



Minimal Models of Some Differential Graded Modules

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Abstract

Minimal models of chain complexes associated with free torus actions on spaces have been extensively studied in the literature. In this paper, we discuss these constructions using the language of operads. The main goal of this paper is to define a new Koszul operad that has projections onto several of the operads used in these minimal model constructions.

Keywords Rank conjecture · Operads · Minimal Hirsch–Brown model

Mathematics Subject Classification 16E45 · 18D50

1 Introduction

Let k be an algebraically closed field of characteristic 2 and G an elementary abelian 2-group of rank r . Considering the chain complexes associated with free G -spaces, one obtains an algebraic conjecture stronger than the Halperin–Carlsson rank conjecture about 2-torus actions (G -actions). For any chain complex C of k -modules, we denote the homology of C by $H(C)$.

Conjecture 1 If C is a finite chain complex of free kG -modules with $H(C) \neq 0$, then $\dim_k H(C) \geq 2^r$.

Considering the polynomial ring $S := k[x_1, \dots, x_r]$, an equivalent algebraic conjecture is given by Proposition 11.1 and 11.2 in [5]. We say (M, ∂) is a *differential graded S -module* (dg- S -module) if M is an S -module, and ∂ is an S -linear endomorphism of M that has degree

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-1 and satisfies $\partial^2 = 0$. Moreover, we say a dg- S -module is *free* if its underlying graded S -module is free.

Conjecture 2 [6, Conjecture II.8] Let $S = k[x_1, \dots, x_r]$ be the polynomial algebra in r variables of degree -1 with coefficients in an algebraically closed field of characteristic 2. If (M, ∂) is a free, finitely generated dg- S -module with $0 < \dim_k H(M) < \infty$, then $\text{rank}_S M \geq 2^r$.

In the literature, bounds for the dimension of $H(C)$ in Conjecture 1 and for the rank of M in Conjecture 2 are obtained by studying minimal models of C and M . In [7], Carlsson showed the existence of minimal models of certain free differential graded S -modules. Here we give an explicit construction of these minimal models using operad theory. Again using operads we construct minimal models of chain complexes of Borel constructions of spaces with a free G -action. These minimal models are equivalent to minimal Hirsch-Brown models given by Allday-Puppe [1].

Note that Conjecture 1 holds if we further assume that the Euler characteristic of C is non-zero. More precisely,

$$\chi(C) = |G| \chi(k \otimes_{kG} C) = \chi(H(C)) = \sum_{i \geq 0} (-1)^i \dim_k H_i(C) \neq 0.$$

Hence the dimension of total homology $\dim_k H(C) \geq |G| = 2^r$. Due to the equivalence of conjectures one could ask if a similar result holds for Conjecture 2. We prove the conjecture in the following case:

Theorem 1 *Conjecture 2 holds if every integer n, m have the same parity whenever $H_n(M) \neq 0$ and $H_m(M) \neq 0$. In fact, $\chi(H(M)) := \sum_{i \geq 0} (-1)^i \dim_k H_i(M) \neq 0$ implies Conjecture 2.*

When the characteristic of the field is odd, a result analogous to Theorem 1 is proved by Walker [15], [16].

Puppe [14] asserted that, given a certain multiplicative structure on the minimal Hirsch-Brown model for the equivariant cohomology of a space with a free torus action, these bounds can be tightened to verify the Halperin–Carlsson rank conjecture. The main goal of this paper is to put a multiplicative structure on minimal Hirsch-Brown models of G -spaces. Note that the group algebra kG is an exterior algebra since k is a characteristic 2 field. First we consider the group algebra kG and the polynomial algebra $S = k[x_1, \dots, x_r]$ as algebraic operads where all non-trivial operations are unary operations. Then to put multiplicative structures on our minimal models we define a new Koszul operad.

Theorem 2 *Let k be an algebraically closed field of characteristic 2 and G an elementary abelian 2-group of rank r . Then there exists an algebraic operad $\mathcal{P}$ in the category of differential graded modules over k such that $\mathcal{P}$ has the following properties:*

- (i) *The unary operations of $\mathcal{P}$ with the composition of $\mathcal{P}$ considered as multiplication is isomorphic to the group algebra kG ;*
- (ii) *$\mathcal{P}$ has an associative binary operation μ ;*
- (iii) *$\mathcal{P}$ is a Koszul operad;*
- (iv) *The Koszul dual operad of $\mathcal{P}$ has projections onto the associative operad As and the polynomial algebra $S = k[x_1, \dots, x_r]$ where we consider S as an operad whose all nontrivial operations are unary;*

- (v) For every G -space X the singular cochain complex $C^\bullet(X; k)$ has a $\mathcal{P}$ -algebra structure whose restriction to unary operations of $\mathcal{P}$ gives the natural kG -module structure on $C^\bullet(X; k)$ and the action of μ is same as the dual of the Alexander-Withney diagonal map.

Let $\mathcal{P}$ be the operad in Theorem 2 and $\iota : \mathcal{P} \rightarrow \Omega\mathcal{P}$ be the universal twisting morphism. Given a space X that admits a free G -action, we will consider the bar construction $B_\iota H(C^\bullet(X; k))$ as the minimal Hirsch-Brown model of X ; see Sect. 4.2.

Throughout this paper, k is an algebraically closed field of characteristic 2 and all (co)operads are non-symmetric (co)operads in the category of dg-modules over k . In Sect. 2, we recall Puppe's method to find lower bounds on total homology dimension of complexes with a free G -action and give an outline of our method. In Sect. 3, we recall definitions, notation, and well-known results about algebraic (co)operads. In Sect. 4, we discuss constructions of minimal models and prove Theorem 1. In Sect. 5, we prove the our main result Theorem 2 and its applications.

2 The outline of an application of Theorem 2

Assume that r is a positive integer and m is a nonnegative integer. Let S denote the polynomial algebra $k[x_1, \dots, x_r]$ with $\deg(x_i) = -1$ and Λ_m denote the exterior algebra $\Lambda(z_1^{(m)}, \dots, z_r^{(m)})$ with $\deg(z_i^{(m)}) = -m$ for all i in $\{1, \dots, r\}$. Note that according to our degree conventions Puppe [14] defines the Koszul complex $K_r(m)$ corresponding to the regular sequence $(x_1^{m+1}, \dots, x_r^{m+1})$ in S as the differential graded algebra $K_r(m) = (S \tilde{\otimes} \Lambda_m, \partial)$, where the differential ∂ determined by $\partial(x_i \otimes 1) = 0$ and $\partial(1 \otimes z_i^{(m)}) = x_i^{m+1}$. Then Puppe considers dg- S -module morphisms $\gamma : K_r(m) \rightarrow K_r(0)$ which lift the projection

$$\epsilon : H(K_r(m)) \cong S/(x_1^{m+1}, \dots, x_r^{m+1}) \longrightarrow S/(x_1, \dots, x_r) \cong H(K_r(0)).$$

Puppe denotes the rank of γ by $\text{rank}(\gamma)$ which is the rank of the localization of $F \otimes \gamma$ where F is the field of fraction of S . In [14, Lemma 2.1.a], Puppe shows that if γ also preserves the multiplicative structure, then $\text{rank}(\gamma) = 2^r$. In [14, Lemma 2.1.b], Puppe asserts that $\text{rank}(\gamma) \geq 2r$ without the assumption about the multiplicative structure.

Let G be an elementary abelian 2-group of rank r generated by $g_1, g_2, \dots, g_r$. Puppe defined a minimal Hirsch-Brown model $M = S \tilde{\otimes} H^\bullet(X; k)$ associated to a free action of G on X , where X is a finite dimensional CW -complex. Then in [14, Proposition 4.1], Puppe showed that there exists a map $\alpha : K_r(m) \rightarrow M$ that induces a surjective map on the zeroth homology of these complexes for large enough m . Moreover, in [14, Proposition 4.2], Puppe showed that there exists a map $\beta : M \rightarrow K_r(0)$ that induces a surjective map on the zeroth homology. Now notice that the rank of $\gamma = \beta \circ \alpha$ is less than or equal to the total homology dimension $\sum_{i=0}^\infty \dim_k H_i(X; k)$ of X . Hence Puppe used this idea to put lower bounds on the total cohomology dimension of complexes with a free G -action.

