & R \\ -P & W \end{bmatrix}, \tag{3.4}$$

where $P \in \mathbf{S}^{p\times q}$, $R \in \mathbf{S}^{q\times r}$, $W \in \mathbf{S}^{p\times r}$, $Q_{11} \in \mathbf{S}^{q\times q}$ and biproper. We say that Π is *complete* (modulo stable modes) if the Smith canonical form of Π over $\mathbf{S}$ contains at least $q = size(Q_{11})$ unit invariant factors.

To clarify the terminology in the subsequent sections we note that the following two statements are alternatively used: Π is complete, (P, Q_{11}, R, W) is complete. Also, in case Π is complete and $W = 0$ we equivalently say that (P, Q_{11}, R) is complete. The following lemma is concerned with the properties of completeness (see also [37]).

Lemma (3.1). *The matrix Π in (3.4) is complete if and only if $rank\, \Pi(z) \geq q$ for all $z \in \mathcal{C}_+$.*

Lemma (3.2) is used in the proof of Lemma (3.1).[1]

Lemma (3.2). *Let $D \in \mathbf{S}^{q\times q}$, $A \in \mathbf{S}^{p\times q}$, and $B \in \mathbf{S}^{p\times r}$, where D is biproper. Assume that*

$$rank \begin{bmatrix} D & 0 \\ A & B \end{bmatrix} (z) \geq q$$

for all $z \in \mathcal{C}_+$. Then, there exists $X \in \mathbf{S}^{r\times q}$ such that $(D, A + BX)$ is right coprime.

Proof. We start with a fact, whose simple proof is omitted.

Let $\bar{A} \in \mathcal{C}^{a\times b}$, $\bar{B} \in \mathcal{C}^{a\times c}$, and $rank([\bar{A}\ \ \bar{B}]) \geq l$, with $b \geq l$. Then, there exists $X \in \mathcal{C}^{c\times b}$ such that $rank(\bar{A} + \bar{B}X) \geq l$.

Let $\mathbf{D} := \{z \in \mathcal{C}_+ |\ det(D(z)) = 0\}$. Suppose that $\mathbf{D}$ is composed of some distinct complex numbers $z_1, \ldots, z_t$ such that

$$z_i = \begin{cases} r_i \in \mathcal{R} & i = 1, \ldots, p \\ c_i \in \mathcal{C} & i = p+1, \ldots, p+l \\ c_i^* \in \mathcal{C} & i = p+l+1, \ldots, p+2l \end{cases}$$

where $c_i \neq c_j^*$, $i, j = 1, \ldots, l$ and $t = p + 2l$.

[1] Although there is a more straightforward proof of Lemma (3.1) using the Smith form of Π, we employ Lemma (3.2) as it yields a useful construction in the proof of Theorem (3.1).

Fix any $z_i \in \mathrm{D}$ where $i \in \{1, ..., p+l\}$. Assume that rank$D(z_i) = q - l_i$ for some integer l_i. Multiplying from left by a nonsingular matrix $C \in \mathcal{C}^{q\times q}$, $D(z_i)$ becomes

$$\begin{bmatrix} \bar{D}_1 & \bar{D}_2 \\ 0 & 0 \end{bmatrix},$$

where $\bar{D}_1 \in \mathcal{C}^{q-l_i\times q-l_i}$, and $\bar{D}_2 \in \mathcal{C}^{q-l_i\times l_i}$. There also exists a nonsingular matrix $E \in \mathcal{C}^{q\times q}$, such that $[\bar{D}_1 \;\; \bar{D}_2]E = [\bar{D} \;\; 0]$. where $\bar{D} \in \mathcal{C}^{q-l_i\times q-l_i}$ and nonsingular. Let $\bar{A} = [\bar{A}_1 \;\; \bar{A}_2] := A(z_i)E$, where $\bar{A}_1 \in \mathcal{C}^{p\times q-l_i}$, and $\bar{A}_2 \in \mathcal{C}^{p\times l_i}$. By the hypothesis rank$[\bar{A}_2 \;\; B(z_i)] \geq l_i$. From the above fact there exists $\hat{X} \in \mathcal{C}^{r\times l_i}$ such that rank$(\bar{A}_2 + B(z_i)\hat{X}) = l_i$. Letting $X_i := [\tilde{X} \;\; \hat{X}]E^{-1}$, where $\tilde{X} \in \mathcal{C}^{r\times q-l_i}$ is arbitrary, rank$[D'(z_i) \; (A(z_i) + B(z_i)X_i)']' = q$. Repeating this process for all z_i where $i \in \{1, ..., p-l\}$ we obtain $X_i \in \mathcal{C}^{r\times q}$, $i \in \{1, ..., p+l\}$ so that rank$[D'(z_i)$ $(A(z_i) + B(z_i)X_i)']' = q$. $i \in \{1, ..., p+l\}$.

We will now construct $X \in \mathbf{S}^{p\times q}$ such that $(A + BX, D)$ is right coprime. Construct $x_{11} \in \mathbf{S}$. the $(1,1)$ element of X using Lemma (2.3) such that $x_{11}(z_i)$ equals the $(1,1)$ element of X_i, $i \in \{1, ..., p+l\}$. The other elements of X are constructed similarly so that $X(z_i) = X_i$, $i \in \{1, ..., p+l\}$. This shows that rank$[D'(z_i) \; (A(z_i) + B(z_i)X_i)']' = q$, $i \in \{1, ..., p+l\}$. Hence, rank$[D'(z)$ $(A(z) + B(z)X)']' = q$. for all $z \in \mathcal{C}_+$. This implies that $(A + BX, D)$ is right coprime.□

Proof of Lemma (3.1). Necessity part is obvious from the rank conditions. To show sufficiency let $Q_{11}^{-1}R = \bar{Q}^{-1}\bar{R}$ for a left coprime pair of matrices $(\bar{Q}, \bar{R})$ over $\mathbf{S}$. Then, there exists unimodular

$$\Psi := \begin{bmatrix} K & \bar{L} \\ L & \bar{K} \end{bmatrix}$$

such that $[\bar{Q} \;\; \bar{R}]\Psi = [I \;\; 0]$. Multiplying from right by Ψ, H becomes

$$\Gamma := \begin{bmatrix} D & 0 \\ -PK & -P\bar{L} \end{bmatrix}$$

for some $D \in \mathbf{S}^{q\times q}$, which is nonsingular because of the fact that Q_{11} is nonsingular. Obviously *rank* $\Gamma(z) \geq q$ for all $z \in \mathcal{C}_+$. Applying Lemma (3.2) there exists $X \in \mathbf{S}^{r\times q}$ such that $(D, -P(K+\bar{L}X))$ is left coprime. Thus, there exists a unimodular matrix $U = [U_{ij}]$, $i, j = 1, 2$ such that

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} D \\ -P(K+\bar{L}X) \end{bmatrix} = \begin{bmatrix} I_q \\ 0 \end{bmatrix},$$

where U_{22} is nonsingular. Then multiplying from left and right respectively, by U and

$$V := \begin{bmatrix} K+\bar{L}X & \bar{L}+(K+\bar{L}X)U_{12}P\bar{L} \\ L+\bar{K}X & \bar{K}+(L+\bar{K}X)U_{12}P\bar{L} \end{bmatrix}.$$

both of which are unimodular, Π becomes

$$\begin{bmatrix} I_q & 0 \\ 0 & -U_{22}P\bar{L} \end{bmatrix},$$

which implies by definition that Π is complete.□

3.2 Solution of Decentralized Stabilization Problem

We first state the solution of DSP for 2-channel systems (see also [37]).

Theorem (3.1). *Given the plant (3.1) with $N = 2$, DSP (and equivalently SCCP) is solvable if and only if* (P_2, Q, R_1) and (P_1, Q, R_2) *are complete.*

The synthesis procedure of Theorem (3.1) consists of solving SCCP through the application of a compensator at the second channel. As the closed loop system obtained is stabilizable and detectable, any stabilizing compensator at the first channel solves DSP. The same approach will be followed in the constructive proof of Theorem (3.2) for N-channel systems. It must be noted that for *strongly connected* systems, a similar procedure of solving DSP via obtaining a solution to SCCP is proposed in [10].

The proof of Theorem (3.1) requires the lemmata (3.3)-(3.5) which are concerned with the several genericity properties of the ring $\mathbf{S}$.

Lemma (3.3). *Let $E \in \mathbf{S}^{k\times d}$ be nonzero. The set of X such that (X, E) is left coprime is generic in $\mathbf{S}^{k\times k}$.*

Proof. This is a straightforward generalization of Proposition 7.6.15 in [66].□

Lemma (3.4) *Let $A \in \mathbf{S}^{k\times k}$ and $B \in \mathbf{S}^{k\times c}$ be such that (A, B) is left coprime. Assume that $E \in \mathbf{S}^{c\times k}$ is nonsingular. The set of X such that $(A + BX, E)$ is left coprime is generic in $\mathbf{S}^{c\times k}$.*

Proof. Lemma (3.4) is Lemma 2.1 of [37].□

Lemma (3.5). *Let $A \in \mathbf{S}^{k\times k}$ and $B \in \mathbf{S}^{k\times c}$ be such that (A, B) is left coprime. Assume that $E \in \mathbf{S}^{k\times d}$ is nonzero. The set of X such that $(A + BX, E)$ is left coprime is generic in $\mathbf{S}^{c\times k}$.*

Proof. We prove the lemma for the case A is nonsingular. The extension of the proof to the general case is straightforward, since the set of X for which $A + BX$ is nonsingular, is generic [66, Lemma 5.2.11].

Let U be a unimodular matrix such that $UE = [\bar{E}' \ 0]'$, where $\bar{E}$ is full row rank. There exists a unimodular matrix V such that

$$UAV = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}.$$

Clearly A_{11} and A_{22} are nonsingular. Also let $UB = [B_1' \ B_2']'$ and $XV = [X_1 \ X_2]$. Since $[A \ B]$ is left unimodular, for any X_1, $(A_{11} + B_1X_1, B_1)$ and $(A_{22}, A_{21} + B_2X_1, B_2)$ are left coprime. This shows that if $[A_{21} \ B_2] = 0$ then A_{22} is unimodular. Now define $\hat{A}_{11} := A_{11} + B_1X_1$, $\hat{A}_{21} := A_{21} + B_2X_1$, and $\hat{A}_{22} := A_{22} + B_2X_2$.

Case 1. $[A_{21} \ B_2] = 0$. In this case A_{22} is unimodular. Also from Lemma (3.4) for almost all X_1 $(\hat{A}_{11}, \bar{E})$ is left coprime. Fix one such X_1. Let $X = [X_1 \ X_2]V^{-1}$, where X_2 is arbitrary. By unimodular operations, it holds that $[A + BX \ E]$ is left unimodular if and only if so is

$$\begin{bmatrix} \hat{A}_{11} & 0 & \bar{E} \\ 0 & A_{22} & 0 \end{bmatrix},$$

which is clearly left unimodular. Since X_1 is almost arbitrary, X_2 is arbitrary and $X = [X_1 \;\; X_2]V^{-1}$, we have that for almost all X $(A + BX, E)$ is left coprime.

Case 2. $[A_{21} \;\; B_2] \neq 0$. Then, it is easy to verify that $A_{21} + B_2X_1 \neq 0$ for almost all X_1. So, for almost all X_1 (i) $(\hat{A}_{11}, \bar{E})$ is left coprime, and (ii) $\hat{A}_{21} \neq 0$. Choose one such X_1. There exist matrices $K, L, \bar{A}_{11}, \bar{B}_1, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6$ such that

$$[\hat{A}_{11} \;\; B_1] \begin{bmatrix} K & -\bar{B}_1 \\ L & \bar{A}_{11} \end{bmatrix} = [I \;\; 0] \tag{3.5}$$

$$\begin{bmatrix} \hat{A}_{11} & \bar{E} \\ \Psi_5 & \Psi_6 \end{bmatrix} \begin{bmatrix} \Psi_1 & \Psi_3 \\ \Psi_2 & \Psi_4 \end{bmatrix} = I. \tag{3.6}$$

It can be verified that $[A \;\; B]$ is equivalent over **S** to

$$\begin{bmatrix} I & 0 & 0 \\ 0 & A_{22} & B_2\bar{A}_{11} - \hat{A}_{21}\bar{B}_1 \end{bmatrix},$$

which implies that $(A_{22}, B_2\bar{A}_{11} - \hat{A}_{21}\bar{B}_1)$ is left coprime. This shows that $(A_{22}, (B_2 \bar{A}_{11} - \hat{A}_{21}\bar{B}_1) + \hat{A}_{21}\Psi_3\Psi_5\bar{B}_1, \hat{A}_{21}\Psi_3)$ is left coprime. From (3.5) and (3.6), $(B_2\bar{A}_{11} - \hat{A}_{21}\bar{B}_1) + \hat{A}_{21}\Psi_3\Psi_5\bar{B}_1 = (B_2 - \hat{A}_{21}\Psi_1B_1)\bar{A}_{11}$. This implies that $(A_{22}, B_2 - \hat{A}_{21}\Psi_1B_1, \hat{A}_{21}\Psi_3)$ is left coprime.

On the other hand, let $X = [X_1 \;\; X_2]V^{-1}$, where X_2 is arbitrary. Unimodular operations yield that $[A + BX \;\; E]$ is left unimodular if and only if $(A_{22} + (B_2 - \hat{A}_{21}\Psi_1B_1)X_2, \hat{A}_{21}\Psi_3)$ is left unimodular. Let $D_l := gclf(A_{22}, B_2 - \hat{A}_{21}\Psi_1B_1)$, such that $A_{22} = D_l\check{A}$ and $B_2 - \hat{A}_{21}\Psi_1B = D_l\check{B}$ for a left coprime pair of matrices $(\check{A}, \check{B})$. Since A_{22} is nonsingular, D_l and $\check{A}$ are nonsingular. Let $D_l^{-1}\hat{A}_{21}\Psi_3 = \check{E}\check{D}^{-1}$ for a right coprime pair of matrices $(\check{E}, \check{D})$. Since $\bar{E}$ is full row rank, so is Ψ_3. This, and the fact that $\hat{A}_{21} \neq 0$ imply $\check{E} \neq 0$. Also $(A_{22} + (B_2 - \hat{A}_{21}\Psi_1B_1)X_2, \hat{A}_{21}\Psi_3)$ is left coprime if and only if $(\check{A} + \check{B}X_2, \check{E})$ is left coprime. This is the same type of equation as the one we started with, except that now the number of rows of A is reduced at least by one. Applying the same arguments repeatedly, we either terminate at Case 1, at some step, or terminate at Case 2, with the number of rows of $\check{A}$ is 1. In this case $\check{E}$ is full row rank and applying Lemma (3.4) completes the proof.□

Proof of Theorem (3.1).

[Only If] Suppose that the matrix (3.2) is unimodular and let (P_1, Q, R_2) not be complete. Then, from Lemma (3.1), for some $z \in \mathcal{C}_+$

$$rank \begin{bmatrix} Q & R_2 \\ -P_1 & 0 \end{bmatrix} (z) < q = size(Q).$$

This implies

$$rank \begin{bmatrix} Q & R_2 P_{c2} \\ -P_1 & 0 \end{bmatrix} (z) = rank \left(\begin{bmatrix} Q & R_2 \\ -P_1 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P_{c2} \end{bmatrix} \right)(z) =: \bar{q} < q.$$

Let for some nonsingular matrix $K \in \mathcal{C}^{q+p_2 \times q+p_2}$ we have

$$\begin{bmatrix} Q & R_2 P_{c2} \\ -P_1 & 0 \end{bmatrix} (z) K = \begin{bmatrix} H_1 & 0 \\ H_2 & 0 \end{bmatrix}$$

where $H_1 \in \mathcal{C}^{q \times \bar{q}}$, $H_2 \in \mathcal{C}^{p_1 \times \bar{q}}$ and $rank\, [H_1' \; H_2']' = \bar{q}$. Observe that

$$rank \begin{bmatrix} Q & R_1 P_{c1} & R_2 P_{c2} \\ -P_1 & Q_{c1} & 0 \\ -P_2 & 0 & Q_{c2} \end{bmatrix} (z) \;\; = rank \begin{bmatrix} Q & R_2 P_{c2} & R_1 P_{c1} \\ -P_1 & 0 & Q_{c1} \\ -P_2 & Q_{c2} & 0 \end{bmatrix} (z)$$

$$= rank \begin{bmatrix} H_1 & 0 & R_1 P_{c1}(z) \\ H_2 & 0 & Q_{c1}(z) \\ H_3 & H_4 & 0 \end{bmatrix}$$

where $H_3 \in \mathcal{C}^{p_2 \times \bar{q}}$ and $H_4 \in \mathcal{C}^{p_2 \times q - p_2 - \bar{q}}$. It holds that

$$rank \begin{bmatrix} H_1 & 0 \\ H_2 & 0 \\ H_3 & H_4 \end{bmatrix} \leq \bar{q} + p_2 < q + p_2$$

implying

$$rank \begin{bmatrix} Q & R_1 P_{c1} & R_2 P_{c2} \\ -P_1 & Q_{c1} & 0 \\ -P_2 & 0 & Q_{c2} \end{bmatrix} (z) = rank \begin{bmatrix} H_1 & 0 & R_1 P_{c1}(z) \\ H_2 & 0 & Q_{c1}(z) \\ H_3 & H_4 & 0 \end{bmatrix} < q + p_2 + p_1.$$

This shows that

$$\Theta := \begin{bmatrix} Q & R_1P_{c1} & R_2P_{c2} \\ -P_1 & Q_{c1} & 0 \\ -P_2 & 0 & Q_{c2} \end{bmatrix}$$

is not unimodular, since $z \in \mathcal{C}_+$ is a zero of $det(\Theta)$. In other words, the completeness of (P_1, Q, R_2) is necessary for DSP to be solvable. The completeness of (P_2, Q, R_1) follows by dual arguments.

[If] Assume that (P_2, Q, R_1) and (P_1, Q, R_2) are complete. Using the procedure described in the proof of Lemma (3.1) construct unimodular matrices $U = [U_{ij}]$, $V = [V_{ij}]$, $\hat{U} = [\hat{U}_{ij}]$ and $\hat{V} = [\hat{V}_{ij}]$, $i, j = 1, 2$, such that U_{22} and $\hat{U}_{22}$ are nonsingular and

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} Q & R_1 \\ -P_2 & 0 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & \Psi \end{bmatrix}, \qquad (3.7)$$

$$\begin{bmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{21} & \hat{V}_{22} \end{bmatrix} \begin{bmatrix} Q & R_2 \\ -P_1 & 0 \end{bmatrix} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & \hat{\Psi} \end{bmatrix}$$

for some $\Psi \in \mathbf{S}^{p_2 \times r_1}$, and $\hat{\Psi} \in \mathbf{S}^{p_1 \times r_2}$.

Step 1. We will construct a compensator $Z_{c2} = P_{c2}Q_{c2}^{-1} = \bar{Q}_{c2}^{-1}\bar{R}_{c2}$, for a left coprime pair of matrices $(\bar{Q}_{c2}, \bar{R}_{c2})$ and a right coprime pair of matrices (Q_{c2}, P_{c2}) such that

$$\Phi_l := \begin{bmatrix} Q & R_2P_{c2} & R_1 \\ -P_2 & Q_{c2} & 0 \end{bmatrix} \quad \text{and} \quad \Phi_r := \begin{bmatrix} Q & R_2 \\ -\bar{R}_{c2}P_2 & \bar{Q}_{c2} \\ -P_1 & 0 \end{bmatrix}$$

are left and right unimodular, respectively.

Multiplying from left and right respectively, by

$$U \text{ and } \begin{bmatrix} V_{11} & -V_{11}(U_{11}R_2P_{c2} + U_{12}Q_{c2}) & V_{12} \\ 0 & I & 0 \\ V_{21} & -V_{21}(U_{11}R_2P_{c2} + U_{12}Q_{c2}) & V_{22} \end{bmatrix},$$

both of which are unimodular, Φ_l becomes

$$\begin{bmatrix} I & 0 & 0 \\ 0 & U_{22}Q_{c2} + U_{21}R_2P_{c2} & \Psi \end{bmatrix}.$$

On the other hand, multiplying from left and right respectively, by

$$\begin{bmatrix} \hat{V}_{11} & 0 & \hat{V}_{12} \\ (\bar{R}_{c2}P_2\hat{U}_{11} - \bar{Q}_{c2}\hat{U}_{21})\hat{V}_{11} & I & (\bar{R}_{c2}P_2\hat{U}_{11} - \bar{Q}_{c2}\hat{U}_{21})\hat{V}_{12} \\ \hat{V}_{21} & 0 & \hat{V}_{22} \end{bmatrix}$$

and $\hat{U}$, both of which are unimodular, Φ_r becomes

$$\begin{bmatrix} I & 0 \\ 0 & -\bar{R}_{c2}P_2\hat{U}_{12} + \bar{Q}_{c2}\hat{U}_{22} \\ 0 & \hat{\Psi} \end{bmatrix}.$$

One concludes that Φ_l is left unimodular if and only if $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is left coprime, and Φ_r is right unimodular if and only if $(-\bar{R}_{c2}P_2\hat{U}_{12} + \bar{Q}_{c2}\hat{U}_{22}, \hat{\Psi})$ is right coprime.

Let $\mathcal{Z}_c(Z_{22})$ be the set of all stabilizing compensators of Z_{22}. We will now show that (a) the class of Z_{c2} for which $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is left coprime and $(-\bar{R}_{c2}P_2\hat{U}_{12} + \bar{Q}_{c2}\hat{U}_{22}, \hat{\Psi})$ is right coprime is open and dense in $\mathcal{Z}_c(Z_{22})$, and (b) in case Ψ and $\hat{\Psi}$ are nonzero the class of Z_{c2} for which $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is left coprime and $(-\bar{R}_{c2}P_2\hat{U}_{12} + \bar{Q}_{c2}\hat{U}_{22}, \hat{\Psi})$ is right coprime is open and dense in $\mathbf{P}^{r_2 \times p_2}$ (with respect to the *Graph Topology* [66]).

First, we will prove statement (a). If $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is left coprime, under sufficiently small perturbations on Q_{c2} and P_{c2} that property is still preserved, because the set of unimodular matrices over S is open [66]. Similarly, under sufficiently small perturbations on $\bar{Q}_{c2}$ and $\bar{P}_{c2}$ the right coprimeness of $(-\bar{R}_{c2}P_2\hat{U}_{12} + \bar{Q}_{c2}\hat{U}_{22}, \hat{\Psi})$ is preserved. We thus conclude that the set of controllers in $\mathcal{Z}_c(Z_{22})$ for which $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is left coprime and $(-\bar{R}_{c2}P_2\hat{U}_{12} + \bar{Q}_{c2}\hat{U}_{22}, \hat{\Psi})$ is right coprime is open.

On the other hand,

$$U\begin{bmatrix} Q & R_1 & R_2 & 0 \\ -P_2 & 0 & 0 & I \end{bmatrix}\begin{bmatrix} V & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & U_{11}R_2 & U_{12} \\ 0 & \Psi & U_{21}R_2 & U_{22} \end{bmatrix}$$

is a left unimodular matrix, since the matrix in the middle at the left hand side is left unimodular. This implies that $(U_{22}, U_{21}R_2, \Psi)$ is left coprime. If $\Psi = 0$ then it holds that $(U_{22}, U_{21}R_2)$ is left coprime. Also $U_{22}^{-1}U_{21}R_2 = Z_{22}$. (This can be shown as follows. From (3.7) we have $U_{22}^{-1}U_{21} = P_2Q^{-1}$. Hence $Z_{22} = P_2Q^{-1}R_2 = U_{22}^{-1}U_{21}R_2$.) We conclude that $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is left coprime for all coprime fractions $P_{c2}Q_{c2}^{-1} \in \mathcal{Z}_c(Z_{22})$ as $U_{22}Q_{c2} + U_{21}R_2P_{c2}$ is unimodular. (It is also true that in this case $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is left coprime only if $P_{c2}Q_c^{-1} \in \mathcal{Z}_c(Z_{22})$.) We now investigate the case that $\Psi \neq 0$ and $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is not a left coprime pair. Let some left and right coprime fractions of Z_{22} over S be given by $Z_{22} = D_l^{-1}N_l = N_rD_r^{-1}$ so that (2.6) holds. In this case $Z_c \in \mathcal{Z}_c(Z_{22})$ if and only if (2.7) holds. Let $P_{c2} = S_r + D_rX_0$, $Q_{c2} = T_r - N_rX_0$ for some X_0. Define

$$\begin{bmatrix} \tilde{A} & \tilde{B} \end{bmatrix} = \begin{bmatrix} U_{22} & U_{21}R_2 \end{bmatrix}\begin{bmatrix} T_r & -N_r \\ S_r & D_r \end{bmatrix}.$$

Let $G := gclf(\tilde{A}, \tilde{B})$. Then, (G, Ψ) is left coprime. Let $G^{-1}\Psi = E\bar{G}^{-1}$ for a right coprime pair of matrices $(\bar{G}, E)$ over S. Also let $\tilde{A} = GA$, $\tilde{B} = GB$. From Lemma (3.5) there exists ΔX with arbitrarily small norm such that $(A + B(X_0 + \Delta X)$, $E)$ and consequently $(\tilde{A} + \tilde{B}(X_0 + \Delta X), \Psi)$ are left coprime pairs. Now letting $\bar{P}_{c2} := S_r + D_r(X_0 + \Delta X)$, $\bar{Q}_{c2} := T_r - N_r(X_0 + \Delta X)$ it holds that $(U_{22}\bar{Q}_{c2} + U_{21}R_2\bar{P}_{c2}, \Psi)$ is left coprime. This shows that the set of $Z_{c2} = P_{c2}Q_{c2}^{-1}$ for which $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is left coprime is dense in $\mathcal{Z}_c(Z_{22})$. Similar arguments yield that the set of $Z_{c2} = \tilde{Q}_{c2}^{-1}\tilde{R}_{c2}$ for which $(-\tilde{R}_{c2}P_2\hat{U}_{12} + \tilde{Q}_{c2}\hat{U}_{22}, \hat{\Psi})$ is right coprime is dense in $\mathcal{Z}_c(Z_{22})$. Hence, the class of $Z_{c2} = P_{c2}Q_{c2}^{-1} = \tilde{Q}_{c2}^{-1}\tilde{R}_{c2}$ for which $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is left coprime and $(\hat{\Psi}, -\tilde{R}_{c2}P_2\hat{U}_{12} + \tilde{Q}_{c2}\hat{U}_{22})$ is right coprime is open and dense in $\mathcal{Z}_c(Z_{22})$[2]. This proves statement (a). The

[2]We implicitly use the fact that if a property holds true for almost all elements of $\mathcal{Z}_c(Z_{22})$

proof of (b) follows the same arguments except that we replace $\mathcal{Z}_c(Z_{22})$ by $\mathbf{P}^{r_2\times p_2}$ and consider only the cases $\Psi \neq 0$ and $\hat{\Psi} = 0$. Note that, in case $\Psi = 0$, which holds if and only if $Z_{21} \doteq 0$. $(U_{22}Q_{c2} + U_{21}R_2P_{c2}, \Psi)$ is left coprime if and only if $P_{c2}Q_{c2}^{-1} \in \mathcal{Z}_c(Z_{22})$. Similarly, in case $\hat{\Psi} = 0$, which holds if and only if $Z_{12} = 0$), $(-\tilde{R}_{c2}P_2\hat{U}_{12} + \tilde{Q}_{c2}\hat{U}_{22}, \hat{\Psi})$ is right coprime if and only if $\tilde{Q}_{c2}^{-1}\tilde{R}_{c2} \in \mathcal{Z}_c(Z_{22})$.

Now fix one $Z_{c2} = P_{c2}Q_c^{-1} = \tilde{Q}_c^{-1}\tilde{R}_c$ which ensures that Φ_l and Φ_r are left and right unimodular, respectively.

Step 2. The right unimodularity of Φ_r implies that

$$\begin{bmatrix} Q & R_2P_{c2} \\ -P_2 & Q_{c2} \\ -P_1 & 0 \end{bmatrix}$$

is right unimodular. (This can be shown as follows. There exist matrices L_1, L_2, L_3, L_4 over S such that $L_1Q_{c2} + L_2P_{c2} = I$ and

$$L_3\begin{bmatrix} Q & R_2 \\ -\tilde{R}_{c2}P_2 & \tilde{Q}_{c2} \end{bmatrix} + L_4\begin{bmatrix} -P_1 & 0 \end{bmatrix} = I.$$

Then it can be verified that

$$\left(\begin{bmatrix} I & 0 \\ L_1P_2 & L_2 \end{bmatrix} L_3 \begin{bmatrix} I & 0 \\ 0 & \tilde{R}_{c2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & L_1 \end{bmatrix}\right)\begin{bmatrix} Q & R_2P_{c2} \\ -P & Q_{c2} \end{bmatrix} + \begin{bmatrix} I & 0 \\ L_1P_2 & L_2 \end{bmatrix} L_4[-P_1\ 0] = I$$

implying our claim.)

We now have $[P_1\ 0]\Sigma^{-1}[R_1'\ 0']'$ is a bicoprime fraction, where

$$\Sigma = \begin{bmatrix} Q & R_2P_{c2} \\ -P_2 & Q_{c2} \end{bmatrix}.$$

Let $Q_l^{-1}R_l = [P_1\ 0]\Sigma^{-1}[R_1'\ 0']'$ be a left coprime fraction, so that for some $Q_{c1} \in \mathbf{S}^{p_1\times p_1}$ and $P_{c1} \in \mathbf{S}^{r_1\times p_1}$, $Q_lQ_{c1} + R_lP_{c1} = I$, with Q_{c1} is biproper. Then, the compensator $diag\{P_{c1}Q_{c1}^{-1}, P_{c2}Q_{c2}^{-1}\}$ solves DSP. This completes the proof.□

with respect to the topology induced by left coprime fractions, then so it does with respect to the topology induced by right coprime fractions and vice versa.

Remark (3.1). The proof of the theorem leads us to the following observations. Let DSP for Z be solvable. We see that if Z_{12} and Z_{21} are both nonzero then SCCP is solvable for almost all compensators in $\mathcal{Z}_c(Z_{22})$ and for almost all compensators in $\mathbf{P}^{r_2 \times p_2}$. If at least one of Z_{12} and Z_{21} is zero then SCCP is solvable for some Z_{c2} if and only if $Z_{c2} \in \mathcal{Z}_c(Z_{22})$. In case Z_{12} and Z_{21} are both nonzero, the set of compensators solving SCCP is reduced to a left unimodularity and a right unimodularity relation in terms of two compensator parameters. This is useful in pinpointing the *nongeneric* cases for the solution of SCCP. (See also Theorem (3.3).)•

To obtain the solution of N-channel DSP we use the following lemma which gives conditions for a closed loop system matrix to be complete.

Lemma (3.6). *Consider the triple*

$$\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, Q_{11}, [S_1 \quad S_2]\right).$$

Define $\tilde{Z}_{11} := T_1 Q_{11}^{-1} S_1 \in \mathbf{P}^{p \times r}$.

Let $(T_2, Q_{11}, [S_1 \quad S_2])$ *and* $\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, Q_{11}, S_2\right)$ *be complete. Then the following statements hold.*

(1) For almost all $Z_c \in \mathcal{Z}_c(\tilde{Z}_{11})$

$$\left([T_2 \quad 0], \begin{bmatrix} Q_{11} & S_1 P_c \\ -T_1 & Q_c \end{bmatrix} \begin{bmatrix} S_2 \\ 0 \end{bmatrix}\right) \tag{3.8}$$

is complete, where $P_c Q_c^{-1}$ *is a right coprime fractional representation of* Z_c.

(2) For almost all $Z_c \in \mathbf{P}^{r \times p}$ *the triple in (3.8) is complete if and only if at least one of* $\tilde{Z}_{12} := T_1 Q_{11}^{-1} S_2$, $\tilde{Z}_{21} := T_2 Q_{11}^{-1} S_1$, *and* $\tilde{Z}_{22} := T_2 Q_{11}^{-1} S_2$ *is nonzero, where* $Z_c = P_c Q_c^{-1}$ *is a right coprime fractional representation of* Z_c.

The proof of Lemma (3.6) requires the lemmata (3.7)-(3.9) which consider some genericity arguments of the ring $\mathbf{S}$.

Lemma (3.7). *Let* $A \in \mathbf{S}^{k \times k}$ *and* $B \in \mathbf{S}^{k \times c}$ *be such that the pair* (A, B) *is* *left coprime. Assume that $E \in \mathbf{S}^{k\times d}$ is nonzero. The set of $[X_1' \;\; X_2']'$ such that $(AX_1 + BX_2, E)$ is left coprime is generic in $\mathbf{S}^{k+c\times k}$.*

Proof. It is enough to prove the Lemma when $E \in \mathbf{S}^{k\times 1}$. If $B = 0$ we can obtain the solution by using Lemma (3.3), because in this case A is unimodular and the lemma reduces to showing that the set of X for which (X, E) is left coprime, is open and dense in $\mathbf{S}^{k\times k}$. Now assume that $B \neq 0$. It can be shown, by using Lemma (3.3) that the set of X_1 for which (AX_1, B) is left coprime is open and dense in $\mathbf{S}^{k\times k}$. Fix one such X_1. Then, from Lemma (3.4), the set of X_2 for which $(AX_1 + BX_2, E)$ is left coprime, is open and dense in $\mathbf{S}^{c\times k}$. So, the set of $[X_1' : X_2']'$ for which $(AX_1 + BX_2, E)$ is left coprime is open and dense in $\mathbf{S}^{k+c\times k}$.$\Box$

Lemma (3.8). *The set of biproper matrices is dense in $\mathbf{S}^{k\times k}$.*

Proof. Let $A \in \mathbf{S}^{k\times k}$ not be biproper so that $A = A_0 + \tilde{A}$ where $A_0 \in \mathcal{R}^{k\times k}$ is the *zero*th coefficient matrix in the formal power series expansion $A = \sum_{i=1}^{\infty} A_i z^{-i}$ of A and $\tilde{A} := A - A_0$ is strictly proper. Given $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{R}^{k\times k}$ with $||A_\varepsilon|| < \varepsilon$ such that $A_0 + A_\varepsilon$ is nonsingular. Here, we used the fact that the set of nonsingular matrices is dense in $\mathcal{R}^{k\times k}$. Then, $B := A_\varepsilon + A$ is biproper and $||B - A||_\infty = ||A_\varepsilon||_\infty = ||A|| < \varepsilon$.$\Box$

Lemma (3.9). *Let $A \in \mathbf{S}^{k\times k}$ and $B \in \mathbf{S}^{k\times c}$ be such that the pair (A, B) is left coprime. Assume that $E \in \mathbf{S}^{k\times d}$ is nonzero. Express $Z \in \mathbf{P}^{c\times k}$ as $Z = ND^{-1}$, where (N, D) is right coprime. The set of $Z = ND^{-1}$ for which $(AD + BN, E)$ is left coprime is open and dense in $\mathbf{P}^{c\times k}$.*

Proof. To show that the set of such Z is open let $Z = ND^{-1} \in \mathbf{P}^{c\times k}$, with (N, D) is right coprime and $(AD + BN, E)$ is left coprime. From Lemma (3.7), we know that there exists $\delta > 0$, such that $\left|\left| \begin{matrix} D - X_1 \\ N - X_2 \end{matrix} \right|\right| < \delta$ implies that $(AX_1 + BX_2, E)$ is left coprime.

Let $\mu(N, D) \in \mathcal{R}_+ - \{0\}$ be such that $\varepsilon < \mu(N, D)$ implies $\bar{X}_1$ is biproper and $(\bar{X}_1, \bar{X}_2)$ is right coprime [66]. Consider any basic neighborhood of Z over $\mathbf{P}^{c\times k}$

defined as

$$\{\bar{X}_2\bar{X}_1^{-1} \mid \left\| \begin{array}{c} D - \bar{X}_1 \\ N - \bar{X}_2 \end{array} \right\| < \varepsilon\}, \ \ \varepsilon < \mu(N, D)$$

Then, the set $\mathcal{T} := \{\bar{N}\bar{D}^{-1} \in \mathbf{P}^{c\times k} \mid \left\| \begin{array}{c} D - \bar{D} \\ N - \bar{N} \end{array} \right\| < \min(\varepsilon, \delta)\}$ is an open set in the subset topology of $\mathbf{P}^{c\times k}$, containing ND^{-1}. It is also true that if $\bar{N}\bar{D}^{-1} \in \mathcal{T}$, then $(A\bar{D} + B\bar{N}, E)$ is left coprime. This shows that the set of such Z is open.

To show that the set of such Z is dense in $\mathbf{P}^{c\times k}$, consider $Z = ND^{-1} \in \mathbf{P}^{c\times k}$, (N, D) is right coprime, and $(AD + BN, E)$ is not left coprime. For any $\delta > 0$, there exists a basic neighborhood of ND^{-1} over $\mathbf{P}^{c\times k}$ defined as

$$\mathcal{T} = \{\bar{X}_2\bar{X}_1^{-1} \mid \left\| \begin{array}{c} D - \bar{X}_1 \\ N - \bar{X}_2 \end{array} \right\| < \varepsilon\}, \ \ \varepsilon < \min(\mu(N, D), \delta).$$

From Lemma (3.7), on the other hand, the above set contains some $X_2X_1^{-1}$ such that $(AX_1 + BX_2, E)$ is left coprime. There also exists $\alpha > 0$ such that for all $\bar{X}_1, \bar{X}_2$ such that $\left\| \begin{array}{c} D - \bar{X}_1 \\ N - \bar{X}_2 \end{array} \right\| < \alpha$, $(A\bar{X}_1 + B\bar{X}_2, E)$ is left coprime. We can assume that $\alpha < \varepsilon/2$. So,

$$\mathcal{T}' := \{\bar{X}_2\bar{X}_1^{-1} \mid \left\| \begin{array}{c} X_1 - \bar{X}_1 \\ X_2 - \bar{X}_2 \end{array} \right\| < \alpha\} \subseteq \mathcal{T}.$$

From Lemma (3.8) there exists $\hat{X}_1$ such that $X_2\hat{X}_1^{-1} \in \mathbf{P}^{c\times k}$ and $\|X_1 - \hat{X}_1\|$ can be made arbitrarily small. Hence, we can assume $X_2\hat{X}_1^{-1} \in \mathcal{T}' \subseteq \mathcal{T}$. But then,

$$\{\bar{X}_2\bar{X}_1^{-1} \in \mathbf{P}^{c\times k} \mid \left\| \begin{array}{c} D - \bar{X}_1 \\ N - \bar{X}_2 \end{array} \right\| < \varepsilon\}$$

is open in $\mathbf{P}^{c\times k}$ and contains $X_2\hat{X}_1^{-1}$, for which $(A\hat{X}_1 + BX_2, E)$ is left coprime. Since the choice of $\mathcal{T}$ is possible for arbitrary $\delta > 0$, this shows that the set of such Z is dense in $\mathbf{P}^{c\times k}$.□

Proof of Lemma (3.6). First note that (3.8) is complete if and only if

$$([T_2 \ \ 0], \begin{bmatrix} Q_{11} & S_1 \\ -\bar{R}_c T_1 & \bar{Q}_c \end{bmatrix}, \begin{bmatrix} S_2 \\ 0 \end{bmatrix}), \qquad (3.9)$$

is complete, where $P_c Q_c^{-1} = \bar{Q}_c^{-1}\bar{R}_c$ for some left coprime pair of matrices $(\bar{Q}_c, \bar{R}_c)$. (This can be shown as follows.

$$\begin{bmatrix} Q_{11} & S_1 & S_2 \\ -\bar{R}_c T_1 & \bar{Q}_c & 0 \\ -T_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & P_c & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & \bar{R}_c & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} Q_{11} & S_1 P_c & S_2 \\ -T_1 & Q_c & 0 \\ -T_2 & 0 & 0 \end{bmatrix}$$

implying that the system matrices associated with

$$([T_2 \ \ 0], \begin{bmatrix} Q_{11} & S_1 P_c \\ -T_1 & Q_c \end{bmatrix}, \begin{bmatrix} S_2 \\ 0 \end{bmatrix}) \ and \ ([T_2 \ \ 0], \begin{bmatrix} Q_{11} & S_1 P_c \\ -T_1 & Q_c \end{bmatrix}, \begin{bmatrix} S_2 \\ 0 \end{bmatrix})$$

are Fuhrmann equivalent over $\mathbf{P}_s$. The result then follows from Lemma (2.2) and Lemma (3.1) via applying various rank inequalities.)

