Colliding gravitational plane waves in dilaton gravity

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The collision of plane waves in dilaton gravity theories and the low energy limit of string theory are considered. The formulation of the problem and some exact solutions are presented.

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I. INTRODUCTION

Plane wave geometries are not only important in classical general relativity but also in string theory. It is now very well known that these geometries are the exact classical solutions of the string theory at all orders of the string tension parameter [1–3]. It is also interesting that plane wave metrics in higher dimensions when dimensionally reduced lead to exact extreme black hole solution in string theory [4].

In this work we shall be interested in the head on collisions of these plane waves in the framework of Einstein-Maxwell-dilaton theories with one U(1) and two U(1) Abelian gauge fields [5]. Our formulation of the problem will also cover the low energy limit of string theory for some fixed values of the dilaton coupling constants. Hence the solutions we present in this work are also exact solutions of the low energy limit of string theories. We give the complete data for the colliding plane-shock waves. We formulate the collision of plane waves and give solutions for the collinear case. When the dilaton coupling constant vanishes one of our solutions reduces to the well-known Bell-Szekeres solution [6] in Einstein-Maxwell theory.

For the collision problem in general relativity, spacetime is divided into four regions with respect to the null coordinates u and v. The second and third (incoming) regions are the Cauchy data (characteristic initial data) for the field equations in the interaction region (region IV). For this purpose the specification of the data is quite important in the formulation of the collision problem [6–15]. We show that the future closing singularities appearing in classical solutions exist also in dilaton gravity and in the low energy limit of the string theory. This is due to focusing effect of the plane waves [16].

It is an open question whether this classical treatment of the collision of plane waves can be extended to all orders in the string tension parameter [17,18]. One of the limiting cases of the solutions in Sec. II is the Bell-Szekeres solution [6]. This solution seems to be a candidate for an exact solution at all orders. The Bell-Szekeres solution in the interaction region is diffeomorphic to the Bertotti-Robinson spacetime [19,15]. It is known that string theory preserves the form of Bertotti-Robinson metric at all orders of the string parameter [20,21]. This does not necessarily lead to a conclusion that the Bell-Szekeres solution is an exact solution of the string theory. The reason is that the diffeomorphism is valid only in the interaction region ($u > 0, v > 0$) and hence the field equations (at higher orders of the string parameter) may not be satisfied on the hyperplanes $u = 0$ and $v = 0$. The Weyl tensor and its covariant derivatives suffer from δ-function and derivatives of the δ-function type of singularities on the hyperplanes $u = 0$ and $v = 0$. It is unlikely that these singular terms cancel each other in the field equations at all orders. If there exists an exact solution representing the collision of plane waves in the full string theory then its low energy limit should be contained in our solutions in the second and third sections. The proof of this conjecture is of course not easy.

In the next sections we shall give the form of the metrics in the incoming regions. These will constitute the data for field equations in the interaction region. In the second section we give the formulation of the problem for one U(1) Abelian gauge field with a solution generalizing the Bell-Szekeres solution in general relativity. In the third section we consider two Abelian U(1) gauge fields and give some interesting exact solutions of the collision of the plane wave problem. In the Appendix we reduce the Maxwell dilaton field equations, in the collision of plane waves, to the two-dimensional Ernst equation.

II. DILATON GRAVITY WITH ONE U(1) VECTOR FIELD

Einstein-Maxwell-dilaton gravity is derivable from a variational principle with the Lagrangian density

$$L = \sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{2}{\kappa^2}(\nabla\psi)^2 - \frac{1}{4}\, e^{-a\psi}F^2\right], \qquad (1)$$

where a is the dilaton coupling constant. The field equations are

$$G_{\mu\nu} = 4\left[\partial_\mu\psi\,\partial_\nu\psi - \frac{1}{2}(\nabla\psi)^2 g_{\mu\nu}\right] + \kappa^2 e^{-a\psi}\left[F^\alpha_\mu F_{\nu\alpha} - \frac{1}{4}F^2 g_{\mu\nu}\right], \qquad (2)$$

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$$\nabla_\mu (e^{-a\psi} F^{\mu\nu}) = 0, \tag{3}$$

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \psi) + \frac{\kappa^2\, a\sqrt{-g}}{16} e^{-a\psi} F^2 = 0. \tag{4}$$

