Scaling Stratified Stochastic Gradient Descent for Distributed Matrix Completion

Nabil Abubaker, M. Ozan Karsavuran, and Cevdet Aykanat

***Abstract*—Stratified SGD (SSGD) is the primary approach for achieving serializable parallel SGD for matrix completion. State-of-the-art parallelizations of SSGD fail to scale due to large communication overhead. During an SGD epoch, these methods send data proportional to one of the dimensions of the rating matrix. We propose a framework for scalable SSGD through significantly reducing the communication overhead via exchanging point-to-point messages utilizing the sparsity of the rating matrix. We provide formulas to represent the essential communication for correctly performing parallel SSGD and we propose a dynamic programming algorithm for efficiently computing them to establish the point-to-point message schedules. This scheme, however, significantly increases the number of messages sent by a processor per epoch from $\mathcal{O}(K)$ to $\mathcal{O}(K^2)$ for a K-processor system which might limit the scalability. To remedy this, we propose a Hold-and-Combine strategy to limit the upper-bound on the number of messages sent per processor to $\mathcal{O}(K \lg K)$. We also propose a hypergraph partitioning model that correctly encapsulates reducing the communication volume. Experimental results show that the framework successfully achieves a scalable distributed SSGD through significantly reducing the communication overhead. Our code is publicly available at: github.com/nfabubaker/CESSGD**

***Index Terms*—Bandwidth cost, combinatorial algorithms, communication cost minimization, collaborative filtering, HPC, hypergraph partitioning, latency cost, matrix completion, recommender systems, SGD.**

I. Introduction

RECOMMENDER systems are omnipresent in e-commerce as well as social, professional and academic networks. These systems help businesses improve profit by targeted advertisements to interested parties, facilitate the recruitment process by matching more relevant candidates to jobs, and help academics explore cross-disciplinary research works as well as expand their collaboration networks. Recommender systems can involve one or more techniques, among which Collaborative Filtering (CF) is the most widely used.

CF approaches recommend an *item* to a target *user* by using other users' ratings given that those other users and the target user have rated some other items similarly. The rating data produced nowadays, whether by social networks or e-commerce, is rather huge and change very often. Recommender systems for such huge data are usually implemented on distributed-memory systems that might involve multiple data centers. Therefore, the CF component should be performant and scalable to utilize the provided computational resources as well as the high-speed networks.

Low-rank matrix factorization have been successfully used in CF via revealing feature vectors that represent the users and the items (latent factors). In matrix-factorization-based CF methods, the known ratings are stored as a sparse matrix, rows of which represent users and columns of which represent items. The sparse matrix is factorized into two dense matrices representing the feature vectors of items and users, and these dense matrices are then used to predict missing ratings in the original rating matrix. This use of matrix factorization is commonly referred to as *matrix completion*. The matrix factorization can be computed with different methods, including stochastic gradient descent (SGD), alternating least squares (ALS), cyclic coordinate descent (CCD) and more.

SGD is very efficient and usually achieves high completion accuracy compared to other methods [1]. However, given its sequential nature it has been a challenge to efficiently parallelize while maintaining accuracy and convergence guarantee. For this reason, serializable parallel SGD algorithms are most desired. Serializability of parallel SGD refers to the existence of an equivalent serially-executed SGD algorithm with the same update order. Serializability guarantees the convergence and assures that no two processors update the same feature vector at the same time (race condition) thus leading to faster convergence [2]. Stratified SGD [3] is the de-facto algorithm for achieving a serializable parallel SGD.

The state-of-the-art methods implementing SSGD (such as DSGD [3], DSGD++ [4], and NOMAD [5]) achieve the inter-processor communication necessary for the correctness of the SSGD through sending/receiving feature vectors with sizes proportional to one of the dimensions of the input rating matrix. In other words, these methods perform dense communications without exploiting the sparse nature of the rating matrix, leading to a huge amount of unnecessary communication especially when the nonzero density of the rating matrix is low. The extra

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communication did not pose a concern because these methods are tested on a relatively small number of processors (up to 64) in distributed setting. At such small scale, the SGD runtime is expected to be dominated by computation; and investing in improving the communication component does not drastically affect the overall running time.

On large scale (hundreds and thousands of processors), the communication component becomes dominant, and therefore reducing the communication overhead becomes essential for the scalability of the SSGD algorithm. The dense communications of the state-of-the-art methods implementing SSGD prohibit their scalability. This is empirically confirmed by us (Section VI) as well as by another work [6] that runs DSGD on thousands of processors (cf. Fig. 8 in [6]).

In this work, we propose a communication-efficient framework for the SSGD algorithm. Our framework starts with exploiting the sparsity of the rating matrix by performing sparse communications instead of dense communications. This is achieved with efficiently finding the essential feature vectors to be communicated between processors and communicating them through point-to-point (P2P) messages. This approach, although invaluable for reducing the volume of communication performed, has the down side of increasing the number of messages sent per processor from $\mathcal{O}(K)$ (as in DSGD) to $\mathcal{O}(K^2)$.

Inter-processor communication cost ideally consists of latency term and bandwidth term. The latency term is proportional to the number of messages sent, whereas the bandwidth term is proportional to the volume of data transferred. If the number of messages is high, the latency cost might dominate the overall communication component since each message's startup time might be higher than that of sending a few kilobytes of data [7]. The $\mathcal{O}(K^2)$ bound of the new sparse communication method has the potential to increase the latency overhead and possibly affecting the scalability as K increases, which makes it *latency-unsafe*. To remedy this, we propose a novel approach called hold and combine that reduces the upper bound on the number of messages from $\mathcal{O}(K^2)$ to $\mathcal{O}(K \lg K)$ which renders the new sparse communication method *latency-safe*.

The volume of the sparse communication in parallel SSGD is also affected by how the ratings are distributed to different processors. This property indicates that there is a room for reducing the volume of communication combinatorialy via intelligent partitioning methods. We propose a partitioning method utilizing a hypergraph partitioning model that correctly encapsulates the total volume of communication between processors. In this method, the objective of reducing the cutsize of the hypergraph model partition also corresponds to reducing the total volume of communication in an SSGD epoch.

The rest of the paper is organized as follows: Section II gives the essentials of using and parallelizing SGD for matrix completion. In Section III, the communication requirement in parallel SSGD is studied in detail. In Section IV, the proposed framework for scaling P2P SSGD, including the hold-and-combine scheme, is presented. In Section V, the proposed hypergraph partitioning (HP) method is presented. Section VI contains the experiments conducted on an HPC system along with the results and discussions. Related works are discussed in Section VII and the paper is concluded in Section VIII.

II. Background

A. Matrix Completion With SGD

We define the matrix completion problem in the context of collaborative filtering as follows: Given a set $\mathcal{U}$ of N users, a set $\mathcal{I}$ of M items, and a set Ω of ratings as the known entries of a sparse rating matrix $\mathbf{R} \in \mathbb{R}^{N \times M}$. The problem is to find two dense factor matrices $\mathbf{W} \in \mathbb{R}^{N \times F}$ and $\mathbf{H} \in \mathbb{R}^{M \times F}$ such that a low-rank approximation $\mathbf{R} \approx \mathbf{W}\mathbf{H}^\top$ is achieved. Here, $F \ll M, N$ is called the dimension or the rank of the factorization. The approximation $\hat{r}_{ij}$ of rating r_{ij} can then be calculated as

$$\hat{r}_{ij} = \mathbf{w}_i \mathbf{h}_j^\top, \tag{1}$$

where $\mathbf{w}_i$ and $\mathbf{h}_j$ respectively denote the ith row of $\mathbf{W}$ and the jth row of $\mathbf{H}$. The quality of the approximation is usually measured by an application-dependent loss function $\mathcal{L}$, thus the problem becomes $\operatorname{argmin}_{\mathbf{W},\mathbf{H}} \mathcal{L}(\mathbf{R}, \mathbf{W}, \mathbf{H})$. For collaborative filtering, $\mathcal{L}$ is usually the eculedian distance and thus the problem becomes

$$\operatorname*{argmin}_{\mathbf{W},\mathbf{H}} \sum_{(i,j) \ni r_{ij} \in \Omega} \left((r_{ij} - \hat{r}_{ij})^2 + \gamma(\|\mathbf{w}_i\|^2 + \|\mathbf{h}_j\|^2\right), \tag{2}$$

where γ is a regularization parameter to avoid over-fitting, and $\hat{r}_{ij}$ is computed with (1).

Since the minimization problem in (2) has two unknowns $\mathbf{W}$ and $\mathbf{H}$, $\mathcal{L}$ is a non-convex function [1]. SGD has been widely used to optimize (minimize) such functions due to its ability to escape local minimas. At an SGD epoch, each rating $r_{ij} \in \Omega$ is used to update the objective function's parameters. The gradient of the objective function at point r_{ij} is calculated $(\nabla_{r_{ij}} \mathcal{L}_{r_{ij}}(\mathbf{R}, \mathbf{W}, \mathbf{H}))$ and the corresponding $\mathbf{w}_i$ and $\mathbf{h}_j$ rows are updated as

$$\mathbf{w}_i = \mathbf{w}_i - \epsilon[(r_{ij} - \hat{r}_{ij})\mathbf{h}_j + \gamma \mathbf{w}_i], \tag{3}$$

$$\mathbf{h}_j = \mathbf{h}_j - \epsilon[(r_{ij} - \hat{r}_{ij})\mathbf{w}_i + \gamma \mathbf{h}_j] \tag{4}$$

where ϵ is the step size.

It is clear from (3) and (4) that SGD is sequential in nature, thus parallelizing it requires communicating up-to-date $\mathbf{W}$- and $\mathbf{H}$-matrix rows. Trivially, the up-to-date $\mathbf{W}$- and $\mathbf{H}$-matrix rows should be communicated after each SGD update which enforces very high communication and synchronization overheads. Otherwise, some SGD updates will be performed on stale versions of $\mathbf{W}$- and $\mathbf{H}$-matrix rows which may drastically affect the learning process and the convergence guarantee. The parallel SGD methods that allow updating on stale $\mathbf{W}$- and $\mathbf{H}$-matrix rows (i.e., allow staleness) are called *asynchronous*. These methods are usually non-serializable. Simple parallelizations of the SGD-based matrix completion, such as row-wise or column-wise partitioning of the rating matrix, are examples of asynchronous SGD (see Fig. 1).