Let $\bar{U}$ and $\bar{V}$ be unimodular matrices such that

$$\bar{U} \begin{bmatrix} Q_{11} & S_2 \\ -T_2 & 0 \end{bmatrix} \bar{V} = \begin{bmatrix} \Lambda & 0 \\ 0 & \Psi \end{bmatrix}, \tag{3.10}$$

where the matrix on the right hand size is the Smith normal form of the matrix at the left so that $size(\Lambda) = size(Q_{11})$. Partition $\bar{U}$ and $\bar{V}$ as $\bar{U} = [\bar{U}_{ij}]$, $\bar{V} = [\bar{V}_{ij}]$, $i, j = 1, 2$. It holds that

$$\bar{U} \begin{bmatrix} Q_{11} & S_2 & S_1 \\ -T_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{V} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & \bar{U}_{11} S_1 \\ 0 & \Psi & \bar{U}_{21} S_1 \end{bmatrix},$$

where the completeness of $(T_2, Q_{11}, [S_1 \ S_2])$ implies that the matrix at the right hand size has rank no less than $size(\Lambda)$. In this case Lemma (3.2) implies the existence of some matrix X_1 over $\mathbf{S}$ such that $(\Lambda, X_1\Psi + (\bar{U}_{11} + X_1\bar{U}_{21})S_1)$ is left coprime. Since $diag\{\Lambda, \Psi\}$ is in the Smith canonical form, every entry of Λ divides every entry of Ψ. Thus, $X_1\Psi = -\Lambda Y_1$ for some Y_1 over $\mathbf{S}$, implying that $(\Lambda, (\bar{U}_{11} + X_1\bar{U}_{21})S_1)$ is left coprime. Now,

$$\begin{bmatrix} I & X_1 \\ 0 & I \end{bmatrix} \bar{U} \begin{bmatrix} Q_{11} & S_2 \\ -T_2 & 0 \end{bmatrix} \bar{V} \begin{bmatrix} I & Y_1 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ 0 & \Psi \end{bmatrix}$$

Define

$$\tilde{U} = \begin{bmatrix} I & X_1 \\ 0 & I \end{bmatrix} \ and \ \tilde{V} = \bar{V} \begin{bmatrix} I & Y_1 \\ 0 & I \end{bmatrix}.$$

Then

$$\begin{bmatrix} \tilde{U} & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} Q_{11} & S_2 \\ -T_2 & 0 \\ -T_1 & 0 \end{bmatrix} \tilde{V} = \begin{bmatrix} \Lambda & 0 \\ 0 & \Psi \\ -T_{11}\tilde{V}_{11} & -T_1\tilde{V}_{12} \end{bmatrix}$$

where $\tilde{V}_{11}$ and $\tilde{V}_{12}$ have obvious definitions. Using the completeness of $([T_1' \ T_2']', Q_{11}, S_2)$ and Lemma (3.2) we can construct X_2 such that $(\Lambda, T_1(\tilde{V}_{11} + \tilde{V}_{12}X_2))$ is right coprime. In this case

$$\begin{bmatrix} I & 0 \\ Y_2 & I \end{bmatrix} \tilde{U} \begin{bmatrix} Q_{11} & S_2 \\ -T_2 & 0 \end{bmatrix} \tilde{V} \begin{bmatrix} I & 0 \\ X_2 & I \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ 0 & \Psi \end{bmatrix}$$

where Y_2 satisfies $Y_2\Lambda = -\Psi X_2$. Define

$$U = \begin{bmatrix} I & 0 \\ Y_2 & I \end{bmatrix} \tilde{U} \ and \ V = \tilde{V} \begin{bmatrix} I & 0 \\ X_2 & I \end{bmatrix}$$

Observe that $U_{11} = \tilde{U}_{11} = \tilde{U}_{11} + X_1\tilde{U}_{21}$ and $V_{11} = \tilde{V}_{11} + \tilde{V}_{12}X_2$. Hence, $(\Lambda, U_{11}S_1)$ is left coprime and $(\Lambda, T_{11}V_{11})$ is right coprime.

It now follows that (3.8) is complete if and only if

$$([0 \ \ -U_{21}S_1P_c], \begin{bmatrix} \Lambda & U_{11}S_1P_c \\ -T_1V_{11} & Q_c \end{bmatrix}, \begin{bmatrix} 0 \\ -T_1V_{12} \end{bmatrix}, \Psi) \tag{3.11}$$

is complete. Similarly (3.9) is complete if and only if

$$([0 \ \ -U_{21}S_1], \begin{bmatrix} \Lambda & U_{11}S_1 \\ -\bar{R}_cT_1V_{11} & \bar{Q}_c \end{bmatrix}, \begin{bmatrix} 0 \\ -\bar{R}_cT_1V_{12} \end{bmatrix}, \Psi) \tag{3.12}$$

is complete. There exist matrices $\Phi_1, \Phi_2, \Phi_3, \Phi, \bar{\Phi}_4, \bar{\Phi}_3$ and $\Theta_1, \Theta_2, \Theta_3, \Theta, \bar{\Theta}_4, \bar{\Theta}_3$, with Θ and Φ are nonsingular, such that

$$\begin{bmatrix} \Theta_1 & -\Theta_2 \\ \Theta_3 & \Theta \end{bmatrix} \begin{bmatrix} \Lambda & \bar{\Theta}_4 \\ -T_1V_{11} & \bar{\Theta}_3 \end{bmatrix} = I \tag{3.13}$$

and

$$\begin{bmatrix} \Lambda & U_{11}S_1 \\ -\bar{\Phi}_4 & \bar{\Phi}_3 \end{bmatrix} \begin{bmatrix} \Phi_1 & -\Phi_3 \\ \Phi_2 & \Phi \end{bmatrix} = I. \tag{3.14}$$

Unimodular operations yield that (3.11) is complete if and only if

$$([0 \quad -U_{21}S_1P_c], \begin{bmatrix} I & 0 \\ 0 & \Theta_3U_{11}S_1P_c + \Theta Q_c \end{bmatrix}, \begin{bmatrix} 0 \\ -\Theta T_1V_{12} \end{bmatrix}, \Psi) \tag{3.15}$$

is complete, and (3.12) is complete if and only if

$$([0 \quad -U_{21}S_1\Phi], \begin{bmatrix} I & 0 \\ 0 & \tilde{R}_cT_1V_{11}\Phi_3 + \tilde{Q}_c\Phi \end{bmatrix}, \begin{bmatrix} 0 \\ -\tilde{R}_cT_1V_{12} \end{bmatrix}, \Psi) \tag{3.16}$$

is complete.

Assume that (2.5) and (2.6) hold for $\tilde{Z}_{11}$. Let

$$[A \quad B] := [\Theta_3U_{11}S_1 \qquad \Theta] \begin{bmatrix} S_r & D_r \\ T_r & -N_r \end{bmatrix} \tag{3.17}$$

and

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} := \begin{bmatrix} S_l & T_l \\ D_l & -N_l \end{bmatrix} \begin{bmatrix} T_1V_{11}\Phi_3 \\ \Phi \end{bmatrix}. \tag{3.18}$$

From (3.13) and (3.14), it follows that (A, B) is left coprime and $(\tilde{A}, \tilde{B})$ is right coprime. Consider the alternative descriptions of P_c, Q_c, $\tilde{R}_c$, $\tilde{Q}_c$ below

$$\begin{bmatrix} Q_c \\ P_c \end{bmatrix} = \begin{bmatrix} T_r & -N_r \\ S_r & D_r \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \tag{3.19}$$

$$\begin{bmatrix} \tilde{Q}_c & \tilde{R}_c \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \begin{bmatrix} T_l & S_l \\ -N_l & D_l \end{bmatrix} \tag{3.20}$$

where X_1, X_2, Y_1, Y_2 are matrices over S of suitable dimensions. Then,

$$\Theta_3U_{11}S_1P_c + \Theta Q_c = AX_1 + BX_2$$

$$\tilde{R}_cT_1V_{11}\Phi_3 + \tilde{Q}_c\Phi = Y_1\tilde{A} + Y_2\tilde{B}.$$

Let us define

$$\Gamma := -T_1V_{12}, \quad \Omega := U_{21}S_1. \tag{3.21}$$

With this new notation, we remind that (3.8) is complete if and only if

$$(-\Omega(S_rX_1 + D_rX_2), AX_1 + BX_2, \Theta\Gamma, \Psi) \tag{3.22}$$

is complete, and (3.9) is complete if and only if

$$(-\Omega\Phi, Y_1\dot{A} + Y_2\dot{B}, (Y_1S_l + Y_2D_l)\Gamma, \Psi) \tag{3.23}$$

is complete. Also notice that (3.8) is complete for almost all $Z_c \in \mathcal{Z}_c(\bar{Z}_{11})$, if and only if for almost all X_2 (3.22) is complete, with $X_1 = I$. This can be verified by using the definition of the topology over $\mathcal{Z}_c(\bar{Z}_{11})$ and equation (2.7). As a dual result, (3.9) is complete for almost all $Z_c \in \mathcal{Z}(\bar{Z}_{11})$, if and only if for almost all Y_2 (3.23) is complete, with $Y_1 = I$. On the other hand, (3.8) is complete for almost all $Z_c \in \mathbf{P}^{r\times p}$, if and only if for almost all $Z \in \mathbf{P}^{r\times p}$, with $Z = X_2X_1^{-1}$ for some right coprime pair of matrices (X_1, X_2), (3.22) is complete. Dually, (3.9) is complete for almost all $Z_c \in \mathbf{P}^{r\times p}$, if and only if for almost all $Z \in \mathbf{P}^{r\times p}$, with $Z = Y_1^{-1}Y_2$ for some left coprime pair of matrices (Y_1, Y_2), (3.23) is complete. These results can be verified by using the topology on $\mathbf{P}^{r\times p}$ and equations (3.19) and (3.20).

We now proceed by investigating three cases.

Case 1. At least one of Γ and Ω is nonzero. If Γ is nonzero, since Θ is nonsingular, $\Theta\Gamma$ is nonzero. Then, applying Lemma (3.5) gives us that for almost all X_2, $(A+BX_2, \Theta\Gamma)$ is left coprime. This implies that for almost all $Z_c \in \mathcal{Z}_c(\bar{Z}_{11})$ (3.22) is complete. Also applying Lemma (3.9) yields that for almost all $Z_c \in \mathbf{P}^{r\times p}$ (3.22) is complete. If Ω is nonzero, on the other hand, then $\Omega\Phi$ is nonzero, because of the nonsingularity of Φ. So, applying the dual of Lemma (3.5) we observe that for almost all Y_2, $(\Omega\Phi, \dot{A} + Y_2\dot{B})$ is right coprime. This implies that for almost all $Z_c \in \mathcal{Z}_c(\bar{Z}_{11})$ (3.23) is complete.

Case 2. $\Gamma = 0$, $\Omega = 0$, $\Psi \neq 0$. In this case (3.22) is complete if and only if $(0, AX_1 + BX_2, 0, \Psi)$ is complete. Clearly, there exists a matrix K over $\mathbf{S}$ of appropriate size such that $K\Psi$ is nonzero and $(0, AX_1 + BX_2, 0, \Psi)$ is equivalent to $(0, AX_1 + BX_2, K\Psi, \Psi)$ over $\mathbf{S}$. Repeating Case 1 yields that for almost all $Z_c \in \mathcal{Z}_c(\bar{Z}_{11})$ and for almost all $Z_c \in \mathbf{P}^{r\times p}$ (3.22) is complete.

Case 3. $\Gamma = 0$, $\Omega = 0$, $\Psi = 0$. In this case (3.11) (and, therefore (3.8)) is complete if and only if

$$\begin{bmatrix} \Lambda & U_{11}S_1P_c \\ -T_1V_{11} & Q_c \end{bmatrix} \tag{3.24}$$

is unimodular. Consider

$$U\begin{bmatrix} Q_{11} & S_2 & S_1 \\ -T_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & U_{11}S_1 \\ 0 & \Psi & U_{21}S_1 \end{bmatrix} \tag{3.25}$$

$$\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Q_{11} & S_2 \\ -T_2 & 0 \\ -T_1 & 0 \end{bmatrix} V = \begin{bmatrix} \Lambda & 0 \\ 0 & \Psi \\ -T_1V_{11} & -T_1V_{12} \end{bmatrix} \tag{3.26}$$

From (3.25) we have

$$T_2 = 0 \Rightarrow \Psi = 0,\ \Omega = U_{21}S_1 = 0$$
$$T_2Q_{11}^{-1}[S_1\ S_2] = 0 \Rightarrow \Psi = 0,\ \Omega = 0$$

From (3.26) we have

$$S_2 = 0 \Rightarrow \Psi = 0,\ \Gamma = T_1V_{12} = 0$$
$$\begin{bmatrix} T_1' & T_2' \end{bmatrix}' Q_{11}^{-1}S_2 = 0 \Rightarrow \Psi = 0,\ \Gamma = 0$$

Observe that Ψ, Ω and Γ are all zero if and only if $T_2Q_{11}^{-1}[S_1\ S_2]$ and $[T_1'\ T_2']'Q_{11}^{-1}S_2$ are both zero. Let $\hat{U}U = I$ and $V\hat{V} = I$. Partition $\hat{U}$ and $\hat{V}$ as $\hat{U} = [\hat{U}_{ij}]$, $\hat{V} = [\hat{V}]_{ij}$, $i,j = 1,2$. In this case $Q_{11} = \hat{U}_{11}\Lambda\hat{V}_{11}$, $S_1 = \hat{U}_{11}U_{11}S_1$ (from (3.25)) and $T_1 = T_1V_{11}\hat{V}_{11}$ (from (3.26)). This shows that $\bar{Z}_{11} = T_1Q_{11}^{-1}S_1 = T_1V_{11}\Lambda^{-1}U_{11}S_1$. Since the right hand side of the equation is bicoprime, this implies that (3.24) is unimodular if and only if $Z_c \in \mathcal{Z}_c(\bar{Z}_{11})$. The proof of (1) of Lemma (3.6) is thus completed. To complete the proof of (2) just observe that $\mathcal{Z}_c(\bar{Z}_{11})$ is not dense in $\mathbf{P}^{r\times p}$ (see the proof of Theorem (3.3)).□

The constructive proof of the following theorem is one of the main contributions of this chapter.

Theorem (3.2). *DSP (and equivalently SCCP) is solvable if and only if* $(P_{\mathrm{N-r}}, Q, R_{\mathrm{r}})$ *is complete for all* $\mathrm{r} \in C_N$.

Proof.

[If] The proof of the "If" part is established by induction. Let $N = 2$. The statement reduces to 2-channel DSP in which case $C_N = \{\{1\},\{2\}\}$ and the hypothesis implies (P_2, Q, R_1) and (P_1, Q, R_2) are complete. So, using Theorem (3.1) the solution is obtained.

Assume that the theorem is true for $N = H \geq 2$. Define $L := H + 1$.

It will be shown that by a suitable choice of $Z_c = P_c Q_c^{-1}$, for a right coprime pair of matrices (Q_c, P_c), the following holds.

i. $\left(\begin{bmatrix} P_{\mathrm{H-r}} & 0 \end{bmatrix}, \begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathrm{r}} \\ 0 \end{bmatrix} \right)$ is complete for all $\mathrm{r} \in C_H$.

ii. $\left(\begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathrm{H}} \\ 0 \end{bmatrix} \right)$ is left coprime.

iii. $\left(\begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} P_{\mathrm{H}} & 0 \end{bmatrix} \right)$ is right coprime.

Then, from **ii** and **iii**

$$\hat{Z} := \begin{bmatrix} P_{\mathrm{H}} & 0 \end{bmatrix} \begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}^{-1} \begin{bmatrix} R_{\mathrm{H}} \\ 0 \end{bmatrix}$$

is a bicoprime fraction which, via **i** and the inductive hypothesis implies that DSP for the plant $\hat{Z}$ is solvable for some compensator $diag\{Z_{c1}, ..., Z_{cH}\}$. This clearly implies that DSP for Z is solvable by the compensator $diag\{Z_{c1}, ..., Z_{cH}, Z_c\}$, completing the proof of "If" part.

To show that **i**, **ii** and **iii** hold for some compensator Z_c, observe that the hypothesis of Theorem implies $(P_{\mathrm{H-r}}, Q, \begin{bmatrix} S_{\mathrm{r}} & S_L \end{bmatrix})$ and $(\begin{bmatrix} P_{\mathrm{L}}' & P_{\mathrm{H-r}}' \end{bmatrix}', Q, S_{\mathrm{r}})$ are complete for all $\mathrm{r} \in C_H$.

Fix any $\mathrm{r} \in C_H$ and let $Q_{11} := Q$, $T_1 := P_L$, $T_2 := P_{\mathrm{H-r}}$, $S_1 := R_L$, and $S_2 := R_{\mathrm{r}}$. Applying Lemma (3.6) we have that

$$(\begin{bmatrix} P_{\mathrm{H-r}} & 0 \end{bmatrix}, \begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathrm{r}} \\ 0 \end{bmatrix})$$

is complete for almost all $Z_c \in \mathcal{Z}_c(Z_{LL})$. Let $\mathcal{Z}_c^{\mathrm{r}}$ denote the set of these compensators, which is open and dense in $\mathcal{Z}_c(Z_{LL})$. Since r is fixed but otherwise arbitrary, it holds that $\cup_{\mathrm{r}\in\mathcal{C}_H}\mathcal{Z}_c^{\mathrm{r}}$ is open and dense in $\mathcal{Z}_c(Z_{LL})$. In other words, i holds for almost all $Z_c \in \mathcal{Z}_c(Z_{LL})$.

Now let $Q_{11} := Q$, $T_1 := P_L$, $T_2 := 0$, $S_1 := R_L$, and $S_2 := R_{\mathrm{H}}$ and apply Lemma (3.6). The facts that (Q, R_{L}) is coprime and (P_L, Q, R_{H}) is complete give us that

$$([0\ \ 0], \begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathrm{H}} \\ 0 \end{bmatrix})$$

is complete for almost all Z_c included in $\mathcal{Z}_c(Z_{LL})$. In other words ii holds for almost all $Z_c \in \mathcal{Z}_c(Z_{LL})$. Dual arguments yield that iii holds for almost all $Z_c \in \mathcal{Z}_c(Z_{LL})$. Since the intersection of open and dense subsets is open and dense, we conclude that for almost all $Z_c \in \mathcal{Z}_c(Z_{LL})$ properties i, ii and iii hold. Hence, we can find at least one Z_c for which i, ii and iii hold. This completes the proof of the "If" part.

[Only If] Let DSP for Z be solvable. Fix $\mathrm{r} \in \mathcal{C}_N$. Observe that DSP for the 2-channel plant

$$\begin{bmatrix} Z_{\mathrm{r}\,\mathrm{r}} & Z_{\mathrm{r}\,\mathrm{N-r}} \\ Z_{\mathrm{N-r}\,\mathrm{r}} & Z_{\mathrm{N-r}\,\mathrm{N-r}} \end{bmatrix}$$

is solvable. This implies from Theorem (3.1) that $(P_{\mathrm{N-r}}, Q, R_{\mathrm{r}})$ is complete. Since, r is fixed but otherwise arbitrary we obtain the fact that $(P_{\mathrm{N-r}}, Q, R_{\mathrm{r}})$ is complete for all $\mathrm{r} \in \mathcal{C}_N$. This completes the proof.□

Using [2] and Lemma (2.2), it is not difficult to show that $z \in \mathcal{C}_+$ is a decentralized fixed mode of Z if and only if

$$rank \begin{bmatrix} Q & R_{\mathrm{r}} \\ -P_{\mathrm{N-r}} & 0 \end{bmatrix}(z) < q$$

for some $\mathrm{r} \in \mathcal{C}_N$, in which case the completeness of $(P_{\mathrm{N-r}}, Q, R_{\mathrm{r}})$ is violated.

Assume that the completeness conditions of Theorem (3.2) hold. The design methodology in the theorem is to apply a compensator to Channel N such that the closed loop system (with the remaining $N-1$ channels) satisfies the following two conditions:

A. The $N-1$-channel system is jointly stabilizable and detectable.

B. All complementary subsystems including Channel 1 of the $N-1$-channel system are complete.

The synthesis procedure continues inductively, and ends up with the first channel, from which the closed loop system is now stabilizable and detectable. By applying to the first channel a stabilizing compensator for the closed loop system, the synthesis procedure is terminated. This is a *hierarchically stable synthesis procedure*, since at each step the local compensator is chosen as an stabilizing compensator of the respective channel in the closed-loop.

3.3 Characterization Results

We start with a definition. Consider the plant transfer matrix Z of the previous section with a bicoprime fraction as in (3.1). Let DSP for Z be solvable and define $L = N - 1$.

It is said that Z_c *is an admissible local compensator for Channel* N, if there exist compensators $Z_{c1}, \ldots, Z_{cL}$, such that the decentralized compensator $diag\{Z_{c1}, \ldots, Z_{cL}, Z_c\}$ stabilizes Z.

In this section the synthesis procedure of Theorem (3.2) will be utilized to characterize the class of all admissable compensators of a specified channel. This also yields a characterization of all decentralized stabilizing compensators of the plant in the following way. For simplicity let $N = 2$. One can obtain the characterization of admissible local compensators for Channel 2. (This also yields the characterization of all compensators solving SCCP.) After a fixed compensator is applied around the 2nd channel, the class of all stabilizing compensators for the single channel system can be obtained by known methods [66]. This procedure can be repeated for all admissible compensators of the second channel, and hence all decentralized stabilizing compensators can be obtained by repeating the process. Alternative characterizations of decentralized stabilizing controllers are available in the literature (see, for example, [22]). On comparing with the one in [22] our characterization seems to be more convenient for obtaining the set of all admissible controllers associated with a fixed channel, because, as can be seen from I of Theorem (3.3), the characterization of admissible local compensators proposed here is given in terms of only two parameters (independent of N) which satisfy certain coprimeness and completeness relations. A characterization of all admissible controllers using the parametrization in [22], however, would require the solution of a multiparameter (depending on N) unimodularity equation.

In II of Theorem (3.3) we give certain connectivity conditions under which the class of admissible local compensators is generic among all compensators. By the statement III of Theorem (3.3) if these conditions fail to hold then the class of admissible local compensators is precisely the set of stabilizing compensators of the corresponding channel. We remind that from the proof of Theorem (3.2) any stabilizing compensator of a channel *independent of connectivity conditions* is generically an admissible compensator.

A rigorous definition of the set of admissable controllers for channel N is given by

$$\mathcal{Z}_{cN} := \{Z_c \in \mathbf{P}^{r_N \times p_N} | \text{ There exists } \{Z_{c1}, ..., Z_{cL}\} \in \mathbf{P}^{r_1 \times p_1} \times \\ ... \times \mathbf{P}^{r_{N-1} \times p_{N-1}}, \text{ such that } \{Z_{c1}, ..., Z_{cN-1}, Z_c\} \text{ solves DSP}\},$$

Thus, $\mathcal{Z}_{cN}$ is the set of compensators $Z_c = P_c Q_c^{-1}$ such that i, ii and iii in the proof of Theorem (3.2) are satisfied with $H = N - 1$. The characterization of $\mathcal{Z}_{cN}$ depends heavily on various quantities defined in the proof of Lemma (3.6). Let $H := N - 1$ and consider the conditions i, ii and iii in the proof of Theorem (3.1).

Let $Z_c = P_c Q_c^{-1} \in \mathcal{Z}_{cN}$ where P_c, Q_c are parametrized as in (3.19) in terms of X_1 X_2, such that $X_2 X_1^{-1}$ is proper.

Now fix any $\mathbf{r} \in \mathcal{C}_H$. Letting $Q_{11} := Q$, $T_1 := P_N$, $T_2 := P_{\mathbf{H}-\mathbf{r}}$, $S_1 := R_N$, $S_2 := R_{\mathbf{r}}$, and following the arguments in the proofs of Theorem (3.2) and Lemma (3.6) it is seen that there exist $A_{\mathbf{r}}$, $B_{\mathbf{r}}$, given by (3.17), $\Psi_{\mathbf{r}}$, given by (3.10), $\Theta_{\mathbf{r}}$, given by (3.13), and $\Omega_{\mathbf{r}}$, $\Gamma_{\mathbf{r}}$, given by (3.21) such that i holds for $\mathbf{r}$ if and only if

$$(-\Omega_{\mathbf{r}}(S_r X_1 + D_r X_2), A_{\mathbf{r}} X_1 + B_{\mathbf{r}} X_2, \Theta_{\mathbf{r}} \Gamma_{\mathbf{r}}, \Psi_{\mathbf{r}})$$

is complete.

In the special case $\mathbf{r} = \mathbf{H}$ letting $Q_{11} := Q$, $T_1 := P_N$, $T_2 := 0$, $S_1 := R_N$, $S_2 := R_{\mathbf{H}}$ and following Theorem (3.2) and Lemma (3.6) there exist $A_{\mathbf{H}}$, $B_{\mathbf{H}}$, $\Theta_{\mathbf{H}}$, and $\Gamma_{\mathbf{H}}$ such that ii holds if and only if

$$(A_{\mathbf{H}}X_1 + B_{\mathbf{H}}X_2, \Theta_{\mathbf{H}}\Gamma_{\mathbf{H}})$$

is left coprime. Similarly, in the special case $\mathbf{r} = \emptyset$ letting $Q_{11} := Q$, $T_1 := P_N$, $T_2 := P_{\mathbf{H}}$, $S_1 := R_N$, $S_2 := 0$ and following Theorem (3.2) and Lemma (3.6) there exist $A_\emptyset$, $B_\emptyset$, $\Phi_\emptyset$, and $\Omega_\emptyset$ such that iii holds if and only if

$$(-\Omega_\emptyset\Phi_\emptyset, A_\emptyset X_1 + B_\emptyset X_2)$$

is right coprime.

We summarize these results in Theorem (3.3) below where $H := N - 1$.

Theorem (3.3). *Let DSP for Z be solvable.*

I. *$\mathcal{Z}_{cN}$ consists of $Z_c = P_cQ_c^{-1}$ where P_c, Q_c are parametrized as in (3.19) in terms of X_1, X_2 such that $P_cQ_C^{-1}$ is proper and (a), (b) and (c) below simultaneously hold:*

(a)

$$(-\Omega_{\mathbf{r}}(S_{\mathbf{r}}X_1 + D_{\mathbf{r}}X_2), A_{\mathbf{r}}X_1 + B_{\mathbf{r}}X_2, \Theta_{\mathbf{r}}\Gamma_{\mathbf{r}}, \Psi_{\mathbf{r}})$$

is complete for all $\mathbf{r} \in \mathcal{C}_H$,

(b)

$$(A_{\mathbf{H}}X_1 + B_{\mathbf{H}}X_2, \Theta_{\mathbf{H}}\Gamma_{\mathbf{H}})$$

is left coprime,

(c)

$$(-\Omega_\emptyset\Phi_\emptyset, A_\emptyset X_1 + B_\emptyset X_2)$$

is right coprime.

II. *$\mathcal{Z}_{cN}$ is an open and dense subset of $\mathbf{P}^{r_N \times p_N}$ if and only if (a) and (b) below simultaneously hold*

(a) $Z_{N,\mathbf{H}} = P_NQ^{-1}R_{\mathbf{H}} \neq 0$ and $Z_{\mathbf{H},N} = P_{\mathbf{H}}Q^{-1}R_N \neq 0$

(b) For each $\mathrm{r} \in C_H$,

$$\dot{Z}_{(N\cup \mathrm{H})-\mathrm{r},\mathrm{r}} \neq 0 \quad \text{or} \quad Z_{\mathrm{H}-\mathrm{r},N\cup\mathrm{r}} \neq 0.$$

III. *If one of (a) or (b) of* **II** *is violated, then* $\mathcal{Z}_{cN} = \mathcal{Z}_c(Z_{NN})$.

For the proofs of statements II and III in Theorem (3.3) we need the technical lemma below.

Lemma (3.10) *Consider the triple*

$$\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, Q_{11}, [S_1 \quad S_2]\right)$$

where $(Q_{11}, [S_1 \quad S_2])$ *is left and* $(Q_{11}, [T_1' \quad T_2']')$ *is right coprime pairs. Also let* (T_1, Q_{11}, S_2) *and* (T_2, Q_{11}, S_1) *be complete. Consider*

$$\begin{bmatrix} Q_{11} & S_1 P_{c1} & S_2 P_{c2} \\ -T_1 & Q_{c1} & 0 \\ -T_2 & 0 & Q_{c2} \end{bmatrix} \tag{3.27}$$

where (P_{c1}, Q_{c1}) *and* (P_{c2}, Q_{c2}) *are coprime.*

In case one of $\bar{Z}_{12} := T_1 Q_{11}^{-1} R_2$ *or* $\bar{Z}_{21} := T_2 Q_{11}^{-1} R_1$ *is zero, the matrix in (3.27) is unimodular if and only if* $(\bar{Z}_{11}, P_{c1} Q_{c1}^{-1})$ *and* $(\bar{Z}_{22}, P_{c2} Q_{c2}^{-1})$ *are stable, where* $\bar{Z}_{11} := T_1 Q_{11}^{-1} R_1$ *and* $\bar{Z}_{22} := T_2 Q_{11}^{-1} R_2$.

Lemma (3.10) states that the decentralized compensator $diag\{Z_{c1}, Z_{c2}\}$ solves the decentralized stabilization problem for a 2-channel not-strongly-connected plant with no unstable decentralized fixed modes if and only if Z_{c1} and Z_{c2} stabilize Channels 1 and 2, respectively.

Proof. We assume without loss of generality that $\bar{Z}_{12} = 0$. Let a left coprime fraction of $[T_1' \; T_2']' Q_{11}^{-1}$ be given by $\tilde{Q}^{-1} [\tilde{T}_1' \; \tilde{T}_2']'$ where

$$\tilde{Q} = \begin{bmatrix} \tilde{Q}_{11} & 0 \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix}$$

It holds that the matrix (3.27) is unimodular if and only if so is

$$\begin{bmatrix} \bar{Q}_{11}Q_{c1} & 0 \\ \bar{Q}_{21}Q_{c1} & \bar{Q}_{22}Q_{c2} \end{bmatrix} + \begin{bmatrix} \bar{T}_1S_1P_{c1} & \bar{T}_1S_2P_{c2} \\ \bar{T}_2S_1P_{c1} & \bar{T}_2S_2P_{c2} \end{bmatrix}$$

where $\bar{T}_1S_2P_{c2} = 0$, since $\bar{Z}_{12} = 0$. Note that $\bar{Z}_{11} = \bar{Q}_{11}^{-1}\bar{T}_1S_1$ and $\bar{Z}_{22} = \bar{Q}_{22}^{-1}\bar{T}_2S_2$, where both fractions are coprime. Then, the matrix (3.27) is unimodular if and only if $\bar{Q}_{11}Q_{c1} + \bar{T}_1S_1P_{c1}$ and $\bar{Q}_{22}Q_{c2} + \bar{T}_2S_2P_{c2}$ are unimodular, i.e., if and only if $(\bar{Z}_{11}, P_{c1}Q_{c1}^{-1})$ and $(\bar{Z}_{22}, P_{c2}Q_{c2}^{-1})$ are stable.□

Proof of Theorem (3.3). Proof of **I** follows from the discussion preceeding the theorem. We will now prove the "If" part of **II**. Assume that for all $\mathrm{r} \in \mathcal{C}_H$, at least one of Γ_{r}, Ω_{r} and Ψ_{r} is nonzero. Then, (2) of Lemma (3.6) and the fact that the union of open and dense sets is open and dense, reveal that for almost all $Z_c \in \mathbf{P}^{r_N \times p_N}$, i in the proof of Theorem (3.2) holds. Similarly, if $\Gamma_{\mathbf{H}}$ is nonzero, for almost all $Z_c \in \mathbf{P}^{r_N \times p_N}$ ii holds, and if $\Omega_\emptyset$ is nonzero, for almost all $Z_c \in \mathbf{P}^{r_N \times p_N}$ iii holds. On the other hand, a closer inspection at the proof of Lemma (3.6) reveals that for some $\mathrm{r} \in \mathcal{C}_H$, Γ_{r}, Ω_{r} and Ψ_{r} are all zero if and only if

$$Z_{\mathbf{H}-\mathrm{r},\mathrm{r}} = 0, \quad Z_{\mathcal{N},\mathrm{r}} = 0, \quad Z_{\mathbf{H}-\mathrm{r},\mathcal{N}} = 0$$

or, equivalently

$$Z_{(\mathcal{N}\cup\mathbf{H})-\mathrm{r},\mathrm{r}} = 0, \quad Z_{\mathbf{H}-\mathrm{r},\mathcal{N}\cup\mathrm{r}} = 0.$$

Also $\Gamma_{\mathbf{H}} = 0$ if and only if $Z_{\mathcal{N},\mathbf{H}} = 0$ and $\Omega_\emptyset = 0$ if and only if $Z_{\mathbf{H},\mathcal{N}} = 0$. This completes the "If" part of the proof.

Now, we will prove **III** and the "Only If" part of **II**. Assume, $Z_{(\mathcal{N}\cup\mathbf{H})-\mathrm{r},\mathrm{r}} = 0$ and $Z_{\mathbf{H}-\mathrm{r},\mathcal{N}\cup\mathrm{r}} = 0$ for some $\mathrm{r} \in \mathcal{C}_H$. Then, by a suitable permutation at the inputs and outputs, the transfer matrix structure of Z takes the following form.

$$\begin{array}{cccc} & \mathbf{H}-\mathrm{r} & \mathcal{N} & \mathrm{r} \\ \mathbf{H}-\mathrm{r} & \times & 0 & 0 \\ \mathcal{N} & \times & \times & 0 \\ \mathrm{r} & \times & \times & \times \end{array}$$

where the $\times$ subblocks are not important for our discussion. In this case applying Lemma (3.10) repeatedly, first by letting

$$\begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} \end{bmatrix} := \begin{bmatrix} Z_{(N\cup\mathbf{H})-\mathbf{r},(N\cup\mathbf{H})-\mathbf{r}} & Z_{(N\cup\mathbf{H})-\mathbf{r},\mathbf{r}} \\ Z_{\mathbf{r},(N\cup\mathbf{H})-\mathbf{r}} & Z_{\mathbf{r},\mathbf{r}} \end{bmatrix},$$

and then letting

$$\begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} \end{bmatrix} := \begin{bmatrix} Z_{\mathbf{H}-\mathbf{r},\mathbf{H}-\mathbf{r}} & Z_{\mathbf{H}-\mathbf{r},N} \\ Z_{N,\mathbf{H}-\mathbf{r}} & Z_{N,N} \end{bmatrix}.$$

we conclude that $\mathcal{Z}_{cN} = \mathcal{Z}_c(Z_{NN})$. In case $Z_{N,\mathbf{H}} = 0$ applying Lemma (3.10) by letting

$$\begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} \end{bmatrix} := \begin{bmatrix} Z_{\mathbf{H},\mathbf{H}} & Z_{\mathbf{H},N} \\ Z_{N,\mathbf{H}} & Z_{N,N} \end{bmatrix},$$

we conclude that $\mathcal{Z}_{cN} = \mathcal{Z}_c(Z_{NN})$. Dual arguments follow for the case when $Z_{\mathbf{H},N}$ is zero. This completes the proof of **III**. Now note that $\mathcal{Z}_c(Z_{NN})$ is *not* dense in $\mathbf{P}^{r_N \times p_N}$. To see this let $Z_{c_0} \in \mathbf{P}^{r_N \times p_N}$ be such that the closed loop characteristic polynomial of (Z_{NN}, Z_{c_0}) has unstable zeros other than zero. Then, for all Z_c belonging to a sufficiently small open ball around Z_{c_0}, the closed loop characteristic polynomial of (Z_{NN}, Z_c) still contains unstable zeros, which implies that $\mathcal{Z}_c(Z_{NN})$ is not dense in $\mathbf{P}^{r_N \times p_N}$ [66, Proposition 7.2.41]. This completes the proof the "Only If" part of **II**.$\Box$

Example (3.1).

Consider the 3-channel system below:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{(z+1)^2} & \frac{(z-1)}{(z+1)^3} & \frac{1}{(z+1)^2} \\ \frac{(2z-5)}{(z+1)(z-2)(z-3)} & \frac{1}{(z-2)(z+1)} & \frac{1}{(z-2)(z+1)} \\ \frac{(2z-3)}{(z-1)(z+1)(z-2)} & \frac{(2z-1)}{(z+1)^2(z-2)} & \frac{(2z-3)}{(z+1)(z-1)(z-2)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = Zu.$$

Obtaining a bicoprime representation of Z over $\mathbf{S}$ we have $y = [P_1' \; P_2' \; P_3']' Q^{-1} [R_1 \; R_2 \; R_3] u$, where $P_1 = [\frac{(z-1)}{(z+1)^2} \;\; 0 \;\; 0]$, $P_2 = [0 \;\; \frac{1}{(z+1)} \;\; \frac{1}{(z+1)}]$, $P_3 = [\frac{1}{(z+1)} \;\; \frac{1}{(z+1)} \;\; 0]$,

$$R_1' = \begin{bmatrix} \frac{1}{(z+1)} & \frac{1}{(z+1)} & \frac{1}{(z+1)} \end{bmatrix}', \quad R_2' = \begin{bmatrix} \frac{(z-1)}{(z+1)^2} & \frac{1}{(z+1)} & 0 \end{bmatrix}',$$

$$R_3' = \begin{bmatrix} \frac{1}{(z+1)} & \frac{1}{(z+1)} & 0 \end{bmatrix}',$$

and $Q = diag\{\frac{(z-1)}{(z+1)}, \frac{(z-2)}{(z+1)}, \frac{(z-3)}{(z+1)}\}$.

Let $H = 2$, $C_H = \{1\}$, and $r = \{1\}$. We now determine $Z_{c3} = P_{c3}Q_{c3}^{-1} \in \mathbf{P}$, for coprime (P_{c3}, Q_{c3}) such that the closed loop system under feedback law $u_3 = -Z_{c3}y_3$ satisfies

$$i.([P_2\ 0], \begin{bmatrix} Q & R_3P_{c3} \\ -P_3 & Q_{c3} \end{bmatrix}, \begin{bmatrix} R_1 \\ 0 \end{bmatrix}) \text{ is complete}$$

$$ii.(\begin{bmatrix} Q & R_3P_{c3} \\ -P_3 & Q_{c3} \end{bmatrix}, \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}) \text{ is left coprime}$$

$$i'.([P_1\ 0], \begin{bmatrix} Q & R_3P_{c3} \\ -P_3 & Q_{c3} \end{bmatrix}, \begin{bmatrix} R_2 \\ 0 \end{bmatrix}) \text{ is complete}$$

$$ii'.(\begin{bmatrix} P_1 & 0 \\ P_2 & 0 \end{bmatrix}, \begin{bmatrix} Q & R_3P_{c3} \\ -P_3 & Q_{c3} \end{bmatrix}) \text{ is right coprime.}$$

Following Theorem (3.3) and the preceeding statements one can verify that i and ii hold for all $Z_{c3} \in \mathbf{P}$, whereas i' holds if and only if $Z_{c3}(1) \neq 0$ and $[Q_{c3}\ \ P_{c3}]_{z=3}[1\ \ -\frac{1}{4}]'_{z=3} \neq 0$. and ii' holds if and only if $Z_{c3}(1) = 0$. So, by combining these results we conclude the following: $Z_{c3} = P_{c3}Q_{c3}^{-1} \in \mathbf{P}$, *for coprime* (P_{c3}, Q_{c3}) *such that i, ii, i' and ii' hold, if and only if* $P_{c3}(1) \neq 0$ *and* $[Q_{c3}\ \ P_{c3}]_{z=3}[1\ \ -\frac{1}{4}]'_{z=3} \neq 0$.

In order to achieve a hierarchically stable design we choose $P_{c3} = \frac{(97z-113)}{(z+1)}$ and $Q_{c3} = \frac{(z^2+7z-169)}{(z+1)^2}$. In this case $Z_{c3} = P_{c3}Q_{c3}^{-1}$ is a minimal order stabilizing compensator for Z_{33}. With this choice of Z_{c3} it can also be verified that i, ii, i'and ii' hold.

Repeating similar arguments for the resulting 2-channel system $\hat{Z}$ we obtain $Z_{c2} = 65$, which stabilizes the second channel of $\check{Z}$. We finally get $Z_{c1} = P_{c1}Q_{c1}^{-1}$ where

$$P_{c1} = \frac{65536(65z^6 + 390z^5 + 976z^4 + 1307z^3 + 805z^2 + 577z + 8)}{317(z+1)^6},$$

and

$$Q_{c1} = \frac{(317z^8 + 3804z^7 - 4237016z^6 - 25463940z^5 + 7629021 38z^4 - 633438348z^3 - 2207193504z^2 + 692117428z + 1415227969)}{317(z+1)^8}.$$

The resulting decentralized compensator has total order 10. It can be shown following the approach in [10] that by using constant feedback compensators around the third and second channels and a 7'th order compensator around the third channel a decentralized compensator of total order 7 could also be utilized to solve DSP. This, however, would not lead to a hierarchically stable design. Hence, the hierarchically stable design is achieved at the expense of increased compensator order.Δ

We now consider the class of compensators solving SCCP. Theorem (3.4) below states that once the solvability conditions are satisfied then the class of compensators solving SCCP is open and dense if and only if the plant is strongly connected.

