SPT-FUNCTION AND ITS PROPERTIES

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By

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July, 2014

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

SPT-FUNCTION AND ITS PROPERTIES

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In this thesis, we survey some properties of the *spt*-function. We start with providing some background information for q-series and the partition function $p(n)$. Then we define the *spt*-function and study its generating function. Our aim is to prove parity result for the *spt*-function. We also obtain a congruence relation between *spt*-function and a certain mock theta function.

Keywords: Partition Theory, q-series, *spt*-function.

ÖZET

SPT-FONKSİYONU VE ÖZELLİKLERİ

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Bu tezde, *spt*-fonksiyonu ve onun bazı özellikleri üzerine çalışacağız. Öncelikle q-serileri ve bölüşüm fonksiyonuna değineceğiz. Daha sonra *spt*-fonksiyonu ve bu fonksiyonun üretici fonksiyonunu inceleyeceğiz. Amacımız *spt*-fonksiyonunun bir parite özelliğini ispatlamaktır. Ayrıca *spt*-fonksiyonunun bir mock teta fonksiyonu ile kongruans ilişkisini de ispatlayacağız.

Anahtar sözcükler: Bölüşüm Teorisi, *spt*-fonksiyonu, q-Serileri.

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Chapter 1

Introduction

We define a partition λ of a natural number n as a set of positive integers such that sum of these positive integers is equal to n. A partition λ is denoted by $\lambda = \lambda_1 + \lambda_2 + \cdots \lambda_n$, λ_i for $i = 1, 2, \cdots, n$ is called a part of the partition λ. As an elementary illustration, 3 has 3 different partitions:
3,
2+1,
1+1+1.

In 2008, George Andrews introduced a partition function which he called the smallest part function denoted by *spt*-function. He defined $spt(n)$ in [2] by the sum of the total appearances of the smallest part in each partition of n. Throughout this thesis, we survey properties of the *spt*-function.

The thesis is organized as follows. Chapter 2 begins with introducing q-series and presenting properties of the partition function $p(n)$. We prove some identities which are helpful for further results. We focus on the smallest part function denoted by *spt*-function and state its congruence properties. At the end of this chapter, a related partition statistics called the rank of a partition is also introduced to give some properties of the generating function of the *spt*-function.

In Chapter 3, we define and survey some properties of vector partitions. We study some concepts of vector partitions in order to give the relation of these with the *spt*-function and its generating function.

In Chapter 4, we begin with self-conjugate partitions and present parity of the generating function for self-conjugate partitions. We need self-conjugate partitions to obtain parity results for the *spt*-function. Lastly, we obtain a congruence relation modulo 4 between *spt*-function and a certain mock theta function.

Chapter 2

Introduction to Partition Theory

In this chapter, we are concerned with q-series and partition theory. At the begining, we present some definitions and emphasize some important techniques that are used throughout this thesis such as taking limits of q-series. Chapter 2 continues with the smallest part function and properties of this function. We finish this chapter by defining the rank of a partition and stating its congruence properties. The content of this chapter is mainly taken from [1], [9], [10] and [11].

2.1 q-Series and The Partition Function $p(n)$

After mentioning the definition of a partition in the previous chapter, we restrict our attention to q-series and the partition function $p(n)$ which counts number of partitions of any nonnegative integer n.

We start with definition of q-series. We assume that q is a complex number with $|q| < 1$.
We adopt the usual notation.

Definition 2.1.1. Define

$$(a)_0 := (a;q)_0 := 1 \ , \ (a)_n = (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k) \quad n \geq 1,$$

$$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty} (1-aq^k)\ .$$

For $-\infty < n < 0$, we define $(a;q)_n$ by

$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

We continue by definition of the partition function $p(n)$.

Definition 2.1.2. For any positive integer n, we define *the partition function* $p(n)$ as the number of unrestricted partitions of n. For convenience, we define $p(0) = 1$.

The generating function for $p(n)$ is given by

Theorem 2.1.3. *[1]*

$$\sum_{n=0}^{\infty} p(n)q^j = \prod_{i=1}^{\infty} (1-q^i)^{-1} = \frac{1}{(q;q)_\infty}.$$

Proof. [1] Observe the following geometric series

$$\frac{1}{1-q^k} = \sum_{i=0}^{\infty} q^{ik}.$$

Therefore, by arguing informally,

$$\begin{aligned}\prod_{i=0}^{\infty} \frac{1}{1-q^i} =&(1+q+q^2+\cdots)(1+q^2+q^4+\cdots)\cdots(1+q^i+q^{2i}+\cdots)\cdots\\ =&\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\sum_{a_3=0}^{\infty}\cdots q^{1a_1+2a_2+3a_3+\cdots}.\end{aligned}$$

We see that power of q gives us a partition λ such that

$$\lambda = \overbrace{1+\cdots+1}^{a_1-many}+\overbrace{2+\cdots+2}^{a_2-many}+\overbrace{3+\cdots+3}^{a_3-many}\cdots$$

Hence the coefficients of q^n in the q-series expansion of $\frac{1}{(q;q)_\infty}$ generates all partitions of the natural number n.
Next, we give a formal proof.
For this aim, let h_n be a sequence such that

$$h_n = \prod_{i=1}^{n} (1 - q^i) \quad n \geq 1.$$

Since h_n contains finitely many absolutely convergent series. Therefore h_n is itself absolutely convergent. For simplicity, let q be a real number such that $0 < q < 1$. Thus,

$$\sum_{j=0}^{n} p(j)q^j \leq \prod_{i=1}^{n} (1 - q^i)^{-1} \leq \prod_{i=1}^{\infty} (1 - q^i)^{-1} < \infty \ .$$

Hence, $\sum_{j=0}^{n} p(j)q^j$ is a bounded increasing sequence in $\mathbb{R}$. Therefore the series

$$\sum_{j=0}^{\infty} p(j)q^j$$

must converge by Monotone Convergence Theorem.
On the other hand,

$$\sum_{j=0}^{\infty} p(j)q^j \geq \prod_{i=1}^{n} (1 - q^i)^{-1} \to \prod_{i=1}^{\infty} (1 - q^i)^{-1} \quad as \quad n \to \infty \ .$$

Thus,

$$\sum_{n=0}^{\infty} p(n)q^j = \prod_{i=1}^{\infty} (1 - q^i)^{-1} = \frac{1}{(q;q)_\infty}$$

as desired. □

An immediate corollary of Theorem 2.1.3 can be given by

Corollary 2.1.4. *[1] Let A be a non-empty subset of $\mathbb{N}$ and let $p_A(n)$ be the number of partitions of n such that every part of the partition of n is an element of A. Then,*

$$\sum_{n\geq 1} p_A(n)q^n = \prod_{n\in A} \frac{1}{1 - q^n}.$$

Observe that the generating function for the number of partitions of n with distinct parts, namely $p_d(n)$, is given by

$$\sum_{n=0}^{\infty} p_d(n)q^n = (-q;q)_\infty = (1+q)(1+q^2)(1+q^3)\cdots(1+q^k)\cdots \ .$$

We are now ready to state famous theorem of Euler. Proving this theorem helps us to understand importance of series manipulation which is one of the basic tools for proving any result about q-series.

Theorem 2.1.5. *(Euler)[11, p.4] The number of partitions of positive integer n into distinct parts equals the number of partitions of n into odd parts, denoted by $p_0(n)$.*

Proof. [11] As we have just stated above, the generating function for $p_d(n)$ is given by

$$\sum_{n=0}^{\infty} p_d(n)q^n = (-q;q)_\infty = (1+q)(1+q^2)(1+q^3)\cdots(1+q^k)\cdots \ . \tag{2.1}$$

Multiply and divide (2.1) by $(q;q)_\infty$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} p_d(n)q^n &= \frac{(1+q)(1-q)(1+q^2)(1-q^2)...(1-q^k)(1+q^k)\cdots}{(1-q)(1-q^2)\cdots(1-q^{2k-1})(1-q^{2k})(1-q^{2k+1})\cdots} \\
&= \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots(1-q^{2k-1})\cdots} \\
&= \sum_{n=0}^{\infty} p_0(n)q^n.
\end{aligned}$$

$\square$

This section continues by congruence properties of $p(n)$. We begin with the following definition.

Definition 2.1.6. Let $f(q) = \sum a_n q^n$ and $g(q) = \sum b_n q^n$ be power series in q. If $a_n \equiv b_n \pmod{m}$ for every integer n, we say that

$$f(q) \equiv g(q) \pmod{m}.$$

The most celebrated congruence relation in the theory of q-series are Ramanujan's relation for the partition function $p(n)$. These congruences stated in [19], [20] are

$$p(5n+4) \equiv 0 \pmod 5, \tag{2.2}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{2.3}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{2.4}$$

(2.2) and (2.3) were proved by Ramanujan in [19]. Later, Ramanujan mentioned that he also found a proof for (2.4) in [21]. In addition, Winquist gave the most elementary proof for (2.4) in [25]. In their most general Ramanujan's congruence relations can be stated by;
For $\alpha \geq 1$;

$$p(5^{\alpha}n + \delta_{5,\alpha}) \equiv 0 \pmod{5^{\alpha}}, \tag{2.5}$$

$$p(7^{\alpha}n + \delta_{7,\alpha}) \equiv 0 \pmod{7^{(\alpha+2)/2}}, \tag{2.6}$$

$$p(11^{\alpha}n + \delta_{11,\alpha}) \equiv 0 \pmod{11^{\alpha}} \tag{2.7}$$

where $\delta_{t,\alpha}$ is the reciprocal modulo t^{α} of 24. (2.5) and (2.6) were proved by G.N. Watson [24] in 1938 and (2.7) was proved by A.O.L. Atkin [9] in 1967.

We emphasize a fundamental theorem about q-series.

Theorem 2.1.7. *(q-analogue of the binomial theorem)[11, p.8] For $|q|,|z| < 1$,*

$$\sum_{n=0}^{\infty} \frac{(a)_n z^n}{(q)_n} = \frac{(az)_{\infty}}{(z)_{\infty}}.$$

It is important to prove the following corollary of Theorem 2.1.7 in order to see usefulness of taking limit in q-series.