We will use an idea similar to Puppe's idea discussed above. In our setting Koszul complexes are dual of the ones considered by Puppe. Now we give the definitions in our setting. Assume G acts on the product of r equidimensional spheres $\mathbb{S}^m \times \dots \times \mathbb{S}^m$ such that g_i acts on the i th sphere with the antipodal action. Then we will denote the minimal Hirsch-Brown model $B_\iota H(C^\bullet(X; k))$ associated to this action by $\tilde{K}_r(m)$ and call it a Koszul complex for our operad in Theorem 2. We use this terminology because when the above bar construction is done using the suboperad of $\mathcal{P}$ generated by its unary operations, we obtain the usual dual Koszul complex $S^* \tilde{\otimes} \Lambda_m$ which we still denote by $K_r(m)$ by abuse of notation. Hence from now on,

$$K_r(m) = (S^* \tilde{\otimes} \Lambda_m, \partial),$$

where the differential ∂ determined by $\partial(x_i \otimes 1) = 0$ and

$$\partial(x_i^n \otimes z_i^{(m)}) = \begin{cases} 0 & \text{if } n < m + 1 \\ x_i^{n-m-1} & \text{if } n \geq m + 1. \end{cases}$$

For the rest of the paper i_m will denote the natural inclusion of $\mathbb{S}^0 \times \dots \times \mathbb{S}^0$ in $\mathbb{S}^m \times \dots \times \mathbb{S}^m$ where each $\mathbb{S}^0$ is sent to the south and north poles of the corresponding $\mathbb{S}^m$. This continuous function induces coalgebra morphisms $i_m^* : \tilde{K}_r(m) \rightarrow \tilde{K}_r(0)$ and $i_m^* : K_r(m) \rightarrow K_r(0)$. Hence, in [14, Lemma 2.1.a] and [14, Lemma 2.1.b], Puppe gave lower bounds on the rank of certain maps between Koszul complexes $K_r(m)$ and $K_r(0)$ which induces the same map as i_m does between the homology of these complexes. Puppe in [14, Corollary 5.2] asserted that if the minimal Hirsch-Brown model of a finite dimensional space X with a free G -action carries differential graded algebra structure, then $\sum_{i=0}^{\infty} \dim_k H^i(X; k) \geq 2^r$. Note that the multiplicative structure is not considered in [14, Proposition 4.1] and [14, Proposition 4.2]. However, in the proof of [14, Corollary 5.2] one needs extensions of [14, Proposition 4.1 and 4.2] to differential graded algebras. More precisely, the morphism α must be constructed in a way compatible with the multiplicative structure. However, the almost random selections of images of α in the homological proofs given in the literature for these results do not take the multiplicative structure into consideration. For example, some selections of α fail for the associated minimal model of the free $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on $\mathbb{R}P^3$ induced by quotienting out the center of the free action of Q_8 on $Sp(1) \cong \mathbb{S}^3$ by the left multiplication. Here we partially fill the gap by the following Proposition.

Proposition 1 *Let G be an elementary abelian 2-group and act freely on a finite-dimensional simplicial set X . Assume $\mathcal{P}$ is the operad in Theorem 2. Let $\tilde{M}$ denote the minimal Hirsch-Brown model of X ; in other words $\tilde{M} = B_i(H(C^\bullet(X; k)))$. Then there exists a positive integer m and $\mathcal{P}^i$ -coalgebra morphisms α, β such that the composition*

$$\tilde{K}_r(m) \xrightarrow{\alpha} \tilde{M} \xrightarrow{\beta} \tilde{K}_r(0),$$

sends $K_r(m)$ to $K_r(0)$ and which induces the same map from $H(K_r(m))$ to $H(K_r(0))$ as i_m^ does.*

However, using the above proposition we cannot confirm the main result of Puppe using [14, Lemma 2.1.a] as our multiplicative structure does not have all the properties Puppe used in the proof of [14, Lemma 2.1.a]. On the other hand [14, Lemma 2.1.b] is proved by only using the differential graded S -module structure on these Koszul complexes. Hence to improve the results about bounds for the total dimension of the cohomology of a space that admits a free G -action, it is enough to consider maps between $\tilde{K}_r(m)$ and $\tilde{K}_r(0)$ and prove analogs of [14, Lemma 2.1.b]. The following proposition is similar to [14, Lemma 2.1.b] but proved using our setting where we fix an isomorphism $\tilde{K}_r(m) \cong \mathcal{P}^i \circ \Lambda_m$ with the composition $\circ$ defined as in [11, Section 5.9.1].

Proposition 2 *Let $\mathcal{P}$ denote the operad in Theorem 2, m be a positive integer and $\gamma : K_r(m) \rightarrow K_r(0)$ be a S^* -coalgebra morphism which induces the same map from $H(K_r(m))$ to $H(K_r(0))$ as i_m^* does. Assume γ extends to a $\mathcal{P}^i$ -coalgebra morphism $\tilde{\gamma} : \tilde{K}_r(m) \rightarrow \tilde{K}_r(0)$ whose restriction to $\mathcal{P}^i \circ 1$ is induced by the identity on $\mathcal{P}^i$. Then the linear map $F \otimes_S \gamma^*$ has rank at least $2r$ where F denotes the field of fractions of the ring S .*

As discussed above Proposition 2 can be used to give lower bounds for the total homology of the finite dimensional complexes with a free G -action.

3 Definitions and Notation

We will take most of definitions and notation from [11] and [9].

3.1 Free (co)Operads

A dg- $\mathbb{N}$ -module M is a sequence of differential graded k -modules

$$M = (M(0), M(1), M(2), \dots).$$

A free operad over the dg- $\mathbb{N}$ -module M is an operad $\mathcal{T}(M)$ together with a dg- $\mathbb{N}$ -module morphism $i : M \rightarrow \mathcal{T}(M)$ such that if $\mathcal{P}$ is an operad and $f : M \rightarrow \mathcal{P}$ is an dg- $\mathbb{N}$ -module morphism then there exists a unique operad morphism $\tilde{f} : \mathcal{T}(M) \rightarrow \mathcal{P}$ with $f = \tilde{f} \circ i$.

There exists a free operad $\mathcal{T}(M)$ over every dg- $\mathbb{N}$ -module M , see [11, Section 5.9.6]. Let $n(v)$ denote the number of leaves of a vertex v in a tree and $\tau(M)$ be the tensor product of $M(n(v))$'s as v ranges over the vertices of a tree τ . As a dg- $\mathbb{N}$ -module, $\mathcal{T}(M)$ is the direct sum of $\tau(M)$'s where τ ranges over all planar trees. The operad composition of $\mathcal{T}(M)$ is given by grafting trees. Hence, as an operad $\mathcal{T}(M)$ is generated by $\mathcal{B}^{\mathcal{T}}(M)$, that is,

$$\left\{ \begin{array}{c} \textcircled{b} \\ | \\ \text{---} \end{array} \middle| b \in \mathcal{B}_0 \right\} \cup \left\{ \begin{array}{c} | \\ \textcircled{b} \\ | \\ \text{---} \end{array} \middle| b \in \mathcal{B}_1 \right\} \cup \left\{ \begin{array}{c} \diagup \quad \diagdown \\ | \\ \textcircled{b} \\ | \\ \text{---} \end{array} \middle| b \in \mathcal{B}_2 \right\} \cup \dots$$

where $\mathcal{B}_j$ is a basis for $M(j)$ as a k -vector space. The dg- $\mathbb{N}$ -module $\mathcal{T}(M)$ is always equipped with an extra grading, called the weight-grading. If M itself has no such extra grading, then the trees in $\mathcal{T}(M)$ with exactly n -vertices are said to have weight-grading n . If M already has weight-grading, then the sum of weight-grades of elements in M used to label the vertices of a tree in $\mathcal{T}(M)$ is the weight-grade of that tree. Hence we have a decomposition of $\mathcal{T}(M)$ indexed by the weight-grading

$$\mathcal{T}(M) = \bigoplus_{n \geq 0} \mathcal{T}(M)^{(n)},$$

where each $\mathcal{T}(M)^{(n)}$ is a dg- $\mathbb{N}$ -module.

Dually, we let $\mathcal{T}^c(M)$ denote the cofree cooperad over M . $\mathcal{T}^c(M)$ is isomorphic to $\mathcal{T}(M)$ as a weight-graded k -vector space, while as a cooperad $\mathcal{T}^c(M)$ is cogenerated by the generators of $\mathcal{T}(M)$ mentioned above.

Let sM denote the dg- $\mathbb{N}$ -module M whose degree is shifted by 1, i.e., $sM_i(n) = M_{i-1}(n)$ for $n \in \mathbb{N}$ and $i \in \mathbb{Z}$. More generally, for any integer m , $s^m M$ denotes the dg- $\mathbb{N}$ -module M whose degree is shifted by m .

