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On the size of two families of unlabeled bipartite graphs

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Abstract

Let $B_u(n, r)$ denote the set of unlabeled bipartite graphs whose edges connect a set of n vertices with a set of r vertices. In this paper, we provide exact formulas for $|B_u(2, r)|$ and $|B_u(3, r)|$ using Polya's Counting Theorem. Extending these results to $n \geq 4$ involves solving a set of complex recurrences and remains open. In particular, the number of recurrences that must be solved to compute $|B_u(n, r)|$ is given by the number of partitions of n that is known to increase exponentially with n by Ramanujan–Hardy–Rademacher's asymptotic formula.

Keywords: Bipartite graph isomorphism; Closed-form formula; Polya's Counting Theorem; Unlabeled bipartite graph

1. Introduction

This paper focuses on the number of unlabeled bipartite graphs. While a few results have been reported on counting series of unlabeled bipartite graphs [1–4], no closed-form expression is known for the exact number of such graphs in the literature. It was established in [5] that this problem is equivalent to the enumeration of binary matrices that remain distinct under row and column permutations. The problem is formally stated as follows. Let (I, O, E) denote a graph with two disjoint sets of vertices, I and a set of vertices, O, where each edge in E connects a vertex in I with a vertex in O. We let $n = |I|$, $r = |O|$, and refer to such a graph as an (n, r)-bipartite graph. Let $G_1 = (I, O, E_1)$ and $G_2 = (I, O, E_2)$ be two (n, r)-bipartite graphs, and $\alpha : I \to I$ and $\beta : O \to O$ be both bijections. The bijection pair (α, β) is an isomorphism between G_1 and G_2 provided that $(\alpha(v_1), \beta(v_2)) \in E_2$ if and only if $(v_1, v_2) \in E_1$, $\forall v_1 \in I, \forall v_2 \in O$. The set of 2^{nr} (n, r)-bipartite graphs is partitioned into equivalence classes under such bijection pairs. Let $B_u(n, r)$ denote any set of (n, r)-bipartite graphs, formed by including exactly one such graph from each of the equivalence classes. Determining $|B_u(n, r)|$ amounts to an enumeration of non-isomorphic (n, r)-bipartite graphs that will henceforth be referred to as unlabeled (n, r)-bipartite graphs. In [5], Harrison used Pólya's counting theorem to obtain an expression for the number of distinct $n \times r$ binary matrices. He further established that this expression also

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enumerates the number of unlabeled (n, r)-bipartite graphs. However, Harrison's expression involves a nested sum whose argument includes factorial, exponentiation and greatest common divisor operations, and it cannot be simplified into a closed-form expression even when n is fixed to small numbers such as 2 and 3. Clearly, $|B_u(1, r)| = r + 1$. Deriving closed-form formulas for $n = 2$ and $n = 3$ is the focus of the present work.

2. A closed-form formula for $|B_u(2, r)|$

We use Polya's counting theorem (See [6]), in particular Harrison's cycle index formulation in [5] to compute $|B_u(2, r)|$. Let S_n denote the symmetric group of permutations of degree n acting on set $N = \{1, 2, \ldots, n\}$. Suppose that the $n!$ permutations in S_n are indexed by $1, 2, \ldots, n!$ in some arbitrary, but fixed manner. The cycle index polynomial of S_n is defined as follows ([7],see p.35, Eqn. 2.2.1):

$$Z_{S_n}(x_1, x_2, \ldots, x_n) = \frac{1}{n!}\sum_{m=1}^{n!}\prod_{k=1}^{n} x_k^{p_{m,k}} \tag{1}$$

where $p_{m,k}$ denotes the number of cycles of length k in the disjoint cycle representation of the m^{th} permutation in S_n, and $\sum_{k=1}^{n} k p_{m,k} = n, \forall m = 1, 2, \ldots, n!$.