Theorem (3.4). *Let SCCP be solvable. The set of compensators $\{Z_{c2}, ..., Z_{cN}\}$, where $Z_{ci} = P_{ci}Q_{ci}^{-1}$, (P_{ci}, Q_{ci}) is right coprime $i = 2, ..., N$, such that*

$$[P_1\ 0 \ldots 0]\ \bar{\Sigma}^{-1}\ [R_1'\ 0\ \ldots\ 0]' \tag{3.28}$$

is bicoprime, where $\bar{\Sigma}$ is given by (3.3), is open and dense in $\mathbf{P}^{r_2 \times p_2} \times \ldots \times \mathbf{P}^{r_N \times p_N}$ (with respect to the product topology induced by $\mathbf{P}^{r_i \times p_i}$, $i = 1, ..., N$) if and only if the plant is strongly connected.

The proof of Theorem (3.4) requires the following lemma which gives necessary and sufficient conditions for a closed loop transfer matrix to be nonzero.

Lemma (3.11). *Consider the triple $([T_1'\ T_2']', Q_{11}, [S_1\ \ S_2])$ where $T_1Q_{11}^{-1}S_1 \in \mathbf{P}^{p \times r}$. Then,*

$$[T_2\ \ 0]\begin{bmatrix} Q_{11} & S_1P_c \\ -T_1 & Q_c \end{bmatrix}^{-1}\begin{bmatrix} S_2 \\ 0 \end{bmatrix} \neq 0 \tag{3.29}$$

for some right coprime (Q_c, P_c) such that $Z_c = P_cQ_c^{-1} \in \mathbf{P}^{r \times p}$ if and only if

$$\bar{Z}_{2,\{1,2\}} \neq 0, \text{ and } \bar{Z}_{\{1,2\},2} \neq 0, \tag{3.30}$$

where $\tilde{Z}_{2,\{1,2\}} := T_2 Q_{11}^{-1}[S_1 \ \ S_2]$, *and* $\tilde{Z}_{\{1,2\},2} := [T_1' \ T_2']' Q_{11}^{-1} S_2$.

Moreover, if (3.30) holds then the set of $Z_c = P_c Q_c^{-1}$ *for which (3.29) holds is an open and dense subset of* $\mathbf{P}^{r\times p}$.

Proof.

We omit the "Only If" part of the proof as it is straightforward. For the "If" part let $S_1 \in \mathbf{S}^{q\times r}$, $T_1 \in \mathbf{S}^{p\times q}$ and observe that (3.29) holds for some P_c, Q_c described by (3.19), if

$$rank \begin{bmatrix} Q_{11} & S_1 P_c & S_2 \\ -T_1 & Q_c & 0 \\ -T_2 & 0 & 0 \end{bmatrix} \geq q + p + 1, \tag{3.31}$$

where $q :=$ size(Q). Repeating the arguments in the proof of Lemma (3.6) (3.31) holds if and only if

$$rank \begin{bmatrix} AX_1 + BX_2 & \Theta\Gamma \\ \Omega(S_r X_1 + D_r X_2) & \Psi \end{bmatrix} \geq p + 1. \tag{3.32}$$

Writing (3.32) explicitly we have that (3.32) holds if and only if

$$rank(\begin{bmatrix} \Theta_3 U_{11} S_1 & \Theta & \Theta\Gamma \\ \Omega & 0 & \Psi \end{bmatrix} \begin{bmatrix} S_r & D_r & 0 \\ T_r & -N_r & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ X_2 & 0 \\ 0 & I \end{bmatrix}) \geq p + 1. \tag{3.33}$$

The hypothesis implies that $[\Omega : \Psi]$ and $[\Gamma' : \Psi']'$ are nonzero. This fact and Θ is nonsingular imply that the first matrix in (3.33) has rank no less than $p + 1$.

Write $C := \Theta\Gamma$, $D := \Omega S_r$, $E := \Omega D_r$. The conclusion above and the fact that the middle matrix in (3.33) is unimodular, imply

$$rank \begin{bmatrix} A & B & C \\ D & E & \Psi \end{bmatrix} \geq p + 1. \tag{3.34}$$

Let $\check{U}$ be a unimodular matrix such that

$$\begin{bmatrix} \check{U}_{11} & \check{U}_{12} \\ \check{U}_{21} & \check{U}_{22} \end{bmatrix} \begin{bmatrix} C \\ \Psi \end{bmatrix} = \begin{bmatrix} \hat{C} \\ 0 \end{bmatrix} \tag{3.35}$$

where $\hat{C}$ is a full row rank matrix. Also let

$$\begin{bmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ \tilde{U}_{21} & \tilde{U}_{22} \end{bmatrix} \begin{bmatrix} A & B \\ D & E \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{D} & \hat{E} \end{bmatrix}$$

for some matrices $\hat{A}$, $\hat{B}$, $\hat{D}$, $\hat{E}$. It follows from (3.34) and (3.35) that the rank of $[\hat{D} : \hat{E}]$ is no less than $p + 1 - c$ where $c :=$ size$(\hat{C}) \geq 1$. Observe that (3.31) holds if and only if

$$rank[\hat{D} : \hat{E}] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \geq p + 1 - c. \tag{3.36}$$

Now, it is not difficult to show by straightforward manipulations that the set of X_1, X_2 for which (3.36) and thus (3.31) holds is generic in

$$\{X_1 \in \mathbf{S}^{p\times p} \textit{ and nonsingular},\ X_2 \in \mathbf{S}^{r\times p} |\ X_2 X_1^{-1} \in \mathbf{P}^{r\times p}\}.$$

This completes the proof.□

Proof of Theorem (3.4).

[Only If] Assume that for some $\mathrm{r} \in \mathcal{C}_N$, $Z_{\mathbf{N-r,r}} = 0$. If $\mathrm{r} = \mathbf{H}$ with $H := N - 1$, or $\mathrm{r} = \{N\}$ then Theorem (3.3) states that $\mathcal{Z}_{cN}$ is only an open and dense subset of $\mathcal{Z}_c(Z_{NN})$. Otherwise Lemma (3.11) reveals that

$$[P_{\mathbf{H-r'}}\ \ 0] \begin{bmatrix} Q & R_N P_c \\ -P_N & Q_c \end{bmatrix}^{-1} \begin{bmatrix} R_{\mathrm{r'}} \\ 0 \end{bmatrix} = 0.$$

for some $\mathrm{r'} \in \mathcal{C}_H$. (This can be shown as follows. If $\mathbf{r} \neq \mathbf{H}$ and $\mathbf{r} \neq \{N\}$ then two cases are possible; either $\mathrm{r} \in \mathcal{C}_H$ or $\mathrm{r} = N \cup \mathrm{r'}$, for some $\mathrm{r'} \in \mathcal{C}_H$.) Repeating this inductively untill $N = 1$, it is observed that at some step $\tilde{Z}_{N,\mathbf{H}} = 0$ or $\tilde{Z}_{\mathbf{H},N} = 0$, where $\tilde{}$ denotes the closed loop transfer matrix. In this case $\mathcal{Z}_{cN}$ is an open and dense subset of $\mathcal{Z}_c(\tilde{Z}_{NN})$, because of Theorem (3.3). On the other hand, it can be shown that $\mathcal{Z}_c(Z_{NN})$ is *not* dense in $\mathbf{P}^{r_N\times p_N}$. (See the proof of Theorem (3.3).) This completes the proof of the necessity part.

[If] If the hypothesis is true, (a) and (b) in **II** of Theorem (3.3) hold. Hence, $\mathcal{Z}_{cN}$ is open and dense in $\mathbf{P}^{r_N\times p_N}$. Also applying Lemma (3.11) it is seen that $\tilde{Z}_{\mathbf{H-r,r}} \neq 0$ for all $\mathbf{r} \in \mathcal{C}_H$, for almost all compensators applied to the N'th channel. This gives us that $\tilde{Z}_{\mathbf{H}-\mathbf{r},\mathbf{r}} \neq 0$ for all $\mathbf{r} \in \mathcal{C}_H$, for almost all $Z_c \in \mathcal{Z}_{cN}$. Repeating these arguments inductively untill $N = 1$, at each step the set $\mathcal{Z}_{cN}$ holds to be generic in $\mathbf{P}^{r_N \times p_N}$. It is easy to see using the definitions that $\{Z_{c2}, ..., Z_{cN} | \ Z_{ci}$ is open and dense $\in \mathbf{P}^{r_i \times p_i}$ $i = 2, ..., N\}$ is generic in the product topology of $\mathbf{P}^{r_2 \times p_2} \times ... \times \mathbf{P}^{r_N \times p_N}$. This completes the proof.□

Remark (3.2). For those plants which are not strongly connected we can use Lemma (3.10) to classify the class of compensators solving SCCP. In this case the plant can be decomposed into its strongly connected components, where the class of compensators solving DSP can be considered for each of the subsystems independently. Also note that the "If" part of Theorem (3.4) is implicit in Theorem 1 of [10].•

Chapter 4

DECENTRALIZED STRONG STABILIZATION PROBLEM

In this chapter we first introduce the notion of decentralized blocking zeros. Then, the following questions are addressed: Let Z be a given N-channel plant. **(a)** Does there exist a *stable* decentralized stabilizing controller for the plant Z? **(b)** If a stable decentralized stabilizing controller for Z does not exist what is the minimum number of unstable poles, counted with multiplicites, that any decentralized stabilizing controller for Z must have? **(c)** Can these unstable poles be arbitrarily distributed among the local controllers?

The problem posed by (a) is the "Decentralized Strong Stabilization Problem" (DSSP) where the objective is to stabilize a plant using a stable decentralized controller. DSSP turns out to be the core problem of "Decentralized Concurrent Stabilization Problem" which is defined and solved in Chapter 5 of this thesis. Problem (b) is a generalization of DSSP. A complete solution to problem (b) yields a solution to DSSP and in the cases where DSSP has no solution it gives a lower bound for the minimum number of poles that any decentralized stabilizing controller must have. Problem (c) is concerned with the distribution of controller complexity in decentralized controllers [3].

In case of centralized controllers the analogue problems of (a) and (b) above have already been solved [77], [67], [66]. The solutions of these problems are given in terms of a parity interlacing property [77] among the real unstable poles and real unstable blocking zeros of the plant. An H_∞ approach to DSSP has been made in [62] where a sufficient solvability condition is given. For a class of 2×2 plants the solution of DSSP has been investigated in [30]. In this thesis we show that solutions to problems (a) and (b) exist if and only if some parity interlacing properties are satisfied. These properties, however, are now to be satisfied among the real unstable poles and real unstable decentralized blocking zeros. The decentralized blocking zeros of a plant are the union of those zeros at which the transfer matrix is upper block triangular for any symmetric permutations of block rows and block columns. The notion of decentralized blocking zeros is an important concept which plays a crucial role in the solution of a number of synthesis problems for large-scale systems [38], [59].

An outline of the chapter and a summary of its main results can be given as follows. In the next section we introduce a preliminary result. Section 4.2 contains the definition of decentralized blocking zeros and an investigation of their properties. Section 4.3 includes the main results of the chapter. Theorem (4.2) gives a solution to problem (b). It can be regarded as the counterpart of Theorem 5.3.1 (See Theorem (4.1) in Section 4.1) of [66], which considers the same problem for centralized controllers. Corollary (4.1) gives a solution to DSSP. The synthesis procedure of Theorem (4.2) also answers the question (c) affirmatively. We note that, as the reader may expect from its centralized counterpart, the proof of Theorem (4.2) is quite involved. In Theorem (4.3), it is shown that DSSP is a generically solvable problem.

4.1 A Preliminary Result

Let Ψ be the set of $\mathcal{R}_{+e}$-blocking zeros of $Z \in \mathbf{P}^{p\times r} - \{0\}$. Let $\sigma_1, \sigma_2, \ldots, \sigma_t$ denote the elements of Ψ arranged in ascending order. Let η_i denote the number of poles of Z counted with multiplicities in the interval (σ_i, σ_{i+1}), $i \in \{1, 2, \ldots, t-1\}$. Also let η be the number of odd integers in the set $\{\eta_1, \ldots, \eta_{t-1}\}$.

The following theorem is based on Theorem 5.3.1 of [66].

Theorem (4.1). (i). *Every stabilizing controller Z_c for Z has at least η poles in C_+ with multiplicities.* (ii)(a). *Given any integer $n \geq \eta$ where $n-\eta$ is an even number, there exists a stabilizing controller Z_c for Z which has exactly n poles in C_+ with multiplicities.* (ii)(b). *Given any integer $n \geq \eta$ where $n-\eta$ is an odd number, there exists a stabilizing controller Z_c for Z which has exactly n poles in C_+ with multiplicities if and only if $\sigma_1 \neq 0$ or $\sigma_t \neq \infty$.*

Proof. Statement (i) follows directly from [66, Theorem 5.3.1]. For the proof of statement (ii) let a left coprime fraction of Z over S be given by $Z = Q^{-1}R$. Let $c \in C_+$ be a nonreal number such that $R(c) \neq 0$. We will first prove (ii)(a). Define $\alpha \in$ S as follows

$$\alpha = [(\frac{z-c}{z+1})(\frac{z-c^*}{z+1})]^{(n-\eta)/2}$$

where c^* is the complex conjugate of c. Construct $\tilde{Q}_c \in \mathrm{S}^{p\times p}$ such that (a) $det(\tilde{Q}_c) = \alpha$ and (b) $(Q\tilde{Q}_c, R)$ is a left coprime pair. Observe that for any $i \in \{1, ..., t-1\}$, $det(Q).det(\tilde{Q}_c)$ has as many zeros as $det(Q)$ has with multiplicities in the interval (σ_i, σ_{i+1}). Then, from [66, Theorem 5.3.1] there exists $\tilde{Z}_c$ with η poles in C_+ with multiplicities such that $((Q\tilde{Q}_c)^{-1}R, \tilde{Z}_c)$ is stable. In this case (Z, Z_c) is stable and Z_c has n poles in C_+ with multiplicities where $Z := \tilde{Z}_c\tilde{Q}_c^{-1}$. This completes the proof of (ii)(a). For the proof of (ii)(b) we first prove the only if statement by contradiction. It will be shown that if $\sigma_1 = 0$ and $\sigma_t = \infty$ then $n-\eta$ must be an even number. This immediately implies that in case $n-\eta$ is odd $\sigma_1 \neq 0$ or $\sigma_t \neq \infty$ must hold. So, assume that $\sigma_1 = 0$, $\sigma_t = \infty$ and let (Z, Z_c) be a stable pair where Z_c has n poles in C_+ with multiplicities. Let $Z_c = P_cQ_c^{-1}$ be a right coprime fraction of Z_c over S. Since $QQ_c + RP_c$ is unimodular, $det(Q).det(Q_c)$ takes the same sign at 0 and ∞, which is the case only if $det(Q).det(Q_c)$ has an even number of $\mathcal{R}_+$ zeros in $(0, \infty)$ with multiplicities. Consequently, if η is an even (odd) number then $det(Q_c)$ has an even (odd) number of zeros in $(0, \infty)$ with multiplicities. Since $det(Q_c)$ has an even number of nonreal zeros, $n-\eta$ must be an even number. This completes the proof of the only if part via the above discussion. For the proof of the if part of (ii)(b) we assume that $\sigma_1 \neq 0$. If $\sigma_1 = 0$ and $\sigma_t \neq \infty$ the below proof can be applied by replacing β below with any positive real number greater σ_t. Define $\alpha \in \mathbf{S}$ as follows

$$\alpha = [(\frac{z-c}{z+1})(\frac{z-c^*}{z+1})]^{(n-\eta-1)/2}.$$

Also let $\beta = 0$. Construct $\tilde{Q}_c \in \mathbf{S}^{p\times p}$ such that (a) $\det(\tilde{Q}_c) = \alpha.\frac{(z-\beta)}{(z+1)}$ and (b) $(Q\tilde{Q}_c, R)$ is a left coprime pair. Observe that for any $i \in \{1, ..., t-1\}$ $det(Q).det(\tilde{Q}_c)$ has as many zeros as $det(Q)$ has with multiplicities in the interval (σ_i, σ_{i+1}). Then, from [66, Theorem 5.3.1] there exists $\tilde{Z}_c$ with η poles in C_+ with multiplicities such that $((Q\tilde{Q}_c)^{-1}R, \tilde{Z}_c)$ is stable. In this case (Z, Z_c) is stable and Z_c has n poles in C_+ with multiplicities where $Z_c := \tilde{Z}_c\tilde{Q}_c^{-1}$. This completes the proof of statement (ii).□

The *Strong Stabilization Problem* ([77], [66]) is defined as determining a stable controller Z_c, i.e., a controller having all entries over $\mathbf{S}$, such that (Z, Z_c) is stable. From Theorem (4.1) we conclude that the strong stabilization problem is solvable if and only if there are an even number of poles of Z between each pair of its blocking zeros; equivalently, the set of unstable real poles of Z and the set Ψ satisfy the parity interlacing property.

4.2 Decentralized Blocking Zeros

The purpose of this section is to introduce the "decentralized blocking zeros" of a multi-channel system and examine how these zeros are influenced by feedback at one or more channels.

We first state the following three results which concern the identification of the (centralized) blocking zeros of Z from the system matrix associated with a fractional representation over $\mathbf{S}$.

Let $Z \in \mathbf{P}^{p\times r}$ and let

$$Z = PQ^{-1}R \tag{4.1}$$

be a fractional representation of Z over $\mathbf{S}$ with Q of size $q \times q$.

Lemma (4.1). *For any $z_0 \in C_{+e}$ for which $Z(z_0) = 0$, one has*

$$rank \begin{bmatrix} Q & R \\ -P & 0 \end{bmatrix} (z_0) \leq q,$$

where equality is achieved if either (P, Q) is right coprime or (Q, R) is left coprime over $\mathbf{S}$.

Lemma (4.2). *If (4.1) is a bicoprime fraction over* $\mathbf{S}$, *then for any $z_0 \in C_{+e}$*

$$rank \begin{bmatrix} Q & R \\ -P & 0 \end{bmatrix} (z_0) = q,$$

if and only if $Z(z_0) = 0$.

Lemma (4.3). *For any $z_0 \in C_{+e}$ such that $det(Q)(z_0) \neq 0$ and*

$$rank \begin{bmatrix} Q & R \\ -P & 0 \end{bmatrix} (z_0) = q,$$

it holds that $Z(z_0) = 0$.

Proofs of Lemmata (4.1)-(4.3). Let $\Omega_l := gclf(Q, R)$, so that $Q = \Omega_l \tilde{Q}$, $R = \Omega_l \tilde{R}$, for a left coprime pair $(\tilde{Q}, \tilde{R})$. Also let $\Omega_r := gcrf(\tilde{Q}, P)$ so that $\tilde{Q} = \bar{Q}\Omega_r$, $P = \bar{P}\Omega_r$, for a right coprime pair $(\bar{Q}, \bar{P})$. Then, a bicoprime fraction of Z over $\mathbf{S}$ is given by $\bar{P}\bar{Q}^{-1}\tilde{R}$. Also, the matrix equality

$$\begin{bmatrix} \Omega_l \bar{Q} & 0 \\ -\bar{P} & I_p \end{bmatrix} \begin{bmatrix} I_q & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} \Omega_r & \bar{Q}^{-1}\tilde{R} \\ 0 & I_r \end{bmatrix} = \begin{bmatrix} Q & R \\ -P & 0 \end{bmatrix} \tag{4.2}$$

holds. Note that if z_0 is a blocking zero of Z, then $\bar{Q}(z_0)$ is nonsingular since blocking zeros are distinct from poles. Let z_0 be a C_{+e} blocking zero of Z and note that the rank at z_0 of the left hand side of the above equality is less than or equal to q. If either (P, Q) is right coprime or (Q, R) is left coprime then the rank at z_0 of the right hand side of (4.2) is greater or equal to q. The proof of Lemma (4.1) and the "if" part of Lemma (4.2) follow from these two statements. If the fraction $PQ^{-1}R$ is bicoprime, then there exist matrices X, Y, P_r, Q_r where Q_r is nonsingular such that $[Q\ R]\Phi = [I_q\ 0]$ where

$$\Phi = \begin{bmatrix} X & -P_r \\ Y & Q_r \end{bmatrix}$$

and is unimodular. If the rank at z_0 of the left hand side is q, then the rank at z_0 of the matrix at the right hand side in the below equation

$$\begin{bmatrix} Q & R \\ -P & 0 \end{bmatrix} \Phi = \begin{bmatrix} I_q & 0 \\ -PX & PP_r \end{bmatrix}$$

is also q from which we obtain $PP_r(z_0) = 0$. Since $Z = PP_rQ_r^{-1}$ where the fraction is coprime, it holds that $Z(z_0) = 0$ proving the "only if" part of Lemma (4.2). Finally, if z_0 in C_{+e} is such that the rank at z_0 of the right hand side of (4.2) is q and $Q(z_0)$ is nonsingular, then all of $\Omega_l(z_0)$, $\bar{Q}(z_0)$, and $\Omega_r(z_0)$ are nonsingular. From this it again follows that $Z(z_0) = 0$. This proves Lemma (4.3). □

Let Z be the transfer matrix of an N-channel system ($N > 1$) so that it is in the partitioned form $Z = [Z_{ij}]$, where $Z_{ij} \in \mathbf{P}^{p_i \times r_j}$, $i, j \in \mathbf{N}$ such that $\sum_{i=1}^N p_i = p$ and $\sum_{i=1}^N r_i = r$. An element z of C_e is called a *decentralized blocking zero* of Z if, when evaluated at z, all the entries of plant transfer matrix below the main diagonal blocks and the entries in the main diagonal blocks become zero (after a suitable symmetric permutation of the block rows and columns). More precisely, z is a decentralized blocking zero of Z if for some permutation $\{i_1, ..., i_N\}$ of $\mathbf{N}$ the following holds:

$$Z_{i_k i_l}(z) = 0, \; k = 1, ..., N, \; l = 1, ..., k.$$

The set of decentralized blocking zeros of Z is denoted by $\mathcal{S}_Z$. It follows that

$$\mathcal{S}_Z = \{z \in C_e | \quad \textit{There exists a permutation } \{i_1, i_2, ..., i_N\} \textit{ of } \mathbf{N} \textit{ such that}$$

$$\begin{bmatrix} Z_{i_1 i_1} & 0 & 0 & & 0 \\ Z_{i_2 i_1} & Z_{i_2 i_2} & 0 & & 0 \\ Z_{i_3 i_1} & Z_{i_3 i_2} & Z_{i_3 i_3} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ Z_{i_N i_1} & Z_{i_N i_2} & Z_{i_N i_3} & & Z_{i_N i_N} \end{bmatrix} (z) = 0\}.$$

For convenience, in the case $N = 1$ (the centralized case), we define the decentralized blocking zeros as the centralized blocking zeros. (We note that as in the case of centralized blocking zeros, [17], the term "blocking" can be justified through a blocking-property of these zeros against certain structured inputs.)

An equivalent description for the set $\mathcal{S}_Z$ can be given as follows. Define

$\mathcal{S}_Z^{diag} := \{z \in \mathcal{C}_e | Z_{ii}(z) = 0, \ i \in \mathrm{N}\}$.

$$\mathcal{S}_Z^{comp} := \{ \ z \in \mathcal{C}_e | \ \textit{There exists a permutation } \{i_1, ..., i_N\} \textit{ of } \mathrm{N} \textit{ such that } z \textit{ is a blocking zero of all the complementary transfer matrices below}$$

$$\begin{bmatrix} Z_{i_2 i_1} \\ Z_{i_3 i_1} \\ \vdots \\ Z_{i_N i_1} \end{bmatrix}, \begin{bmatrix} Z_{i_3 i_1} & Z_{i_3 i_2} \\ \vdots & \vdots \\ Z_{i_N i_1} & Z_{i_N i_2} \end{bmatrix}, \ldots, [Z_{i_N i_1} \ Z_{i_N i_2} \ldots Z_{i_N i_{N-1}}]\}$$

It easily follows that

$$\mathcal{S}_Z = \mathcal{S}_Z^{diag} \cap \mathcal{S}_Z^{comp}. \tag{4.3}$$

That is, every decentralized blocking zero is a common blocking zero of all the main diagonal transfer matrices and various complementary transfer matrices. In the simplest case of two channels, these alternative descriptions yield the following expressions for $\mathcal{S}_Z$:

$$\begin{aligned} \mathcal{S}_Z &= \{z \in \mathcal{C}_e | Z_{11}(z) = 0, Z_{21}(z) = 0, \ \textit{and} \ Z_{22}(z) = 0\} \cup \{z \in \mathcal{C}_e | Z_{22}(z) = 0, \\ & \quad Z_{12}(z) = 0, \ \textit{and} \ Z_{11}(z) = 0\} \\ &= \{z \in \mathcal{C}_e | Z_{11}(z) = 0 \ \textit{and} \ Z_{22}(z) = 0\} \cap \{z \in \mathcal{C}_e | Z_{21}(z) = 0 \ \textit{or} \ Z_{12}(z) = 0\}. \end{aligned}$$

Note that, any (centralized) blocking zero is clearly a decentralized blocking zero and in fact $\mathcal{S}_Z$ can be a much larger set than the set $\{z \in \mathcal{C}_e | Z(z) = 0\}$ of blocking zeros.

As stated in [16], [17], the blocking zeros block out the transmission of various modes in the arbitrary inputs. A similar dynamical interpretation for decentralized blocking zeros can be given as they block the corresponding modes in the structured inputs where certain entries are restricted to be zero.

Despite the fact that the $\mathcal{C}_{+e}$ centralized blocking zeros are disjoint with the poles of Z, in general the decentralized blocking zeros and the poles are not disjoint.

Example (4.1). Consider the two (scalar) channel transfer matrix

$$Z = \begin{bmatrix} \frac{z}{z-1} & \frac{1}{z} \\ \frac{z}{z-1} & \frac{z}{z-1} \end{bmatrix}.$$

The poles are $\{0, 1, 1\}$ and the only decentralized blocking zero is $\{0\}$. The common element 0 is actually a decentralized fixed mode of Z.

Lemma (4.4). *Let an N-channel transfer matrix $Z = [Z_{ij}]$ be free of C_{+e} decentralized fixed modes. Then, the set of poles of Z and $S_Z \cap C_{+e}$ are disjoint.*

Proof. The proof is based on the following fact.

Fact (4.1). *Let $K = [K_{ij}]$, $K_{ij} \in \mathbf{P}^{t_i \times s_j}$, $i, j \in \mathbf{N}$, be given. Assume that DSP for K is solvable. Let a bicoprime fraction of K be given by $[T_1' \dots T_N']O^{-1}[S_1 \dots S_N]$ such that $O \in \mathbf{S}^{g \times g}$, $T_i \in \mathbf{S}^{t_i \times g}$ and $S_i \in \mathbf{S}^{g \times s_i}$, $i \in \mathbf{N}$. Let $z \in C_{+e}$ be such that*

$$rank \begin{bmatrix} O & S_{i_1} & \dots & S_{i_j} \\ -T_{i_j} & 0 & \ddots & 0 \\ & \vdots & & \vdots \\ -T_{i_N} & 0 & & 0 \end{bmatrix} (z) \leq g, \forall j \in \mathbf{N} \tag{4.4}$$

for some permutation $\{i_1, \dots, i_N\}$ of $\mathbf{N}$. Then, $O(z)$ is nonsingular.

Proof. We will prove the statement by assuming that $i_j = j$, $j \in \mathbf{N}$. For any other permutation the below proof can be applied by appropriate modifications on the indices.

Let a left coprime fraction of K be given by $K = \tilde{O}^{-1}\tilde{S}$ where $\tilde{O} = [\tilde{O}_{ij}]$, $\tilde{O}_{ij} \in \mathbf{S}^{t_i \times t_j}$, $i, j \in \mathbf{N}$, $\tilde{S} = [\tilde{S}_{ij}]$, $\tilde{S}_{ij} \in \mathbf{S}^{t_i \times s_j}$, $i, j \in \mathbf{N}$. We can choose $\tilde{Q}$ as upper triangular so that $\tilde{O}_{ij} = 0$, $i = 2, \dots, N$, $j = 1, \dots, i-1$. It follows that for any $z \in C_{+e}$ (4.4) holds if and only if

$$rank \begin{bmatrix} \tilde{O} & \tilde{S}_1 \dots \tilde{S}_j \\ -diag\{I_{t_j}, \dots, I_{t_N}\} & 0 \end{bmatrix} (z) \leq t, \ \ \forall j \in \mathbf{N} \tag{4.5}$$

where $t := size(\tilde{O})$ and $\tilde{S}_i \in \mathbf{S}^{t \times s_i}$ denotes the ith column of $\tilde{S}$. Unimodular operations yield that (4.5) holds only if $\tilde{S}_1(z) = 0$. Now, let $Z_c = diag\{Z_{c1}, \dots, Z_{cN}\}$

solve DSP for K. Let a right coprime fraction of Z_{ci} be given by $Z_{ci} = P_{ci}Q_{ci}^{-1}$, $P_{ci} \in \mathbf{S}^{s_i \times t_i}$, $Q_{ci} \in \mathbf{S}^{t_i \times t_i}$, $i \in \mathbf{N}$. Then,

$$rank \begin{bmatrix} \tilde{O} & \tilde{S}_1 P_{c1} \dots \tilde{S}_N P_{cN} \\ -I & diag\{Q_{c1}, \dots, Q_{cN}\} \end{bmatrix}$$

is a unimodular matrix and is therefore nonsingular when evaluated at any $z \in C_{+\epsilon}$. Let $z = z_0$ satisfy (4.5). The fact that $\tilde{S}_1(z_0) = 0$ implies via the above discussion that $\tilde{O}_{11}(z_0)$ is nonsingular. In this case, going back to (4.5) and applying unimodular operations we conclude that $\tilde{S}_{j2}(z_0) = 0$, $j = 2, \dots, N$. It then follows that $\tilde{O}_{22}(z_0)$ is nonsingular. Repeating this process it holds that $\tilde{O}_{jj}(z_0)$ is also nonsingular, $j = 3, \dots, N$. Then, $\tilde{O}(z_0)$ is nonsingular. Since $det(\tilde{O})$ and $det(O)$ are associates, $O(z_0)$ is also nonsingular. Since $z = z_0 \in C_{+\epsilon}$ satisfying (4.5) is fixed but otherwise arbitrary, the proof follows.Δ

We now continue the proof of Lemma (4.4).

Letting $K := Z$ and using Fact (4.1) we conclude that the set of unstable zeros of $det(Q)$ and $\mathcal{S}_Z \cap C_{+\epsilon}$ are disjoint. Since every unstable zero of $det(Q)$ is an unstable pole of Z, this completes the proof of Lemma (4.4).$\square$

Lemma (4.2) above characterizes the $C_{+\epsilon}$ blocking zeros of Z in terms of the system matrix associated with a bicoprime fraction of Z. We now give a similar result for decentralized blocking zeros under the assumption that the N-channel transfer matrix has no unstable decentralized fixed modes.

Lemma (4.5). *Let $Z = [Z_{ij}]$ be given in a bicoprime fractional representation*

$$Z = \begin{bmatrix} P_1 \\ \\ P_N \end{bmatrix} Q^{-1} \begin{bmatrix} R_1 & & R_N \end{bmatrix}, \tag{4.6}$$

where $Z_{ij} = P_i Q^{-1} R_j$ for $i, j = 1, \dots N$. If $Z = [Z_{ij}]$ is free of unstable decentral-

ized fixed modes, then

$\mathcal{S}_Z \cap \mathcal{C}_{+e} = \{z \in \mathcal{C}_{+e}|$ *There exists a permutation* $\{i_1, ..., i_N\}$ *of* N *such that*

$$rank \begin{bmatrix} Q & R_{i_1} & R_{i_2} & \dots & R_{i_j} \\ -P_{i_j} & 0 & 0 & & 0 \\ -P_{i_{j+1}} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & & \\ -P_{i_N} & 0 & 0 & & 0 \end{bmatrix} (z) = q, \ \forall j \in \mathbf{N}\}.$$

Proof. Let

$\mathcal{T} := \{z \in \mathcal{C}_{+e}|$ *There exists a permutation* $\{i_1, ..., i_N\}$ *of* N *such that*

$$rank \begin{bmatrix} Q & R_{i_1} & R_{i_2} & & R_{i_j} \\ -P_{i_j} & 0 & 0 & & 0 \\ -P_{i_{j+1}} & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -P_{i_N} & 0 & 0 & & 0 \end{bmatrix} (z) = q, \ \forall j \in \mathbf{N}\}.$$

If $z_0 \in \mathcal{S}_Z \cap \mathcal{C}_{+e}$, then Lemma (4.1) implies that $z_0 \in \mathcal{T}$. On the other hand, if $z_0 \in \mathcal{T}$ then by Fact (4.1) $Q(z_0)$ is nonsingular which, via Lemma (4.3), implies that $z_0 \in \mathcal{S}_Z \cap \mathcal{C}_{+e}$. This shows that $\mathcal{T} = \mathcal{S}_Z \cap \mathcal{C}_{+e}$.□

Now we will discuss some interpretations for decentralized blocking zeros in terms of invariant zeros and transmission zeros.

Let a permutation $P = \{i_1, ..., i_N\}$ of N and $j \in \mathbf{N}$ be fixed. Then, $z_0 \in \mathcal{C}_{+e}$ is called an unstable *invariant zero* associated with the l'th invariant factor of system $([P'_{i_j} \ \dots \ P'_{i_N}]', \ Q, \ [R_{i_1} \ \dots \ R_{i_j}])$ where $1 \le l \le rank \ [P'_{i_j} \ \dots \ P'_{i_N}]' \ Q^{-1}[R_{i_1} \dots R_{i_j}] + q$, if

$$rank \begin{bmatrix} Q & R_{i_1} & & R_{i_j} \\ -P_{i_j} & 0 & & 0 \\ \vdots & \vdots & & \\ -P_{i_N} & 0 & & 0 \end{bmatrix} (z_0) < l.$$

Let $\mathbf{N}_P$ be a subset of $\mathbf{N}$ such that $j \in \mathbf{N}_P$ if and only if $[P'_{i_j} \ldots P'_{i_N}]' Q^{-1} [R_{i_1} \ldots R_{i_j}] \neq 0$. Assume that $\mathcal{S}_Z$ is a finite set (see page 69). From Lemma (4.5) and its proof (see Fact (4.1)) one can draw the following conclusion: *$z \in \mathcal{C}_{+e}$ is a decentralized blocking zero of a plant Z which has no $\mathcal{C}_+$ decentralized fixed modes if and only if there exists a permutation $P = \{i_1, ..., i_N\}$ of $\mathbf{N}$ such that z is a common invariant zero associated with the $q+1$'st invariant factor of systems $([P'_{i_j} \ldots P'_{i_N}]', Q, [R_{i_1} \ldots R_{i_j}])$, $j \in \mathbf{N}_P$.*

Referring to Section 2.1, a transmission zero $z \in \mathcal{C}$ of Z is not a pole of Z then $Z(z) \in \mathcal{C}^{p \times r}$ and $rank\, Z(z) < rank\, Z$. Conversely, if $z \in \mathcal{C}_{+e}$ is such that z is not a pole of Z and $rank\, Z(z) < rank\, Z$ then z is a transmission zero of Z. Now let Z be full rank and be free of $\mathcal{C}_{+e}$ decentralized fixed modes. If $z \in \mathcal{S}_Z \cap \mathcal{C}_{+e}$ then z is not a pole of Z (Lemma (4.4)) and $rank\, Z(z) < rank\, Z$. As a result, we conclude the following.

Let Z be full rank and be free of $\mathcal{C}_{+e}$ decentralized fixed modes. Then, every $\mathcal{C}_{+e}$ decentralized blocking zero of Z is also a transmission zero of Z.

Note that if Z is not full rank the above statement does not hold in general. For example

$$Z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is free of $\mathcal{C}_{+e}$ decentralized fixed modes but is not full rank. Although Z has no transmission zeros, every $z \in \mathcal{C}$ is a decentralized blocking zero.

A different characterization of $\mathcal{C}_{+e}$ decentralized blocking zeros can be given by viewing them as the intersection of the set of blocking zeros of any fixed but otherwise arbitrary channel and a set of zeros pertaining to the remaining channels. Let $L := N - 1$ and define

$$\check{\Psi} = \{z \in \mathcal{C}_{+e} |\ \textit{There exists a permutation } \{i_1, i_2, \ldots, i_L\} \textit{ of } \mathbf{L} \textit{ such that for each } j \in \mathbf{L}$$

$$\textit{either} \begin{bmatrix} Z_{i_1 i_1} & Z_{i_1 i_j} & Z_{i_1 N} \\ \vdots & \vdots & \vdots \\ Z_{i_L i_1} & Z_{i_L i_j} & Z_{i_L N} \end{bmatrix}(z) = 0$$

$$\textit{or} \begin{bmatrix} Z_{i_1 i_1} & Z_{i_1 i_j} \\ \vdots & \vdots \\ Z_{i_L i_1} & Z_{i_L i_j} \\ Z_{N i_1} & Z_{N i_j} \end{bmatrix}(z) = 0\}.$$

Lemma (4.6). $\{z \in \check{\Psi} |\ Z_{NN}(z) = 0\} = \mathcal{S}_Z \cap \mathcal{C}_{+e}$.

Proof. The proof is based on the following fact.

Fact (4.2). *Let* $G = [G_{ij}]$, $i, j \in \mathbf{N}$ *be a matrix over* $\mathbf{P}$. *Define* $L = N - 1$. *Then, for any* $z \in \mathcal{C}_{+e}$ *satisfying*

$$\textit{For each } j \in \mathbf{L}$$

$$\textit{either} \begin{bmatrix} G_{j1} & G_{jj} & G_{jN} \\ \vdots & \vdots & \vdots \\ G_{L1} & G_{Lj} & G_{LN} \end{bmatrix}(z) = 0 \textit{ or } \begin{bmatrix} G_{j1} & G_{jj} \\ \vdots & \vdots \\ G_{L1} & G_{Lj} \\ G_{N1} & G_{Nj} \end{bmatrix}(z) = 0, \tag{4.7}$$

one of the following holds

$$\begin{aligned} G_{(1,2,\ldots,L,N)}(z) &= 0 \\ G_{(1,2,\ldots,N,L)}(z) &= 0 \\ &\vdots \\ G_{(1,2,\ldots,N,\ldots,L)}(z) &= 0 \\ \\ G_{(1,N,2,\ldots,L)}(z) &= 0 \\ G_{(N,1,2,\ldots,L)}(z) &= 0 \end{aligned} \tag{4.8}$$

provided $G_{NN}(z) = 0$, *where by definition*

$$G_{(i_1,\ldots,i_N)}(z) = 0 \Leftrightarrow \begin{bmatrix} G_{i_j i_1} & & G_{i_j i_j} \\ \vdots & & \vdots \\ G_{i_N i_1} & \ldots & G_{i_N i_j} \end{bmatrix}(z) = 0, \ \forall j \in \mathbf{N}. \qquad (4.9)$$

Proof. We prove the statement by induction. Let $N = 2$. Then, $L = 1$ and z satisfies (4.7) if and only if

$$[G_{11}\ G_{12}](z) = 0 \text{ or } [G'_{11}\ G'_{21}]'(z) = 0.$$

If z further satisfy $G_{22}(z) = 0$, then it is easy to see that the statement holds. This proves the inductive argument for $N = 2$. Now assume that the fact is true for L. Let $N = L + 1$. Let z satisfy that $G_{NN}(z) = 0$ and (4.7) holds. Observe that (i), and from the inductive hypothesis (ii) below hold.

(i) One of the equalities below holds.

$$[G_{L1} \ldots G_{LL}\ G_{LN}](z) = 0, \quad \begin{bmatrix} G_{L1} & & G_{LL} \\ G_{N1} & \ldots & G_{NL} \end{bmatrix}(z) = 0$$

(ii) One of the equalities below holds

$$\check{G}_{(1,2,\ldots,L-1,L)}(z) = 0, \ \check{G}_{(1,2,\ldots,L,L-1)}(z) = 0, \ \ldots, \ \check{G}_{(1,L,2,\ldots,L-1)}(z) = 0,$$

$$\check{G}_{(L,1,2,\ldots,L-1)}(z) = 0$$

where $\check{G} = [\check{G}_{ij}]$, $i, j \in \mathbf{L}$ is the submatrix of G obtained by deleting its L-th block row and column such that $\check{G}_{ij} = G_{ij}$, $i, j = 1, \ldots, L-1$, $\check{G}_{iL} = G_{iN}$, $i = 1, \ldots, L-1$, $\check{G}_{Lj} = G_{Nj}$, $j = 1, \ldots, L-1$, $\check{G}_{LL} = G_{NN}$, and $\check{G}_{(\cdot,\cdot,\ldots,\cdot)}$ is defined as in (4.9).

Observe the following.