3.2 Quadratic (co)Operads

A pair (M, R) is called an *operadic quadratic data pair* if M is a dg- $\mathbb{N}$ -module and R is a sub-dg- $\mathbb{N}$ -module of $\mathcal{T}(M)^{(2)}$. The *quadratic operad* associated to the quadratic data pair (M, R) is

$$\mathcal{P}(M, R) := \mathcal{T}(M)/(R),$$

where (R) is the operadic ideal generated by $R \subseteq \mathcal{T}(M)^{(2)}$. In other words, $\mathcal{P}(M, R)$ is the largest quotient operad $\mathcal{P}$ of $\mathcal{T}(M)$ for which the composite

$$R \mapsto \mathcal{T}(M)^{(2)} \mapsto \mathcal{T}(M) \twoheadrightarrow \mathcal{P}$$

is zero.

Dually, the *quadratic cooperad* $\mathcal{C}(M, R)$ associated to the quadratic data pair (M, R) is the largest subcooperad of $\mathcal{T}^c(M)$ for which the composite

$$\mathcal{C} \hookrightarrow \mathcal{T}^c(M) \rightarrow \mathcal{T}^c(M)^{(2)} \rightarrow \mathcal{T}^c(M)^{(2)}/R$$

is zero, see [11, Section 7.1] and [3, Section 6.3.1].

The *Koszul dual cooperad* of a quadratic operad $\mathcal{P} = \mathcal{P}(M, R)$ is

$$\mathcal{P}^i := \mathcal{C}(sM, s^2R),$$

where s^2R is the image of R under the natural map $\mathcal{T}(M)^{(2)} \rightarrow \mathcal{T}(sM)^{(2)}$. Similarly, the *Koszul dual operad* of a quadratic cooperad $\mathcal{C} = \mathcal{C}(M, R)$ is

$$\mathcal{C}^i := \mathcal{P}(s^{-1}M, s^{-2}R),$$

where $s^{-2}R$ is the image of R under the map $\mathcal{T}(M)^{(2)} \rightarrow \mathcal{T}(s^{-1}M)^{(2)}$ induced by the natural degree 1 dg- $\mathbb{N}$ -module morphism M to sM , see [11, Section 7.4.7].

3.3 The (co)Bar Construction

For an operad $\mathcal{P}$, let $\overline{\mathcal{P}}$ be the cokernel of the unit map $I \rightarrow \mathcal{P}$. If $\mathcal{P} = I \oplus \overline{\mathcal{P}}$ as dg- $\mathbb{N}$ -modules, the *bar construction* $B\mathcal{P}$ of $\mathcal{P}$ is the dg-cooperad $\mathcal{T}^c(s\overline{\mathcal{P}})$ with differential $d_1 + d_2$, where d_1 and d_2 are as in [11, Section 6.5.1].

Similarly, for a cooperad $\mathcal{C}$, let $\overline{\mathcal{C}}$ denote the kernel of the counit map $\mathcal{C} \rightarrow I$. If $\mathcal{C} = I \oplus \overline{\mathcal{C}}$ as dg- $\mathbb{N}$ -modules, the *cobar construction* $\Omega\mathcal{C}$ of $\mathcal{C}$ is the dg-operad $\mathcal{T}(s^{-1}\overline{\mathcal{C}})$ with differential $d_1 + d_2$, where d_1 and d_2 are as in [11, Section 6.5.2].

Let (M, R) be an operadic quadratic data pair. The quadratic operad $\mathcal{P} = \mathcal{P}(M, R)$ is *Koszul* if the natural dg-cooperad morphism $\mathcal{P}^i \rightarrow B\mathcal{P}$ is a quasi-isomorphism of dg-cooperads, see [11, Theorem 7.4.2]. When $\mathcal{P}$ is Koszul, we define the operad $\mathcal{P}_\infty := \Omega\mathcal{P}^i$.

3.4 (Co)Algebras Over (co)Operads

Let $\mathcal{P}$ be an operad. A $\mathcal{P}$ -*algebra* is a differential graded k -module A together with an operad morphism $\mathcal{P} \rightarrow \text{End}_A$, where $\text{End}_A(n) = \text{Hom}(A^{\otimes n}, A)$. Dually, for a cooperad $\mathcal{C}$, a $\mathcal{C}$ -*coalgebra* is a differential graded k -module C together with an operad morphism $\mathcal{C}^* \rightarrow \text{coEnd}_C$, where $\mathcal{C}^*$ is the dual of $\mathcal{C}$ and $\text{coEnd}_C(n) = \text{Hom}(C, C^{\otimes n})$. For differential graded k -module C , we define an operad morphism

$$\psi_C : \text{coEnd}_C \rightarrow \text{End}_{C^*}$$

which sends $\alpha : C \rightarrow C^{\otimes n}$ to the composition $(C^*)^{\otimes n} \xrightarrow{i_C} (C^{\otimes n})^* \xrightarrow{\alpha^*} C^*$ where i_C is defined by

$$i_C(f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \cdots f_n(v_n)$$

for every $f_1, \dots, f_n \in C^*$ and $v_1, \dots, v_n$ in C . Given a dg- $\mathcal{C}$ -coalgebra C , we get dg- $\mathcal{C}^*$ -algebra structure on C^* by the composition $\mathcal{C} \rightarrow \text{coEnd}_C \xrightarrow{\psi_C} \text{End}_{C^*}$.

A cooperad $\mathcal{C}$ is called *coaugmented* if its counit map has a right inverse. Let C be a coalgebra over coaugmented cooperad $\mathcal{C}$. For $x \in C$, we define $x_1, x_2, \dots$ by

$$\Delta_C(x) = (x_1, x_2, \dots) \in \prod_{n \geq 1} (\mathcal{C}(n) \otimes C^{\otimes n}),$$

where Δ_C denotes the structure map of the coalgebra C . We filter the coalgebra C by $F_r C := \{x \in C \mid x_i = 0 \text{ for any } i > r\}$ for $r \geq 1$. If $C = \bigcup_{r \geq 1} F_r C$, then C is conilpotent, see [11, Section 5.8.4].

Let $\mathcal{C}$ be a dg-cooperad, $\mathcal{P}$ a dg-operad, and $\varphi : \mathcal{C} \rightarrow \mathcal{P}$ a twisting morphism as in [11, Section 11.1.1]. The *bar construction* B_φ is a functor from the category of dg- $\mathcal{P}$ -algebras to the category of conilpotent dg- $\mathcal{C}$ -coalgebras, defined on a dg- $\mathcal{P}$ -algebra A by

$$B_\varphi A := (\mathcal{C} \circ_\varphi \mathcal{P}) \circ_{\mathcal{P}} A,$$

where $\circ_\varphi$ denotes the right-twisted composite product and $\circ_{\mathcal{P}}$ denotes the relative composite product over $\mathcal{P}$, see [11, Sections 6.4.7 and 11.2.1].

Dually, the *cobar construction* Ω_φ is a functor from the category of conilpotent dg- $\mathcal{C}$ -coalgebras to the category of dg- $\mathcal{P}$ -algebras, defined on a conilpotent dg- $\mathcal{C}$ -algebra C by

$$\Omega_\varphi C := (\mathcal{P} \circ_\varphi \mathcal{C}) \circ_{\mathcal{C}} C,$$

where $\circ_\varphi$ denotes the left-twisted composite product and $\circ_{\mathcal{C}}$ denotes the relative composite product over $\mathcal{C}$, see [11, Sections 6.4.7 and 11.2.1].

Let W, V be two $\mathcal{P}_\infty$ -algebras. Then an ∞ -morphism $f : W \rightarrow V$ is a dg- $\mathbb{N}$ -module morphism $\mathcal{P}^i \rightarrow \text{End}_V^W$, where $\text{End}_V^W(n) = \text{Hom}(W^{\otimes n}, V)$. Moreover, f is an ∞ -quasi-isomorphism if f sends the counit in $\mathcal{P}^i$ to a quasi-isomorphism in $\text{End}_V^W(1)$.

3.5 Homotopy Operadic Algebras

Let (W, d_W) and (V, d_V) be chain complexes that are dg- k -modules. Assume i and p are chain maps and h is chain homotopy as in the diagram

$$h \begin{array}{c} \curvearrowright \\ (V, d_V) \end{array} \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{i} \end{array} (W, d_W).$$

W is a *homotopy retract* of V if $\text{Id}_V - i \circ p = d_V \circ h + h \circ d_V$ and i is a quasi-isomorphism. Moreover, W is a *deformation retract* of V if we also have $\text{Id}_W = p \circ i$.