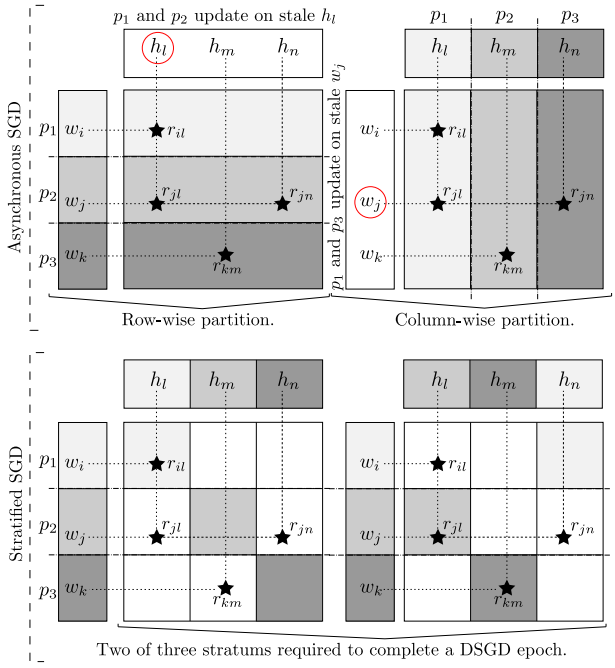


Fig. 1. Stale updates in simple row- or column-wise partitions (upper part) versus stale-free DSGD (bottom). In the row-wise partition of $\mathbf{R}$, the rows of $\mathbf{W}$ are partitioned conformably and thus each $\mathbf{W}$-matrix row is accessed by one processor. However, this is not the case for $\mathbf{H}$-matrix rows. For instance, ratings r_{il} and r_{jl} are respectively assigned to p_1 and p_2 and both used to update $\mathbf{h}_l$ possibly at the same time thus either p_1 or p_2 will update on a stale $\mathbf{h}_l$. A similar discussion holds for column-wise partition in a dual manner regarding r_{jl}, r_{jn} and $\mathbf{w}_j$. Black starts are known ratings.

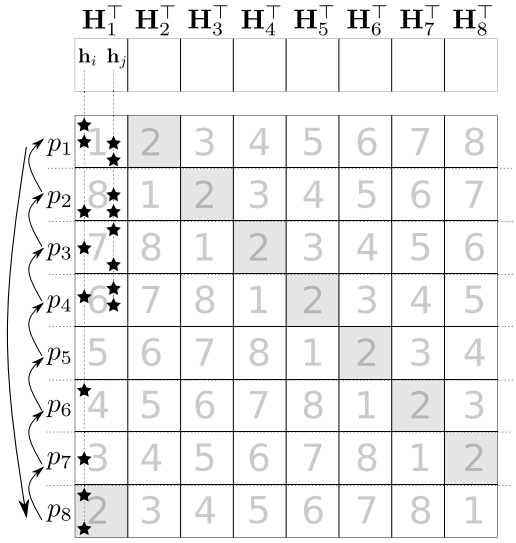


Fig. 2. The numbers identify the sub-matrix blocks that constitute a stratum in a ring strata with $seed=1$. Stratum $\mathcal{S}_2$ is highlighted. Side arrows show the processor update order of $\mathbf{h}_i$ and $\mathbf{h}_j$ in $\mathbf{H}_1^\top$.

B. Stratified SGD (SSGD) and its Parallelization

1) SSGD: The SSGD method is proposed by Gemulla et al. [3] in order to mitigate the staleness problem. In SSGD, the rating matrix is divided into K^2 2D blocks using K-way mutually exclusive and exhaustive partitions on the rows $\Pi^R = \{R_1, \ldots, R_K\}$ and columns $\Pi^C = \{C_1, \ldots, C_K\}$ of $\mathbf{R}$. The rows of the dense matrices $\mathbf{W}$ and $\mathbf{H}$ are partitioned conformably with Π^R and Π^C, respectively. We denote the row blocks of $\mathbf{W}$ and $\mathbf{H}$ that respectively conform with R_α and C_β as $\mathbf{W}_\alpha$ and $\mathbf{H}_\beta$. We denote a block of $\mathbf{R}$ with rows in R_α and columns in C_β as $\mathbf{R}_{\alpha\beta}$.

In SSGD, a set of K 2D non-overlapping sub-matrix blocks are called a *stratum* (denoted by $\mathcal{S}$ hereafter). Two 2D sub-matrix blocks are said to be non-overlapping if they do not share any row or column. A set of K stratums $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_K\}$ that exhausts all of the K^2 sub-matrix blocks is called correct strata. Fig. 2 shows the strata $S = \langle \mathcal{S}_1 = \{\mathbf{R}_{1,1}, \mathbf{R}_{2,2}, \ldots, \mathbf{R}_{8,8}\}, \mathcal{S}_2 = \{\mathbf{R}_{1,2}, \mathbf{R}_{2,3}, \ldots, \mathbf{R}_{8,1}\}, \ldots, \mathcal{S}_8 = \{\mathbf{R}_{1,8}, \mathbf{R}_{2,1}, \ldots, \mathbf{R}_{8,7}\}\rangle$.

Given correct strata to be used in an SSGD epoch, each stratum is processed in a separate mini epoch (called sub-epoch), and the order in which these sub-epochs are executed can be random. Although the SSGD algorithm is serial, its distinguishing property is that no ratings in different blocks of a stratum can update the same row of the factor matrices $\mathbf{W}$ and $\mathbf{H}$, which makes it suitable for stale-free parallelization.

2) Parallel SSGD: In [3], the parallel algorithm that utilizes SSGD is called the Distributed Stochastic Gradient Descent (DSGD) algorithm. In DSGD, each stratum is executed in parallel in one sub-epoch, where the $\mathbf{W}$- and $\mathbf{H}$-matrix rows are updated with the ratings in the stratum according to (3) and (4). Then, inter-processor communications are performed to synchronize all updated rows of factor matrices. If a row-parallel execution is chosen, that is the $\mathbf{R}$ matrix is partitioned row-wise such that each row block is executed by a single processor, then communication is restricted to the $\mathbf{H}$-matrix rows. Row-parallel execution is usually preferred because the number of items is generally much less than the number of users which means the amount of data to be communicated ($\mathbf{H}$) is small compared to $\mathbf{W}$. In row-parallel execution, we abuse the stratum notation $\mathcal{S}$ to also be viewed as a mapping function $\mathcal{S} : [K] \rightarrow [K]$ (where $[K]$ is used to denote the set $\{1, \ldots, K\}$ hereafter) from a processor p_k to the index β of a column block C_β. For instance, $\mathcal{S}_2(p_4) = 5$ means that during sub-epoch 2, processor p_4 will exclusively update the rows of the $\mathbf{H}_5$ sub-matrix. We also use $\mathcal{S}_\beta^{-1}(p_x)$ to retrieve the sub-epoch at which p_x updates $\mathbf{H}_\beta$. As mentioned previously in the introduction, DSGD performs dense communications. We will utilize the parallelization style of DSGD in our methods while changing how the communication is performed. Hereafter, we will refer to the parallelization style of DSGD as "parallel SSGD," and we will use the name "DSGD" to distinguish the algorithm that performs dense communication.

3) Generating Correct Strata: There are several ways to generate a correct strata that covers the whole dataset and schedule the strata to sub-epochs. For simplicity, we consider a simple form of scheduling as follows: at sub-epoch 1, processor p_x, for $x = 1, 2, \ldots, K$, processes the ratings in $\mathbf{R}_{xx}$ to update the rows in $\mathbf{W}_x$ and $\mathbf{H}_x$; at sub-epoch k, processor p_x processes the ratings in $\mathbf{R}_{x\beta}$ to update the rows in $\mathbf{W}_x$ and $\mathbf{H}_\beta$, where $\beta = 1 + (x + k - 2) \mod K$. We refer to this scheduling as "ring scheduling" or "ring strata" hereafter. A general form of the ring scheduling consists of a $seed$, where $1 \leq seed \leq K$. At

sub-epoch k, the processor p_{seed} processes the ratings in $\mathbf{R}_{seed,k}$ to update the rows in $\mathbf{W}_{seed}$ and $\mathbf{H}_k$. At sub-epoch k, processor p_x processes the ratings in $\mathbf{R}_{x\beta}$ to update the rows in $\mathbf{W}_x$ and $\mathbf{H}_\beta$, where

$$\beta = 1 + (k + (seed + x - 1) \mod K - 2) \mod K. \quad (5)$$

III. Communication in Parallel SSGD

In this section, we analyze the communication requirement of parallel SSGD. We define the essential required communication in an SSGD epoch that utilizes the data sparsity, and compare it with the dense communication of DSGD.

Given strata S where each stratum is to be processed in a sub-epoch in row-parallel execution. For an $\mathbf{H}$-matrix row block $\mathbf{H}_\beta$, we define the sequence

$$\Upsilon_\beta = \langle p_{i_1}, p_{i_2}, \ldots, p_{i_K} \rangle,$$

of processors that compute the gradient using ratings in the column block C_β according to S. That is, p_{i_1} updates the rows of $\mathbf{H}_\beta$ in the first sub-epoch, p_{i_2} in the second sub-epoch and so forth. Furthermore, we define a distance metric d_β^{xy} between two processors p_x and p_y updating $\mathbf{H}_\beta$ as

$$d_\beta^{xy} = \mathcal{S}_\beta^{-1}(p_x) - \mathcal{S}_\beta^{-1}(p_y). \quad (6)$$

This distance quantifies the number of sub-epochs elapsed after p_x updates rows in $\mathbf{H}_\beta$ and before p_y does so.

