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Two classes of Quadratic Assignment Problems that are solvable as Linear Assignment Problems

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ABSTRACT

The Quadratic Assignment Problem is one of the hardest combinatorial optimization problems known. We present two new classes of instances of the Quadratic Assignment Problem that can be reduced to the Linear Assignment Problem and give polynomial time procedures to check whether or not an instance is an element of these classes.

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1. Introduction

The Quadratic Assignment Problem (QAP) is the problem of determining a one-to-one and onto assignment between two sets, each consisting of n objects (e.g. n facilities and n locations) so as to minimize the sum of the costs associated with pairs of assignments. The initial formulation is due to Koopmans and Beckmann [1], where the cost of assigning facility i to location j and of facility k to location l is $f_{ik}d_{jl}$ with f_{ik} denoting the material flow per unit time between facilities i and k and d_{jl} denoting the distance between locations j and l. Define x_{ij} to be 1 if facility i is assigned to location j, and 0 otherwise. The Koopmans–Beckmann formulation of the QAP is as follows:

$$\text{minimize} \sum_{i,j,k,l=1}^{n} f_{ik}d_{jl}x_{ij}x_{kl} \tag{1}$$

subject to

$$\sum_{k=1}^{n} x_{ik} = 1, \quad \forall i \in \{1, \dots, n\} \tag{2}$$

$$\sum_{i=1}^{n} x_{ik} = 1, \quad \forall k \in \{1, \dots, n\} \tag{3}$$

$$x_{ik} \in \{0, 1\}, \quad \forall i, k \in \{1, \dots, n\}. \tag{4}$$

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Lawler [2] studied the case with general cost coefficients, where $f_{ik}d_{jl}$ is replaced by a general cost term C_{ijkl}, which is the cost of assigning facilities i and k to locations j and l, respectively. The Koopmans–Beckmann form requires, as input, a flow matrix $F = [f_{ik}]$ and a distance matrix $D = [d_{jl}]$ resulting in an input of $O(n^2)$ while the general form with costs C_{ijkl} $(i, j, k, l = 1, \ldots, n)$ requires the specification of $O(n^4)$ cost terms.

The QAP is one of the hardest combinatorial optimization problems known. It is NP-Hard in the strong sense [3]. While theoretical, algorithmic, and technological developments have led to significant advances in solvable sizes of many well-known NP-Hard problems, QAP has remained as a stand alone class that seems to defy all solution attempts except for very limited sizes. The largest solved instance in the QAPLIB [4] is of size 36 [5,6] while that of the Traveling Salesman Problem, for example, is around 85 900 cities [7].

Our focus in this paper is on certain polynomial time solvable classes of the QAP that can be solved as a Linear Assignment Problem (LAP). Given the costs c_{ij}, $(i, j \in \{1, \ldots, n\})$ the LAP is the problem of minimizing $\sum_{i,j=1}^{n} c_{ij}x_{ij}$ subject to (2)–(4). It is well known that the LAP is solvable in polynomial time. Akgül [8] gives a review of many of the polynomial time algorithms that solve the LAP. Reducing an instance of the QAP to an instance of the LAP in polynomial time implies polynomial time solvability of the QAP instance on hand. Consequently, the three classes proposed in this paper are polynomial time solvable.

Special structures that have received attention in the literature in the context of the QAP seems to be rather limited. We refer the reader to the survey by Burkard et al. [9] and the books by Burkard [10] and Çela [11] for a complete exposition to the polynomially solvable classes of the QAP. Prior to these studies, Chen [12] proposed three special cases of the general form of the QAP that can be represented as parametric LAPs. The complexity status of these classes is open, but computational results have been reported by Chen for test problems up to size 50. Burkard et al. [13] provided three polynomial time solvable classes of the Koopmans–Beckmann form, where one input matrix is monotone anti-Monge while the other is either symmetric Toeplitz generated by a benevolent (or a k-benevolent) function, or symmetric with bandwidth 1. They show that certain assignments qualify as optimal for these cases. Deineko and Woeginger [14] provided another polynomially solvable class for the Koopmans–Beckmann form with one matrix being Kalmanson and the other being symmetric decreasing circulant. Burkard et al. [9] analyzed in their survey, coefficient matrices with special properties (sum, product, Monge, anti-Monge, Kalmanson, Toeplitz and circulant) and gave complexity results for many of the resulting cases. The two classes we propose in this paper are new and not derived from the aforementioned classes.

