EQUITABLE DECISION MAKING APPROACHES OVER ALLOCATIONS OF MULTIPLE BENEFITS TO MULTIPLE ENTITIES

A THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

MASTER OF SCIENCE

IN

INDUSTRIAL ENGINEERING

By

Sema Nur Kaynar Keleş

July 2017

EQUITABLE DECISION MAKING APPROACHES OVER ALLOCATIONS OF MULTIPLE BENEFITS TO MULTIPLE ENTITIES
By Sema Nur Kaynar Keleş
July 2017

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Özlem Karsu (Advisor)

Firdevs Ulus

Barbaros Yet

Approved for the Graduate School of Engineering and Science:

Ezhan Karaşan
Director of the Graduate School

ABSTRACT

EQUITABLE DECISION MAKING APPROACHES OVER ALLOCATIONS OF MULTIPLE BENEFITS TO MULTIPLE ENTITIES

Sema Nur Kaynar Keleş
M.S. in Industrial Engineering
Advisor: Özlem Karsu
July 2017

In this study, we develop decision support tools for policy makers that will help them make choices among a set of allocation alternatives. We assume that alternatives are evaluated based on their benefits to different users and that there are multiple benefit (output) types to consider. We assume that the policy maker has both efficiency (maximizing total output) and equity (distributing outputs across different users as fair as possible) concerns. This problem is a multicriteria decision making problem where the alternatives are represented with matrices rather than vectors.

We develop interactive algorithms that guide a policy maker to her most preferred solution (a set of most preferred solutions), which are based on utility additive (UTA) and convex cone methods. Our computational experiments demonstrate the satisfactory performance of the algorithms. We believe that such decision support tools may be of great use in practice and help in moving towards fair and efficient allocation decisions.

Keywords: Interactive Approaches, Additive Utility, Convex Cone Method, Fairness, Equity.

ÖZET

BİRDEN FAZLA ÇIKTININ BİRDEN FAZLA KULLANICIYA DAĞILIMLARINA KARŞILIK GELEN ALTERNATİFLER ARASINDA EŞİTLİKÇİ KARAR VERME YÖNTEMLERİ

Sema Nur Kaynar Keleş
Endüstri Mühendisliği, Yüksek Lisans
Tez Danışmanı: Özlem Karsu
Temmuz 2017

Bu çalışmada, karar vericilere proje alternatifleri arasında başlatılacak olanı seçerken yardımcı olacak karar destek sistemleri geliştirilmiştir. Proje alternatiflerinin birden fazla çıktının birden fazla kullanıcıya farklı dağılımlarına karşılık geldiği bu problemde, karar vericilerin alternatifleri sadece dağıtılan çıktının toplam miktarına göre değil, çıktıların kullanıcılara nasıl dağıtıldığına göre de değerlendirdiği varsayılmaktadır. Bu problem, alternatiflerin vektörler yerine matrislerle gösterildiği, çok kriterli bir karar verme problemidir.

Karar vericiyi, en çok tercih ettiği alternatife yönlendirecek iki ayrı interaktif yöntem geliştirilmiştir. Bu yöntemlerden ilki, toplanabilir fayda skorları yöntemine (UTA) dayanırken, diğeri konveks koni yöntemi kullanılarak geliştirilmiştir. Elde ettiğimiz sonuçlar doğrultusunda iki yöntemin de yeterli performansa sahip olduğu gözlemlenmiştir. Bu tür karar destek sistemlerinin politika yapım süreçlerinde karar vericileri eşitlikçi ve verimli alternatiflere yönlendirmekte faydalı olacağına inanıyoruz.

Anahtar sözcükler: İnteraktif Yöntemler, Toplanabilir Fayda, Konveks Koni Yöntemi, Eşitlikçilik.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In many decision making settings the decision makers have to choose among a set of given alternatives, considering multiple criteria. Due to the trade-offs that exist between different criteria, this problem is a challenging problem, which motivated the design of many decision support tools (see [2] for a survey). Some example settings from the literature involve decision making problems in the areas of energy planning [3], finance [4], and sustainable development [5].

In a classical *multicriteria decision making (MCDM)* setting, each alternative is associated with a vector whose elements show the performance of that alternative with respect to each criterion. These problems are multiple criteria evaluation problems where a finite set of alternatives is given explicitly. For evaluation problems, one may try to: identify the best alternative or a small subset of most preferred alternatives, rank the alternatives, or sort the alternatives into predefined groups [6].

In some of the real life examples, alternatives correspond to different distributions of a single criterion/output to multiple users/entities. These cases lead to a special multi-criteria choice problem where the alternatives are allocation vectors, in which each element corresponds to the amount of a resource/benefit/good that a user/entity/beneficiary enjoys. In these problems, the users are considered to be anonymous which means that their identities do not affect the decision. To exemplify, in a two entity setting the policy maker would be indifferent between the following two allocations (7, 8) and (8, 7). Equity is of great importance for these problems as an expected result of the anonymity assumption. The policy maker needs to consider the trade-off between equity and efficiency for such allocation problems before choosing one. The trade-off between these two concerns makes the problem challenging.

To see the trade-off between equity and efficiency, let us consider a hospital site selection problem where the entities are different neighbourhoods and the travel times from the neighbourhoods to the hospital is the output. Each alternative location will lead to a different distribution of travel times to the hospital. The policy maker may want to choose the site considering both efficiency (minimizing the total travel time) and equity (treating different neighbourhoods as equitably as possible). To illustrate the trade-off between these two concerns, consider the following two allocations where the vectors correspond to the travel times of three different neighbourhoods to two alternative hospital locations: (3, 5, 6) and (5, 5, 5). While the total travel time is less in the first allocation, second allocation is more equitable. Note that there is no obvious choice given this pair, different DMs may choose different allocations depending on how inequity-averse they are.

In this thesis, we consider a set of allocations that distribute multiple outputs among multiple users and the problem is determining the most preferred one (or a set of most preferred alternatives). This problem is a special type of multiple criteria choice problem since the alternatives correspond to matrices rather than vectors. It involves equity concerns for users alongside the usual trade-off between different criteria. The described problem will be called *multi-dimensional equitable choice problem* throughout the thesis.

For example, let us consider a healthcare policy selection problem where a policy maker is given a set of alternatives. Assume that each alternative corresponds to different allocations of two outputs, increase in quality adjusted life time and protection from health related out-of-pocket expenditures, to two population groups who are anonymous. In most setting the alternatives will reflect the trade-off between equity and efficiency; hence finding a most preferred allocation will not be easy. We develop interactive approaches to help the DM to find her most preferred alternative in such settings.

Multi-dimensional equitable choice problem has different characteristics than classical MCDM problems, hence it reveals the need of developing different approaches than the classical methods in the literature. In this thesis, we discuss the different properties of these problems and develop solution methodologies for such problems.

The rest of this thesis is composed of the following chapters:

Chapter 2: We describe the problem setting in detail and then provide the dominance rules in line with the impartiality assumption used in equitable choice problems.

Chapter 3: We first provide a brief review of the related literature on MCDM problems and main solution methodologies. We then discuss the interactive approaches that have been used in the literature and that motivated our algorithms. We conclude our literature review chapter by discussing relevant work from the group decision making literature.

Chapter 4: We use a value function based approach and assume that the decision maker (DM)'s preferences can be represented by a value function that is not known. Hence we make use of interactive algorithms that gather preference information from the DM. We present the defined value functions (marginal value, user value, and social welfare functions) that constitute the basis of our solution approaches. We then discuss the main assumptions on the forms of the value functions by the two approaches that we suggest (UTA-based and convex cone-based approaches) alongside the different information retrieval procedures.

Chapter 5: We provide an extension of well-known UTA method which assumes an additive value functions and discuss the UTA-based interactive algorithm designed for the multidimensional equitable choice problems.

Chapter 6: We discuss the use of convex cone method for multi-dimensional equitable choice problem and present convex cone-based algorithm designed for our problem setting.

Chapter 7: We provide the result of computational experiments that we perform to check the computational feasibility and the quality of the results for the proposed algorithms.

Chapter 8: We provide the concluding remarks.

Chapter 2

Problem Definition

In this chapter, we provide the problem definition in detail and discuss how this problem extends the current literature on multi-criteria decision making. Then, we address the multidimensional equity concerns of the policy maker and define some dominance rule relations that will be used throughout the thesis.

Consider an example healthcare project selection problem in which the policy maker is to choose a project to initiate among a set of projects. In this problem, we are given a set of alternatives $A = \left\{a^1, a^2, ..., a^N\right\}$ and a typical member shows the distribution of multiple (n) outputs over multiple (m) users. In the matrix representation, the rows and columns correspond to different users (population groups) and outputs, respectively as follows:

$$a^k = \begin{array}{cc} & n \text{ Outputs} \\ m \text{ Users} & \begin{bmatrix} a^k_{11} & a^k_{12} & \cdots & a^k_{1n} \\ a^k_{21} & a^k_{22} & \cdots & a^k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a^k_{m1} & a^k_{m2} & \cdots & a^k_{mn} \end{bmatrix} \end{array}$$

where for a given alternative a^k, a^k_{ij} represents the level of output j allocated to user i. We assume that the decision maker is trying to select the best alternative in line with her preferences.

This problem can be considered as a multicriteria choice problem, in which alternatives are explicitly given and the problem is determining the most preferred one. However, it is an MCDM problem of a special type. It is different than the classical MCDM problems discussed in the literature in the sense that the alternatives distribute multiple outputs to multiple users that is, they correspond to matrices rather than vectors.

Moreover, unlike a classical MCDM problem, this problem involves a fairness factor alongside the usual trade-off between different outputs. That is, how we distribute an output is also of concern to the decision maker. Such multi-dimensional equitable choice problems are commonly encountered in public sector where equity concerns have significant importance. For example, it is important to treat all the users as equitably as possible for some public sector problems such as healthcare project initialization, R&D project selection, or resource allocation in different organization units. Most of the alternatives that the policy maker considers may reflect the trade-off between equity and efficiency. Choosing among them is a difficult task hence it is great importance of designing decision support tools that aid the DM.

We will try to explain the relation and the possible trade-off between equity and efficiency by using a small example.

Example 1 *Consider a problem in which a DM is faced with a set of alternatives showing distributions of two goods (outputs) to two users. When we increase efficiency while keeping the equity level same, we obtain a better alternative. For example, when we have* $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$ *and* $\begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix}$ *as two alternatives, the DM will choose* $\begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix}$ *over* $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$*, since*$\begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix}$ *distributes higher amounts of outputs to the users and both alternatives have complete equality. This example illustrates the efficiency concern of the DM.*

When we have a more equitable allocation in both goods while keeping the efficiency level same, we obtain a better alternative. For example, when we have $\begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix}$ *and* $\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ *as two alternatives, the DM will choose* $\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ *over* $\begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix}$*. Both alternatives distribute 8 units of output 1 and 8 units of output 2, but alternative 2 provides a more equitable allocation for each of the outputs.*

In the first example, only efficiency levels change and in the second example, only equity levels change. Therefore, they do not reflect the trade-off between the two concerns that many real life examples come along with. Choosing between alternatives where both efficiency and equity levels change can be a cognitively challenging task. For example,we cannot say which alternative would be chosen between $\left(\begin{smallmatrix} 4 & 5 \\ 6 & 7 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 3 & 4 \\ 8 & 9 \end{smallmatrix}\right)$*. In this example, the alternative that has higher efficiency level is worse in terms of equity. The trade-off between equity and efficiency can be observed here.*

In this study we assume a non-hierarchical relation among the users. We assume that changing the bundles over the users does not affect the social welfare value that an alternative brings. (A bundle is a distribution of benefits to a single user and corresponds to a row in our matrix notation). This is the so-called *impartiality* assumption defined in equitable preferences [7]. For example, we assume that the DM will be indifferent between two alternatives $\left(\begin{smallmatrix} 4 & 5 \\ 3 & 6 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 3 & 6 \\ 4 & 5 \end{smallmatrix}\right)$.

Since we consider situations in which multiple benefits are distributed to multiple entities, the fairness concerns are of a multidimensional nature. There are well-known results in the economics literature on single benefit (income) distributions to multiple individuals that discuss various axioms and link these to dominance relations such as Lorenz dominance [8] and Generalized Lorenz dominance [9]. However, it is considerably harder to obtain such rules and equivalence results in a multidimensional framework [10]. A pioneering work that touches upon these dominance issues in the multidimensional settings is due to [11]. We also provide dominance rules in line with the assumptions (impartiality and monotonicity) that we make on the preference model of the central DM.

These dominance rules are obtained by extending vector dominance relations for alternatives that are represented by matrices. We will first give the definition of (weak) dominance relation over vectors and then, discuss the corresponding extensions.

Definition 1 *Given two alternatives* z^k, $z^{k'} \in \mathbb{R}^n$ *where* n *is the number of outputs (criteria) and* $J = \{1,2,...,n\}$,

$z^k \preceq_d z^{k'}$ *(*$z^{k'}$ *weakly dominates* z^k*)* $\Longleftrightarrow z^k_j \leq z^{k'}_j$ *for all* $j \in J$.

A simple extension of *Definition* 1 for our problem setting would be the following:

Definition 2 *Given two alternatives a^k, $a^{k'} \in \mathbb{R}^{(m \times n)}$ where m and n are the number of users and the number of outputs, respectively, let us define the following sets $I = \{1,2,...,m\}$ and $J = \{1,2,...,n\}$*

$a^k \preceq_d \; a^{k'}$ ($a^{k'}$ weakly dominates a^k) $\Longleftrightarrow a^k_{ij} \leq a^{k'}_{ij}$ for all $i \in I$, $j \in J$.

Consider two alternatives $a^k = \left(\begin{smallmatrix} 5 & 4 \\ 4 & 3 \end{smallmatrix}\right)$ and $a^{k'} = \left(\begin{smallmatrix} 6 & 5 \\ 4 & 3 \end{smallmatrix}\right)$. Since $a^k_{ij} \leq a^{k'}_{ij}$ for all i, j, we say that $a^{k'}$ dominates a^k. Alternative k' brings greater value to the *first user* for each criterion than alternative k while the *second user* gets the same bundle in both alternatives. Here, the users are called as *first* and *second* just to provide an ease in the expression. Their usage do not imply any superiority relation. Let us consider a scenario where alternative k becomes $a^k = \left(\begin{smallmatrix} 4 & 3 \\ 5 & 4 \end{smallmatrix}\right)$. From the *impartiality* assumption, the DM is indifferent between $\left(\begin{smallmatrix} 5 & 4 \\ 4 & 3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 4 & 3 \\ 5 & 4 \end{smallmatrix}\right)$. However, the dominance rule introduced in *Definition* 2 fails to acknowledge this relation when the row ordering of the users changes. Hence, we modify this dominance rule to handle the impartiality assumption.

Definition 3 *For an alternative $a^k \in \mathbb{R}^{(m \times n)}$ where m and n represent the number of users and the number of outputs, respectively, let $\pi(a^k)$ be the set of all different row permutations of a^k and $R = \{1,2,...,m!\}$. Given two alternatives a^k and $a^{k'} \in \mathbb{R}^{(m \times n)}$,*

$a^k \preceq_{em} \; a^{k'}$ ($a^{k'}$ equitably matrix weak dominates (em-dominates) a^k) $\Longleftrightarrow$ $\pi_r(a^k) \preceq_d \; a^{k'}$ for at least one $r \in R$.