Corollary 2.1.8. *[11, p.9] For* $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty} \quad |z| < 1 \tag{2.8}$$

and

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{\frac{n(n-1)}{2}}}{(q)_n} = (z)_\infty \quad |z| < \infty \ . \tag{2.9}$$

Proof. [11] We can prove (2.8) by letting $a = 0$ in Theorem 2.1.7.
For (2.9), write a/b instead of a and write bz instead of z in Theorem 2.1.7. Then we have for $|bz| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a/b)_n (bz)^n}{(q)_n} = \frac{(az)_\infty}{(bz)_\infty}. \tag{2.10}$$

Letting $b \to 0$, we get

$$\lim_{b\to 0} (a/b)_n b^n = \lim_{b\to 0} \left(1 - \frac{a}{b}\right)\left(1 - \frac{aq}{b}\right)\cdots\left(1 - \frac{aq^{n-1}}{b}\right) b^n = (-a)^n q^{\frac{n(n-1)}{2}}.$$

Next, we put $a = 1$ and let $b \to 0$ in (2.10) we have

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q)_n} = (z)_\infty.$$

Since we are taking limits of an infinite series, we should justify our calculations. In other words, we have to show that

$$\sum_{n=0}^{\infty} \frac{(a/b)_n (bz)^n}{(q)_n}$$

converges uniformly when $b \to 0$. For this aim, choose a real number M such that $|q| < M < 1$. Let ϵ be given such that $0 < 2\epsilon < 1 - M$ and $|b| \le \epsilon$ be fixed. Let N_0 be the unique positive integer such that

$$|b| + |a|M^k \ge 2\epsilon \quad for \quad 0 \le k \le N_0$$

and

$$|b| + |a|M^{N_0} < 2\epsilon.$$

Thus,

$$\begin{aligned}
\left| \frac{(a/b)_n b^n}{(q)_n} \right| &= \left| \frac{\left(1 - \frac{a}{b}\right)\left(1 - \frac{aq}{b}\right) \cdots \left(1 - \frac{aq^{n-1}}{b}\right) b^n}{(q)_n} \right| \\
&= \left| \frac{(b-a)(b-aq)\cdots(b-aq^{n-1})}{(q)_n} \right| \\
&\leq \frac{(|b|+|a|)(|b|+|a|M)\cdots(|b|+|a|M^{n-1})}{(1-M)^n} \\
&\leq \frac{(2\epsilon)^{n-N_0}(\epsilon+|a|)^{N_0}}{(1-M)^n} \\
&= \left(\frac{\epsilon+|a|}{2\epsilon}\right)^{N_0} \left(\frac{2\epsilon}{1-M}\right)^n .
\end{aligned}$$

Since $\epsilon > 0$ such that $2\epsilon < 1 - M$. Therefore,

$$\sum_{n=0}^{\infty} \left(\frac{2\epsilon}{1-M}\right)^n < \infty \ .$$

We can conclude that

$$\sum_{n=0}^{\infty} \frac{(a/b)_n (bz)^n}{(q)_n}$$

absolutely converges by Weierstrass M-Test when $b \to 0$. In other words, we are allowed to take limits under summation sign in q-series. $\square$

Next we introduce a famous theorem called Jacobi Triple Product Identity. It was introduced by German mathematician Jacobi [18] in 1829. For brevity, we use q-series notation of this theorem.

Theorem 2.1.9. *(Jacobi Triple Product Identıty)[18] For $z \neq 0$ and $|q| < 1$,*

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty \ . \tag{2.11}$$

A fundamental theorem which is an immediate consequence of Theorem 2.1.9 is given by

Theorem 2.1.10. *(Jacobi's Identity)[11]*

$$\sum_{n=0}^{\infty} (-1)^n(2n+1)q^{n(n+1)/2} = (q;q)_\infty^3 \ .$$

Proof. [11] By replacing z by z^2q in (2.11), we find that

$$\sum_{n=-\infty}^{\infty} z^{2n}q^{n^2+n} = (-z^2q^2;q^2)_\infty(-1/z^2;q^2)_\infty(q^2;q^2)_\infty \ . \tag{2.12}$$

By dividing both sides of (2.12) by $1+1/z^2$ we have

$$\frac{\sum_{n=-\infty}^{\infty} z^{2n+1}q^{n^2+n}}{z+1/z} = (-z^2q^2;q^2)_\infty(-q^2/z^2;q^2)_\infty(q^2;q^2)_\infty. \tag{2.13}$$

On the other hand it is easy to observe that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n} = 0. \tag{2.14}$$

We take limit of both sides of (2.13) as $z \to i$ to deduce that

$$\begin{aligned} \lim_{z\to i} \frac{\sum_{n=-\infty}^{\infty} z^{2n+1}q^{n^2+n}}{z+1/z} =& \frac{\sum_{n=-\infty}^{\infty} (-1)^n(2n+1)q^{n^2+n}}{2} \\ & \text{(by (2.14) and L'Hospital's Rule)} \\ =& \lim_{z\to i} (-z^2q^2;q^2)_\infty(-q^2/z^2;q^2)_\infty(q^2;q^2)_\infty \\ =& (q^2;q^2)_\infty(q^2;q^2)_\infty(q^2;q^2)_\infty. \end{aligned} \tag{2.15}$$

Hence, we conclude from (2.15) that

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} (-1)^n(2n+1)q^{n(n+1)} = (q^2;q^2)_\infty^3 \ . \tag{2.16}$$

If we divide the infinite sum in (2.16) into two parts, it is easy to see that

$$\sum_{n=-\infty}^{-1} (-1)^n(2n+1)q^{n(n+1)} = \sum_{n=0}^{\infty} (-1)^n(2n+1)q^{n(n+1)}. \tag{2.17}$$

Finally, replacing q^2 by q in (2.16) and using (2.17), we have

$$\sum_{n=0}^{\infty} (-1)^n(2n+1)q^{n(n+1)/2} = (q;q)_\infty^3.$$

□

From now on, we are concerned with smallest part function and its properties.

2.2 The *spt*-Function and its Congruence Properties

In this section, we begin with recalling definition of the smallest part function. Our main goal is to emphasize interesting congruence properties and the generating function of the smallest part function.

Let n be any nonnegative integer. The smallest part function is defined by the sum of the total appearances of the smallest part in each partition of n. *The smallest part function* is called as *spt*-function.

We present an example to clarify the definition of $spt(n)$.

Examples 2.2.1. The partitions of 5 are 5, 4+1, 3+2, 3+1+1, 2+1+1+1, 2+2+1, 1+1+1+1+1. The smallest parts of these partitions appear 1, 1, 1, 2, 3, 1, 5 times respectively. Therefore, sum of appearances of smallest parts is 14. That is, $spt(5) = 14$.

The generating function of $spt(n)$ is given by [2]

$$\sum_{n=1}^{\infty} spt(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty}. \tag{2.18}$$

Note that, we can observe the generating function of the *spt*-function by investigating an arbitray term of the right hand side of (2.18). First we know that

$$\sum_{m=0}^{\infty} (m+1)q^{nm} = \frac{1}{(1-q^n)^2}\ . \tag{2.19}$$

If we put (2.19) into (2.18), we have the following;

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} mq^{nm} \prod_{k=n+1}^{\infty} \frac{1}{(1-q^k)}\ . \tag{2.20}$$

Thus an arbitrary term of (2.20) is as follows:

$$mq^{\overbrace{n+\cdots+n}^{m-many}}(1+q^{n+1}+q^{(n+1)+(n+1)}+\cdots)(1+q^{n+2}+q^{(n+2)+(n+2)}+\cdots)\cdots \tag{2.21}$$

A typical term in (2.21) is

$$mq^{\overbrace{n+\cdots+n}^{m-many}+(n+1)+(n+1)+(n+2)}.$$

It is easy to observe that coefficients of the second sum in the right hand side of (2.20) give how many times the smallest part n appears in a partition which is generated by (2.21). On the other hand, the product in (2.20) provides remaining parts of the partition which contain parts bigger than n. Therefore, right hand side of (2.18) gives generating function of the $spt(n)$ for integer $n \geq 1$.

A fundamental theorem about *spt*-function which was proved by George Andrews in [2] is given by

Theorem 2.2.2. *[2]*

$$\sum_{n\geq 1} spt(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{1}{(q,q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(3n+1)}{2}}(1+q^n)}{(1-q^n)^2}\ . \tag{2.22}$$

Before proving this theorem we state two lemmas and a theorem in order to use them in the proof of Theorem 2.2.2.

Lemma 2.2.3. *[2] Let $f(z)$ be any function which is at least twice differentiable at $z = 1$. Then,*

$$-\frac{1}{2}\left[\frac{d^2}{dz^2}(1-z)(1-z^{-1})f(z)\right]_{z=1} = f(1).$$

Next lemma is concerned with differentiation of q-series.

Lemma 2.2.4. *[2] For $|q| < 1$,*

$$-\frac{1}{2}\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty\right]_{z=1} = (q;q)_\infty^2 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}\ .$$

Proof. [2] Replace z by $-zq^{-1/2}$ and q by $q^{1/2}$ in Theorem 2.1.9. We have

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{n(n-1)/2} = (z;q)_\infty(q/z;q)_\infty(q;q)_\infty\ . \tag{2.23}$$

Note that the formula for finite sum of geometric series is given by

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}. \tag{2.24}$$

Therefore,

$$\begin{aligned}
&-\frac{1}{2}\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty\right]_{z=1} \\
&= -\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\frac{\sum_{n=-\infty}^{\infty}(-z)^n q^{n(n-1)/2}}{1-z}\right]_{z=1} \quad \text{(by (2.23))} \\
&= -\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\sum_{n=0}^{\infty}(-z)^{-n}q^{n(n+1)/2}\left(\frac{1-z^{2n+1}}{1-z}\right)\right]_{z=1} \qquad (2.25) \\
&= -\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sum_{j=0}^{2n} z^{-n+j}\right]_{z=1} \quad \text{(by (2.24))} \\
&= -\frac{1}{2(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sum_{j=0}^{2n}(-n+j)(-n+j-1).
\end{aligned}$$

It is easy to see that,

$$\sum_{j=0}^{2n}(-n+j)(-n+j-1)=\sum_{j=-n}^{n}j^2-\sum_{j=-n}^{n}j=\sum_{j=-n}^{n}j^2=\frac{n(n+1)(2n+1)}{3}\ . \tag{2.26}$$

Thus

$$\begin{aligned}
-\frac{1}{2}\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty\right]_{z=1} &= -\frac{1}{2(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\frac{1}{3}n(n+1)(2n+1)\\
&\quad \text{(by (2.25) and (2.26))}\\
&= -\frac{q}{3(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}-1}\frac{n(n+1)}{2}(2n+1)\\
&= -\frac{q}{3(q;q)_\infty}\frac{d}{dq}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}\\
&= -\frac{q}{3(q;q)_\infty}\frac{d}{dq}(q;q)_\infty^3 \quad \text{(by Theorem 2.1.10).}
\end{aligned}$$

By using logarithmic differentiation, we can see that

$$\frac{d}{dq}(q;q)_\infty^3=-3(q;q)_\infty^3\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^n}\ .$$

Therefore,

$$\begin{aligned}
-\frac{1}{2}\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty\right]_{z=1} &= -\frac{q}{3(q;q)_\infty}\frac{d}{dq}(q;q)_\infty^3\\
&=(q;q)_\infty^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.
\end{aligned}$$

□

Before stating our next theorem, we need a definition.