2. New classes of polynomially solvable instances

In this section, we present our analytical results based on decompositions of the cost coefficients.

2.1. Additive decomposition

The first class we propose is what we refer to as the additively decomposable class for general cost terms. The proposed class is a significant generalization of an earlier class proposed by Burkard et al. [9] for the Koopmans–Beckmann form. Denote the flow and distance matrices by $F = [f_{ik}]$ and $D = [d_{jl}]$ for the Koopmans–Beckmann form, respectively. Burkard et al. [9] showed that if $2n$ numbers f_i^r, f_i^c $(i \in \{1, \ldots, n\})$ can be found associated, respectively, with the n rows and the n columns of the flow matrix such that $f_{ik} = f_i^r + f_k^c$ $\forall i, k \in \{1, \ldots, n\}$, the problem is reducible to the LAP. The result is also valid if a similar decomposition is available for D.

The additive decomposition we propose here is a more general one that works for the case of general costs C_{ijkl} and relies on solving a linear system of equations with $O(n^3)$ variables and $O(n^4)$ equations. Because the linear equation system is overdetermined, it may or may not have a solution. Whenever there exists a solution, the QAP on hand is solved as a LAP in polynomial time.

To define the additively decomposable class of interest, let $I = \{1, 2, \ldots, n\}$ and let I^k be the k-fold Cartesian product of I by itself. For $k = 4$, denote by $q = ijkl$ any quadruplet in I^4. We define a quadruplet $q = ijkl$ to be *incompatible* if either $i = k$ and $j \neq l$ or $j = l$ and $i \neq k$, and define it to be *compatible* otherwise. Incompatible quadruplets correspond to the cases where either two distinct facilities are assigned to the same location or the same facility is assigned to two distinct locations. Such assignments are infeasible in the QAP. Compatible quadruplets refer to the cases where either two distinct facilities are assigned to two distinct locations or a facility is assigned to a single location (i.e. q is of the form $ijij$). Define $\bar{I}$ to be the subset of I^4 consisting of compatible quadruplets. Note that there are $n^4 - 2n^3 + 2n^2$ compatible quadruplets. We write C_q to mean the cost C_{ijkl} for which $q = ijkl$. For a nonempty subset s of $\{1, 2, 3, 4\}$, we define $q(s)$ to be the ordered $|s|$-tuple obtained from the quadruplet q by retaining the indices in q that correspond to positions in s while deleting all other indices. For example, if $q = k_1k_2k_3k_4$ and $s = \{1, 2, 4\}$, then $q(s) = k_1k_2k_4$. If $s = \{2, 4\}$, then $q(s) = k_2k_4$. Define also $q(\phi) = 0$ $\forall q \in I^4$.

Corresponding to each nonempty proper subset s of $\{1, 2, 3, 4\}$ and each $t \in I^{|s|}$, define a variable u_t^s. Additionally, for $s = \phi$, we take $t = 0$ and define an additional variable u_0^{ϕ}. For example, if $n = 5$ and $s = \{1, 2, 4\}$, then each element of $\{1, 2, 3, 4, 5\}^3$ gives an ordered triplet $t = ijk$ for which a variable $u_t^{\{1,2,3\}}$ is defined. In general, the number of u_t^s variables is $4n^3 + 6n^2 + 4n + 1$. Let A be a matrix of 0s and 1s with rows corresponding to compatible quadruplets and columns corresponding to (s, t) pairs. A has $|\bar{I}| = n^4 - 2n^3 + 2n^2$ rows and $4n^3 + 6n^2 + 4n + 1$ columns. Denote the element in row q and column (s, t) of A by $a_q^{s,t}$. We define $a_q^{s,t} = 1$ if $q(s) = t$ and 0 otherwise. For example, if $n = 5$, $s = \{1, 2, 4\}$, $t = 115$,