Em-dominance enables us to make further inferences compared to the previous dominance relations. Let us take the example where $a^k = \left(\begin{smallmatrix} 4 & 3 \\ 5 & 4 \end{smallmatrix}\right)$ and $a^{k'} = \left(\begin{smallmatrix} 6 & 5 \\ 4 & 3 \end{smallmatrix}\right)$ and $\pi(a^1) = \left\{\left(\begin{smallmatrix} 4 & 3 \\ 5 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & 4 \\ 4 & 3 \end{smallmatrix}\right)\right\}$. Since $\pi_2(a^k) = \left\{\left(\begin{smallmatrix} 5 & 4 \\ 4 & 3 \end{smallmatrix}\right)\right\}$ and $\pi_2(a^k) \preceq_d \; a^{k'}$, $a^{k'}$ *em-dominates* a^k.

The *em-dominance* relation will help us eliminate some alternatives. However, in most real life cases, we will have trade-offs and using dominance relations will not be sufficient to make decisions. Hence we propose decision support tools, that will help the DM choose her most preferred alternative in a set of *em-efficient* alternatives (an alternative is *em-efficient* if there is no other alternative that *em-dominates* it).

Chapter 3

Literature Review

In this study, we address multi-criteria decision making (MCDM) problems where the alternatives correspond to distributions of multiple goods/outputs to multiple users/entities. Hence the alternatives are not vectors but matrices, the columns and rows of which show the allocated outputs and the users that enjoy these goods, respectively. Each element of the matrix shows the level of an output a user receives. Due to this property, the problem of concern is a special type of MCDM problem. We tackle this problem by proposing two interactive approaches that are based on the assumption that the policy maker has fairness concerns as well efficiency concerns. Therefore, in our literature review, we will first discuss MCDM problems and main solution approaches, and show where our problem fits in the literature. We then provide a brief review of the interactive approaches that have been used in the literature and also motivated our algorithms. Since the problem involves fair and efficient allocation of benefits to multiple entities, it is also related to the group decision making settings. We conclude our literature review chapter by mentioning some of the noteworthy studies in group decision making literature.

3.1 Multiple Criteria Choice Problems

MCDM is concerned with the problems that involve multiple criteria to be considered. A typical MCDM model can be formulated as follows:

$$\text{"Max"} \ \{z_1(x), z_2(x), \ldots, z_n(x)\}$$
$$\text{s.t. } x \in X$$

where x is a generic decision vector and X is the feasible decision space. $\{z_1(x), z_2(x), \ldots, z_n(x)\}$ is the vector of n objective functions where $z_j(x)$ corresponds to the value of j^{th} objective function for x. Using quotation marks implies that the maximization of a vector is not a well-defined mathematical operation. Unless there exists a solution that maximizes all of the objective values simultaneously, the problem will have multiple Pareto optimal solutions. A solution vector is called a *Pareto optimal/efficient* solution if it is not possible to improve the value of one of the objectives without making some of the other objective values worse.

MCDM problems are often categorized into two categories based on the structure of the decision space X: i.e. *multiple criteria design problems* and *multiple criteria evaluation problems*. In multiple criteria design problems, the decision space X is defined by a set of constraints implicitly. In the design problems, the DMs deal with either a continuous decision space or an exponentially growing discrete space of alternatives [12]. In multiple criteria evaluation problems, a finite set of alternatives is given explicitly [12] and the DMs deal with a discrete decision space. For example, finding the most preferred house among the houses for sale in a specific region is a multiple criteria evaluation problem. Note that the DM may not always try to find the most preferred alternative given a set of alternatives. She may want to [13]:

1. identify the best alternative or a small subset of most preferred alternatives ;

2. rank the alternatives;

3. sort the alternatives into predefined groups.

Accordingly, the MCDM evaluation problem could be a *finding the best, ranking or sorting problem*. In this thesis, we consider the first problematique and assume that the DM is trying to determine a set of most preferred alternatives among a given set of alternatives. This is also called a multi-criteria choice problem.

Multi-criteria choice problems generally assume that the criteria are not comparable (they assume different criteria). In such settings, each alternative is associated with a vector whose elements show the performance of that alternative with respect to each criterion.

A line of the MCDM literature considers problems where the alternatives are allocation vectors, in which each element corresponds to the amount of an output that a beneficiary enjoys and the DM has equity concerns alongside the efficiency concerns [14]. This problem is called as *equitable choice problem* and the tradeoff between these two concerns makes the problem challenging. A distinguishing feature of such choice problems is the impartiality property. In such a setting, the decision maker is indifferent between an allocation and any permutation of that allocation, making the problem and the solution approaches different than their counterparts in the classical multi-criteria decision making literature.

[15] introduces the definition of equitable efficiency, which is used in this thesis with an extension, and provides techniques to find the set of equitably efficient solutions of resource allocation problems. [16] defines equitable aggregation functions of the criteria for both linear and nonlinear multiple optimization problems to select equitable efficient solutions. They argue that the aggregation function has to be strictly increasing for each criterion in a minimization setting to guarantee the consistency of the results. [17] provides sorting algorithms for single outcome allocations where the DM's preferences are assumed to be in line with impartiality and convexity assumptions.

In this thesis, we consider a set of alternatives that distribute multiple outputs among multiple users and the problem is determining the most preferred one. This problem is a special type of multiple criteria choice problem since the alternatives correspond to matrices rather than vectors. Hence, it has multidimensional equity concerns alongside the usual trade-off between different criteria/outputs.

Figure 3.2 provides the categorization of MCDM problems as evaluation and design problems and describes the main problematiques of each category. It also summarizes equitable choice and multidimensional equitable choice problems.

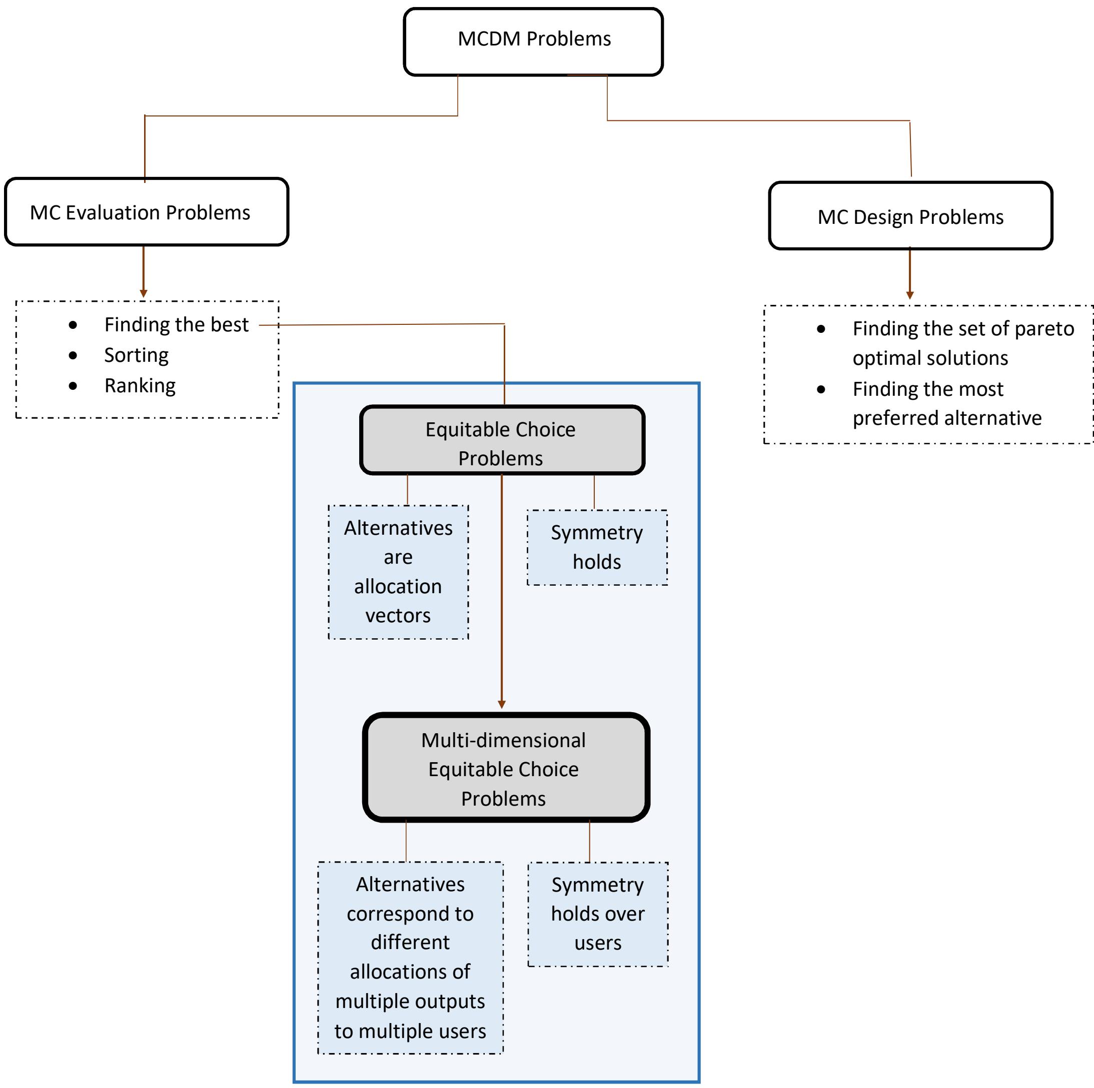

Figure 3.1: Summary of the MCDM problems

3.2 Interactive Approaches (UTA and Convex Cone)

The methods used to solve multiple criteria design (MCDM) problems are generally categorized according to the involvement of DM in information retrieval process as priori methods, posteriori methods and interactive methods. In *priori methods*, first the preference information of DM is taken and then the solution set that satisfies the provided information is found. Goal programming, value function, and lexicographic ordering methods take the preference information as priori to the solution process. Posteriori methods deal with finding the whole set of pareto optimal solutions and presenting them to the DM for further evaluation. Weighted sum scalarization and ε -constraint methods are examples of posteriori methods. Another solution approach is to obtain preference information from DM iteratively until the best alternative or a small subset of alternatives with predefined cardinality is found. This method is called *interactive approach*. When multiciteria choice problems are considered, interactive approaches are one of the most popular approaches.

In this thesis, we propose two interactive approaches for mutidimensional equitable choice problems, that are motivated from *UTA* and *convex cone methods*. Both approaches are value function based approaches that assume that the preferences of the DM is consistent with an underlying value function. Both UTA and convex cone approaches incorporate preference information of the DM into mathematical models to infer the underlying utility function of the DM or to eliminate the alternatives that cannot be the best alternative for the set of considered functions. Figure 3.2 demonstrates the steps of these methods.

We now discuss the two interactive approaches that we extend in detail.

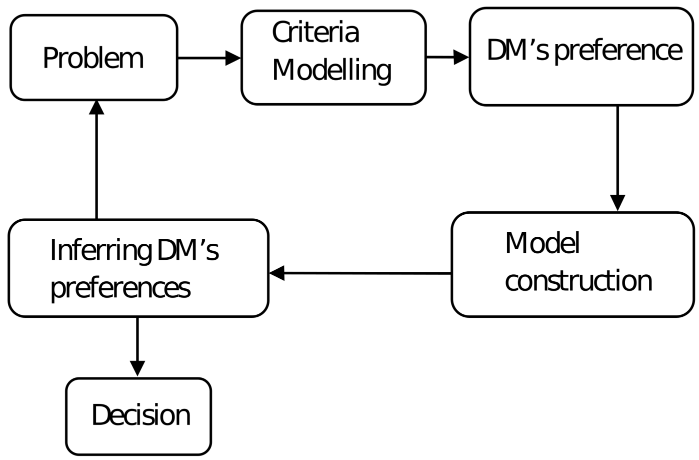

Figure 3.2: Underlying setting of the UTA and the convex cone methods [1]

3.2.1 UTA Method

One of the well-known solution methods which uses additive value functions is the UTA method. It is introduced by Jacquet-Lagrèze and Siskos ([18], [19]). This methods assumes that the overall utility of an alternative can be found by summing up the marginal values that the alternative brings with respect to criteria. It generally assumes piecewise linear marginal value functions and exploits linear programming techniques to incorporate the preference information of the DM. A set of value functions which are compatible with the provided preference information can be assessed using UTA methods.

There are several variants of the UTA method, which use different optimality criteria to achieve the value functions. There are also variations specifically designed for the three problematiques mentioned before: finding the best alternative, sorting (classification) and ranking.

[20] improves UTA approach to find the best alternative and propose a method called UTASTAR. This method introduces error terms for both overestimation and under estimation for each pair of consecutive alternatives in the ranking. They find

additive value functions that are as consistent as possible with the provided preference information by minimizing the summation of the error terms.

[21] extends UTASTAR method for classification of the alternatives by defining UTADIS method. This method also uses two error terms to measure the differences between the obtained results and the real classification of the reference alternatives. The error terms are calculated as the violations of lower and upper bounds of the groups by the reference alternatives. Several new variants for the UTADIS method with different objective functions can be found in the literature ([22], [23]).

In UTA methods, even though the value functions are required to be compatible with the preference information, there may still exist many such value functions. Typically, only one value function is selected arbitrarly to make recommendations to the DM among many other value functions. The idea of evaluating all functions that are compatible with the preference information was firstly introduced in robust ordinal regression method called UTA^{GMS} [24]. They define necessary and possible consequences for the provided preference information to make recommendations to the DM. This method provides two different rankings (necessary and possible rankings) for the alternatives. Necessary ranking is obtained by considering all value functions compatible with the preference information whereas possible ranking is obtained by checking the existence of at least one value function compatible with the preference information.

The UTA-based methods discussed in the literature have been used for the classical MCDM problems, where each alternative corresponds to a vector. [17] extends the approach by modifying and using it on a equitable resource allocation setting, where the alternatives correspond to distributions of a single benefit to multiple entities and the DM has fairness concerns, i.e. there is symmetry.

We extend these two lines of research by proposing UTA-based algorithm that exploits UTA^{GMS} method for multidimensional equitable choice problems where multidimensional fairness and efficiency concerns exist.

3.2.2 Convex Cone Method

Convex cone method is another way of representing the DMs preference information when the underlying value function of the DM is assumed to be quasi-concave. [25] provides an interactive method to find the best alternative, where the DM is asked pairwise comparison questions and the underlying value function of the DMs is assumed to be a non-decreasing and quasi concave value function. They use the pairwise comparison information to generate the corresponding cones and eliminate the inferior alternatives which are inferior to any of these cones. [26] and [27] also exploit convex cone method by aiming to increase the strength of cones generated by creating artificial alternatives or developing ways to choose the cone generators. The authors discuss the ways of obtaining appropriate artificial alternatives. They managed to ask smaller number of pairwise comparison questions to find the best alternative compared to [25].