A. Defining Essential Required Communication

The communication of $\mathbf{H}$-matrix rows required for correctly executing SSGD in a distributed fashion is described according to the following definition:

Definition 1 (d-gap rows): During parallel SSGD, if a row $\mathbf{h}_j \in \mathbf{H}_\beta$ is updated by both p_{i_x} and $p_{i_{x+1}}$, then $\mathbf{h}_j$ is called a zero-gap row. If $\mathbf{h}_j$ is updated by both p_{i_x} and $p_{i_{x+2}}$ but not $p_{i_{x+1}}$, then $\mathbf{h}_j$ is called a one-gap row. Then for the general case, consider two nonadjacent processors in Υ_β: p_{i_x} and p_{i_y} such that $x < y$. $\mathbf{h}_j$ is called a d-gap row if it is updated by both p_{i_x} and p_{i_y} but not any of the $d = d_\beta^{xy}$ processors in-between (that is, in $\langle p_{i_{x+1}}, \ldots, p_{i_{y-1}} \rangle$). The set of all such d-gap rows between p_{i_x} and p_{i_y} in $\mathbf{H}_\beta$ is given by

$$\mathbf{H}_\beta^{i_x i_y} = \{\mathbf{h}_j \mid (\exists r_{ij})[r_{ij} \in \mathbf{R}_{x\beta} \cap \mathbf{R}_{y\beta} \wedge r_{ij} \notin \mathbf{R}_{(x+1)\beta} \cup \cdots \cup \mathbf{R}_{(y-1)\beta}]\}. \quad (7)$$

Communicating $\mathbf{H}_\beta^{i_x i_y}$ from p_{i_x} to p_{i_y} after p_{i_x} processes the ratings in 2D block $\mathbf{R}_{i_x\beta}$ and before p_{i_y} starts processing the ratings in $\mathbf{R}_{i_y\beta}$ guarantees a correct distributed row-parallel SSGD execution.

B. Communication in DSGD

In the original DSGD algorithm [3], after processor p_x updates row block $\mathbf{H}_\beta$ in sub-epoch k, it sends the rows in $\mathbf{H}_\beta$ to the the processor that will update $\mathbf{H}_\beta$ in sub-epoch $k+1$. Therefore, at each sub-epoch, each processor sends a whole row block of $\mathbf{H}$ to exactly one processor. For instance, assuming the DSGD is executed according to the ring strata given in Fig. 2, after sub-epoch 1 is completed p_1 sends $\mathbf{H}_1$ to p_8, p_8 sends $\mathbf{H}_8$ to p_7 and so forth.

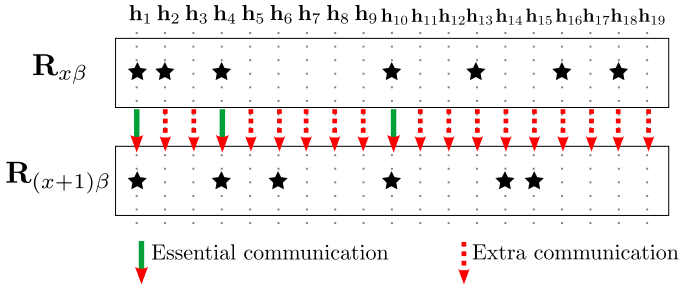


Fig. 3. Illustrating the extra communication of DSGD. The figure shows two blocks of $\mathbf{R}$ that belong to processors p_x and p_{x+1} such that p_{x+1} updates column block $\mathbf{H}_\beta$ right after p_x. A row $\mathbf{h}_j \in \mathbf{H}_\beta$ shas to be sent from p_x to p_{x+1} if both processors contain a nonzero with column index equal to $\mathbf{h}_j$ because p_{x+1} has to know the up-to-date version of $\mathbf{h}_j$ after p_x updates it. When either only one of p_x or p_{x+1} has such nonzero, or neither of them, the communication of $\mathbf{h}_j$ at this stage is considered extra and can be avoided.

The communication scheme of DSGD guarantees the correctness of the SSGD algorithm since up-to-date $\mathbf{h}_j \in \mathbf{H}_\beta^{i_x i_y}$ will eventually reach p_{i_y} from p_{i_x}, assuming $x < y$ in Υ_β, via forwarding through $\langle p_{i_{x+1}}, \ldots, p_{i_{y-1}} \rangle$. Furthermore, the communication scheme of the DSGD has the nice property of very low latency overhead since it restricts the number of messages sent by any processor at any sub-epoch to one. However, this scheme suffers from increasing the bandwidth overhead (communication volume) due to forwarding the $\mathbf{H}$-matrix rows. For each epoch, the communication volume sent by all processors is equal to $F \times M \times K$ words as each processor sends approximately M/K dense $\mathbf{H}$-matrix rows each of size F words during each of the K sub-epochs. Especially for highly sparse rating matrices, it is clear that the volume of communication performed is much more than the required, and the increased bandwidth overhead due to forwarding can be prohibitive as K increases, see Fig. 3.

In Fig. 2, the update sequence for row block $\mathbf{H}_1$ is $\Upsilon_1 = \langle p_1, p_8, p_7, p_6, p_5, p_4, p_3, p_2 \rangle$. The communication of $\mathbf{h}_i \in \mathbf{H}_1$ through the subsequence/subchain $p_1 \rightarrow p_8 \rightarrow p_7 \rightarrow p_6$ does not incur any extra volume since each of these processors update $\mathbf{h}_i$. However, p_5 does not update $\mathbf{h}_i$ but still p_5 needs to receive the up-to-date $\mathbf{h}_i$ from p_6 and forward it to p_4 in the next sub-epoch. In this case, $\mathbf{h}_i$ incurs F words of forwarding overhead. In the case of $\mathbf{h}_j \in \mathbf{H}_1$, the first processor to update it after p_1 is p_4. Therefore, four forwarding communications, each of size F, are incurred due to $\mathbf{h}_j$ in $p_1 \rightarrow p_8 \rightarrow p_7 \rightarrow p_6 \rightarrow p_5$.

Let $\lambda(\mathbf{h}_j)$ denote the number of processors that update $\mathbf{h}_j \in \mathbf{H}_\beta$, then the amount of forwarding overhead of $\mathbf{h}_j$ in DSGD is $F(K - \lambda(\mathbf{h}_j))$. The total amount of forwarding overhead per epoch then becomes $F(MK - \sum_{\mathbf{h}_j \in \mathbf{H}} \lambda(\mathbf{h}_j))$. The clear difference between the communication in DSGD and the essential required communication is that the former is a direct factor of K and M, whereas the latter is upper bounded by the number of nonzeros (nnz) of the rating matrix. This can be shown as follows: at sub-epoch k, processor p_x sends $nnz(\mathbf{R}_{x,\mathcal{S}_k(p_x)})$ $\mathbf{H}$-matrix rows at the worst case. This means that, at worst case,

Algorithm 1: Point-to-Point Parallel SSGD on Processor p_x.

Require: Rating matrix $\mathbf{R}$, Processor count K
1: Initialize local factor matrices $\mathbf{W}$ and $\mathbf{H}$ randomly
2: **repeat**
3: Receive strata S from p_1 through `Bcast`.
4: Construct P2P communication according to S
5: **for** $k = 1$ to K **do** ▷For each sub-epoch
6: $\beta_{\text{prev}} \leftarrow \mathcal{S}_{k-1}(p_x)$
7: $\beta_{\text{curr}} \leftarrow \mathcal{S}_k(p_x)$;
8: **for** each $p_y \in SendSet^k(p_x)$ **do**
9: Send $\mathbf{H}^{xy}_{\beta_{\text{prev}}}$ to p_y
10: **end for**
11: **for** each $p_z \in RecvSet^k(p_x)$ **do**
12: Receive $\mathbf{H}^{zx}_{\beta_{\text{curr}}}$ from p_z
13: **end for**
14: **for** each $r_{ij} \in \mathbf{R}_{x\beta_{\text{curr}}}$ **do**
15: $\mathbf{w}_i = \mathbf{w}_i - \epsilon[(r_{ij} - \hat{r}_{ij})\mathbf{h}_j + \gamma \mathbf{w}_i]$
16: $\mathbf{h}_j = \mathbf{h}_j - \epsilon[(r_{ij} - \hat{r}_{ij})\mathbf{w}_i + \gamma \mathbf{h}_j]$
17: **end for**
18: **end for**
19: **until** convergence or max. number of epochs reached

the total volume of communication sent by all processors per an SSGD epoch is equal to $F \sum_{x\in[K]} \sum_{k\in[K]} nnz(\mathbf{R}_{x,\mathcal{S}_k(p_x)}) = F \times nnz(\mathbf{R})$.

IV. A Framework for Scaling SSGD

A. Communicating d-Gap Rows Through P2P Messages

We propose to avoid the forwarding overhead by sending an updated $\mathbf{H}$-matrix row to the processor that updates it next directly through P2P communications. At the beginning of sub-epoch k, processor p_x sends P2P messages to a set of processors $SendSet^k(p_x)$ and receives from $RecvSet^k(p_x)$. These two sets can be respectively constructed as

$$SendSet^k(p_x) = \{p_y \mid \mathbf{H}^{xy}_{\mathcal{S}_{k-1}(p_x)} \neq \emptyset\}, \tag{8}$$

$$RecvSet^k(p_x) = \{p_y \mid \mathbf{H}^{yx}_{\mathcal{S}_k(p_x)} \neq \emptyset\}. \tag{9}$$

For example, in Fig. 2, at the beginning of the second sub-epoch p_1 sends $\mathbf{h}_i$ to p_8 and $\mathbf{h}_j$ to p_4.

Algorithm 1 presents the P2P-based parallel SSGD algorithm for processor p_x. At line 3, processor p_1 picks strata S and broadcasts to all other processors. At line 4, p_x determines the communication requirement according to (7) and constructs the send/receive information of the P2P messages according to (8) and (9). Then, the up-to-date rows required in the current sub-epoch are communicated at lines 8-13 through P2P messages. The SGD updates are performed at lines 15 and 16 respectively according to (3) and (4).