Let $S_n \times S_r$ denote the direct product of symmetric groups S_n and S_r acting on $N = \{1, 2, \ldots, n\}$ and $R = \{1, 2, \ldots, r\}$, respectively, where n and r are positive integers such that $n < r$. It can be inferred from Harrison ([8], Lemma 4.1 and Theorem 4.2) that the cycle index polynomial of $S_n \times S_r$ is given by

$$Z_{S_n\times S_r}(x_1, x_2, \ldots, x_{nr}) = Z_{S_n}(x_1, x_2, \ldots, x_n) \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r), \tag{2}$$

where $\boxtimes$ is a particular polynomial multiplication that distributes over ordinary addition, and in which the multiplication $X_m \bigodot X_t$ of two product terms, $X_m = x_1^{p_{m,1}} x_2^{p_{m,2}} \cdots x_n^{p_{m,n}}$ and $X_t = x_1^{q_{t,1}} x_2^{q_{t,2}} \cdots x_r^{q_{t,r}}$ in Z_{S_n} and Z_{S_r}, respectively, is defined as[1]

$$X_m \bigodot X_t = \prod_{k=1}^{n}\prod_{j=1}^{r} x_{\operatorname{lcm}(k,j)}^{p_{m,k} q_{t,j} \gcd(k,j)}. \tag{3}$$

Harrison further proved that [5]

$$|B_u(n, r)| = Z_{S_n\times S_r}(\underbrace{2, 2, .., 2}_{nr}) \tag{4}$$

when[2] $n \neq r$.

We need one more fact that can be found in Harary ([7], p. 36) in order to compute $|B_u(2, r)|$:

$$Z_{S_r}(x_1, x_2, \ldots\ldots, x_r) = \frac{1}{r}\sum_{i=1}^{r} x_i Z_{S_{r-i}}(x_1, x_2, \ldots\ldots, x_{r-i}) \tag{5}$$

where $Z_{S_0}() = 1$.

We now calculate $|B_u(2, r)|$ as follows.[3]

$$|B_u\,(2, r)| = Z_{S_2\times S_r}(2, 2, \ldots, 2), \tag{6}$$

$$= \left[Z_{S_2}(x_1, x_2) \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)\right](2, 2, \ldots, 2), \tag{7}$$

$$= \left[\left(\frac{1}{2}\left(x_1^2 + x_2\right)\right) \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)\right](2, 2, \ldots, 2), \tag{8}$$

$$= \frac{1}{2}\left[x_1^2 \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r) + x_2 \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)\right](2, 2, \ldots, 2), \tag{9}$$

[1] The $\operatorname{lcm}(a, b)$ and $\gcd(a, b)$ denote least common multiple and greatest common divisor of a and b.

[2] As noted in [5], $n = r$ case involves a different cycle index polynomial and will be omitted here as well.

[3] Note that the zero powers of x_1, x_2, are not shown in the cycle index polynomial Z_{S_2}. We will use the same convention for all other cycle index polynomials throughout the paper.

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$$= \frac{1}{2}\left\{\left[x_1^2 \boxtimes \frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,\ldots,2) + \left[x_2 \boxtimes \frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,\ldots,2),\right\}, \tag{10}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!} x_1^2 \bigodot \prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,\ldots,2) + \left[\frac{1}{r!}\sum_{t=1}^{r!} x_2 \bigodot \prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,\ldots,2)\right\}, \tag{11}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_{\mathrm{lcm}(1,j)}^{2q_{t,j}\gcd(1,j)}\right](2,2,\ldots,2) + \left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_{\mathrm{lcm}(2,j)}^{q_{t,j}\gcd(2,j)}\right](2,2,\ldots,2).\right\}, \tag{12}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{2q_{t,j}}\right](2,2,\ldots,2) + \left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_{\mathrm{lcm}(2,j)}^{q_{t,j}\gcd(2,j)}\right](2,2,\ldots,2)\right\}, \tag{13}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} 2^{2q_{t,j}}\right] + \left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} 2^{q_{t,j}\gcd(2,j)}\right]\right\}, \tag{14}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} (2^2)^{q_{t,j}}\right] + \left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{\text{odd } j} 2^{q_{t,j}} \prod_{\text{even } j} (2^2)^{q_{t,j}}\right]\right\}, \tag{15}$$

$$= \frac{1}{2}\left\{\left[Z_{S_r}(2^2,2^2,\ldots,2^2)\right] + \left[Z_{S_r}(2,2^2,2,2^2,\ldots)\right]\right\}. \tag{16}$$

Thus, we have reduced the computation of $|B_u(2,r)|$ to computing the two terms in (16). These computations are carried out in the next two lemmas.