(a)

$$\check{G}_{(1,2,\ldots,L-1,L)}(z) = 0 \text{ and } \begin{bmatrix} G_{L1} & & G_{LL} \\ G_{N1} & \ldots & G_{NL} \end{bmatrix}(z) = 0 \Rightarrow G_{(1,2,\ldots,L-1,L,N)}(z) = 0,$$

(b)

$$\hat{G}_{(1,2,\ldots,L,\ldots,L-1)}(z) = 0 \ and \ \begin{bmatrix} G_{L1} & G_{LL} & G_{LN} \end{bmatrix}(z) = 0$$
$$\Rightarrow G_{(1,2,\ldots,N,\ldots,L-1,L)}(z) = 0$$

where L in $\hat{G}_{(1,2,\ldots,L,\ldots,L-1)}$ and N in $G_{(1,2,\ldots,N,\ldots,L-1,L)}$ are at the same position from the beginning. This completes the proof.Δ

We now continue the proof of Lemma (4.6).

Let $z \in \tilde{\Psi}$ and $Z_{NN}(z) = 0$. Then, there exists a permutation $\{i_1, \ldots, i_L\}$ of $\mathbf{L}$ such that

$$For\ each\ j \in \mathbf{L}$$
$$either \begin{bmatrix} Z_{i_1 i_1} & Z_{i_1 i_j} & Z_{i_1 N} \\ \vdots & \vdots & \vdots \\ Z_{i_L i_1} & Z_{i_L i_j} & Z_{i_L N} \end{bmatrix}(z) = 0 \ or \begin{bmatrix} Z_{i_1 i_1} & Z_{i_1 i_j} \\ \vdots & \vdots \\ Z_{i_L i_1} & Z_{i_L i_j} \\ Z_{N i_1} & Z_{N i_j} \end{bmatrix}(z) = 0.$$

Let G be defined as $G_{lk} = Z_{i_l i_k}$, $G_{Nk} = Z_{N i_k}$ and $G_{lN} = Z_{i_l N}$, $l, k \in \mathbf{L}$. Applying Fact (4.2) we have that one of the equalities in (4.8) holds. This implies that $z \in \mathcal{S}_Z \cap \mathcal{C}_{+e}$. Since $z \in \tilde{\Psi}$ is arbitrary we have $\{z \in \tilde{\Psi} | \ Z_{NN}(z) = 0\} \subset \mathcal{S}_Z \cap \mathcal{C}_{+e}$. Conversely, let $z \in \mathcal{S}_Z \cap \mathcal{C}_{+e}$. There exists a permutation $\{i_1, \ldots, i_N\}$ of $\mathbf{N}$ such that

$$\begin{bmatrix} Z_{i_1 i_1} & Z_{i_1 i_j} \\ \vdots & \vdots \\ Z_{i_N i_1} & Z_{i_N i_j} \end{bmatrix}(z) = 0, \ \forall j \in \mathbf{N}.$$

Let $i_l = N$ for some $l \in \mathbf{N}$. It holds that

$$\begin{bmatrix} Z_{i_1 i_1} & \cdots & Z_{i_1 i_j} \\ \vdots & & \vdots \\ Z_{i_N i_1} & & Z_{i_N i_j} \end{bmatrix}(z) = 0, \ \forall j \in \mathbf{N} - \{l\}, \tag{4.10}$$

$$\begin{bmatrix} Z_{i_j N} \\ \\ Z_{i_N N} \end{bmatrix}(z) = 0, \ \forall j \in \{l+1, \ldots, N\} \tag{4.11}$$

and

$$\begin{bmatrix} Z_{Ni_1} & \quad & Z_{Ni_j} \end{bmatrix}(z) = 0,\ \forall j \in \{1, \ldots, l-1\}\ (in\ case\ l > 1). \tag{4.12}$$

Define a new set of integers $\{i'_1, \ldots, i'_L\}$ as follows.

$$i'_j = \begin{cases} i_{j+1} & ,\ j \geq l \\ i_j & ,\ otherwise \end{cases} ,\ j \in \mathbf{L}$$

From (4.10) it is easy to see that

$$\begin{bmatrix} Z_{i'_j i'_1} & & Z_{i'_j i'_j} \\ \vdots & & \vdots \\ Z_{i'_L i'_1} & & Z_{i'_L i'_j} \end{bmatrix}(z) = 0,\ \forall j \in \mathbf{L}.$$

Moreover, from (4.11)

$$\begin{bmatrix} Z_{i'_j N} \\ \vdots \\ Z_{i'_L N} \end{bmatrix}(z) = 0,\ l \leq j \leq L$$

and from (4.12) $[Z_{Ni'_1} \ldots Z_{Ni'_j}](z) = 0$, $1 \leq j < l$. Then, for any $j \in \mathbf{L}$

$$\begin{bmatrix} Z_{i'_j i'_1} & & Z_{i'_j i'_j} & Z_{i'_j N} \\ \vdots & & \vdots & \vdots \\ Z_{i'_L i'_1} & & Z_{i'_L i'_j} & Z_{i'_L N} \end{bmatrix}(z) = 0 \ or \begin{bmatrix} Z_{i'_j i'_1} & & Z_{i'_j i'_j} \\ \vdots & & \vdots \\ Z_{i'_L i'_1} & & Z_{i'_L i'_j} \\ Z_{Ni'_1} & & Z_{Ni'_j} \end{bmatrix}(z) = 0.$$

This implies that $z \in \tilde{\Psi}$. Since $z \in \mathcal{S}_Z \cap \mathcal{C}_{+e}$ is arbitrary, one has $\mathcal{S}_Z \cap \mathcal{C}_{+e} \subset \tilde{\Psi}$. On the other hand, by definition, $z \in \mathcal{S}_Z \cap \mathcal{C}_{+e}$ implies $Z_{NN}(z) = 0$. Hence, $\mathcal{S}_Z \cap \mathcal{C}_{+e} \subset \tilde{\Psi} \cap \{z \in \mathcal{C}_{+e} | \ Z_{NN}(z) = 0\}$. This completes the proof.□

We now examine how dynamic feedback at one channel affects the unstable decentralized blocking zeros. This is done for feedbacks which do not introduce any unstable decentralized fixed modes in the resulting $(N-1)$-channel system.

Lemma (4.7). *Let $Z_{cN} = P_{cN}Q_{cN}^{-1}$ be a coprime fraction over* **S** *of a compensator at the N-th channel of (4.6) such that the resulting fraction*

$$\hat{Z}(Z_{cN}) := \begin{bmatrix} P_1 & 0 \\ \vdots & \vdots \\ P_L & 0 \end{bmatrix} \begin{bmatrix} Q & R_N P_{cN} \\ -P_N & Q_{cN} \end{bmatrix}^{-1} \begin{bmatrix} R_1 & & R_L \\ 0 & & 0 \end{bmatrix} \tag{4.13}$$

of the L-channel system is a bicoprime fraction and (if $L > 1$) $\hat{Z}(Z_{cN})$ is free of unstable decentralized fixed modes. Then,

$$\mathcal{S}_Z \cap \mathcal{C}_{+\epsilon} \subset \mathcal{S}_{\hat{Z}(Z_{cN})} \cap \mathcal{C}_{+\epsilon}$$

where $\mathcal{S}_{\hat{Z}(Z_{cN})}$ is the set of decentralized blocking zeros of $\hat{Z}(Z_{cN})$.

Proof. Note by Lemma (4.5) that

$\mathcal{S}_{\hat{Z}(Z_{cN})} \cap \mathcal{C}_{+\epsilon} :=$ $\{z \in \mathcal{C}_{+\epsilon} |$ *There exists a permutation* $\{i_1, ..., i_L\}$ *of* **L** *such that*

$$rank \begin{bmatrix} Q & R_N P_{cN} & R_{i_1} & & R_{i_j} \\ -P_N & Q_{cN} & 0 & & 0 \\ -P_{i_j} & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & & 0 \\ P_{i_N} & 0 & 0 & & 0 \end{bmatrix} = q + p_N, \forall j \in \mathbf{L}\}. \tag{4.14}$$

Let $z_0 \in \mathcal{S}_Z \cap \mathcal{C}_{+\epsilon}$. By Lemma (4.5), there exists a permutation $\{i_1, i_2, ..., i_N\}$ of **N** such that

$$rank \begin{bmatrix} Q & R_{i_1} & & R_{i_j} \\ -P_{i_j} & 0 & & 0 \\ \vdots & \vdots & & \\ -P_{i_N} & 0 & & 0 \end{bmatrix} (z_0) = q, \ \forall j \in \mathbf{N}. \tag{4.15}$$

It follows by (4.15) that

$$rank \begin{bmatrix} Q & R_N P_{cN} & R_{i_1} & & R_{i_j} \\ -P_N & Q_{cN} & 0 & & 0 \\ -P_{i_j} & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & & 0 \\ P_{i_N} & 0 & 0 & & 0 \end{bmatrix} \leq q + p_N, \quad \forall j \in \mathbf{N} \tag{4.16}$$

as we are adding p_N rows and columns to the matrices in (4.15). Consider the inequalities in (4.16) for $j \in \mathbf{N}$ such that $i_j \neq N$. Define $\{i'_1, ..., i'_L\}$ as follows. Let M be such that $i_M = N$ and let

$$i'_j = \begin{cases} i_{j+1}, & if\ j \geq M \\ i_j, & otherwise \end{cases}, \quad j \in \mathbf{L}.$$

Then by (4.16) we have

$$rank \begin{bmatrix} Q & R_N P_{cN} & R_{i'_1} & & R_{i'_j} \\ -P_N & Q_{cN} & 0 & & 0 \\ -P_{i'_j} & 0 & 0 & & 0 \\ \vdots & & \vdots & & 0 \\ P_{i'_L} & 0 & 0 & & 0 \end{bmatrix} \leq q + p_N, \quad \forall j \in \mathbf{L} \tag{4.17}$$

as we are deleting certain block rows or columns. By hypothesis, $\hat{Z}(Z_{cN})$ is free of unstable decentralized fixed modes and each matrix in (4.16) contains a system matrix associated with a complementary subsystem of (4.13) as its submatrix. By the fact that the plant is free of unstable decentralized fixed modes, the inequalities in (4.17) are actually equalities. Therefore, $z_0 \in \mathcal{S}_{\hat{Z}(Z_{cN})} \cap \mathcal{C}_{+e}$ by the description of the set $\mathcal{S}_{\hat{Z}(Z_{cN})} \cap \mathcal{C}_{+e}$ given in (4.14). □

Note that $\mathcal{S}_Z$ is a finite set if and only if for every permutation $\{i_1, ..., i_N\}$ of $\mathbf{N}$ the matrix

$$\begin{bmatrix} Z_{i_1 i_1} & 0 & 0 & & 0 \\ Z_{i_2 i_1} & Z_{i_2 i_2} & 0 & & 0 \\ Z_{i_3 i_1} & Z_{i_3 i_2} & Z_{i_3 i_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \cdot 0 \\ Z_{i_N i_1} & Z_{i_N i_2} & Z_{i_N i_3} & & Z_{i_N i_N} \end{bmatrix}$$

is different than zero (over **P**). It also holds that if Z is strongly connected then $\mathcal{S}_Z$ is a finite set. Define

$$\Psi = \mathcal{S}_Z \cap \mathcal{R}_{+e}$$

which is the set of decentralized blocking zeros of Z lying in the extended right half real line. Let $\sigma_1, \sigma_2, \ldots, \sigma_t$ denote the elements of Ψ arranged in the ascending order. Also let η_i denote the number of poles of Z counted with multiplicities in the interval (σ_i, σ_{i-1}), $i \in \{1, 2, \ldots, t-1\}$. Define η to be the number of odd integers in the set $\{\eta_1, \ldots, \eta_{t-1}\}$.

The following lemma is a key result which is used in the constructive part of Theorem (4.2) in the next section. Briefly, it says that given any nonnegative integer $n_N \leq \eta$ one can construct a local controller around any fixed but otherwise arbitrary channel (the Nth channel below without loss of generality) which has n_N poles in $\mathcal{C}_+$ with multiplicities, and ensures that DSP for the resulting $L = N-1$ channel plant $\hat{Z}(Z_{cN})$ is solvable and satisfies an appropriate interlacing property between the set of real unstable poles and the set of real unstable decentralized blocking zeros. In this lemma we assume the following (see also the next section)

(A1) Z is strongly connected,

(A2) *rank* $Z_{ij} \geq 2$ *or rank* $Z_{ji} \geq 2$, $\forall i, j \in \mathbf{N}$, $i \neq j$.

Lemma (4.8). *Let $Z = [Z_{ij}]$ be free of $\mathcal{C}_+$ decentralized fixed modes. Let a nonnegative integer $n_N \leq \eta$ be given. There exists $Z_{cN} = P_{cN} Q_{cN}^{-1} \in \mathbf{P}^{r_N \times p_N}$ for a right coprime pair of matrices (Q_{cN}, P_{cN}) over* **S** *such that*

(a) *Z_{cN} has n_N $\mathcal{C}_+$ poles counted with multiplicities*

(b) *The fraction (4.13) of $\hat{Z}(Z_{cN})$ is bicoprime*

(c) *Denoting by $\mathcal{S}_{\hat{Z}(Z_{cN})}$ the set of decentralized blocking zeros of $\hat{Z}(Z_{cN})$ and letting $\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{\bar{t}}$ denote the elements of*

$$\Psi_L(\hat{Z}) := \mathcal{S}_{\hat{Z}(Z_{cN})} \cap \mathcal{R}_{+e}, \tag{4.18}$$

arranged in the ascending order and denoting by $\bar{\eta}_i$ the number of poles of $\hat{Z}(Z_{cN})$ counted with multiplicities in the interval $(\bar{\sigma}_i, \bar{\sigma}_{i+1})$, $i \in \{1, 2, \ldots, \bar{t}-1\}$, it holds that $\bar{\eta} = \eta - n_N$ where $\bar{\eta}$ is the number of odd integers in the sequence $\bar{\eta}_1, \ldots,$

$\bar{\eta}_{\bar{l}-1}$.

(d) (If $L = N - 1 > 1$) *DSP for* $\hat{Z}(Z_{cN})$ *is solvable,* $\hat{Z}(Z_{cN})$ *is strongly connected and satisfies*

$$rank\hat{Z}_{ij} \geq 2 \text{ or } rank\hat{Z}_{ji} \geq 2 \;\; \forall i, j \in \mathbf{L},\, i \neq j$$

where $\hat{Z}_{ij} \in \mathbf{P}^{p_i \times r_j}$ *denotes the* i, j*th submatrix of* $\hat{Z}(Z_{cN})$.

Proof.

The following facts are used in the proof of Lemma (4.8).

Fact (4.3). *Let* $S_1 \in \mathbf{S}^{p\times r}$, $S_2 \in \mathbf{S}^{p\times n}$ *and* $S_3 \in \mathbf{S}^{m\times r}$ *be such that either* $rank\, S_2 \geq 2$ *or* $rank\, S_3 \geq 2$. *Then, there exists an open and dense subset* $\mathcal{X}$ *of* $\mathbf{S}^{n\times m}$ *such that, for any fixed but otherwise arbitrary* $X \in \mathcal{X}$

$$(S_1 + S_2XS_3)(z) = 0$$
$$\Longrightarrow$$
$$\begin{bmatrix} S_1 & S_2 \end{bmatrix}(z) = 0 \text{ or } \begin{bmatrix} S_1' & S_3' \end{bmatrix}'(z) = 0,$$

for all $z \in \mathcal{C}_{+e}$.

Fact (4.4). *Let* $T_1 \in \mathbf{P}^{p\times r}$, $T_2 \in \mathbf{P}^{p\times n}$ *and* $T_3 \in \mathbf{P}^{m\times r}$ *be such that either* $rank\, T_2 \geq 2$ *or* $rank\, T_3 \geq 2$. *Then, there exists an open and dense subset* $\mathcal{X}$ *of* $\mathbf{S}^{n\times m}$ *such that, for any fixed but otherwise arbitrary* $X \in \mathcal{X}$

$$(T_1 + T_2XT_3)(z) = 0$$
$$\Longrightarrow$$
$$\begin{bmatrix} T_1 & T_2 \end{bmatrix}(z) = 0 \text{ or } \begin{bmatrix} T_1' & T_3' \end{bmatrix}'(z) = 0,$$

for all $z \in \mathcal{C}_{+e}$.

Fact (4.5). *Let* $Z_1 \in \mathbf{R}^{p\times r}$, $Z_2 \in \mathbf{R}^{p\times n}$ *and* $Z_3 \in \mathbf{R}^{m\times r}$ *be such that either* $rank\, Z_2 \geq 2$ *or* $rank\, Z_3 \geq 2$. *Also let* $K_1 \in \mathbf{S}^{n\times m}$ *and* $K_2 \in \mathbf{S}^{n\times n}$ *be such that* K_2 *is biproper. Define* $\mathbf{K} = \{z \in \mathcal{C}_+ |\; det(K_2)(z) = 0\}$. *Then, there exists an open and dense subset* $\hat{\mathcal{X}}$ *of* $\mathbf{S}^{n\times m}$ *such that for any fixed but otherwise arbitrary* $X \in \hat{\mathcal{X}}$

$$(Z_1 + Z_2(K_1 + K_2X)Z_3)(z) = 0$$
$$\Longrightarrow$$
$$\begin{bmatrix} Z_1 & Z_2 \end{bmatrix}(z) = 0 \text{ or } [Z_1' \; Z_3']'(z) = 0,$$

for all $z \in C_{+e} - \mathrm{K}$.

Proof of Fact (4.3).

First consider the following statement and its proof.

Let $\tilde{A} \in \mathbf{S}^{\tilde{p}\times\tilde{r}}$, $\tilde{B} \in \mathbf{S}^{\tilde{p}\times\tilde{n}}$ *and* $\tilde{C} \in \mathbf{S}^{\tilde{m}\times\tilde{r}}$ *be such that the smallest invariant factor (sif) of* $[\tilde{A}\ \tilde{B}]$ *and the sif of* $[\tilde{A}'\ \tilde{C}']'$ *are units, and either* $rank\tilde{B} \geq 2$ *or* $rank\tilde{C} \geq 2$. *Then, for almost all* $X \in \mathbf{S}^{\tilde{n}\times\tilde{m}}$, $sif(\tilde{A} + \tilde{B}X\tilde{C})$ *is unit.*

We can assume neither $\tilde{B}$ nor $\tilde{C}$ equals zero, because otherwise $sif\tilde{A}$ is unit, and the statement holds trivially. If X_0 is such that $sif(\tilde{A} + \tilde{B}X_0\tilde{C})$ is unit, by choosing the norm of Δ small enough, $sif(\tilde{A} + \tilde{B}(X_0 + \Delta)\tilde{C})$ is still a unit, since the set of units are open in $\mathbf{S}$. To show that the class of such X is dense, assume X_0 is such that $sif(\tilde{A} + \tilde{B}X_0\tilde{C})$ is not a unit of $\mathbf{S}$. Let U_l and U_r be unimodular matrices of suitable size, such that

$$U_l\tilde{B} = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}, \text{ and } \tilde{C}U_r = \begin{bmatrix} \hat{C} & 0 \end{bmatrix},$$

where $\hat{B} \in \mathbf{S}^{\hat{p}\times\tilde{n}}$ and full row rank, and $\hat{C} \in \mathbf{S}^{\tilde{m}\times\hat{r}}$ and full column rank. By assumption either $rank\tilde{B} \geq 2$ or $rank\tilde{C} \geq 2$. We assume $rank\tilde{B} \geq 2$. Otherwise $rank\tilde{C} \geq 2$ and the dual of the proof below follows. Clearly, $rank\hat{B} \geq 2$ implies $\hat{p} \geq 2$. Let b and c be the smallest invariant factors of $\hat{B}$ and $\hat{C}$ respectively. Define $B_l = \hat{B}/b$ and $C_r = \hat{C}/c$. There exist unimodular matrices V_l and V_r such that $V_lB_l = B$, $C_rV_r = C$, where the first row of B is left unimodular and the first column of C is right unimodular. Further define

$$\bar{A} = \begin{bmatrix} V_l & 0 \\ 0 & I \end{bmatrix} U_l\tilde{A}U_r \begin{bmatrix} V_r & 0 \\ 0 & I \end{bmatrix}.$$

Partition $\bar{A}$ as follows:

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix},$$

where $\bar{A}_{11} \in \mathbf{S}^{\hat{p}\times\hat{r}}$, $\bar{A}_{12} \in \mathbf{S}^{\hat{p}\times\tilde{r}-\hat{r}}$, $\bar{A}_{21} \in \mathbf{S}^{\tilde{p}-\hat{p}\times\hat{r}}$ and $\bar{A}_{22} \in \mathbf{S}^{\tilde{p}-\hat{p}\times\tilde{r}-\hat{r}}$. Clearly, $sif(\tilde{A} + \tilde{B}(X_0 + \Delta)\tilde{C})$ equals the sif of

$$\bar{A} + bc\begin{bmatrix} B \\ 0 \end{bmatrix}(X_0 + \Delta)\begin{bmatrix} C & 0 \end{bmatrix},$$

for any $\Delta \in \mathbf{S}^{\tilde{n}\times\tilde{m}}$. Define $A = \tilde{A}_{11} + bcBX_0C$. Let us assume, without loss of generality that the first column of A is nonzero, because otherwise there exists a perturbation Δ_1 on X_0 with arbitrarily small norm such that the first column of A in nonzero with X_0 is replaced by $X_0 + \Delta_1$ (This is guaranteed by the fact that B and the first column of C are nonzero). Also note that for any $z \in C_{+e}$, $(bc) = 0$ and $\tilde{A}_{11} = 0$ imply

$$\begin{bmatrix} 0 & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \neq 0,$$

because of the hypothesis that $sif[\tilde{A}\ \tilde{B}]$ and $sif[\tilde{A}'\ \tilde{C}']'$ are units. Let $\sum_{i=1}^{\tilde{n}} \beta_i b_{1i} = 1$, and $\sum_{i=1}^{\tilde{m}} c_{i1}\alpha_i = 1$, for some α_i, $i = 1, ..., \tilde{m}$ and β_i, $i = 1, ..., \tilde{n}$, where b_{1i}, $i = 1, ..., \tilde{n}$ denote the first row elements of B, and c_{i1}, $i = 1, ..., \tilde{m}$ denote the first column elements of C. Define $\theta_j = \sum_{i=1}^{\tilde{n}} \beta_i b_{ji}$, $j = 1, ..., \hat{p}$, and $\gamma_j = \sum_{i=1}^{\tilde{m}} c_{ij}\alpha_i$, $j = 1, ..., \hat{r}$, where b_{ji} denotes the (j,i)'th element of B and c_{ij} denotes the (i,j)'th element of C. Note that, $\theta_1 = 1$ and $\gamma_1 = 1$. By the fact that $\hat{p} \geq 2$, $\gamma_1 = 1$ and the first column of A is nonzero, we can assume that for at least one index pair (i,j), $a_{ij} \neq \theta_i\gamma_j a_{11}$. (We omit the simple proof of the construction of such β_i and α_j, $i = 1, ..., \tilde{n}$, and $j = 1, ..., \tilde{m}$.) Now let Σ be a nonempty set of index pairs so that $\Sigma = \{(i_1, j_1), (i_2, j_2), ..., (i_v, j_v)\}$ where $v = min(\hat{p}, \hat{r})$, satisfying $a_{ij} \neq \theta_i\gamma_j a_{11}$, whenever $(i,j) \in \Sigma$, and $a_{ij} = \theta_i\gamma_j a_{11}$, whenever $(i,j) \notin \Sigma$. Define $q_{ij} = gcf(a_{ij}, bc)$, $i = 1, ..., \hat{p}$, $j = 1, ..., \hat{r}$, such that $a_{ij} = q_{ij}\hat{a}_{ij}$ and $bc = q_{ij}\hat{q}_{ij}$, for coprime pairs $(\hat{a}_{ij}, \hat{q}_{ij})$. If $a_{11} = 0$, let δ satisfy (δ, a_{ij}) are coprime for all $(i,j) \in \Sigma$. If $a_{11} \neq 0$, let δ satisfy $(\hat{a}_{11} + \delta\hat{q}_{11}, a_{ij} - \theta_i\gamma_j a_{11})$ are coprime for all $(i,j) \in \Sigma$. The norm of δ can be chosen arbitrarily small in both cases. By letting

$$\Delta = \begin{bmatrix} \beta_1\delta\alpha_1 & \dots & \beta_1\delta\alpha_{\tilde{m}} \\ \vdots & & \vdots \\ \beta_{\tilde{n}}\delta\alpha_1 & & \beta_{\tilde{n}}\delta\alpha_{\tilde{m}} \end{bmatrix},$$

we have $(A + bcB\Delta C)_{ij} = a_{ij} + \delta bc\theta_i\gamma_j$, $i = 1, ..., \hat{p}$, $j = 1, ..., \hat{r}$. If $a_{11} = 0$, the choice of Δ yields

$$gcf_{i=1,...,\hat{p},\, j=1,...,\hat{r}}[(A + bcB\Delta C)_{ij}] = gcf_{(i,j)\in\Sigma}(a_{ij}, bc).$$

In the case that $a_{11} \neq 0$, the choice of Δ yields

$$gcf_{i=1,\ldots,\hat{p},j=1,\ldots,\hat{r}}[(A + bcB\Delta C)_{ij}] = gcf_{(i,j)\in\Sigma}(a_{ij}, q_{11}).$$

(This latter statement can be seen more clearly as follows. Observe that whenever $(i,j) \notin \Sigma$, $gcf[(A + B\Delta Cbc)_{11}, (A + B\Delta Cbc)_{ij}] = a_{11} + bc\delta$. So,

$$gcf_{i=1,\ldots,\hat{p},j=1,\ldots,\hat{r}}[(A + bcB\Delta C)_{ij}] = gcf(a_{11} + bc\delta, gcf_{(i,j)\in\Sigma}(A + bcB\Delta C)_{ij}).$$

$(\hat{a}_{11} + \delta\hat{q}_{11}) = 0$, $z \in C_{+e}$ imply $\delta(z) = -\hat{a}_{11}/\hat{q}_{11}$, where $\hat{q}_{11}$ is nonzero because of the coprimeness of $(\hat{a}_{11}, \hat{q}_{11})$. In this case $(a_{ij} + bc\delta\theta_i\gamma_j) = (a_{ij} - \theta_i\gamma_j a_{11}) \neq 0$. Hence

$$\begin{aligned} gcf(a_{11} + bc\delta, gcf_{(i,j)\in\Sigma}(A + B\Delta Cbc)_{ij}) &= \\ gcf(q_{11}, gcf_{(i,j)\in\Sigma}(A + B\Delta Cbc)_{ij}) &= \quad gcf_{(i,j)\in\Sigma}(q_{11}, a_{ij}).) \end{aligned}$$

In both cases $sif(A + B\Delta Cbc)$ is coprime with

$$sif \begin{bmatrix} 0 & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}.$$

Since the norm of Δ can be made smaller than any prespecified positive number by choosing δ suitably, the proof of the statement is completed.

Now let $\alpha := sif([S_1 \;\; S_2])$ such that $S_1 = \bar{A}\alpha$ and $S_2 = \tilde{B}\alpha$ for some matrices $\bar{A}$ and $\tilde{B}$ over $\mathbf{S}$. Also let $\beta := sif([\bar{A}' \;\; S_3']'$ such that $\bar{A} = \check{A}\beta$ and $S_3 = \tilde{C}\beta$. It holds that

$$S_1 + S_2XS_3 = \alpha\beta(\check{A} + \check{B}X\tilde{C})$$

for every X. Applying the above statement one has

$$\begin{gathered} (S_1 + S_2XS_3)(z) = 0 \\ \Longrightarrow \\ \alpha(z) = 0 \; or \; \beta(z) = 0 \end{gathered}$$

for all $z \in C_{+e}$. This completes the proof.Δ

Proof of Fact (4.4). Define α_i: *least common multiple of the denominator polynomials of* T_i, $i \in \mathbf{3}$. Let d_i denote the degree of α_i. We define $S_i = T_i\alpha_i/(z +$

$1)^{d_i}$; $i \in \mathbf{3}$, which are matrices on $\mathbf{S}$. From Fact (4.3) there exists an open and dense subset $\mathcal{X}_1$ of $\mathbf{S}^{n\times m}$ such that for any fixed but otherwise arbitrary $X \in \mathcal{X}_1$

$$\begin{gathered}(\tfrac{\alpha_2 \cdot \alpha_3}{(z+1)^{d_2+d_3}} S_1 + \tfrac{\alpha_1}{(z+1)^{d_1}} S_2 X S_3)(z) = 0 \\ \Longrightarrow \\ \left[\tfrac{\alpha_2 \cdot \alpha_3}{(z+1)^{d_2+d_3}} S_1 \quad \tfrac{\alpha_1}{(z+1)^{d_1}} S_2 \right](z) = 0 \ or \ [\tfrac{\alpha_2 \cdot \alpha_3}{(z+1)^{d_2+d_3}} S_1' \quad S_3']'(z) = 0,\end{gathered} \tag{4.19}$$

for all $z \in C_{+e}$. Now define

$$\mathcal{T}_1 = \{z \in C_{+e} | [T_1 \ T_2](z) = 0 \ or \ [T_1' \ T_3']'(z) = 0\}.$$

$$\mathcal{T}_2 = \{C_{+e} \ poles \ of \ T_1\} \cup \{C_{+e} \ poles \ of \ T_2\} \cup \{C_{+e} \ poles \ of \ T_3\}.$$

It can be easily shown that the set of X for which the set of C_{+e}-blocking zeros of $T_1 + T_2XT_3$ is disjoint from $\mathcal{T}_2 - (\mathcal{T}_2 \cap \mathcal{T}_1)$ is open and dense in $\mathbf{S}^{n\times m}$. We call this set $\mathcal{X}_2$ and let $\mathcal{X} := \mathcal{X}_1 \cap \mathcal{X}_2$, which is open and dense in $\mathbf{S}^{n\times m}$. Fix an arbitrary element X of $\mathcal{X}$. For any $z \in C_{+e} - \mathcal{T}_2$, $\alpha_1(z)$, $\alpha_2(z)$ and $\alpha_3(z)$ are all nonzero, and therefore

$$(T_1 + T_2XT_3)(z) = \frac{(z+1)^{d_1+d_2+d_3}}{\alpha_1 \cdot \alpha_2 \cdot \alpha_3}(\frac{\alpha_2 \cdot \alpha_3}{(z+1)^{d_2+d_3}} S_1 + \frac{\alpha_1}{(z+1)^{d_1}} S_2XS_3)(z) = 0.$$

implies

$$(\frac{\alpha_2 \cdot \alpha_3}{(z+1)^{d_2+d_3}} S_1 + \frac{\alpha_1}{(z+1)^{d_1}} S_2XS_3)(z) = 0,$$

yielding that (4.19), and consequently

$$[T_1 \ T_2](z) = 0 \ or \ [T_1' \ T_2']'(z) = 0 \tag{4.20}$$

hold. On the other hand, if $z \in \mathcal{T}_2$ is such that $(T_1 + T_2XT_3)(z) = 0$, then the construction of $\mathcal{X}_2$ ensures that $z \in \mathcal{T}_1 \cap \mathcal{T}_2$, i.e. (4.20) holds. This completes the proof of Fact (4.4).Δ

Proof of Fact (4.5). Define $T_1 = Z_1 + Z_2K_1Z_3$, $T_2 = Z_2K_2$ and $T_3 = Z_3$. From Fact (4.4), there exists an open and dense subset $\mathcal{X}_1$ of $\mathbf{S}^{n\times m}$ such that for any fixed but otherwise arbitrary $X \in \mathcal{X}_1$

$$(T_1 + T_2XT_3)(z) = 0$$

$$\Longrightarrow$$

$$[T_1\ T_2](z) = 0 \ or \ [T_1'\ T_2']'(z) = 0, \tag{4.21}$$

for all $z \in C_{+\epsilon}$. Define $\mathcal{T} = \{z \in C_{+\epsilon} - \mathbf{K} |\ (4.21)\ holds\ but\ (Z_2K_1Z_3)(z) \neq 0\}$. There also exists an open and dense subset $\mathcal{X}_2 \in \mathbf{S}^{n\times m}$ such that for any fixed but otherwise arbitrary $X \in \mathcal{X}_2$ $(Z_2K_2XZ_3)(z) \neq 0$, for all $z \in \mathcal{T}$. Let $\hat{\mathcal{X}} := \mathcal{X}_1 \cap \mathcal{X}_2$, which is open and dense. Now fix any arbitrary element X of $\hat{\mathcal{X}}$. Let $z_0 \in C_{+\epsilon} - \mathbf{K}$. If $(T_1 + T_2XT_3)(z_0) = 0$ then by the choice of $\mathcal{X}_1$ we have that equation (4.21) holds. We claim that $(Z_2K_1Z_3)(z_0) = 0$. To see this, observe that if $(Z_2K_1Z_3)(z_0) = 0$ then by the choice of X_2 we have $(Z_2K_2XZ_3)(z_0) \neq 0$, which contradicts that $(T_1 + T_2XT_3)(z_0) = 0$. Therefore $(Z_2K_1Z_3)(z_0) = 0$. This implies via (4.21) that

$$[Z_1\ Z_2K_2](z_0) = 0 \ or \ [Z_1'\ Z_3']'(z_0) = 0.$$

Since $K_2(z_0)$ is nonsingular by the definition of $\mathbf{K}$, it holds that

$$[Z_1\ Z_2](z_0) = 0 \ or \ [Z_1'\ Z_3']'(z_0) = 0.$$

Since $z_0 \in C_{+\epsilon} - \mathbf{K}$ is arbitrary, the proof is completed.Δ

The proof of Lemma (4.8) is given below.

Assume that some left and right coprime fractions of Z_{NN} over $\mathbf{S}$ are given by $Z_{NN} = D_l^{-1}N_l = N_rD_r^{-1}$. Let $\Omega_l := gclf(Q, R_N)$, so that $Q = \Omega_l\tilde{Q}$, $R_N = \Omega_l\tilde{R}_N$, for a left coprime pair of matrices $(\tilde{Q}, \tilde{R}_N)$. Also let $\Omega_r := gcrf(\tilde{Q}, P_N)$ so that $\tilde{Q} = \ddot{Q}\Omega_r$, $P_N = \tilde{P}_N\Omega_r$, for a right coprime pair of matrices $(\ddot{Q}, \tilde{P}_N)$. Then, a bicoprime fraction of Z_{NN} over $\mathbf{S}$ is given by $\tilde{P}_N\ddot{Q}^{-1}\tilde{R}_N$. Also note that $det(D_l) = det(D_r) = det(\ddot{Q})$. Determine a biproper $\tilde{Q}_{cN} \in \mathbf{S}^{p_N\times p_N}$ such that
(P1) $det(\tilde{Q}_{cN})$ has n_N C_+ zeros with multiplicities none of which is included in $\Psi \cup \{0\}$
(P2) the number of sign changes of $det(Q).det(\tilde{Q}_{cN})$ in the sequence σ_1, σ_2, ..., σ_t is equal to $\eta - n_N$
(P3) in case Z_{NN} is not identically zero $det(\tilde{Q}_{cN})$ and $sif(N_l)$ are coprime.
(Such a $\tilde{Q}_{cN}$ can always be constructed easily. The simplest form for $\tilde{Q}_{cN}$ is

given by $diag\{g_1(z), ..., g_{p_N}(z)\}$ where $g_i(z)$'s are biproper and $\Pi_{i=1}^{p_N} g_i(z)$ has n_N $\mathcal{R}_+$ zeros with multiplicities which are distributed among the poles of Z_{NN} and the elements of Ψ to satisfy the desired requirements.)

We proceed by the following fact.

Given any $\delta \in \mathcal{R}_+ - \{0\}$ there exists $\Delta \in \mathcal{R}^{p_N \times p_N}$ for which $||\Delta|| < \delta$, $\check{Q}_{cN} + \Delta$ is biproper and (a), (b), (c) below are satisfied for almost all $P_c \in \mathbf{S}^{r_N \times p_N}$

(a) $((\check{Q}_{cN} + \Delta)D_l, P_c)$ is right coprime, (D_r, P_c) is left coprime

(b) Letting $Z_{cN} := P_c(\check{Q}_{cN} + \Delta)^{-1}$ the fraction (4.13) of $\hat{Z}(Z_{cN})$ is bicoprime where Q_{cN} and P_{cN} are replaced by $\check{Q}_{cN} + \Delta$ and P_c, respectively

(c) (If $L > 1$) DSP for $\hat{Z}(Z_{cN})$ is solvable, $\hat{Z}(Z_{cN})$ is strongly connected and satisfies

$$rank \hat{Z}_{ij} \geq 2 \text{ or } rank \hat{Z}_{ji} \geq 2 \quad \forall i, j \in \mathbf{L},\ i \neq j$$

Note that the existence of Δ and the fact that the set of P_c satisfying (a), (b) and (c) is open and dense in $\mathbf{S}^{r_N \times p_N}$ follows from [66, Proposition 7.6.15] and [56, Lemma A.2] for part (a), from [56, Theorem 3.2] for part (b), and from [56, Theorem 4.1, Lemma 4.2] with appropriate modifications for part (c). In each case we utilize the facts that under sufficiently small perturbations on $\check{Q}_{cN}$ the properties P1, P2 in the construction of $\check{Q}_{cN}$ still holds.

We now continue the proof of Lemma (4.8). There exists $\delta \in \mathcal{R}_+ - \{0\}$ such that every $\Delta \in \mathcal{R}^{p_N \times p_N}$ with $||\Delta|| < \delta$ satisfies that $\check{Q}_{cN} + \Delta$ is biproper and the properties P1, P2, P3 still hold with $\check{Q}_{cN}$ replaced by $\check{Q}_{cN} + \Delta$. For that value of δ using the fact above construct a matrix $\Delta \in \mathcal{R}^{p_N \times p_N}$ such that $Q_{cN} := \check{Q}_{cN} + \Delta$ is biproper, the properties P1, P2, P3 hold with $\check{Q}_{cN}$ replaced by Q_{cN} and for some open and dense subset $\mathcal{X}$ of $\mathbf{S}^{r_N \times p_N}$, $P_c \in \mathcal{X}$ implies that (a), (b), (c) of the fact hold.

We will now construct P_{cN} such that $Z_{cN} = P_{cN} Q_{cN}^{-1}$ satisfies (a), (b), (c), (d) of Lemma (4.8).

Let

$$\begin{aligned}
\Omega &:= \{z \in \mathcal{R}_{+e} | (det(\Omega_l).det(\Omega_r))(z) \neq 0\},\\
\mathrm{D} &:= \{z \in \mathcal{R}_{+e} | (det(D_l).det(Q_{cN}))(z) = 0\},\\
\hat{\Psi} &:= \Omega \cap \{\mathrm{D} \cup \bar{\Psi}\},\\
\hat{\Psi}_1 &:= \{z \in \hat{\Psi} | Z_{NN}(z) = 0\} \ \mathit{and}\\
\hat{\Psi}_2 &:= \hat{\Psi} - \hat{\Psi}_1
\end{aligned}$$

where $\bar{\Psi} = \tilde{\Psi} \cap \mathcal{R}_{+e}$ (see Lemma (4.6)). Note that Ω is the set of extended real numbers excluding the input-output decoupling zeros of (P_N, Q, R_N) and $\mathbf{D}$ is the union of the sets of unstable real poles of Z_{NN} and the unstable real zeros of $det(Q_{cN})$. Since, via P3, $det(Q_{cN})$ and $sif(N_l)$ are coprime, $z \in \hat{\Psi}_1$ implies that $z \in \bar{\Psi}$. From Lemma (4.6) we have $\hat{\Psi}_1 \subset \Psi$. On the other hand, from Lemma (4.6) $\Psi \subset \bar{\Psi}$. From Lemma (4.4) $\Psi \subset \mathbf{\Omega}$ and therefore $\hat{\Psi}_1 = \Psi$.