Theorem 3 [11, Theorem 10.3.1] *Let $\mathcal{P}$ be a Koszul operad and (W, d_W) a homotopy retract of (V, d_V) . Any $\mathcal{P}_\infty$ -algebra structure on V can be transferred to a $\mathcal{P}_\infty$ -algebra structure on W such that i extends to an ∞ -quasi-isomorphism.*

This theorem, known as the Homotopy Transfer Theorem, is a generalization of [10, Theorem 1] and will be used in Sects. 4 and 5 to construct minimal Hirsch-Brown models and minimal models discussed by Carlsson. In these constructions, we also use the following property of the bar construction:

Theorem 4 [11, Proposition 11.2.3] *Let $\varphi : \mathcal{C} \rightarrow \mathcal{P}$ be an operadic twisting morphism and A, A' dg- $\mathcal{P}$ -algebras. If $f : A \rightarrow A'$ is a quasi-isomorphism, then f induces a quasi-isomorphism between the dg- $\mathcal{C}$ -coalgebras $B_\varphi A$ and $B_\varphi A'$.*

The bar and cobar constructions form adjoint functor pair.

Proposition 3 [11, Corollary 11.3.5] *Let $\mathcal{P}$ be a Koszul operad with canonical twisting morphism $\kappa : \mathcal{P}^i \rightarrow \mathcal{P}$. For every dg- $\mathcal{P}$ -algebra A , the counit of the adjunction*

$$\epsilon_\kappa : \Omega_\kappa B_\kappa A \rightarrow A$$

is a quasi-isomorphism of dg- $\mathcal{P}$ -algebras. Dually, for every conilpotent dg- $\mathcal{P}^i$ -coalgebra C , the unit of the adjunction

$$v_K : C \rightarrow B_K \Omega_K C$$

is a quasi-isomorphism of dg- $\mathcal{P}^i$ -coalgebras.

The relation between ∞ -quasi-isomorphisms and quasi-isomorphisms is given by the following:

Theorem 5 [11, Theorem 11.4.9] *Let $\mathcal{P}$ be a Koszul operad and A, A' dg- $\mathcal{P}_\infty$ -algebras. There exists an ∞ -quasi-isomorphism of dg $\mathcal{P}_\infty$ -algebras $A \rightarrow A'$ if and only if there exists a zigzag of quasi-isomorphisms of dg- $\mathcal{P}_\infty$ -algebras $A \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \dots \rightarrow A'$.*

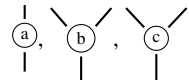
Such a zigzag of quasi-isomorphism will be written $A \rightsquigarrow A'$.

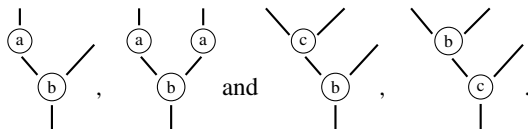
3.6 The Poincaré-Birkhoff-Witt Basis

We already defined Koszul duality for an operad by using bar construction. The non-derived Koszul duality was introduced by Priddy [13] for algebras and generalized by Hoffbeck [9] for operads by considering the existence of a certain basis, called Poincaré-Birkhoff-Witt (PBW) basis. Hoffbeck's criterion asserts that an operad is Koszul if it admits a PBW basis.

In order to define a PBW basis for an operad $\mathcal{P}$, we consider the path sequence of the tree monomials of $\mathcal{P}$ as in [3, Definition 3.4.1.2]. Then we order the path sequences corresponding to the trees by the graded path lexicographic order [3, Definition 3.4.1.7]. Briefly, given two path sequences p and p' of the same arity, we have $p < p'$ if and only if either

- (i) the longer sequence is bigger, or
- (ii) if they have the same length, then we compare the first (leftmost) letters where they differ.

For instance, suppose that we have tree monomials  equipped with a monomial order $a < b < c$. Let us consider the following tree monomials:



The path sequences correspond to the tree monomials have the order $(ba, b) < (ba, ba)$ and $(bc, bc, b) < (cb, cb, c)$. In other words, $(b; a, 1) < (b; a, a)$ and $(b; c, c, 1) < (c; b, b, 1)$. For more details, see [3, Chapter 2.3, 3.4].

For the next notion, we refer the reader to [9]. For every tree τ , let $\mathcal{B}_\tau^{\mathcal{T}(M)}$ be a monomial basis of $\tau(M)$ such that each element is a tensor product of elements in $\mathcal{B}^M$. A PBW basis of a non-symmetric quadratic operad $\mathcal{P}$ is a set $\mathcal{B}^{\mathcal{P}} \subset \mathcal{T}(M)$ of representatives of a base of the module $\mathcal{P}$, containing 1 and $\mathcal{B}^M$ and for all tree τ a subset $\mathcal{B}_\tau^{\mathcal{P}} \subset \mathcal{B}_\tau^{\mathcal{T}(M)}$ satisfying the following conditions:

- For $\alpha \in \mathcal{B}_\sigma^{\mathcal{P}}$ and $\beta \in \mathcal{B}_\tau^{\mathcal{P}}$, either the partial composition product $\alpha \circ_i \beta$ is in $\mathcal{B}_{\sigma \circ_i \tau}^{\mathcal{P}}$ or the elements of the basis $\gamma \in \mathcal{B}^{\mathcal{P}}$ which appear in the unique decomposition $\alpha \circ_i \beta = \sum_\gamma c_\gamma \gamma$, satisfy $\gamma > \alpha \circ_i \beta$ in $\mathcal{T}(M)$.

- Suppose that $\alpha|_{\tau_e}$ denotes the restriction of a treewise tensor α to a subtree τ_e generated by an edge e ; in other words $\alpha|_{\tau_e}$ is the smallest piece of tree that includes the edge e and so it has 2 vertices. A treewise tensor α is in $\mathcal{B}_{\tau}^{\mathcal{P}}$ if and only if for every internal edge e of τ , the restricted treewise tensor $\alpha|_{\tau_e}$ lie in $\mathcal{B}_{\tau_e}^{\mathcal{P}}$

Moreover, by the second condition, a treewise tensor is in the basis if and only if every subtensor generated by an edge is in the basis. Hence, it is enough to set the quadratic part of the basis to determine the basis completely. Then we have the following result:

Theorem 6 [9, Theorem 6.6] *A non-symmetric operad endowed with a non-symmetric PBW basis is Koszul.*

We use this fact in the proof of Koszulness of the operad in Theorem 2.

4 Minimal Models

In this section, r denotes a positive integer. Here we discuss Hirsch-Brown Models in view of the Homotopy Transfer Theorem.

4.1 Unary Quadratic (co)Operads

Let (M, R) be the quadratic data pair

$$M = (0, kv_1 \oplus kv_2 \oplus \cdots \oplus kv_r, 0, 0, \dots)$$

and

$$R = \{ v_i \otimes v_i \mid 1 \leq i \leq r \} \cup \{ v_i \otimes v_j + v_j \otimes v_i \mid 1 \leq i < j \leq r \}.$$

We define a quadratic operad $\mathcal{W}$ and a quadratic cooperad $\mathcal{S}$ as follows:

$$\mathcal{W} := \mathcal{P}(M, R) \text{ and } \mathcal{S} := \mathcal{C}(sM, s^2R).$$

Then considering the identifications

$$t_i = \begin{array}{c} | \\ \bigcirc \\ | \end{array} v_i \quad \text{and} \quad x_i^* = \begin{array}{c} | \\ \bigcirc \\ | \end{array} sv_i$$

for i in $\{1, 2, \dots, r\}$, the operad $\mathcal{W}$ is isomorphic to $(0, \Lambda, 0, 0, \dots)$ where Λ is the exterior algebra $\Lambda(t_1, t_2, \dots, t_r)$ and the cooperad $\mathcal{S}$ is isomorphic to $(0, S^*, 0, 0, \dots)$ where S is the polynomial algebra $k[x_1, \dots, x_r]$.

4.2 Minimal Hirsch–Brown Models

First note that we consider cochain complexes as chain complexes after multiplying the grading by -1 . In other words, there exists a categorical isomorphism between the category of chain complexes and the category of cochain complexes by identifying a chain complex (C, ∂) and a cochain complex (D, ζ) if $C_{-i} = D^i$ and $\partial_{-i} = \zeta^i$. From now on we only work with chain complexes. The cohomology of a simplicial set X will be denoted by $H^\bullet(X; k)$ which corresponds to $\bigoplus_{m=0}^{\infty} H^m(X; k)$ under the isomorphism mentioned above. Hence, $H^\bullet(X; k)$ is trivial in all positive degrees.