Lemma 1. $Z_{S_r}(2^2,2^2,\ldots,2^2) = \binom{r+3}{r}$.

Proof. Using (5), we have

$$rZ_{S_r}(2^2,2^2,\ldots,2^2) = \sum_{i=1}^{r} 2^2 Z_{S_{r-i}}(2^2,2^2,\ldots,2^2), \tag{17}$$

$$(r-1)Z_{S_{r-1}}(2^2,2^2,\ldots,2^2) = \sum_{i=1}^{r-1} 2^2 Z_{S_{r-1-i}}(2^2,2^2,\ldots,2^2). \tag{18}$$

Subtracting the second equation from the first one and simplifying it gives

$$rZ_{S_r}(2^2,2^2,\ldots,2^2) - (r-1)Z_{S_{r-1}}(2^2,2^2,\ldots,2^2) = 4Z_{S_{r-1}}(2^2,2^2,\ldots,2^2), \tag{19}$$

$$Z_{S_r}(2^2,2^2,\ldots,2^2) = (\frac{r+3}{r})Z_{S_{r-1}}(2^2,2^2,\ldots,2^2). \tag{20}$$

Expanding the last equation recursively, we obtain

$$Z_{S_r}(2^2,2^2,\ldots,2^2) = (\frac{r+3}{r})(\frac{r+2}{r-1})Z_{S_{r-2}}(2^2,2^2,\ldots,2^2), \tag{21}$$

$$= (\frac{r+3}{r})(\frac{r+2}{r-1})(\frac{r+1}{r-2})\ldots(\frac{4}{1})Z_{S_0}(). \tag{22}$$

Noting that $Z_{S_0}() = 1$ proves the statement, i.e.,

$$Z_{S_r}(2^2,2^2,\ldots,2^2) = \binom{r+3}{r}. \quad \square$$

Lemma 2.

$$Z_{S_r}(2,2^2,2,2^2,\ldots) = \frac{2r^2+8r+7+(-1)^r}{8}. \tag{23}$$

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Proof. By (5),

$$rZ_{S_r}(2,2^2,\ldots)=\sum_{\text{odd } i}^{r-\beta_1}2Z_{S_{r-i}}(2,2^2,\ldots)+\sum_{\text{even } i}^{r-\beta_2}2^2Z_{S_{r-i}}(2,2^2,\ldots), \tag{24}$$

where $\beta_1=1$, $\beta_2=0$ if r is even and $\beta_1=0$, $\beta_2=1$ if r is odd. Similarly, for $r-2$,

$$(r-2)Z_{S_{r-2}}(2,2^2,\ldots)=\sum_{\text{odd } i}^{r-2-\beta_1}2Z_{S_{r-2-i}}(2,2^2,\ldots)+\sum_{\text{even } i}^{r-2-\beta_2}2^2Z_{S_{r-2-i}}(2,2^2,\ldots). \tag{25}$$

Subtracting the second equation from the first one and rearranging the terms give

$$rZ_{S_r}(2,2^2,\ldots)=2Z_{S_{r-1}}(2,2^2,\ldots)+(r+2)Z_{S_{r-2}}(2,2^2,\ldots). \tag{26}$$

We now use induction and this recurrence to prove that (23) holds.