Note that for any $z \in \hat{\Psi}_2$, $N_l(z)$ is nonzero. Let $\gamma_1, \gamma_2, ..., \gamma_l$ denote the elements of $\hat{\Psi}$ in the ascending order. From the proof of Theorem 2.2 in [57] given any $z \in \mathcal{R}_{+e}$ for which $N_l(z) \neq 0$ and $(det(\Omega_l).det(\Omega_r))(z) \neq 0$, we can find $X \in \mathcal{R}^{r_N \times p_N}$ such that $(det(\Omega_l).det(\Omega_r).\ det(D_lQ_{cN} + N_lX))(z)$ is nonzero and has any desired sign. For each i where $\gamma_i \in \hat{\Psi}_2$, let X_i be such that $(det(\Omega_l).det(\Omega_r).\ det(D_lQ_{cN} + N_lX_i))(\gamma_i)$ is nonzero and has the same sign with $det(QQ_{cN})(\alpha_i)$ where $\alpha_i = \infty$ if $\Psi = \emptyset$, and

$$\alpha_i = \begin{cases} \textit{minimum of the all elements of } \Psi \textit{ which are greater than } \gamma_i, & \textit{if there exists an element of } \Psi \textit{ which is greater than } \gamma_i \\ \textit{maximum of the all elements of } \Psi \textit{ which are less than } \gamma_i, & \textit{otherwise} \end{cases}$$

if Ψ is nonempty. Construct $\tilde{P}_{cN} \in \mathbf{S}^{r_N \times p_N}$ using standard interpolation techniques such that $\tilde{P}_{cN}(\gamma_i) = X_i$ for all $\gamma_i \in \hat{\Psi}_2$. This ensures that $det(\Omega_l).det(\Omega_r).\ det(D_lQ_{cN} + N_l\tilde{P}_{cN})$ takes nonzero values with appropriate signs in the sequence $\gamma_1, \gamma_2, ..., \gamma_l$ such that the number of sign changes of $det(\Omega_l).det(\Omega_r).det(D_lQ_{cN} + N_l\tilde{P}_{cN})$ in this sequence is equal no $\eta - n_N$. Since sufficiently small perturbations on $\tilde{P}_{cN}$ do not deteriorate the above property, we can assume that $\tilde{P}_{cN} \in \mathcal{X}$, since $\mathcal{X}$ is an open and dense subset of $\mathbf{S}^{r_N \times p_N}$. We will now construct Δ_c such that letting $P_{cN} := \tilde{P}_{cN} + \Delta_c$ the compensator $Z_{cN} = P_{cN}Q_{cN}^{-1}$ satisfies that the set $\Psi_L(\hat{Z})$ defined in (4.18) is contained in $\hat{\Psi}$. The norm of Δ_c will be chosen to be sufficiently small so that $P_{cN} \in \mathcal{X}$ (the properties (a), (b), (c) of Lemma (4.8) still hold where P_c is replaced by P_{cN}) and the number of sign changes of $det(\Omega_l).det(\Omega_r).det(D_lQ_{cN}+N_lP_{cN})$ in the sequence $\gamma_1, \gamma_2, ..., \gamma_l$ is equal to $\eta - n_N$. Let $\bar{\gamma}_1, ..., \bar{\gamma}_{\bar{l}}$ be the elements of $\Psi_L(\hat{Z})$ in the ascending order. Since $\Psi_L(\hat{Z}) \subset \hat{\Psi}$, the number of sign changes of $det(\Omega_l).det(\Omega_r).det(D_lQ_{cN} + N_l\tilde{P}_{cN})$ in $\bar{\gamma}_1, ..., \bar{\gamma}_{\bar{l}}$ is less than or equal to $\eta - n_N$. On the other hand, by Lemma (4.7) $\Psi \subset \Psi_L(\hat{Z})$. Therefore the number of sign changes of $det(\Omega_l).det(\Omega_r).det(D_lQ_{cN} + N_lP_{cN})$ in this sequence is no less than $\eta - n_N$. Hence, we conclude that the number of sign changes of $det(\Omega_l).det(\Omega_r).det(D_lQ_{cN}+N_lP_{cN})$ in the sequence $\bar{\gamma}_1, ..., \bar{\gamma}_{\bar{l}}$ is equal to $\eta - n_N$. Then, the fact that Q_{cN} has n_N C_+ poles with multiplicities implies (a) of Lemma (4.8). Statements (b) and (d) of Lemma (4.8) are implied by (b) and (c) of the fact. Finally, statement (c) of Lemma (4.8) is implied by the underlined statement above and the fact that every unstable zero of $det(\Omega_l).det(\Omega_r).det(D_lQ_{cN} + N_lP_{cN})$ is an unstable pole of $\hat{Z}(Z_{cN})$ with the same multiplicity and vice versa.

The perturbation matrix Δ_c will now be constructed. Define $Q_{cN}^{-1}N_r = \hat{N}_r\hat{Q}_{cN}^{-1}$ for a right coprime pair of matrices $(\hat{Q}_{cN}, \hat{N}_r)$. (Note that if $N_r = 0$ then $\hat{Q}_{cN}$ is unimodular.) Let $T := \tilde{P}_{cN}Q_{cN}^{-1}(I + Z_{NN}\tilde{P}_{cN}Q_{cN}^{-1})^{-1}$ and let $T_1^{-1}T_2 = T$ be a left coprime fraction of T over $\mathbf{S}$. It holds that $\tilde{P}_{cN} = D_r(T_1D_r - T_2N_r)^{-1}T_2Q_{cN}$. Since $(D_r, \tilde{P}_{cN})$ is left coprime, $(T_1D_r - T_2N_r)D_r^{-1}$ is over $\mathbf{S}$, i.e., $T_2 = \tilde{T}_2D_l$ for some matrix $\tilde{T}_2$ over $\mathbf{S}$. Let $T_1^{-1}\tilde{T}_2 = \hat{T}_2\hat{T}_1^{-1}$, for a right coprime pair of matrices $(\hat{T}_2, \hat{T}_1)$. It follows that $\tilde{P}_{cN} = \hat{T}_2(\hat{T}_1 - N_l\hat{T}_2)^{-1}D_lQ_{cN}$. By the left coprimeness of $(\hat{T}_1 - N_l\hat{T}_2, \hat{T}_2)$ and by the right coprimeness of $(Q_{cN}D_l, \tilde{P}_{cN})$ it easily follows that $D_lQ_{cN} = (\hat{T}_1 - N_l\hat{T}_2)V$ for some unimodular V over $\mathbf{S}$ and $\tilde{P}_{cN} = \hat{T}_2V$. Observe that if $N_r \neq 0$ for any $\Delta \in \mathbf{S}^{r_N \times p_N}$ satisfying $||\Delta|| < 1/||VN_r||$, $V^{-1} - N_r\Delta$ is unimodular. Let $\{i_1, i_2, ..., i_L\}$ be a fixed permutation of the elements in $\mathbf{L}$. For a fixed $j \in \mathbf{L}$ define

$$Z_1 = \begin{bmatrix} Z_{i_ji_1} & & Z_{i_ji_j} \\ \vdots & & \vdots \\ Z_{i_Li_1} & \cdots & Z_{i_Li_j} \end{bmatrix}, \; Z_2 = \begin{bmatrix} Z_{i_jN} \\ \vdots \\ Z_{i_LN} \end{bmatrix}, \; Z_3 = \hat{T}_1^{-1}D_l[Z_{Ni_1} \ldots Z_{Ni_j}],$$

$K_1 = \hat{T}_2$ and $K_2 = D_r\hat{Q}_{cN}$. From Fact (4.5) and the connectivity assumption **(A2)**, there exists an open and dense subset of $\mathbf{S}^{r_N \times p_N}$ such that for any fixed but otherwise arbitrary Δ in this set

$$\begin{bmatrix} Z_{i_j i_1} & Z_{i_j i_j} \\ \vdots & \vdots \\ Z_{i_L i_1} & Z_{i_L i_j} \end{bmatrix} - \begin{bmatrix} Z_{i_j N} \\ \vdots \\ Z_{i_L N} \end{bmatrix} (\hat{T}_2 + D_r\hat{Q}_{cN}\Delta)\hat{T}_1^{-1}D_l[Z_{Ni_1} \; \dots \; Z_{Ni_j}])(z) = 0$$

$$\Longrightarrow$$

$$\begin{bmatrix} Z_{i_j i_1} & Z_{i_j i_j} & Z_{i_j N} \\ \vdots & \vdots & \vdots \\ Z_{i_L i_1} & Z_{i_L i_j} & Z_{i_L N} \end{bmatrix} (z) = 0 \; or \begin{bmatrix} Z_{i_j i_1} & Z_{i_j i_j} \\ \vdots & \vdots \\ Z_{i_L i_1} & Z_{i_L i_j} \\ Z_{N i_1} & Z_{N i_j} \end{bmatrix} (z) = 0, \; \forall z \in C_{+e} - \mathbf{D}. \tag{4.22}$$

Since the union of open and dense subsets is open and dense, repeating the above argument we conclude that there exists an open and dense subset of $\mathbf{S}^{r_N \times p_N}$ such that for every Δ in this set the implication (4.22) holds for all $j \in \mathbf{L}$. Repeating for all permutations of $\mathbf{L}$ and taking the union of open and dense subsets we can construct an open and dense subset $\tilde{\mathcal{X}}$ of $\mathbf{S}^{r_N \times p_N}$ such that for any $\Delta \in \tilde{\mathcal{X}}$ the implication in (4.22) holds for all $j \in \mathbf{L}$ and for all permutations of $\mathbf{L}$, represented by $\{i_1, ..., i_L\}$. Now choose $\tilde{\Delta} \in \tilde{\mathcal{X}}$ with sufficiently small norm such that $(V^{-1} - \hat{N}_r\tilde{\Delta})$ is unimodular, and the norm of $\Delta_c := -\hat{T}_2 V + (\hat{T}_2 + D_r\hat{Q}_{cN}\tilde{\Delta})(V^{-1} - \hat{N}_r\tilde{\Delta})^{-1}$ is sufficiently small to ensure that $P_{cN} := \tilde{P}_{cN} + \Delta_c = \hat{T}_2 V + \Delta_c \in \mathcal{X}$ and the number of sign changes of $det(\Omega_l).det(\Omega_r).det(D_l Q_{cN} + N_l P_{cN})$ in the sequence $\gamma_1, \gamma_2, ..., \gamma_l$ is equal to $\eta - n_N$. Then,

$$\begin{aligned} \hat{Z}(Z_{cN}) &= \begin{bmatrix} Z_{11} & Z_{1L} \\ \vdots & \vdots \\ Z_{L1} & Z_{LL} \end{bmatrix} - \begin{bmatrix} Z_{1N} \\ \vdots \\ Z_{LN} \end{bmatrix} P_{cN}Q_{cN}^{-1}(I + Z_{NN}P_{cN}Q_{cN}^{-1})^{-1}[Z_{N1} \; \dots \; Z_{NL}] \\ &= \begin{bmatrix} Z_{11} & Z_{1L} \\ \vdots & \vdots \\ Z_{L1} & Z_{LL} \end{bmatrix} - \begin{bmatrix} Z_{1N} \\ \vdots \\ Z_{LN} \end{bmatrix} (\hat{T}_2 + D_r\hat{Q}_{cN}\tilde{\Delta})\hat{T}_1^{-1}D_l[Z_{N1} \; \dots \; Z_{NL}]. \end{aligned}$$

(This can be proved as follows:

$$\begin{aligned}
& D_l Q_{cN} = (\hat{T}_1 - N_l \hat{T}_2)V \\
\Longrightarrow\quad & \hat{T}_1 = D_l Q_{cN}(V^{-1} - \hat{N}_r \hat{\Delta}) + N_l(\hat{T}_2 + D_r \hat{Q}_{cN} \hat{\Delta}) \\
\Longrightarrow\quad & (\hat{T}_2 + D_r \hat{Q}_{cN} \hat{\Delta})(V^{-1} - \hat{N}_r \hat{\Delta})^{-1} = (\hat{T}_2 + D_r \hat{Q}_{cN} \hat{\Delta}). \\
& \qquad\qquad (\hat{T}_1^{-1} D_l Q_{cN} + \hat{T}_1^{-1} N_l(\hat{T}_2 + D_r \hat{Q}_{cN} \Delta) \\
& \qquad\qquad\qquad (V^{-1} - \hat{N}_r \tilde{\Delta})^{-1})\ , \\
\Longrightarrow\quad & P_{cN} = (\hat{T}_2 + D_r \hat{Q}_{cN} \hat{\Delta})\hat{T}_1^{-1}(D_l Q_{cN} + N_l(\hat{T}_2 + D_r \hat{Q}_{cN} \hat{\Delta})(V^{-1} - \hat{N}_r \tilde{\Delta})^{-1}) \\
\Longrightarrow\quad & P_{cN} = (\hat{T}_2 + D_r \hat{Q}_{cN} \hat{\Delta})\hat{T}_1^{-1}(D_l Q_{cN} + N_l P_{cN}) \\
\Longrightarrow\quad & P_{cN} = (\hat{T}_2 + D_r \hat{Q}_{cN} \hat{\Delta})\hat{T}_1^{-1} D_l (I + Z_{22} P_{cN} Q_{cN}^{-1}) Q_{cN} \\
\Longrightarrow\quad & P_{cN} Q_{cN}^{-1}(I + Z_{NN} P_{cN} Q_{cN}^{-1})^{-1} = (\hat{T}_2 + D_r \hat{Q}_{cN} \hat{\Delta})\hat{T}_1^{-1} D_l
\end{aligned}$$

implying the equality above.)

Now observe by Lemma (4.4) that $\Psi_L(\hat{Z})$ is disjoint from the poles of $\hat{Z}(Z_{cN})$. Since the C_+ input decoupling zeros and the output decoupling zeros of (4.6) are included among the C_+ poles of $\hat{Z}(Z_{cN})$, it follows that $\Psi_L(\hat{Z}) \subset \mathbf{\Omega}$. By the equation (4.22) and the above discussion, it holds that $\Psi_L(\hat{Z}) \subset \hat{\Psi}$. This completes the proof.□

4.3 Least Number of Unstable Controller Poles

In this section we consider the synthesis of decentralized stabilizing controllers with minimum number of unstable poles. As a particular case, we obtain the solution of decentralized strong stabilization problem. In terms of the notation of Section 4.2, a more precise definition of decentralized strong stabilization problem can be given as follows.

Decentralized Strong Stabilization Problem (DSSP). *Let $Z = [Z_{ij}]$, $Z_{ij} \in \mathbf{P}^{p_i \times r_j}$, $i, j \in \mathbf{N}$ be the transfer matrix of a given plant. Determine stable local controllers $Z_{ci} \in \mathbf{S}^{r_i \times p_i}$, $i \in \mathbf{N}$ such that the pair $(Z, diag\{Z_{c1}, ..., Z_{cN}\})$ is stable.*

We assume throughout this section that

(A1) Z is strongly connected, and **(A2)** *rank* $Z_{ij} \geq 2$ *or rank* $Z_{ji} \geq 2$, $\forall i, j \in$

N, $i \neq j$

hold. The assumption **(A1)** is introduced since the construction of decentralized stabilizing compensators is more straightforward under this assumption. If the assumption **(A1)** fails, then Z can be decomposed into its strongly connected components and DSP can be considered for each strongly connected subsystem independently ([10], [22, Chapter 4], Lemma (3.10), Theorem (3.3)). For the problem of synthesizing a least unstable decentralized stabilizing controller and for DSSP, the case where **(A1)** fails can be handled similarly (see Remark (4.1) below). The assumption **(A2)** is made because of technical reasons. It allows us to carry out various genericity arguments in the synthesis of local controllers. It does exclude some important cases such as a two (scalar) input/output plant. (However, see Remark (4.2) below.)

We can now state the main result.

Theorem (4.2). *Let $Z = [Z_{ij}]$ be free of C_{+e} decentralized fixed modes.* (i) *Every decentralized stabilizing controller $Z_c = diag\{Z_{c1}, \ldots, Z_{cN}\}$, $Z_{ci} \in \mathbf{P}^{r_i \times p_i}$, $i \in$ N for Z has at least η poles in C_+ with multiplicities.* (ii) *Given any nonnegative integers n_i, $i \in$ N where $\sum_{i=1}^{N} n_i - \eta$ is a nonnegative and even number, there exists a decentralized stabilizing controller $Z_c = diag\{Z_{c1}, \ldots, Z_{cN}\}$, $Z_{ci} \in \mathbf{P}^{r_i \times p_i}$, $i \in$ N for Z where Z_{ci} has exactly n_i poles in C_+ with multiplicities, $i \in$ N.*

Proof. Let a bicoprime fraction of Z over $\mathbf{S}$ be given by $Z = [P_1' \; P_2' \; \ldots \; P_N']'$ Q^{-1} $[R_1 \; R_2 \; \ldots \; R_N]$, where $Q \in \mathbf{S}^{q \times q}$, $R_i \in \mathbf{S}^{q \times r_i}$ and $P_i \in \mathbf{S}^{p_i \times q}$, $i \in$ N.

(i) The proof will be given by induction. Let $N = 2$ and note that

$$\Psi = \{z \in \mathcal{R}_{+e} | [Z_{11}' \; Z_{21}']'(z) = 0 \text{ and } Z_{22}(z) = 0\} \cup \{z \in \mathcal{R}_{+e} | [Z_{11} \; Z_{12}](z) = 0, \text{ and } Z_{22}(z) = 0\}.$$

If $z \in \Psi$ satisfies $[Z_{11}' \; Z_{21}']'(z) = 0$, then applying Lemma (4.1) with $Z :=$ $[Z_{11}' \; Z_{21}']'$, $P := [P_1' \; P_2']'$, and $R := R_1$ we have

$$rank \begin{bmatrix} Q & R_1 \\ -P_1 & 0 \\ -P_2 & 0 \end{bmatrix} (z) = q, \tag{4.23}$$

where strict equality holds by the fact that (Q, P_1, P_2) is right coprime. If $z \in \Psi$ satisfies $[Z_{11}\ Z_{12}](z) = 0$, then applying Lemma (4.1) with $Z := [Z_{11}\ Z_{12}]$, $P := P_1$, and $R := [R_1\ R_2]$ we have

$$rank \begin{bmatrix} Q & R_1 & R_2 \\ -P_1 & 0 & 0 \end{bmatrix} (z) = q, \tag{4.24}$$

where the strict equality holds since (Q, R_1, R_2) is left coprime.

Let $Z_{ci} \in \mathbf{P}^{r_i \times p_i}$, $i = 1, 2$ be the transfer matrices of some compensators with the number of unstable poles n_1 and n_2, respectively, counted with multiplicities. Also assume that $diag\{Z_{c1}, Z_{c2}\}$ solves DSP for Z. Let $Z_{c2} = P_{c2}Q_{c2}^{-1}$ be a coprime representation over S. Then, Theorem 3.2 of [37] and Theorem (3.1) imply that

$$\hat{Z}(Z_{c2}) := [P_1\ 0] \begin{bmatrix} Q & R_2 P_{c2} \\ -P_2 & Q_{c2} \end{bmatrix}^{-1} \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \tag{4.25}$$

is a bicoprime fraction and $(\hat{Z}(Z_{c2}), Z_{c1})$ is stable. For any $z \in \mathcal{R}_{+e}$ for which (4.23) or (4.24) holds, it is easy to see that

$$rank \begin{bmatrix} Q & R_2 P_{c2} & R_1 \\ -P_2 & Q_{c2} & 0 \\ -P_1 & 0 & 0 \end{bmatrix} (z) = q + p_2.$$

Using the bicoprimeness of the fraction (4.25) and applying Lemma (4.2) to $\hat{Z}(Z_{c2})$, we have that every $z \in \Psi$ is an $\mathcal{R}_{+e}$-blocking zero of $\hat{Z}(Z_{c2})$. From the proof of Theorem 1 in [66] Z_{c1} stabilizes $\hat{Z}(Z_{c2})$ only if the number of sign changes of

$$det(\begin{bmatrix} Q & R_2 P_{c2} \\ -P_2 & Q_{c2} \end{bmatrix}) \tag{4.26}$$

in the sequence σ_1, σ_2, ..., σ_t is not greater than n_1, the number of unstable poles Z_{c1}. (Since each σ_i is an $\mathcal{R}_{+e}$-blocking zero of $\hat{Z}(Z_{c2})$, the determinant in (4.26) is nonzero when evaluated at any σ_i and therefore its sign in the sequence σ_1, σ_2, ..., σ_t is well-defined.) On the other hand, for any $z \in \Psi$ it holds that $Z_{22}(z) = 0$. Therefore, the number of sign changes of the determinant in (4.26) and that of $det(Q).det(Q_{c2})$ in the sequence σ_1, σ_2, ..., σ_t are equal. It follows that the number of sign changes of $det(Q)$ in this sequence equals η (the number of odd integers in the set $\{\eta_1, \eta_2, \ldots, \eta_l\}$). Then, $det(Q).det(Q_{c2})$ has at least $\eta - n_2$ sign changes in the sequence $\sigma_1, \sigma_2, \ldots, \sigma_l$. In other words, for Z_{c1} to stabilize $\hat{Z}(Z_{c2})$ it must hold that $\eta - n_2 \leq n_1$. This establishes the basis of induction for $N = 2$.

Now assume that the statement holds true for L. We establish the statement for $N := L + 1$. Let Z_{ci} with n_i unstable poles for $i \in \mathbf{N}$ solve DSP for Z. Let $Z_{cN} = P_{cN}Q_{cN}^{-1}$ be a right coprime fraction over $\mathbf{S}$ of Z_{cN} and consider $\hat{Z}(Z_{cN})$ and its induced fraction in (4.13). By Theorem (3.2), (4.13) is a bicoprime fraction and DSP for $\hat{Z}(Z_{cN})$ is solvable. Let $\Psi_L(\hat{Z})$, namely the set of real unstable decentralized blocking zeros of $Z(\hat{Z}_{cN})$, be as defined by (4.18). By Lemma (4.7), we have $\Psi \subset \Psi_L(\hat{Z})$ and, by Lemma (4.4), the elements of $\Psi_L(\hat{Z})$ and the poles of $\hat{Z}(Z_{cN})$ are disjoint. Let $\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{\bar{l}}$ denote the elements of $\Psi_L(\hat{Z})$ arranged in the ascending order. Also let $\bar{\eta}_i$ denote the number of poles of $\hat{Z}(Z_{cN})$ counted with multiplicities in the interval $(\bar{\sigma}_i, \bar{\sigma}_{i+1})$, $i \in \{1, 2, \ldots, \bar{l} - 1\}$. (Clearly, every unstable pole of $\hat{Z}(Z_{cN})$ is an unstable zero of

$$det(\begin{bmatrix} Q & R_N P_{cN} \\ -P_N & Q_{cN} \end{bmatrix}) \tag{4.27}$$

with the same multiplicity and vice versa.) By the inductive hypothesis the number of odd integers in the sequence $\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_{\bar{l}-1}$ is less than or equal to $\sum_{i=1}^{L} n_i$. In this case the number of sign changes of the determinant (4.27) in the sequence $\sigma_1, \sigma_2, \ldots, \sigma_l$ is not greater than $\sum_{i=1}^{L} n_i$. Also in this sequence (4.27) and $det(Q).det(Q_{cN})$ takes the same sign as every decentralized blocking zero z of Z satisfies $Z_{NN}(z) = 0$. The number of sign changes of $det(Q).det(Q_{cN})$ in this sequence is no less than $\eta - n_N$, where η is the number of sign changes of $det(Q)$ in $\sigma_1, \sigma_2, \ldots, \sigma_l$, which is precisely the number of odd integers in the set $\{\eta_1, \eta_2, \ldots, \eta_l\}$. That is $\eta - n_N \leq n_1 + n_2 + \ldots + n_L$. Since the number of unstable poles of Z_c is equal to $\sum_{i=1}^{N} n_i$ the proof of the first statement is completed.

(ii) For the proof of the second statement we first consider the simplest case where $\sum_{i=1}^{N} n_i = \eta$. Applying Lemma (4.8) inductively we obtain compensators $Z_{cN}, \ldots, Z_{c2}$ with $n_N, \ldots, n_2$ C_+ poles counted with multiplicities, respectively, such that the following fraction of the closed-loop single channel plant is bicoprime

$$\tilde{Z} := \begin{bmatrix} P_1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} Q & R_N P_{cN} & & R_2 P_{c2} \\ -P_N & Q_{cN} & & 0 \\ \vdots & \vdots & & \\ -P_2 & 0 & & Q_{c2} \end{bmatrix}^{-1} \begin{bmatrix} R_1 \\ 0 \\ \\ 0 \end{bmatrix}$$

and has the following property: If $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, ..., $\tilde{\sigma}_{\tilde{l}}$ denote the $\mathcal{R}_{+e}$ blocking zeros of $\tilde{Z}$ arranged in the ascending order and if $\tilde{\eta}_i$ denotes the number of poles of $\tilde{Z}$ counted with multiplicities in the interval $(\tilde{\sigma}_i, \tilde{\sigma}_{i+1})$, $i \in \{1, 2, ..., \tilde{l} - 1\}$, it holds that $\tilde{\eta} = \eta - \sum_{i=1}^{L} n_i$ where $\tilde{\eta}$ is the number of odd integers in the sequence $\tilde{\eta}_1$, ..., $\tilde{\eta}_{\tilde{l}-1}$. Then, $n_1 - \tilde{\eta} = 0$ and (ii) of Theorem (4.1) implies the existence of Z_{c1} such that Z_{c1} has n_1 C_+ poles counted with multiplicities and $(\tilde{Z}; Z_{c1})$ is stable. Consequently, $diag\{Z_{c1}, ..., Z_{cN}\}$ is a solution to DSP for Z. Moreover the compensator Z_{ci} has n_i C_+ poles counted with multiplicities, $i \in \mathrm{N}$.

The general case where $\sum_{i=1}^{N} n_i - \eta$ is a nonnegative even number is treated similarly, however a modification on Lemma (4.8) is needed. Due to its complex nature, we omit the modified version of Lemma (4.8) and give only a sketch of the proof for the case $N = 2$. The case $N \geq 2$ can be handled similarly.

Let $n_1 + n_2 - \eta$ be a nonnegative real number. A local compensator Z_{c2} around channel 2 can be found such that the induced fraction (4.25) of $\tilde{Z} = \hat{Z}(Z_{c2})$ is bicoprime and Z_{c2} has n_2 poles in C_+ with multiplicities. These poles are allocated in such a way that $\tilde{n}_2$ of them are real whereas the others are nonreal where $\tilde{n}_2 \leq min(\eta, n_2)$ is the maximum integer satisfying $n_2 - \tilde{n}_2$ is an even number. Moreover, if $\tilde{\sigma}_1, ..., \tilde{\sigma}_{\tilde{l}}$ denote the $\mathcal{R}_{+e}$ blocking zeros of $\tilde{Z}$ in the ascending order and if $\tilde{\eta}_i$ denotes the number of poles of $\tilde{Z}$ counted with multiplicities in the interval $(\tilde{\sigma}_i, \tilde{\sigma}_{i+1})$, $i \in \{1, ..., \tilde{l} - 1\}$, it holds that $\tilde{\eta} = \eta - \tilde{n}_2$ where $\tilde{\eta}$ is the number of odd integers in the sequence $\tilde{\eta}_i$, $i = 1, ..., \tilde{l} - 1$. Observe that if $n_2 \leq \eta$ then $\tilde{n}_2 = n_2$, if $n_2 > \eta$ and $n_2 - \eta$ is even then $\tilde{n}_2 = \eta$, and if $n_2 > \eta$ and $n_2 - \eta$ is odd then $\tilde{n}_2 = \eta - 1$. In all cases $n_1 + \tilde{n}_2 - \eta = n_1 - \tilde{\eta}$ is a nonnegative even number as $n_1 + n_2 - \eta$ is even. Applying (ii) of Theorem (4.1) we obtain a compensator Z_{c1} which has n_1 poles in C_+ with multiplicities and $(\hat{Z}(Z_{c2}), Z_{c1})$ is stable. This completes the proof.□

Remark (4.1). If the plant Z is not strongly connected, it can be decomposed into its strongly connected subsystems using known procedures [10]. In this case, for each strongly connected subsystem DSP can be considered independently of the other strongly connected subsystems. Therefore, assuming that DSP for Z is solvable, the synthesis of a decentralized stabilizing controller with minimum number of unstable poles can be achieved by applying the procedure in Theorem (4.2) to the strongly connected subsystems of Z separately.•

Remark (4.2). Note that the connectivity assumptions **(A1)**, **(A2)** are used only in the proof of part (ii). Therefore, part (i) of Theorem (4.2) is valid even in the absence of these assumptions. It is our belief that even part (ii) is valid in the absence of assumption **(A2)** as the notion of decentralized blocking zeros seems to be very natural for those plants where the assumption **(A2)** fails.•

Remark (4.3). On comparing Theorems (4.1) and (4.2), we now conclude that the "least possible" unstable order (McMillan degree) of centralized and decentralized stabilizing controllers are determined, respectively, by the number of odd distributions of $\mathcal{R}_+$ poles among $\mathcal{R}_{+e}$-blocking zeros of Z and among the $\mathcal{R}_{+e}$ decentralized blocking zeros of Z. Since the set of decentralized blocking zeros may be a much larger set than the set of centralized blocking zeros, the least unstable order of a centralized controller is usually much smaller than the least unstable order of a decentralized controller.•

We can now state a solution to DSSP. The result is immediately obtained on noting that $\eta = 0$ is a necessary condition for the solvability of DSSP by part (i) of Theorem (4.2).

Corollary (4.1). *DSSP is solvable if and only if Z is free of unstable decentralized fixed modes and there are an even number of real unstable poles of Z between each pair of zeros in the set Ψ.*

By Remark (4.3), the solvability of DSP together with the strong centralized stabilizability is in general not enough for the solvability of DSSP. This is illustrated by the following example.

Example (4.2). Let a 2×2 transfer matrix be given by

$$Z = \begin{bmatrix} \frac{(z-1)(z-3)}{(z+1)(z-2)(z-4)} & \frac{(z-1)(z-3)}{(z+1)(z-2)(z-4)^2} \\ \\ \frac{1}{(z+1)} & \frac{(z-1)(z-3)}{(z+1)^3} \end{bmatrix}.$$

It is easily checked that [1] Z is free of unstable decentralized fixed modes. We have $\Psi = \{1, 3, \infty\}$. $\eta_1 = 1$ (corresponding to the pole at $z = 2$) and $\eta_2 = 1$ (corresponding to the pole at $z = 4$). Theorem (4.2) and Remark (4.2) yield that Z is *not* decentralized strong stabilizable, and that any decentralized stabilizing controller of Z has at least $\eta_1 + \eta_2 = 2$ unstable poles with multiplicities. On the other hand, since Z has no $\mathcal{R}_{+e}$-blocking zeros except $z = \infty$, it is (centralized) strong stabilizable.Δ

By using various different characterizations of the $\mathcal{R}_{+e}$ decentralized blocking zeros given in Section 4.2, it is possible to obtain many interesting sufficient conditions for the solvability of DSSP. One obvious condition is that Ψ has at most one element since then any set of $\mathcal{R}_+$ poles will have parity interlacing property with Ψ. We state four less obvious conditions below: condition (a) follows by (4.3) and (b) by the definition of $\mathcal{S}_Z$ and by the fact that any symmetric permutation of block rows and columns will include either Z_{ij} or Z_{ji} in its lower triangular for any $i \neq j$. Condition (c) follows by the fact that every decentralized blocking zero of Z is actually a common blocking zero of various complementary transfer matrices. (See Section 4.2.) Conditions (d), (e) are consequences of the conclusion following Lemma (4.5).

Corollary (4.2). *Let $Z = [Z_{ij}]$ be free of $\mathcal{C}_{+e}$ decentralized fixed modes. Each of the following conditions implies the solvability of DSSP for Z:*

(a) *There exist $i \in \mathbf{N}$ for which Z_{ii} has no $\mathcal{R}_+$ decentralized blocking zeros.*

(b) *There exist $i, j \in \mathbf{N}$ with $i \neq j$ for which Z_{ij} and Z_{ji} each has at most one $\mathcal{R}_{+e}$ decentralized blocking zero.*

(c) *Every complementary transfer matrix of Z is free of $\mathcal{R}_+$ blocking zeros.*

(d) *There exists $i \in \mathbf{N}$ such that the $q + 1$'st invariant factor of system*

(P_i, Q, R_i) has no $\mathcal{R}_{+e}$ zeros except possibly zeros at ∞, i.e., equivalently

$$rank\left(\begin{bmatrix} Q & R_i \\ -P_i & 0 \end{bmatrix}(z)\right) \geq q+1, \ \forall z \in \mathcal{R}_+.$$

(e) *The plant Z is full rank and has no $\mathcal{R}_{+e}$ transmission zeros.*

The following example illustrates the determination of a solution to DSSP.

Example (4.3). Let Z below be the transfer matrix of a 2-channel system.

$$Z = \left[\begin{array}{c|cc} \frac{1}{(z+1)^2} & \frac{(z-1)}{(z+1)^3} & \frac{1}{(z+1)^2} \\ \frac{(2z-5)}{(z+1)(z-2)(z-3)} & \frac{1}{(z-2)(z+1)} & \frac{1}{(z-2)(z+1)} \\ \hline \frac{(2z-3)}{(z-1)(z+1)(z-2)} & \frac{(2z-1)}{(z+1)^2(z-2)} & \frac{(2z-3)}{(z+1)(z-1)(z-2)} \end{array}\right]$$

where $Z_{11} \in \mathbf{P}^{2\times 1}$, $Z_{12} \in \mathbf{P}^{2\times 2}$, $Z_{21} \in \mathbf{P}$ and $Z_{22} \in \mathbf{P}^{1\times 2}$. The plant Z is free of unstable decentralized fixed modes and $\Psi = \{\infty\}$. That is, Z is decentralized strong stabilizable. A bicoprime fraction of Z over $\mathbf{S}$ is given by $[P_1' \ P_2']'Q^{-1}[R_1 \ R_2]$ where

$$P_1 = \begin{bmatrix} \frac{(z-1)}{(z+1)^2} & 0 & 0 \\ 0 & \frac{1}{(z+1)} & \frac{1}{(z+1)} \end{bmatrix}, \quad P_2 = \begin{bmatrix} \frac{1}{(z+1)} & \frac{1}{(z+1)} & 0 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} \frac{1}{(z+1)} \\ \frac{1}{(z+1)} \\ \frac{1}{(z+1)} \end{bmatrix}, \quad R_2 = \begin{bmatrix} \frac{(z-1)}{(z+1)^2} & \frac{1}{(z+1)} \\ \frac{1}{(z+1)} & \frac{1}{(z+1)} \\ 0 & 0 \end{bmatrix}$$

and $Q = diag\{(z-1)/(z+1), (z-2)/(z+1), (z-3)/(z+1)\}$. Following the procedure in Lemma (4.8) we obtain $Z_{c2} = [0 \ \ 1]'$ which is such that the following fraction of the single channel closed loop plant is bicoprime:

$$\hat{Z}(Z_{c2}) := \left[\begin{array}{cc} P_1 & 0 \end{array}\right] \left[\begin{array}{ccc|c} \frac{(z-1)}{(z+1)} & 0 & 0 & \frac{1}{(z+1)} \\ 0 & \frac{(z-2)}{(z+1)} & 0 & \frac{1}{(z+1)} \\ 0 & 0 & \frac{(z-3)}{(z+1)} & 0 \\ \hline -\frac{1}{(z+1)} & -\frac{1}{(z+1)} & 0 & 1 \end{array}\right]^{-1} \left[\begin{array}{c} R_1 \\ 0 \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{(z-1)(z-2)}{(z+1)(z^3-2z^2+z-1)} \\ \frac{(2z^3-5z^2+2)}{(z+1)(z^3-2z^2+z-1)(z-3)} \end{array}\right]$$

Since $\hat{Z}(Z_{c2})$ has no $\mathcal{R}_+$ blocking zeros, Theorem (4.1) implies that it can be stabilized by some $Z_{c1} \in \mathbf{S}^{1\times 2}$. In particular,

$$\left(\hat{Z}(Z_{c2}), \left[\begin{array}{cc} -\frac{(53z^3-56z^2+42z-29)(668z^2-835z-2648)}{22(z+10)(z+1)^4} & \frac{(53z^3-56z^2+42z-29)(3345z^2-12004z+7955)}{22(z+10)(z+1)^4} \end{array}\right]\right)$$

is a stable pair. Thus, the stable decentralized controller

$$\left[\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & -\frac{(53z^3-56z^2+42z-29)(668z^2-835z-2648)}{22(z+10)(z+1)^4} & \frac{(53z^3-56z^2+42z-29)(3345z^2-12004z+7955)}{22(z+10)(z+1)^4} \end{array}\right]$$

stabilizes Z. $\triangle$

It is known that strong stabilization problem is generically solvable for non-scalar systems. We can prove the following analogue result for decentralized strong stabilization problem. Let $\bar{\mathbf{P}}^{p\times r}$ be a subset of $\mathbf{P}^{p\times r}$ such that $Z \in \bar{\mathbf{P}}^{p\times r}$ if and only if **(A1)**, **(A2)** hold for Z and DSP for Z is solvable.

Theorem (4.3). *For almost all $Z \in \bar{\mathbf{P}}^{p\times r}$ DSSP is solvable, where the quantifier "almost all" is with respect to the subset topology induced by the graph topology.*

Proof. If DSSP for $Z \in \bar{P}^{p\times r}$ is solvable then there exists a stable decentralized controller which stabilizes all the plants contained in a sufficiently small neighborhood around Z in $\bar{\mathbf{P}}^{p\times r}$. This proves that the set of plants for which DSSP is solvable is open in $\bar{\mathbf{P}}^{p\times r}$. Now let $Z \in \bar{\mathbf{P}}^{p\times r}$ be such that DSSP for Z is not solvable. Let $\varepsilon > 0$ be given. Assume that the fractional representation in (4.6) holds. Let $i_0, j_0 \in \mathbf{N}$, $i_0 \neq j_0$ be fixed. One can construct matrices Δ_{P1}, Δ_{P2}, Δ_Q, Δ_{R1}, Δ_{R2} of appropriate sizes over S such that (i) $\|[\Delta'_{P1} \;\; \Delta'_{P2}]'\| < \varepsilon$, $\|\Delta_Q\| < \varepsilon$, $\|[\Delta_{R1} \;\; \Delta_{R2}]\| < \varepsilon$, (ii) $(Q + \Delta_Q, P_{i_0} + \Delta_{P1})$, $(Q + \Delta_Q, P_{j_0} + \Delta_{P2})$ right coprime and $(Q + \Delta_Q, R_{j_0} + \Delta_{R1})$, $(Q + \Delta_Q, R_{i_0} + \Delta_{R2})$ are left coprime pairs [66]. Furthermore, they satisfy that (iii) $(P_{i_0} + \Delta_{P1})(Q + \Delta_Q)^{-1}\,(R_{j_0} + \Delta_{R1})$ and $(P_{j_0} + \Delta_{P2})(Q + \Delta_Q)^{-1}\,(R_{i_0} + \Delta_{R2})$ have no unstable blocking zeros except possibly zeros at ∞ [67].

Then, define $Z + \Delta_Z$ as the plant whose a bicoprime fractional representation is given by (4.6) where $P_{i_0} \rightarrow P_{i_0} + \Delta_{P1}$, $P_{j_0} \rightarrow P_{j_0} + \Delta_{P2}$, $R_{i_0} \rightarrow R_{i_0} + \Delta_{R2}$, $R_{j_0} \rightarrow R_{j_0} + \Delta_{R1}$, $Q \rightarrow Q + \Delta_Q$. By keeping ε small enough one can ensure that **(A1)**, **(A2)** hold for $Z + \Delta_Z$ and $Z + \Delta_Z$ is free of unstable decentralized fixed modes, i.e., $Z + \Delta_Z$ belongs to $\bar{\mathbf{P}}^{p\times r}$ [56]. Furthermore, $(Z + \Delta_Z)_{i_0 j_0}$ and $(Z+\Delta_Z)_{j_0 i_0}$ each has at most one $\mathcal{R}_{+e}$ decentralized blocking zero. From Corollary (4.2) (b) we conclude that DSSP for $Z + \Delta_Z$ is solvable. This shows that the set of Z for which DSSP is solvable is dense in $\bar{\mathbf{P}}^{p\times r}$ and terminates the proof.□

The statement (ii) of Theorem (4.2) answers the question (c) at the beginning of Chapter 4 affirmatively and provides a partial solution to the problem of distributing the controller complexity among the local controllers, [3]. In [3], the controller complexity refers to the McMillan degree of the controller. We have shown that the *unstable* McMillan degree of the controller can nearly arbitrarily be distributed among the local controllers such that every local controller has a prespecified number of unstable poles with the exception that an arbitrary one of the controllers may have to posess one extra pole. (This constraint is due to Theorem (4.1) (iii).) Note, however, that an arbitrary distribution of unstable poles among the local controllers might yield an undesired distribution of stable poles among the controllers since no attempt has been made in the synthesis procedure of Theorem (4.2) to allocate the stable compensator poles.

Chapter 5

DECENTRALIZED CONCURRENT STABILIZATION PROBLEM

The objective of this chapter is to rigorously establish the relationship between the notion of Decentralized Blocking Zeros, Decentralized Strong Stabilization Problem (DSSP), Decentralized Concurrent Stabilization Problem (DCSP) and the applications of DCSP in the decentralized synthesis problems.

The motivation of DCSP, which is a special decentralized simultaneous stabilization problem [58], arises from the controller synthesis problems for large-scale systems. In the following sections we will be dealing with three special problems concerning large-scale systems, namely **(p1)** stabilization of composite systems using locally stabilizing subsystem controllers, **(p2)** stabilization of composite systems via the stabilization of diagonal transfer matrices and **(p3)** reliable decentralized stabilization problem. All these problems will be formulated and solved in the DSSP and DCSP framework under a mild connectivity assumption. For a discussion and brief overview of these problems the reader is referred to Chapter 1.