[28] provides an experimental study which analyses the effects of selecting cone generations, deciding the number of alternatives used for cone generation, and the order in which pairwise comparisons are asked to the DM. [29] introduces *p cones* concept to improve the performance of the interactive algorithms by reducing the preference information requirements. *P cones* concept is used to demonstrate the closeness of an alternative to being cone dominated and they obtained satisfactory performance of *p cones* concept on minimizing the burden of the DM. [30] provides an interactive procedure to place alternatives into acceptable and unacceptable classes. The underlying value function of the DM is assumed to be quasi-concave and non-decreasing. They employ convex cone and polyhedra methods to partition the alternatives into these classes. [31] exploits convex cone method in optimization of multi-objective knapsack problems. They provide an interactive multi-objective evolutionary algorithm, genetic algorithm, where the solutions are obtained with a reasonable number of comparison questions. [32] proposes an algorithm to determine a preference-based strict partial order for a given number of alternatives when the underlying value function of the DM is assumed to be quasi-concave.

[33] extends the use of convex cones for allocation problems where a single benefit is distributed to multiple users and impartiality holds. Since the preference model of

the DM is assumed to be equitable, impartiality holds, which implies that the value function of the DM is symmetric quasi-concave. Impartiality assumption implies that the DM would be indifferent between an allocation vector and any permutation of that vector and this brings a computational complexity. The study develops a method to handle the complexity resulting from the impartiality assumption.

We will extend this work by proposing a convex cone based algorithm for multi-dimensional equitable choice problems.

3.3 Group Decision Making

The problem of allocating multiple benefits to multiple users can be considered as related to the group decision making problem, where alternatives that have different consequences for a number of entities (individuals) are evaluated, typically by the group of entities itself. Hence in such settings, one of the main concerns is constructing a social welfare function whose arguments are the individual utilities. The suggested decision support methods include assessments of the preferences of individuals and a rule for aggregating these preferences to determine group preferences [34] [35]. The pioneering studies that deal with aggregation of cardinal utilities are due to [34], [36], [37], [38], [39].

One of the important concerns in group decision making is equity (fairness) of the group decision [40], [41], [42], [43]. In line with this, [41] structures a framework in which an individual's utility depends on what others receive. Group members' approach to equity is reflected through individual utility functions, which are functions of the distribution vector. Similar to the previous studies, the authors consider a linear aggregation rule. [42] considers equity in distributions of risk and [43] extends this discussion by considering preferences on trade-offs and develops notions of inequity neutrality and inequity aversion. He discusses different conditions and links them to various forms of group value functions.

In most of the group decision making studies, individuals have different preference models (represented by different individual utility functions) and the aim is aggregating these preferences into a group preference model. However, in the problem settings we consider, we assume that there is a single policy maker (DM) hence we do not have the concern of aggregating individual preferences. In group decision making, since each individual's utility is usually considered as a function of what he receives (independent of what others get), assuming an additive social welfare function may be realistic. As we will elaborate later, we try to relax the preferential separability assumption, which is common to many group decision making settings, since we assume that the policy maker's preferences involve equity concerns (hence will depend on how a benefit is distributed) alongside efficiency concerns. Even when separability is assumed, we structure the framework so as to encourage equity in the distributions of benefits.

Chapter 4

Solution Approaches

In this chapter, we define the value functions that constitute the basis of our solution approaches and discuss their relations with each other. Then we discuss the main assumptions of UTA-based and convex cone-based approaches that we propose and the information retrieval procedures used in these two approaches.

Recall that we consider the problem of selecting the best alternative among a finite set of alternatives. In the literature, different approaches such as outranking relations and multi-attribute value theory approaches are used for this problem type [44]. We propose value function based approaches to this problem. Such approaches assume that the DM's preferences can be represented by value functions and incorporate preference information of the DM into the mathematical models to infer more about the DM's preferences.

We construct our approaches by defining three different value functions: marginal value function (MVF), user value function (UVF) and social welfare function (SWF).

For each output a MVF is defined, which assigns value scores to different levels of the output. Let $MV_j(.)$ be the non-decreasing marginal value function for output j. $MV_j(a^k_{ij})$ represents the value derived by the DM (policy maker) from the allocation of the jth output of alternative k to any user i. Marginal value that is obtained from an output is independent from the users, hence MVF depends only on the output type and not the user enjoying it.

Another function that can be defined is the user value function (UVF). Let $UV(b_i^k)$ be the social value (as perceived by the DM) derived by providing a user with bundle b_i^k (this is the i^{th} row in alternative k). In other words, $UV(.)$ assigns a total value score to the bundles (vectors showing levels of output with respect to all output types). Again, due to impartiality, we assume that given the same bundle, user values obtained from that bundle does not change for different users. Since the users are indistinguishable for the DM, this value is independent of the users' identities.

We also define a social welfare function (SWF) for the alternatives. Let $SW(a^k)$ be the total social welfare that alternative k brings. It is used to evaluate overall values of the alternatives to the DM. The interactive algorithms that are developed for the problem introduced above aim to find the alternative which has the highest social welfare value for the DM. The social welfare value of an alternative is not independent from the corresponding user values and marginal values hence $SW(.)$ may be considered as a function of $UV(.)$ and $MV_j(.)$. SWF can be assumed to have different forms in different solution methodologies.

Example 2 *To illustrate the three functions described above, let us consider an example problem where a healthcare policy maker aims to choose the best alternative among the six alternatives provided below. Let the alternatives correspond to different distributions of two outcomes, the resulting quality adjusted life time and out-of-pocket expenditures, to two user groups. Recall that MVFs are defined for the outputs. Hence two MVFs will be defined: one for the quality adjusted life time and one for the out-of-pocket expenditures ($MV_1(.)$ and $MV_2(.)$ for short.)*

$$a^1 = \begin{pmatrix} 2 & 8 \\ 3 & 4 \end{pmatrix} a^2 = \begin{pmatrix} 5 & 5 \\ 6 & 2 \end{pmatrix} a^3 = \begin{pmatrix} 4 & 6 \\ 3 & 5 \end{pmatrix}$$

$$a^4 = \begin{pmatrix} 5 & 5 \\ 4 & 6 \end{pmatrix} a^5 = \begin{pmatrix} 3 & 5 \\ 8 & 2 \end{pmatrix} a^6 = \begin{pmatrix} 6 & 4 \\ 3 & 7 \end{pmatrix}$$

In this example, the observed levels for outputs 1 and 2 are (2, 3, 4, 5, 6, 8) and (2, 4, 5, 6, 7, 8), respectively. MVFs convert these levels into their value correspondences. For example, $MV_1(6)$ and $MV_2(5)$ represent the values obtained from getting 6 and 5 units from the first and the second outputs, respectively. Notice that the functions do not take into account the user information. They only consider the output types and the level of outputs.

In all the alternatives, each user receives a bundle of two outputs (e.g. user 1 gets (2, 8) in the first alternative and (5, 5) in the second alternative) and UVFs calculate the total value (as perceived by DM) that a bundle brings to a user. Recall that bundles show the distribution levels of the outputs to a single user. For example UV(2, 8) returns the total value obtained from providing 2 units from the first output and 8 units from the second output to a user.

Similarly, SWF assigns total values to the alternative. For example $SW\left(\begin{smallmatrix} 2 & 8 \\ 3 & 4 \end{smallmatrix}\right)$ gives the social welfare value that first alternative brings. As indicated above, SWF maybe a function of UVFs. Hence, $SW\left(\begin{smallmatrix} 3 & 4 \\ 2 & 8 \end{smallmatrix}\right)$ can also be represented as SW(UV(3,4), UV(2,8)). Since the users are anonymous to the DM (recall the impartiality assumption), $\left(\begin{smallmatrix} 3 & 4 \\ 2 & 8 \end{smallmatrix}\right)$ should have the same social welfare value with the $\left(\begin{smallmatrix} 2 & 8 \\ 3 & 4 \end{smallmatrix}\right)$. This implies that SW(UV(3,4), UV(2,8)) and SW(UV(2,8),UV(3,4)) have to be the same. Therefore, SWFs have to be a symmetric function of the user values.

The methods we discuss below are based on different assumptions on the forms of these marginal value, user value, and social welfare functions, which are summarized in Table 4.1.

Table 4.1: Summary of the solution approaches

Approach	MVF	UVF	SWF	Preference information
UTA-based	Concave	Additive	Additive	Vector comparisons and/or holistic comparisons
Cone-based	Linear	Additive	S. quasi-concave	Holistic comparisons and vector comparisons: optional

The first approach exploits basic UTA techniques. It uses concave MVFs for the outputs. UVFs are assumed to be additive, i.e. the total value that a user acquires through an alternative is the sum of the values that she obtains from each output. Furthermore, social welfare that an alternative brings is assumed to be the sum of user values i.e. SWFs are additive. Note that use of concave functions encourages equitable distribution of an output since for any concave MVF, the social value of an alternative increases when the the levels of each output get closer to each other which reflects the DM's concern on fairness.

The second approach (convex cone based approach) assumes linear MVFs and additive UVFs. User values are calculated as weighted aggregations of the marginal values of the outputs. This approach also assumes SWFs are symmetric quasi-concave and hence it relaxes the additivity assumption of the first approach.

We design interactive algorithms that take preference information from the DM iteratively by asking pairwise comparison questions. In the UTA-based approach, we ask the DM to compare two different bundles of outputs (vectors) while the convex cone based approach asks the DM to compare alternatives holistically.

For example, when ***holistic comparison*** method is employed, the DM is asked to compare two alternatives from the given set of alternatives such as $\left(\begin{smallmatrix} 2 & 8 \\ 3 & 4 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 5 & 5 \\ 6 & 2 \end{smallmatrix}\right)$ whereas when ***vector (bundle) comparison*** method is employed, the DM is asked to compare the following bundles (3, 4) and (6, 2), which represent different distributions of outputs to only one user. In UTA-based method, one can also use holistic comparison method, too. On the other hand, in order to generate the corresponding cones, one need to use holistic comparison method in cone-method.

Holistic comparison is more challenging in terms of its cognitive requirements. This is because comparing alternatives requires to analyse trade-offs between the alternatives in both user and output directions. Especially when the number of outputs and the number of users are increased, it would get more difficult for the DM to compare the alternatives holistically. However one can eliminate the alternative which is not preferred, permanently from the set by asking holistic comparison questions

On the other hand, vector comparison questions are not that cognitively challenging for the DM since they do not require the DM to consider the trade-offs between/among output levels of different users. This question type only requires comparing the trade-offs between multiple the outputs given to a single user. Although, vector comparison questions are easier for the DM, we cannot eliminate any alternative directly based on such questions.

Following two chapters are devoted to discussion of UTA-based and convex cone-based methods in detail.

Chapter 5

UTA-based Method

In this part, we discuss the interactive approach based on the well-known UTA method introduced by Jacquet-Lagrèze and Siskos ([18], [19]). In a classical MCDM problem, this method assumes an additive unweighted global value function, which is the sum of the marginal value scores and assigns values to alternatives in line with the preferences of a DM by using linear programming techniques ([45], [46]).

Although the UTA method forces marginal value functions to be compatible with the preference information, there may still exist many such value functions. The idea of evaluating all value functions that are compatible with the preference information (instead of chosing one) was firstly introduced in UTA^{GMS} method [24]. Our approach also follows this idea of evaluating all compatible value functions.

UTA-based methods are introduced and exploited for the multi-criteria evaluation problems where the alternatives are represented as vectors instead of matrices. We extend the use of these methods for the settings where alternatives are represented as matrices rather than vectors. We further extend the use of UTA-methods in which the DM has fairness concerns.

In the UTA-based method, we propose that the value that is obtained from an output vector (bundle), which is defined as user value, is the sum of the marginal values acquired from each output level in that bundle (we assume preference independence).

Moreover the social welfare function is assumed to be sum of the users' total values (i.e we use a utilitarian framework) in that alternative. These functions are in the following forms:

$UV(b_i^k) = \sum_{j=1}^{n} MV_j(a_{ij}^k)$ where $b_i^k = (a_{i1}^k, a_{i2}^k, ..., a_{in}^k)$ $\quad \forall i = 1,2,...,m, \quad \forall k = 1,2,...,N$

$SW(a^k) = \sum_{i=1}^{m} \sum_{j=1}^{n} MV_j(a_{ij}^k) \quad \forall k = 1,2,...,N$

with the normalization constraints below,

$MV_j(a_{j_*})$=0, $\quad \sum_{j=1}^{n} MV_j(a_j^*)$=1, $\quad MV_j(a_{ij}) \geq 0 \quad \forall j = 1,2,...,n, \ \forall i = 1,2,...,m$

where a_{j_*} and a_j^* are the least and most preferable levels of output j, respectively. With the additivity assumption, inferring the marginal value functions of the outputs will be sufficient to calculate the user value scores and also social welfare scores of the alternatives.

There are different UTA applications in the literature, each with its own assumptions on the shape of the marginal value functions. They can be linear ([47] , [26]), piecewise linear ([48], [21], [19]), or monotone [49]. In our problem setting, we assume that all $MV_j(.)$s are concave and approximate them in our mathematical models using piecewise linear approximation. This assumption is used to (partially) reflect the fairness concerns of the DM into the model. When concave marginal value functions are used, the total value increases as the levels of an output distributed to users get closer. (Everything else being the same the total value that the alternative brings becomes higher.)

Figure 5.1 illustrates a concave marginal value function for an output with four different levels and the corresponding piecewise linear function estimated by the UTA-based method. As seen in the figure, increasing output level from 3 to 4 would increase the marginal value score obtained from that output. Increasing the level from 4 to 5 also increases the marginal value score but in a smaller amount. That is, increasing the output from a higher level results in a smaller increase in the marginal value score.

This observation illustrates the effect of having concave marginal value functions on promoting a fair allocation for each output to the users.

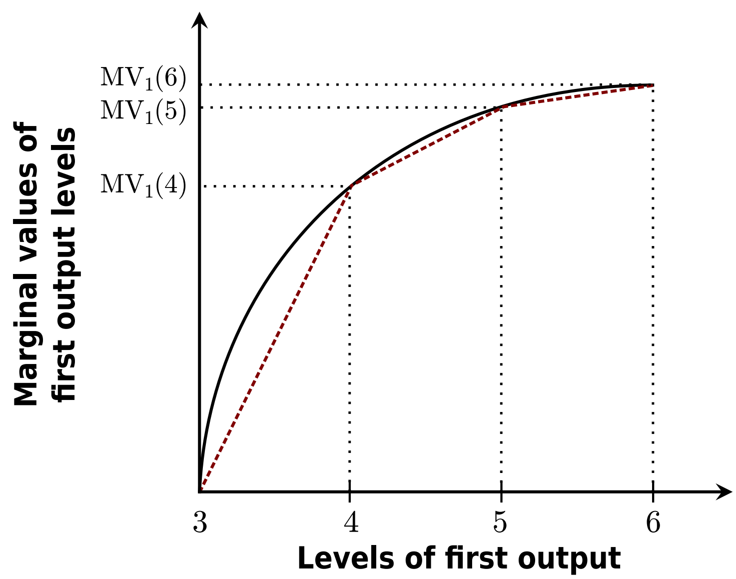

Figure 5.1: Concave marginal value function and its piecewise linear approximation

We now introduce the UTA-based model which considers all marginal value functions compatible with the preference information that the DM provides. Then, we discuss the proposed algorithm that uses this model to find the best alternative (or a small subset of most preferred alternatives), followed by a simple numerical example. We finalize this chapter with a discussion of the UTA-based approach.