Basis $r=0$. Substituting $r=0$ in (23) gives 1 as it should since $Z_{S_0}()=1$.
$r=1$. Substituting $r=1$ in (23) gives

$$Z_{S_1}(2)=\frac{2(1)^2+8(1)+7+(-1)^1}{8}=2, \tag{27}$$

and this agrees with (5), i.e., $Z_{S_1}(2)=\frac{1}{1}\left(2Z_{S_0}()\right)=2$.

Induction Step: Suppose that (23) holds for $r-2$ and $r-1$. Then by (26), we have

$$\begin{aligned}rZ_{S_r}(2,2^2,\ldots)&=2Z_{S_{r-1}}(2,2^2,\ldots)+(r+2)Z_{S_{r-2}}(2,2^2,\ldots),\\ &=2\frac{2(r-1)^2+8(r-1)+7+(-1)^{r-1}}{8}+(r+2)\frac{2(r-2)^2+8(r-2)+7+(-1)^{r-2}}{8}, &&(28)\\ &=r\frac{2r^2+8r+7+(-1)^r}{8}, &&(29)\end{aligned}$$

that agrees with (23). □

Finally, by combining Lemmas 1 and 2, we have

Theorem 1.

$$|B_u(2,r)|=\frac{2r^3+15r^2+34r+22.5+1.5(-1)^r}{24}. \quad \square \tag{30}$$

3. A closed-form formula for $|B_u(3,r)|$

We proceed as in the computation of $|B_u(2,r)|$.

$$\begin{aligned}|B_u(3,r)|&=Z_{S_3\times S_r}(2,2,\ldots\ldots,2), &&(31)\\ &=\left[Z_{S_3}(x_1,x_2,x_3)\boxtimes Z_{S_r}(x_1,x_2,\ldots\ldots,x_r)\right](2,2,\ldots,2), &&(32)\\ &=\left[\left(\frac{1}{6}\left(x_1^3+3x_1x_2+2x_3\right)\right)\boxtimes Z_{S_r}(x_1,x_2,\ldots\ldots,x_r)\right](2,2,\ldots,2), &&(33)\\ &=\frac{1}{6}\left[x_1^3\boxtimes Z_{S_r}(x_1,x_2,\ldots\ldots,x_r)\right](2,2,\ldots,2)+\\ &\quad\frac{1}{6}\left[3x_1x_2\boxtimes Z_{S_r}(x_1,x_2,\ldots\ldots,x_r)\right](2,2,\ldots,2)+\\ &\quad\frac{1}{6}\left[2x_3\boxtimes Z_{S_r}(x_1,x_2,\ldots\ldots,x_r)\right](2,2,\ldots,2), &&(34)\\ &=\frac{1}{6}\left\{\left[x_1^3\boxtimes\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,2,\ldots,2)+\left[3x_1x_2\boxtimes\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,2,\ldots,2)+\right.\end{aligned}$$

$$\left[2x_3 \boxtimes \frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,\ldots,2)\Bigg\}, \tag{35}$$

$$= \frac{1}{6}\Bigg\{\left[\frac{1}{r!}\sum_{t=1}^{r!} x_1^3 \bigodot \prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,\ldots,2) + \left[\frac{3}{r!}\sum_{t=1}^{r!} x_1x_2 \bigodot \prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,\ldots,2) +$$

$$\left[\frac{2}{r!}\sum_{t=1}^{r!} x_3 \bigodot \prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,\ldots,2)\Bigg\}, \tag{36}$$

$$= \frac{1}{6}\Bigg\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_{\mathrm{lcm}(1,j)}^{3q_{t,j}\gcd(1,j)}\right](2,2,\ldots,2) + \left[\frac{3}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_{\mathrm{lcm}(1,j)}^{q_{t,j}\gcd(1,j)} x_{\mathrm{lcm}(2,j)}^{q_{t,j}\gcd(2,j)}\right](2,2,\ldots,2) +$$

$$\left[\frac{2}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_{\mathrm{lcm}(3,j)}^{q_{t,j}\gcd(3,j)}\right](2,2,\ldots,2)\Bigg\}, \tag{37}$$