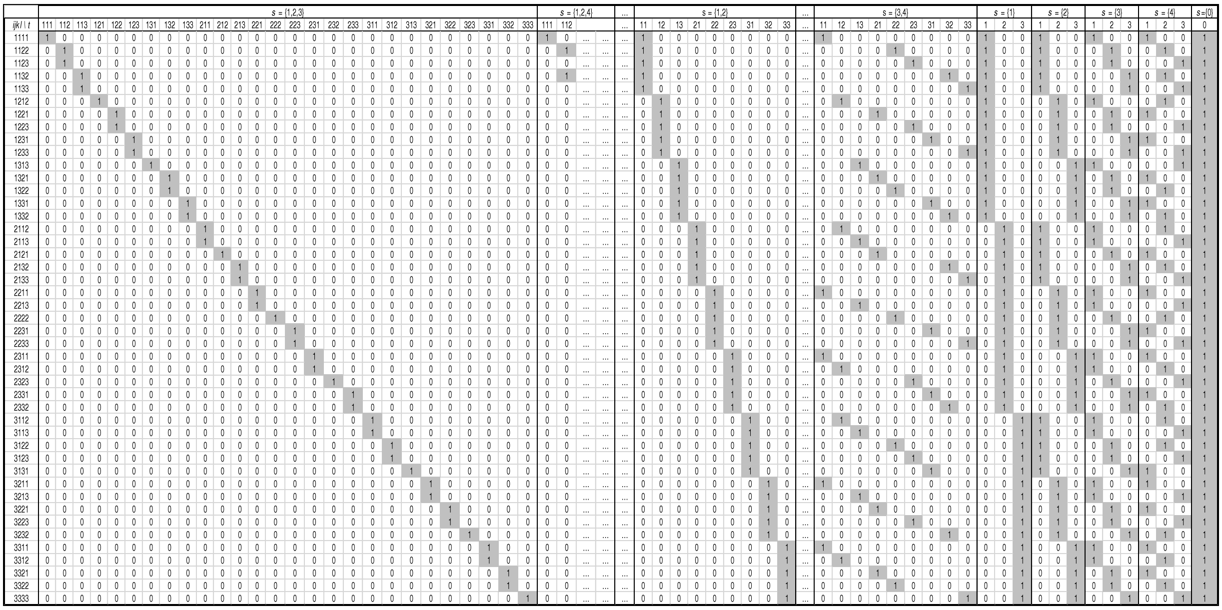

Fig. 1. An example of the A matrix for $n = 3$. Columns in the middle are omitted due to space limitation.

then each of the choices $q = 1125$, $q = 1135$, $q = 1145$, and $q = 1155$ gives $q(s) = 115 = t$ so that $a_q^{s,t} = 1$ while any other choice of q gives $a_q^{s,t} = 0$. Let $A = [a_q^{s,t}]$ and u be the vector of u_t^s values where the columns of A and the elements of u are identically ordered by (s, t). Let C be the vector of costs C_{ijkl}, $ijkl \in I^4$, and $\bar{C}$ be the vector obtained from C by deleting all cost components C_{ijkl} corresponding to incompatible quadruplets $ijkl \in I^4$. We assume that the rows of A and the elements of $\bar{C}$ are identically ordered by $q \in \bar{I}$. An example of the A matrix for $n = 3$ is depicted in Fig. 1.

Theorem 1. *If the linear equality system*

$$Au = \bar{C} \tag{5}$$

has a solution, then the instance of the QAP defined by C can be solved as a LAP.