5.1 UTA-based Model

UTA-based model introduced below checks if alternative a^k can have higher social welfare value than alternative $a^{k'}$ considering the DM's preference information.

UTA-based model

Sets:

I: the set of users $\{1,\ldots,m\}$.

J: the set of outputs $\{1,\ldots,n\}$.

Q: the pairwise comparison information gathered so far $\{(p,p') : p \textit{ is preferred over } p' \ \& \ p,p' \in \mathbb{R}^n\}$.

C_j: the vector that stores unique values of output j in an increasingly ordered manner.

Parameters:

L_j: the number of different levels in output j.

T_{ijk}: The rank of a^k_{ij} in set C_j where $i \in I$, $j \in J$ and $a^k \in A$.

T_{pj}: The rank of p_j in set C_j where $p \in Q$, $j \in J$.

ε: a small positive number to ensure the MVFs are increasing.

γ: a small positive number to ensure the MVFs are strictly concave.

Ω: a small positive number to incorporate strict preference information.

Variables:

MV_{jt}: the value of the t^{th} minimum level in output j.

UV_p: the total value achieved from the bundle p where $p = \{p_1,\ldots,p_n\}$ and $p \in Q$.

SW_k: the total social welfare that alternative k brings where $k \in A$.

$$\text{Maximize } 0 \tag{5.1}$$

$$\text{subject to } MV_{j,t+1} - MV_{jt} \geq \varepsilon \qquad \forall j \in J,\ t \in \{1, \ldots, L_j - 1\} \tag{5.2}$$

$$\frac{MV_{j,t+1} - MV_{jt}}{C_{j,t+1} - C_{jt}} - \frac{MV_{j,t+2} - MV_{j,t+1}}{C_{j,t+2} - C_{j,t+1}} \geq \gamma \quad \forall j \in J,\ t \in \{1, \ldots, L_j - 2\} \tag{5.3}$$

$$\sum_{j \in J} MV_{jL_j} = 1 \tag{5.4}$$

$$MV_{j1} = 0 \qquad \forall j \in J \tag{5.5}$$

$$UV_p = \sum_{j \in J} MV_{jT_{pj}} \qquad \forall p \in Q \tag{5.6}$$

$$UV_p - UV_{p'} \geq \Omega \qquad \forall (p, p') \in Q \tag{5.7}$$

$$SW_k = \sum_{j \in J, i \in I} MV_{jT_{ijk}} \tag{5.8}$$

$$SW_{k'} = \sum_{j \in J, i \in I} MV_{jT_{ijk'}} \tag{5.9}$$

$$SW_k - SW_{k'} \geq 0 \tag{5.10}$$

$$MV_{jt} \geq 0 \qquad \forall j \in J, t \in \{1, \ldots, L_j\} \tag{5.11}$$

The model is a feasibility check model which controls if alternative k can bring higher social welfare value than alternative k'. It tries to assign values to output levels in such a way that social value score of alternative k will be greater than social value score of alternative k'. Constraint set (5.2) ensures that the marginal value functions will be increasing. That is, for each output the assigned marginal value has to increase when the level of the output increases. Constraint set (5.3) is used to have concave marginal value functions. Parameter γ determines the concavity levels of the marginal value functions. This constraint set ensures that the increments will be awarded more when the output levels are small. Constraint sets (5.4) and (5.5) are for normalization. Constraint set (5.4) guarantees that the user values are in the range [0-1] and hence the social welfare values of all alternatives are in the range [0-m]. Constraint set (5.5) ensures that the marginal value scores assigned to the minimum levels of outputs will be 0. Constraint set (5.6) assigns a user value score to each bundle p in the preference information set using the marginal value scores assigned to the observed output levels in that bundle. Constraint set (5.7) incorporates the provided information by the DM to into the model. They ensure that the user values of preferred bundles will be greater than the user values of bundles which are not preferred. Constraint sets (5.8) and (5.9) assign social welfare values to alternatives a^k and $a^{k'}$, respectively using the assigned marginal value scores for the output levels. Constraint (5.10) checks if alternative a^k can bring higher social welfare value than alternative $a^{k'}$ considering the UTA-assumptions (increasing and concave marginal value functions and additivity assumption on user values and social welfare values) and the provided preference information.

Parameter selection is an important part of UTA-based method. Using high values for the parameters (ε, γ, and Ω) would decrease the number of questions asked to the DM (since the parameters also narrow the possible region for the MVFs); however it may also result misrepresentation of the underlying value function of the DM. Hence one may end up with the alternative which is not the real best alternative. On the other hand setting parameters so small may lead to increase in the number of questions asked to the DM. For example, if concavity parameter γ or strict preference information parameter Ω are chosen so small, this may increase the number of questions that are asked to the DM.

Observing the possible scores of the marginal value, the user value, and the social welfare functions, one may have an insight into the parameter values. For example, the marginal values of the outputs and the user values of the bundles are in [0-1]. So setting ε or Ω to 0.5 would highly restrict the MVFs and UVFs.

5.2 UTA-based Algorithm

We now discuss the interactive UTA-based algorithm for the problem setting described above, which finds the best alternative (or a small subset of most preferred ones) using basic UTA principles.

At each iteration of this algorithm, preference information of the DM is gathered by asking vector (bundle) comparison questions. These comparison questions limit the possible region for the MVFs in line with the preference information of the DM. Then the algorithm checks for all alternatives which are not eliminated if there exists any other alternative that brings higher social welfare value for all possible marginal value function assignments that are compatible with given preference information and makes the necessary eliminations. If the number of remaining alternatives is greater than a predetermined threshold value K, the DM is asked another bundle comparison question. Obtaining preference information is key to eliminate alternatives since they narrow the possible region of MVFs in line with the provided information. This loop is repeated until the number of remaining alternatives is less than or equal to a predetermined threshold value K.

Algorithm 1 *Step 1: Initialization. Set REMAIN=*$\{a^1, a^2, ..., a^N\}$*. Find the em-dominated alternatives and remove them from the set REMAIN.*

Step 2: UTA eliminations. For all pairs of alternatives, make pairwise comparisons using the ***Comparison*** *subroutine. Make the necessary eliminations and update set REMAIN accordingly.*

Step 3: Take new preference information from the DM by using ***Vectorpreference-info*** *subroutine. Add the information as a constraint to the UTA-based model and go to Step 4.*

Step 4: Preference Information Eliminations. For all pairs of alternatives, make pairwise comparisons using the ***Comparison*** *subroutine. Make the necessary eliminations and update set REMAIN accordingly. If all pairs in REMAIN have not been checked yet, repeat this step. Otherwise, if the number of alternatives in set REMAIN is higher than K, go to step 3. If not, go to Step 5.*

Step 5: Stop and report REMAIN.

Let us now explain *Vectorpreferenceinfo* and *Comparison* subroutines in more detail.

Vectorpreferenceinfo

This subroutine is used to select the vectors to be asked to the DM. We tried three alternative methods to choose these vectors, which are based on random selection, distance to an ideal vector and pairwise distances between the set of vectors and analysed their performances in terms of the number of questions asked to the DM.

Random selection: We randomly choose two vectors/bundles from the provided set of alternatives and the DM is asked to compare these two alternatives. Note that the algorithm ensures that the DM will not be asked to compare the same bundles again.

Ideal: We create an *ideal vector* $((ideal_1, ideal_2, ..., ideal_n)$ where $ideal_j = \max_{\forall k \in A, \forall i \in I} a_{ij}^k)$ and calculate the Euclidean distances between the *ideal vector* and each vector (bundle) in the given set of alternatives. The vectors to be asked are chosen starting from the ones which have smallest distance to the *ideal vector*. After asking the two bundles with the smallest distances, we eliminate one of the bundles from the set of bundles to be asked to the DM in order not to ask the same question to the DM with a predetermined rule.

The rule permits to eliminate the preferred bundle from the set because keeping the strongest vector in the set decreases the set of possible questions that may be asked to the DM. For example, suppose that the DM is asked to compare (9, 8) and (5, 9) and (9, 8) is preferred. However, (9,8) may dominate many bundles and hence keeping it in the set of bundles to be asked to the DM may decrease the possible number of questions can be asked to the DM. Therefore, we permit (9,8) to be eliminated from the set of bundles to be asked to the DM even though it is the closest bundle to the *ideal* vector.

Minimum pairwise distance: We calculate the Euclidean distances between all pairs of vectors in the given set of alternatives. For example let us consider the following alternatives with two outputs and two users: $\begin{pmatrix} a_{11}^k & a_{12}^k \\ a_{21}^k & a_{22}^k \end{pmatrix}$ and $\begin{pmatrix} a_{11}^{k'} & a_{12}^{k'} \\ a_{21}^{k'} & a_{22}^{k'} \end{pmatrix}$. The distance is calculated as follows: $\sqrt{\sum_{j \in J, i \in I} |a_{ij}^k - a_{ij}^{k'}|^2}$. Then the vectors/bundles that have the minimum distance to each other are chosen to ask the DM.The pair asked is removed from the set.

Comparison

This subroutine is used to compare the social welfare values of the alternatives with each other. For any two alternatives a^k and $a^{k'}$, we check if there exists a feasible solution where $SW(a^k) \geq SW(a^{k'})$ using *UTA-based model*. In *Step 2*, the alternatives are compared only considering the assumptions on the form of marginal value functions, i.e. being increasing and concave. Then, in *Step 3* after obtaining a new preference information, we make all possible comparisons again. If $SW(a^k)$ cannot be higher than $SW(a^{k'})$, then we conclude that $a^{k'}$ is preferred over a^k. However if $SW(a^k)$ can be higher than $SW(a^{k'})$, then we need to check if $SW(a^{k'})$ can be higher than $SW(a^k)$. If $SW(a^{k'})$ can be higher than $SW(a^k)$, we remain inconclusive. If $SW(a^{k'})$ cannot be higher than $SW(a^k)$, then we can conclude that a^k is preferred over $a^{k'}$.

Table 5.1 provided below summarizes the elimination rules used in the UTA-based algorithm.

Table 5.1: Elimination rules for UTA-based algorithm

Is $SW(a^k) \geq SW(a^{k'})$ possible?	Is $SW(a^{k'}) \geq SW(a^k)$ possible?	Results
Yes	Yes	Inconclusive
Yes	No	Eliminate alternative $a^{k'}$
No	Yes	Eliminate alternative a^k

Given preference information, if $SW(a^k) \geq SW(a^{k'})$ is possible but $SW(a^{k'}) \geq SW(a^k)$ is not possible, then we conclude that the alternative k is better. If both cases are possible we do not make any eliminations.

Figure 5.2 illustrates the general overview of the UTA-based algorithm.

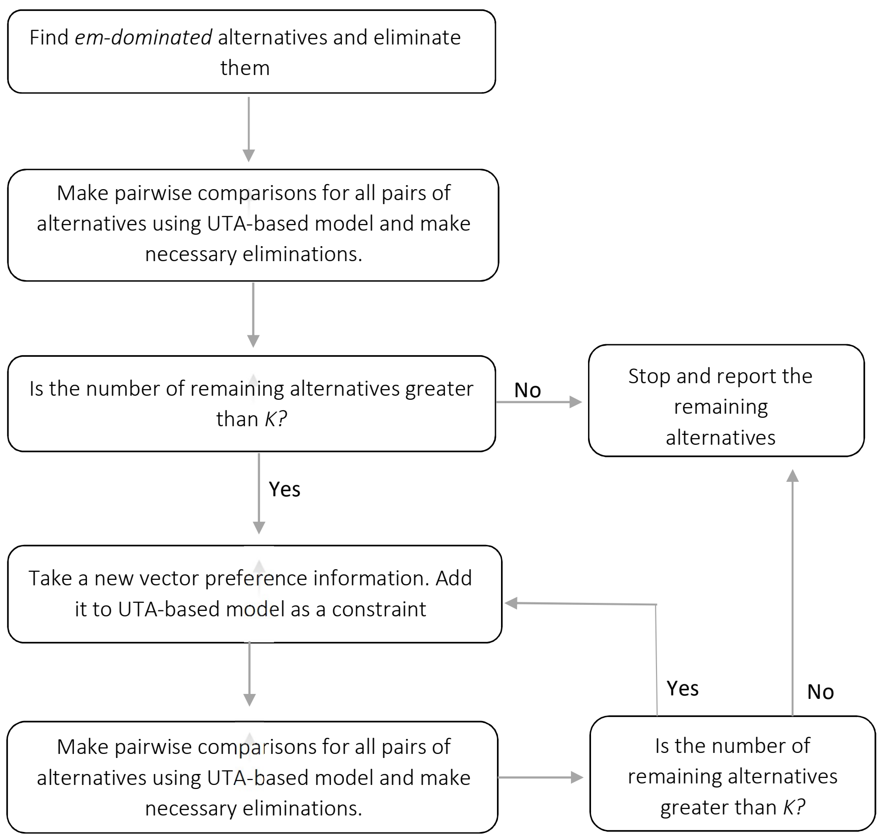

Figure 5.2: General overview of the UTA-based algorithm

5.3 Numerical Example

Let us now review the steps of the Algorithm 1 for the provided example (Example 2) where the DM tries to find the best alternative among six alternatives.

Step 1. Checks the *em-dominance relation* among the alternatives. It eliminates a^3 since a^4 *em-dominates* a^3.

Step 2. Compares the alternatives just considering the assumptions on MVFs (being increasing and concave) without obtaining any preference information. At this step a^5 is eliminated (the total levels of both outputs that are distributed by a^2 and a^5 are the same and a^2 distributes first output more equally than a^5). The same relation is observed between a^4 and a^6, hence a^6 is also eliminated.

Step 3. The DM is asked to compare bundles (4, 6) and (5, 5) (assume that the question selection method is based on the distance from the ideal vector). Assume that the DM chooses (5, 5). This preference information is added to UTA-based model as a constraint (constraint 5.7).

Step 4. Through solving the related mathematical models, we conclude that a^1 can not be better than a^4, hence it is eliminated. The remaining alternatives are a^2 and a^4.

Step 3. The DM is asked to compare bundles (8, 2) and (3, 7). This preference information is added to UTA-based model as a constraint (constraint 5.7).

Step 4. Through solving the related mathematical models, a^2 is eliminated.

Step 5. The algorithm returns a^4 as the solution.

5.4 Discussion of UTA-based Method

UTA-based approach introduces concave marginal value functions hence encourages a more equitable distribution for each of the outputs over users regardless of the levels of other outputs that they receive. That is, using additivity over users, we make this underlying assumption that the outputs are not substitutable and hence a more equitable distribution is always desired regardless of users' positions with respect to the other outputs.