$$= \frac{1}{6}\Bigg\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{3q_{t,j}}\right](2,2,\ldots,2) + \left[\frac{3}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{q_{t,j}} x_{\mathrm{lcm}(2,j)}^{q_{t,j}\gcd(2,j)}\right](2,2,\ldots,2) +$$

$$\left[\frac{2}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_{\mathrm{lcm}(3,j)}^{q_{t,j}\gcd(3,j)}\right](2,2,\ldots,2)\Bigg\}, \tag{38}$$

$$= \frac{1}{6}\Bigg\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} 2^{3q_{t,j}}\right] + \left[\frac{3}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} 2^{q_{t,j}}2^{q_{t,j}\gcd(2,j)}\right] + \left[\frac{2}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} 2^{q_{t,j}\gcd(3,j)}\right]\Bigg\}, \tag{39}$$

$$= \frac{1}{6}\Bigg\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}(2^3)^{q_{t,j}}\right] + 3\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{\text{odd } j}(2^2)^{q_{t,j}}\prod_{\text{even } j}(2^3)^{q_{t,j}}\right]$$

$$+2\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j \bmod 3=0}(2^3)^{q_{t,j}}\prod_{j \bmod 3\neq 0}2^{q_{t,j}}\right]\Bigg\}, \tag{40}$$

$$= \frac{1}{6}\Big\{\left[Z_{S_r}(2^3,2^3,\ldots,2^3)\right] + 3\left[Z_{S_r}(2^2,2^3,2^2,2^3,\ldots)\right] + 2\left[Z_{S_r}(2,2,2^3,2,2,2^3,\ldots)\right]\Big\}. \tag{41}$$

Thus, we have reduced the computation of $|B_u(3,r)|$ to computing the three terms in (41). These computations are carried out in the next three lemmas.

Lemma 3. $Z_{S_r}(2^3,2^3,\ldots,2^3) = \binom{r+7}{r}$.

Proof. Using (5), we have

$$rZ_{S_r}(2^3,2^3,\ldots,2^3) = \sum_{i=1}^{r} 2^3 Z_{S_{r-i}}(2^3,2^3,\ldots,2^3), \tag{42}$$

$$(r-1)Z_{S_{r-1}}(2^3,2^3,\ldots,2^3) = \sum_{i=1}^{r-1} 2^3 Z_{S_{r-1-i}}(2^3,2^3,\ldots,2^3). \tag{43}$$

Subtracting the second equation from the first one and simplifying it give

$$rZ_{S_r}(2^3,2^3,\ldots,2^3) - (r-1)Z_{S_{r-1}}(2^3,2^3,\ldots,2^3) = 8Z_{S_{r-1}}(2^3,2^3,\ldots,2^3), \tag{44}$$

$$Z_{S_r}(2^3,2^3,\ldots,2^3) = (\frac{r+7}{r})Z_{S_{r-1}}(2^3,2^3,\ldots,2^3). \tag{45}$$

Expanding the last equation recursively, we obtain

$$Z_{S_r}(2^3, 2^3, \ldots, 2^3) = (\frac{r+7}{r})(\frac{r+6}{r-1})Z_{S_{r-2}}(2^3, 2^3, \ldots, 2^3), \quad (46)$$

$$= (\frac{r+7}{r})(\frac{r+6}{r-1})(\frac{r+5}{r-2})\ldots(\frac{8}{1})Z_{S_0}(). \quad (47)$$

Noting that $Z_{S_0}() = 1$ proves the statement, i.e.,

$$Z_{S_r}(2^3, 2^3, \ldots, 2^3) = \binom{r+7}{r} = \binom{r+7}{7}. \quad \square$$

Lemma 4.

$$Z_{S_r}(2^2, 2^3, 2^2, 2^3, \ldots) = \frac{(r+4)\left(2r^4 + 32r^3 + 172r^2 + 352r + 15(-1)^r + 225\right)}{960}. \quad (48)$$

Proof. We consider two cases:

Case 1: $r \bmod 2 = 0$.