Let G be an elementary abelian 2-group of rank r with generator set $\{g_1, \dots, g_r\}$. Then we can identify the group algebra kG with the exterior algebra Λ by identifying $1 + g_i$ with t_i . Moreover, the group cohomology $H^\bullet(BG; k)$ is isomorphic to the polynomial algebra S as graded k -algebras. Assume that G acts freely on a simplicial set X . The goal of this section is to use the techniques discussed in the previous section to give a construction of the minimal Hirsch-Brown model of X which is equivalent to the one constructed in [1]. In other words, we define a differential graded S -module denoted by $H^\bullet(BG; k) \tilde{\otimes} H^\bullet(X; k)$ so that $H^\bullet(BG; k) \tilde{\otimes} H^\bullet(X; k)$ is isomorphic to $H^\bullet(BG; k) \otimes H^\bullet(X; k)$ as a left S -module, and there exists a zig zag of quasi-isomorphisms between $H^\bullet(BG; k) \tilde{\otimes} H^\bullet(X; k)$ and a differential graded S -module which is chain homotopy equivalent to the cochain complex of the Borel construction $EG \times_G X$.

The chain complex $C = C(X; k)$ is a dg- $\mathcal{W}$ -algebra by the morphism from $kG \otimes C$ to C which sends $g \otimes \sigma$ to $g\sigma$ for any $g \in G$, $\sigma \in C$. Moreover, we have $\mathcal{S} = \mathcal{W}^i$ and $\mathcal{W}_\infty = \Omega \mathcal{S}$. Let j denote the inclusion of dg- $\mathcal{W}$ -algebras into dg- $\Omega \mathcal{S}$ -algebras. Since $H(C)$ is a deformation retract of $j(C)$ as dg- k -modules, by Theorem 3 there exists a $\Omega \mathcal{S}$ -algebra structure on $H(C)$ such that $H(C)$ and $j(C)$ are ∞ -quasi-isomorphic as $\Omega \mathcal{S}$ -algebras. We know that $\mathcal{W}$ is also a Koszul operad. Then by Theorem 5, there exists a zigzag of quasi-isomorphisms of dg- $\Omega \mathcal{S}$ -algebras $H(C) \rightsquigarrow j(C)$.

Note that $\mathcal{S}$ is a connected cooperad, so it is conilpotent. Let $\iota : \mathcal{S} \rightarrow \Omega \mathcal{S}$ be the universal twisting morphism. By Theorem 4, there is a zigzag of quasi-isomorphisms $B_i H(C) \rightsquigarrow (B_i \circ j)(C)$ as $\mathcal{S}$ -coalgebras. As graded $\mathbb{N}$ -modules, we have the following isomorphism:

$$B_i H(C) = (\mathcal{S} \circ_i \Omega \mathcal{S}) \circ_{\Omega \mathcal{S}} H(C) \cong S^* \otimes H(C).$$

This isomorphism induces a differential on $S^* \otimes H(C)$. We denote the new differential graded $\mathbb{N}$ -module by $S^* \tilde{\otimes} H(C)$.

We consider the $\mathcal{S}^*$ -algebra $(B_i H(C))^*$ as a version of the minimal Hirsch-Brown model because

$$\begin{aligned} (B_i H(C))^* &\cong (S^* \tilde{\otimes} H(C))^* \\ &\cong S \tilde{\otimes} (H(C))^* \\ &\cong H^\bullet(BG; k) \tilde{\otimes} H^\bullet(X; k). \end{aligned}$$

Let $\kappa : \mathcal{S} \rightarrow \mathcal{W}$ be the canonical twisting morphism. Note that $C(EG; k)$ is kG -chain homotopy equivalent to $S^* \tilde{\otimes} kG := \mathcal{S} \circ_\kappa \mathcal{W}$, where both are considered as differential graded $\mathbb{N}$ -modules. Also we have $(B_i \circ j) = B_{\kappa}$. Hence

$$\begin{aligned} ((B_i \circ j)(C))^* &\cong (S^* \tilde{\otimes} kG \otimes_{kG} C)^* \\ &\simeq (C(EG; k) \otimes_{kG} C)^* \\ &\simeq C(EG \times_G X; k)^* \\ &= C^\bullet(EG \times_G X; k) \end{aligned}$$

where the last equality is due to our conventions about cochain complexes and the second homotopy equivalence follows from the homotopy equivalence $C(EG \times_G X; k) \simeq C(EG; k) \otimes_{kG} C$ proved in [1, proof of Theorem 1.2.8] and [8, VI.12].

We can also consider the chain complex $C = C(X; k)$ as a dg- $\mathcal{W}^*$ -coalgebra by the morphism from C to $kG \otimes C$ which send σ to $\sum_{g \in G} g^* \otimes g^{-1} \sigma$. Hence C^* is a dg- $\mathcal{W}$ -algebra as discussed in Section 3.4. Hence the $\mathcal{S}$ -coalgebra $B_i H(C^*)$ is another version of the minimal Hirsch-Brown model. In fact this second version is what we use in Sect. 5.

4.3 The Minimal Model of Carlsson

Let N be a differential graded S -module, so it is a $\text{dg-}\mathcal{S}^*$ -algebra. We view N as a $\text{dg-}\mathcal{S}$ -coalgebra. The goal of this section is to construct Carlsson's minimal model [7] for N . We construct a $\text{dg-}\mathcal{S}$ -coalgebra that is quasi-isomorphic to N and has zero differential when tensored with k over $\mathcal{S}$.

We have $F = F_2(N)$ in the filtration from Sect. 3.4, so the coalgebra N is conilpotent. As a $\text{dg-}k$ -module $H(N)$ is a deformation retract of N . We obtain the following deformation retract of $\text{dg-}k$ -modules by applying the functor Ω_κ , where $\kappa : \mathcal{S} \rightarrow \mathcal{S}^1$ is the canonical twisting morphism

$$\Omega_\kappa(h) \hookrightarrow (\Omega_\kappa N) \begin{array}{c} \xrightarrow{\Omega_\kappa(p)} \\ \xleftarrow{\Omega_\kappa(i)} \end{array} (\Omega_\kappa H(N)) .$$

By Theorem 3, $\Omega_\kappa N \xleftarrow{\Omega_\kappa(i)} \Omega_\kappa H(N)$ extends to an ∞ -quasi-isomorphism of $\text{dg-}\Omega\mathcal{S}$ -algebras. Furthermore, we have another deformation retract

$$(h') \hookrightarrow (\Omega_\kappa H(N)) \begin{array}{c} \xrightarrow{p'} \\ \xleftarrow{i'} \end{array} (H(\Omega_\kappa H(N)))$$

and so $\Omega_\kappa H(N) \xleftarrow{i'} H(\Omega_\kappa H(N))$ extends to an ∞ -quasi-isomorphism of $\text{dg-}\Omega\mathcal{S}$ -algebras by Theorem 3. Combining these two ∞ -quasi-isomorphisms, we have an ∞ -quasi-isomorphism of $\text{dg-}\Omega\mathcal{S}$ -algebras $\Omega_\kappa N \leftarrow H(\Omega_\kappa H(N))$. Thus by Theorem 5, there is a zigzag of quasi-isomorphisms as $\text{dg-}\Omega\mathcal{S}$ -algebras

$$\Omega_\kappa N \rightsquigarrow H(\Omega_\kappa H(N)).$$

Then by Theorem 4, we have a zigzag of quasi-isomorphisms of $\text{dg-}\mathcal{S}$ -coalgebras

$$B_\kappa \Omega_\kappa N \rightsquigarrow B_\kappa H(\Omega_\kappa H(N)).$$

There is a quasi-isomorphism of $\text{dg-}\mathcal{S}$ -coalgebras $N \rightarrow B_\kappa \Omega_\kappa N$ by Proposition 3. Therefore, we obtain a zigzag of quasi-isomorphisms of $\text{dg-}\mathcal{S}$ -coalgebras

$$N \rightsquigarrow B_\kappa H(\Omega_\kappa H(N)).$$

Note that $k \otimes_S B_\kappa H(\Omega_\kappa H(N))$ has zero differential. Hence we call the $\text{dg-}\mathcal{S}$ -coalgebra $B_\kappa H(\Omega_\kappa H(N))$ the Carlsson minimal model of N .