To illustrate, let us consider the following two alternatives in Example 2: $a^2 = \left(\begin{smallmatrix} 5 & 5 \\ 6 & 2 \end{smallmatrix}\right)$ and $a^5 = \left(\begin{smallmatrix} 3 & 5 \\ 8 & 2 \end{smallmatrix}\right)$. The first output is distributed in a more equitable manner in a^2 but this occurs at the cost of making the second user, who was worse off with respect to second output, have less of first output compared to a^5. In UTA-based approach, a^2 is considered better since, everything else being the same, the first output is distributed in a more equitable manner. However, one can argue that redistribution is only meaningful and social welfare increasing when one user is definitely underprivileged and redistribution alleviates this under-privilege, which is not the case in this example. In such cases, the UTA-based approaches will not be of use and the preference model of a DM who would prefer e.g. $\left(\begin{smallmatrix} 3 & 5 \\ 8 & 2 \end{smallmatrix}\right)$ over $\left(\begin{smallmatrix} 5 & 5 \\ 6 & 2 \end{smallmatrix}\right)$ on the grounds that the outputs may be substitutable can not be taken into account. This is due to the additivity assumption of the UTA-based approach.

There exists a large body of work in the economics literature discussing inequality in single good distributions like income. When a single good is distributed, most of the literature agrees on the suitability of using nonadditive social welfare functions rather than assuming separability ([50], [51], [52]). The convex cone based approach, which we discuss in the next chapter, alleviates some drawbacks of the UTA-based approach as it (partially) relaxes the additivity assumption for the social welfare function used. That is, the convex cones approach will allow a DM to prefer the first distribution over the second in the above example as we will elaborate in the next chapter.

Chapter 6

Convex Cone-based Approach

In this chapter, we discuss the convex cone based approach, which is widely used in the MCDM literature ([53], [25], [27], [54], [28]). Convex cones are used in MCDM problems to incorporate preference information in the model. This method assumes that the underlying value function of the DM is quasi-concave and is based on eliminating the alternatives that are inferior with respect to the cones generated based on preference information that she provides [25].

We first summarize the main findings in classical MCDM choice problems. Then we discuss an extension of the approach to cases where each alternative shows the allocation of a single output over multiple users and the DM has an equitable preference model (discussed in [55]). Finally, we provide the extension we suggest for problems where the alternatives are defined as matrices.

6.1 Use of Convex Cones in Classical MCDM Choice Problems

Most of the literature using convex cones deals with the classical MCDM choice problems where equity is not a concern of the DM. In this section, we give the main definitions and results used in the classical MCDM choice problems, where alternatives are vectors.

Definition 4 *Given a set of k vectors, such that $z^1,\ldots,z^k \in \mathbb{R}^m$, the cone $C(z^1,\ldots,z^{k-1};z^k)$ is defined, where $z^\ell : \ell \neq k$ are the upper generators and z^k is the lower generator as follows:*

$$C(z^1,\ldots,z^{k-1};z^k) = \left\{z \mid z = z^k + \textstyle\sum_{\ell\neq k}\mu_\ell(z^k - z^\ell), \mu_\ell \geq 0\right\}.$$

The cone dominated region of $C(z^1,\ldots,z^{k-1};z^k)$ is denoted by $CD(z^1,\ldots,z^{k-1};z^k)$ and defined as follows:

$$CD(z^1,\ldots,z^{k-1};z^k) = \left\{z' \mid z' \leq z \text{ where } z \in C(z^1,\ldots,z^{k-1};z^k)\right\}.$$

If the value function of the DM (SW(.)) is quasi-concave, the following holds [25],

Lemma 1 *For any $z^c \in C(z^1,\ldots,z^{k-1};z^k)$, $SW(z^c) \leq SW(z^k)$. Also, for any $z' \in CD(z^1,\ldots,z^{k-1};z^k)$, $SW(z') \leq SW(z^k)$.*

Each point $z' \in CD(z^1,\ldots,z^{k-1};z^k)$ is called *cone dominated*.

Cone domination enables us to make eliminations in line with the preference information that the DM provides. Figure 6.1(a)-(b) illustrates the relation between vector and cone dominance. Figure 6.1(a) shows the vector dominated region by (2,6) without obtaining any preference information. Now suppose that the DM is asked to select one of the following alternatives (2,6) and (3,4) and the DM prefers (3,4) over (2,6). Figure 6.1(b) shows the 2-point cone generated by these alternatives. The solid line represents $C((3,4);(2,6))$ and the filled area is the cone-dominated region, $CD((3,4);(2,6))$. Any alternative in this region is cone dominated. As shown in the

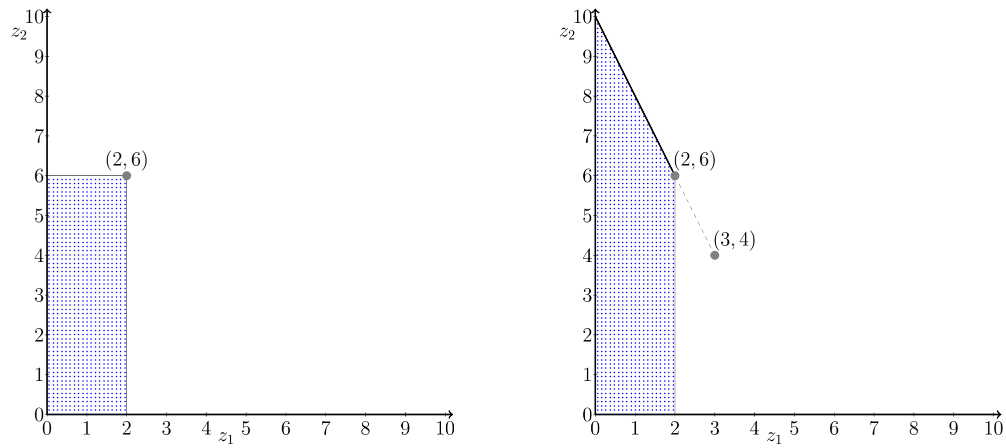

Figure 6.1: (a)The vector dominated region by (2,6). (b) $C((3,4);(2,6))$ and cone dominated region

figure, cone dominance can make further eliminations compared to vector dominance relation.

In interactive algorithms using convex cone, preference information is taken from the DM iteratively. Based on this preference information, cones are generated. For any candidate alternative in the set, one checks whether the alternative is cone dominated and if so eliminates the alternative from further consideration, iteratively reducing the set of candidate solutions.

Linear programming models can be used to check if an alternative is in the cone dominated region. For an alternative z^I, the following model checks if z^I is in the cone dominated region $CD(z^1,...,z^{k-1};z^k)$. The right hand side of constraint set (6.2) corresponds to a point on $C(z^1,...,z^{k-1};z^k)$, which (vector) dominates z^I. If this model is feasible, $z^I \in CD(z^1,...,z^{k-1};z^k)$.

$$\text{Minimize } 0 \tag{6.1}$$

$$\text{subject to } z_i^I \leq z_i^k + \sum_{\ell=1}^{k-1} \mu_\ell (z_i^k - z_i^\ell), \quad for\ i = 1,...,m \tag{6.2}$$

$$\mu_\ell \geq 0, \qquad for\ \ell = 1,...,k-1 \tag{6.3}$$

6.2 Use of Convex Cones in Equitable MCDM Choice Problems

A large body of the literature using convex cones in MCDM problems do not touch upon the concept of equitability. [55] extends the use of convex cones for allocation settings where a single output is distributed to multiple users and impartiality holds. Since the preference model of the DM is assumed to be equitable, impartiality holds, which implies that the value function of the DM is symmetric quasi-concave. This assumption implies that each vector (allocation) of size *m* will have *m!* permutations and the DM is indifferent to all these permutations. Hence given single pairwise preference information, one can generate multiple cones considering various permutations of the upper and lower generators.

[55] also involves using a different dominance relation than the vector dominance relation, namely the *generalized Lorenz dominance* (also called *equitable dominance*) relation, which is defined below.

Definition 5 *Let $\vec{z}^k$ denote the permutation of z^k such that $\vec{z}^k$: $\vec{z}^k_1 \le \vec{z}^k_2 \le ... \le \vec{z}^k_m$ where m is the number of users. $\vec{z}^k$ is called the ordered vector of z^k. Let $\bar{Q}(z^k)$ denote the cumulative ordered vector of z^k defined as follows:*

$$\bar{Q}(z^k) = (\bar{Q}_1(z^k), \bar{Q}_2(z^k), ..., \bar{Q}_m(z^k)) \text{ where } \bar{Q}_i(z^k) = \sum_{t=1}^{i} \vec{z}^k \ \ \forall i \in I, \ I = \{1, 2, ..., m\}.$$

That is, $\bar{Q}_i(z^k)$ shows the total output amount provided to the poorest i users in the distribution.

Theorem 1 *Given two alternatives z^1, $z^2 \in \mathbb{R}^m$,*

$$z^1 \preceq_{GL} z^2 \text{ (z^2 generalized Lorenz dominates z^1)} \Longleftrightarrow \bar{Q}_i(z^1) \le \bar{Q}_i(z^2) \ \ \forall\, i \in I \text{ [16].}$$

Generalized Lorenz dominance is introduced as an extension of the widely-known Lorenz dominance concept used in the economics literature [9]. It can be used to compare distribution vectors over anonymous users even when the means of the distributions are not equal. Moreover, pairs of alternatives for which vector dominance remains inconclusive, could be compared using generalized Lorenz dominance. For example, assume that we have three alternatives where z^1=(12, 7, 3, 18), z^2=(2, 7, 12, 18), and z^3=(9, 7, 15, 5). None of the vectors is dominated in the vector dominance sense. However, since $\bar{Q}(z^1)$=(3, 10, 22, 40) and $\bar{Q}(z^2)$=(2, 9, 21, 39), $z^2 \preceq_{GL} z^1$. Figure 6.2 shows the generalized Lorenz curves of the alternatives provided. It is seen that the cumulative output amount given to the poorest i users in z^1 is always higher than that of z^2; hence the generalized Lorenz curve of z^1 is always above that of z^2. However, there is no dominance between z^3 and z^1 since the two curves intersect.

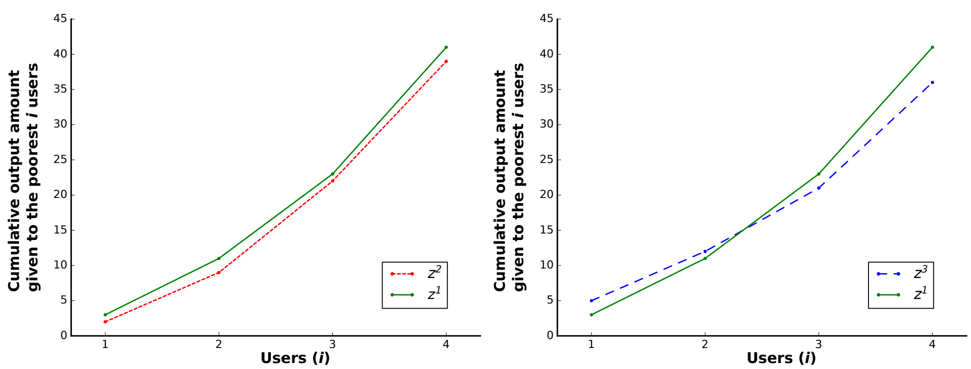

Figure 6.2: Generalized Lorenz dominance illustration

Let us reconsider the example described above in Figure 6.1. When the DM has equity concerns, the dominated region by vector (2,6) can be defined using generalized Lorenz dominance relations. Figure 6.3(a) shows the generalized Lorenz dominated region by (2,6) without obtaining any preference information.

If the DM prefers (3,4) over (2,6), impartiality implies that the DM prefers any permutation of $(3,4)$ over any permutation of $(2,6)$. So in addition to $C((3,4);(2,6))$ we can generate the cones $C((4,3);(2,6))$, $C((4,3);(6,2))$ and $C((3,4);(6,2))$ and eliminate the alternatives which are inferior to any of these cones. Figure 6.3(b) shows all the 2-point cones generated by (2,6) and (4,3) and the dominated region by these cones. As shown in the figure, when the impartiality over users is assumed, cone dominance enables us to make further eliminations than generalized Lorenz dominance

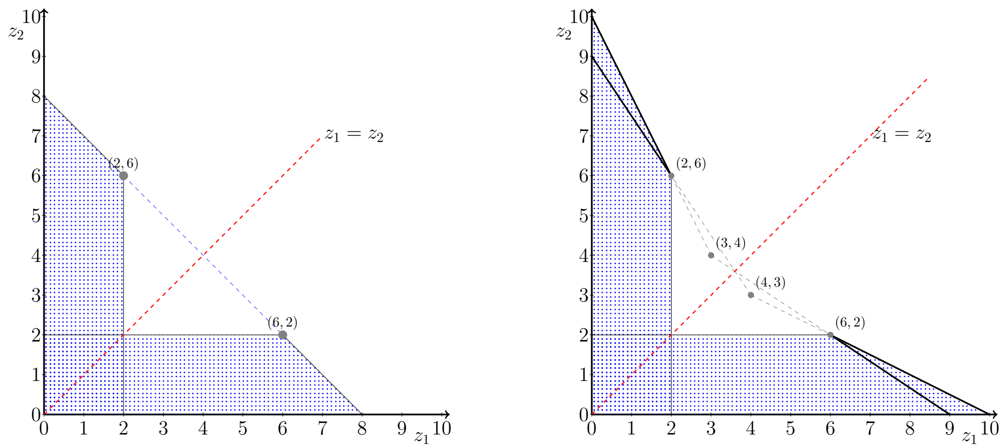

Figure 6.3: (a)Generalized Lorenz dominated region by (2,6). (b) $C((3,4);(2,6))$ and its equitably cone dominated region

relations.

Considering multiple permutation cones increases the amount of inference one can make from the preference information. However, note that, when one has a preference information of n vectors of size m, the number of permutation cones to be considered becomes $m!^n$. [55] introduces results to handle this complexity.

When dealing with single benefit distributions, [55] eliminates an alternative if it is generalized Lorenz dominated by any of the permutation cones. It is proved that, rather than considering all the permutation cones, it is sufficient to use the cone generated by the ordered versions of the generators. In order to check if an alternative z^I is (generalized Lorenz) dominated by any of the permutation cones the following model is solved ([55]):

$$\text{Maximize} \sum_{h=1}^{m} hr_h - \sum_{h=1}^{m}\sum_{i=1}^{m} d_{hi} \tag{6.4}$$

$$\text{subject to } z_i^c - \sum_{\ell=1}^{k-1} \mu_\ell(\vec{z}_i^{\,k} - \vec{z}_i^{\,\ell}) = \vec{z}_i^{\,k} \qquad for\ i = 1,...,m \tag{6.5}$$

$$r_h - d_{hi} - z_i^c \leq 0 \qquad for\ i,h = 1,...,m \tag{6.6}$$

$$\sum_{j=1}^{k} \vec{z}_j^I \leq hr_h - \sum_{i=1}^{m} d_{hi} \qquad for\ h = 1,...,m \tag{6.7}$$

$$d_{hi} \geq 0 \qquad for\ i,h = 1,...,m \tag{6.8}$$

$$\mu_\ell \geq 0 \qquad for\ \ell = 1,...,k-1 \tag{6.9}$$

where r_h and d_{hi} are auxiliary variables used to ensure that cumulative ordered vector of z^c is found (at optimality, $hr_h^* - \sum_{i=1}^{m} d_{hi}^* = \bar{Q}_h(z^c)$. Note that the model has alternate optima, $r_h^* = \vec{z}_h^c + g$, where g is a scalar and $d_{hi}^* = 0$ for $i : z_i^c > \vec{z}_h^c$ and $d_{hi}^* = \vec{z}_h^c - z_i^c + g$ for $i : z_i^c \leq \vec{z}_h^c$. These ensure that at optimality the difference term $hr_h^* - \sum_{i=1}^{m} d_{hi}^* = \bar{Q}_h(z^c)$ (Please see [56] for more information). This model checks if there exist $z^c \in C(z^1,...,z^{k-1};z^k)$ such that $\bar{Q}(z^I) \leq \bar{Q}(z^c)$. Constraint set (6.5) creates z^c such that $z^c \in C(z^1,...,z^{k-1};z^k)$. Constraint set (6.6) together with the objective function ensures that at optimality, $hr_h^* - \sum_{i=1}^{m} d_{hi}^* = \bar{Q}_h(z^c)$ and constraint set (6.7) guarantees that $\bar{Q}(z^I) \leq \bar{Q}(z^c)$.