We now state a summary of the main results presented in this chapter. Section 5.1 considers the solution of DCSP. In Theorem (5.1) we obtain a solution to DCSP by transforming it to a Decentralized Strong Stabilization Problem. Proposition (5.3) investigates the set of decentralized blocking zeros of a subsidiary plant associated with Z and T_i, $i \in \mathbf{N}$ and establish a relation between this set of zeros and the set of invariant zeros of the complementary subsystems associated with Z. (See also Remarks (5.1), (5.2).) Theorem (5.2) states a solution to DCSP in a special case. Theorem (5.3) states that DCSP is a generically solvable problem. Section 5.2 is concerned with the solution of problem (p1). Theorems (5.4), (5.10) and (5.14) give solutions to the problem and Theorems (5.9), (5.13) and (5.17) state that the problem is generically solvable in the state-feedback, output feedback and dynamic interconnection cases, respectively. Section 5.3 considers problem (p2). Theorem (5.18) gives a solution to the problem by formulating it in the DCSP setup. Theorems (5.19)-(5.21) investigate the problem in the special cases (i) the diagonal plants are stable (ii) the plant is stabilizable and detectable from all channels and (iii) the off-diagonal plants are stable, respectively. Theorem (5.22) states that the problem is generically solvable. In Section 5.4 problem (p3) is considered. We formulate the problem by generalizing the reliable decentralized stabilization problem considered in [57] to N-channel systems. Theorem (5.23) gives a solution to the problem in the DCSP framework. Theorems (5.24) and (5.25) investigate the problem in some special cases. Theorem (5.26) states that the problem is generically solvable. Theorem (5.27) considers a more special reliable stabilization problem, namely the "multiple controller reliable synthesis problem" (MCRSP) for 2-channel decentralized systems and states the solution of the problem using the results of Section 5.3. We note that some partial results were recently reported on MCRSP using a similar technique in [54] where various *sufficient* solvability conditions are given. Here, under a mild connectivity assumption we provide a complete solution to the problem in terms of a parity interlacing property among the subplant zeros and poles in Theorem (5.27).

5.1 Decentralized Concurrent Stabilization Problem

In this section decentralized concurrent stabilization problem and its relations with the decentralized strong stabilization problem will be investigated.

Decentralized Concurrent Stabilization Problem (DCSP). *Let* $Z = [Z_{ij}]$, $Z_{ij} \in \mathbf{P}^{p_i \times r_j}$, $i, j \in \mathbf{N}$ *be the transfer matrix of a given plant where* $p := \sum_{i=1}^{N} p_i$, $r := \sum_{i=1}^{N} r_i$. *Also let some plants* $T_i \in \mathbf{P}^{p_i \times r_i}$, $i \in \mathbf{N}$ *be given. Determine local controllers* $Z_{ci} \in \mathbf{P}^{r_i \times p_i}$, $i \in \mathbf{N}$ *such that the pairs* (T_i, Z_{ci}), $i \in \mathbf{N}$ *are stable and the pair* $(Z, diag\{Z_{c1}, ..., Z_{cN}\})$ *is stable.*

Observe that DCSP is actually a special decentralized simultaneous stabilization problem (see [58]).

The solution of DCSP is obtained by transforming it to a decentralized strong stabilization problem. To do this, we first give some definitions.

Let some left and right coprime fractions of T_i, $i \in \mathbf{N}$ be given as

$$T_i = D_{li}^{-1} N_{li} = N_{ri} D_{ri}^{-1}, \ i \in \mathbf{N}. \tag{5.1}$$

There exist matrices $K_i \in \mathbf{S}^{p_i \times p_i}$, $L_i \in \mathbf{S}^{r_i \times p_i}$, $\bar{K}_i \in \mathbf{S}^{r_i \times r_i}$, $\bar{L}_i \in \mathbf{S}^{r_i \times p_i}$, $i \in \mathbf{N}$ such that

$$\begin{bmatrix} D_{li} & N_{li} \\ \bar{L}_i & -\bar{K}_i \end{bmatrix} \begin{bmatrix} K_i & N_{r_i} \\ L_i & -D_{r_i} \end{bmatrix} = I, \ i \in \mathbf{N} \tag{5.2}$$

where L_i, $i \in \mathbf{N}$ are strictly proper. Then, (T_i, Z_{ci}) is a stable pair if and only if

$$Z_{ci} = (L_i - D_{ri} X_i)(K_i + N_{ri} X_i)^{-1} \tag{5.3}$$

for some X_i over $\mathbf{S}$. Also let a coprime fraction of Z be given by $Q^{-1}[R_1 \ ... \ R_N]$ where $Q \in \mathbf{S}^{p \times p}$, $R_i \in \mathbf{S}^{p \times r_i}$, $i \in \mathbf{N}$. Define $P_i \in \mathbf{S}^{p_i \times p}$, $i \in \mathbf{N}$ as follows:

$$\begin{bmatrix} P_1 \\ \vdots \\ P_N \end{bmatrix} = I_p. \tag{5.4}$$

It follows that $Z_{ij} = P_i Q^{-1} R_j$, $i, j \in \mathbf{N}$ and $diag\{Z_{c1}, ..., Z_{cN}\}$ stabilizes Z, where Z_{ci} is given by (5.3), if and only if

$$\begin{bmatrix} Q & R_1(L_1 - D_{r1}X_1) & R_2(L_2 - D_{r2}X_2) & R_N(L_N - D_{rN}X_N) \\ -P_1 & (K_1 + N_{r1}X_1) & 0 & 0 \\ -P_2 & 0 & (K_2 + N_{r2}X_2) & 0 \\ \vdots & & & 0 \\ -P_N & 0 & 0 & (K_N + N_{rN}X_N) \end{bmatrix} \tag{5.5}$$

is unimodular over **S**. Define

$$Q_{11} = \begin{bmatrix} Q & R_1L_1 & R_2L_2 & & R_NL_N \\ -P_1 & K_1 & 0 & & 0 \\ -P_2 & 0 & K_2 & & 0 \\ \vdots & \vdots & \vdots & & 0 \\ -P_N & 0 & 0 & & K_N \end{bmatrix},$$

$$R = \begin{bmatrix} -R_1D_{r1} & -R_2D_{r2} & & -R_ND_{rN} \\ N_{r1} & 0 & & 0 \\ 0 & N_{r2} & & 0 \\ \vdots & \vdots & & 0 \\ 0 & 0 & & N_{rN} \end{bmatrix} \tag{5.6}$$

and

$$P = \begin{bmatrix} & P_1 \\ 0_p & \vdots \\ & P_N \end{bmatrix}. \tag{5.7}$$

Further, define $T_d = diag\{T_1, ..., T_N\}$, $D_d = diag\{D_{r1}, ..., D_{rN}\}$, $N_d = diag\{N_{r1}, ..., N_{rN}\}$, $K_d = diag\{K_1, ..., K_N\}$, $L_d = diag\{L_1, ..., L_N\}$, $\bar{D}_d = diag\{D_{l1}, ..., D_{lN}\}$, $\bar{N}_d = diag\{N_{l1}, ..., N_{lN}\}$, $\bar{K}_d = diag\{\bar{K}_1, ..., \bar{K}_N\}$, $\bar{L}_d = diag\{\bar{L}_1, ..., \bar{L}_N\}$, $\bar{R} = [R_1 \; ... \; R_N]$. Various coprimeness relations yield that

$$\bar{Z} := PQ_{11}^{-1}R \tag{5.8}$$

is a bicoprime fraction where the nonsingularity of Q_{11} is ensured by the fact that L_i, $i \in \mathbf{N}$ are strictly proper. With this notation the matrix in (5.5) is unimodular if and only if so is $Q_{11} + R\, diag\{X_1, ..., X_N\}\, P$. The following theorem states the solution of DCSP.

Theorem (5.1). *DCSP is solvable for Z and T_i, $i \in \mathbf{N}$ if and only if DSSP for the plant $\bar{Z}$ is solvable.*

Proof. If DCSP is solvable, then by the problem definition the matrix (5.5) is unimodular for some X_i, $i \in \mathbf{N}$ which implies that $(\bar{Z}, diag\{X_1, ..., X_N\})$ is a stable pair. Conversely, if $(\bar{Z}, diag\{X_1, ..., X_N\})$ is stable for some X_i, $i \in \mathbf{N}$, then $Q_{11} + R\, diag\{X_1, ..., X_N\}\, P$ is unimodular, which implies via the unimodularity of the matrix in (5.5) and equation (5.3) that DCSP for Z and T_i, $i \in \mathbf{N}$ is solvable.□

It is clear from the problem definitions that for DSSP to be solvable Z must be free of unstable decentralized fixed modes. The following result states that if Z is free of unstable decentralized fixed modes then so is $\bar{Z}$.

Proposition (5.1). *Let Z be free of unstable decentralized fixed modes. Then, for all K_i, L_i, $\bar{K}_i$, $\bar{L}_i$, $i \in \mathbf{N}$ satisfying (5.2) $\bar{Z}$ given by (5.6), (5.7), (5.8) is also free of unstable decentralized fixed modes.*

Proof. Fix arbitrary K_i, L_i, $\bar{K}_i$, $\bar{L}_i$ $i \in \mathbf{N}$ satisfying (5.2) where L_i, $\bar{L}_i$, $i \in \mathbf{N}$ are strictly proper. Define $T = QK_d + \bar{R}L_d$ and $S = QN_d - \bar{R}D_d$. Observe that $T \in \mathbf{S}^{p\times p}$ and is nonsingular, and $S \in \mathbf{S}^{p\times r}$. Let $S_i \in \mathbf{S}^{p\times r_i}$ denote the i'th block-column of S for $i \in \mathbf{N}$. Simple manipulations on the equation (5.8) yield that a coprime fraction of $\bar{Z}$ is given by $\bar{Z} = T^{-1}S$. Define

$$A = \begin{bmatrix} \bar{D}_d & N_d \\ \bar{L}_d & -\bar{K}_d \end{bmatrix} \in \mathbf{S}^{p+r\times p+r} \tag{5.9}$$

which is unimodular. For any proper subset $\mathbf{r} = \{i_1, ..., i_\mu\}$ of $\mathbf{N}$ define $A_\mathbf{r} \in \mathbf{S}^{p_{i_1}+...+p_{i_\mu}+r_{i_1}+...+r_{i_\mu}\times p+r}$ to be the submatrix of A consisting of block rows $i_1, ..., i_\mu, N + i_1, ..., N + i_\mu$. Also let $I_\mathbf{r} \in \mathbf{S}^{p_{i_1}+...+p_{i_\mu}\times p}$ denote the matrix whose kj'th submatrix equals

$$\begin{cases} I_{p_j} & if\ j = i_k \\ 0 & otherwise \end{cases}$$

for $k = 1, ..., \mu$, $j = 1, ..., N$. It holds that (Theorem (3.2), [22], [37]) $\bar{Z} = T^{-1}S$ is free of unstable decentralized fixed modes if and only if for every proper subset $\mathbf{r} = \{i_1, ..., i_\mu\}$ of $\mathbf{N}$

$$rank \begin{bmatrix} T & S_{i_{\mu+1}} \cdots S_{i_N} \\ I_\mathbf{r} & 0 \end{bmatrix}(z) \geq p = size(T),\ \forall z \in \mathcal{C}_+. \tag{5.10}$$

Equation (5.10) holds if and only if

$$rank \begin{bmatrix} T & S \\ I_{\mathbf{r}} & 0 \\ 0 & \bar{I}_{\mathbf{r}} \end{bmatrix} (z) \geq p + \sum_{i \in \mathbf{r}} r_i, \ \forall z \in \mathcal{C}_+$$

where $\bar{I}_{\mathbf{r}} \in \mathbf{S}^{r_{i_1}+\ldots+r_{i_\mu} \times r}$ and whose kj'th submatrix equals

$$\begin{cases} I_{r_j} & if\ j = i_k \\ 0 & otherwise \end{cases}$$

for $k = 1, \ldots, \mu$, $j = 1, \ldots, N$. It can be verified that

$$\begin{bmatrix} T & S \\ I_{\mathbf{r}} & 0 \\ 0 & \bar{I}_{\mathbf{r}} \end{bmatrix} A = \begin{bmatrix} Q & \bar{R} \\ & A_{\mathbf{r}} \end{bmatrix}.$$

Since A is unimodular we conclude that (5.10) holds if and only if

$$rank \begin{bmatrix} Q & \bar{R} \\ & A_{\mathbf{r}} \end{bmatrix} (z) \geq p + \sum_{i \in \mathbf{r}} r_i, \ \forall z \in \mathcal{C}_+. \qquad (5.11)$$

Applying unimodular operations, equation (5.11) holds if and only if

$$rank \begin{bmatrix} Q & \bar{R} \\ I_{\mathbf{r}} & 0 \\ 0 & \bar{I}_{\mathbf{r}} \end{bmatrix} (z) \geq p + \sum_{i \in \mathbf{r}} r_i, \ \forall z \in \mathcal{C}_+,$$

or equivalently

$$rank \begin{bmatrix} Q & R_{i_{\mu+1}} \ldots R_{i_N} \\ I_{\mathbf{r}} & 0 \end{bmatrix} (z) \geq p, \ \forall z \in \mathcal{C}_+. \qquad (5.12)$$

Since Z is free of unstable decentralized fixed modes, for every proper subset $\{i_1, \ldots, i_\mu\}$ of $\mathbf{N}$ the inequality stated in (5.12) holds (Theorem (3.2), [22], [56].) This completes the proof.□

Proposition (5.2). *If the following condition holds*

$$rank\ Z_{ij} \geq 2 \ or\ rank\ Z_{ji} \geq 2, \ \forall i, j \in \mathbf{N}, \ i \neq j \qquad (5.13)$$

and Z is strongly connected then there exist K_i, L_i, $\bar{K}_i$, $\bar{L}_i$, $i \in \mathrm{N}$ satisfying (5.2) such that $\bar{Z}$ given by (5.6), (5.7), (5.8) satisfies

$$rank\,\bar{Z}_{ij} \geq 2 \text{ or } rank\,\bar{Z}_{ji} \geq 2,\ \forall i,j \in \mathrm{N},\ i \neq j \tag{5.14}$$

where $\bar{Z}_{ij}$ denotes the ij'th submatrix of $\bar{Z}$. Further, $\bar{Z}$ is strongly connected.

Proof. Define a subset $\mathcal{Y}$ of $N \times N$ such that $(i,j) \in \mathcal{Y}$ if and only if $i \neq j$ and $rank\,Z_{ij} \geq 2$. Let $\mathcal{T}$ be the subset of $\mathcal{C}_{+e}$ excluding the poles of T_i, $i \in \mathrm{N}$. Determine a positive real number $z_0 \in \mathcal{T}$ satisfying that for all $(i,j) \in \mathcal{Y}$, $rank\,Z_{ij}(z_0) \geq 2$. (Such a z_0 can be found easily, since $\mathcal{T}$ is an open and dense subset of $\mathcal{C}_{+e}$.) Given $\tilde{K}_i$, $\tilde{L}_i$, $\tilde{\bar{K}}_i$, $\tilde{\bar{L}}_i$, $i \in \mathrm{N}$ where $\tilde{L}_i$, $\tilde{\bar{L}}_i$, $i \in \mathrm{N}$ are strictly proper and

$$\begin{bmatrix} D_{li} & N_{li} \\ \tilde{\bar{L}}_i & -\tilde{\bar{K}}_i \end{bmatrix} \begin{bmatrix} \tilde{K}_i & N_{ri} \\ \tilde{L}_i & -D_{ri} \end{bmatrix} = I,\ i \in \mathrm{N},$$

determine Θ_i over S satisfying that $\Theta_i(z_0) = D_{ri}^{-1}(z_0)\tilde{L}_i(z_0)$, $i \in \mathrm{N}$ where the nonsingularity of $D_{ri}(z_0)$, $i \in \mathrm{N}$ is ensured by the fact that $z_0 \in \mathcal{T}$. Define

$$K_i = \tilde{K}_i + N_{ri}\Theta_i,\ \ L_i = \tilde{L}_i - D_{ri}\Theta_i,\ \ \bar{K}_i = \tilde{\bar{K}}_i + \Theta_i N_{li},\ \ \bar{L}_i = \tilde{\bar{L}}_i - \Theta_i D_{li},\ i \in \mathrm{N}. \tag{5.15}$$

Obtain Q_{11}, R and $\bar{Z} = PQ_{11}^{-1}R$. It follows that, for $i \neq j$, $\bar{Z}_{ij} = -K_i^{-1}P_i\bar{Q}^{-1}R_j$ $(D_{rj} + L_jK_j^{-1}N_{rj})$ where $\bar{Q} := Q + \sum_{i=1}^{N} R_iL_iK_i^{-1}P_i$. Since L_i, $i \in \mathrm{N}$ are strictly proper, it holds that $rank\,\bar{Z}_{ij} \geq 2$ if $rank\,P_i\bar{Q}^{-1}R_j \geq 2$. For any $(i,j) \in \mathcal{Y}$ (5.15) yields by the construction of Θ_i that $L_i(z_0) = 0$, $i \in \mathrm{N}$. In other words, $\bar{Q}(z_0) = Q(z_0)$, therefore $rank\,P_i\bar{Q}^{-1}R_j \geq 2$. This shows that $rank\,\bar{Z}_{ij} \geq 2$. Since $(i,j) \in \mathcal{Y}$ is arbitrary, we have the inequalities stated in (5.14) which also imply that $\bar{Z}$ is strongly connected.□

We hereafter assume that

$$\begin{aligned} &\text{(i)} && Z \text{ is free of unstable decentralized fixed modes} \\ &\text{(ii)} && Z \text{ is strongly connected} \\ &\text{(iii)} && rank\,Z_{ij} \geq 2 \text{ or } rank\,Z_{ji} \geq 2,\ \forall i,j \in \mathrm{N},\ i \neq j \end{aligned} \tag{5.16}$$

The following procedure summarizes the solution of DCSP. First obtain left and right coprime fractions of T_i as in (5.1). Then, determine initial compensators

$L_i K_i^{-1} = \bar{K}_i^{-1} \bar{L}_i$ in (5.2) such that $\bar{Z}$ in (5.8) is strongly connected and (5.14) holds where Q_{11}, R are given by (5.6) and P is given by (5.7). Determine the solvability of DSSP for $\bar{Z}$ using Corollary (4.1). If DSSP is solvable construct X_i following the proof of Theorem (4.2). This yields the compensators Z_{ci} in (5.3) which solve DCSP.

The solution of DCSP is obtained via a transformed decentralized strong stabilization problem on the auxiliary plant $\bar{Z}$. Note that in the solution of DCSP one can obtain infinitely many auxiliary plants for which the solvability of DSSP implies the solvability of DCSP and vice versa. In the sequel we will be dealing with some special choices of the auxiliary plants which would enable us to obtain more transparent solvability conditions.

The next result is concerned with the unstable decentralized blocking zeros of the auxiliary plant $\bar{Z}$. Define

$$\bar{\mathcal{Z}} = \{\bar{Z} \mid \ \bar{Z} \textit{ is given by } (5.6),\ (5.7),\ (5.8) \textit{ for some } K_i,\ L_i,\ \bar{K}_i,\ \bar{L}_i,\ i \in \mathbf{N} \textit{ satisfying } (5.2)\}.$$

In other words, $\bar{\mathcal{Z}}$ is the set of all auxiliary plants obtained via (5.6), (5.7), (5.8). For any $\bar{Z} \in \bar{\mathcal{Z}}$ let $\mathcal{S}_{\bar{Z}}$ be the set of unstable decentralized blocking zeros of $\bar{Z}$. Also define $Q_i \in \mathbf{S}^{p \times p_i}$, $i \in \mathbf{N}$ to be the i'th block column of Q, i.e., $Q = [Q_1 \ \dots \ Q_N]$.

Proposition (5.3). *The following equality holds: For every* $\bar{Z} \in \bar{\mathcal{Z}}$, $\mathcal{S}_{\bar{Z}} = \hat{\Psi}$

where

$$\hat{\Psi} := \{z \in \mathcal{C}_{+e} | \text{ There exists a permutation } \{i_1, ..., i_N\} \text{ of } \mathbf{N} \text{ such that}$$

$$rank \begin{bmatrix} Q_{i_1} & R_{i_1} \\ D_{li_1} & N_{li_1} \end{bmatrix}(z) = p_{i_1},$$

$$rank \begin{bmatrix} Q_{i_1} & Q_{i_2} & R_{i_1} & R_{i_2} \\ 0 & D_{li_2} & 0 & N_{li_2} \end{bmatrix}(z) = p_{i_1} + p_{i_2},$$

$$rank \begin{bmatrix} Q_{i_1} & Q_{i_2} & & Q_{i_N} & R_{i_1} & R_{i_2} & & R_{i_N} \\ 0 & 0 & & D_{li_N} & 0 & 0 & & N_{li_N} \end{bmatrix}(z) = p\}. \tag{5.17}$$

Proof. Fix any arbitrary $\tilde{Z} \in \tilde{\mathcal{Z}}$. Recall from Lemma (4.5) that the following holds

$$S_{\tilde{Z}} = \{z \in \mathcal{C}_{+e} | \text{ There exists a permutation } \{i_1, ..., i_N\} \text{ of } \mathbf{N} \text{ such that}$$

$$rank \begin{bmatrix} QK_d + \tilde{R}L_d & [QN_d - \tilde{R}D_d]_{\{i_1,...,i_j\}} \\ & \\ -P_{i_j} & \\ \vdots & 0 \\ -P_{i_N} & \end{bmatrix}(z) = p, \ \forall j \in \mathbf{N}\}$$

where $[QN_d - \tilde{R}D_d]_{\{i_1,...,i_j\}} \in \mathbf{S}^{p \times r_{i_1} + ... + r_{i_j}}$ is the matrix consisting of block columns $\{i_1, ..., i_j\}$ of $QN_d - \tilde{R}D_d$. Now let $z \in \mathcal{C}_{+e}$ be such that

$$rank \begin{bmatrix} QK_d + \tilde{R}L_d & [QN_d - \tilde{R}D_d]_{\{1,...,j\}} \\ & \\ -P_j & \\ & 0 \\ -P_N & \end{bmatrix}(z) = p, \ \forall j \in \mathbf{N}. \tag{5.18}$$

Postmultiplying the above matrix by a suitable submatrix of A in (5.9) and applying further unimodular operations we obtain that (5.18) holds if and only if

$$rank \begin{bmatrix} Q_1 & Q_2 & & Q_j & R_1 & R_2 & & R_j \\ 0 & 0 & & D_{lj} & 0 & 0 & & N_{lj} \end{bmatrix}(z) = p_1 + p_2 + \ldots + p_j, \ \forall j \in \mathbf{N}.$$

By modifying the indices appropriately and repeating the above arguments one can show that for any permutation $\{i_1, \ldots, i_N\}$ of N and for any $z \in C_{+\epsilon}$

$$rank \begin{bmatrix} QK_d + \bar{R}L_d & [QN_d - \bar{R}D_d]_{\{i_1,\ldots,i_j\}} \\ & \\ -P_{i_j} & \\ \vdots & 0 \\ -P_{i_N} & \end{bmatrix}(z) = p, \ \forall j \in \mathbf{N}$$

holds if and only if

$$rank \begin{bmatrix} Q_{i_1} & Q_{i_2} & & Q_{i_j} & R_{i_1} & R_{i_2} & & R_{i_j} \\ 0 & 0 & & D_{li_j} & 0 & 0 & \ldots & N_{li_j} \end{bmatrix}(z) = p_{i_1} + p_{i_2} + \ldots + p_{i_j}, \ \forall j \in \mathbf{N}.$$

This shows that $z \in S_{\bar{Z}}$ implies $z \in \hat{\Psi}$ in (5.17) and vice versa. Since $\bar{Z} \in \bar{\mathcal{Z}}$ is arbitrary, this completes the proof.□

Remark (5.1). From Proposition (5.3) we conclude that the set of unstable decentralized blocking zeros of any auxiliary plant in $\bar{\mathcal{Z}}$ is independent of the initial compensators; it depends only on the plants Z and T_i, $i \in \mathbf{N}$. Therefore, it constitutes an invariant set associated with Z and T_i, $i \in \mathbf{N}$.•

Let us now investigate the set $\hat{\Psi}$ in detail. The following proposition states that the zeros in the set $\hat{\Psi}$ are among the set of zeros of invariant factors associated with the complementary subsystems of $([P_1' \ldots P_N']', Q, [R_1 \ldots R_N])$. (See Remark (5.2).)

Proposition (5.4). *Define*

$$\Gamma = \{z \in C_{+e} |\ z \text{ is a zero of the } p+1\text{'st invariant factor associated with the system } \left(\begin{bmatrix} P_{i_{\mu+1}} \\ \vdots \\ P_{i_N} \end{bmatrix}, Q, \begin{bmatrix} R_{i_1} & & R_{i_\mu} \end{bmatrix}\right) \text{ for some proper subset } \{i_1, ..., i_\mu\} \text{ of } \mathbf{N}\} \tag{5.19}$$

Then, the following inclusion holds: $\hat{\Psi} \subseteq \Gamma$.

Proof. Recall from Chapter 2 that for any $z \in C_{+e}$

$$rank \begin{bmatrix} Q & R_{i_1} & R_{i_\mu} \\ -P_{i_{\mu+1}} & & \\ \vdots & 0 & \\ -P_{i_N} & & \end{bmatrix}(z) = p$$

holds if and only if z is a C_{+e} zero of the $p+1$'st invariant factor associated with the system

$$\left(\begin{bmatrix} P_{i_{\mu+1}} \\ \vdots \\ P_{i_N} \end{bmatrix}, Q, \begin{bmatrix} R_{i_1} & & R_{i_\mu} \end{bmatrix}\right).$$

That is, the following equality holds:

$$\Gamma = \{z \in C_{+e} |\ \text{For some proper subset } \{i_1, ..., i_\mu\} \text{ of } \mathbf{N} \quad rank \begin{bmatrix} Q & R_{i_1} & R_{i_\mu} \\ -P_{i_{\mu+1}} & & \\ \vdots & 0 & \\ -P_{i_N} & & \end{bmatrix}(z) = p\}$$

Let $z_0 \in C_{+e}$ satisfy that

$$rank \begin{bmatrix} Q_1 & Q_2 & & Q_j & R_1 & R_2 & & R_j \\ 0 & 0 & \dots & D_{lj} & 0 & 0 & & N_{lj} \end{bmatrix}(z_0) = p_1 + p_2 + \dots + p_j, \tag{5.20}$$

for some $j \in \{2, ..., N\}$. Since Z is free of unstable decentralized fixed modes, it holds that $rank[Q_1 \ ... \ Q_{j-1} \ R_1 \ ... R_{j-1}](z) \geq p_1 + ... + p_{j-1}$, $\forall z \in C_{+\epsilon}$. Then, for some unimodular matrices U and V of appropriate size

$$U[Q_1 \ ... \ Q_{j-1} \ R_1 \ ... \ R_{j-1}]V = \begin{bmatrix} I_{\bar{p}} & 0 \\ 0 & \Lambda \end{bmatrix} \tag{5.21}$$

where $\bar{p} := p_1 + ... + p_{j-1}$, $\Lambda \in \mathbf{S}^{p-\bar{p} \times r_1 + ... + r_{j-1}}$ and the matrix at the right hand size of (5.21) is the Smith canonical form of the middle matrix at the left. Equation (5.20) holds if and only if

$$rank \begin{bmatrix} \bar{Q}_j^2 & \Lambda & \bar{R}_j^2 \\ D_{lj} & 0 & N_{lj} \end{bmatrix} (z_0) = p_j \tag{5.22}$$

where

$$\begin{bmatrix} \bar{Q}_j^1 & \bar{R}_j^1 \\ \bar{Q}_j^2 & \bar{R}_j^2 \end{bmatrix} := U[Q_j \ R_j]$$

so that $\bar{Q}_j^1 \in \mathbf{S}^{\bar{p} \times p_j}$, $\bar{R}_j^1 \in \mathbf{S}^{\bar{p} \times r_j}$, $\bar{Q}_j^2 \in \mathbf{S}^{p-\bar{p} \times p_j}$ and $\bar{R}_j^2 \in \mathbf{S}^{p-\bar{p} \times r_j}$. Equation (5.22) holds if and only if $[\bar{R}_j^2 D_{rj} - \bar{Q}_j^2 N_{rj} \ \ \Lambda](z_0) = 0$. The fact that $\Lambda(z_0) = 0$ implies

$$rank \begin{bmatrix} Q & R_1 & ... & R_{j-1} \\ -P_j & & & \\ & & 0 & \\ -P_N & & & \end{bmatrix} (z_0) = p.$$

i.e, $z_0 \in \Gamma$. Now let $\{i_1, ..., i_j\}$ be some permutation of $\mathbf{N}$. Modifying the indices appropriately and applying the arguments similar to those above, it can be shown that for any $z_0 \in C_{+\epsilon}$ for which

$$rank \begin{bmatrix} Q_{i_1} & Q_{i_j} & R_{i_1} & R_{i_j} \\ 0 & D_{li_j} & 0 & N_{li_j} \end{bmatrix} (z_0) = p_{i_1} + ... + p_{i_j}$$

holds for some $j \in \{2, ..., N\}$ only if

$$rank \begin{bmatrix} Q & R_{i_1} & ... & R_{i_{j-1}} \\ -P_{i_j} & & & \\ \vdots & & 0 & \\ -P_{i_N} & & & \end{bmatrix} (z_0) = p.$$

This implies by definition that every $z \in \hat{\Psi}$ is contained in Γ, completing the proof.□

Remark (5.2). Consider the cases where $\Gamma = \emptyset$ or Γ contains only one element. In these cases, it follows from Proposition (5.4) and Theorem (5.1) that DCSP for Z and T_i, $i \in \mathbf{N}$ is solvable. This sufficient condition will be used in the following sections where we consider the synthesis of special decentralized controllers for large-scale systems.•

We now give a necessary and sufficient condition for the solvability of DCSP in a special case. Let $\mathcal{Z}$ and $\mathcal{T}_d$ denote the sets of C_+ poles of Z and T_d, respectively, with multiplicities.

Theorem (5.2). *Let $\mathcal{T}_d \cap \mathcal{Z} = \emptyset$ and $\mathcal{T}_d \cap \hat{\Psi} = \emptyset$. Then, DCSP is solvable if and only if DSSP for $T - Z$ is solvable.*

Proof. The strong connectedness of Z implies that the transfer matrix $T - Z$ is also strongly connected. Since $\mathcal{T}_d \cap \mathcal{Z} = \emptyset$, $(Q, \bar{D}_d)$ is a left coprime pair. Let $Q\bar{D}_d^{-1} = \tilde{D}_d^{-1}\tilde{Q}$ for a left coprime pair of matrices $(\tilde{D}_d, \tilde{Q})$. Then, a left coprime fraction of $T - Z$ is given by $Q^{-1}\tilde{D}_d^{-1}(\tilde{Q}\bar{N}_d - \tilde{D}_d\bar{R})$. Define

$$\mathbf{D} = \{z \in C_{+e} \mid z \text{ is a decentralized blocking zero of } T - Z\} \cup \hat{\Psi}$$

From Lemma (4.4) $z \in \mathbf{D}$ implies $D_d(z)$ is nonsingular. Following the proof of Proposition (5.2) let us choose L_d such that $L_d(z) = 0\ \forall z \in \mathbf{D}$ and $\bar{Z} = (QK_d + \bar{R}L_d)^{-1}\ (QN_d - \bar{R}D_d)$ satisfies that the relation (5.14) holds and $\bar{Z}$ is strongly connected. With this choice of L_d, if $z \in \mathbf{D}$ then $(QK_d + \bar{R}L_d)(z) = (QK_d)(z) = (Q\bar{D}_d^{-1})(z)$. It now holds that

$$\begin{aligned} z \in \hat{\Psi} \implies \bar{Z}(z) &= [(QK_d + \bar{R}L_d)(z)]^{-1}(QN_d - \bar{R}D_d)(z) \\ &= (\bar{D}_d Q^{-1} Q(T - Z)D_d)(z) \\ &= (\bar{D}_d(T - Z)D_d)(z) \end{aligned}$$

Since $\bar{D}_d$ and D_d are block diagonal, z is a decentralized blocking zero of $T - Z$. Conversely, if z is an C_{+e} decentralized blocking zero of $T - Z$ then the same arguments yield that z is a decentralized blocking zero of $\bar{Z}$ as well. Hence, the set of C_{+e} decentralized blocking zeros of $T - Z$ is precisely $\hat{\Psi}$. Note that

$det(QK_d + \bar{R}L_d)$ takes the same sign at all $z \in \hat{\Psi} \cap \mathcal{R}_{+e}$ if and only if so does $det(Q).det(D_d)$, since for each $z \in \hat{\Psi}$, $det(QK_d + \bar{R}L_d)(z) = (det(Q).det(K_d))(z)$ $= (det(Q).det^{-1}(D_d))(z)$. This completes the proof.□

The assumption that $\mathcal{T}_d \cap \mathcal{Z} = \emptyset$ and $\mathcal{T}_d \cap \hat{\Psi} = \emptyset$ generically holds in $\bar{\mathbf{P}}^{p\times r}$ $\times \mathbf{P}^{p_1\times r_1}$ ... $\times \mathbf{P}^{p_N\times r_N}$ with respect to the product topology induced by the graph topology where $\bar{\mathbf{P}}^{p\times r}$ denotes the set of transfer matrices Z in $\mathbf{P}^{p\times r}$ satisfying that (5.16) holds. From Theorem (5.2) we conclude that *for almost all plants Z, $T = diag\{T_i;\ i \in \mathbf{N}\}$, a solution to DCSP exists if and only if DSSP for the difference plant $T - Z$ is solvable.*

We will now show that DCSP is a generically solvable problem.

Theorem (5.3). *The set of $N+1$-tuples $(Z, T_1, ..., T_N)$ for which DCSP is solvable is open and dense in $\bar{\mathbf{P}}^{p\times r}$ $\times \mathbf{P}^{p_1\times r_1}$ ... $\times \mathbf{P}^{p_N\times r_N}$ (with respect to the product topology induced by the graph topology).*

Proof. Let DCSP be solvable for some $(Z, T_1, ..., T_N)$ by a set of local controllers $Z_{c1}, ..., Z_{cN}$. Under sufficiently small perturbations on Z and T_i's it holds that the pairs $(Z + \Delta, diag\{Z_{c1}, ..., Z_{cN}\})$, $(T_1 + \Delta_1, Z_{c1})$, ..., $(T_N + \Delta_N, Z_{cN})$ are still stable with Δ and Δ_i, $i \in \mathbf{N}$ denoting the perturbations over $\mathbf{P}$. This proves that the solvability of DCSP is an open property. Now suppose that DCSP is not solvable for some $(Z, T_1, ..., T_N)$. We will show that by an arbitrarily small perturbation $\Delta \in \mathbf{P}^{p\times r}$ on Z the matrix $Z + \Delta$ belongs to $\bar{\mathbf{P}}^{p\times r}$ and the set of $\mathcal{C}_{+e}$ decentralized blocking zeros associated with $Z + \Delta$ and T_i, $i \in \mathbf{N}$ denoted by $\hat{\Psi}_\Delta$ satisfies $\hat{\Psi}_\Delta \cap \mathcal{R}_{+e} \subseteq \{\infty\}$, i.e., it contains at most only one $\mathcal{R}_{+e}$ element. In this case Remark (5.2) states that DCSP for $Z + \Delta$ and $T_i, i \in \mathbf{N}$ is solvable. This shows that the set of $(Z, T_1, ..., T_N)$ for which DCSP is solvable is dense. To prove the existence of such perturbations we proceed as follows. Let $\bar{Z}$ be given by (5.2), (5.6), (5.7), (5.8). We remind that for $i \in \mathbf{N}$, $S_i \in \mathbf{S}^{p\times r_i}$ denotes the i'th block column of $S = QN_d - \bar{R}D_d$. One can find arbitrarily small strictly proper perturbations $\bar{\Delta}_i \in \mathbf{S}^{p\times r_i}$ on S_i's such that $\{\ \mathcal{R}_{+e}\ zeros\ of\ sif\,(S_i + \bar{\Delta}_i)\} \subseteq \{\infty\}$, $i \in \mathbf{N}$. Since (D_d, N_d) is a right coprime pair we can find strictly proper matrices $\tilde{\Delta}_1 \in \mathbf{S}^{p\times p}$, $\tilde{\Delta}_2 \in \mathbf{S}^{p\times r}$ such that $\tilde{\Delta}_1 N_d - \tilde{\Delta}_2 D_d = [\bar{\Delta}_1\ ...\ \bar{\Delta}_N]$. Define

$\Delta = (Q + \tilde{\Delta}_1)^{-1}(\bar{R} + \tilde{\Delta}_2) - Z$. It can be ensured by choosing the relevant norms sufficiently small that $Z + \Delta = (Q + \tilde{\Delta}_1)^{-1}(\bar{R} + \tilde{\Delta}_2)$ is a coprime fraction and $Z + \Delta$ is a matrix over $\bar{\mathbf{P}}^{p \times r}$. The choice of Δ reveals that $\hat{\Psi}_\Delta \cap \mathcal{R}_{+e}$ contains at most one element: $z = \infty$, because every unstable decentralized blocking zero z of

$$[(Q + \tilde{\Delta}_1)K_d + (\bar{R} + \tilde{\Delta}_2)L_d]^{-1}[(Q + \tilde{\Delta}_1)N_d - (\bar{R} + \tilde{\Delta}_2)D_d]$$

satisfies $(S_i + \tilde{\Delta}_i)(z) = 0$ for some $i \in \mathbf{N}$. This and the above discussion complete the proof.□

Before closing this section we give a necessary condition for the solvability of DCSP. (See also Section 5.3.) Define

$$\Theta = \{z \in \mathcal{R}_{+e} | T_i(z) = 0, i \in \mathbf{N}\},$$

$$\Psi = \{z \in \mathcal{R}_{+e} | \text{ There exists a permutation } \{i_1, ..., i_N\} \text{ of } \mathbf{N} \text{ such that}$$

$$\begin{bmatrix} Z_{i_1 i_1} & 0 & & 0 \\ Z_{i_2 i_1} & Z_{i_2 i_2} & & 0 \\ \vdots & \vdots & & \vdots \\ Z_{i_N i_1} & Z_{i_N i_2} & & Z_{i_N i_N} \end{bmatrix} (z) = 0\},$$

i.e., Ψ is the set of $\mathcal{R}_{+e}$ decentralized blocking zeros of Z.

Proposition (5.5). *The problem DCSP for Z and T_i, $i \in \mathbf{N}$ is solvable only if there are an even number of real elements of $\mathcal{T}_d \cup \mathcal{Z}$ between each pair of elements in the set $\Theta \cap \Psi$, where the union $\mathcal{T}_d \cup \mathcal{Z}$ is taken with multiplicities.*

Proof. From Lemma (4.4) every $z \in \Theta \cap \Psi$ implies $Q(z) \neq 0$. Then, we can choose L_d such that $L_d(z) = 0$ for all $z \in \Theta \cap \Psi$, $\bar{Z}$ satisfies that the relation (5.14) holds and $\bar{Z}$ is strongly connected. Let $z_0 \in \Theta \cap \Psi$ be fixed. Observe that $D_d(z_0)$ and $K_d(z_0)$ are nonsingular. It holds that

$$\begin{aligned} (T^{-1}S)(z_0) &= (QK_d)^{-1}(z_0)(QN_d - \bar{R}D_d)(z_0) \\ &= K_d^{-1}(z_0)(N_d D_d^{-1} - Q^{-1}\bar{R})(z_0)D_d(z_0) \\ &= -K_d^{-1}(z_0)Z(z_0)D_d(z_0). \end{aligned}$$

Since K_d and D_d are block diagonal matrices this latter equality shows that $z_0 \in \hat{\Psi}$. This concludes us that $\Theta \cap \Psi \subseteq \hat{\Psi}$. On the other hand $det(T)(z) = det(QK_d)(z)$. From Proposition (5.3) and Theorem (5.1) DCSP is solvable only if $det(T)(z)$ takes the same sign at the $\mathcal{R}_{+e}$ elements of the set $\hat{\Psi}$ which holds, by the fact that $\Theta \cap \Psi \subset \hat{\Psi}$, only if $det(Q).det(D_d)$ takes the same sign at all $z \in \Theta \cap \Psi$. This completes the proof. Note that in Proposition (5.5) the plant Z does not need to satisfy (iii) of (5.16), since we consider only a necessary condition for the solvability of DSSP (Remark (4.2)).□

Corollary (5.1). *Let $T_i = Z_{ii}$, $i \in$ N. Then, DCSP is solvable only if there are an even number of real elements of $\mathcal{T}_d \cup \mathcal{Z}$ between each pair of $\mathcal{R}_{+e}$-decentralized blocking zeros of Z, where the union is taken with multiplicities.*

Proof. The proof follows from the fact that in this special case $\Theta \cap \Psi = \Psi$.□

5.2 Locally Stabilizing Subsystem Controllers

Consider a collection of linear time-invariant finite-dimensional systems described by

$$\Sigma_i : \begin{array}{rl} \dot{x}_i = & A_i x_i + B_i v_i + u_i \\ y_i = & C_i x_i \end{array}, \quad i \in \mathbf{N} \tag{5.23}$$

where $A_i \in \mathcal{R}^{n_i \times n_i}$, $B_i \in \mathcal{R}^{n_i \times r_i}$ and $C_i \in \mathcal{R}^{p_i \times n_i}$ corresponding to states, inputs and outputs, respectively. Assume that these systems are interconnected according to the rule $u_i = \sum_{j=1}^{N} A_{ij} x_j$, $i \in$ N. Then, the composite (interconnected) system can be described as

$$\Sigma : \begin{array}{rl} \dot{x} = & Ax + Bv \\ y = & Cx \end{array} \tag{5.24}$$

where $x := [x_1' \; ... \; x_N']'$,

$$A := \begin{bmatrix} A_1 + A_{11} & A_{12} & & A_{1N} \\ A_{21} & A_2 + A_{22} & & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & & A_N + A_{NN} \end{bmatrix}, \; B := diag\{B_1, ..., B_N\},$$

$$C := diag\{C_1, ..., C_N\}, \; y := [y_1' \; ... \; y_N']' \; and \; v := [v_1' \; ... \; v_N']'. \tag{5.25}$$

It is assumed that the subsystems $\Sigma_i = (C_i, A_i, B_i)$, $i \in \mathrm{N}$ and the composite system $\Sigma = (C, A, B)$ are stabilizable and detectable. We let $n := \sum_{i=1}^{N} n_i$.