4.4 A Special Case of Carlsson's Conjecture

The following is equivalent to Theorem 1:

Theorem 7 *Let k be an algebraically closed field of characteristic 2 and S the polynomial algebra in r variables of degree -1 with coefficients in k . Assume (M, ∂) is a free $\text{dg-}S$ -module and $0 < \dim_k H(M) < \infty$. Further assume that $\chi(H(M)) := \sum_{i \geq 0} (-1)^i \dim_k H_i(M)$ is non-zero. Then $2^r \leq \text{rank}_S M$.*

Proof We can consider M as a $\text{dg-}\mathcal{S}$ -coalgebra. As in Sect. 4.3, we have a zigzag of quasi-isomorphism of $\text{dg-}\mathcal{S}$ -coalgebras

$$M \rightsquigarrow B_\kappa H(\Omega_\kappa H(M)),$$

where each middle term in this zigzag is free.

If $f : K \rightarrow L$ is a quasi-isomorphism of bounded-below complexes of free modules, then the mapping cone of f is a bounded-below acyclic complex of free modules. Therefore, the mapping cone is contractible and f is split, so f is a homotopy equivalence [4, Proposition 0.3, Proposition 0.7]. This implies the following zigzag of quasi-isomorphism:

$$k \otimes_S M \leftarrow k \otimes_S B_K H(\Omega_K H(M)) \cong H(\Omega_K H(M)).$$

Also notice that

$$\chi(H(\Omega_K H(M))) = \chi(\Omega_K H(M)) = 2^r \chi(H(M)) \neq 0,$$

Thus,

$$2^r \leq \dim_k(H(\Omega_K H(M))) = \dim_k(H(k \otimes_S M)) \leq \dim_k(k \otimes_S M) = \text{rank}_S(M).$$

□

5 Multiplicative Structures on Minimal Hirsch–Brown Models

In this section, we will prove that the operad $\tilde{\mathcal{W}}$ defined in Sect. 5.2 is an operad that satisfies the properties listed in Theorem 2.

5.1 The Operad $\tilde{\mathcal{W}}$ in the Case $r = 1$

Consider the associative operad that is generated by a binary operation μ_0 , that satisfies the associativity relation $(\mu_0; \mu_0, 1) = (\mu_0; 1, \mu_0)$. In terms of trees, we have

$$\mu_0 \bullet = \text{tree with } \mu_0 \text{ node} \quad \text{with the relation} \quad \begin{array}{c} \mu_0 \bullet \\ \diagdown \\ \mu_0 \end{array} = \begin{array}{c} \mu_0 \bullet \\ \diagup \\ \mu_0 \end{array}.$$

Similarly, for an exterior algebra of a single variable, we have an unary operation t ;

$$t \bullet = \text{tree with } t \text{ node} \quad \text{with the relation} \quad \begin{array}{c} t \bullet \\ | \\ t \bullet \end{array} = 0.$$

In the case $r = 1$, we define a quadratic operad $\tilde{\mathcal{W}}$ by setting generating operations as

$$t \bullet = \text{tree with } t \text{ node}, \quad \mu_0 \bullet = \text{tree with } \mu_0 \text{ node} \quad \text{and} \quad \mu_1 \bullet = \text{tree with } \mu_1 \text{ node}$$

The relations of $\tilde{\mathcal{W}}$ are as follows :

$$R_1: \begin{array}{c} t \bullet \\ | \\ t \bullet \end{array} = 0, \quad R_2: \begin{array}{c} t \bullet \\ \diagdown \\ \mu_1 \end{array} = 0,$$

$$R_3: \begin{array}{c} \mu_0 \bullet \\ \diagdown \\ \mu_0 \end{array} = \begin{array}{c} \mu_0 \bullet \\ \diagup \\ \mu_0 \end{array}, \quad R_4: \begin{array}{c} \mu_1 \bullet \\ \diagdown \\ \mu_1 \end{array} = \begin{array}{c} \mu_1 \bullet \\ \diagup \\ \mu_1 \end{array},$$

$$R_5: \begin{array}{c} \mu_0 \\ | \\ t \end{array} = \begin{array}{c} t \\ | \\ \mu_0 \end{array} + \begin{array}{c} t \\ / \backslash \\ \mu_0 \end{array} + \begin{array}{c} t \\ / \backslash \\ \mu_1 \end{array},$$

$$R_6: \begin{array}{c} \mu_1 \\ / \backslash \\ \mu_0 \end{array} = \begin{array}{c} \mu_0 \\ / \backslash \\ \mu_1 \end{array}, \quad R_7: \begin{array}{c} \mu_1 \\ | \\ t \end{array} = \begin{array}{c} t \\ / \backslash \\ \mu_1 \end{array},$$

$$R_8: \begin{array}{c} \mu_0 \\ / \backslash \\ \mu_1 \end{array} = \begin{array}{c} \mu_1 \\ / \backslash \\ \mu_0 \end{array} + \begin{array}{c} \mu_1 \\ / \backslash \\ \mu_0 \end{array} + \begin{array}{c} \mu_1 \\ / \backslash \\ \mu_1 \end{array}.$$

Please see Lemma 1 in Sect. 5.4 to understand where these relations come from.

Now consider the graded path lex order on all quadratics; firstly there is only one 1-ary quadratic operation and it is represented by $\begin{array}{c} t \\ | \\ t \end{array}$. Secondly, we sort all 2-ary operations:

$$\begin{array}{c} t \\ / \backslash \\ \mu_0 \end{array} < \begin{array}{c} t \\ / \backslash \\ \mu_1 \end{array} < \begin{array}{c} \mu_0 \\ | \\ t \end{array} < \begin{array}{c} \mu_1 \\ | \\ t \end{array} < \begin{array}{c} t \\ / \backslash \\ \mu_0 \end{array}.$$

Correspondingly, path sequences of the planar rooted trees are

$$(\mu_0, \mu_0 t) < (\mu_1, \mu_1 t) < (t\mu_0, t\mu_0) < (t\mu_1, t\mu_1) < (\mu_0 t, \mu_0).$$

Then we sort all 3-ary operations:

$$\begin{array}{c} \mu_0 \\ / \backslash \\ \mu_0 \end{array} < \begin{array}{c} \mu_1 \\ / \backslash \\ \mu_0 \end{array} < \begin{array}{c} \mu_0 \\ / \backslash \\ \mu_1 \end{array} < \begin{array}{c} \mu_1 \\ / \backslash \\ \mu_1 \end{array} < \begin{array}{c} \mu_0 \\ / \backslash \\ \mu_0 \end{array} < \begin{array}{c} \mu_1 \\ / \backslash \\ \mu_0 \end{array} < \begin{array}{c} \mu_0 \\ / \backslash \\ \mu_1 \end{array} < \begin{array}{c} \mu_1 \\ / \backslash \\ \mu_1 \end{array}.$$

Correspondingly, path sequences of the planar rooted trees are

$$\begin{aligned} &(\mu_0, \mu_0^2, \mu_0^2) < (\mu_0, \mu_0\mu_1, \mu_0\mu_1) < (\mu_1, \mu_1\mu_0, \mu_1\mu_0) < (\mu_1, \mu_1^2, \mu_1^2) \\ &< (\mu_0^2, \mu_0^2, \mu_0) < (\mu_0\mu_1, \mu_0\mu_1, \mu_0) < (\mu_1\mu_0, \mu_1\mu_0, \mu_1) < (\mu_1^2, \mu_1^2, \mu_1). \end{aligned}$$

Hence, the quadratic part of a non-symmetric PBW basis is given by

$$\begin{array}{c} t \\ / \backslash \\ \mu_0 \end{array}, \begin{array}{c} t \\ / \backslash \\ \mu_1 \end{array}, \begin{array}{c} \mu_0 \\ | \\ t \end{array}, \begin{array}{c} \mu_0 \\ / \backslash \\ \mu_0 \end{array}, \begin{array}{c} \mu_1 \\ / \backslash \\ \mu_0 \end{array}, \begin{array}{c} \mu_0 \\ / \backslash \\ \mu_1 \end{array}, \begin{array}{c} \mu_1 \\ / \backslash \\ \mu_1 \end{array}.$$

Correspondingly, path sequences of the quadratic part of the basis is given by

$$\begin{aligned} &(\mu_0, \mu_0 t) < (\mu_1, \mu_1 t) < (t\mu_0, t\mu_0) < (\mu_0, \mu_0^2, \mu_0^2) < (\mu_0, \mu_0\mu_1, \mu_0\mu_1) \\ &< (\mu_1, \mu_1\mu_0, \mu_1\mu_0) < (\mu_1, \mu_1^2, \mu_1^2). \end{aligned}$$

The other way around those trees correspond to the elements;

$$(\mu_0; 1, t) < (\mu_1; 1, t) < (t; \mu_0) < (\mu_0; 1, \mu_0) < (\mu_0; 1, \mu_1) < (\mu_1; 1, \mu_0) < (\mu_1; 1, \mu_1).$$