6.3 Use of Convex Cones in Multi-dimensional Equitable MCDM Choice Problems

We suggest a further extension of the convex cone method to problems where the alternatives are defined as matrices rather than vectors. We assume that social welfare is a symmetric quasi-concave function of the user values, which are assumed to be additive.

We assume the DM has an equitable preference model over the distribution vector of these user values, hence use the convex cones method discussed in [55] (with the generalized Lorenz dominance relation). The user values are calculated as the weighted sum of the scaled output levels. The scaled matrix a^{k^s} for an alternative a^k is generated as follows: $a_{ij}^{k^s} = (a_{ij}^k - \min_{i\in I,k\in A} a_{ij}^k)/(\max_{i\in I,k\in A} a_{ij}^k - \min_{i\in I,k\in A} a_{ij}^k)$. For the sake of simplicity, from now on we use a^k for the scaled levels, too. $UV(b_i^k)$ is calculated as $UV(b_i^k)$=$\sum_{j\in J}(w_j a_{ij}^k)$, where w_j is the weight given to j^{th} output.

Then, the previous model becomes,

$$\text{Maximize} \sum_{h=1}^{m} hr_h - \sum_{h=1}^{m}\sum_{i=1}^{m} d_{hi} \tag{6.10}$$

$$\text{subject to } z_i^c - \sum_{\ell=1}^{k-1} \mu_\ell((\overrightarrow{wa^k})_i - (\overrightarrow{wa^\ell})_i) = (\overrightarrow{wa^k})_i \qquad for\ i = 1,...,m \tag{6.11}$$

$$r_h - d_{hi} - z_i^c \leq 0 \qquad for\ i,h = 1,...,m \tag{6.12}$$

$$\sum_{j=1}^{h} (\overrightarrow{wa^I})_j \leq hr_h - \sum_{i=1}^{m} d_{hi} \qquad for\ h = 1,...,m \tag{6.13}$$

$$\sum_{j=1}^{n} w_j = 1 \tag{6.14}$$

$$d_{hi} \geq 0 \qquad for\ i,h = 1,...,m \tag{6.15}$$

$$w_j \geq 0 \qquad j = 1,...,n \tag{6.16}$$

$$\mu_\ell \geq 0 \qquad for\ \ell = 1,...,k-1 \tag{6.17}$$

This model checks if there exists any z^c vector on $C(wa^1, wa^2, ..., wa^{k-1}; wa^k)$, that generalized Lorenz dominates a given alternative a^I (wa^I) for any weight value (w). Since the weight vectors are also unknown, the model discussed above is non-linear. Moreover, even when the above model is feasible we can not eliminate an alternative, since it could have been cone dominated for some w vector and not dominated for others. To be affirmative, one should ensure that alternative a^I is cone dominated over the entire feasible weight space. In order to handle this non-linearity and be conclusive, we use discretization and perform a parametric search over the entire (discretized) feasible weight region.

We now introduce the Cone-based model developed for multi-dimensional equitable choice problems. Then, we discuss the proposed algorithm that uses this model to find the best alternative (or a small subset of most preferred alternatives), followed by a simple numerical example. We finalize this chapter providing a discussion of the convex cone-based approach.

6.3.1 Cone-based Model

Cone-based model introduced below checks if an alternative is in the cone dominated region for a given weight vector. Assume that the DM is asked to choose between two alternatives and a^U represents the alternative that the DM prefers and a^L represents the alternative that the DM does not prefer and we want to check if a^I is in the cone dominated region of the cone generated by these alternatives. Remember that our alternatives are represented by matrices. We first calculate the user value vectors for the alternatives by using weighted sum of their output levels. After we obtain user value vectors for the alternatives, we use Cone-based model to check if a^I is cone dominated.

The following model checks if alternative a^I is in the cone dominated region generated by a^U and a^L.

Parameters:

$\vec{V}^L$: the vector $(\vec{V}_1^L, \vec{V}_2^L, ..., \vec{V}_m^L) \in \mathbb{R}^m$ that stores ordered UVs of a^L (for the given weight values) in an ascending manner.

$\vec{V}^U$: the vector $(\vec{V}_1^U, \vec{V}_2^U, ..., \vec{V}_m^U) \in \mathbb{R}^m$ that stores ordered UVs of a^U (for the given weight values) in an ascending manner.

$\vec{V}^I$: the vector $(\vec{V}_1^I, \vec{V}_2^I, ..., \vec{V}_m^I) \in \mathbb{R}^m$ that stores ordered UVs of a^I (for the given weight values) in an ascending manner.

Variables:

μ: the scalar for C($\vec{V}^U$; $\vec{V}^L$).

V^c: a vector $\in \mathbb{R}^m$: $V^c \in$ C($\vec{V}^U$; $\vec{V}^L$).

r_h: auxiliary variables used to ensure that the cumulative ordered vector of V^c is found.

d_{hi}: auxiliary variables used to ensure that the cumulative ordered vector of V^c is found.

$$\text{Minimize} \sum_{h=1}^{m} hr_h - \sum_{h=1}^{m}\sum_{i=1}^{m} d_{hi} \tag{6.18}$$

$$\text{subject to } V_i^c - \mu(\vec{V}_i^L - \vec{V}_i^U) = \vec{V}_i^L \quad for\ i = 1,...,m \tag{6.19}$$

$$r_h - d_{hi} - V_i^c \leq 0 \qquad for\ i,h = 1,...,m \tag{6.20}$$

$$hr_h - \sum_{i=1}^{m} d_{hi} \geq \sum_{j=1}^{h} \vec{V}_j^I \qquad for\ h = 1,...,m \tag{6.21}$$

$$d_{hi} \geq 0 \qquad for\ i,h = 1,...,m \tag{6.22}$$

$$\mu \geq 0 \tag{6.23}$$

Constraint set (6.19) creates a V^c vector in $C(V^U;\ V^L)$. Constraint sets (6.20) and (6.21) ensure that the created V^c generalized Lorenz dominates V^I by using r_h and d_{hi} auxiliary variables. Constraints (6.22) and (6.23) are non-negativity constraints.

6.3.2 The Convex Cone-based Algorithm

We now describe the convex cone-based algorithm we use for our problem setting. We will explain the algorithm for problems with two outputs and for the case where only 2-point cones (these are cones with only two generators) are used. It is straightforward to generalize the algorithm for problems with more than two outputs with an appropriate discretization of the feasible weight space. The algorithm can easily be modified if one wants to use k-point cones (cones with $k-1$ upper generators and one lower generator).

We assume that there are N alternatives and m users as before. In addition to the set

REMAIN, which keeps the alternatives not eliminated so far, we define the following sets: the set *CONES* stores all the alternative pairs on which the DM provides preference information. The set *POSW*1 stores the possible weight values for the first output, which are compatible with the preference information that the DM provided. Recall that we discretize the weight space.

Algorithm 2 *Step 1: Initialization. CONES*=$\emptyset$*. REMAIN*=$\{a^1, a^2, ..., a^N\}$*. Find the em-dominated alternatives and remove them from REMAIN. POSW1*=$\{0, 0.05, ..., 0.95, 1\}$

Step 2: Take new preference information from the DM using ***Holisticpreferenceinfo*** *subroutine and let* a^U *and* a^L *indicate preferred and not-preferred alternatives, respectively. Remove* a^L *from REMAIN. If the number of alternatives in the set REMAIN is greater than K, narrow the possible weight interval by using* ***Narrowweight*** *subroutine (if possible) and go to Step 3. Otherwise, STOP.*

Step 3: Update CONES=$\{CONES\} \cup (a^U; a^L)$ *and remove the cone dominated alternatives from REMAIN by using* ***Conedominancecheck*** *subroutine. If the number of alternatives in the set REMAIN is greater than K, go to Step 2. Otherwise, STOP.*

Let us now explain each subroutine in more detail.

Holisticpreferenceinfo

This subroutine is used to determine the alternatives to ask the DM for pairwise comparison. It creates an *ideal alternative*, *IDEAL*, such that $IDEAL_{ij} = \max_{\forall k \in A, \forall i \in I}$ a^k_{ij} and calculates the Euclidean distance between each alternative in set *REMAIN* and *IDEAL*. Then the DM is asked to choose between two alternatives that have the minimum distances.

Narrowweight

This subroutine is used to narrow the possible weight interval of the first output in line with the preference information. Suppose that the DM is asked to choose between two alternatives in $\mathbb{R}^{(m \times n)}$. Let a^U be the preferred alternative and a^L be the alternative which is not preferred. We eliminate the weights that satisfy the following inequality $\bar{Q}(a^U[\begin{smallmatrix} w_1 \\ \cdots \\ w_n \end{smallmatrix}]) \leq \bar{Q}(a^L[\begin{smallmatrix} w_1 \\ \cdots \\ w_n \end{smallmatrix}])$ based on Remark 1.

Remark 1 *If the DM prefers a^U over a^L, then a^L cannot generalized Lorenz dominate a^U. Then, from the definition of generalized Lorenz dominance, we are sure that the following inequality $\bar{Q}(a^U[\begin{smallmatrix} w_1 \\ \cdots \\ w_n \end{smallmatrix}]) \leq \bar{Q}(a^L[\begin{smallmatrix} w_1 \\ \cdots \\ w_n \end{smallmatrix}])$ cannot hold. The weight values that satisfy the above inequality should be eliminated as they would lead to a less preferred alternative to generalized Lorenz dominate a more preferred one, contradicting with the assumptions made on the preference model.*

Conedominancecheck

This subroutine is used to find the cone dominated alternatives in the set *REMAIN*. The subroutine checks if a^I (wa^I) is cone dominated by any $C(wa^{U'};wa^{L'})$ such that $(a^{U'},a^{L'}) \in CONES\ \forall w \in POSW1$ where $a^I \in REMAIN$ using cone-based model. To eliminate an alternative, it is sufficient to ensure that for any weight level possible, there exists a cone dominating the alternative. If so, that alternative is removed from the set *REMAIN*. This is repeated for all the alternatives in the set *REMAIN*.

Figure 6.4 illustrates the main steps that the cone-based algorithm follows in a compact way.

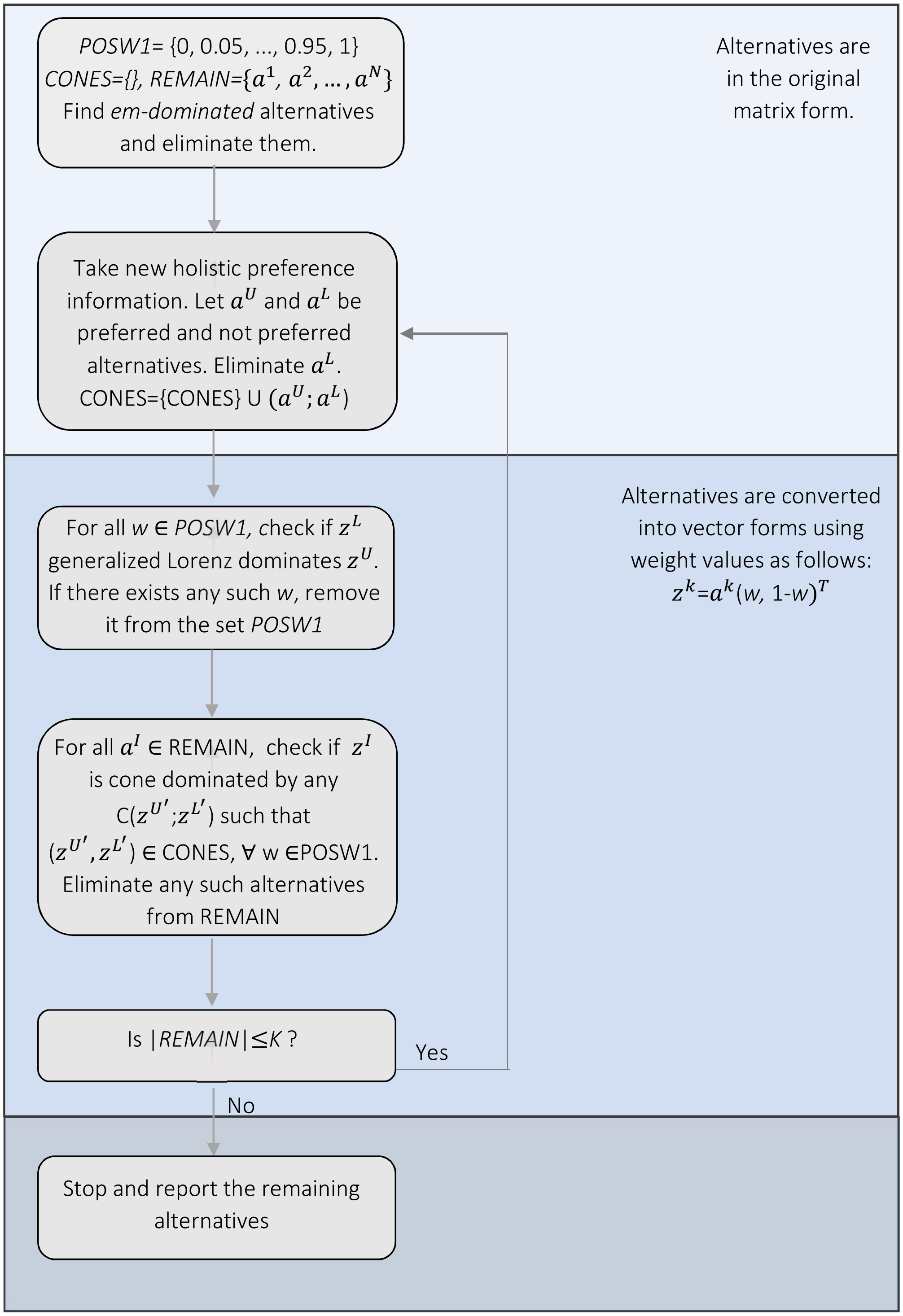

Figure 6.4: Steps of the cone-based algorithm

6.3.3 Numerical Example

Let us now review the steps of the Algorithm 2 for the provided example (Example 2) where the DM tries to find the best alternative among six alternatives. We assume that the underlying social welfare function of the DM is $SW(a^k) = (0.7a^k_{11} + 0.3a^k_{12})(0.7a^k_{21} + 0.3a^k_{22})$.