Algorithm 2: Find d-gap $\mathbf{H}$-matrix Rows on Processor p_x

Require: Rating matrix $\mathbf{R}$, Processor count K, Strata S
1: **for** each $\mathbf{H}_\beta \in \mathbf{H}$ **do**
2: Compute $\Upsilon^{p_x}_\beta$
3: $mask \leftarrow B^{\Upsilon^{p_x}_\beta[2]}_\beta$
4: **for** $k = 2$ to K **do**
5: $p_y \leftarrow \Upsilon^{p_x}_\beta[k]$
6: $\Psi^{p_x,p_y}_\beta \leftarrow B^x_\beta \wedge B^y_\beta \oplus (B^x_\beta \wedge B^y_\beta \wedge mask)$
7: $\mathbf{H}^{xy}_\beta \leftarrow \{\mathbf{h}_i \in \mathbf{H}_\beta \mid \Psi^{p_x,p_y}_\beta[i] = 1\}$
8: $mask \leftarrow mask \vee B^y_\beta$
9: **end for**
10: **end for**

B. Efficiently Constructing d-Gap Row Sets

Computing d-gap $\mathbf{H}$-matrix rows using (7) has recurring computations for different instances. For example, computing $\mathbf{H}^{i_x i_y}_\beta$ and $\mathbf{H}^{i_x i_{y+1}}_\beta$ would require computing the same $\mathbf{R}_{(x+1)\beta} \cup \cdots \cup \mathbf{R}_{(y-1)\beta}$ term twice. For an efficient computation, we devise an algorithm that utilizes a dynamic programming formulation leveraging efficient bulk bit-wise operations.

Consider a binary string $B^{i_x}_\beta \in \{0,1\}^{n_\beta}$ of length $n_\beta = |\mathbf{H}_\beta|$, such that the bth entry of $B^{i_x}_\beta$ is set to '1' if p_{i_x} updates the bth row in $\mathbf{H}_\beta$, and set to '0' otherwise. Then, the indices of the rows to be communicated between p_{i_x} and p_{i_y} are the indices of the 1-bits in

$$\Psi^{i_x,i_y}_\beta = B^{i_x}_\beta \wedge B^{i_y}_\beta \oplus (B^{i_x}_\beta \wedge B^{i_y}_\beta \wedge (B^{i_x+1}_\beta \vee \cdots \vee B^{i_y-1}_\beta)), \tag{10}$$

where $\oplus$, $\wedge$ and $\vee$ respectively denote logical exclusive OR (XOR), logical AND and logical OR operations. The term $(B^{i_x+1}_\beta \vee \cdots \vee B^{i_y-1}_\beta)$ in (10) can be computed incrementally thanks to the associativity property of the $\vee$ operation.

Given $\mathbf{H}_\beta$ and Υ_β, we define $\Upsilon^{p_x}_\beta$ as the sequence of processors updating $\mathbf{H}_\beta$ starting from p_x. $\Upsilon^{p_x}_\beta$ can be obtained from Υ_β by left-rotating the sequence until p_x is at the first index. Algorithm 2 presents the efficient dynamic-programming-based computation of the d-gap $\mathbf{H}$-matrix rows between p_x and the other $K-1$ processors. For each $\mathbf{H}_\beta$, the order of processors updating $\mathbf{H}_\beta$ starting from p_x according to strata S is maintained in $\Upsilon^{p_x}_\beta$ (line 3). Then, in lines 4-9, p_x constructs the d-gap rows one by one according to this order leveraging the bottom-up construction of the term $(B^{i_x+1}_\beta \vee \cdots \vee B^{i_y-1}_\beta)$.

C. Hold & Combine Strategy for Reducing Latency

Using P2P messages to communicate the updated rows without forwarding is indispensable for reducing the bandwidth overhead of the communication. However, it has a high potential of increasing the latency overhead via increasing the number of messages performed per epoch compared to DSGD. In DSGD, a processor sends K messages per epoch (one message to one processor at each sub-epoch), whereas using the P2P requires sending at most $K \times (K-1)$ messages per epoch (up to $K-1$ messages from each of the K processors at each sub-epoch).

We propose the hold and combine (H&C) strategy to reduce the upper-bound on the number of messages sent per epoch to $\mathcal{O}(K \lg K)$.

Definition 2: Fixed-distance strata is any strata that satisfies

$$d^{xy}_{\alpha} = d^{xy}_{\beta} \quad \text{for any pair of } \mathbf{H}\text{-matrix rows } \alpha \text{ and } \beta. \tag{11}$$

That is, the fixed-distance strata have the property of constant distance between any two processors regardless of the $\mathbf{H}$-matrix row block they are updating. We refer to the distance between two processors p_x and p_y in a fixed-distance strata as d^{xy}. Any ring strata scheduled with (5) is a fixed-distance strata.

During an SGD epoch, the communication of $\mathbf{H}^{xy}_{\beta}$ should be performed after p_x updates $\mathbf{H}_{\beta}$ in sub-epoch k and before p_y starts updating $\mathbf{H}_{\beta}$. This means that $\mathbf{H}^{xy}_{\beta}$ can be sent at the beginning of any sub-epoch between $k+1$ and $k+d^{xy}_{\beta}$. Now consider the communication of $\mathbf{H}^{xy}_{\beta}$ at sub-epoch k in a fixed-distance strata. Observe that when sub-epoch $k+d^{xy}$ is reached, all the rows in $\mathbf{H}^{xy}_{\mathcal{S}_{k+1}(p_x)}, \mathbf{H}^{xy}_{\mathcal{S}_{k+2}(p_x)}, \ldots, \mathbf{H}^{xy}_{\mathcal{S}_{k+d^{xy}-1}(p_x)}$ are already updated by p_x and ready to be sent to p_y. So, these rows can be held by p_x and sent all at once in one message to p_y in sub-epoch $k+d^{xy}$ along with $\mathbf{H}^{xy}_{\beta}$.

Utilizing fixed-distance strata, we propose to hold P2P messages and combine them as follows: If $d^{xy} \geq K/2$, then the messages between p_x and p_y in an epoch can be combined into two or more P2P messages. This is because if $d^{xy} = K-1$ then one message is needed for $K-1$ $\mathbf{H}$-matrix row blocks and another message needed for the last block. Otherwise if $d^{xy} < K/2$, then the messages between p_x and p_y can be combined in $\lceil K/d^{xy} \rceil$ P2P messages. Therefore, the number of messages sent per processor per epoch can be computed by

$$\sum_{i=1}^{\frac{K}{2}} 2 + \sum_{i=1}^{\frac{K-1}{2}} \frac{K}{i}.$$

The second summation is a harmonic series which can be approximated by $\ln(K-1)/2+1$, thus

$$\sum_{i=1}^{\frac{K}{2}} 2 + \sum_{i=1}^{\frac{K-1}{2}} \frac{K}{i} \approx K + K\ln(K-1)/2 \leq K \lg K. \tag{12}$$

is the upper bound on the number of messages sent per processor per epoch.

To facilitate the presentation of the H&C strategy, we assume that each processor constructs a tabular-shaped message schedule (TSMS). In the TSMS of p_x, rows are the $K-1$ processors that p_x communicates with during an epoch, and columns represent sub-epochs as well as the corresponding $\mathbf{H}$-matrix row blocks updated by p_x. Each table entry TSMS(p_y, $\mathbf{H}_{\beta}$) represents the sub-epoch $\mathcal{S}^{-1}_{\beta}(p_y)$.

Fig. 4 shows a TSMS for p_3 using strata with $seed = 5$. In the figure, the circled TSMS entries denote the messages ($\mathbf{H}$-matrix row blocks) that can be combined. For instance, the communication requirement between p_3 and p_7 during an SGD epoch can be done with two messages. The first message, required at the beginning of sub-epoch 5, consists of $\mathbf{H}^{3,7}_7 \cup \mathbf{H}^{3,7}_8 \cup \mathbf{H}^{3,7}_1 \cup \mathbf{H}^{3,7}_2$. The second message, required at the beginning of sub-epoch 1 of the next SGD epoch, consists of $\mathbf{H}^{3,7}_3 \cup \mathbf{H}^{3,7}_4 \cup \mathbf{H}^{3,7}_5 \cup \mathbf{H}^{3,7}_6$. Observe that the sub-epoch at which the combined message should be sent is decided by the first $\mathbf{H}$-matrix block of the combined message. For instance, the first message to p_7 must arrive before p_7 starts updating rows in $\mathbf{H}_7$ which is sub-epoch 5.

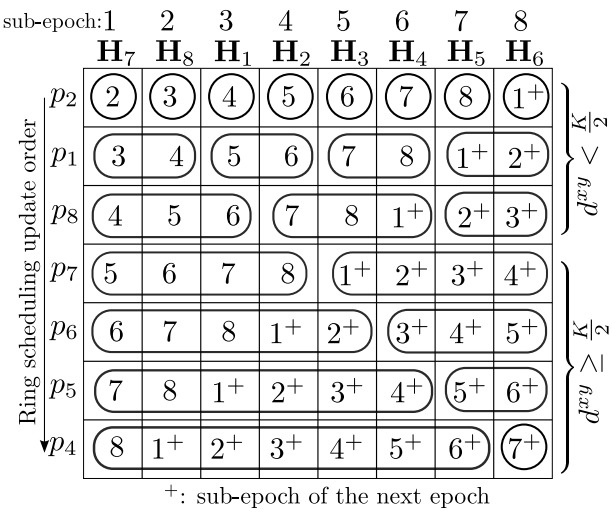


Fig. 4. An example TSMS for p_3. The rows are the processors that p_3 communicates with sorted according to their distance from p_3. The columns represent both the sub-epochs and the $\mathbf{H}$-matrix blocks to be updated at each sub-epoch. An entry (p_y, $\mathbf{H}_{\beta}$) gives the sub-epoch at which p_y updates $\mathbf{H}_{\beta}$ after p_3 does (note that this sub-epoch might be in the next epoch). The circles show the messages that can be combined.

Algorithm 3: Message Combining Strategy on Processor p_x.