Definition 2.2.5. *The Heine's series* or *the basic hypergeometric series* is denoted by

$${}_m\phi_n\left(\begin{array}{c}a_1,a_2,\cdots,a_m;q,z\\ b_1,b_2,\cdots,b_n\end{array}\right):=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_m)_k z^k}{(b_1)_k(b_2)_k\cdots(b_n)_k(q)_k}\ .$$

where $|z| < 1$, $|q| < 1$ and $b_i \neq q^{-n}$ for any nonnegative integer n.

Next we state an important theorem about hypergeometric series.

Theorem 2.2.6. *(q-analog of Whipple's Theorem)[16]*

$$ {}_8\phi_7 \left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-N}; q, \frac{a^2q^{2+N}}{bcde} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, aq^{N+1} \end{matrix} \right) = \frac{(aq;q)_N(\frac{aq}{de};q)_N}{(\frac{aq}{d};q)_N(\frac{aq}{e};q)_N} {}_4\phi_3 \left(\begin{matrix} \frac{aq}{bc}, d, e, q^{-N}; q, q \\ \frac{aq}{b}, \frac{aq}{c}, \frac{deq^{-N}}{a} \end{matrix} \right). $$

Proof. See [16], page 42. □

Now we are ready to give the proof of Theorem 2.2.2.

Proof. [2](Proof of Theorem 2.2.2) Let

$$ f(z) = \sum_{n=0}^{\infty} \frac{(z;q)_n(z^{-1};q)_n q^n}{(1-z)(1-z^{-1})(q;q)_n}. $$

We know that

$$ (q;q)_n = \frac{(q;q)_\infty}{(q^{n+1};q)_\infty} \tag{2.27} $$

for any $n \geq 1$. Therefore

$$
\begin{aligned}
&-\frac{1}{2}\left[\frac{d^2}{dz^2}\left(\sum_{n=0}^{\infty}\frac{(z;q)_n(z^{-1};q)_nq^n}{(q;q)_n}\right)\right]_{z=1}\\
=&-\frac{1}{2}\left[\frac{d^2}{dz^2}\left(1+(1-z)(1-z^{-1})\sum_{n=1}^{\infty}\frac{(z;q)_n(z^{-1};q)_nq^n}{(q;q)_n(1-z)(1-z^{-1})}\right)\right]_{z=1}\\
=&-\frac{1}{2}\left[\frac{d^2}{dz^2}(1-z)(1-z^{-1})\left(\sum_{n=1}^{\infty}\frac{(z;q)_n(z^{-1};q)_nq^n}{(q;q)_n(1-z)(1-z^{-1})}\right)\right]_{z=1}\\
=&\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}q^n}{(1-q^n)} \quad \text{(by Lemma 2.2.3)}\\
=&\sum_{n=1}^{\infty}\frac{(q;q)_nq^n}{(1-q^n)^2}\\
=&(q;q)_\infty\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty} \quad \text{(by (2.27))}\ .
\end{aligned}
\tag{2.28}
$$

From (2.28) we see that

$$
-\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\left(\sum_{n=0}^{\infty}\frac{(z;q)_n(z^{-1};q)_nq^n}{(q;q)_n}\right)\right]_{z=1}=\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty}\ .
\tag{2.29}
$$

On the other hand, by setting $d=e^{-1}=z$, and then letting $b,c,N\to\infty$ and $a\to 1$ in Theorem 2.2.6, we have

$$
\sum_{n=0}^{\infty}\frac{(z;q)_n(z^{-1};q)_nq^n}{(q;q)_n}=\frac{(zq;q)_\infty(z^{-1}q;q)_\infty}{(q;q)_\infty^2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^nq^{\frac{n(3n+1)}{2}}(1+q^n)(z;q)_n(z^{-1};q)_n}{(q/z;q)_n(zq;q)_n}\right).
\tag{2.30}
$$

Note that

$$
\begin{aligned}
\sum_{n\geq 1} spt(n)q^n =&\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty} \quad \text{(by (2.18))}\\
=&-\frac{1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\left(\sum_{n=0}^{\infty}\frac{(z;q)_n(z^{-1};q)_nq^n}{(q;q)_n}\right)\right]_{z=1} \quad \text{(by (2.29))}.
\end{aligned}
\tag{2.31}
$$

Therefore,

$$\begin{aligned}
&\sum_{n\geq 1} spt(n)q^n\\
=&\frac{-1}{2(q;q)_\infty}\left[\frac{d^2}{dz^2}\frac{(zq;q)_\infty(z^{-1}q;q)_\infty}{(q;q)_\infty^2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(3n+1)}{2}}(1+q^n)(z)_n(z^{-1})_n}{(q/z)_n(zq)_n}\right)\right]_{z=1}\\
&\text{(by (2.30) and (2.31))}\\
=&\frac{1}{(q,q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+\frac{1}{(q,q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(3n+1)}{2}}(1+q^n)}{(1-q^n)^2}\\
&\text{(by Lemma 2.2.3 and 2.2.4).}
\end{aligned}$$

□

In [2], Andrews also gave congruence properties of *spt*-function which are very similar to congruence properties of $p(n)$.

Theorem 2.2.7. *[2]*

$$spt(5n+4) \equiv 0 \pmod 5,$$

$$spt(7n+5) \equiv 0 \pmod 7,$$

$$spt(13n+6) \equiv 0 \pmod{13}.$$

Proof. See [2], page 8. □

Examples 2.2.8. The partitions of 4 can be stated as follows;
4,
3+1,
2+2,
2+1+1,
1+1+1+1.
Therefore, $spt(4) = 10 \equiv 0 \pmod 5$.

Next, we continue by defining rank of a partition. The concept of our next section was given by F. J. Dyson [12].

2.3 The Rank of a Partition

Our aim in this section is to introduce congruence properties and generating function of the rank of a partition. We begin with presenting a definition.

Definition 2.3.1. Let λ be a partition. *The rank of a partition* λ is defined as largest part of λ minus number of parts of λ.

Examples 2.3.2. A partition $5+3+1+1$ of 10 has rank $5-4=1$.

Let $N(m,n)$ denote the number of partitions of n with rank m. We give the generating function of $N(m,n)$ by [17]

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}\ . \tag{2.32}$$

Another description for the left hand side of (2.32) can be stated as follows.

Proposition 2.3.3. *[3]*

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)}\ .$$

Proof. [3] We know that

$$z = re^{i\theta} = rcos(\theta) + risin(\theta)$$

where r is modulus of z and angle θ is argument of z. Therefore

$$cos(\theta) = \frac{z+z^{-1}}{2}\ . \tag{2.33}$$

Watson [23] gave following equation by using Theorem 2.2.6 so that

$$\begin{aligned} &1+\sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)(2-2cos(\theta))q^{n(3n+1)/2}}{1-2q^n cos(\theta)+q^{2n}} \\ =&\prod_{r=1}^{\infty}(1-q^r)\left[1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{\prod_{m=1}^{n}(1-2q^m cos(\theta)+q^{2m})}\right]. \end{aligned} \tag{2.34}$$

Hence

$$
\begin{aligned}
&1+\sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)(2-2cos(\theta))q^{n(3n+1)/2}}{1-2q^n cos(\theta)+q^{2n}} \\
=&1+\sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)(2-z-z^{-1})q^{n(3n+1)/2}}{1-q^n(z+z^{-1})+q^{2n}} \quad \text{(by (2.33))} \\
=&1+\sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)(1-z)(1-z^{-1})q^{n(3n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)} \\
=&\prod_{r=1}^{\infty}(1-q^r)\left[1+\sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{m=1}^{n}(1-zq^m)(1-z^{-1}q^m)}\right] \quad \text{(by (2.34))} \\
=&(q)_\infty \left[1+\sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}\right] \\
=&(q)_\infty \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n \quad \text{(by (2.32))}.
\end{aligned} \tag{2.35}
$$

Therefore, we obtain from (2.35) and after some elementary manipulations that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n =&\frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)(1-z)(1-z^{-1})q^{n(3n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}\right) \\
=&\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)}.
\end{aligned}
$$

□

After Dyson defined the rank of a partition, he left following congruence properties of the rank of a partition as conjectures in [12]. These were proven by Atkin and Swinnerton-Dyer in [10]. Their methods for proving Dyson's conjectures are heavily dependent on the theory of modular forms.

Theorem 2.3.4. *[10] Let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. Then we have,*

$$N(k,5,5n+4) = \frac{p(5n+4)}{5} \quad for \quad 0 \le k \le 4,$$

$$N(k,7,7n+5) = \frac{p(7n+5)}{7} \quad for \quad 0 \le k \le 6.$$

Let us consider partitions of 4 and rank of these partitions as follows:

$$\begin{aligned}
&\lambda_1 = 4 \quad rank \;\; of \;\; \lambda_1 = 4 - 1 = 3\\
&\lambda_2 = 3 + 1 \quad rank \;\; of \;\; \lambda_2 = 3 - 2 = 1\\
&\lambda_3 = 2 + 2 \quad rank \;\; of \;\; \lambda_3 = 2 - 2 = 0\\
&\lambda_4 = 2 + 1 + 1 \quad rank \;\; of \;\; \lambda_4 = 2 - 3 = -1\\
&\lambda_5 = 1 + 1 + 1 + 1 \quad rank \;\; of \;\; \lambda_5 = 1 - 4 = -3.
\end{aligned} \tag{2.36}$$

As an example for $n = 0$ in Theorem 2.3.4 we see from (2.36) that

$$N(0, 5, 4) = N(1, 5, 4) = \cdots = N(4, 5, 4) = \frac{p(4)}{5} = 1.$$

Chapter 3

Vector Partitions

In this chapter, we introduce vector partitions. At the beginning, we give construction of vector partitions and clarify our subject by giving an example. Later, we work on a sum $N_V(m, n)$ which is defined in a set V and state its congruence properties. The final topic in Chapter 3 is vector partitions over a subset S of V and a sum $N_S(m, n)$ defined in S. The content of this chapter is mainly based on [7] and [14].