Step 1. Checks the *em-dominance relation* among the alternatives. It eliminates a^3 since a^4 *em-dominates* a^3.

Step 2. The DM is asked to compare a^2 and a^4 and prefers a^2 to a^4. This preference information eliminates a^4 and narrows the possible weight interval for the first output to [0.7-1].

Step 3. C($wa^2;wa^4$) is generated for all the possible discretized weights (0.7, 0.75, 0.8, 0.85, 0.9, 0.95, 1). The remaining alternatives (a^1,a^5,a^6) are checked if any of them is dominated using the cone-based model. In this example, none of the alternatives are dominated so no elimination can be made.

Step 2. The DM is asked to compare a^2 and a^6 and prefers a^2 to a^6. This information eliminates a^6 but does not narrow the possible weight interval any further.

Step 3. Cones C($wa^2;wa^6$) are generated and the remaining alternatives (a^1,a^5) are checked for cone dominance using the cone-based model. a^1 is dominated by C($wa^2;wa^6$) for all w, hence it is eliminated.

Step 2. The DM is asked to compare a^2 and a^5 and prefers a^2 to a^5. STOP. The algorithm returns a^2.

6.4 Discussion of Convex Cone-based Method

As discussed in Chapter 5, assuming a symmetric quasi-concave social welfare function partially handles the issue of preferential independence by relaxing the additivity assumption of the UTA-based approach. To elaborate, recall the previous example consisting of alternatives $a^2 = \left(\begin{smallmatrix} 5 & 5 \\ 6 & 2 \end{smallmatrix}\right)$ and $a^5 = \left(\begin{smallmatrix} 3 & 5 \\ 8 & 2 \end{smallmatrix}\right)$, in which the UTA-based approach always gives a^2 more social value due to the assumption that allocating one output in a more equitable manner is always desired regardless of the user's position with respect to the other outputs.

However, there may be symmetric quasi-concave function forms representing different comparisons. Consider the following types of social welfare functions, which are symmetric quasi-concave (note that they are not necessarily additive in the way assumed in UTA): additive, multiplicative, Rawlsian (maximizing the minimum), and ordered weighted averaging (a rank-based function which gives more weights to worse-off entities and returns a weighted sum). For these two alternatives, the utility vectors become $(0.5, 2w/3)$ and $((0.5-2w/6), w)$, where w is the weight of the first output. Note that we used scalarized matrices when calculating these utility vectors. An additive function, which assumes that the social welfare is the sum of user values, would consider the two options as equally good; however, the results for the other functions would change depending on the weight parameter, allowing more flexibility. For example, when the underlying social welfare function is taken as multiplication of user values, the DM would prefer a^5 over a^2 when $w = 0.2$. The convex cone based approach can take such preferences into account.

Chapter 7

Computational Experiments

In this chapter, we provide the results of computational experiments performed to check the computational efficiency and the quality of the results for proposed algorithms. We first discuss the main results of UTA-based and convex cone-based approaches and then provide a comparison between these two methods.

We generate problem instances with two outputs ($n = 2$). We use two values for the number of alternatives ($N = 25$ and $N = 50$) and two values for the number of users ($m = 3$ and $m = 5$). We create 10 problem instances for each parameter setting. The output levels are randomly generated in the range [10-100]. The algorithms stop when the number of remaining alternatives is less than or equal to a pre-specified threshold value. In our experiments, we set this threshold value K as $K = 0.05N$ ($K = 1$ for $N = 25$ and $K = 2$ for $N = 50$).

The algorithms are coded in MATLAB and solved by a dual core (Intel Core i5 2.40GHz) computer with 8 GB RAM. All models are solved by CPLEX 12.6 and the solution times are expressed in central processing unit (CPU) seconds.

7.1 Results of the UTA-based Algorithm

In this section, we discuss the results of the UTA-based algorithm for different question selection strategies. We then discuss the effects of parameters used in the UTA-based model on the number of questions asked and solution time. We report the average and maximum values for the number of questions asked and the solution time (in seconds). We also report the accuracy of the results, which is calculated as the percentage of instances in which the actual best alternative is in the set of alternatives returned by the algorithm.

We assume that the underlying marginal value functions are the square root function of the levels of the outputs, i.e. $MV(a_{ij}^k) = \sqrt{a_{ij}^k}$ and simulate the responses of the DM accordingly for all computational experiments provided in this section.

7.1.1 Question Selection Strategy

In this part, we provide a comparison of three different question selection strategies used in UTA-based approach: Random, Ideal and Minimum pairwise distance explained in Section 5.2. Table 7.1 summarizes the results of our experiments for these three question selection strategies. In these experiments, we set the parameter values as follows: Ω=0.005, $\gamma = 10^{-5}$, and ε=0.002.

Table 7.1: Results for UTA-based algorithm

			# of questions		sol. time		
N	m	Method	avg.	max	avg.	max	accuracy
25	3	random	56.8	97	17.34	34.22	90%
		ideal	16.77	42	4.88	14.08	90%
		min dist	28.9	59	7.18	19.66	90%
	5	random	39.12	83	12.82	34.99	100%
		ideal	19.8	45	10	23.3	100%
		min dist	20.3	52	16.8	37.7	100%
50	3	random	28.3	83	9.64	26.2	100%
		ideal	9.4	31	9.67	21.76	100%
		min dist	7.6	29	8.46	19.3	100%
	5	random	60.44	102	40.36	96.74	80%
		ideal	17.9	47	20.34	58.46	80%
		min dist	19.4	51	22.83	60.32	80%

Table 7.1 reveals the importance of the question selection strategy both on the number of questions asked to the DM and on the solution times. It is seen that, selecting the vectors to be asked to the DM based on their distance to an ideal vector (generally) outperforms other two strategies in both performance measures. This indicates that because the information obtained from the comparison of bundles with higher output values is more effective in finding the best alternative.

7.1.2 Parameter Selection

Recall that we use three parameters in the UTA-based model: ε, γ, and Ω. ε is used to ensure having increasing MVFs, γ is used to ensure having concave MVFs, and Ω is used to incorporate strict preference information of the DM to the model.

In order to observe the effects of the parameters, we performed computational experiments using different combinations of the parameter values. For each parameter, we use two levels as shown in Table 7.2. In these experiments, we use *Ideal* vector selection strategy.

Table 7.2: Different values used for each parameter

ε	0.002	0.004
γ	10^{-5}	2.10^{-5}
Ω	0.005	0.01

Table 7.3 shows the results of our experiments. As seen in Table 7.3, when a parameter set that implies a smaller set of possible MVFs (i.e. when $\varepsilon, \gamma, \Omega$ are set to higher levels), the number of questions asked decreases with a few exceptions. On the other hand, using large parameter values may deteriorate the accuracy of the results. Note that this inaccuracy is due to the possible inconsistency between the chosen parameters dictating the functional form (γ, ε) and the underlying social welfare function used for simulating the responses.

Another inconsistency may result from using large values for Ω. When the DM prefers one bundle over another, we assume that the difference between these two bundles is greater than or equal to Ω. However, this may not be consistent with the underlying value functions of the DM.

Table 7.3: Results for the UTA-based algorithm for different parameter values

					# of questions		sol. time		
N	m	Ω	ε	γ	avg.	max	avg.	max	accuracy
25	3	0.005	0.002	1.10^{-5}	16.77	42	4.88	14.08	90%
				2.10^{-5}	15.77	42	4.85	14.08	90%
			0.004	1.10^{-5}	12	40	3.79	14.03	90%
				2.10^{-5}	4.66	14	2.03	13.78	90%
		0.01	0.002	1.10^{-5}	16.12	41	4.26	12.3	90%
				2.10^{-5}	14.66	34	4.2	11.87	90%
			0.004	1.10^{-5}	13.11	40	3.75	12.2	90%
				2.10^{-5}	7.2	38	2.79	10.8	90%
	5	0.005	0.002	1.10^{-5}	19.8	45	10	23.3	100%
				2.10^{-5}	17.4	45	9.1	23.3	100%
			0.004	1.10^{-5}	10.6	31	2.5	10.3	100%
				2.10^{-5}	6.77	18	1.35	3.85	100%
		0.01	0.002	1.10^{-5}	8.77	19	4.92	15.98	100%
				2.10^{-5}	7.44	19	4.5	16.46	100%
			0.004	1.10^{-5}	7.22	19	2.02	8.49	100%
				2.10^{-5}	6.11	15	1.37	6.09	100%
50	3	0.005	0.002	1.10^{-5}	9.4	31	9.67	21.76	100%
				2.10^{-5}	8.6	26	8.26	18.35	100%
			0.004	1.10^{-5}	8.5	30	8.11	20.6	100%
				2.10^{-5}	8.1	24	7.26	16.33	100%
		0.01	0.002	1.10^{-5}	5.6	9	6.31	15.72	70%
				2.10^{-5}	4.3	8	5.31	14.03	70%
			0.004	1.10^{-5}	5.1	8.7	5.94	15.01	70%
				2.10^{-5}	4.1	8	5.11	13.52	70%
	5	0.005	0.002	1.10^{-5}	17.9	47	20.34	58.46	80%
				2.00E-05	14.8	45	14.6	35.44	70%
			0.004	1.00E-05	3.2	14	0.87	3.79	70%
				2.10^{-5}	0.9	7	0.23	1.78	70%
		0.01	0.002	1.10^{-5}	8.5	30	9.76	30.89	70%
				2.10^{-5}	7.4	21	7.3	21	70%
			0.004	1.10^{-5}	1.2	6	0.36	1.63	70%
				2.10^{-5}	0.8	6	0.2	1.43	70%

7.2 Results of the Convex Cone-based Algorithm

In this section we provide the computational experiments for the convex cone-based method. For this method, we assume 5 different underlying symmetric quasi-concave social welfare function forms to simulate the responses of the DM as follows:

1. The sum of user values (UVs): $SW_s(a^k) = \sum_{i=1}^{m} UV(b_i^k)$
2. The multiplication of UVs: $SW_p(a^k) = \prod_{i=1}^{m} UV(b_i^k)$
3. The minimum of UVs (Rawlsian): $SW_m(a^k) = \min_{\forall i \in I} UV(b_i^k)$
4. Sum of pairwise minima of UVs: $SW_{sp}(a^k) = \sum_{i,i' \in I: i \neq i'} min(UV(b_i^k), UV(b_{i'}^k))$.
5. Ordered Weighted Average (OWA) of UVs: $SW_o(a^k) = \sum_{i=1}^{m} w_i^{owa} \overrightarrow{U}_i^k$, where U^k is the vector of user values of a^k and $w^{owa} \in \mathbb{R}^m$ is a nonincreasing weight vector.

For the OWA function we take $w^{owa} = (0.5,\ 0.3,\ 0.2)$ for $m = 3$ and $w^{owa} = (0.4,\ 0.3,\ 0.2,\ 0.06,\ 0.04)$ when $m = 5$. In all these settings, a^k corresponds to the scaled matrix obtained by scalarizing the elements of the original matrix. Recall that in this approach, UVs are calculated as $UV(b_i^k)$=$(w_1 a_{i1}^k) + (1 - w_1) a_{i2}^k$. We assume 3 different underlying weights for the outputs: $w^1 = (0.15,\ 0.85)$, $w^2 = (0.5,\ 0.5)$, and $w^3 = (0.85,\ 0.15)$.

Tables 7.4 and 7.5 summarize the results of our experiments for the convex cone based algorithm. We report the average and maximum values for the number of questions asked and the solution time (in seconds), for each parameter setting. We also report the average and maximum reduction (as percentage) in the weight interval for the first output. Recall that in the convex cone based approach, we narrow down the possible weight interval of the user value function from initial interval [0-1] based on the preference information. In these experiments we used the question selection strategy that chooses two closest alternatives to an ideal alternative, due to its superior performance in the experiments on the UTA-based algorithm.

Table 7.4: Convex cone results for additive, Rawlsian, and multiplicative social welfare functions

SWF form	N	m	w_1	# of quest avg.	# of quest max	sol. time avg.	sol. time max	weight red. (%) avg.	weight red. (%) max
$SW_s(a^k) = \sum_{i=1}^m UV(b_i^k)$	25	3	0.15	8.3	13	40.7	69.8	61	85
			0.5	5.8	9	29.5	57.1	51	90
			0.85	5.4	9	24.6	54.9	52	80
		5	0.15	9.9	13	44.2	95.0	73	95
			0.5	11.1	15	83.8	127.2	56	90
			0.85	10.2	13	61.9	126.7	68	90
	50	3	0.15	6	9	43.38	93.82	64	85
			0.5	5.7	9	43.7	91.89	58	90
			0.85	6.8	10	49.05	94.4	60	95
		5	0.15	15.4	24	304.3	634.1	74	95
			0.5	15.3	20	376.6	697.0	80	100
			0.85	14.9	18	307.0	676	65	95
$SW_m(a^k) = \min_{\forall i \in I} UV(b_i^k)$	25	3	0.15	8	11	38.4	68.7	64	85
			0.5	6.3	9	31.2	52	58	95
			0.85	6	10	29.5	56.3	46	65
		5	0.15	11	15	64.9	122.4	70	90
			0.5	10.9	14	83.9	144.1	69	95
			0.85	11	14	68.0	111.3	61	100
	50	3	0.15	6.2	9	41.09	73.19	63	95
			0.5	6.9	11	59.5	102.4	66	95
			0.85	7.8	11	65.1	100.9	61	95
		5	0.15	16.3	20	275.9	637.7	77	100
			0.5	15.4	19	339.7	634.6	77	95
			0.85	14.9	17	237.7	483.6	75	90
$SW_p(a^k) = \prod_{i=1}^m UV(b_i^k)$	25	3	0.15	8.3	12	38.8	67.48	58	75
			0.5	5.9	9	32.6	59.8	50	90
			0.85	5.9	10	29.3	68.1	49	70
		5	0.15	11.2	15	77.5	123.3	62	80
			0.5	11.2	14	93.5	139.1	62	90
			0.85	10.3	13	70.9	127.5	65	85
	50	3	0.15	6.8	10	57.3	110.8	59	80
			0.5	6.3	9	58.5	112.6	56	85
			0.85	6.1	9	43.5	79.8	64	85
		5	0.15	17.2	24	377.1	666.7	63	75
			0.5	16.4	20	394.0	681.6	71	90
			0.85	15.6	18	337.6	667.4	64	80

Table 7.5: Convex cone results for sum of pairwise minima and ordered weighted averaging (OWA) social welfare functions

SWF form	N	m	w_1	# of quest avg.	# of quest max	sol. time avg.	sol. time max	weight red.(%) avg.	weight red.(%) max
$SW_{sp}(a^k) = \sum_{\substack{i,i' \in I \\ s.t. \\ i \neq i'}} min(UV(b_i^k), UV(b_{i'}^k))$	25	3	0.15	8.2	11	42.0	69.7	58	75
			0.5	6.5	9	38.7	60.6	51	95
			0.85	6	10	39.2	84.7	46	65
		5	0.15	10.9	13	92.0	146.8	66	80
			0.5	11.3	14	87.8	124.5	53	95
			0.85	10.6	13	62.8	111.4	65	90
	50	3	0.15	6.3	10	50.4	78.4	61	80
			0.5	7.4	11	75.2	112.7	55	80
			0.85	7.8	11	67.0	105.9	55	80
		5	0.15	17.6	24	532.3	843.6	61	75
			0.5	16.3	20	805.6	1329.8	64	85
			0.85	15.7	18	541.9	1176.1	63	80
$SW_o(a^k) = \sum_{i=1}^{m} w_i^{owa} \overrightarrow{U}_i^k$	25	3	0.15	8.3	12	39.0	69.0	58	75
			0.5	6.7	9	33.6	56.9	47	80
			0.85	5.9	10	34.6	73.8	46	65
		5	0.15	10.9	15	78.3	113.9	66	80
			0.5	11.3	14	93.9	129.9	53	95
			0.85	10.6	13	59.9	103.6	66	90
	50	3	0.15	6.2	9	43.8	80.2	61	80
			0.5	7.3	11	70.6	105812.0	57	80
			0.85	6.8	10	60.8	111.6	59	80
		5	0.15	17.6	24	385.4	638.7	61	75
			0.5	16.4	20	424.9	731.2	65	90
			0.85	15.7	18	329.5	632.4	63	80

7.3 Discussion on the UTA-based and the Convex Cone-based Methods

The previous sections discuss the results of the UTA-based and convex cone-based algorithms. One could attempt to compare these approaches with respect to the number of questions asked to the DM and the solution times. However, note that the type of questions asked in these two algorithms differ in nature as one uses vector comparison questions and the other uses matrix type comparison questions. This makes a direct comparison impossible.