Require: $SendSet^k(p_x)$, $\mathbf{H}^{xy}$ $\forall k, y \in [K] \wedge y \neq x, S$

1: **for** $k = 1$ to K **do**
2: **for** each $p_y \in SendSet^k(p_x)$ **do**
3: $m^{\text{id}} \leftarrow \lceil \frac{k}{d^{xy}} \rceil$ ▷get the message ID
`//get the` $\mathbf{H}$`-matrix block that` p_x `updates at SE` k
4: $\beta \leftarrow \mathcal{S}_k(p_x)$
`//add the` d`-gap rows` $\mathbf{H}^{xy}_{\beta}$ `to Msg` m^{id}
5: $M^{xy}_{m^{\text{id}}} \leftarrow M^{xy}_{m^{\text{id}}} \cup \mathbf{H}^{xy}_{\beta}$
`//When does` p_x `update the first block of` m^{id} `?`
6: $t \leftarrow (m^{\text{id}} - 1) * d^{xy} + 1$
`//get the` $\mathbf{H}$`-matrix block that` p_x `updates at SE` t
7: $\eta \leftarrow \mathcal{S}_t(p_x)$
`//get the sub-epoch` s `at which` p_y `updates` $\mathbf{H}_{\eta}$
8: $s \leftarrow \mathcal{S}^{-1}_{\eta}(p_y)$
9: $cSendSet^s(p_x) \leftarrow cSendSet^s(p_x) \cup \{p_y\}$
10: **end for**
11: **end for**

Algorithm 3 shows the procedure to construct combined messages from P2P messages at p_x. Given the $SendSet^k(p_x)$ $\forall k \in \{1, \ldots, K\}$ and the d-gap rows between p_x and $\{p_y \mid y \neq x\}$, the combined messages are constructed as follows: There are $\lceil K/d^{xy} \rceil$ possible messages to p_y each of which is identified by m^{id}. For each $p_y \in SendSet^k(p_x)$ the rows in $\mathbf{H}^{xy}_{\beta}$ are assigned to a combined message $M^{xy}_{m^{\text{id}}}$ (lines 3 and 4). Then, p_y is added to the new send set of the sub-epoch at which message m^{id} is sent (lines 5-9).

Algorithm 1 can be modified to accommodate the H&C strategy as follows: After constructing the P2P communication (line 4), Algorithm 3 is used to combine the messages. Then, lines 8-13 can be replaced with the sending/receiving of combined messages; for each p_y in $cSendSet^k(p_x)$ a combined message is identified using $m^{\text{id}} = \lceil k/d^{xy} \rceil$ and sent to p_y, and similarly so for receiving from each p_z in $cRecvSet^k(p_x)$.

It is important to make sure that the $K \lg K$ messages sent per epoch are uniformly distributed over K sub-epochs. Otherwise, some sub-epochs will constitute a performance bottleneck due to high number of messages. We show that utilizing Algorithm 3 for combining the messages has the nice property of limiting the expected number of messages sent by each processor at each sub-epoch to $\mathcal{O}(\lg K)$.

Theorem 1: Using the H&C strategy, the expected number of messages sent by each processor at each sub-epoch is $\mathcal{O}(\lg K)$.

Proof: Consider a set $\mathcal{F}^{xy}$ that consists of all sub-epochs wherein a message is sent from p_x to p_y. For each sub-epoch k, the function

$$\sigma(k, p_x, p_y) = \begin{cases} 1 & k \in \mathcal{F}^{xy} \\ 0 & \text{otherwise} \end{cases},$$

defines if there is a message to be sent from p_x to p_y in k.

We can prove that $\mathcal{O}(\lg K)$ messages are sent by each processor at each sub-epoch as follows. The number of messages sent by p_x per sub-epoch is equal to the number of occurrences of that sub-epoch in $\biguplus_{y \in [K] \wedge y \neq x} \mathcal{F}^{xy}$. For each processor p_y with distance d^{xy}, the probability that k is one of the sub-epochs in which a message is sent to p_y is equal to $1/d^{xy}$. In other words, given K sub-epochs, the probability that sub-epoch k will be used to send one of the $\lceil K/d^{xy} \rceil$ messages is $1/d^{xy}$. Then, the expected number of messages from p_x to p_y at sub-epoch k is

$$\mathbf{E}[\sigma(k, p_x, p_y)] = \mathbf{Pr}(\sigma(k, p_x, p_y) = 1) \times 1 + \mathbf{Pr}(\sigma(k, p_x, p_y) = 0) \times 0 = \frac{1}{d^{xy}}.$$

Using linearity of expectation, the expected total number of messages sent by p_x at sub-epoch k is

$$\mathbf{E}\left[\sum_{p_y \neq p_x} \sigma(k, p_x, p_y)\right] = \sum_{i=1}^{K} \frac{1}{i} \approx \ln K + 1 \leq \lg K + 1. \tag{13}$$

V. HP Model for Reducing Bandwidth Cost

There exists two hypergraph models for 1D partitioning of sparse matrices for SpMV-like kernels; namely the column-net model for rowwise partitioning and the row-net model for columnwise partitioning [8]. In these models, the "connectivity−1" metric [8] is utilized for partitioning objective of reducing the communication volume in SpMV-like kernels, whereas the partitioning constraint is maintaining computational balance among processors. As mentioned earlier, rating matrices usually have larger number of rows than columns, hence we mainly focus on rowwise partitioning of rating matrix $\mathbf{R}$. The hypergraph model discussed here is topologically similar to the column-net model, however the cutsize metric utilized in the partitioning objective is different.

In the hypergraph model $\mathcal{H}_{\mathbf{R}} = (\mathcal{V}, \mathcal{N})$, there exists a vertex $v_i \in \mathcal{V}$ for each row $\mathbf{r}_i$ of $\mathbf{R}$ and a net (hyperedge) $n_j \in \mathcal{N}$ for each column $\mathbf{c}_j$ of $\mathbf{R}$. Each net n_j connects the vertices corresponding to the $\mathbf{R}$-matrix rows that contain nonzeros in column $\mathbf{c}_j$. That is, $Pins(n_j) = \{v_i \in \mathcal{V} \mid r_{ij} \neq 0\}$. Each vertex v_i is associated with a weight equals to the number of nonzeros in row $\mathbf{r}_i$. Each net is associated with a cost F.

A K-way partition $\Pi(\mathcal{H}_{\mathbf{R}}) = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_K\}$ is decoded as a K-way rowwise partition of $\mathbf{R}$, where the rows corresponding to the vertices in part $\mathcal{V}_\alpha$ constitute the row block $\mathbf{R}_\alpha$, for $\alpha = 1, 2, \ldots, K$. Without loss of generality, row block $\mathbf{R}_\alpha$ is assigned to processor p_α for $\alpha = 1, 2, \ldots, K$. The $\mathbf{W}$-matrix rows are partitioned conformably with the $\mathbf{R}$-matrix row partition. That is, $\mathbf{W}$-matrix rows in $\mathbf{W}_\alpha$ correspond to the $\mathbf{R}$-matrix rows in $\mathbf{R}_\alpha$.

In partition $\Pi(\mathcal{H}_{\mathbf{R}})$, the weight of each part is equal to the sum of the weights of the vertices in that part. Hence, the partitioning constraint of maintaining balance on the part weights encodes maintaining balance on the nonzero counts of the $\mathbf{R}$-matrix row blocks. This in turn corresponds to maintaining balance on the computational loads of the processors.

In partition $\Pi(\mathcal{H}_{\mathbf{R}})$, a net n_j is said to connect a part $\mathcal{V}_\alpha$ if it connects at least one vertex in part $\mathcal{V}_\alpha$, that is, $Pins(n_j) \cap \mathcal{V}_\alpha \neq \emptyset$. The connectivity set $\Lambda(n_j)$ of a net n_j is defined as the set of the parts that net n_j connects, whereas the connectivity $\lambda(n_j)$ denotes the number of parts connected by net n_j, that is $\lambda(n_j) = |\Lambda(n_j)|$. A net n_j is said to be cut if $\lambda(n_j) > 1$ and uncut otherwise. The partitioning objective is to minimize the cutsize which is defined over the cut nets.

In this model, $\Lambda(n_j)$ also represents the set of $\mathbf{R}$-matrix row blocks that has at least one nonzero in column $\mathbf{c}_j$ of $\mathbf{R}$. Hence, the connectivity set of net n_j denotes the set of processors that update the $\mathbf{H}$-matrix row $\mathbf{h}_j$. Consider the $\mathbf{H}$-matrix row $\mathbf{h}_j$ corresponding to a cut net n_j in the P2P communication scheme. Also consider $\mathbf{h}_j$ update sequence defined using the connectivity set and strata. For each epoch, each processor except the last processor in the sequence should send its updated $\mathbf{h}_j$ value once to the next processor in the sequence. The last processor sends its updated $\mathbf{h}_j$ value to the first processor for the next iteration. Hence, each cut net n_j incurs a communication volume of $F\lambda(n_j)$. On the other hand, uncut nets incurs no communication. Therefore, cutsize which encapsulates the total communication volume during an SSGD epoch can be computed as

$$\sum_{n_j \ni \lambda(n_j) > 1} F\lambda(n_j). \tag{14}$$

Among the various cutsize metrics in the literature, cutsize (14) is called as the sum of external degrees (SOED) [9].

There exists several successful hypergraph partitioning tools that utilize multilevel recursive bipartitioning (RB) algorithms. Among these partitioning tools, to our knowledge, only *hMETIS* [9] supports the SOED metric via direct multi-way partitioning [10]. In fact Karypis and Kumar [10] clearly indicates that RB framework does not allow directly optimizing the SOED

metric. Here, we propose an RB framework that encodes the minimization of the SOED metric correctly.

In the RB framework, a given hypergraph $\mathcal{H}$ is recursively bipartitioned until K parts are obtained, assuming K is a power of two without loss of generality. At each RB step, a bipartition $\Pi_2 = \{\mathcal{V}_L, \mathcal{V}_R\}$ on the current hypergraph forms two vertex-induced subhypergraphs $\mathcal{H}_L = (\mathcal{V}_L, \mathcal{N}_L)$ and $\mathcal{H}_R = (\mathcal{V}_R, \mathcal{N}_R)$. Here, $\mathcal{V}_L$ and $\mathcal{V}_R$ are respectively used to refer to the left and right parts of the bipartition. The net sets $\mathcal{N}_L$ and $\mathcal{N}_R$ are constructed through cut net splitting method [8] as follows: Internal nets of $\mathcal{V}_L$ and $\mathcal{V}_R$ are respectively included in $\mathcal{N}_L$ and $\mathcal{N}_R$. A cut net n_j in Π_2 is split into two subnets n'_j and n''_j, where $Pins(n'_j) = Pins(n_j) \cap \mathcal{V}_L$ and $Pins(n''_j) = Pins(n_j) \cap \mathcal{V}_R$.