Proof. Assume that $\hat{u} = (\hat{u}_t^s)$ solves (5). Then $A\hat{u} = \bar{C}$ implies that

$$\sum_{t:q(s)=t} \hat{u}_t^s = C_q, \quad q \in \bar{I}. \tag{6}$$

Using (6), the objective function value of the QAP for any feasible solution $X = (x_{ij})$ can be rewritten as:

$$\sum_{ijkl \in I^4} C_{ijkl} x_{ij} x_{kl} = \sum_{ijkl \in \bar{I}} C_{ijkl} x_{ij} x_{kl} = \sum_{ijkl \in \bar{I}} \begin{pmatrix} \hat{u}_{ijk}^{123} + \hat{u}_{ijl}^{124} + \hat{u}_{ikl}^{134} + \hat{u}_{jkl}^{234} + \hat{u}_{ij}^{12} + \hat{u}_{ik}^{13} + \hat{u}_{il}^{14} \\ + \hat{u}_{jk}^{23} + \hat{u}_{jl}^{24} + \hat{u}_{kl}^{34} + \hat{u}_i^1 + \hat{u}_j^2 + \hat{u}_k^3 + \hat{u}_l^4 + \hat{u}_0^0 \end{pmatrix} x_{ij} x_{kl} \tag{7}$$

$$= \sum_{ijkl \in \bar{I}} \hat{u}_{ijk}^{123} x_{ij} x_{kl} + \cdots + \sum_{ijkl \in \bar{I}} \hat{u}_0^0 x_{ij} x_{kl} \tag{8}$$

where the first equality follows from the fact that feasibility ensures $x_{ij} x_{kl} = 0$ for any incompatible quadruplet $ijkl$. Each of the fifteen summations in (8) can be written in such a way as to separate out the omitted index (indices) from u_t^s terms. For example, the first summation gives

$$\sum_{ijk \in I^3} \hat{u}_{ijk}^{123} x_{ij} \sum_{l \in I} x_{kl} = \sum_{ijk \in I^3} \hat{u}_{ijk}^{123} x_{ij} \tag{9}$$

where the equality follows from (2). The other summations can be similarly processed using (2)–(4) to obtain the following equality:

$$\sum_{ijkl\in I^4} C_{ijkl}x_{ij}x_{kl} = \sum_{ijk\in I^3} \hat{u}^{123}_{ijk}x_{ij} + \sum_{ijl\in I^3} \hat{u}^{124}_{ijl}x_{ij} + \sum_{ikl\in I^3} \hat{u}^{134}_{ikl}x_{kl} + \sum_{jkl\in I^3} \hat{u}^{234}_{jkl}x_{kl} + n\sum_{ij\in I^2} \hat{u}^{12}_{ij}x_{ij} + \sum_{ik\in I^2} \hat{u}^{13}_{ik} + \sum_{il\in I^2} \hat{u}^{14}_{il} + \sum_{jk\in I^2} \hat{u}^{23}_{jk} + \sum_{jl\in I^2} \hat{u}^{24}_{jl} + n\sum_{kl\in I^2} \hat{u}^{34}_{kl}x_{kl} + n\sum_{i\in I} \hat{u}^1_i + n\sum_{j\in I} \hat{u}^2_j + n\sum_{k\in I} \hat{u}^3_k + n\sum_{l\in I} \hat{u}^4_l + n^2\hat{u}^0_0. \tag{10}$$

The resulting LAP has the following cost coefficient for the variable x_{ij}:

$$\hat{c}_{ij} = \sum_{r\in I} \hat{u}^{123}_{ijr} + \sum_{r\in I} \hat{u}^{124}_{ijr} + \sum_{r\in I} \hat{u}^{134}_{rij} + \sum_{r\in I} \hat{u}^{234}_{rij} + n.\hat{u}^{12}_{ij} + n.\hat{u}^{34}_{ij} \tag{11}$$

so that the objective function $\sum_{ijkl\in I^4} C_{ijkl}x_{ij}x_{kl}$ of the QAP is equal to the objective function $\sum_{ij\in I^2} \hat{c}_{ij}x_{ij}$ of the resulting LAP plus the constant $\hat{K}$ where

$$\hat{K} = \sum_{ik\in I^2} \hat{u}^{13}_{ik} + \sum_{il\in I^2} \hat{u}^{14}_{il} + \sum_{jk\in I^2} \hat{u}^{23}_{jk} + \sum_{jl\in I} \hat{u}^{24}_{jl} + n\left(\sum_{i\in I} \hat{u}^1_i + \sum_{i\in I} \hat{u}^2_i + \sum_{i\in I} \hat{u}^3_i + \sum_{i\in I} \hat{u}^4_i + n.\hat{u}^0_0\right). \tag{12}$$

Thus, the instance of the QAP defined by C is solvable as a LAP whenever the system $Au = \bar{C}$ has a solution.