3.1 Introduction to Vector Partitions

Let π be a partition. We denote number of parts of π as $\#(\pi)$ and sum of parts of π as $\sigma(\pi)$. Note that $\#(\emptyset) = \sigma(\emptyset) = 0$ for the empty partition $\emptyset$ of 0.

Let $\mathcal{D}$ be the set of partitions into distinct parts and let $\mathcal{P}$ be the set of partitions with unrestricted parts. A set V is defined so that

$$V := \mathcal{D} \times \mathcal{P} \times \mathcal{P}.$$

In other words,

$$V = \Big\{(\pi_1, \pi_2, \pi_3)| \quad \pi_1 \text{ is a partition with distinct parts}$$
$$\pi_2 \text{ and } \pi_3 \text{ are partitions with unrestricted parts}\Big\} .$$

We call elements of V as *vector partitions*. For $\pi = (\pi_1, \pi_2, \pi_3)$ in V, define sum of parts s, weight ω, and crank r by,

$$s(\pi) := \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3) \ ,$$

$$\omega(\pi) := (-1)^{\#(\pi_1)} \ ,$$

$$r(\pi) := \#(\pi_2) - \#(\pi_3).$$

If $s(\pi) = n$ then we call π as a vector partition of n.

Examples 3.1.1. Let $\pi = (4+5, 4+3+1+1, 2+2)$. Then $s(\pi) = 22$, $\omega(\pi) = (-1)^2 = 1$ and $r(\pi) = 4 - 2 = 2$. Note that, π is a vector partition of 22.

Next, we restrict our attention to introducing a sum $N_V(m,n)$ which counts weight of vector partitions of n with crank m.

3.2 Properties of the Sum $N_V(m, n)$

In this section, our aim is to present basic properties of the sum $N_V(m,n)$ and its congruence properties. We begin with some notations and definitions.

We denote the number of vector partitions of n with crank m counted according to weight ω as $N_V(m,n)$ such that

$$N_V(m,n) := \sum_{\substack{\pi \in V \\ s(\pi)=n \\ r(\pi)=m}} \omega(\pi).$$

Let $N_V(k,t,n)$ be the number of vector partitions of n with crank congruent to k modulo t counted according to weight ω. Thus,

$$N_V(k,t,n) := \sum_{m=-\infty}^{\infty} N_V(mt+k,n) := \sum_{\substack{\pi\in V \\ s(\pi)=n \\ r(\pi)\equiv k \pmod t}} \omega(\pi).$$

Weight of a vector partition $\pi = (\pi_1, \pi_2, \pi_3)$ only depends on the number of parts of π_1. Therefore transforming partitions π_2 and π_3 does not effect $N_V(m,n)$. Hence,

$$N_V(m,n) = N_V(-m,n). \tag{3.1}$$

Since $t - m \equiv -m \pmod t$. From (3.1), we can readily say that,

$$N_V(t-m,t,n) = N_V(m,t,n).$$

The generating function for $N_V(m,n)$ is given by [14]

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_V(m,n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)} = \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty}. \tag{3.2}$$

Note that, we can observe the generating function of $N_V(m,n)$ by investigating an arbitray exponent in the product (3.2) given by

$$\overbrace{(1-q^1)(1-q^2)(1-q^3)...}^{Part-1} \overbrace{(1+zq^1+z^2q^{1+1}+\cdots)...(1+z^{-1}q^1+z^{-2}q^{1+1}+\cdots)...}^{Part-2} \tag{3.3}$$

Let $\pi = (\pi_1, \pi_2, \pi_3)$. Part-1 in (3.3) generates partitions into distinct parts with minus sign for partitions containing odd number of parts and plus sign for partitions containing even number of parts. Choose a partition $\pi_1 = \pi_1^1 + \pi_1^2 + \cdots + \pi_1^n$ among partitions generated by Part-1. Then, the coefficient of $q^{\pi_1^1+\pi_1^2+\cdots+\pi_1^n}$ in Part-1 gives $\omega(\pi)$.

On the other hand, Part-2 gives difference of number of parts of two partitions. That is, if we choose $\pi_2 = \pi_2^1 + \pi_2^2 + \cdots + \pi_2^k$ and $\pi_3 = \pi_3^1 + \pi_3^2 + \cdots + \pi_3^m$ among partitions generated by $\frac{1}{(zq;q)_\infty}$ and $\frac{1}{(z^{-1}q;q)_\infty}$ respectively, we see that power of z in the term from Part-2

$$z^{k-m}q^{\pi_2^1+\pi_2^2+\cdots+\pi_2^k+\pi_3^1+\pi_3^2+\cdots+\pi_3^m}$$

gives the difference of number of parts of π_2 and π_3 which is defined as $r(\pi)$.

Another description for the generating function (3.2) of $N_V(m,n)$ can be given by

Lemma 3.2.1. *[14]*

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_V(m,n)z^m q^n = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)}.$$

Proof. [14] In order to prove this lemma, use the limiting form of Jackson's theorem ([22]). This theorem can be stated as

$$\begin{aligned}&{}_6\phi_5\left(\begin{matrix} z, q\sqrt{z}, -q\sqrt{z}, a_1, a_2, a_3; q, \frac{zq}{a_1a_2a_3} \\ \sqrt{z}, -\sqrt{z}, \frac{zq}{a_1}, \frac{zq}{a_2}, \frac{zq}{a_3}\end{matrix}\right) \\ &= \prod_{n=1}^{\infty}\frac{(1-zq^n)(1-za_1^{-1}a_2^{-1}q^n)(1-za_1^{-1}a_3^{-1}q^n)(1-za_2^{-1}a_3^{-1}q^n)}{(1-za_1^{-1}q^n)(1-za_2^{-1}q^n)(1-za_3^{-1}q^n)(1-za_1^{-1}a_2^{-1}a_3^{-1}q^n)}.\end{aligned} \tag{3.4}$$

It is easy to observe that

$$\lim_{z\to 1}\frac{(z)_n}{(\sqrt{z})_n} = \lim_{z\to 1}\frac{(1-z)(1-qz)\cdots(1-q^{n-1}z)}{(1-\sqrt{z})(1-q\sqrt{z})\cdots(1-q^{n-1}\sqrt{z})} = \lim_{z\to 1}(1+\sqrt{z}) = 2 \tag{3.5}$$

and

$$\lim_{a_3\to\infty}\frac{(a_3)_n}{a_3^n} = \frac{(1-a_3)(1-qa_3)\cdots(1-q^{n-1}a_3)}{a_3^n} = (-1)^n q^{n(n-1)/2}\ . \tag{3.6}$$

By setting $a_1 = z$, $a_2 = z^{-1}$ and letting $z \to 1$, $a_3 \to \infty$ in (3.4) we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(z)_n(q\sqrt{z})_n(-q\sqrt{z})_n(a_1)_n(a_2)_n(a_3)_n \left(\frac{zq}{a_1a_2a_3}\right)^n}{(\sqrt{z})_n(-\sqrt{z})_n \left(\frac{zq}{a_1}\right)_n \left(\frac{zq}{a_2}\right)_n \left(\frac{zq}{a_3}\right)_n (q)_n} \\
=&\sum_{n=0}^{\infty} \frac{2(-1)^n(q)_n(-q)_n(z)_n(z^{-1})_n q^{n(n+1)/2}}{(-1)_n(zq)_n(z^{-1}q)_n(q)_n} \quad \text{(by (3.5) and (3.6))} \\
=&1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} \\
=&\frac{(q;q)_\infty^2}{(zq;q)_\infty(z^{-1}q;q)_\infty}. \quad \text{(by (3.4))}
\end{aligned} \tag{3.7}$$

Therefore

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_V(m,n)z^m q^n =&\prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)} \quad \text{(by (3.2))} \\
=&\frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \\
=&\frac{1}{(q;q)_\infty}\left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right] \\
&\text{(by (3.7))}.
\end{aligned} \tag{3.8}$$

On the other hand,

$$1+\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} = \sum_{n=-\infty}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}. \tag{3.9}$$

Hence, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_V(m,n)z^m q^n =&\frac{1}{(q;q)_\infty}\left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right] \\
&\text{(by (3.8))} \\
=&\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} \quad \text{(by (3.9))}.
\end{aligned}$$

$\square$

Next we restrict our attention to congruence properties of $N_V(m,n)$. Garvan proved following lemma which is necessary for stating congruence properties of $N_V(m,n)$.

Lemma 3.2.2. *[14]*

$$N_V(0,t,tn+\delta_t) = N_V(1,t,tn+\delta_t) = \cdots = N_V(t-1,t,tn+\delta_t) = \frac{p(tn+\delta_t)}{t} \quad (3.10)$$

for $t = 5, 7, 11$ where δ_t is reciprocal of 24 modulo t.
For t prime, (3.10) is equivalent to the coefficient of $q^{tn+\delta_t}$ in

$$\prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-\zeta_t q^n)(1-\zeta_t^{-1}q^n)}$$

being zero, where

$$\zeta_t = \exp(2\pi i/t).$$

Proof. See [14], page 54. □

Garvan proved his main result in [14] by using Lemma 3.2.2 and some q-series identities.

Theorem 3.2.3. *[14]*

$$N_V(k,5,5n+4) = \frac{p(5n+4)}{5} \quad for \quad 0 \le k \le 4,$$

$$N_V(k,7,7n+5) = \frac{p(7n+5)}{7} \quad for \quad 0 \le k \le 6,$$

$$N_V(k,11,11n+6) = \frac{p(11n+6)}{11} \quad for \quad 0 \le k \le 10.$$

Let us clarify Theorem 3.2.3 by giving an example.