In order to encode the SOED metric (14), we propose the following strategy during the RB framework. We assign a cost of $2F$ to each net of the initial hypergraph. Then, after each RB step, internal nets inherit their cost, whereas splitted nets are assigned a cost of F. That is, a net holds its cost of $2F$ until it becomes cut for the first time, then a cost of F is assigned to each of split subnets and they inherit their cost of F through the further RB steps until the end of the partitioning. Hence, when a net becomes cut for the first time it incurs $2F$ to the cutsize, then whenever its subnets become cut they incur F to the cutsize. In this way, the sum of all cut net costs encountered during the overall RB algorithm becomes equal to the SOED metric (14).

VI. Experimental Evaluations

A. Experimental Framework

We evaluate the contributions proposed in this work through comparing three methods implementing parallel SSGD using six real-world rating matrices. The first method, DSGD, is the algorithm proposed in the original work of Gemulla et al. [3]. DSGD performs block-wise communication of $\mathbf{H}$-matrix row blocks in each sub-epoch. The second method, P2P, uses P2P messages as in Algorithm 1. The third, H&C, uses combined P2P messages (Algorithm 3) for communication.

In all three methods, column-to-stratum assignments are done randomly in such a way that the number of columns per stratum differs by at most one. Row-to-processor assignments are obtained either randomly in a way similar to that of column-to-stratum assignments, or using the HP method discussed in Section V. Whenever the former is used, the method will be prefixed by RAND, whereas if the latter is used the method will be prefixed by HP. The HP method is implemented according to the RB framework described in Section V to encapsulate the SOED metric. In order to obtain two-way partitions on the (sub)hypergraphs at each RB level, we use the HP tool PaToH [8] with default parameters in `SPEED` mode.

We implemented the parallel SSGD code that includes DSGD, P2P and H&C in C and used MPI for inter-process communications. We perform our experiments on an HPC system with AMD EPYC 7742 processors and a high-speed HDR InfiniBand network with 200 Gb/s bandwidth.

We compare the three methods in terms of communication cost metrics as well as SGD iteration time. The communication cost metrics consist of bandwidth-oriented metrics: *sum-max vol* and *tot vol*, and latency-oriented metrics: *sum-max msgs* and *tot msgs*. *sum-max msgs* is calculated as follows: at each sub-epoch, the number of messages sent by the bottleneck processor (the processor that sends highest number of messages) is obtained. Then, the summation is taken over all K sub-epochs. That is,

$$\textit{sum-max msgs} = \sum_{k=1}^{K} \max_{x \in [K]} (|SendSet^k(p_x)|).$$

In a similar way, *sum-max vol* is computed as

$$\textit{sum-max vol} = \sum_{k=1}^{K} \max_{x \in [K]} (SendVol^k(p_x)).$$

tot msgs and *tot vol* are respectively computed as

$$\textit{tot msgs} = \sum_{k=1}^{K} \sum_{x \in [K]} (|SendSet^k(p_x)|),$$

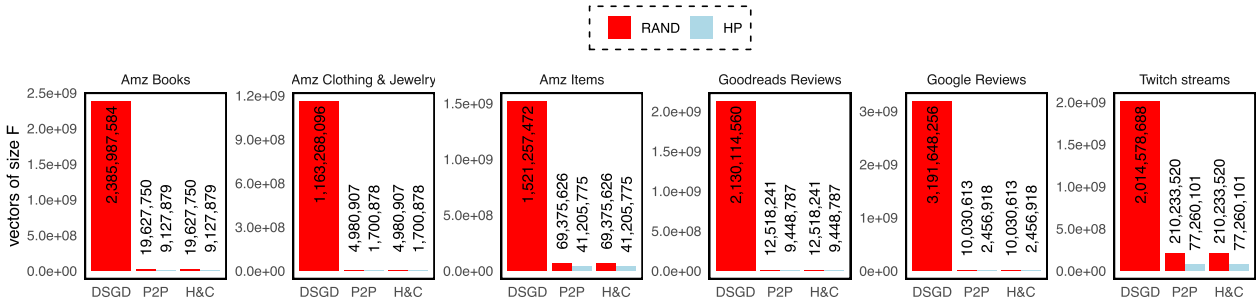
$$\textit{tot vol} = \sum_{k=1}^{K} \sum_{x \in [K]} (SendVol^k(p_x)).$$

Here, $SendVol^k(p_x) = |\mathbf{H}_{\mathcal{S}_{k-1}(p_x)}|$ if DSGD is used, and $SendVol^k(p_x) = \sum_{p_y} |\mathbf{H}^{xy}_{\mathcal{S}_{k-1}(p_x)}|$ if P2P or H&C are used. Whenever the values for the volume of communication are presented, these values are normalized with respect to F. This uncoupling of F from the volume values helps evaluate the proposed methods and model for any F value.

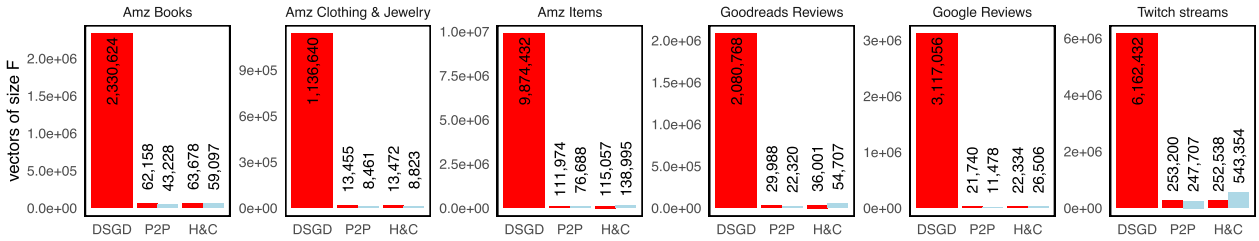
TABLE I
Properties of Matrices in the Dataset

Matrix	#rows	#cols	#nnz	density
`Amz Items`	21.177M	9.874M	82.677M	3.95E-07
`Amz Books`	8.026M	2.330M	22.50M	1.20E-06
`Amz Clothing & Jewelry`	3.117M	1.136M	5.75M	1.62E-06
`Goodreads Reviews`	0.465M	2.080M	15.740M	1.63E-05
`Google Reviews`	5.055M	3.117M	11.454M	7.27E-07
`Twitch`	15.524M	6.162M	474.677M	4.96E-06

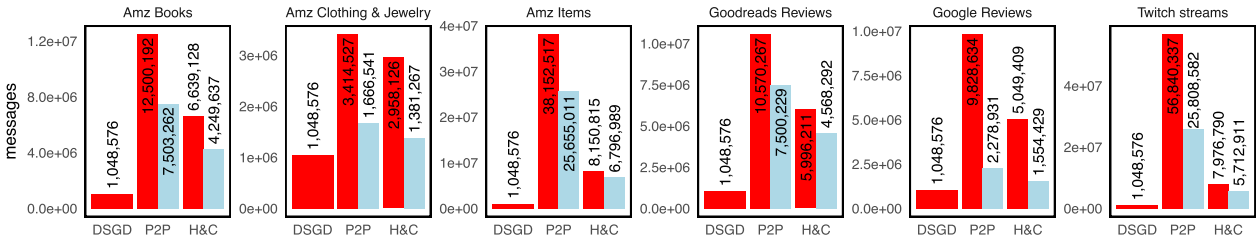
Table I shows the real-world matrices used to evaluate the proposed methods and their properties. `Amz Items` contains product reviews from Amazon between May 1996 - July 2014 [11] with aggressive duplicate removal. The other two amazon datasets, Books and Clothing, are category-based subsets of the original comprehensive reviews. `Goodread Reviews` contains user ratings of books from the Goodreads website [12]. `Google Reviews` contains user ratings/reviews of local businesses from the Google Maps website [13], [14]. `Twitch` contains ratings relative to how much time a user spent on a stream in the Twitch streaming website [15]. The original data does not contain any explicit ratings. We modified the dataset to represent (*user, stream, rating*) such that the rating value is proportional to the amount of time the user spent in the specific stream.