The problem of stabilizing the composite system Σ using locally stabilizing subsystem controllers, denoted by (p1), *is defined as synthesising local controllers Σ_{ci}, $i \in \mathrm{N}$ around subsystems Σ_i such that* (i) *when the interconnections do not exist (Σ_i, Σ_{ci}), $i \in \mathrm{N}$ are stable and* (ii) *when the interconnections A_{ij} exist the composite closed-loop system becomes stable.* In the control theory there is an enormous literature concerning this problem. When the states of the subsystems are directly measurable, there is a variety of solution procedures employing the vector Lyapunov functions [23], [40], [52], high gain controllers [75], [23], [45], special interconnection structures [24], [46], [53] etc.. In case where the subsystem states are not directly measurable the problem is attempted to solve by observing the subsystem states and, in some cases, decentralized state feedback laws using local controllers [69], [52], [50], [71], [25]. We note that all these methods give only some sufficient solvability conditions for the problem. In fact, as indicated in [52], the problem is a decentralized simultaneous stabilization problem which can be formulated and solved in the DCSP framework.

Let Λ_i and Λ_A be the sets of C_+ eigenvalues of A_i, $i \in \mathrm{N}$ and A, respectively, with multiplicities. Define $\Lambda = (\cup_{i \in \mathrm{N}} \Lambda_i) \cup \Lambda_A$, where the unions are taken with multiplicities.

5.2.1 Dynamic State Feedback

Let the subsystem states be directly measurable by the corresponding controller. Define

$$Z = (sI - A)^{-1}B \text{ and } T_i = (sI - A_i)^{-1}B_i,\ i \in \mathrm{N}, \tag{5.26}$$

where the plant Z is assumed to satisfy (5.16) (see [15]). In the special case (5.26) above $n_i = p_i$, $i \in \mathrm{N}$ and $C_i = I_{p_i}$, $i \in \mathrm{N}$. Then, the problem is to determine controllers Z_{ci}, $i \in \mathrm{N}$ such that the pairs (T_i, Z_{ci}), $i \in \mathrm{N}$ are stable and the pair $(Z, diag\{Z_{c1}, ..., Z_{cN}\})$ is stable. We have the following result whose proof follows from the problem definition.

Theorem (5.4). *Let Z and T_i, $i \in \mathrm{N}$ be defined according to (5.26). Then, (p1) is solvable using state feedback if and only if DCSP for Z and T_i, $i \in \mathrm{N}$ is solvable.*

Although the above theorem gives a complete solution to the problem, some further analysis concerning the decentralized blocking zeros of the auxiliary plant associated with Z and T_i, $i \in \mathrm{N}$ will now be made.

Proposition (5.6). *The set of C_{+e} decentralized blocking zeros of the auxiliary plant $\tilde{Z}$ associated with Z and T_i, $i \in \mathrm{N}$ denoted by $\hat{\Psi}$ is given as follows.*

$$\hat{\Psi} = \{\infty\} \cup \{z \in C_+ \mid \textit{There exists a permutation } \{i_1, ..., i_N\} \textit{ of } \mathrm{N} \textit{ such that}$$

$$rank \begin{bmatrix} zI - A_{i_1} - A_{i_1 i_1} & B_{i_1} \\ -A_{i_2 i_1} & 0 \\ \vdots & \vdots \\ -A_{i_N i_1} & 0 \\ zI - A_{i_1} & B_{i_1} \end{bmatrix}(z) = p_{i_1},$$

$$rank \begin{bmatrix} zI - A_{i_1} - A_{i_1 i_1} & -A_{i_1 i_2} & B_{i_1} & 0 \\ -A_{i_2 i_1} & zI - A_{i_2} - A_{i_2 i_2} & 0 & B_{i_2} \\ -A_{i_3 i_1} & -A_{i_3 i_2} & 0 & 0 \\ & & \vdots & \vdots \\ -A_{i_N i_1} & -A_{i_N i_2} & 0 & 0 \\ 0 & zI - A_{i_2} & 0 & B_{i_2} \end{bmatrix}(z) = p_{i_1} + p_{i_2},$$

$$rank \begin{bmatrix} zI - A_{i_1} - A_{i_1 i_1} & -A_{i_1 i_2} & -A_{i_1 i_N} & B_{i_1} & 0 & 0 \\ -A_{i_2 i_1} & zI - A_{i_2} - A_{i_2 i_2} & -A_{i_2 i_N} & 0 & B_{i_2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -A_{i_N i_1} & -A_{i_N i_2} & zI - A_{i_N} - A_{i_N i_N} & 0 & 0 & B_{i_N} \\ 0 & 0 & zI - A_{i_N} & 0 & 0 & B_{i_N} \end{bmatrix}(z) = p\}.$$

Proof. We let $Q = (zI - A).\frac{1}{(z+1)}$, $\tilde{R} = B.\frac{1}{(z+1)}$, $D_{li} = (zI - A_i).\frac{1}{(z+1)}$, $N_{li} = B_i.\frac{1}{(z+1)}$, $i \in \mathbf{N}$. With these particular choices of coprime fractions of Z and T_i's, the special form of $\hat{\Psi}$ above follows from the definition. We note that $z = \infty$ belongs to $\hat{\Psi}$ as $S = QN_d - \tilde{R}D_d$ is a strictly proper rational matrix.□

Utilizing the above proposition we below give two sufficient conditions in Theorems (5.5) and (5.6) for the solution of DCSP in terms of various system matrices associated with the composite system $\Sigma = (C, A, B)$ and the susbsystems $\Sigma_i = (C_i, A_i, B_i)$, $i \in \mathbf{N}$. Note that for $i \in \mathbf{N}$

$$\Pi_i := \begin{bmatrix} zI - A_i & B_i \\ -A_{1i} & 0 \\ -A_{2i} & 0 \\ \vdots & \vdots \\ -A_{Ni} & 0 \end{bmatrix}$$

is a system matrix associated with the system consisting of the state matrix A_i, input matrix B_i and output matrix $[A'_{1i}\ A'_{2i}\ \dots\ A'_{Ni}]'$. Also, for a proper subset $\{i_1, \dots, i_\mu\}$ of $\mathbf{N}$ the matrix

$$\Pi_{\{i_1,\dots,i_\mu\}} := \begin{bmatrix} zI - A_{i_1} - A_{i_1 i_1} & -A_{i_1 i_2} & -A_{i_1 i_\mu} & B_{i_1} & 0 & 0 \\ -A_{i_2 i_1} & zI - A_{i_2} - A_{i_2 i_2} & -A_{i_2 i_\mu} & 0 & B_{i_2} & 0 \\ \vdots & & & \vdots & \vdots & \vdots \\ -A_{i_\mu i_1} & -A_{i_\mu i_2} & zI - A_{i_\mu} - A_{i_\mu i_\mu} & 0 & 0 & B_{i_\mu} \\ -A_{i_{\mu+1} i_1} & -A_{i_{\mu+1} i_2} & -A_{i_{\mu+1} i_\mu} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -A_{i_N i_1} & -A_{i_N i_2} & -A_{i_N i_\mu} & 0 & 0 & 0 \end{bmatrix}$$

is a system matrix associated with the system consisting of the state, input and output matrices, respectively, below.

$$\begin{bmatrix} A_{i_1} + A_{i_1 i_1} & A_{i_1 i_2} & & A_{i_1 i_\mu} \\ A_{i_2 i_1} & A_{i_2} + A_{i_2 i_2} & & A_{i_2 i_\mu} \\ \vdots & & & \vdots \\ A_{i_\mu i_1} & A_{i_\mu i_2} & & A_{i_\mu} + A_{i_\mu i_\mu} \end{bmatrix},$$

$$diag\{B_{i_j}; j \in \mathrm{N}\}. \begin{bmatrix} A_{i_{\mu+1} i_1} & A_{i_{\mu+1} i_2} & & A_{i_{\mu+1} i_\mu} \\ \vdots & \vdots & & \vdots \\ A_{i_N i_1} & A_{i_N i_2} & & A_{i_N i_\mu} \end{bmatrix}.$$

Theorem (5.5). *Let* $rank\,\Pi_i(z) > p_i$, $\forall z \in C_+$. *Then, (p1) is always solvable using state feedback.*

Proof. If $z_0 \in \hat{\Psi}$, then $rank\,\Pi_i(z_0) = p_i$, for some $i \in \mathrm{N}$. If the hypothesis of the theorem is satisfied it holds that $\hat{\Psi} = \{\infty\}$. In this case Theorem (5.1) states that DCSP is solvable and the proof is completed.□

Theorem (5.6). *Let* $rank\,\hat{\Pi}_{\{i_1,\ldots,i_\mu\}}(z) > p_{i_1} + p_{i_2} + \ldots + p_{i_\mu}$, $\forall z \in C_+$, *for all proper subsets* $\{i_1, \ldots, i_\mu\}$ *of* N. *Then, (p1) is always solvable using state feedback.*

Proof. Observe from the proof of Proposition (5.6) that

$$\Gamma = \{\infty\} \cup \left\{ z \in C_{+\epsilon} \;\middle|\; \begin{array}{c} For\ some\ proper\ subset\ \{i_1, \ldots, i_\mu\}\ of\ \mathrm{N} \\ rank\,\hat{\Pi}_{\{i_1,\ldots,i_\mu\}}(z) = p_{i_1} + p_{i_2} + \ldots + p_{i_\mu} \end{array} \right\}.$$

Then, the result follows from Theorem (5.1).□

Corollary (5.2). Consider the special case where the composite system (5.24) is symmetrically interconnected [51] (see also [32]) so that $A_i = A_o$,

$$A_{ij} = \begin{cases} H & , i \neq j \\ 0 & , i = j \end{cases},$$

$B_i = B_o$ and $p_i = p_o$ $i, j \in \mathrm{N}$ for some matrices A_o, H, B_o. Theorem (5.5) states that (p1) for the symmetrically interconnected system is solvable using state feedback if

$$rank \begin{bmatrix} zI - A_o & B_o \\ -H & 0 \end{bmatrix} (z) > p_o, \ \forall z \in C_+$$

which holds if and only if the $p_o + 1$'st invariant factor of the system $(H, zI - A_o, B_o)$ has no unstable zeros.□

As an application of Theorem (5.2) we have the following result.

Theorem (5.7). *Assume that* $(\cup_{i\in\mathbf{N}}\Lambda_i) \cap \Lambda_A = \emptyset$ *and* $(\cup_{i\in\mathbf{N}}\Lambda_i) \cap \hat{\Psi} = \emptyset$. *Then, (p1) is solvable using state feedback if and only if DSSP for $T - Z$ is solvable.*

We now investigate a previously established fact using our setup [45], [52] (see also the references in [52]). Let the input matrices B_i, $i \in \mathbf{N}$ be full-column rank.

Theorem (5.8). *Assume that range $A_{ij} \subseteq$ rangeB_i, $i, j \in \mathbf{N}$. Then, (p1) is always solvable using state feedback.*

Proof. Let D_{li}, N_{li}, $i \in \mathbf{N}$, Q and $\bar{R}$ be as in the proof of Proposition (5.6). We also obtain D_{ri}, N_{li}, K_i, L_i, $\bar{K}_i$, $\bar{L}_i$ $i \in \mathbf{N}$ defined by (5.1) and (5.2) such that $\bar{Z}$ given by (5.6), (5.7), (5.8) satisfies (5.16). By assumption we have

$$\begin{bmatrix} A_{11} & A_{12} & & A_{1N} \\ A_{21} & A_{22} & & A_{2N} \\ \vdots & & & \vdots \\ A_{N1} & A_{N2} & & A_{NN} \end{bmatrix} \doteq - \begin{bmatrix} B_1 & 0 & & 0 \\ 0 & B_2 & & 0 \\ & \vdots & & \vdots \\ 0 & 0 & & B_N \end{bmatrix} E$$

for some matrix $E = [E_{ij}]$ of appropriate size where $E_{ij} \in \mathcal{R}^{r_i \times p_j}$, $i, j \in \mathbf{N}$. It holds that $Q = \bar{D}_d + \bar{N}_d E$. Hence $QK_d + \bar{R}L_d = I + \bar{N}_d E K_d$ and $QN_d - \bar{R}D_d = \bar{N}_d E N_d$. We have $\bar{Z} = (I + \bar{N}_d E K_d)^{-1} \bar{N}_d E N_d = \bar{N}_d (I + E K_d \bar{N}_d)^{-1} E N_d$. Since B_i, $i \in \mathbf{N}$ are full column rank $\hat{N}_d \bar{N}_d = I_r \cdot \frac{1}{(z+1)}$ for some $\hat{N}_d$ of appropriate size. Then, z is an unstable decentralized blocking zero of $\bar{Z}$ if and only if it is an unstable decentralized blocking zero of $(I + EK_d\bar{N}_d)^{-1}EN_d$. The identity

$$\begin{bmatrix} \bar{D}_d & \bar{N}_d \\ -\bar{L}_d & \bar{K}_d \end{bmatrix} \begin{bmatrix} K_d & -N_d \\ L_d & D_d \end{bmatrix} = I$$

implies that $K_d\bar{N}_d = N_d\bar{K}_d$. As a result, DSSP for $\bar{Z}$ above is solvable if and only if DSSP for $\hat{Z} := (I + EN_d\bar{K}_d)^{-1}EN_d$ is solvable. It will now be shown that DSSP for $\hat{Z}$ is solvable. Let an unstable decentralized blocking zero z_0 of $\hat{Z}$ be such that

$$rank \left[\begin{array}{cc} I + EN_d\bar{K}_d & E_1N_{r1} \ \dots \ E_jN_{rj} \\ \begin{bmatrix} 0 & 0 & -I_{r_j} & 0 & & 0 \\ 0 & 0 & 0 & -I_{r_{j+1}} & & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -I_{r_N} \end{bmatrix} & 0 \end{array} \right] (z_0) = r, \ \forall j \in \mathbf{N}, \tag{5.27}$$

where $E_i \in \mathbf{S}^{r \times p_i}$, $i \in \mathbf{N}$ denote the i'th block column of E. Using (5.27) above one can show that $(E_{ij}N_{rj})(z_0) = 0, i = 1, ..., N, j = 1, ..., i$. Observe that in this case $det(I + EN_d\bar{K}_d)(z_0) = det(I) = 1$. Modifying the indices appropriately and repeating for all $\mathcal{R}_{+e}$ decentralized blocking zeros of $\hat{Z}$ we conclude that for any $\mathcal{R}_{+e}$ decentralized blocking zero z of $\hat{Z}$, $det(I + EN_d\bar{K}_d)(z) = det(I) = 1$. This shows that DSSP for $\bar{Z}$ is solvable. The proof is then completed via the above discussion.□

The following result states that (p1) is generically solvable in terms of the interconnection matrices. Consider the following condition

$$\begin{array}{c} for\ each\ proper\ subset\ \{i_1, ..., i_\mu\}\ of\ \mathbf{N} \\ rank\, diag\{C_{i_{\mu+1}}, ..., C_N\} \geq 2\ or\ rank\, diag\{B_{i_1}, ..., B_\mu\} \geq 2 \end{array} \tag{5.28}$$

which is already implied by (iii) of (5.16) when $N \geq 3$.

Theorem (5.9). *For almost all*

$$A_c := \begin{bmatrix} A_{11} & A_{12} & & A_{1N} \\ A_{21} & A_{22} & & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & & A_{NN} \end{bmatrix} \in \mathcal{R}^{p \times p}$$

(p1) is solvable using state feedback.

The proof is based on the following lemmata.

Lemma (5.1). *Let $\bar{A} \in \mathcal{R}^{n\times n}$, $\bar{B} \in \mathcal{R}^{n\times r} - \{0\}$ and $\bar{C} \in \mathcal{R}^{p\times n} - \{0\}$ be given such that $rank\bar{B} \geq 2$ or $rank\bar{C} \geq 2$. Given each $\varepsilon \in \mathcal{R}_- - \{0\}$ there exists $\Delta \in \mathcal{R}^{n\times n}$ such that $||\Delta|| < \varepsilon$ and*

$$rank \begin{bmatrix} zI - (\bar{A} + \Delta) & \bar{B} \\ \bar{C} & 0 \end{bmatrix} (z) > n, \ \forall z \in C_-.$$

The proof of Lemma (5.1) is based on the following lemma.

Lemma (5.2). *Let E_1, $E_2 \in \mathcal{R}^{n+1\times n}$ be given. Given each $\bar{\varepsilon} \in \mathcal{R}_+ - \{0\}$ there exists $\bar{\Delta} \in \mathcal{R}^{n+1\times n}$ such that $||\bar{\Delta}|| < \bar{\varepsilon}$ and $rank(zE_1 - E_2 - \bar{\Delta})(z) \geq n$, $\forall z \in C_+$.*

Proof. The proof is given by induction. For $n = 1$ let $E_1 = [e_1\ e_2]'$, $E_2 = [\bar{e}_1\ \bar{e}_2]'$ where $e_1, e_2, \bar{e}_1, \bar{e}_2 \in \mathcal{R}$. It is clear that with arbitrarily small perturbations δ_1, $\delta_2 \in \mathcal{R}$ the polynomials $ze_1 - \bar{e}_1 - \delta_1$ and $ze_2 - \bar{e}_2 - \delta_2$ can be made coprime, proving the claim for $n = 1$.

Now assume that the lemma holds true for $l \geq 1$. Let $n = l + 1$. Define

$$E_1 = \begin{bmatrix} e_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \ E_2 = \begin{bmatrix} \bar{e}_{11} & \bar{E}_{12} \\ \bar{E}_{21} & \bar{E}_{22} \end{bmatrix}$$

where $e_{11}, \bar{e}_{11} \in \mathcal{R}$, $E_{12}, \bar{E}_{12}\ \mathcal{R}^{1\times l}$, $E_{21}, \bar{E}_{21} \in \mathcal{R}^{l+1\times 1}$, $E_{22}, \bar{E}_{22} \in \mathcal{R}^{l+1\times l}$. By the inductive hypothesis, there exists $\Delta_{22} \in \mathcal{R}^{l+1\times l}$ with norm less than $\bar{\varepsilon}/3$ such that $rank[zE_{22} - \bar{E}_{22} - \Delta_{22}](z) \geq l$, $\forall z \in C_+$. There exists a unimodular polynomial matrix $T \in \mathbf{R}^{l+1\times l+1}$ such that

$$T(zE_{22} - \bar{E}_{22} - \Delta_{22}) = \begin{bmatrix} I_l \\ 0 \end{bmatrix}.$$

Define $[\check{T}'\ \bar{T}']' = T$ such that $\check{T} \in \mathbf{R}^{l\times l+1}$, $\bar{T} \in \mathbf{R}^{1\times l+1}$. Further define $E = \check{T}(zE_{21} - \bar{E}_{21})$ and $e = \bar{T}(zE_{21} - \bar{E}_{21})$. Since $\bar{T}$ is nonzero, there exists $\Delta_{21} \in \mathcal{R}^{l+1\times 1}$ with norm less than $\bar{\varepsilon}/3$ such that $e - \bar{T}\Delta_{21}$ is a nonzero polynomial. There also exists $\delta_{11} \in \mathcal{R}$ satisfying $|\delta_{11}| < \bar{\varepsilon}/3$ such that the polynomials

$$ze_{11} - \bar{e}_{11} - (zE_{12} - \bar{E}_{12})(E - \check{T}\Delta_{21}) - \delta_{11}, \quad e - \bar{T}\Delta_{21} \tag{5.29}$$

are coprime. Observe that the norm of

$$\bar{\Delta} = \begin{bmatrix} \delta_{11} & 0 \\ \Delta_{21} & \Delta_{22} \end{bmatrix}$$

is less than $\bar{\varepsilon}$. We will now verify that $rank(zE_1 - E_2 - \bar{\Delta})(z) \geq l+1$, $\forall z \in \mathcal{C}_+$. For any $z \in \mathcal{C}_+$ it holds that

$$rank(zE_1 - E_2 - \bar{\Delta})(z) = rank(\begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix}(zE_1 - E_2 - \bar{\Delta}))(z)$$

$$= rank(\begin{bmatrix} ze_{11} - e_{11} - \delta_{11} - (zE_{12} - \bar{E}_{12})(E - \tilde{T}\Delta_{21}) & 0 \\ 0 & I_l \\ e - \tilde{T}\Delta_{21} & 0 \end{bmatrix})(z)$$

Since the polynomials in (5.29) are coprime $rank(zE_1 - E_2 - \bar{\Delta})(z) \geq l+1$ $\forall z \in \mathcal{C}_+$. This completes the proof.□

Proof of Lemma (5.1). We assume without loss of generality that $r = 1$, $p = 2$. There exist nonsingular real matrices U and V such that $\bar{C}V = [I_2\ 0]$, $U\bar{B} = [1\ 0']'$. Let

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & E_1 \end{bmatrix} := UV, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & E_2 \end{bmatrix} := U\bar{A}V$$

so that A_1, $\tilde{A}_1 \in \mathcal{R}^{1\times 2}$, A_2, $\tilde{A}_2 \in \mathcal{R}^{1\times n-2}$, A_3, $\tilde{A}_3 \in \mathcal{R}^{n-1\times 2}$ and E_1, $E_2 \in \mathcal{R}^{n-1\times n-2}$. From Lemma (5.2) there exists $\bar{\Delta} \in \mathcal{R}^{n-1\times n-2}$ with norm less than $\varepsilon/(||U^{-1}||.||V^{-1}||)$ such that $rank(zE_1 - E_2 - \bar{\Delta}) \geq n-2$, $\forall z \in \mathcal{C}_+$. Define

$$\Delta = U^{-1}\begin{bmatrix} 0 & 0 \\ 0 & \bar{\Delta} \end{bmatrix}V^{-1}.$$

Observe that $||\Delta|| < \varepsilon$. On the other hand,

$$rank\begin{bmatrix} zI - (\bar{A}+\Delta) & \bar{B} \\ -\bar{C} & 0 \end{bmatrix}(z) = rank(\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} zI - (\bar{A}+\Delta) & \bar{B} \\ -\bar{C} & 0 \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix})(z)$$

$$= rank\begin{bmatrix} 0 & 0 & 1 \\ 0 & zE_1 - E_2 - \bar{\Delta} & 0 \\ I_2 & 0 & 0 \end{bmatrix}(z),$$

$\forall z \in \mathcal{C}_+$. By the choice of $\bar{\Delta}$

$$rank \begin{bmatrix} 0 & 0 & 1 \\ 0 & zE_1 - E_2 - \bar{\Delta} & 0 \\ I_2 & 0 & 0 \end{bmatrix} (z) \geq n+1, \ \forall z \in \mathcal{C}_+$$

which implies

$$rank \begin{bmatrix} zI - (\bar{A} + \Delta) & \bar{B} \\ -\bar{C} & 0 \end{bmatrix} (z) \geq n+1, \ \forall z \in \mathcal{C}_+.$$

This completes the proof.□

Proof of Theorem (5.9). For Q and $\bar{R}$ we again refer to the proof of Proposition (5.6). We remind also that $[P_1' \ \dots \ P_N']' = I_p$, so that $P_i \in \mathbf{S}^{p_i \times p}$, $i \in \mathbf{N}$.

Step 1. Since

$$(diag\{C_1, ..., C_N\}, zI - A, diag\{B_1, ..., B_N\}), \ (\begin{bmatrix} P_1 \\ \vdots \\ P_N \end{bmatrix}, Q, \begin{bmatrix} R_1 & & R_N \end{bmatrix})$$

are two stabilizable and detectable realizations of Z, they are Fuhrmann equivalent over $\mathbf{P}_s$ [27]. Fix any proper subset $\{i_1, ..., i_\mu\}$ of $\mathbf{N}$. From Lemma (2.1) the systems

$$(diag\{C_{i_{\mu+1}}, ..., C_{i_N}\}, zI - A, diag\{B_{i_1}, ..., B_{i_\mu}\}), \ (\begin{bmatrix} P_{i_{\mu+1}} \\ \vdots \\ P_{i_N} \end{bmatrix}, Q, \begin{bmatrix} R_{i_1} & & R_{i_\mu} \end{bmatrix})$$

are also Fuhrmann equivalent over $\mathbf{P}_s$. From Lemma (2.2) we conclude that

$$rank \begin{bmatrix} zI - A & diag\{B_{i_1}, ..., B_{i_\mu}\} \\ diag\{-C_{i_{\mu+1}}, ..., -C_{i_N}\} & 0 \end{bmatrix} (z) > p, \ \forall z \in \mathcal{C}_+ \quad (5.30)$$

if and only if

$$rank \begin{bmatrix} Q & R_{i_1} & R_{i_\mu} \\ -P_{i_{\mu+1}} & & \\ \vdots & 0 & \\ -P_{i_N} & & \end{bmatrix} (z) > p, \ \forall z \in \mathcal{C}_+. \quad (5.31)$$

If (5.31) holds for all proper subsets $\{i_1, ..., i_\mu\}$ of N then by definition $\Gamma \subseteq \{\infty\}$. Thus, if we can show that (5.30) holds for almost all A_c and for all proper subsets $\{i_1, ..., i_\mu\}$ of N for which $(zI - A)^{-1}B$ satisfies (5.16) then the proof will be completed via Proposition (5.4) and Remark (5.2). (Recall that $A = diag\{A_1, ..., A_N\} + A_c$.)

Step 2. Fix any proper subset $\{i_1, ..., i_\mu\}$ of N. If A_c is such that (5.30) holds this means that the $p+1$'st invariant factor of the system matrix associated with

$$(diag\{C_{i_{\mu+1}}, ..., C_{i_N}\}, zI - A, diag\{B_{i_1}, ..., B_{i_\mu}\})$$

has only stable zeros, which is a robust property under sufficiently small perturbations on A_c. On the other hand, if A_c is such that (5.30) fails, i.e., if for some $z \in \mathcal{C}_+$

$$rank \begin{bmatrix} zI - A & diag\{B_{i_1}, ..., B_{i_\mu}\} \\ diag\{-C_{i_{\mu+1}}, ..., -C_{i_N}\} & 0 \end{bmatrix} (z) = p,$$

Lemma (5.1) reveals an arbitrarily small perturbation on A_c such that (5.30) is made to be satisfied with A modified accordingly. (Note that (iii) of (5.16) ensures $rank\ diag\{C_{i_{\mu+1}}, ..., C_{i_N}\} \geq 2$ or $rank\ diag\{B_{i_1}, ..., B_{i_\mu}\} \geq 2$.) Hence, the set of A_c for which (5.30) holds is open and dense in $\mathcal{R}^{p\times p}$. Repeating for all proper subsets of N and using the fact that the intersection of open and dense subsets is also open and dense we conclude that for all proper subsets $\{i_1, ..., i_N\}$ of N (5.30) holds, for almost all A_c. Also note that the set of A_c for which (5.16) is satisfied is open. These arguments, together with the conclusion of Step 1 above complete the proof. (In the above proof the dependence of A, Q and Γ on the interconnection matrix A_c has not been indicated for the notational convenience.)□

5.2.2 Dynamic Output Feedback

In case only the subsystem outputs are available to the local controllers, we define

$$Z = C(sI - A)^{-1}B \text{ and } T_i = C_i(sI - A_i)^{-1}B_i,\ i \in \mathrm{N} \tag{5.32}$$

so that $Z_{ij} = [0 \ldots C_i \ldots 0]\ (zI - A)^{-1}\ [0' \ldots B_j' \ldots 0']'$. We assume that Z satisfies (5.16).

Theorem (5.10). *Let Z and T_i, $i \in \mathrm{N}$ be defined according to (5.32). Then, (p1) is solvable using output feedback if and only if DCSP for Z and T_i, $i \in \mathrm{N}$ is solvable.*

As in the case of state feedback we will investigate the solvability of DCSP in detail. We first give the set of C_{+e} decentralized blocking zeros $\hat{\Psi}$ of the auxiliary plant $\tilde{Z}$ associated with Z and T_i, $i \in \mathrm{N}$.

Proposition (5.7). *The set $\hat{\Psi}$ associated with Z and T_i, $i \in \mathrm{N}$ is given as follows:*

$$\hat{\Psi} = \{\infty\} \cup \{z \in C_+ \mid \textit{There exists a permutation } \{i_1, \ldots, i_N\} \textit{ of } \mathrm{N} \textit{ such that}$$

$$\mathrm{rank} \left[\begin{array}{cccc|cccccc|ccccc}
0 & 0 & 0 & 0 & & & & & & & B_{i_1} & & & 0 & 0 \\
0 & 0 & 0 & 0 & \multicolumn{6}{c|}{zI \quad - \quad A} & 0 & B_{i_2} & & 0 & 0 \\
\vdots & & \vdots & \vdots & & & & & & & \vdots & & & \vdots & \vdots \\
0 & 0 & 0 & 0 & & & & & & & 0 & 0 & \ldots & B_{i_j} & 0 \\
\hline
I_{p_{i_1}} & 0 & \ldots & 0 & -C_{i_1} & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & I_{p_{i_2}} & & 0 & 0 & -C_{i_2} & & 0 & & 0 & 0 & 0 & & 0 & 0 \\
\vdots & & & \vdots & \vdots & \vdots & & & & \vdots & \vdots & & & & \vdots \\
0 & 0 & & I_{p_{i_j}} & 0 & 0 & & -C_{i_j} & & 0 & 0 & 0 & & 0 & 0 \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots & & \vdots & & & & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & -C_{i_N} & 0 & 0 & \ldots & 0 & 0 \\
\hline
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & \ldots & B_{i_j} & zI - A_{i_j} \\
0 & 0 & & I_{p_{i_j}} & 0 & 0 & & 0 & & 0 & 0 & 0 & & 0 & -C_{i_j}
\end{array} \right] (z)$$

$$= p_{i_1} + \quad + p_{i_j} + q + q_{i_j}, \ \forall j \in \mathrm{N} \}$$

Proof. Let $Z = \tilde{P}\tilde{Q}^{-1}\tilde{R}$ be a bicoprime fraction of Z over S such that $\tilde{Q} \in \mathrm{S}^{q \times q}$, $\tilde{P} = [\tilde{P}_1' \ldots \tilde{P}_N']'$, $\tilde{P}_i \in \mathrm{S}^{p_i \times q}$, $\tilde{R} = [\tilde{R}_1 \ldots \tilde{R}_N]$, $\tilde{R}_i \in \mathrm{S}^{q \times r_i}$, $i \in \mathrm{N}$. Also let $T_i = \bar{P}_i\bar{Q}_i^{-1}\bar{R}_i$ be a bicoprime fraction of T_i, $i \in \mathrm{N}$, where $\bar{Q}_i \in \mathrm{S}^{q_i \times q_i}$, $\bar{P}_i \in \mathrm{S}^{p_i \times q_i}$, $\bar{R}_i \in \mathrm{S}^{q_i \times r_i}$, $i \in \mathrm{N}$. Also recall that $Z = Q^{-1}R$ and $T_i = D_{li}N_{li}$, $i \in \mathrm{N}$ be some left and right coprime fractions of Z and T_i, $i \in \mathrm{N}$. Using unimodular operations it holds that

$$rank \begin{bmatrix} Q_{i_1} & R_{i_1} \\ D_{li_1} & N_{li_1} \end{bmatrix} (z) = p_{i_1}$$

for some $z \in \mathcal{C}_+$ if and only if

$$
rank \left[\begin{array}{c|c|cc}
0 & \tilde{Q} & \tilde{R}_{i_1} & 0 \\
\hline
I & -\tilde{P}_{i_1} & 0 & 0 \\
0 & -\tilde{P}_{i_2} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & -\tilde{P}_{i_N} & 0 & 0 \\
\hline
0 & 0 & \bar{R}_{i_1} & \bar{Q}_{i_1} \\
I & 0 & 0 & -\bar{P}_{i_1}
\end{array} \right] (z) = p_{i_1} + q + q_{i_1}.
$$

Similarly, for any $j \in \mathbf{N}$

$$
rank \begin{bmatrix} Q_{i_1} & & Q_{i_j} & R_{i_1} & & R_{i_j} \\ 0 & & D_{i_j} & 0 & & N_{i_j} \end{bmatrix} (z) = p_{i_1} + p_{i_2} + \ldots + p_{i_j}
$$

for some $z \in \mathcal{C}_+$ if and only if

$$
rank \left[\begin{array}{cccc|c|ccccc}
0 & 0 & \ldots & 0 & \tilde{Q} & \tilde{R}_{i_1} & \tilde{R}_{i_2} & \ldots & \tilde{R}_{i_j} & 0 \\
\hline
I_{p_{i_1}} & 0 & & 0 & -\tilde{P}_{i_1} & 0 & 0 & & 0 & 0 \\
0 & I_{p_{i_2}} & & 0 & -\tilde{P}_{i_2} & 0 & 0 & & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & & I_{p_{i_j}} & -\tilde{P}_{i_j} & 0 & 0 & & 0 & 0 \\
0 & 0 & & 0 & -\tilde{P}_{i_{j+1}} & 0 & 0 & & 0 & 0 \\
 & & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 0 & -\tilde{P}_{i_N} & 0 & 0 & \ldots & 0 & 0 \\
\hline
0 & 0 & & 0 & 0 & 0 & 0 & & \bar{R}_{i_j} & \bar{Q}_{i_j} \\
0 & 0 & & I & 0 & 0 & 0 & & 0 & -\bar{P}_{i_j}
\end{array} \right] (z)
$$

$$
= p_{i_1} + p_{i_2} + \ldots + p_{i_j} + q + q_{i_j}.
$$

We define $\tilde{Q} = (zI - A).\frac{1}{(z+1)}$, $\tilde{P} = C$, $\tilde{R} = B.\frac{1}{(z+1)}$, $\bar{Q}_i = (zI - A_i).\frac{1}{(z+1)}$, $\bar{P}_i = C_i$, $\bar{R}_i = B_i.\frac{1}{(z+1)}$, $i \in \mathbf{N}$. The result now follows from the above discussion. Note that $z = \infty$ belongs to $\hat{\Psi}$ as Z and T_i are strictly proper.□

As an application of Remark (5.2), the following theorem states a sufficient condition for the solvability of DCSP.

Theorem (5.11). *Let*

$$rank \begin{bmatrix} zI - A & diag\{B_{i_1}, ..., B_{i_\mu}\} \\ diag\{-C_{i_{\mu+1}}, ..., -C_{i_N}\} & 0 \end{bmatrix} (z) > n, \ \forall z \in C_+$$

for all proper subsets $\{i_1, ..., i_\mu\}$ of $\mathbf{N}$. *Then, (p1) is always solvable using output feedback.*

Proof. Following similar arguments to the proof of Proposition (5.4) it can be shown that the set Γ associated with (C, A, B) takes the following form

$$\Gamma = \{\infty\} \cup \{z \in C_+ | For\ some\ proper\ subset\ \{i_1, ..., i_\mu\}\ of\ \mathbf{N}$$
$$rank \begin{bmatrix} zI - A & diag\{B_{i_1}, ..., B_{i_\mu}\} \\ diag\{-C_{i_{\mu+1}}, ..., -C_{i_N}\} & 0 \end{bmatrix} (z) = n \quad \}.$$

If the hypothesis of the theorem holds then $\Gamma = \{\infty\}$. The result now follows from Remark (5.2).□

Our next result is the extension of Theorem (5.7) to output feedback case.

Theorem (5.12). *Assume that* $(\cup_{i \in \mathbf{N}} \Lambda_i) \cap \Lambda_A = \emptyset$ *and* $(\cup_{i \in \mathbf{N}} \Lambda_i) \cap \hat{\Psi} = \emptyset$. *Then, (p1) is solvable using dynamic output feedback if and only if DSSP for $T - Z$ is solvable.*

The final result for the output feedback case is given by the next theorem which is concerned with the genericity of solution in terms of the interconnection matrices. The proof of Theorem (5.13) follows the same arguments as that of Theorem (5.9) and is therefore omitted.

Theorem (5.13). *For almost all*

$$A_c := \begin{bmatrix} A_{11} & A_{12} & & A_{1N} \\ A_{21} & A_{22} & & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & & A_{NN} \end{bmatrix} \in \mathcal{R}^{n \times n}$$

(p1) is solvable using output feedback.

5.2.3 Dynamic Interconnections

A more general version of the above problem can be stated in terms of dynamic interconnections [41]. Let $\dot{z}_i = \sum_{j=1}^{N} \bar{A}_{ij} z_j + \sum_{j=1}^{N} \bar{B}_{ij} x_j$, $i \in \mathbf{N}$ describe the interconnection dynamics. Assume that the subsystems (5.23) are interconnected by $u_i = \sum_{j=1}^{N} \bar{C}_{ij} z_j + \sum_{j=1}^{N} A_{ij} x_j$, $i \in \mathbf{N}$. Then, the composite system can be described as

$$\Sigma : \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A_e \begin{bmatrix} x \\ z \end{bmatrix} + B_e v \qquad\qquad (5.33)$$

$$y = C_e \begin{bmatrix} x \\ z \end{bmatrix}$$

where

$$A_e = \begin{bmatrix} A_1 + A_{11} & A_{12} & & A_{1N} & \bar{C}_{11} & \bar{C}_{12} & & \bar{C}_{1N} \\ A_{21} & A_2 + A_{22} & & A_{2N} & \bar{C}_{21} & \bar{C}_{22} & & \bar{C}_{2N} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & & A_N + A_{NN} & \bar{C}_{N1} & \bar{C}_{N2} & & \bar{C}_{NN} \\ \bar{B}_{11} & \bar{B}_{12} & & \bar{B}_{1N} & \bar{A}_{11} & \bar{A}_{12} & & \bar{A}_{1N} \\ \bar{B}_{21} & \bar{B}_{22} & & \bar{B}_{2N} & \bar{A}_{21} & \bar{A}_{22} & & \bar{A}_{2N} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \bar{B}_{N1} & \bar{B}_{N2} & & \bar{B}_{NN} & \bar{A}_{N1} & \bar{A}_{N2} & & \bar{A}_{NN} \end{bmatrix},$$

$$B_e = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

$C_e = [C\ 0]$, and B, C, y and v are as in (5.25). It is assumed that $\Sigma = (C_e, A_e, B_e)$ is stabilizable and detectable. The problem (p1) is now to design local controllers Σ_{ci}, $i \in \mathbf{N}$ around subsystems Σ_i, which yield that the pairs (Σ_i, Σ_{ci}), $i \in \mathbf{N}$ are stable when the interconnections do not exist. It is further required that when the interconnections do exist the composite closed-loop system is stable. Let

$$Z := C_e(sI - A_e)^{-1} B_e \ \textit{and}\ T_i := C_i(sI - A_i)^{-1} B_i,\ i \in \mathbf{N}. \qquad (5.34)$$

We assume that Z satisfies (5.16).