Examples 3.2.4. We can write vector partitions of 5 with crank congruent to 1 modulo 7 as follows:

$$
\begin{aligned}
&\pi_1 = (\emptyset, 2+2, 1) \qquad \omega(\pi_1) = (-1)^0 = 1 \\
&\pi_2 = (\emptyset, 1+1+1, 1+1) \qquad \omega(\pi_2) = (-1)^0 = 1 \\
&\pi_3 = (\emptyset, 5, \emptyset) \qquad \omega(\pi_3) = (-1)^0 = 1 \\
&\pi_4 = (\emptyset, 1+1, 3) \qquad \omega(\pi_4) = (-1)^0 = 1 \\
&\pi_5 = (\emptyset, 3+1, 1) \qquad \omega(\pi_5) = (-1)^0 = 1 \\
&\pi_6 = (\emptyset, 2+1, 2) \qquad \omega(\pi_1) = (-1)^0 = 1 \\
&\pi_7 = (1, 2+1, 1) \qquad \omega(\pi_7) = (-1)^1 = -1 \\
&\pi_8 = (1, 4, \emptyset) \qquad \omega(\pi_8) = (-1)^1 = -1 \\
&\pi_9 = (1, 1+1, 2) \qquad \omega(\pi_9) = (-1)^1 = -1 \\
&\pi_{10} = (2, 1+1, 1) \qquad \omega(\pi_{10}) = (-1)^1 = -1 \\
&\pi_{11} = (2, 3, \emptyset) \qquad \omega(\pi_{11}) = (-1)^1 = -1 \\
&\pi_{12} = (3, 2, \emptyset) \qquad \omega(\pi_{12}) = (-1)^1 = -1 \\
&\pi_{13} = (4, 1, \emptyset) \qquad \omega(\pi_{13}) = (-1)^1 = -1 \\
&\pi_{14} = (2+1, 2, \emptyset) \qquad \omega(\pi_{14}) = (-1)^2 = 1 \\
&\pi_{15} = (3+1, 1, \emptyset) \qquad \omega(\pi_{15}) = (-1)^2 = 1.
\end{aligned} \tag{3.11}
$$

We see from (3.11) that

$$N_V(1,7,5) = \sum_{\substack{\pi \in V \\ s(\pi)=5 \\ r(\pi)\equiv 1 \pmod 7}} \omega(\pi) = \frac{p(5)}{7} = 1.$$

Another important property of $N_V(m,n)$ can be stated as follows.

Theorem 3.2.5. *[15]*

$$N_V(m,n) \geq 0$$

for all $(m,n) \neq (0,1)$.

Proof. See [15], page 42-43. $\square$

Next, we are concerned with a subset S of V.

3.3 Vector Partitions over Subset S of V

Our aim in this section is to introduce vector partitions defined on a subset S of V. This concept is given by Andrews, Garvan and Liang in [7].

For any partition π, let $s(\pi)$ be the smallest part of π. Define $s(\emptyset) = \infty$ for the empty partition. A subset S of the set of vector partitions V is given by

$$S := \{ \ \pi = (\pi_1, \pi_2, \pi_3) \in V \ \ : \ \ 1 \le s(\pi_1) < \infty \ \mathit{and} \ s(\pi_1) \le \mathit{min}\{ \ s(\pi_2), s(\pi_3)\} \ \} \ .$$

Let $\pi = (\pi_1, \pi_2, \pi_3)$ and $|\pi_i|$ be the sum of all parts of π_i for $i = 1, 2, 3$. Define

$$|\pi| := |\pi_1| + |\pi_2| + |\pi_3|,$$

$$\omega_1(\pi) := (-1)^{\#(\pi_1)-1} \ ,$$

$$crank(\pi) := \#(\pi_2) - \#(\pi_3).$$

The number of vector partitions of n over S with crank m counted according to weight ω_1 is denoted by $N_S(m, n)$ and it can be showed as

$$N_S(m, n) := \sum_{\substack{\pi \in S \\ |\pi| = n \\ crank(\pi) = m}} \omega_1(\pi). \tag{3.12}$$

Let $N_S(m, t, n)$ be the number of vector partitions of n in S with crank congruent to m modulo t counted according to weight ω_1. We have

$$N_S(m, t, n) := \sum_{k=-\infty}^{\infty} N_S(kt + m, n) = \sum_{\substack{\pi \in S \\ |\pi| = n \\ crank(\pi) \equiv m \pmod t}} \omega_1(\pi).$$

In a similar way with crank of vector partitions over V, we have

$$N_S(m,n) = N_S(-m,n)$$

and

$$N_S(m,t,n) = N_S(t-m,t,n).$$

Note that there are several descriptions of generating function for $N_S(m,n)$. One of these descriptions and its immediate corallary are given below.

Theorem 3.3.1. *[7] Let*

$$S(z,q) = \sum_{n=1}^{\infty}\sum_{m} N_S(m,n)z^m q^n. \tag{3.13}$$

Then we have,

$$S(z,q) = \sum_{n=1}^{\infty} \frac{q^n(q^{n+1};q)_\infty}{(zq^n;q)_\infty(z^{-1}q^n;q)_\infty}\ . \tag{3.14}$$

Corollary 3.3.2. *[7] For $n \geq 1$,*

$$\sum_{\substack{\pi\in S\\|\pi|=n}} \omega_1(\pi) = \sum_{m} N_S(m,n) = spt(n). \tag{3.15}$$

Proof. [7] Note that, we can find an equation for $spt(n)$ by letting $z = 1$ in (3.14) so that

$$\begin{aligned}
S(1,q) &= \sum_{n=1}^{\infty}\sum_{m} N_S(m,n)q^n \\
&= \sum_{n=1}^{\infty} \frac{q^n(q^{n+1};q)_\infty}{(q^n;q)_\infty(q^n;q)_\infty} \quad \text{(by Theorem 3.3.1)} \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(q^{n+1};q)_\infty(1-q^n)^2} \\
&= \sum_{n=1}^{\infty} spt(n)q^n \quad \text{(by (2.18))}.
\end{aligned}$$

Therefore we have from (3.12) that

$$\sum_{\substack{\pi\in S\\|\pi|=n\\crank(\pi)=m}} \omega_1(\pi) = \sum_m N_S(m,n) = spt(n).$$

□

Another important property of $N_S(m,n)$ can be given as follows.

Theorem 3.3.3. *[7]*

$$N_S(m,n) \geq 0$$

for all (m,n).

Proof. See [7], page 13. □

Next we focus congruence properties of $N_S(m,n)$.

3.4 Congruence Properties of $N_S(m,n)$

Our aim in this section is to prove congruence properties of $N_S(m,n)$. At the begining, we start with obtaining a relation between generating functions of $N_S(m,n)$, $N_V(m,n)$ and $N(m,n)$. We complete this section with stating some congruence properties of $N_S(m,n)$ and giving an example. The content of this section is mainly based on [7].

Andrews, Garvan and Liang concluded following theorem by using Bailey's Lemma.

Theorem 3.4.1. *[7]*

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-zq^n)(1-z^{-1}q^n)} \frac{(q^{n+1};q)_\infty}{(zq^{n+1};q)_\infty(z^{-1}q^{n+1};q)_\infty}$$
$$= \frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1}q^{n(n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)} - \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1}q^{n(3n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}\right).$$

Once we have Theorem 3.4.1, the relation between generating functions of $N_S(m,n)$, $N_V(m,n)$ and $N(m,n)$ is given by

Theorem 3.4.2. *[7]*

$$\begin{aligned}
S(z,q) =& \sum_{n=1}^{\infty}\sum_{m} N_S(m,n) z^m q^n \\
=& \frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{(1-zq^n)(1-z^{-1}q^n)} - \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(3n+1)}{2}}}{(1-zq^n)(1-z^{-1}q^n)}\right) \\
=& \frac{-1}{(1-z)(1-z^{-1})}\left[\sum_{n=0}^{\infty}\sum_{m} N_V(m,n) z^m q^n - \sum_{n=0}^{\infty}\sum_{m} N(m,n) z^m q^n\right].
\end{aligned}$$

Proof. [7] Let us recall equalities in Lemma 3.2.1 and Propostion 2.3.3. We have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n) z^m q^n = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} \tag{3.16}$$

and

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_V(m,n) z^m q^n = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)}. \tag{3.17}$$

Hence

$$\begin{aligned}
S(z,q) =& \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N_S(m,n) z^m q^n \\
=& \sum_{n=1}^{\infty} \frac{q^n}{(1-zq^n)(1-z^{-1}q^n)} \frac{(q^{n+1};q)_\infty}{(zq^{n+1};q)_\infty (z^{-1}q^{n+1};q)_\infty} \quad \text{(by (3.3.1)} \\
=& \frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{(1-zq^n)(1-z^{-1}q^n)} - \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(3n+1)}{2}}}{(1-zq^n)(1-z^{-1}q^n)}\right) \\
& \text{(by Theorem 3.4.1)} \\
=& \frac{-1}{(1-z)(1-z^{-1})}\left[\sum_{n=0}^{\infty}\sum_{m} N_V(m,n) z^m q^n - \sum_{n=0}^{\infty}\sum_{m} N(m,n) z^m q^n\right].
\end{aligned}$$

Last line follows from (3.16) and (3.17). $\square$

We continue by presenting a lemma whose proof is analogous to proof of Lemma 3.2.2.