5.2 The Operad $\tilde{\mathcal{W}}$ in General

For a positive integer r , let (M, R) be the quadratic data pair consists of

$$M = \left(0, \bigoplus_{i=1}^r k v_i, \bigoplus_{L \subseteq T} k \mu_L, \dots \right)$$

and

$$R = \{ R_i^1, R_{i,j}^2, R_{K,L}^3, R_{i,K}^4 \mid i, j \in T \text{ and } K, L \subseteq T \} \cup \{ R_{i,K}^5 \mid i \in K \subseteq T \} \\ \cup \{ R_{i,j,K}^6 \mid j \in K \subseteq T \text{ and } i \notin K \}$$

where $T = \{1, \dots, r\}$ and for $i, j \in T$ and $K, L \subseteq T$ with

$$\begin{aligned} R_i^1: & \begin{array}{c} t_i \\ | \\ t_i \end{array} = 0, \\ R_{i,j}^2: & \begin{array}{c} t_i \\ | \\ t_j \end{array} = \begin{array}{c} t_j \\ | \\ t_i \end{array}, \\ R_{K,L}^3: & \begin{array}{c} \mu_K \\ \diagdown \\ \mu_L \end{array} = \begin{cases} \begin{array}{c} \mu_L \\ \diagup \\ \mu_K \end{array} & \text{if } L = \emptyset \text{ or } K \cap L \neq \emptyset \\ \begin{array}{c} \mu_{K \cup L} \\ \diagdown \\ \mu_0 \end{array} + \begin{array}{c} \mu_L \\ \diagup \\ \mu_K \end{array} + \begin{array}{c} \mu_L \\ \diagdown \\ \mu_{K \cup L} \end{array} & \text{if } L \neq \emptyset \text{ and } K \cap L = \emptyset, \end{cases} \\ R_{i,K}^4: & \begin{array}{c} \mu_K \\ | \\ t_i \end{array} = \begin{cases} \begin{array}{c} t_i \\ \diagup \\ \mu_K \end{array} & \text{if } i \in K \\ \begin{array}{c} t_i \\ \diagdown \\ \mu_K \end{array} + \begin{array}{c} t_i \\ \diagup \\ \mu_K \end{array} + \begin{array}{c} t_i \\ \diagdown \\ \mu_{0 \cup K \cup \{i\}} \end{array} & \text{if } i \notin K \end{cases}, \\ R_{i,K}^5: & \begin{array}{c} t_i \\ \diagdown \\ \mu_K \end{array} = 0 \quad \text{if } i \in K, \\ R_{i,j,K}^6: & \begin{array}{c} t_i \\ \diagdown \\ \mu_K \end{array} = \begin{array}{c} t_j \\ \diagdown \\ \mu_{\{K \setminus \{j\}\} \cup \{i\}} \end{array} \quad \text{if } i \notin K, j \in K. \end{aligned}$$

We define a quadratic operad and a quadratic cooperad as follows:

$$\tilde{\mathcal{W}} := \mathcal{P}(M, R) \text{ and } \tilde{\mathcal{S}} := \mathcal{C}(sM, s^2R).$$

5.3 The PBW Basis of $\tilde{\mathcal{W}}$

In order to define a PBW basis for the operad $\tilde{\mathcal{W}}$, we consider the graded path lexicographic order given in Sect. 3.6. It is straight forward to check that the quadratic part of the basis of

$\tilde{\mathcal{W}}$ is given by the following set of trees:

$$\left\{ \begin{array}{c} \text{---} t_j \\ \diagup \\ \bullet \\ \text{---} \mu_K \end{array} \middle| \begin{array}{l} j \in T \\ \text{and} \\ K \subseteq T \end{array} \right\} \cup \left\{ \begin{array}{c} \bullet \\ \text{---} \mu_K \\ \diagdown \\ t_j \end{array} \middle| \begin{array}{l} j \notin K \\ \text{and} \\ K \subseteq T \end{array} \right\} \cup \left\{ \begin{array}{c} \bullet \\ \text{---} \mu_L \\ \diagup \\ \bullet \\ \text{---} \mu_K \end{array} \middle| K, L \subseteq T \right\}.$$

Note that every composition of the above operations can be rewritten by a unique basis element.

5.4 Multiplicative Structure on Minimal Hirsch–Brown Models

Let G , X , and C be as in Sect. 4.2. Let $\Delta : C \rightarrow C \otimes C$ denote the Alexander–Whitney diagonal map. For g in G , let $g : C \rightarrow C$ denote the multiplication by g from the left. The map Δ is coassociative as in [12] and by naturality we have

$$\Delta \circ g = (g \otimes g) \circ \Delta$$

for any g in G . Hence for g in G , if $t = 1 + g$ then we have

$$\begin{aligned} \Delta \circ t &= \Delta \circ (1 + g) = (\Delta \circ 1) + (\Delta \circ g) = (1 \otimes 1) \circ \Delta + (g \otimes g) \circ \Delta \\ &= ((1 \otimes 1) + (g \otimes g)) \circ \Delta = ((t \otimes 1) + (1 \otimes t) + (t \otimes t)) \circ \Delta. \end{aligned} \quad (1)$$

Hence, we have a operad morphism defined as follows

$$\begin{aligned} \tilde{\mathcal{W}} &\longrightarrow \text{coEnd}_C \\ t_i &\rightarrow 1 + g_i \\ \mu_I &\rightarrow \left(\prod_{i \in I} (1 + g_i), 1 \right) \circ \Delta \end{aligned} \quad (2)$$

for i in T and $I \subseteq T$. By abuse of notation, we will denote the image of t_i and μ_I under this operad morphism by t_i and μ_I . Hence t_i and μ_I will be considered as operations in coEnd_C .

In the following lemma, we assume $r = 1$, and hence $T = \{1\}$. In this case instead of $g_1, t_1, \mu_\emptyset, \mu_T$ we write g, t, μ_0, μ_1 respectively.

Lemma 1 *Assume that $r = 1$. Then the operations t, μ_0 and μ_1 in coEnd_C satisfy the relations $R_1, \dots, R_8$ in Sect. 5.1.*

Proof We need to prove the following equations:

$$\begin{aligned} R_1 : (t; t) &= 0, \\ R_2 : (\mu_1; t, 1) &= 0, \\ R_3 : (\mu_0; \mu_0, 1) &= (\mu_0; 1, \mu_0), \\ R_4 : (\mu_1; \mu_1, 1) &= (\mu_1; 1, \mu_1), \\ R_5 : (t; \mu_0) &= (\mu_0; t, 1) + (\mu_0; 1, t) + (\mu_1; 1, t), \\ R_6 : (\mu_0; \mu_1, 1) &= (\mu_1; 1, \mu_0), \\ R_7 : (t; \mu_1) &= (\mu_1; 1, t), \\ R_8 : (\mu_1; \mu_0, 1) &= (\mu_0; \mu_1, 1) + (\mu_0; 1, \mu_1) + (\mu_1; 1, \mu_1). \end{aligned}$$

First notice we have $\mu_1 = ((1 + g), 1) \circ \Delta = (\mu_0; t, 1)$ in coEnd_C . We will call this equation R_0 .

The first equation $(t; t) = 0$ holds since $t = 1 + g$, $g^2 = 1$ and $(1 + g)^2 = 1 + 2g + g^2 = 0$ in characteristic 2 field. The second equation $(\mu_1; t, 1) = ((\mu_0; t, 1); t, 1) = (\mu_0; t^2, 1) = 0$

follows from R_0 and R_1 . By associativity of the operation μ_0 , we have the equation R_3 . The equation R_5 is given by the Eq. 1.