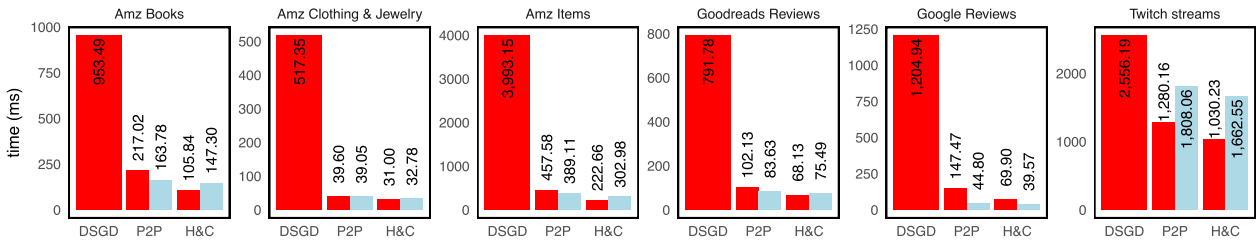
(a) *tot vol* in an SGD epoch



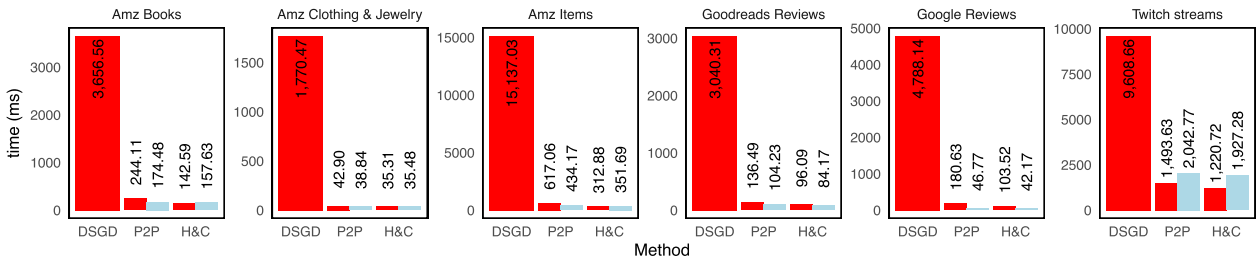
(b) *sum-max vol* in an SGD epoch



(c) *tot msgs* in an SGD epoch

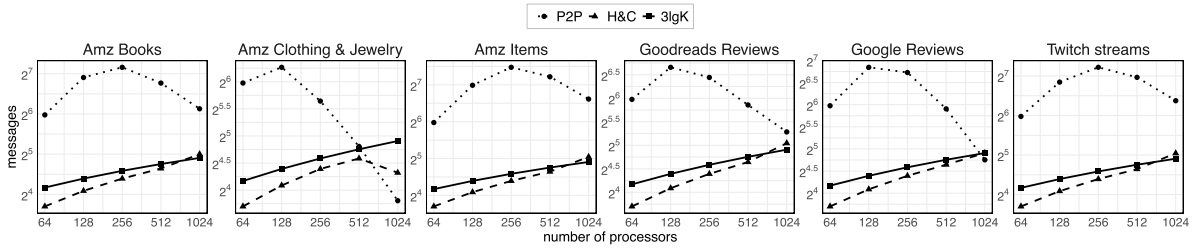


(d) SGD iteration times when $F = 16$

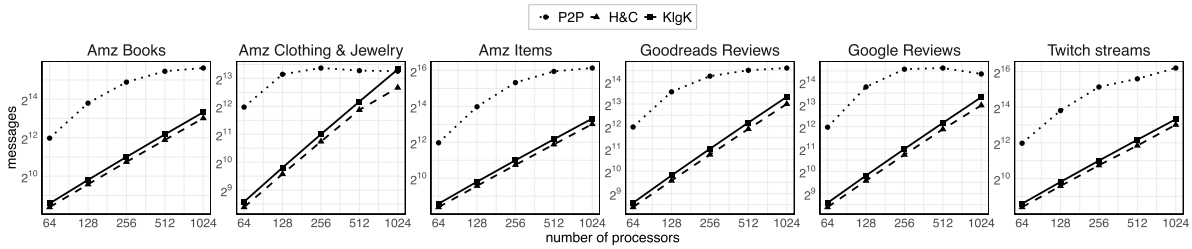


(e) SGD iteration times when $F = 64$

Fig. 5. Comparing RAND- and HP-based P2P and H&C methods against RAND-based DSGD using communication cost metrics (a to c) and SGD iteration time (d and e) using all dataset matrices on $K = 1024$ processors.

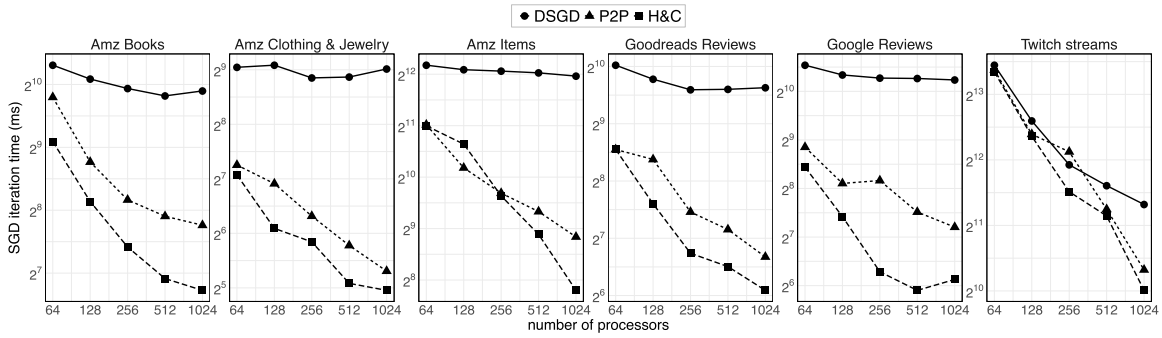


(a) *max-max msgs* in an SGD epoch ($\log_2$-$\log_2$)

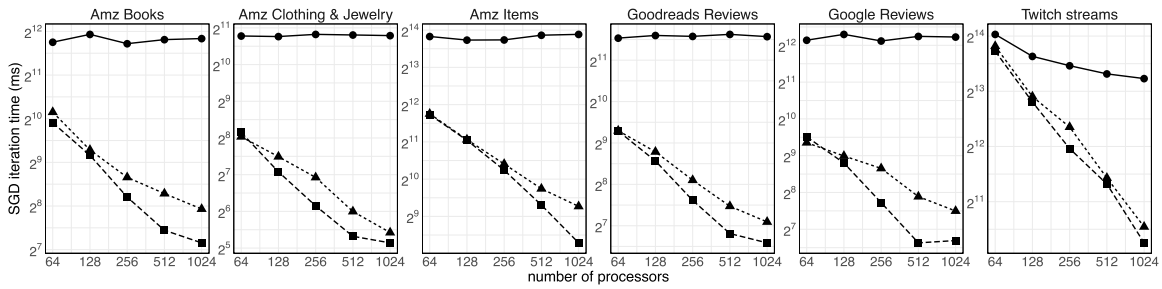


(b) *sum-max msgs* in an SGD epoch ($\log_2$-$\log_2$)

Fig. 6. Showcasing the upper bound of the max-max messages and sum-max messages sent per sub-epoch using the H&C method compared to P2P on $K = \{64, \ldots, 1024\}$ processors.



(a) SGD iteration times when $F = 16$ ($\log_2$-$\log_2$)



(b) SGD iteration times when $F = 64$ ($\log_2$-$\log_2$)

Fig. 7. Strong scaling curves of DSGD, P2P and H&C on $K = \{64, 128, 256, 512, 1024\}$ processors using all dataset matrices with two F values.

B. Evaluations With Communication Cost Metrics

Fig. 5(a), (b) and (c) compare DSGD, P2P and H&C in terms of communication cost metrics *tot vol*, *sum-max vol* and *tot msgs* on $K = 1024$ processors. In the figures, the red bars denote RAND-based methods whereas light blue bars denote HP-based methods. HP does not affect DSGD's communication which is why HP is not applicable for DSGD and hence DSGD has only red bars. Comparison in terms of *sum-max msgs* will be discussed in Fig. 6 .

1) Bandwidth-Oriented Communication Cost Metrics: As seen in Fig. 5(a), both P2P and H&C incur the essential amount of communication volume as defined in (10), without any forwarding overhead. Compared to DSGD, both RAND- and HP-based P2P and H&C methods incur significantly reduced

amount of communication volume per epoch (more than 10x). Compared to RAND, the HP-based P2P and H&C methods incur significantly reduced volume (between 1.4x and 5x).

Fig. 5(b) shows that in all matrix instances P2P and H&C have a significantly reduced *sum-max vol* compared to DSGD (more than 10x). H&C has slightly higher *sum-max vol* compared to P2P. This is because combining the messages disturbs the random volume balancing of P2P. As expected, HP-based P2P incurs less *sum-max vol* compared to RAND-based P2P. HP-based H&C shows a decrease in *sum-max vol* on two matrices (`Amz Books` and `Amz Clothing & Jewelry`), and an increase in other four matrices. This is because the HP method, when used for H&C, does not encapsulate reducing the *sum-max vol* metric.

2) Latency-Oriented Communication Cost Metrics: Fig. 5(c) shows that the H&C method significantly reduces *tot msgs* on all dataset matrices. DSGD always incurs a constant number of messages for each K value, thus *tot msgs* is always equal to $K^2 = (1024)^2 = 1048576$. *tot msgs* of P2P can go up to $K^2 \times (K-1)$. On the other hand, H&C keeps *tot msgs* limited to $\mathcal{O}(K^2 \lg K)$. Depending on the sparsity pattern of the matrix, *tot msgs* of P2P can be very high (e.g., `Amz Books`, `Amz Items` and `Twitch`) or relatively close to the lower bound (e.g., `Amz Clothing & Jewelry`). The H&C method successfully controls the fluctuation in the number of messages thanks to the $\lg K$ factor. The significant reduction in *tot vol* of HP-based P2P and H&C methods compared to those of RAND-based is expected to reflect on the total number of messages, which is the case as shown in the figure.

Fig. 6 showcases the H&C method's regularization of messages sent per epoch over K sub-epochs. In order to experimentally verify the $\mathcal{O}(\lg K)$ bound given in Theorem 1, we introduce the *max-max msgs* metric as the maximum number of messages sent per sub-epoch among all sub-epochs. That is,

$$max\text{–}max\ msgs = max\left\{\max_{x\in[K]}(|SendSet^k(p_x)|) \mid k \in [K]\right\}.$$

As seen in Fig. 6(a), using H&C, *max-max msgs* is empirically found to be $\approx 3 \times \lg K$, which is very close to the expected $\lg K$ bound on the number of messages per sub-epoch given in (13). The figure shows that P2P incurs high *max-max msgs* on $K = 256$, and then the *max-max msgs* values start to decrease as K increases. We believe this is attributed to the ability of random partitioning to balance P2P message counts and volume. In Fig. 6(b), the *sum-max msgs* metric is shown for all matrices in the dataset using P2P and H&C on $K = 64, \ldots, 1024$ processors. The figure shows the success of H&C in keeping the number of messages under the $K \lg K$ theoretical bound. Since P2P's *sum-max msgs* do not decrease as K increases, this means maximum number of messages per sub-epoch are almost equal among all sub-epochs, especially when $K \geq 512$. On the other hand, although the H&C's *max-max msgs* come very close to those of P2P on some instances such as `Goodreads Reviews` and `Google Reviews`, *sum-max msgs* stay significantly less than those of P2P. This means that although the maximum number of messages sent per sub-epoch can reach $3 \lg K$ in very few sub-epochs, it is still equal to or less than the expected $\lg K$ messages.