By (5),

$$rZ_{S_r}(2^2, 2^3, 2^2, 2^3, \ldots) = \sum_{\text{odd } i}^{r-1} 2^2 Z_{S_{r-i}}(2^2, 2^3, 2^2, 2^3, \ldots) + \sum_{\text{even } i}^{r} 2^3 Z_{S_{r-i}}(2^2, 2^3, 2^2, 2^3, \ldots), \quad (49)$$

and

$$(r-2)Z_{S_{r-2}}(2^2, 2^3, 2^2, 2^3, \ldots) = \sum_{\text{odd } i}^{r-3} 2^2 Z_{S_{r-2-i}}(2^2, 2^3, 2^2, 2^3, \ldots) + \sum_{\text{even } i}^{r-2} 2^3 Z_{S_{r-2-i}}(2^2, 2^3, 2^2, 2^3, \ldots). \quad (50)$$

Subtracting the second equation from the first one and rearranging the terms give

$$rZ_{S_r}(2^2, 2^3, 2^2, 2^3, \ldots) = 4Z_{S_{r-1}}(2^2, 2^3, 2^2, 2^3, \ldots) + (r+6)Z_{S_{r-2}}(2^2, 2^3, 2^2, 2^3, \ldots). \quad (51)$$

Case 2: $r \bmod 2 = 1$.

Again by (5),

$$rZ_{S_r}(2^2, 2^3, 2^2, 2^3, \ldots) = \sum_{\text{odd } i}^{r} 2^2 Z_{S_{r-i}}(2^2, 2^3, 2^2, 2^3, \ldots) + \sum_{\text{even } i}^{r-1} 2^3 Z_{S_{r-i}}(2^2, 2^3, 2^2, 2^3, \ldots), \quad (52)$$

$$(r-2)Z_{S_{r-2}}(2^2, 2^3, 2^2, 2^3, \ldots) = \sum_{\text{odd } i}^{r-2} 2^2 Z_{S_{r-2-i}}(2^2, 2^3, 2^2, 2^3, \ldots) + \sum_{\text{even } i}^{r-3} 2^3 Z_{S_{r-2-i}}(2^2, 2^3, 2^2, 2^3, \ldots). \quad (53)$$

Subtracting the second equation from the first one, and rearranging the terms give

$$rZ_{S_r}(2^2, 2^3, 2^2, 2^3, \ldots) = 4Z_{S_{r-1}}(2^2, 2^3, 2^2, 2^3, \ldots) + (r+6)Z_{S_{r-2}}(2^2, 2^3, \ldots). \quad (54)$$

Hence, we obtain the same recurrence for both even and odd r. We now use induction and this recurrence to prove that (48) holds.

Basis $r = 0$. Substituting $r = 0$ in (48) gives 1 as it should since $Z_{S_0}() = 1$.

$r = 1$. Substituting $r = 1$ in (48) gives

$$Z_{S_1}(2^2) = \frac{(1+4)\left(2(1)^4 + 32(1)^3 + 172(1)^2 + 352(1) + 15(-1)^1 + 225\right)}{960} = 4, \quad (55)$$

and this agrees with (5), i.e., $Z_{S_1}(2^2) = \frac{1}{1}\left(2^2 Z_{S_0}()\right) = 2^2 = 4$.

Induction Step:

Suppose that (48) holds for $r-2$ and $r-1$. Then by (54), we have

$$rZ_{S_r}(2^2, 2^3, 2^2, 2^3, \ldots)$$

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$$= 4Z_{S_{r-1}}(2^2, 2^3, 2^2, 2^3, \ldots) + (r+6)Z_{S_{r-2}}(2^2, 2^3, 2^2, 2^3, \ldots), \tag{56}$$

$$= \frac{4(r+3)\left(2(r-1)^4 + 32(r-1)^3 + 172(r-1)^2 + 352(r-1) + 15(-1)^{(r-1)} + 225\right)}{960} + \frac{(r+6)(r+2)\left(2(r-2)^4 + 32(r-2)^3 + 172(r-2)^2 + 352(r-2) + 15(-1)^{(r-2)} + 225\right)}{960}, \tag{57}$$