A spacetime describing the collision of plane waves admits two spacelike Killing vector fields. In the general case these vectors are nonorthogonal but here in this work we consider them to be orthogonal. For such a case an appropriate form of the metric $g_{\mu\nu}$ and U(1) gauge potential A_μ are given by

$$ds^2 = 2e^{-M} du\, dv + e^{-U-V} dy^2 + e^{-U+V} dz^2, \tag{5}$$

$$A_\mu = (0, 0, A, 0), \tag{6}$$

where $M = M(u,v)$, $U = U(u,v)$, $V = V(u,v)$, $A = A(u,v)$ and dilaton field $\psi = \psi(u,v)$. The field equations turn out to be

$$-2A_{,uv} = (V_u - a\,\psi_u)\, A_{,v} + (V_v - a\,\psi_v)\, A_{,u}, \tag{7}$$

$$U_{uv} - U_u U_v = 0, \tag{8}$$

$$2M_{uv} = -2\, U_{uv} + U_u U_v + V_u V_v + 8\psi_u \psi_v, \tag{9}$$

$$2V_{uv} - U_u V_v - U_v V_u - 2\kappa^2 e^{U+V-a\psi} A_{,u} A_{,v} = 0, \tag{10}$$

$$2\psi_{uv} - U_u \psi_v - U_v \psi_u + \frac{a\kappa^2}{4} e^{U+V-a\psi} A_{,u} A_{,v} = 0, \tag{11}$$

$$-2M_u U_u - 2\, U_{uu} + U_u^2 + V_u^2 + 8\psi_u^2 + 2\kappa^2 e^{U+V-a\psi} A_{,u}^2 = 0, \tag{12}$$

$$-2M_v U_v - 2\, U_{vv} + U_v^2 + V_v^2 + 8\psi_v^2 + 2\kappa^2 e^{U+V-a\psi} A_{,v}^2 = 0. \tag{13}$$

Note that (9) can be derived from the other equations. It is not independent. From (10) and (11), letting $E = V - a\psi$ we obtain

$$2E_{uv} - U_u E_v - U_v E_u - \left(2 + \frac{a^2}{4}\right) \kappa^2 e^{U+E} A_{,u} A_{,v}. \tag{14}$$

Letting

$$B = \sqrt{2 + \frac{a^2}{4}}\, \kappa\, A, \tag{15}$$

(7) and (14) become

$$-2\, B_{,uv} = E_u B_{,v} + E_v B_{,u}, \tag{16}$$

$$2E_{uv} - U_u E_v - U_v E_u - e^{U+E}\, B_{,u}\, B_{,v} = 0. \tag{17}$$

The above two equations are the real and imaginary parts of the Ernst equation

$$\mathrm{Re}(\varepsilon)\, \nabla^2 \varepsilon = \nabla\varepsilon\, \nabla\varepsilon, \tag{18}$$

where differential operators in (18) are defined with respect to the metric given by $ds^2 = 2\, du\, dv - e^{-2U}\, d\phi^2$ and

$$\varepsilon = e^{-\frac{1}{2}(E+U)} + i \frac{B}{\sqrt{2}}. \tag{19}$$

The remaining part of the Einstein equations are given as

$$U_{uv} - U_u U_v = 0, \tag{20}$$

$$2\, X_{uv} - U_u X_v - U_v X_u = 0, \tag{21}$$

$$-2M_u U_u - 2\, U_{uu} + U_u^2 + \frac{1}{\alpha}(E_u^2 + 8\, X_u^2) + 2\kappa^2 e^{U+E} A_{,u}^2 = 0, \tag{22}$$

$$-2M_v U_v - 2\, U_{vv} + U_v^2 + \frac{1}{\alpha}(E_v^2 + 8\, X_v^2) + 2\kappa^2 e^{U+E} A_{,v}^2 = 0, \tag{23}$$

where

$$\psi = \frac{1}{\alpha}\left(X - \frac{a}{8} E\right), \quad V = \frac{1}{\alpha}(a\, X + E), \quad \alpha = 1 + \frac{a^2}{8}. \tag{24}$$

Hence a solution of the dilaton gravity field equations depends upon a linear equation (21) and the Ernst equation (18). The integrability of the Ernst equation and its properties are now very well known [22], but the characteristic initial value problem has not been solved yet.

The formulation of the collision of plane waves is as follows: The spacetime is divided into four disjoint regions by the null hyperplanes $u = 0$ and $v = 0$.

The first region ($u \leq 0, v \leq 0$):

$$ds^2 = 2du\, dv + dy^2 + dz^2. \tag{25}$$

This is the flat spacetime with $\psi = A = 0$.