Define Class 1 to be the set of instances of the QAP for which (5) has a solution. The following algorithm checks whether or not an instance belongs to Class 1 and solves it whenever it does. The correctness of the algorithm directly follows from Theorem 1. □

Algorithm 1.

Step 1. Solve $Au = \bar{C}$ to obtain a solution $\hat{u}$, if it exists. If no solution exists, stop. The instance does not belong to Class 1. Else, continue.

Step 2. Define the cost coefficients $\hat{c}_{ij}$ using $\hat{u}$ in (11).

Step 3. Solve the resulting LAP to get an optimal solution $\hat{X} = (\hat{x}_{ij})$. Then $\hat{X}$ solves the QAP instance and its optimal objective value is

$$\sum_{ijkl\in I^4} C_{ijkl}\hat{x}_{ij}\hat{x}_{kl} = \sum_{ij\in I^2} \hat{c}_{ij}\hat{x}_{ij} + \hat{K} \tag{13}$$

where $\hat{K}$ is as defined in (12).

2.2. Multiplicative decomposition

We now propose a second class of instances of the QAP that are solvable as LAPs. This class is based on decomposing general cost coefficients in a multiplicative way and requires solving a nonlinear system of equations with $O(n^2)$ variables and $O(n^4)$ equations. We do provide a polynomial time solution for this system whenever a solution exists. Chen [12] gave a similar decomposition that results in a parametric LAP whose complexity status is open, whereas our decomposition implies polynomial time solvability of the QAP whenever the decomposition proposed in Theorem 2 is valid.

Define first $\underline{z}(c)$ and $\bar{z}(c)$ to be the minimum and maximum objective values of the LAP, respectively, for which the cost data is $c = (c_{ij})$.

Theorem 2. *If there exists $v = (v_{ij}, ij \in I^2)$ that satisfies*

$$v_{ij}v_{kl} = C_{ijkl}, \quad ijkl \in \bar{I}, \tag{14}$$

and if $0 \le \underline{z}(v)$ or $\bar{z}(v) \le 0$, then the instance of the QAP defined by costs C_{ijkl}, $ijkl \in I^4$, is equivalent to the LAP with costs v_{ij}, $ij \in I^2$, for the case $0 \le \underline{z}(v)$, and to the LAP with costs $-v_{ij}$, $ij \in I^2$, for the case $\bar{z}(v) \le 0$.

Proof. Assume that such v_{ij}, $ij \in I^2$, exist. Then the objective function becomes:

$$\sum_{ijkl\in I^4} v_{ij}v_{kl}x_{ij}x_{kl} = \sum_{ijkl\in \bar{I}} v_{ij}v_{kl}x_{ij}x_{kl}. \tag{15}$$

Reorganizing the terms, (15) can be rewritten as:

$$\sum_{ij\in I^2} v_{ij}x_{ij} \sum_{kl\in I^2} v_{kl}x_{kl} = \left(\sum_{ij\in I^2} v_{ij}x_{ij}\right)^2. \tag{16}$$

v

$i \backslash j$	1	2	3
1	1	2	3
2	4	5	6
3	7	8	9

C

	j	1			2			3		
i	k\l	1	2	3	1	2	3	1	2	3
1	1	1	■	■	■	4	■	■	■	9
	2	■	5	6	8	■	12	12	15	■
	3	■	8	9	14	■	18	21	24	■
2	1	■	8	12	5	■	15	6	12	■
	2	16	■	■	■	25	■	■	■	36
	3	■	32	36	35	■	45	42	48	■
3	1	■	14	21	8	■	24	9	18	■
	2	■	35	42	32	■	48	36	45	■
	3	49	■	■	■	64	■	■	■	81

Fig. 2. An example of the decomposable cost matrix and the corresponding decomposition. The cells that have been shaded black are elements of $I^4 \setminus \bar{I}$.