$$= \frac{8r^5 + 120r^4 + 640r^3 + 1440r^2 + [1212 - 60(-1)^r]r - 180(-1)^r + 180}{960} + \frac{2r^6 + 32r^5 + 180r^4 + 400r^3 + [193 + 15(-1)^r]r^2 + [120(-1)^r - 312]r + 180(-1)^r - 180}{960}, \tag{58}$$

$$= \frac{2r^6 + 40r^5 + 300r^4 + 1040r^3 + [1633 + 15(-1)^r]r^2 + [900 + 60(-1)^r]r}{960}, \tag{59}$$

$$= r\frac{(r+4)\left(2r^4 + 32r^3 + 172r^2 + 352r + 15(-1)^r + 225\right)}{960}, \tag{60}$$

that agrees with (48). □

Lemma 5.

$$Z_{S_r}(2, 2, 2^3, 2, 2, 2^3, \ldots) = \begin{cases} \dfrac{(r^3 + 12r^2 + 45r + 54)}{54} & \text{if } r \bmod 3 = 0 \\[2ex] \dfrac{(r^3 + 12r^2 + 45r + 50)}{54} & \text{if } r \bmod 3 = 1 \\[2ex] \dfrac{(r^3 + 12r^2 + 39r + 28)}{54} & \text{if } r \bmod 3 = 2 \end{cases} \tag{61}$$

Proof. We consider three cases:

Case 1: $r \bmod 3 = 0$. Using (5), we have

$$rZ_{S_r}(2, 2, 2^3, \ldots) = \sum_{i \bmod 3=0}^{r} 2^3 Z_{S_{r-i}}(2, 2, 2^3, \ldots) + \sum_{i \bmod 3=1}^{r-2} 2Z_{S_{r-i}}(2, 2, 2^3, \ldots) + \sum_{i \bmod 3=2}^{r-1} 2Z_{S_{r-i}}(2, 2, 2^3, \ldots), \tag{62}$$

$$(r-3)Z_{S_{r-3}}(2, 2, 2^3, \ldots) = \sum_{i \bmod 3=0}^{r-3} 2^3 Z_{S_{r-3-i}}(2, 2, 2^3, \ldots) + \sum_{i \bmod 3=1}^{r-5} 2Z_{S_{r-3-i}}(2, 2, 2^3, \ldots) + \sum_{i \bmod 3=2}^{r-4} 2Z_{S_{r-3-i}}(2, 2, 2^3, \ldots). \tag{63}$$

Subtracting (63) from (62) we get

$$rZ_{S_r}(2, 2, 2^3, \ldots) - (r-3)Z_{S_{r-3}}(2, 2, 2^3, \ldots) = 2Z_{S_{r-1}}(2, 2, 2^3, \ldots) + 2Z_{S_{r-2}}(2, 2, 2^3, \ldots) + 8Z_{S_{r-3}}(2, 2, 2^3, \ldots), \tag{64}$$

$$rZ_{S_r}(2, 2, 2^3, \ldots) = 2Z_{S_{r-1}}(2, 2, 2^3, \ldots) + 2Z_{S_{r-2}}(2, 2, 2^3, \ldots) + (r+5)Z_{S_{r-3}}(2, 2, 2^3, \ldots). \tag{65}$$

Cases 2, 3: $r \bmod 3 = 1$, $r \bmod 3 = 2$. We omit the derivations for these two cases as it is not difficult to show that these two cases also lead to the recurrence in (65).

Now we use the recurrences given in (5) and (65) to prove (61) by induction on r.

Basis ($r = 0$). Substituting $r = 0$ in (61) gives 1 as it should since $Z_{S_0}() = 1$.

($r = 1$). Substituting $r = 1$ in (61) gives 2 as it should since $Z_{S_1}(2) = \frac{1}{1}\left(2Z_{S_0}()\right) = 2$ by (5).