Lemma 3.4.3.

$$N_S(0,t,tn+\delta_t) = N_S(1,t,tn+\delta_t) = \cdots = N_S(t-1,t,tn+\delta_t) = \frac{spt(tn+\delta_t)}{t} \tag{3.18}$$

for $t = 5, 7$ where δ_t is reciprocal of 24 modulo t.
For t prime, (3.18) is equivalent to the coefficient of $q^{tn+\delta_t}$ in

$$S(\zeta_t, q) = \sum_{n=1}^{\infty} \left(\sum_{r=0}^{t-1} N_S(r,t,n)\zeta_t^r \right) q^n$$

being zero, where

$$\zeta_t = \exp(2\pi i/t).$$

Proof. By replacing z with ζ_t in (3.13), we have

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} N_S(m,n)\zeta_t^m q^n &= \sum_{k=0}^{t-1} \sum_{\substack{m\equiv k \\ (\text{mod } t)}} \sum_{n=1}^{\infty} N_S(m,n)\zeta_t^m q^n \\ &= \sum_{k=0}^{t-1} \zeta_t^k \sum_{n=1}^{\infty} \left(\sum_{m\equiv k \pmod t} N_S(m,n) \right) q^n \\ &= \sum_{k=0}^{t-1} \zeta_t^k \sum_{n=1}^{\infty} N_S(k,t,n) q^n. \end{aligned} \tag{3.19}$$

From (3.19), we can see that

$$\sum_{k=0}^{t-1} N_S(k,t,tn+\delta_t)\zeta_t^k$$

is coefficient of $q^{tn+\delta_t}$ in (3.13). Now suppose that (3.18) is true. Hence

$$\begin{aligned} \sum_{k=0}^{t-1} N_S(k,t,tn+\delta_t)\zeta_t^k &= N_S(0,t,tn+\delta_t) \sum_{k=0}^{t-1} \zeta_t^k \\ &= 0 \quad (\text{by definition of } \zeta_t). \end{aligned}$$

Suppose now that coefficient of $q^{tn+\delta_t}$ in is zero. That is

$$\sum_{k=0}^{t-1} N_S(k,t,tn+\delta_t)\zeta_t^k = 0. \tag{3.20}$$

The minimal polynomial of ζ_t over $\mathbb{Q}$ is

$$p(x) = 1 + x + x^2 + \cdots + x^{t-1}.$$

We have from definition of the minimal polynomial of ζ_t and (3.20) that

$$N_S(0,t,tn+\delta_t) = N_S(1,t,tn+\delta_t) = \cdots = N_S(t-1,t,tn+\delta_t). \tag{3.21}$$

Therefore

$$\begin{aligned} spt(tn+\delta_t) =& \sum_{m=-\infty}^{\infty} N_S(m,tn+\delta_t) \ \ \text{(by Corollary 3.3.2)} \\ =& \sum_{k=0}^{t-1} N_S(k,t,tn+\delta_t) \\ =& tN_S(0,t,tn+\delta_t) \ \ \text{(by (3.21))}. \end{aligned}$$

□

Now we are ready to state our next theorem.

Theorem 3.4.4. *[7]*

$$N_S(k,5,5n+4) = \frac{spt(5n+4)}{5} \quad for \quad 0 \le k \le 4,$$

$$N_S(k,7,7n+5) = \frac{spt(7n+5)}{7} \quad for \quad 0 \le k \le 6.$$

Proof. [7] From Lemma 3.4.3, it is enough to show that the coefficient of $q^{tn+\delta_t}$ in

$$S(\zeta_t,q) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N_S(m,n)\zeta_t^m q^n = \sum_{n=1}^{\infty}\left(\sum_{r=0}^{t-1} N_S(r,t,n)\zeta_t^r\right) q^n$$

is zero when $t = 5$, 7 and $\zeta_t = \exp(2\pi i/t)$.

Put ζ_t instead of z in Theorem 3.4.2. Thus

$$\begin{aligned}
S(\zeta_t, q) =& \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N_S(m,t,n)\zeta_t^m q^n = \sum_{n=1}^{\infty} \left(\sum_{r=0}^{t-1} N_S(r,t,n)\zeta_t^r \right) q^n \\
=& \frac{-1}{(1-\zeta_t)(1-\zeta_t^{-1})} \left[\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_V(m,n)\zeta_t^m q^n - \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n)\zeta_t^m q^n \right] \\
& \text{(by Theorem 3.4.2)} \\
=& \frac{-1}{(1-\zeta_t)(1-\zeta_t^{-1})} \left[\sum_{n=0}^{\infty} \left(\sum_{r=0}^{t-1} N_V(r,t,n)\zeta_t^r \right) q^n - \sum_{n=0}^{\infty} \left(\sum_{r=0}^{t-1} N(r,t,n)\zeta_t^r \right) q^n \right].
\end{aligned} \tag{3.22}$$

On the other hand we know from Theorem 2.3.4 that

$$\begin{aligned}
N_V(k,5,5n+4) &= \frac{p(5n+4)}{5} \quad for \quad 0 \le k \le 4, \\
N_V(k,7,7n+5) &= \frac{p(7n+5)}{7} \quad for \quad 0 \le k \le 6.
\end{aligned} \tag{3.23}$$

We also know from Theorem 3.2.3 that

$$\begin{aligned}
N(k,5,5n+4) &= \frac{p(5n+4)}{5} \quad for \quad 0 \le k \le 4, \\
N(k,7,7n+5) &= \frac{p(7n+5)}{7} \quad for \quad 0 \le k \le 6.
\end{aligned} \tag{3.24}$$

Therefore, from (3.22), (3.23) and (3.24) we have for $t = 5$ and 7,

$$\begin{aligned}
& \sum_{r=0}^{t-1} N_S(r,t,tn+\delta_t)q^{tn+\delta_t} \\
=& \frac{-1}{(1-\zeta_t)(1-\zeta_t^{-1})} \left(\sum_{r=0}^{t-1} N_V(r,t,tn+\delta_t)\zeta_t^r - \sum_{r=0}^{t-1} N(r,t,tn+\delta_t)\zeta_t^r \right) q^{tn+\delta_t} = 0.
\end{aligned}$$

□

Examples 3.4.5. If we look at the vector partitions of 5 with crank congruent to 1 modulo 7 in (3.11) then we see that only following partitions

$$\pi_7 = (1, 2+1, 1)$$

$$\pi_8 = (1, 4, \emptyset)$$

$$\pi_9 = (1, 1 + 1, 2)$$

$$\pi_{11} = (2, 3, \emptyset)$$

$$\pi_{14} = (2 + 1, 2, \emptyset)$$

$$\pi_{15} = (3 + 1, 1, \emptyset)$$

lie in the subset S of V. We have $\omega_1(\pi_7) = \omega_1(\pi_8) = \omega_1(\pi_9) = \omega_1(\pi_{11}) = (-1)^{1-1} = (-1)^0 = 1$ and $\omega_1(\pi_{14}) = \omega_1(\pi_{15}) = (-1)^{2-1} = -1$. Therefore,

$$N_S(1, 7, 5) = \sum_{\substack{\pi \in S \\ s(\pi)=5 \\ crank(\pi) \equiv 1 \pmod 7}} \omega_1(\pi) = \frac{spt(5)}{7} = 2.$$

Chapter 4

Mock Theta Functions and Parity of $spt(n)$

In this chapter, our aim is to show that parity of *spt*-function can be determined by certain mock theta functions. At the beginning, we concern with self-conjugate S-partitions and relate it with *spt*-function. Finally, we state some consequences about parity of *spt*-function. The content of this chapter is mainly based on [8] and [23].

4.1 Self-Conjugate Vector Partitions

In this section, we cover properties of self-conjugate partitions which are defined in [8] and relate these partitions with *spt*-function. At the end, we give generating function of such partitions. Note that, results of this section are used in order to state properties of parity of *spt*-function.
Recall the definition of the set S.

$$S := \{\ \pi = (\pi_1, \pi_2, \pi_3) \in V\ :\ 1 \leq s(\pi_1) < \infty \ and\ s(\pi_1) \leq min\{\ s(\pi_2), s(\pi_3)\}\ \}\ .$$

Let l be a map on S given by,

$$l(\pi) = l(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_3, \pi_2).$$

An S-partition $\pi = (\pi_1, \pi_2, \pi_3)$ is a fixed point for l if and only if $\pi_2 = \pi_3$. We call these fixed points as *self-conjugate S-partitions*. We denote the number of self-conjugate S-partitions counted in terms of the weight ω_1 by $N_{SC}(n)$ so that

$$N_{SC}(n) := \sum_{\substack{\pi \in S \\ |\pi|=n \\ l(\pi)=\pi}} \omega_1(\pi).$$

A congruence relation between $N_{SC}(n)$ and $spt(n)$ can be given by

Proposition 4.1.1. *[8]*

$$N_{SC}(n) \equiv spt(n) \pmod 2.$$

Proof. By Corollary 3.3.2, we know that

$$\sum_{\substack{\pi \in S \\ |\pi|=n \\ crank(\pi)=m}} \omega_1(\pi) = \sum_m N_S(m, n) = spt(n).$$

Note that, for any self-conjugate S-partition $\pi = (\pi_1, \pi_2, \pi_3)$ of n, $crank(\pi) = \#(\pi_2) - \#(\pi_3) = 0$.
Another partition $\beta = (\beta_1, \beta_2, \beta_3)$ of n having crank 0 should be in the form such that $\#(\beta_2) = \#(\beta_3)$ but $\beta_2 \neq \beta_3$. In this case, the contribution of such partitions to $N_S(0, n)$ is a multiple of 2, call it $2k$ where k is a nonnegative integer. We have also an equation from previous chapter so that

$$N_S(m, n) = N_S(-m, n).$$

Therefore,

$$spt(n) = 2 \sum_{m \geq 1} N_S(m, n) + 2k + N_{SC}(n).$$

Thus,

$$N_{SC}(n) \equiv spt(n) \pmod 2.$$

□

Andrews, Garvan and Liang gave a generating function for $N_{SC}(n)$ in [8] by

Theorem 4.1.2. *[8]*

$$SC(q) := \sum_{n=1}^{\infty} N_{SC}(n)q^n = \sum_{n=1}^{\infty} q^n \frac{(q^{n+1};q)_\infty}{(q^{2n};q^2)_\infty} = \frac{1}{(-q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{n-1}}{(1-q^n)} .$$

Proof. [8] In order to see the first equality we divide the product

$$\frac{q^n(q^{n+1};q)_\infty}{(q^{2n};q^2)_\infty}$$

into two parts. First, power of q in the part

$$q^n(q^{n+1};q)_\infty = q^n(1-q^{n+1})(1-q^{n+2})\cdots$$

generates a partition π_1 into distinct parts such that n is the smallest part of π_1. Note also that π_1 is an element of the set $\mathcal{D}$. On the other hand, the coefficient of $q^{|\pi_1|}$ is -1 if number of parts of π_1 is even and 1 if number of parts of π_1 is odd.