Theorem (5.14). *Let Z and T_i, $i \in \mathbf{N}$ be defined according to (5.34). Then, (p1) is solvable using output feedback if and only if DCSP for Z and T_i, $i \in \mathbf{N}$ is solvable.*

To investigate $\hat{\Psi}$, the set of unstable decentralized blocking zeros associated with Z and T_i, $i \in \mathbf{N}$ we define $\bar{C} = [\bar{C}_{ij}]$, $i, j \in \mathbf{N}$, $\bar{B} = [\bar{B}_{ij}]$, $i, j \in \mathbf{N}$, $\bar{A} = [\bar{A}_{ij}]$, $i, j \in \mathbf{N}$, where $\bar{A} \in \mathcal{R}^{\bar{n} \times \bar{n}}$. Then, we have the following result.

Proposition (5.8). *The set $\hat{\Psi}$ associated with Z and T_i, $i \in \mathbf{N}$ is given as follows:*

$$\hat{\Psi} = \{\infty\} \cup \{ z \in \mathcal{C}_+ \mid \textit{There exists a permutation } \{i_1, \ldots, i_N\} \textit{ of } \mathbf{N} \textit{ such that}$$

$$\text{rank} \left[\begin{array}{cccc|ccccccc|ccccc}
0 & 0 & & 0 & & & & & & & & B_{i_1} & 0 & & 0 & 0 \\
0 & 0 & & 0 & zI & - & A & & & & -\bar{B} & 0 & B_{i_2} & & 0 & 0 \\
\vdots & & & & & & & & & & & \vdots & \vdots & & & \\
0 & 0 & & 0 & & & & & & & & 0 & 0 & & B_{i_j} & 0 \\
0 & 0 & \cdots & 0 & & -\bar{C} & & & & & zI - \bar{A} & 0 & 0 & \cdots & 0 & 0 \\
\hline
I_{p_{i_1}} & 0 & & 0 & -C_{i_1} & 0 & & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I_{p_{i_2}} & & 0 & 0 & -C_{i_2} & & 0 & & 0 & 0 & 0 & 0 & & 0 & 0 \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & & & \vdots \\
0 & 0 & & I_{p_{i_j}} & 0 & 0 & & -C_{i_j} & & 0 & 0 & 0 & 0 & & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -C_{i_N} & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & B_{i_j} & zI - A_{i_j} \\
0 & 0 & & I_{p_{i_j}} & 0 & 0 & & 0 & & 0 & 0 & 0 & 0 & & 0 & -C_{i_j}
\end{array} \right] (z)$$

$$= p_{i_1} + \cdots + p_{i_j} + n + \bar{n} + n_{i_j}, \ \forall j \in \mathbf{N} \}$$

Proof. We define $\tilde{Q} = (zI - A_e) . \frac{1}{(z+1)}$, $\tilde{P} = C_e$, $\tilde{R} = B_e . \frac{1}{(z+1)}$. The proof can be given similarly to Proposition (5.7).$\Box$

A sufficient condition for the solution of the problem is given next.

Theorem (5.15). *Let*

$$\text{rank} \left[\begin{array}{ccc|ccc}
 & zI - A & -\bar{B} & B_{i_1} & & 0 \\
 & & & 0 & & B_{i_\mu} \\
 & -\bar{C} & zI - \bar{A} & 0 & \cdots & 0 \\
\hline
-C_{i_{\mu+1}} & 0 & 0 & & & \\
\vdots & \vdots & \vdots & & 0 & \\
0 & -C_{i_N} & 0 & & &
\end{array} \right] (z) > n + \bar{n}, \ \forall z \in \mathcal{C}_+$$

for all proper subsets $\{i_1, \ldots, i_\mu\}$ of $\mathbf{N}$. *Then, (p1) is always solvable using output feedback.*

Proof. Similarly to the proof of Theorem (5.11) the set Γ associated with (C_e, A_e, B_e) is given by

$$\Gamma = \{\infty\} \cup \Big\{ z \in C_+ \,\Big|\, \textit{For some proper subset } \{i_1, \ldots, i_N\} \textit{ of } \mathbf{N} \quad \mathrm{rank} \left[\begin{array}{cc|ccc} zI - A & -\bar{B} & B_{i_1} & & 0 \\ & & \vdots & & \vdots \\ & & 0 & & B_{i_\mu} \\ -\bar{C} & zI - \bar{A} & 0 & \ldots & 0 \\ \hline \begin{array}{cc} -C_{i_{\mu+1}} & 0 \\ \vdots & \ddots \\ 0 & -C_{i_N} \end{array} & \begin{array}{c} 0 \\ \\ 0 \end{array} & & 0 & \end{array} \right](z) = n + \bar{n} \Big\}.$$

The result then follows from Remark (5.2).□

We conclude this section by the following results. Theorem (5.16) is an extension of Theorem (5.12) and gives the solution of the problem in a special case. Theorem (5.17) is an extension of Theorem (5.13) and states that the composite system (5.33) can be stabilized using locally stabilizing subsystem controllers for almost all interconnection dynamics. The proofs of these theorems can be given following the proofs of Theorems (5.2) and (5.9), respectively.

Let Λ_{A_e} be the set of C_+ eigenvalues of A_e counted with multiplicities.

Theorem (5.16). *Assume that* $(\cup_{i \in \mathbf{N}} \Lambda_i) \cap \Lambda_{A_e} = \emptyset$ *and* $(\cup_{i \in \mathbf{N}} \Lambda_i) \cap \Gamma = \emptyset$. *Then, (p1) is solvable using dynamic output feedback if and only if DSSP for $T - Z$ is solvable.*

Theorem (5.17). *For almost all* $(\bar{C}, A_c, \bar{B}, \bar{A}) \in \mathcal{R}^{r \times \bar{n}} \times \mathcal{R}^{n \times n} \times \mathcal{R}^{\bar{n} \times n} \times \mathcal{R}^{\bar{n} \times \bar{n}}$ *(p1) is solvable using output feedback.*

5.3 Diagonally Stabilizing Controllers

One of the approaches to the synthesis of controllers for multi input-multi output systems is to generalize the Nyquist and Inverse Nyquist Array methods which were originally developed for single input-single output systems [42]. The tecnique used to achieve this objective is, in general, based on the diagonal dominance of transfer matrices and has many applications to decentralized control ([78], [34], [74], [54], see also the references therein). One of the applications is concerned with the following problem.

(p2): *Let $Z = [Z_{ij}]$, $Z_{ij} \in \mathbf{P}^{p_i \times r_j}$ $i, j \in \mathbf{N}$ be the transfer matrix of a given plant where $p := \sum_{i=1}^{N} p_i$, $r := \sum_{i=1}^{N} r_i$. Determine local controllers Z_{ci}, $i \in \mathbf{N}$ such that* (i) *(Z_{ii}, Z_{ci}), $i \in \mathbf{N}$ are stable and* (ii) *$(Z, diag\{Z_{c1}, \ldots, Z_{cN}\})$ is stable.*

In the abovementioned references several aspects of this problem are considered and some sufficient conditions for its solution are given. Observe that the problem has already been formulated as a decentralized simultaneous stabilization problem and a necessary and sufficient solvability condition for it can be given using the solution of DCSP. Define

$$T_i = Z_{ii}, \; i \in \mathbf{N}. \tag{5.35}$$

We assume that Z satisfies (5.16). Assume that T_i, $i \in \mathbf{N}$ have the left and right coprime fractions as defined by (5.1).

Theorem (5.18). *Let T_i, $i \in \mathbf{N}$ be defined according to (5.35). Then, (p2) is solvable if and only if DCSP for Z and T_i, $i \in \mathbf{N}$ is solvable.*

Let a coprime fraction of Z be given as $Z = Q^{-1}[R_1 \ldots R_N]$ where $Q \in \mathbf{S}^{p \times p}$, $R_i \in \mathbf{S}^{p \times r_i}$, $i \in \mathbf{N}$. Also let $P_i \in \mathbf{P}^{p_i \times p}$, $i \in \mathbf{N}$ be defined as in (5.4). The following result is immediate from Proposition (5.4) and Remark (5.2).

Proposition (5.9). *Let the following set be empty or contains only one* *element*

$$\Gamma = \{z \in C_{+\epsilon} | \text{ For some proper subset } \{i_1, ..., i_\mu\} \text{ of } \mathbf{N}$$
$$rank \begin{bmatrix} Q & R_{i_1} & R_{i_\mu} \\ -P_{i_{\mu+1}} & & \\ \vdots & 0 & \\ -P_{i_N} & & \end{bmatrix} (z) = p, \ \forall z \in C_{+\epsilon} \}$$

Then, (p2) is always solvable.

Theorems (5.19)-(5.21) below investigate three special cases of this problem by extending some of the results in [57] to N-channel case.

Theorem (5.19). *Let Z_{ii}, $i \in \mathbf{N}$ be all stable. Then, (p2) is solvable if and only if DSSP for*

$$\begin{bmatrix} 0 & -Z_{12} & -Z_{1N} \\ -Z_{21} & 0 & -Z_{2N} \\ \vdots & \vdots & \vdots \\ -Z_{N1} & -Z_{N2} & 0 \end{bmatrix} \tag{5.36}$$

is solvable.

Proof. If Z_{ii}, $i \in \mathbf{N}$ be all stable then we can set $T_i = N_{li} = N_{ri}$, $D_{li} = D_{ri} = I$, $K_i = I$, $L_i = 0$, for all $i \in \mathbf{N}$. The matrices Q_{11} and R in (5.6) become

$$Q_{11} = \begin{bmatrix} Q & 0 & 0 & 0 \\ -P_1 & I & 0 & 0 \\ -P_2 & 0 & I & 0 \\ & & & \vdots \\ -P_N & 0 & 0 & I \end{bmatrix}, \quad R = \begin{bmatrix} -R_1 & -R_2 & -R_n \\ N_{r1} & 0 & 0 \\ 0 & N_{r2} & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & N_{rN} \end{bmatrix}.$$

Simple manipulations yield that $\bar{Z}$ in (5.8) is given by equation (5.36). This completes the proof.□

We note that the solvability of DCSP can be more explicitly observed in this special structure. For example if $N = 2$ then DCSP is solvable if and only if $\bar{Z}$ is free of unstable decentralized fixed modes and the ordered pair of sets $(\mathcal{S}_1, \mathcal{S}_2)$ satisfies the parity interlacing property, where $\mathcal{S}_1 := \{$ *the set of* $\mathcal{R}_{+\epsilon}$ *poles of*

Z_{12} with multiplicities $\}$ $\cup$ $\{$ *the set of $\mathcal{R}_{+e}$ poles of Z_{21} with multiplicities* $\}$ and $\mathcal{S}_2 :=$ $\{$ *the set of $\mathcal{R}_{+e}$-blocking zeros of Z_{12}* $\}$ $\cup$ $\{$ *the set of $\mathcal{R}_{+e}$-blocking zeros of Z_{21}* $\}$ where the union in $\mathcal{S}_1$ is taken with multiplicities.□

The interesting result below has various applications in the synthesis of reliable controllers (see also the next section). The result is due to the fact that the determinant of the auxiliary plant considered in DSSP becomes equal to either an even or odd power of a certain determinant when evaluated at decentralized blocking zeros.

Theorem (5.20). *Let Z be stabilizable and detectable from all channels.*

(a) *If N is odd then (p2) is always solvable*

(b) *If N is even then (p2) is solvable if and only if there are an even number of real poles of Z, counted with multiplicities between each pair of $\mathcal{R}_{+e}$ decentralized blocking zeros of the matrix*

$$\check{Z} := \begin{bmatrix} 0 & -Z_{12} & & -Z_{1N} \\ -Z_{21} & 0 & & -Z_{2N} \\ \vdots & \vdots & & \vdots \\ -Z_{N1} & -Z_{N2} & & 0 \end{bmatrix}.$$

Proof. The hypothesis implies that (Q, R_i), $i \in \mathbf{N}$ are left coprime and (Q, P_i), $i \in \mathbf{N}$ are right coprime pairs. Since $N_{ri}D_{ri}^{-1} = P_iQ^{-1}R_i$, $i \in \mathbf{N}$, $det(D_{ri})$ and $det(Q)$ are associates for all $i \in \mathbf{N}$. Let $\hat{\Psi}$ be defined as in (5.17).

Step 1. It will be shown that $\hat{\Psi}$ is precisely the set of $\mathcal{C}_{+e}$ decentralized blocking zeros of $\check{Z}$. Let $z \in \hat{\Psi}$. Then, from Proposition (5.4) there exits a permutation $\{i_1, ..., i_N\}$ of $\mathbf{N}$ such that

$$rank \begin{bmatrix} Q & R_{i_1} & & R_{i_{j-1}} \\ -P_{i_j} & & & \\ \vdots & & 0 & \\ -P_{i_N} & & & \end{bmatrix} (z) = p, \; \forall j \in \{2, ..., N\}. \qquad (5.37)$$

Since $(Q, R_{i_1}, ..., R_{i_{j-1}})$ is left coprime and $(Q, R_{i_j}, ..., R_{i_N})$ is right coprime, Lemma

(4.4) implies that equation (5.37) holds if and only if

$$\begin{bmatrix} Z_{i_j i_1} & \cdots & Z_{i_j i_{j-1}} \\ \vdots & & \vdots \\ Z_{i_N i_1} & & Z_{i_N i_{j-1}} \end{bmatrix}(z) = 0, \ \forall j \in \{2, ..., N\}.$$

This shows that z is an $\mathcal{C}_{+\epsilon}$ decentralized blocking zero of $\tilde{Z}$. Conversely, if z is an $\mathcal{C}_{+\epsilon}$ decentralized blocking zero of $\tilde{Z}$ then Lemma (4.4) implies $Q(z)$ is nonsingular. In this case $D_d(z)$ is also nonsingular. We can choose L_d such that the plant $\bar{Z}$ given by (5.6), (5.7), (5.8) satisfies (5.16). With this choice of $\bar{Z}$ one can show, following the proof of Proposition (5.5), that

$$\begin{aligned} \bar{Z}(z) &= (Q\tilde{D}_d^{-1})^{-1}(z)(N_d D_d^{-1} - Q_d^{-1}\bar{R})(z)D_d(z) \\ &= \tilde{D}_d(z)\hat{Z}(z)D_d(z) \end{aligned} \tag{5.38}$$

We conclude that every $\mathcal{C}_{+\epsilon}$ decentralized blocking zero of $\tilde{Z}$ belongs to $\hat{\Psi}$. Hence $\hat{\Psi}$ is the set of $\mathcal{C}_{+\epsilon}$ decentralized blocking zeros of $\tilde{Z}$.

Step 2. Observe from (5.38) that $z \in \hat{\Psi}$ implies $det(QK_d + \bar{R}L_d)(z) = det(Q)(z).det(D_d)(z)$. If N is even the sign of $det(Q)(z).det(D_d)(z)$ and the sign of $det(Q)(z)$ are the same for any $z \in \mathcal{R}_{+\epsilon}$. If N is odd, on the other hand the sign of $det(Q)(z).det(D_d)(z)$ is positive for all $z \in \mathcal{R}_{+\epsilon}$. The result now folllows from Theorem (5.1).□

Theorem (5.21). *Let Z_{ij}. $i, j \in$ N, $i \neq j$ be all stable. Then, (p2) is always solvable.*

Proof. If Z_{ij}, $i, j \in$ N, $i \neq j$ are all stable, a bicoprime representation $[P_1' \ ... \ P_N']'Q^{-1}[R_1 \ ... \ R_N]$ of Z can be given as follows: $Q = diag\{D_{l1}, D_{l2}, ..., D_{lN}\}$,

$$[R_1 \ R_2 \ ... \ R_N] = \begin{bmatrix} N_{l1} & D_{l1}Z_{12} & D_{l1}Z_{1N} \\ D_{l2}Z_{21} & N_{l2} & D_{l2}Z_{2N} \\ \vdots & \vdots & \vdots \\ D_{lN}Z_{N1} & D_{lN}Z_{N2} & N_{lN} \end{bmatrix},$$

and P is as in (5.4), where we remind that $Z_{ii} = N_{ri}D_{ri}^{-1}$. Referring to (5.6) it holds that

$$Q_{11} = \begin{bmatrix} D_{l1} & 0 & 0 & N_{l1}L_1 & D_{l1}Z_{12}L_2 & & D_{l1}Z_{1N}L_N \\ 0 & D_{l2} & 0 & D_{l2}Z_{21}L_1 & N_{l2}L_2 & & D_{l2}Z_{2N}L_N \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & D_{lN} & D_{lN}Z_{N1}L_1 & D_{lN}Z_{N2}L_2 & & N_{lN}L_N \\ -I & 0 & 0 & K_1 & 0 & & 0 \\ 0 & -I & 0 & 0 & K_2 & \dots & 0 \\ & & & \vdots & \vdots & & \vdots \\ 0 & 0 & -I & 0 & 0 & & K_N \end{bmatrix}$$

such that

$$\det(Q_{11}) = det(\begin{bmatrix} I & D_{l1}Z_{12}L_2 & D_{l1}Z_{1N}L_N \\ D_{l2}Z_{21}L_1 & I & D_{l2}Z_{2N}L_N \\ \vdots & \vdots & \vdots \\ D_{lN}Z_{N1}L_1 & D_{lN}Z_{N2}L_2 & I \end{bmatrix}). \tag{5.39}$$

We now claim that DCSP for Z and Z_{ii}, $i \in \mathbf{N}$ is solvable. Indeed, let an $\mathcal{R}_{+e}$ decentralized blocking zero z of the auxiliary plant $\bar{Z}$ satisfy

$$\begin{bmatrix} \bar{Z}_{i1} & \bar{Z}_{ii} \\ \vdots & \vdots \\ \bar{Z}_{N1} & \bar{Z}_{Ni} \end{bmatrix}(z) = 0, \ \forall i \in \mathbf{N}.$$

Then, $(D_{l2}Z_{21}\ D_{r1})(z) = 0$, $(D_{l3}Z_{31}\ D_{r1})(z) = 0$, $(D_{l3}Z_{32}\ D_{r2})(z) = 0$, ..., $(D_{lN}Z_{N1}\ D_{r1})(z) = 0$, $(D_{lN}Z_{N2}\ D_{r2})(z) = 0$, ..., $(D_{lN}Z_{NN-1}\ D_{r\,N-1})(z) = 0$. In this case, via (5.39) we have $det(Q_{11})(z) = 1$. Repeating for all permutations $\{i_1, ..., i_N\}$ of $\mathbf{N}$ we obtain the result that $\bar{Z}$ is decentralized strong stabilizable. The result now follows from Theorem (5.1).□

Remark (5.3). In [34] and [74] the problem of stabilizing a plant via the stabilization of diagonal transfer matrices is investigated using the block diagonal dominance properties of the plant. In the abovecited references, however, it is assumed that the number of unstable poles of Z and $diag\{Z_{11}, ..., Z_{NN}\}$ are the same. The following example shows that unless that assumption holds, one cannot guarantee the solvability of the problem even when the block diagonal dominance is achieved in the closed loop system. Let

$$Z = \begin{bmatrix} \frac{(z-2)}{(z+1)} & 0 \\ 0 & \frac{(z-4)}{(z+1)} \end{bmatrix}^{-1} \begin{bmatrix} \frac{(z-1)(z-3)(z-2)}{(z+1)^4} & \frac{(z-3)(z-1)}{(z+1)^3}\varepsilon_2 \\ \frac{1}{(z+1)}\varepsilon_1 & \frac{(z-3)(z-1)}{(z+1)^3} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{(z-1)(z-3)}{(z+1)^3} & \frac{(z-3)(z-1)}{(z+1)^2(z-2)}\varepsilon_2 \\ \frac{1}{(z-4)}\varepsilon_1 & \frac{(z-3)(z-1)}{(z+1)^2(z-4)} \end{bmatrix}$$

where ε_1, ε_2 are real numbers. The plant Z is free of unstable decentralized fixed modes [1] and is strongly connected. We note that $\Psi = \{1, 3, \infty\}$, $\mathcal{Z} = \{2, 4\}$ and $\mathcal{T}_d = \{4\}$ where Ψ is the set of $\mathcal{R}_{+e}$ decentralized blocking zeros of Z. Observe that the number of unstable poles of Z and $diag\{Z_{11}, Z_{22}\}$ are not the same. One has $\mathcal{Z} \cup \mathcal{T}_d = \{2, 4, 4\}$. Between 1 and 3 there are an odd number of elements of $\mathcal{Z} \cup \mathcal{T}_d$. Therefore, the composite system Z cannot be stabilized via the stabilization of Z_{11} and Z_{22} regardless of how small ε_1, ε_2 are. It is, however, not difficult to show that the block diagonal dominance for Z is achievable in the closed loop system by choosing ε_1, ε_2 suitably small [74], [34, Thm. 3.15]. (Although Z does not satisfy **(iii)** of (5.16) this does not cause any problems as we consider only a necessary condition for the solvability of DSSP (Remark (4.2).)•

We finally investigate the genericity properties of the problem. The quantifier 'almost all' below is with respect to the graph topology.

Theorem (5.22). *(p2) is solvable for almost all $Z \in \bar{\mathbf{P}}^{p\times r}$.*

Proof. The fact that the set of Z which can be stabilized via the stabilization of diagonal transfer matrices is open in $\bar{\mathbf{P}}^{p\times r}$ can be proved similarly to the proof of Theorem (5.3). The proof of the fact that the set of such Z is dense can be given by applying the the following lemma, where we assume ($p_i \geq 2$ and $r_j \geq 2$) or ($p_j \geq 2$ and $r_i \geq 2$), $i, j \in \mathbf{N}$, $i \neq j$, and Proposition (5.4).□

Lemma (5.3). *For almost all $(Q, [R_1, \ldots R_N]) \in \mathbf{S}^{p\times p} \times \mathbf{S}^{p\times r}$ the set Γ defined by (5.19) is contained in $\{\infty\}$ where P_i, $i \in \mathbf{N}$ are as defined by (5.7).*

Note that the proof of Lemma (5.3) is similar to the proof of Lemma (5.1).

5.4 Reliable Decentralized Stabilization Problem

Let a nominal system be given. Assume that this system is subject to some finite number of discrete variations in its parameters each resulting in a new system. If there exists a controller showing a satisfactory performance (stabilization) for each of the resulting systems, as well as the nominal system, it is called a *reliable* controller. Since reliable controllers have many practical advantages, there has been a continuing interest in the control theory considering the synthesis of reliable controllers [45], [47], [13], [67], [43], [31], [35], [21], [9], [20], [11], [62], [48], [8], [33], [57], [54]. In [35], [62] and [8] decentralized reliable stabilization problem has been investigated and its relations to DSSP and DCSP are discussed (see also [48]). In [57] some particular examples of decentralized reliable stabilization problem have been solved. In this section we formulate and solve the reliable decentralized stabilization problem in the DCSP framework.

We consider a system whose transfer matrix is given by $Z = [Z_{ij}]$, $i, j \in \mathbf{N}$, $Z_{ij} \in \mathbf{P}^{p_i \times r_j}$, $i, j \in \mathbf{N}$ where Z_{ii}, $i \in \mathbf{N}$ are strictly proper and Z satisfies (5.16). It is assumed that the system is subject to a finite number of discrete variations in its open-loop parameters, such as the interconnection breakdowns or on-off type of changes in the physical elements. For each variation we associate an integer i so that $\mathbf{I} = \{1, ..., I\}$ represents the set of all possible variations. These variations yield new physical systems which are given by the transfer matrices $Z^i = [Z^i_{kl}]$, $Z^i_{kl} \in \mathbf{P}^{p_k \times r_l}$, $k, l \in \mathbf{N}$, $i \in \mathbf{I}$. The variations are assumed to have a special form so that associated with $\mathbf{I}$ there exists a set of plants $T_i \in \mathbf{P}^{p_i \times r_i}$, $i \in \mathbf{N}$ where

(a) For each $i \in \mathbf{I}$ $Z^i_{kk} = T_k$, $k \in \mathbf{N}$,

(b) For each $i \in \mathbf{I}$ there exists a permutation $P^i = \{i_1, ..., i_N\}$ of $\mathbf{N}$ satisfying that $Z^i_{i_k i_l} = 0$, $k = 1, ..., N-1$, $l = k+1, ..., N$.

Observe that corresponding to each variation, the main diagonal blocks in the transfer matrix of the resulting system are equal to T_i, $i \in \mathbf{N}$. Moreover, the resulting transfer matrix can be put into a lower triangular form by a symmetric permutation of block rows and columns. It is assumed that Z^i, $i \in \mathbf{I}$ are free of unstable decentralized fixed modes.

The Reliable Decentralized Stabilization Problem (RDSP) is defined as determining controllers Z_{ci}, $i \in \mathbf{N}$ such that $(Z, diag\{Z_{c1}, ..., Z_{cN}\})$ is stable and $(Z^i, diag\{Z_{c1}, ..., Z_{cN}\})$ is stable, for all $i \in \mathbf{I}$.

Example (5.1). We consider RDSP of a feedforward interconnected system [57]. Let $Z = [Z_{ij}]$, $i, j \in \mathbf{3}$ be a nominal plant where the off-diagonal subplants are subject to four different group of discrete variations represented by the set $\mathbf{I} = \{1, 2, 3, 4\}$. It is assumed that Z satisfies (5.16) and Z_{ii}, $i \in \mathbf{3}$ are strictly proper. We let $T_i = Z_{ii}$, $i \in \mathbf{3}$. The plants Z^i, $i \in \mathbf{I}$ are given as follows.

$$Z^1 = \begin{bmatrix} Z_{11} & 0 & 0 \\ Z_{21} & Z_{22} & 0 \\ Z_{31} & 0 & Z_{33} \end{bmatrix}, \ Z^2 = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ 0 & Z_{22} & 0 \\ 0 & Z_{32} & Z_{33} \end{bmatrix}, \ Z^3 = \begin{bmatrix} Z_{11} & 0 & 0 \\ Z_{21} & Z_{22} & 0 \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix}.$$

$$Z^4 = \begin{bmatrix} Z_{11} & 0 & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ 0 & 0 & Z_{33} \end{bmatrix},$$

so that $P^1 = \{1, 3, 2\}$, $P^2 = \{2, 3, 1\}$, $P^3 = \{1, 2, 3\}$, $P^4 = \{3, 1, 2\}$. In RDSP our objective is to determine a decentralized controller $Z_c = diag\{Z_{c1}, Z_{c2}, Z_{c3}\}$ satisfying that (Z, Z_c), (Z^1, Z_c), (Z^2, Z_c), (Z^3, Z_c), (Z^4, Z_c) are all stable.Δ

Example (5.2). In this example we consider RDSP for a feedback interconnected system. Let two systems be given by

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i v_i + u_i \\ y_i &= C_i x_i \end{aligned} \quad , \ i \in \mathbf{2}$$

which are interconnected according to the dynamical rule

$$\dot{z} = \bar{A} z + \bar{B} x_2, \ \ u_1 = \bar{C} z, \ \ u_2 = A_{21} x_1.$$

The composite system is described by

$$\Sigma: \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_1 & 0 & \bar{C} \\ A_{21} & A_2 & 0 \\ 0 & \bar{B} & \bar{A} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix}$$

Let the elements $\bar{B}$ and A_{21} of the composite system Σ be subject to some variations represented by $\mathrm{I} = \{1, 2\}$ such that

i	1	2
A_{21}	0	A_{21}
$\bar{B}$	$\bar{B}$	0

where i represents the corresponding variation. We let

$$\Sigma_1: \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_1 & 0 & \bar{C} \\ 0 & A_2 & 0 \\ 0 & \bar{B} & \bar{A} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix},$$

$$\Sigma_2: \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_1 & 0 & \bar{C} \\ A_{21} & A_2 & 0 \\ 0 & 0 & \bar{A} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix}.$$

It is assumed that Σ, Σ_1, Σ_2 are stabilizable and detectable. Let Z, Z^1, Z^2 denote the transfer matrices from the input $[v_1' \; v_2']'$ to the output $[y_1' \; y_2']'$ associated with systems Σ, Σ_1, Σ_2, respectively. It is not difficult to verify that

$$Z^1 = \begin{bmatrix} C_1(zI - A_1)^{-1}B_1 & C_1(zI - A_1)^{-1}\bar{C}(zI - \bar{A})^{-1}\bar{B}(zI - A_2)^{-1}B_2 \\ 0 & C_2(zI - A_2)^{-1}B_2 \end{bmatrix},$$

$$Z^2 = \begin{bmatrix} C_1(zI - A_1)^{-1}B_1 & 0 \\ C_2(zI - A_2)^{-1}A_{21}(zI - A_1)^{-1}B_1 & C_2(zI - A_2)^{-1}B_2 \end{bmatrix}.$$

We also assume that Z^1, Z^2 are free of unstable decentralized fixed modes and Z satisfies (5.16). In RDSP our objective is to determine $Z_c = diag\{Z_{c1}, Z_{c2}\}$ such that (Z, Z_c), (Z^1, Z_c), (Z^2, Z_c) are all stable. In the RDSP set-up above observe that $P^1 = \{2, 1\}$, $P^2 = \{1, 2\}$ and $T_1 = C_1(zI-A_1)^{-1}B_1$, $T_2 = C_2(zI-A_2)^{-1}B_2$.$\Delta$

The solution of RDSP is given by the following theorem.

Theorem (5.23) *The problem RDSP is solvable if and only if DCSP for Z and T_i, $i \in$ N is solvable.*

Proof. Since for each $i \in$ I, Z^i is free of unstable decentralized fixed modes, any decentralized controller $diag\{Z_1, ..., Z_N\}$ of appropriate size stabilizes Z^i if and only if (Z^i_{kk}, Z_k), $k \in$ N are stable [56], [22]. (See also Chapter 3.) **[If]**: If DCSP for Z and T_i, $i \in$ N is solvable then there exist controllers Z_{ci}, $i \in$ N such that $(Z, diag\{Z_{c1}, ..., Z_{cN}\})$ is stable and (T_i, Z_{ci}) is stable, for all $i \in$ N. The solvability of DCSP and (b) above together imply that $(Z^i, diag\{Z_{c1}, ..., Z_{cN}\})$ is stable for all $i \in$ I. This, by the problem definition, implies that RDSP is solvable.

[Only If]: If RDSP is solvable there exist controllers Z_{ci}, $i \in$ N such that $(Z, diag\{Z_{c1}, ..., Z_{cN}\})$ is stable and $(Z^i, diag\{Z_{c1}, ..., Z_{cN}\})$ is stable, $i \in$ I. From (b) above we conclude that (T_i, Z_{ci}) is stable, for all $i \in$ N. This implies by problem definition that DCSP for Z and T_i, $i \in$ N is solvable.□

The following theorem gives a sufficient condition for the solution of RDSP. We refer to Section 5.3 for the terminology.

Theorem (5.24). *The problem RDSP is solvable if the set Γ given by (5.19) is empty or contains only one element.*

Proof. Follows from Remark (5.2).□

We now state the solution of RDSP in a special case.

Theorem (5.25). *Let $\mathcal{T}_d \cap \mathcal{Z} = \emptyset$ and $\mathcal{T}_d \cap \hat{\Psi} = \emptyset$. Then, RDSP is solvable if and only if DSSP for $T - Z$ is solvable.*

Proof. Follows from Theorem (5.2).□

Example (5.2) (Continued) The applications of Theorems (5.24) and (5.25) will be demonstrated. Assume that Z satisfies $rank\, Z_{12} \geq 2$ or $rank\, Z_{21} \geq 2$ where Z_{12} and Z_{21} are the transfer matrices between $v_1 - y_2$ and $v_2 - y_1$, respectively.

(a) Let $q+1$st invariant factors of the following complementary subsystems have only stable zeros:

$$(\begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} A & 0 & \bar{C} \\ A_{21} & A_2 & 0 \\ 0 & \bar{B} & \bar{A} \end{bmatrix}, \begin{bmatrix} 0 & C_2 & 0 \end{bmatrix}).$$

$$(\begin{bmatrix} 0 \\ B_2 \\ 0 \end{bmatrix}, \begin{bmatrix} A & 0 & \bar{C} \\ A_{21} & A_2 & 0 \\ 0 & \bar{B} & \bar{A} \end{bmatrix}, \begin{bmatrix} C_1 & 0 & 0 \end{bmatrix})$$

where

$$q = size(\begin{bmatrix} A & 0 & \bar{C} \\ A_{21} & A_2 & 0 \\ 0 & \bar{B} & \bar{A} \end{bmatrix}).$$

Then, from Lemma (2.2) the set Γ in (5.19) satisfies that $\Gamma = \{\infty\}$. From Theorem (5.24) we conclude that RDSP is solvable, i.e., there exists a decentralized compensator $Z_c = diag\{Z_{c1}, Z_{c2}\}$ such that (Z, Z_c), (Z^1, Z_c) and (Z^2, Z_c) are all stable.

(b) (This part is independent of part (a) above.) Let A_1 and A_2 have only stable eigenvalues. Then, $T_d = \emptyset$. Consequently, $T_d \cap \mathcal{Z} = \emptyset$, $T_d \cap \hat{\Psi} = \emptyset$. From Theorem (5.25) RDSP is solvable if and only if DSSP for $diag\{Z^1, Z^2\} - Z$ is solvable.Δ

Our final result is concerned with the genericity of solution of RDSP.

Theorem (5.26). *The set of $N+1$-tuples $(Z, T_1, ..., T_N)$ for which RDSP is solvable is open and dense in $\mathbf{P}^{p \times r} \times \mathbf{P}^{p_1 \times r_1} \ldots \times \mathbf{P}^{p_N \times r_N}$ (with respect to the product topology induced by the graph topology).*

Proof. Follows from Theorem (5.3).□

5.4.1 Further Results on Reliable Stabilization

The above results are concerned with the reliable stabilization with respect to subsystem interconnection breakdowns. However, it is possible to extend some of the results in Section 5.3 to obtain a reliable decentralized stabilization procedure against actuator/sensor failures for 2-channel systems. In this context we consider the following problem (see also [47], [57], [54]).

Multiple Controller Reliable Synthesis Problem (MCRSP): *Let* $Z = [Z_{ij}]$, *where* $Z_{ij} \in \mathbf{P}^{p_i \times r_j}$ $i, j \in \mathbf{2}$, *be the transfer matrix of a two-channel plant. Determine compensators* $Z_{ci} \in \mathbf{P}^{r_i \times p_i}$ $i \in \mathbf{2}$ *such that* (a) $(Z, diag\{Z_{c1}, Z_{c2}\})$ *is internally stable,* (b) $(Z, diag\{Z_{c1}, 0\})$ *is internally stable,* (c) $(Z, diag\{0, Z_{c2}\})$ *is internally stable.*

The motivation of the problem can be explained as follows. It is assumed that around each channel there are one actuator and one sensor. Let a_i, s_i, c_i $i \in \mathbf{2}$ denote the actuator, sensor and the compensator respectively, around channels 1 and 2. In the following table six different failure combinations of these elements are shown where '-' indicates that the associated element has a failure (modelled as fixed zero output) and '+' indicates that the associated element is functional.

	a_1	a_2	s_1	s_2	c_1	c_2	Failure Model of Compensator
	-	+	+	+	+	+	$diag\{0, Z_{c2}\}$
	+	-	+	+	+	+	$diag\{Z_{c1}, 0\}$
Type	+	+	-	+	+	+	$diag\{0, Z_{c2}\}$
of	+	+	+	-	+	+	$diag\{Z_{c1}, 0\}$
Failure	+	+	+	+	-	+	$diag\{0, Z_{c2}\}$
	+	+	+	+	+	-	$diag\{Z_{c1}, 0\}$

It follows that if MCRSP is solved the stability of closed loop system is preserved under any failure shown in the table (see also [57], [60], [47]). We assume that Z is stabilizable and detectable from both channels 1 and 2 (which is a necessary condition for the problem to be solvable).

Theorem (5.27). *Suppose that either* $rank Z_{12} \geq 2$ *or* $rank Z_{21} \geq 2$. *Then,* *MCRSP is solvable if and only if Z has even number of real poles between each pair of zeros in the union of the sets of $\mathcal{R}_{+e}$-blocking zeros of Z_{12} and Z_{21}.*

Proof. Follows from Theorem (5.20).□

Chapter 6

CONCLUSIONS

In this chapter, we summarize the results obtained in the thesis. Some research topics for future investigation are also addressed.

In Chapter 3, we have considered the solution of DSP using a stable proper fractional approach. A hierarchically stable synthesis procedure for decentralized stabilizing controllers is proposed where each local controller is chosen as a stabilizing controller for the associated channel in the closed loop system. A characterization of decentralized stabilizing controllers are obtained and several genericity properties of these controllers are investigated.

In Chapter 4, we first introduce the notion of decentralized blocking zeros of a multichannel plant. Various properties of decentralized blocking zeros are investigated. Then, the synthesis of least unstable decentralized stabilizing controllers and the solution of DSSP are considered. It is shown that the least unstable degree of a decentralized stabilizing controller is determined by the number of odd distributions of poles among the real unstable decentralized blocking zeros of the system. It is further shown that the unstable poles of decentralized stabilizing controllers can be nearly arbitrarily spread among the local controllers.

In Chapter 5, we have investigated the Decentralized Concurrent Stabilization Problem (DCSP) for a pair of plants $Z, diag\{T_1, ..., T_N\}$ and the applications of DCSP to the synthesis of decentralized controllers for large-scale systems. DCSP is a special decentralized simultaneous stabilization problem. It is shown that a solution to DCSP exists if and only if DSSP is solvable for an auxiliary plant. Thus, the set of unstable decentralized blocking zeros of the auxiliary plant plays a primary role in the solution of DCSP [38]. Summarizing the results in Chapter 5 we have the following.

(i) The set of decentralized blocking zeros of the auxiliary plant associated with Z and $diag\{T_1, ..., T_N\}$ has been shown to be a subset of the invariant zeros of the complementary subsystems associated with Z. Thus, if that set of invariant zeros is empty or contains only one element DCSP is solvable regardless of the diagonal plants T_i, $i \in \mathbf{N}$.

(ii) DCSP is a generically solvable problem

(iii) If the sets of the unstable poles of Z and $diag\{T_1, ..., T_N\}$ are disjoint then DCSP is solvable if and only if DSSP for the difference plant $diag\{T_1, ..., T_N\} - Z$ is solvable. This is an analogous result to [66, Lemma 4.4.20] in the centralized case.

The following large-scale control problems have been formulated and solved in the DCSP framework: (p1) Stabilization of composite systems using locally stabilizing subsystem controllers, (p2) Stabilization of composite systems via the stabilization of diagonal transfer matrices and (p3) Reliable decentralized stabilization problem. It has been shown that the following properties commonly appear in these problems:

(i) they are generically solvable

(ii) if a set of invariant zeros of the complementary subsystems associated with the composite system Z is stable then they are solvable.

We believe that the solvability conditions obtained for problems (p1) and (p2) provide a considerable progress in the research for large-scale systems as they constitute a suitable framework for the related problems in terms of well-known system invariants such as zeros and poles and the new notion of decentralized blocking zeros. For example, a more general version of problem (p1) is known to be the *the expanding system problem* [14], [53] for which our results yield several necessary conditions.

It should also be noted that although problems (p1) and (p2) have become two main approaches to the synthesis of decentralized stabilizing controllers for large-scale systems, they have not been considered in the same framework so far as the relevant solution techniques for these problems are quite different from each other. The approach in this thesis yields a unified synthesis methodology for these problems by assembling these into DCSP.

Some further research topics related to this thesis can be proposed as follows.

(i) In problem (p2) of Chapter 5 the relation between theorems (5.18)-(5.21) and the sufficient conditions obtained in [34], [74] using diagonal dominance techniques need to be clarified.

(ii) It comes forth that time-varying controllers should be given more emphasis in the controller synthesis problems for large-scale systems, since they have significant advantages in the decentralized stabilization and decentralized concurrent stabilization problems compared to time-invariant controllers [4], [39], [72], [73], [28], [58]. In [58] a time-varying version of DCSP is considered and it is shown that periodic controllers weaken the solvability conditions of DCSP considerably. For example, if Z is strongly connected, DCSP can always be solved using a periodic controller. These results can be extended to continuous-time systems using sampled-data periodic controllers. The abovementioned expanding construction problem of large-scale systems can also be analysed using periodic controllers. The advantages of time-varying controllers in some multipurpose decentralized synthesis problems, such as the servomechanism problem [12], can also be investigated.

(iii) It is possible to extend the results in Chapter 3 to a class of infinite-dimensional systems [61]. One can investigate the solutions of DSSP and DCSP in the same set-up. The extension of the results in chapters 4,5 to infinite-dimensional systems would be quite nontrivial as infinite-dimensional systems may have infinitely many blocking zeros [5], [6], [7].

(iv) Perhaps the most challenging problem that can be addressed for future investigation in this thesis is bringing forth the role of decentralized blocking zeros in design limitations. From the proof of Theorem (4.2) (i), it follows that every $\mathcal{C}_{+e}$ decentralized blocking zero is a *fixed* $\mathcal{C}_{+e}$ blocking zero associated with every single channel in the closed loop system resulting from the application of *any* decentralized stabilizing controller. Since right half plane zeros impose certain performance limitations regarding sensitivity reduction, it is our intuition that decentralized blocking zeros are also pertinent to various design limitations in multivariable systems.

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