If $0 \le \underline{z}(v)$, all feasible assignments induce a nonnegative objective value in the LAP with cost vector $v = (v_{ij})$ so that any feasible assignment that minimizes $\sum_{ij\in I^2} v_{ij}x_{ij}$ also minimizes $\left(\sum_{ij\in I^2} v_{ij}x_{ij}\right)^2$. If $\bar{z}(v) \le 0$, all feasible assignments yield a non-positive objective value in the LAP so that any feasible assignment that minimizes $\sum_{ij\in I^2} -v_{ij}x_{ij}$ also minimizes $\left(\sum_{ij\in I^2} v_{ij}x_{ij}\right)^2$. □

Define Class 2 to be the set of instances of the QAP that fulfills the assumptions of Theorem 2. The corresponding $\boldsymbol{v}$ and $\boldsymbol{C}$ matrices for an element of this set of problems is provided in Fig. 2. Notice that every element of this class must satisfy $v_{ij}^2 = C_{ijij}$ (or equivalently $v_{ij} = \pm\sqrt{C_{ijij}}$), implying that an instance for which $C_{ijij} < 0$ for some $ij \in I^2$ is not an element of Class 2. Note that if all C_{ijij}, $ij \in I^2$, are nonnegative, two possible values can be assigned to each v_{ij} corresponding to the plus or minus roots so that there are 2^{n^2} possible choices of the multipliers (v_{ij}, $ij \in I^2$). Despite the exponential number of possibilities, the following algorithm identifies the correct values of the multipliers in $O(n^2)$ time (followed by an $O(n^4)$ secondary check). The algorithm determines whether or not a given instance belongs to Class 2.

Algorithm 2.

Step 1. Pick an arbitrary facility–location pair ij. Set $v_{ij} = \sqrt{C_{ijij}}$. Note that whenever a multiplicative decomposition with multipliers v_{ij}, $ij \in I^2$ exists, another multiplicative decomposition with multipliers $-v_{ij}$, $ij \in I^2$ also exists. Hence setting $v_{ij} = \sqrt{C_{ijij}}$ for a single pair ij does not result in a loss of generality.

Step 2. For every facility–location pair ab where $i \ne a$ and $j \ne b$, go to (a) or (b) depending on $C_{ijab} < 0$ or $C_{ijab} \ge 0$, respectively.

(a) Case with $C_{ijab} < 0$: Check the equality $v_{ij}.(-\sqrt{C_{abab}}) = C_{ijab}$. If the equality fails, then stop (no multiplicative decomposition exists), else set $v_{ab} = -\sqrt{C_{abab}}$.
(b) Case with $C_{ijab} \ge 0$: Check the equality $v_{ij}.(\sqrt{C_{abab}}) = C_{ijab}$. If the equality fails, then stop (no multiplicative decomposition exists), else set $v_{ab} = \sqrt{C_{abab}}$.

If termination has not occurred for any of the pairs checked in Step 2, continue to Step 3.

Step 3. For the facility–location pairs $il \in I^2$, $l \in I - \{j\}$, pick a facility–location pair $ab \in I^2$, where $a \ne i$ and $b \notin \{j, l\}$. Check the equality$(\sqrt{C_{ilil}}).v_{ab} = C_{ilab}$. If the equality is satisfied, set $v_{il} = \sqrt{C_{ilil}}$. Else, check the equality$(-\sqrt{C_{ilil}}).v_{ab} = C_{ilab}$. If the equality is satisfied, set $v_{il} = -\sqrt{C_{ilil}}$; else, stop (no multiplicative decomposition exists).

If termination has not occurred for any of the pairs checked in Step 3, continue to Step 4.

Step 4. For the facility–location pairs $kj \in I^2$, $k \in I - \{i\}$, pick a facility–location pair $ab \in I^2$, where $a \notin \{i, k\}$ and $b \ne j$. Check the equality $(\sqrt{C_{kjkj}}).v_{ab} = C_{kjab}$. If the equality is satisfied, set $v_{kj} = \sqrt{C_{kjkj}}$. Else, check the equality$(-\sqrt{C_{kjkj}}).v_{ab} = C_{kjab}$. If the equality is satisfied, set $v_{kj} = -\sqrt{C_{kjkj}}$; else, stop (no multiplicative decomposition exists).