Secondly we have

$$\frac{1}{(q^{2n};q^2)_\infty}$$
$$= \left(1 + q^{(n)+(n)} + q^{(n+n)+(n+n)} + \cdots\right)\left(1 + q^{(n+1)+(n+1)} + q^{(n+1+n+1)+(n+1+n+1)}\cdots\right)\ldots$$

In this case, let π_2 be a partition constructed by numbers within the first paranthesis in powers of q. In the same way let π_3 be another partition constructed by numbers within the second paranthesis in powers of q. It is easy to observe that π_2 and π_3 are partitions with unrestricted parts and their smallest parts are bigger than n. In other words, partitions π_2 and π_3 lie in the set $\mathcal{P}$. If we define a partition π such that $\pi = (\pi_1, \pi_2, \pi_3)$ then we conclude that the sum

$$\sum_{n=1}^{\infty} q^n \frac{(q^{n+1};q)_\infty}{(q^{2n};q^2)_\infty}$$

is the generating function for $N_{SC}(n)$.

For the second equality, notice that for $n \geq 1$

$$\begin{aligned}
\frac{(q^{n+1};q)_\infty}{(q^{2n};q^2)_\infty} &= \frac{(1-q^{n+1})(1-q^{n+2})\cdots}{(1-q^{2n})(1-q^{2n+2})(1-q^{2n+4})\cdots} \\
&= \frac{1}{(1-q^{2n})(1+q^{n+1})(1+q^{n+2})\cdots} \\
&= \frac{1}{(1-q^{n})(1+q^{n})(1+q^{n+1})(1+q^{n+2})\cdots} \\
&= \frac{1}{(1-q^{n})(-q^{n};q)_\infty} \\
&= \frac{(-q;q)_{n-1}}{(1-q^{n})(-q;q)_\infty}.
\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} q^n \frac{(q^{n+1};q)_\infty}{(q^{2n};q^2)_\infty} = \frac{1}{(-q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{n-1}}{(1-q^n)} .$$

□

In the next section, we are concerned with a fundamental topic in theory of theta functions.

4.2 Introduction to Mock Theta Functions

In the last letter of Ramanujan to Hardy [23], Ramanujan mentioned notion of mock theta functions. He also gave 17 examples for these functions. We are interested in some of these functions of order 3. The content can be found in [19] and [23].

Here is the complete list of the mock theta functions of order 3.

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})}$$

$$\Psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}$$

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2}$$

$$\upsilon(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1+q)(1+q^3)\cdots(1+q^{2n+1})}$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})}$$

Following identity was stated by Ramanujan and proven by Watson in [23].

$$2\phi(-q) - f(q) = f(q) + 4\Psi(-q) = \upsilon_4(0,q)\prod_{r=1}^{\infty}(1+q^r)^{-1} \tag{4.1}$$

where $\upsilon_4(z,q)$ is a theta function

$$\upsilon_4(z,q) = \sum_{n=-\infty}^{\infty}(-1)^n e^{2\pi i n z} q^{n^2} = (e^{2\pi i z}q;q^2)_\infty(e^{-2\pi i z}q;q^2)_\infty(q^2;q^2)_\infty.$$

$\upsilon_4(z,q)$ also satisfies that (See for example [4])

$$\upsilon_4(0,q)^2 = 1 + 4\sum_{m=0}^{\infty}\frac{(-1)^{m+1}q^{2m+1}}{1+q^{2m+1}}\ . \tag{4.2}$$

Note that we know the equality

$$\frac{1}{(-q;q)_\infty} = (q;q^2)_\infty.$$

Therefore (4.1) can be written as

$$2\phi(-q) - f(q) = f(q) + 4\Psi(-q) = \upsilon_4(0,q)(q;q^2)_\infty. \tag{4.3}$$

This chapter continues with giving properties of parity of $spt(n)$.

4.3 Parity of $spt(n)$

An important property about parity of $spt(n)$ is emphasized in this section. We begin with stating some preliminary lemmas.

Lemma 4.3.1. *[8]*

$$\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{1-q^n} = \sum_{n=0}^{\infty}\frac{1}{(q^2;q^2)_n}((q)_{2n}-(q)_\infty) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n}. \tag{4.4}$$

We also need two theorems in order to prove equalities in Lemma 4.3.1.

Theorem 4.3.2. *[5]*

$$\sum_{n=0}^{\infty}\left(\frac{(t)_\infty}{(a)_\infty} - \frac{(t)_n}{(a)_n}\right)$$

$$= \sum_{n=1}^{\infty}\frac{(q/a)_n(a/t)^n}{(q/t)_n} + \frac{(t)_\infty}{(a)_\infty}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} + \sum_{n=1}^{\infty}\frac{q^nt^{-1}}{1-t^{-1}q^n} - \sum_{n=1}^{\infty}\frac{tq^n}{1-tq^n} - \sum_{n=1}^{\infty}\frac{aq^nt^{-1}}{1-aq^nt^{-1}}\right).$$

Proof. See [5], page 403-404. □

Other theorem is given by the following equality.

Theorem 4.3.3. *[5]*

$$\sum_{n=0}^{\infty}\left(\frac{(a)_\infty(b)_\infty}{(q)_\infty(c)_\infty}-\frac{(a)_n(b)_n}{(q)_n(c)_n}\right)$$
$$=\frac{(b)_\infty(a)_\infty}{(c)_\infty(q)_\infty}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}-\sum_{n=1}^{\infty}\frac{(c/b)_n b^n}{(a)_n(1-q^n)}\right).$$

Proof. See [5], page 404. $\square$

We continue to give a proof of Lemma 4.3.1.

Proof. (Proof of Lemma 4.3.1)[8] Since

$$\frac{(-q;q)_{n-1}}{(1-q^n)}=\frac{(1+q)(1+q^2)\cdots(1+q^{n-1})}{1-q^n}=\frac{(1-q^2)(1-q^4)\cdots(1-q^{2n-2})}{(1-q)(1-q^2)\cdots(1-q^n)}=\frac{(q^2;q^2)_{n-1}}{(q)_n}.$$

We have

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{1-q^n}=\frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(q^2;q^2)_{n-1}}{(q)_n}\ . \tag{4.5}$$

On the other hand, we know that for $n\geq 1$

$$\frac{(q^2;q^2)_{n-1}}{(q^2;q^2)_\infty}=\frac{1}{(q^{2n};q^2)_\infty}\ . \tag{4.6}$$

Therefore,

$$\begin{aligned}\frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(q^2;q^2)_{n-1}}{(q)_n}&=\sum_{n=1}^{\infty}\frac{q^n}{(q)_n(q^{2n};q^2)_\infty}\quad\text{(by (4.6))}\\&=\sum_{n=1}^{\infty}\frac{q^n}{(q)_n}\sum_{k=0}^{\infty}\frac{q^{2nk}}{(q^2;q^2)_k}\quad\text{(by Theorem 2.1.7)}\\&=\sum_{k=0}^{\infty}\frac{1}{(q^2;q^2)_k}\sum_{n=1}^{\infty}\frac{q^{n(2k+1)}}{(q)_n}\\&=\sum_{k=0}^{\infty}\frac{1}{(q^2;q^2)_k}\left(\frac{1}{(q^{2k+1};q)_\infty}-1\right)\quad\text{(by Theorem 2.1.7).}\end{aligned} \tag{4.7}$$

We also know that

$$\frac{(q)_\infty}{(q^{2k+1};q)_\infty} = (q)_{2k}. \tag{4.8}$$

If we multiply both sides of (4.5) by $(q;q)_\infty$, we find that

$$\begin{aligned}\frac{(q;q)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{1-q^n} =&\frac{(q;q)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(q^2;q^2)_{n-1}}{(q)_n}\\ =&(q;q)_\infty\sum_{k=0}^{\infty}\frac{1}{(q^2;q^2)_k}\left(\frac{1}{(q^{2k+1};q)_\infty}-1\right) \quad \text{(by (4.7))}\\ =&\sum_{k=0}^{\infty}\frac{1}{(q^2;q^2)_k}\left(\frac{(q;q)_\infty}{(q^{2k+1};q)_\infty}-(q;q)_\infty\right)\\ =&\sum_{k=0}^{\infty}\frac{1}{(q^2;q^2)_k}\left((q;q)_{2k}-(q)_\infty\right) \quad \text{(by (4.8))}\end{aligned}$$

as desired.

For the second equality in our theorem, consider following limit

$$\lim_{a\to 0}(q^2/a;q^2)_n(a/t)^n = \left(1-\frac{q^2}{a}\right)\left(1-\frac{q^3}{a}\right)\cdots\left(1-\frac{q^{2n}}{a}\right)\frac{a^n}{t^n} = \frac{(-1)^n q^{n(n+1)}}{t^n}.$$

Replace q by q^2, t by q and let $a \to 0$ in Theorem 4.3.2 we find that

$$\sum_{n=0}^{\infty}\big((q;q^2)_\infty-(q;q^2)_n\big) = \sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2}}{(q;q^2)_n} + (q;q^2)_\infty\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\ . \tag{4.9}$$

On the other hand, let $q \to q^2$ and $a = b = c = 0$ in Theorem 4.3.3, we have

$$\sum_{n=0}^{\infty}\left(\frac{1}{(q^2;q^2)_\infty}-\frac{1}{(q^2;q^2)_n}\right) = \frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\ . \tag{4.10}$$

We know that

$$\frac{(q;q)_\infty}{(q^2;q^2)_\infty} = (q;q^2)_\infty, \tag{4.11}$$

and

$$\frac{(q;q)_{2n}}{(q^2;q^2)_n} = (q;q^2)_n. \tag{4.12}$$

Hence

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{(q^2;q^2)_n} \left((q;q)_{2n} - (q;q)_\infty\right) \\
&= \sum_{n=0}^{\infty} \left((q;q^2)_n - (q;q^2)_\infty + (q;q^2)_\infty - \frac{(q;q)_\infty}{(q^2;q^2)_n} \right) \quad \text{(by (4.12)).} \\
&= \sum_{n=0}^{\infty} \left((q;q^2)_n - (q;q^2)_\infty\right) + (q;q)_\infty \sum_{n=0}^{\infty} \left(\frac{1}{(q^2;q^2)_\infty} - \frac{1}{(q^2;q^2)_n} \right) \quad \text{(by (4.11)).} \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} - (q;q^2)_\infty \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} + \frac{(q;q)_\infty}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \quad \text{(by (4.9) and (4.10)).} \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} - (q;q^2)_\infty \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} + (q;q^2)_\infty \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \quad \text{(by (4.11))} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n^2}}{(q;q^2)_n}.
\end{aligned}
$$

□

Note that the infinite sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} \tag{4.13}$$

is a mock theta function. Andrews, Dyson and Hickerson studied (4.13) in [6] and they interpreted (4.13) in terms of partitions in a following way:

Consider partitions of n into odd parts with the condition that if k occurs as a part, then all positive odd numbers less than k also occur. Denote $S^*(n)$ be the number of such partitions with largest part congruent to 3 modulo 4 minus the number of such partitions with largest parts congruent to 1 modulo 4. Andrews, Dyson and Hickerson gave a generating function for $S^*(n)$ in [6] by

$$\sum_{n\geq 1} S^*(n) q^n = \sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} . \tag{4.14}$$

Explicit formula for the coefficients of (4.14) was given in [6]. In order to state the formula, we need a definition from [6].