We observe that the accuracy and the general performance of the UTA-based algorithm depend on how the parameters are chosen. A parameter set reducing the size of the set of possible marginal value functions would make more eliminations at each iteration; hence the algorithm would terminate with relatively less number of questions, sometimes at the expense of accuracy. On the other hand, convex cone method is more stable and also more accurate (robust) in the sense that the true best alternative is always returned. The convex cones based approach considers a larger set of social welfare function forms hence more questions are asked to reach a conclusion.

We also observe that both approaches are satisfactory in terms of solution time as linear programming models of small-sizes are solved. It is also observed that when the number of users increases to 5, the number of questions asked in both algorithms increases with a few exceptions. A similar case occurs when the number of alternatives is increased from 25 to 50 especially for cases where m=5.

Chapter 8

Conclusion

In this study we consider multicriteria evaluation problems in which a decision maker has to choose the best alternative (or a small subset of most preferred alternatives) among a given set of alternatives. Each alternative represents an allocation of multiple types of outputs to multiple users and is associated with a matrix, whose columns and rows correspond to outputs and users, respectively.

This problem is an extension of the classical multiple criteria choice problem, in which alternatives are vectors. Moreover, since there are multiple users, equity in the distribution of the outputs across the users is important as well as efficiency. In that sense, the problem is an extension of the allocation problems that focus on the distribution of a single output.

We design two interactive algorithms that will guide the decision maker to her most preferred alternative. The first algorithm is motivated by the well-known UTA method. It assumes additivity in the social welfare function. The second algorithm is motivated by convex cone method and it (partially) relaxes this additivity assumption by defining the social welfare as a symmetric quasi-concave function of the user values.

The fairness concerns imply special axioms for the underlying preference model of the decision maker such as impartiality, which means that the identities of the users are not important and do not affect the decision, making the problem and the solution approaches different than their counterparts in the classical multicriteria decision making literature.

In the UTA-based algorithm we check whether an alternative can be better than another for all alternative pairs given the preference information via linear programming models. We then reduce the set of alternatives that are candidates to be the most preferred alternative. The second approach is based on an extension of the well-known convex cone approach. Due to the fairness concerns, the method uses generalized Lorenz dominance instead of vector dominance while checking cone dominance. This method checks dominance over the whole set of possible parameters (weights) to make robust conclusions. Both approaches obtain the preference information of the policy maker iteratively and incorporate it into mathematical programming models to infer the best alternative.

We demonstrate the computational feasibility of our approaches by conducting experiments on randomly generated problem instances. It is important to note that the models are not directly comparable since they ask different comparison questions. The computational experiments demonstrate that the parameter selection is an important part for the UTA-based method. As expected, the parameter values have direct effects on the number of questions asked to the policy maker or on the quality of the results. Thus one needs to be careful while selecting the parameter values. Both algorithms show satisfactory performance in terms of solution time, however they are not directly comparable since they use different comparison questions.

As the problem is relevant in many real life decision making settings, more research in this topic awaits further attention. Future research could be performed in a few directions: In the convex cone based approach, we considered problems where the number of outputs and the number of users are not too large. Increasing the number of outputs would significantly affect the solution times due to the discretization process. Further research could be performed for developing algorithms for larger problem instances. One can also consider the multi-criteria design version of this problem, in which the alternatives are implicitly defined by constraints rather than given explicitly.

Acknowledgement

This study was supported by TUBITAK (The Scientific and Technological Research Council of Turkey) under Grant no:215M713.

Bibliography

[1] Y. Siskos and A. Spyridakos, "Intelligent multicriteria decision support: Overview and perspectives," *European Journal of Operational Research*, vol. 113, no. 2, pp. 236–246, 1999.

[2] T. J. Stewart, "A critical survey on the status of multiple criteria decision making theory and practice," *Omega*, vol. 20, no. 5-6, pp. 569–586, 1992.

[3] D. Diakoulaki, C. H. Antunes, and A. G. Martins, "MCDA and energy planning," in *Multiple criteria decision analysis: state of the art surveys*, pp. 859–890, Springer, 2005.

[4] J. Spronk, R. E. Steuer, and C. Zopounidis, "Multicriteria decision aid/analysis in finance," in *Multiple Criteria Decision Analysis*, pp. 1011–1065, Springer, 2016.

[5] H.-K. Chiou, G.-H. Tzeng, and D.-C. Cheng, "Evaluating sustainable fishing development strategies using fuzzy MCDM approach," *Omega*, vol. 33, no. 3, pp. 223–234, 2005.

[6] Ö. Karsu, "Using holistic multicriteria assessments: The convex cones approach," *Wiley Encyclopedia of Operations Research and Management Science*, 2013.

[7] A. J. Culyer and J. P. Newhouse, *Handbook of health economics*, vol. 1. Elsevier, 2000.

[8] M. O. Lorenz, "Methods of measuring the concentration of wealth," *Publications of the American statistical association*, vol. 9, no. 70, pp. 209–219, 1905.

[9] A. F. Shorrocks, "Ranking income distributions," *Economica*, vol. 50, no. 197, pp. 3–17, 1983.

[10] A. Trannoy, “Multidimensional egalitarianism and the dominance approach: a lost paradise,” *Inequality and economic integration. Routledge, London*, pp. 284–302, 2006.

[11] S.-C. Kolm, “Multidimensional egalitarianisms,” *The Quarterly Journal of Economics*, pp. 1–13, 1977.

[12] M. Köksalan and J. Wallenius, “Multiple criteria decision making: Foundations and some approaches,” in *New Directions in Informatics, Optimization, Logistics, and Production*, pp. 171–183, INFORMS, 2012.

[13] B. Roy, “Problems and methods with multiple objective functions,” *Mathematical programming*, vol. 1, no. 1, pp. 239–266, 1971.

[14] Ö. Karsu and A. Morton, “Inequity averse optimization in operational research,” *European Journal of Operational Research*, vol. 245, no. 2, pp. 343–359, 2015.

[15] M. M. Kostreva and W. Ogryczak, “Linear optimization with multiple equitable criteria,” *RAIRO-Operations Research*, vol. 33, no. 3, pp. 275–297, 1999.

[16] M. M. Kostreva, W. Ogryczak, and A. Wierzbicki, “Equitable aggregations and multiple criteria analysis,” *European Journal of Operational Research*, vol. 158, no. 2, pp. 362–377, 2004.

[17] Ö. Karsu, “Approaches for inequity-averse sorting,” *Computers & Operations Research*, vol. 66, pp. 67–80, 2016.

[18] E. Jacquet-Lagrèze and J. Siskos, *Une Méthode de cons truction d’une fonction d’utilité additive, explicative d’une préférence globale*. Université Paris IX-Dauphine, Laboratoire d’analyse et modélisation de systèmes pour l’aide à la décision, 1978.

[19] E. Jacquet-Lagrèze and J. Siskos, “Assessing a set of additive utility functions for multicriteria decision-making, the uta method,” *European journal of operational research*, vol. 10, no. 2, pp. 151–164, 1982.

[20] Y. Siskos and D. Yannacopoulos, “Utastar: An ordinal regression method for building additive value functions,” *Investigaçao Operacional*, vol. 5, no. 1, pp. 39–53, 1985.

[21] J. Devaud, G. Groussaud, and E. Jacquet-Lagreze, “Utadis: Une méthode de construction de fonctions d’utilité additives rendant compte de jugements globaux,” *European Working Group on Multicriteria Decision Aid, Bochum*, 1980.

[22] C. Zopounidis and M. Doumpos, “A multicriteria decision aid methodology for the assessment of country risk,” *European Research on Management and Business Economics*, vol. 3, no. 3, pp. 13–33, 1997.

[23] C. Zopounidis and M. Doumpos, “A preference disaggregation decision support system for financial classification problems,” *European Journal of Operational Research*, vol. 130, no. 2, pp. 402–413, 2001.

[24] S. Greco, R. Słowiński, J. R. Figueira, and V. Mousseau, “Robust ordinal regression,” in *Trends in multiple criteria decision analysis*, pp. 241–283, Springer, 2010.

[25] P. Korhonen, J. Wallenius, and S. Zionts, “Solving the discrete multiple criteria problem using convex cones,” *Management Science*, vol. 30, no. 11, pp. 1336–1345, 1984.

[26] M. M. Köksalan and O. V. Taner, “An approach for finding the most preferred alternative in the presence of multiple criteria,” *European Journal of Operational Research*, vol. 60, no. 1, pp. 52–60, 1992.

[27] M. M. Köksalan, M. H. Karwan, and S. Zionts, “An improved method for solving multiple criteria problems involving discrete alternatives,” *IEEE Transactions on Systems, Man, and Cybernetics*, no. 1, pp. 24–34, 1984.

[28] O. V. Taner and M. M. Köksalan, “Experiments and an improved method for solving the discrete alternative multiple-criteria problem,” *Journal of the Operational Research Society*, vol. 42, no. 5, pp. 383–391, 1991.

[29] S. Y. Prasad, M. H. Karwan, and S. Zionts, “Use of convex cones in interactive multiple objective decision making,” *Management Science*, vol. 43, no. 5, pp. 723–734, 1997.

[30] C. Ulu and M. Köksalan, "An interactive procedure for selecting acceptable alternatives in the presence of multiple criteria," *Naval Research Logistics (NRL)*, vol. 48, no. 7, pp. 592–606, 2001.

[31] J. W. Fowler, E. S. Gel, M. M. Köksalan, P. Korhonen, J. L. Marquis, and J. Wallenius, "Interactive evolutionary multi-objective optimization for quasi-concave preference functions," *European Journal of Operational Research*, vol. 206, no. 2, pp. 417–425, 2010.

[32] A. Dehnokhalaji, P. J. Korhonen, M. Köksalan, N. Nasrabadi, and J. Wallenius, "Convex cone-based partial order for multiple criteria alternatives," *Decision Support Systems*, vol. 51, no. 2, pp. 256–261, 2011.

[33] Ö. Karsu, A. Morton, and N. Argyris, "Incorporating preference information in multicriteria problems with equity concerns," in *Technical Report*, London School of Economics and Political Science, 2012.

[34] J. S. Dyer and R. K. Sarin, "Group preference aggregation rules based on strength of preference," *Management Science*, vol. 25, no. 9, pp. 822–832, 1979.

[35] R. Vetschera, P. Sarabando, and L. Dias, "Levels of incomplete information in group decision models–a comprehensive simulation study," *Computers & Operations Research*, vol. 51, pp. 160–171, 2014.

[36] M. Fleming, "A cardinal concept of welfare," *The Quarterly Journal of Economics*, vol. 66, no. 3, pp. 366–384, 1952.

[37] J. C. Harsanyi, "Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility," *Journal of political economy*, vol. 63, no. 4, pp. 309–321, 1955.

[38] R. L. Keeney and C. W. Kirkwood, "Group decision making using cardinal social welfare functions," *Management Science*, vol. 22, no. 4, pp. 430–437, 1975.

[39] R. L. Keeney, "A group preference axiomatization with cardinal utility," *Management Science*, vol. 23, no. 2, pp. 140–145, 1976.

[40] J. E. Smith and D. Von Winterfeldt, "Anniversary article: decision analysis in management science," *Management science*, vol. 50, no. 5, pp. 561–574, 2004.

[41] J. Eliashberg and R. L. Winkler, "Risk sharing and group decision making," *Management Science*, vol. 27, no. 11, pp. 1221–1235, 1981.

[42] R. L. Keeney, "Utility functions for equity and public risk," *Management Science*, vol. 26, no. 4, pp. 345–353, 1980.

[43] C. M. Harvey, "Decision analysis models for social attitudes toward inequity," *Management science*, vol. 31, no. 10, pp. 1199–1212, 1985.

[44] S. Greco, J. Figueira, and M. Ehrgott, "Multiple criteria decision analysis," *Springer's International series*, 2005.

[45] Y. Siskos, E. Grigoroudis, and N. F. Matsatsinis, "UTA methods," in *Multiple criteria decision analysis: State of the art surveys*, pp. 297–334, Springer, 2005.

[46] M. Doumpos and C. Zopounidis, *Multicriteria decision aid classification methods*, vol. 73. Springer Science & Business Media, 2002.

[47] C. W. Kirkwood and R. K. Sarin, "Ranking with partial information: A method and an application," *Operations Research*, vol. 33, no. 1, pp. 38–48, 1985.

[48] V. Belton and T. Stewart, *Multiple criteria decision analysis: an integrated approach*. Springer Science & Business Media, 2002.

[49] C. A. B. e Costa and J.-C. Vansnick, "Macbeth—an interactive path towards the construction of cardinal value functions," *International transactions in operational Research*, vol. 1, no. 4, pp. 489–500, 1994.

[50] M. Rothschild and J. E. Stiglitz, "Some further results on the measurement of inequality," *Journal of Economic Theory*, vol. 6, no. 2, pp. 188–204, 1973.

[51] P. Dasgupta, A. Sen, and D. Starrett, "Notes on the measurement of inequality," *Journal of economic theory*, vol. 6, no. 2, pp. 180–187, 1973.

[52] J. E. Foster and A. Sen, "On economic inequality," 1997.

[53] G. B. Hazen, "Preference convex unanimity in multiple criteria decision making," *Mathematics of operations research*, vol. 8, no. 4, pp. 505–516, 1983.

[54] B. Malakooti, “Theories and an exact interactive paired-comparison approach for discrete multiple-criteria problems,” *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 19, no. 2, pp. 365–378, 1989.

[55] Ö. Karsu, *Inequity-averse decisions in operational research*. PhD thesis, The London School of Economics and Political Science (LSE), 2014.

[56] W. Ogryczak and T. Śliwiński, “On solving linear programs with the ordered weighted averaging objective,” *European Journal of Operational Research*, vol. 148, no. 1, pp. 80–91, 2003.