If termination has not occurred for any of the pairs checked in Step 4, continue to Step 5. All multipliers v_{pq}, $pq \in I^2$, have now been determined.

Step 5. Check the set of equalities $v_{pq}v_{st} = C_{pqst}$ for any of the quadruplets $pqst$ in $\bar{I}$ not checked yet in the previous steps. If all equations are satisfied, a multiplicative decomposition is on hand (found at the end of Step 4), else no multiplicative decomposition exists with multipliers v_{ij}, $ij \in I^2$.

The steps of the algorithm above take $O(1)$, $O(n^2)$, $O(n)$, $O(n)$, and $O(n^4)$ time, respectively. If a multiplicative decomposition has been found, the next step of the procedure is to solve the LAPs with the objective function $\min \sum_{ij\in I^2} v_{ij}x_{ij}$ and $\min \sum_{ij\in I^2} -v_{ij}x_{ij}$ to get the values $\underline{z}(v)$ and $\bar{z}(v)$, respectively. If $0 \le \underline{z}(v)$ or $\bar{z}(v) \le 0$, then the solution of the corresponding LAP qualifies as optimal for the QAP instance on hand. If the last condition does not hold, then the QAP on hand is equivalent to what we refer to as "the absolute Linear Assignment Problem".

Note also that any QAP with arbitrary costs can be transformed into an equivalent QAP with nonnegative costs by adding a sufficiently large constant to each cost term. If a multiplicative decomposition exists for the transformed costs, the transformed as well as the original QAP on hand are polynomial time solvable. If no multiplicative decomposition exists for the transformed QAP, it is still possible that there exists a multiplicative decomposition for the original QAP with arbitrary costs. In this case, the multipliers may be of mixed signs and it is necessary to check the condition $0 \le \underline{z}(v)$ or $\bar{z}(v) \le 0$. If this condition does not hold, then we have $\underline{z}(v) < 0 < \bar{z}(v)$ and the QAP on hand is equivalent to the minimization of $\left(\sum_{ij\in I^2} v_{ij}x_{ij}\right)^2$ subject to (2)–(4) which is equivalent, in turn, to the minimization of $\left|\sum_{ij\in I^2} v_{ij}x_{ij}\right|$ subject to the same constraints. This last problem, which we refer to as the *absolute LAP*, seeks an assignment where the objective value is as close to 0 in absolute value as possible.

Regarding the complexity status of the absolute LAP, it has been shown that the special case of Koopmans–Beckmann QAP with both coefficient matrices being symmetric product matrices is NP-hard [12]. This class of instances can easily be verified to be a special case of the absolute LAP, namely the case where the coefficient matrix of the LAP is a product matrix:

$$\min \sum_{i,j,k,l=1}^{n} f_{ik}d_{jl}x_{ij}x_{kl} = \min \sum_{i,j,k,l=1}^{n} \alpha_i\alpha_k\beta_j\beta_l x_{ij}x_{kl} = \min \left(\sum_{i,j=1}^{n} \alpha_i\beta_j x_{ij}\right)^2 = \min \left|\sum_{i,j=1}^{n} \alpha_i\beta_j x_{ij}\right|. \tag{17}$$

This relationship proves that the absolute LAP is also NP-Hard.

As a consequence, whenever there is a multiplicative decomposition for which $\underline{z}(v) < 0 < \bar{z}(v)$, the QAP on hand reduces to an absolute LAP which is also NP-Hard. Despite that, it may be easier, on the average, to solve the absolute LAP than the QAP.

3. Conclusion

In this study, we have identified two classes of instances (additively decomposable general costs and a subset of multiplicatively decomposable general costs) that are solvable in polynomial time as LAPs. Using a result from the literature [12], we have also shown that multiplicatively decomposable general cost instances that cannot be solved in polynomial time, remain NP-Hard. The results we have presented suggest new directions to explore for discovering possibly exploitable structures.

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