Definition 4.3.4. Define an arithmetic function $T(m)$ for integers which are congruent to 1 modulo 24 as follows:

Let m be an integer which is congruent to 1 modulo 24. Consider Pell's equation

$$u^2 - 6v^2 = m. \tag{4.15}$$

Notice that if (u, v) is a solution of (4.15) then u is an integer which is congruent to ± 1 modulo 6 and v is even. We call two solutions (u, v) and (u', v') equivalent if

$$u' + v'\sqrt{6} = \pm(5 + 2\sqrt{6})^r (u + v\sqrt{6}).$$

for some integer r. Let $T(m)$ be the number of inequivalent solution of (4.15) with $u + 3v \equiv \pm 1 \pmod{12}$ minus the number of inequivalent solutions of (4.15) with $u + 3v \equiv \pm 5 \pmod{12}$.

Andrews, Dyson and Hickerson proved that $T(m)$ is a multiplicative function. They also proved that

Lemma 4.3.5. *[6] Let p be a prime which is congruent to 1 modulo 6 or the negative of a prime which is congruent to 5 modulo 6, and let $e \geq 1$. Then:*
(a) If p is not congruent to 1 modulo 24 and e is odd, then $T(p^e) = 0$.
(b) If p is not congruent to 1 modulo 24 and e is even, then

$$T(p^e) = \begin{cases} 1 & \text{if } p \equiv 13 \quad \text{or} \quad 19 \pmod{24} \\ (-1)^{e/2} & \text{if } p \equiv 7 \pmod{24}. \end{cases}$$

(c) If $p \equiv 1 \pmod{24}$ then either $T(p) = 2$ or $T(p) = -2$. In the first case, $T(p^e) = e + 1$. In the second case, $T(p^e) = (-1)^e(e + 1)$.

They stated a theorem about the coefficients of (4.14) as follows.

Theorem 4.3.6. *[6] For $n \geq 1$, $2S^*(n) = T(1 - 24n)$.*

Andrews, Dyson and Hickerson emphasized that Theorem 4.3.6 is equivalent to a theorem given by

Theorem 4.3.7. *[6]*

$$\sum_{n\geq 1}\frac{(-1)^n q^{n^2}}{(q;q^2)_n}=\sum_{n\geq 1}(-1)^n q^{n(3n-1)}(1+q^{2n})\sum_{j=0}^{2n-1}q^{-j(j+1)/2}.$$

Now, we are ready to prove a result about parity of $spt(n)$.

Theorem 4.3.8. *[8]*

$$spt(n)\equiv S^*(n) \pmod 2.$$

Proof. By Proposition 4.1.1, we have

$$spt(n)\equiv N_{SC}(n) \pmod 2. \tag{4.16}$$

On the other hand, by Theorem 4.1.2 we have the generating function of $N_{SC}(n)$

$$\sum_{n=1}^{\infty} N_{SC}(n)q^n=\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{(1-q^n)}. \tag{4.17}$$

However, in Lemma 4.3.1, we proved that

$$\sum_{n=1}^{\infty} N_{SC}(n)q^n=\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{(1-q^n)}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n}. \tag{4.18}$$

We also know that

$$\sum_{n=1}^{\infty} S^*(n)q^n=\sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2}}{(q;q^2)_n}\ . \tag{4.19}$$

Thus

$$\begin{aligned}\sum_{n=1}^{\infty} N_{SC}(n)q^n &=\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{(1-q^n)} \quad \text{(by (4.17))}\\ &=-\sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2}}{(q;q^2)_n} \quad \text{(by (4.18))}\\ &=-\sum_{n=1}^{\infty} S^*(n)q^n \quad \text{(by (4.19))}.\end{aligned} \tag{4.20}$$

Equating coefficients of q in (4.20) we have

$$N_{SC}(n) = -S^*(n).$$

Therefore, by (4.16) we get

$$spt(n) \equiv N_{SC}(n) \equiv S^*(n) \pmod 2$$

as desired. □

It now follows from Lemma 4.3.5, Theorem 4.3.6 and Theorem 4.3.8 that

Theorem 4.3.9. *[14]*
(i) $N_{SC}(n) = 0$ if and only if

$$p^e || \ 24n - 1.$$

(ii) $spt(n)$ is odd if and only if $24n - 1 = p^{4a+1}m^2$ for some prime $p \equiv 23 \pmod{24}$ and some integers a, m, where $(p, m) = 1$.

We should remark that parity result for the *spt*-function was firstly given by A. Folsom and K. Ono in [13]. However, their results were corrected as in Theorem 4.3.9.
We finish this chapter by proving one more result about connection between $spt(n)$ and a mock theta function of third order.

4.4 Relation Between $spt(n)$ and Mock Theta Functions

In this section, our aim is to prove a result which relates $spt(n)$ and a mock theta function. Let us start with stating an identity from [23] so that

$$f(q) = \frac{1}{(q)_\infty}\left(1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}\right). \tag{4.21}$$

where $f(q)$ is the mock theta function of order 3 defined by Ramanujan([20]). Now we are ready to state our next theorem.

Theorem 4.4.1. *[8]Let*

$$\Psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} = \sum_{n=1}^{\infty} \psi(n) q^n.$$

Then

$$spt(n) \equiv (-1)^{n-1}\psi(n) \pmod 4.$$

Proof. [8] It is easy to show that

$$\frac{1}{(1-q^n)^2} = \frac{1}{(1+q^n)^2} + 4\frac{q^n}{(1-q^{2n})^2} \ . \tag{4.22}$$

We have

$$\begin{aligned} \frac{1}{(q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n} &= \frac{1}{4(q)_\infty}((q)_\infty f(q) - 1) \quad \text{(by (4.21))} \\ &= -\Psi(-q) + \frac{1}{4}(q;q^2)_\infty v_4(0,q) - \frac{1}{4(q)_\infty} \quad \text{(by (4.3))}. \end{aligned} \tag{4.23}$$

If we consider (4.22) then we see that

$$\begin{aligned} \sum_{n\geq 1} spt(n)q^n = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \frac{1}{(q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2} \quad \text{by (2.2.2)} \\ = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \frac{1}{(q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1+q^n)^2} + 4\frac{1}{(q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{(1-q^n)^2} \\ \equiv \frac{1}{(q)_\infty}\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \frac{1}{(q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n} \pmod 4. \end{aligned} \tag{4.24}$$

On the other hand, it is easy to observe that

$$(q;q^2)_\infty (q^2;q^2)_\infty = (q;q)_\infty. \tag{4.25}$$

We have,

$$\begin{aligned}\frac{v_4(0,q)^2}{(q)_\infty} =& \frac{1}{(q)_\infty}((q;q^2)_\infty(q^2;q^2)_\infty(q;q^2)_\infty)^2 \\ =& \frac{1}{(q)_\infty}((q;q)_\infty(q;q^2)_\infty)^2 \quad \text{(by (4.25))} \\ =& \frac{1}{(q)_\infty}((q)_\infty(q;q^2)_\infty(q^2;q^2)_\infty(q;q^2)_\infty(q;q^2)_\infty) \\ =& v_4(0,q)(q;q^2)_\infty \quad \text{(by defn. of } v_4(0,q)).\end{aligned} \tag{4.26}$$

Therefore,

$$\begin{aligned}\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n} =& -\Psi(-q) + \frac{1}{4}(q;q^2)_\infty v_4(0,q) - \frac{1}{4(q)_\infty} \quad \text{(by (4.23))} \\ =& -\Psi(-q) + \frac{1}{4(q)_\infty}(v_4(0,q)^2 - 1) \quad \text{(by (4.26))} \\ =& -\Psi(-q) + \frac{1}{(q)_\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}q^{2m+1}}{1+q^{2m+1}} \quad \text{(by (4.2))}.\end{aligned} \tag{4.27}$$

Thus we have from (4.24) and (4.27) that

$$\sum_{n\geq 1} spt(n)q^n \equiv \frac{1}{(q)_\infty}\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \sum_{m=0}^{\infty}\frac{(-1)^{m+1}q^{2m+1}}{1+q^{2m+1}}\right) - \Psi(-q) \pmod 4. \tag{4.28}$$

Now, we investigate the infinite sum above in modulo 4.

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}q^{2m+1}}{1+q^{2m+1}} \\
\equiv&\sum_{n=1}^{\infty} \frac{q^{4n+1}}{1-q^{4n+1}} - \sum_{n=1}^{\infty} \frac{q^{4n-1}}{1-q^{4n-1}} + 2\sum_{n=1}^{\infty} \frac{q^{4n+2}}{1-q^{4n+2}} - \sum_{m=0}^{\infty} \frac{q^{4m+1}}{1+q^{4m+1}} \\
&+ \sum_{m=0}^{\infty} \frac{q^{4m-1}}{1+q^{4m-1}} \pmod 4 \\
\equiv&2\sum_{n=0}^{\infty} \frac{q^{8n+2}}{1-q^{8n+2}} - 2\sum_{n=1}^{\infty} \frac{q^{8n-2}}{1-q^{8n-2}} + 2\sum_{n=0}^{\infty} \frac{q^{4n+2}}{1-q^{4n+2}} \pmod 4 \\
\equiv&4\sum_{n=0}^{\infty} \frac{q^{8n+2}}{1-q^{8n+2}} \\
\equiv&0 \pmod 4.
\end{aligned} \tag{4.29}$$

Hence we get from (4.28) and (4.29) that

$$\sum_{n\geq 1} spt(n)q^n \equiv -\Psi(-q) \pmod 4.$$

In other words,

$$spt(n) \equiv (-1)^{n-1}\psi(n) \pmod 4$$

by the definition of function $\Psi(q)$. $\square$

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