The equation R_4 follows from R_0 , R_5 , R_1 and R_3 . More precisely, one can obtain $(\mu_1; \mu_1, 1) = (\mu_0; (t; \mu_1), 1) = (\mu_0; (t; (\mu_0; t, 1)), 1)$ by R_0 . Then

$$\begin{aligned} (\mu_1; \mu_1, 1) &= (\mu_0; (\mu_0; t^2, 1), 1) + (\mu_0; (\mu_0; t, t), 1) + (\mu_0; (\mu_0; t^2, t), 1) \text{ by } R_5, \\ &= (\mu_0; (\mu_0; t, t), 1) \text{ by } R_1, \\ &= ((\mu_0; \mu_0, 1); t, t, 1), \\ &= ((\mu_0; 1, \mu_0); t, t, 1) \text{ by } R_3, \\ &= (\mu_0; t, (\mu_0; t, 1)) = (\mu_1; 1, \mu_1). \end{aligned}$$

The equation R_6 can be seen as

$$\begin{aligned} (\mu_0; \mu_1, 1) &= (\mu_0; (\mu_0; t, 1), 1) = ((\mu_0; \mu_0, 1); t, 1, 1) \text{ by } R_0; \\ &= ((\mu_0; 1, \mu_0); t, 1, 1) \text{ by } R_3, \\ &= (\mu_0; t, (\mu_0; 1, 1)), \\ &= ((\mu_0; t, 1); 1, \mu_0) = (\mu_1; 1, \mu_0). \end{aligned}$$

The equation R_7 follows from R_0 , R_5 and R_1 :

$$(t; \mu_1) = (t; (\mu_0; t, 1)) = (\mu_0; t^2, 1) + (\mu_0; t, t) + (\mu_0; t^2, t) = (\mu_0; t, t) = (\mu_1; 1, t).$$

The last equation R_8 follows from R_0 , R_5 , R_7 , R_3 and R_4 . More precisely,

$$\begin{aligned} (\mu_1; \mu_0, 1) &= ((\mu_0; t, 1); \mu_0, 1) \text{ by } R_0, \\ &= (\mu_0; (t; \mu_0), 1), \\ &= (\mu_0; (\mu_0; t, 1), 1) + (\mu_0; (\mu_0; 1, t), 1) + (\mu_0; (\mu_1; 1, t), 1) \text{ by } R_5, \\ &= (\mu_0; (\mu_0; t, 1), 1) + ((\mu_0; \mu_0, 1); 1, t, 1) + (\mu_0; (t; (\mu_0; t, 1)), 1) \text{ by } R_7, \\ &= (\mu_0; (\mu_0; t, 1), 1) + ((\mu_0; 1, \mu_0); 1, t, 1) + (\mu_0; (t; (\mu_0; t, 1)), 1) \text{ by } R_3, \\ &= (\mu_0; \mu_1, 1) + (\mu_0; 1, \mu_1) + (\mu_1; \mu_1, 1) \text{ by } R_0, \\ &= (\mu_0; \mu_1, 1) + (\mu_0; 1, \mu_1) + (\mu_1; 1, \mu_1) \text{ by } R_4. \end{aligned}$$

□

Lemma 2 *The operations $t_1, t_2, \dots, t_r$ and μ_L for $L \subseteq T = \{1, \dots, r\}$ in coEnd_C satisfy the equations $R_i^1, R_{i,j}^2, R_{K,L}^3, R_{i,K}^4, R_{i,K}^5$ and $R_{i,j,K}^6$ listed in Sect. 5.2.*

Proof In the operad coEnd_C , we have $\mu_L = \left(\mu_0; \prod_{i \in L} t_i, 1 \right)$. Hence Lemma 1 can be repeatedly applied for t_i at a time to prove this lemma. □

Note that by Eq. 2, we can consider C as a $\text{dg-}\tilde{\mathcal{W}}$ -coalgebra and hence C^* as a $\text{dg-}\tilde{\mathcal{W}}$ -algebra. We will call the $\tilde{\mathcal{S}}$ -coalgebra $B_t H(C^*)$ the minimal Hirsch-Brown model as at the end of Sect. 4.2. Now, note that the inclusion $\text{As} \hookrightarrow \tilde{\mathcal{W}}$, induces $\text{As}^i \hookrightarrow \tilde{\mathcal{S}}$, and so it induces $\tilde{\mathcal{S}}^* \twoheadrightarrow (\text{As}^i)^* = \text{As}$. We also have surjective morphism $\tilde{\mathcal{W}} \twoheadrightarrow \text{As}$ obtained by sending t_i to 0 for i in T and μ_L to 0 for $\emptyset \neq L \subseteq T$. This induces $\tilde{\mathcal{S}} \twoheadrightarrow \text{As}^i$, and so it induces $\text{As} = (\text{As}^i)^* \twoheadrightarrow \tilde{\mathcal{S}}^*$. Notice that the composition

$$\text{As} \hookrightarrow \tilde{\mathcal{S}}^* \twoheadrightarrow \text{As}$$

is the identity morphism on $\mathcal{A}s$. So this means we have a multiplicative structure on duals of these minimal Hirsh-Brown models. Unfortunately, this multiplicative structure does not have all the properties of the multiplicative structure used by Puppe, therefore we cannot repeat the proof of [14, Lemma 2.1.a] to obtain an equivalent result. We hope that results stronger than [14, Lemma 2.1.b] can be proved to tighten the bounds mentioned in Sect. 1. The reason for this hope is because we also know that the composition

$$\mathcal{S}^* \rightarrow \tilde{\mathcal{S}}^* \rightarrow \mathcal{S}^*$$

is the identity morphism on $\mathcal{S}^*$ where the first morphism is induced by sending μ_L to 0 for $L \subseteq T$ and the second morphism comes from inclusion $\mathcal{A}s \rightarrow \tilde{\mathcal{W}}$. Hence we have a $\mathcal{S}$ -module structure which is enough to prove [14, Lemma 2.1.b].

Proof of Theorem 2 Take $\mathcal{P} = \tilde{\mathcal{W}}$ where $\tilde{\mathcal{W}}$ is the operad constructed in Sect. 5.2. Hence $\mathcal{P}$ is a Koszul operad by Theorem 6 and Sect. 5.3. Notice that the Koszul dual operad of $\mathcal{P}$ is $\tilde{\mathcal{S}}^*$. Hence the other properties of $\mathcal{P}$ are proved in Sect. 5.4. $\square$

Proof of Proposition 1 Let X be a finite-dimensional free G -simplicial set. Notice that the group G acts freely on products of r many equidimensional spheres where g_i acts on the i th sphere by the antipodal action. Hence G acts freely on $\mathbb{S}^0 \times \dots \times \mathbb{S}^0$ and $\mathbb{S}^m \times \dots \times \mathbb{S}^m$ where $m = \dim(X) + 1$. Let x_0 be a point in X . First, we can define an equivariant map from $\mathbb{S}^0 \times \dots \times \mathbb{S}^0$ to X by sending the south poles $s := (-1, \dots, -1)$ to x_0 and extending the map equivariantly. Second, we can construct a map from X to $\mathbb{S}^m \times \dots \times \mathbb{S}^m$ by sending x_0 to $i_m(s)$ then extending the map by using equivariant obstruction theory; see [2, Chapter II]. Since the associated minimal models of products of equidimensional spheres are Koszul complexes, we obtain the result of the proposition by naturality of our constructions. $\square$

Proof of Proposition 2 Let $\mathcal{P}$ denote the operad in Theorem 2, m be a positive integer and $\gamma : K_r(m) \rightarrow K_r(0)$ be a S^* -coalgebra morphism which induces the same map from $H(K_r(m))$ to $H(K_r(0))$ as i_m^* does. Assume γ extends to a $\mathcal{P}^{\mathbf{i}}$ -coalgebra morphism $\tilde{\gamma} : \tilde{K}_r(m) \rightarrow \tilde{K}_r(0)$ whose restriction to $\mathcal{P}^{\mathbf{i}} \circ 1$ is induced by the identity on $\mathcal{P}^{\mathbf{i}}$. Set $T = \{1, \dots, r\}$. For $U \subseteq T$ and $n \in \{0, m\}$, let $z_U^{(n)}$ denote $\prod_{i \in U} z_i^{(n)}$ and x_U denote $\prod_{i \in U} x_i^m$. Since γ induces the

same map from $H(K_r(m))$ to $H(K_r(0))$ as i_m^* does, γ^* sends $[1 \otimes z_T^{(0)}]$ to $[x_T \otimes z_T^{(m)}]$. Since γ extends to $\mathcal{P}^{\mathbf{i}}$ -coalgebra map whose restriction to $\mathcal{P}^{\mathbf{i}} \circ 1$ is induced by the identity on $\mathcal{P}^{\mathbf{i}}$, we can say that γ^* sends $x_U \otimes z_{T-U}^{(0)}$ to $x_T \otimes z_{T-U}^{(m)} + e$ where e contains only terms in the form $f \otimes z_L$ such that $T - U$ is a proper subset of L and L is a subset of T . Hence $F \otimes \gamma^*$ sends $z_\emptyset, z_{\{1\}}, \dots, z_{\{r\}}, z_{\{1,2\}}, \dots, z_{\{1,r\}}$ to linearly independent vectors in $F \otimes_S (K_r(m))^*$. Then the linear map $F \otimes_S \gamma^*$ has rank at least $2r$ where F denotes the field of fractions of the ring S . $\square$

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Data Availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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