TABLE II
NORMALIZED COST METRICS OF P2P-HP WITH RESPECT TO P2P-RAND ON $K = 1024$ PROCESSORS. A VALUE $v < 1$ MEANS P2P-HP OUTPERFORMS P2P-RAND BY $(1 - V) \times 100\%$

Matrix	total messages	total volume	SGD time F=16	SGD time F=64
`Amz Items`	0.67	0.59	0.85	0.70
`Amz Books`	0.60	0.47	0.75	0.71
`Amz Clothing & Jewelry`	0.49	0.34	0.99	0.91
`Goodreads Reviews`	0.71	0.75	0.82	0.76
`Google Reviews`	0.23	0.24	0.30	0.26
`Twitch streams`	0.45	0.37	1.41	1.37
Avg. (Geometric mean)	**0.50**	**0.43**	**0.78**	**0.71**

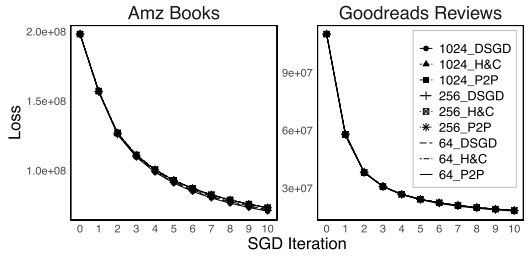
C. Evaluations With SGD Iteration Time

Fig. 5(d) and (e) compare the methods in terms of SGD iteration time on $K = 1024$ processors respectively using $F = 16$ and $F = 64$ values. The figure shows that the P2P improvement over DSGD is significant (more than 4x on all matrices, except for `Twitch` which is 1.4x) when $F = 16$. The improvement grows further as F increases to 64. It becomes more than 15x on all matrices except `Twitch`, and on `Twitch` the improvement becomes at least 4.7x.

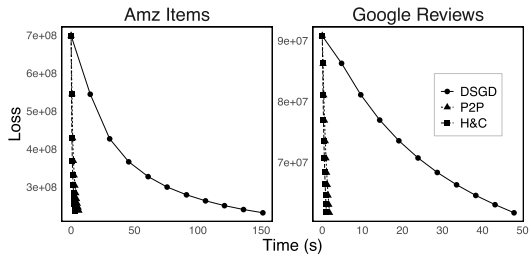
Using RAND, the H&C improvement over P2P is also significant. When $F = 16$, H&C improves the iteration runtime over P2P by 2x, 1.2x, 2x, 1.5x, 2.15x, and 1.25x respectively on `Amz Books`, `Amz Clothing & Jewelry`, `Amz Items`, `Goodreads Reviews`, `Google Reviews` and `Twitch`. When $F = 64$, the respective values become 1.7x, 1.2x, 2x, 1.4x, 1.74x, and 1.22x.

Using HP improves the P2P runtime by 1.3x, 1.17x, 1.22x and 3.35x on `Amz Books`, `Amz Items`, `Goodreads Reviews` and `Google Reviews`, respectively, when $F = 16$. On `Amz Clothing & Jewelry` there is no significant improvement and on `Twitch` there is deterioration by 1.4x. When $F = 64$, HP improves the P2P runtime by 1.4x, 1.42x, 1.3x and 3.9x respectively on `Amz Books`, `Amz Items`, `Goodreads Reviews` and `Google Reviews`. Table II also shows the normalized P2P-HP cost metrics with respect to those of P2P-RAND. On average, HP improves (reduces) the SGD iteration time by 22% when $F = 16$ and 29% when $F = 64$. The increase in the gap between HP and RAND in terms of P2P runtime when F grows from 16 to 64 is expected since the HP method aims at reducing the total volume, effect of which is seen more with higher F values. We observed that the HP method improves the H&C runtime compared to RAND only on `Goodreads Reviews` and `Google Reviews`.

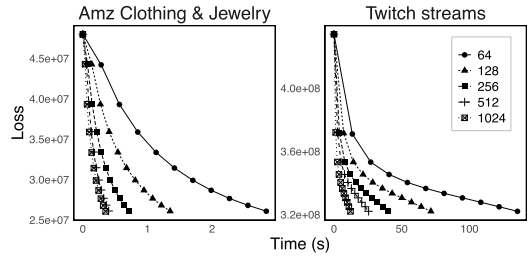
Fig. 7 shows the strong scaling curves of RAND-based DSGD, P2P and H&C using two different F values on $K = \{64, 128, 256, 512, 1024\}$ processors. As seen in the figure, P2P and H&C show superior scaling compared to DSGD. Furthermore, H&C performs significantly better than P2P, especially with smaller F values.



(a) SGD iteration vs. loss for all methods on different K values.



(b) Time vs. loss for all methods on K=1024 processors.



(c) Time vs. loss for H&C on different K values.

Fig. 8. Loss curves.

Increasing the F value is expected to render the SGD communication as bandwidth-bound. Therefore, the effect of the methods that reduce the bandwidth (volume of communication) becomes conspicuous. This is observed in two different cases when moving from $F = 16$ in Fig. 7(a) to $F = 64$ in Fig. 7(b): (i) the performance gap between P2P/H&C and DSGD increases as F becomes larger as a result of the huge reduction in communication volume when using P2P/H&C, and (ii) the difference in performance between P2P and H&C slightly reduces due to the communication overhead leaning towards bandwidth.

D. Evaluations With Loss Values

Since all the methods discussed in this work follow the stratified SGD algorithm, their loss values per iteration is expected to be very similar regardless of the communication strategy used or number of processors. We demonstrate this using Fig. 8(a). The figure shows the loss value (y-axis) following each SGD iteration (x-axis) of `Amz Books` and `Goodreads Reviews` using the RAND-based DSGD, P2P, H&C methods on $K = \{64, 256, 1024\}$ processors. The loss values are very close as expected thus the curves appear to be on top of each other.

Fig. 8(b) shows the amount of time (x-axis) required to reach a certain loss value (y-axis) of `Amz Items` and `Google Reviews` using the RAND-based DSGD, P2P, H&C methods on 1,024 processors. The figure shows that DSGD requires significantly more time to reach a certain loss value compared to P2P and H&C.

Fig. 8(c) shows the scaling behavior of the RAND-based H&C method with `Amz Clothes & Jewelry` and `Twitch` in terms of loss value as the time increases.

VII. Related Work

There exist several works in the literature that adopt the SSGD for parallel matrix completion for shared-memory systems [16], [17], [18] and distributed-memory systems [3], [4], [5]. Here, we focus on the works that involve distributed-memory implementations. The work of Gemulla et at. [3] proposed the SSGD approach as well as the parallel DSGD algorithm discussed in Sections II-B and III-B. Teflioudi et al. [4] proposed DSGD++, an improved DSGD framework for better performance. They use computation and communication overlaying through dividing the input matrix into $K \times 2K$ blocks, and in each of the K sub-epochs DSGD++ performs computation on K blocks while simultaneously communicating the other K blocks. They report up to 2.3x improvement over DSGD in terms of runtime. Yun et al. [5] extend the idea of DSGD++ in their framework, NOMAD, and divide the input matrix into $K \times M$ blocks. Each of the K processors dedicates ℓ threads to update ℓ $\mathbf{H}$-matrix rows, and $M - \ell$ other threads for communication. Once processor p_x updates an $\mathbf{H}$-matrix row, or a set of rows, it sends it/them to another processor p_y that has idle computation threads. DSGD, DSGD++ and NOMAD have the same total communication volume during an SGD epoch which is equals to $F \times M \times K$ as discussed in Section III-B. The number of messages sent per processor during an epoch of DSGD and DSGD++ has an upper bound of $\mathcal{O}(K)$, whereas NOMAD may send up to $\mathcal{O}(M)$ messages. Guo et al. [6] proposed a novel framework, BaPa, for improving the nonzero load balance of DSGD through a novel algorithm for balancing per-processor and per-epoch ratings. Their BaPa-based DSGD shows a significant runtime improvement on small number of processors (< 16). However, their results show that both the original DSGD as well as the BaPa-based DSGD stop scaling after 256 processors.

There are several asynchronous-SGD-based parallel matrix completion algorithms in the literature. ASGD [4] (shown in the upper part of Fig. 1) is the simplest example of such algorithm. During ASGD, it is possible that several processors update the same $\mathbf{H}$-matrix row $\mathbf{h}_j$ at the same time (i.e., stale updates). This results in each processor having a different copy of $\mathbf{h}_j$. These copies are coordinated by sending them to a processor responsible for $\mathbf{h}_j$. This processor takes their average and then sends the up-to-date version of $\mathbf{h}_j$ back to the same set of processors. This type of coordination is done once or more during an SGD epoch [4], [19]. GASGD [19] extends ASGD by utilizing intelligent partitioning for balancing computational loads, reducing communication between processors, and reducing staleness. The authors utilize a bipartite graph model and

propose a partitioning method based on the *balanced K-way vertex-cut* problem [20] to achieve the partitioning goals. Luo et al. [21] proposed a different strategy to facilitate asynchronously computing SGD in parallel which is called alternating SGD. In alternating SGD, each epoch is divided into two sub-epochs where in each sub-epochs one factor matrix is fixed and the other is updated. This approach enables limiting the feature vector updates that use stale data to one of the two factor matrices during a sub-epoch. Recently, Shi et al. [22] proposed a distributed algorithm based on alternating SGD with data-aware partitioning.

VIII. Conclusion

We proposed a framework for scaling stratified SGD through significantly reducing the communication overhead. The framework targets at reducing the bandwidth overhead by efficiently finding the required communication during an SGD epoch, using P2P messages to perform it, and an HP-based method to further reduce the P2P communication volume. The framework targets at reducing the increase in latency overhead through the novel H&C strategy to limit the number of messages sent by a processor per epoch to $\mathcal{O}(K \lg K)$. Our proposed framework achieves scalable distributed SGD, on up to $K = 1024$ processors, without any compromise on convergence rate or any update on stale factors. The proposed framework achieves up to 15x runtime improvement over the state of the art DSGD method, on 1024 processors, using six real-world rating matrices.

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