$(r = 2)$. Substituting $r = 2$ in (61) gives 3 as it should since
$Z_{S_2}(2, 2) = \frac{1}{2}\left(2Z_{S_1}(2) + 2Z_{S_0}()\right) = \frac{4+2}{2} = 3$ by (5).
$(r = 3)$. Substituting $r = 3$ in (61) gives 6 as it should since
$Z_{S_3}(2, 2, 2^3) = \frac{1}{3}\left(2Z_{S_2}(2, 2) + 2Z_{S_1}(2) + 2^3 Z_{S_0}()\right) = \frac{6+4+8}{3} = 6$ by (5).

Induction Step: Suppose that (61) holds for $r-1$, $r-2$, and $r-3$ and $r \bmod 3 = 0$. Then by (65),

$$r Z_{S_r}(2, 2, 2^3, \ldots)$$
$$= 2Z_{S_{r-1}}(2, 2, 2^3, \ldots) + 2Z_{S_{r-2}}(2, 2, 2^3, \ldots) + (r+5)Z_{S_{r-3}}(2, 2, 2^3, \ldots), \tag{66}$$
$$= \frac{2[(r-1)^3 + 12(r-1)^2 + 39(r-1) + 28]}{54} + \frac{2[(r-2)^3 + 12(r-2)^2 + 45(r-2) + 50]}{54}$$
$$+ \frac{(r+5)[(r-3)^3 + 12(r-3)^2 + 45(r-3) + 54]}{54}, \tag{67}$$
$$= \frac{2r^3 + 18r^2 + 36r + 2r^3 + 12r^2 + 18r}{54} + \frac{r^4 + 8r^3 + 15r^2}{54}, \tag{68}$$
$$= \frac{r^4 + 12r^3 + 45r^2 + 54r}{54}, \tag{69}$$
$$= \frac{r(r^3 + 12r^2 + 45r + 54)}{54}, \tag{70}$$

as stated in (61). The other two cases are shown to hold similarly and omitted. □

Combining Lemmas 3–5 we have

Theorem 2.

$$|B_u(3, r)| = \begin{cases} \frac{1}{6}\left[\binom{r+7}{7} + \frac{3(r+4)\left(2r^4 + 32r^3 + 172r^2 + 352r + 15(-1)^r + 225\right)}{960} + \frac{2(r^3 + 12r^2 + 45r + 54)}{54}\right] & \text{if } r \bmod 3 = 0, \\ \frac{1}{6}\left[\binom{r+7}{7} + \frac{3(r+4)\left(2r^4 + 32r^3 + 172r^2 + 352r + 15(-1)^r + 225\right)}{960} + \frac{2(r^3 + 12r^2 + 45r + 50)}{54}\right] & \text{if } r \bmod 3 = 1, \\ \frac{1}{6}\left[\binom{r+7}{7} + \frac{3(r+4)\left(2r^4 + 32r^3 + 172r^2 + 352r + 15(-1)^r + 225\right)}{960} + \frac{2(r^3 + 12r^2 + 39r + 28)}{54}\right] & \text{if } r \bmod 3 = 2. \end{cases}$$ □

Remark 1. The computation method described here can be extended to $|B_u(n, r)|$ for $n \geq 4$, but the solutions of resulting recurrences become significantly more complex to obtain closed form formulas. More significantly, the number of recurrences that must be solved is given by the number of partitions of n that is known to increase exponentially with n by Ramanujan–Hardy–Rademacher's asymptotic formula. We also note that for any integer $n \geq 2$, the solution of one of these recurrences results in $\frac{\binom{r+2^n-1}{r}}{n!}$, and this establishes a lower bound for $|B_u(n, r)|$, $\forall\, r \geq 2$. □

Remark 2. It is noted that $|B_u(2, 2i-2)|$ coincides with the ith hexagonal pyramidal number (see the integer sequence, A002412 in [9]), when $i = 1, 2, 3, \ldots$. □

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