The second region ($u > 0, v \leq 0$):

$$ds^2 = 2e^{-M_2} du\, dv + e^{-U_2 - V_2} dy^2 + e^{-U_2 + V_2} dz^2, \tag{26}$$

where $M_2 = M_2(u)$, $U_2 = U_2(u)$, $V_2 = V_2(u)$, $\psi_2 = \psi_2(u)$, and $A_2 = A_2(u)$ constitutes the data at $v \leq 0$. The only field equation is

$$-2M_{2,u} U_{2,u} - 2\, U_{2,uu} + U_{2,u}^2 + \frac{1}{\alpha}(E_{2,u}^2 + 8\, X_{2,u}^2) + 2\kappa^2\, e^{U_2 + E_2}\, A_{2,u}^2 = 0. \tag{27}$$

The third region ($u \leq 0, v > 0$):

$$ds^2 = 2e^{-M_3} du\, dv + e^{-U_3-V_3} dy^2 + e^{-U_3+V_3} dz^2, \quad (28)$$

where $M_3 = M_3(v)$, $U_3 = U_3(v)$, $V_3 = V_3(v)$, $\psi_3 = \psi_3(v)$, and $A_3 = A_3(v)$ constitutes the data at $u \leq 0$. The only field equation is

$$-2M_{3,v}U_{3,v} - 2\,U_{3,vv} + U_{3,v}^2 + \frac{1}{\alpha}\,(E_{3,v}^2 + 8\,X_{3,v}^2) + 2\kappa^2\, e^{U_3+E_3}\, A_{3,v}^2 = 0. \quad (29)$$

The second and third regions are called the incoming regions and the corresponding spacetimes are the plane wave geometries. Hence the functions $M_2 = M_2(u)$, $U_2 = U_2(u)$, $V_2 = V(u)$, $\psi_2 = \psi_2(u)$, $A_2 = A_2(u)$ and $M_3 = M_3(v)$, $U_3 = U_3(v)$, $V_3 = V_3(v)$, $\psi_3 = \psi_3(v)$, $A_3 = A_3(v)$ should be considered as the data on the hyperplanes $v = 0$ and $u = 0$, respectively.

The fourth region ($u > 0, v > 0$): The metric takes form (5) with $M = M(u,v)$, $U = U(u,v)$, $V = V(u,v)$, $\psi = \psi(u,v)$, and $A = A(u,v)$ such that in the incoming regions ($u \leq 0, v \leq 0$) the metric (5) reduces to the corresponding metrics in the related regions. The field equations are given in Eqs. (18) and (20)–(23).

The problem is to find the solutions of the above equations in such a way that the following conditions must be satisfied:

$$M(u, v \leq 0) = M_2(u), \quad U(u, v \leq 0) = U_2(u), \quad V(u, v \leq 0) = V_2(u), \quad (30)$$
$$\psi(u, v \leq 0) = \psi_2(u), \quad A(u, v \leq 0) = A_2(u), \quad (31)$$

$$M(u \leq 0, v) = M_3(v), \quad U(u \leq 0, v) = U_3(v), \quad V(u \leq 0, v) = V_3(v), \quad (32)$$
$$\psi(u \leq 0, v) = \psi_3(v), \quad A(u \leq 0, v) = A_3(v). \quad (33)$$

An exact solution of the above problem is

$$U = -\ln\cos(P+Q) - \ln\cos(P-Q),$$
$$E = \ln\cos(P+Q) - \ln\cos(P-Q), \quad (34)$$
$$A = \rho\sin(P-Q), \quad (35)$$
$$X = \frac{k_1}{2}\,\ln\,\frac{\cos Q - \sin P}{\cos Q + \sin P} + \frac{k_2}{2}\,\ln\,\frac{\cos P - \sin Q}{\cos P + \sin Q}. \quad (36)$$

Here $P = a_2\, u\, \theta(u)$, $Q = a_3\, v\, \theta(v)$, where θ is the Heaviside step function, a_2 and a_3 are arbitrary constants and

$$\rho^2 = \frac{16}{(8+a^2)\kappa^2}. \quad (37)$$

There are two distinct solutions.

(1) $k_1 = k_2 = k$ and $k^2 = \frac{a^2}{16}$.

Second region $v \leq 0, u > 0$, or $Q = 0$:

$$e^{a\,\psi_2} = \left[\frac{1-\sin P}{1+\sin P}\right]^{\frac{a\,k}{2\,\alpha}}, \quad (38)$$

$$e^{-U_2-V_2} = \cos^2 P\,\left[\frac{1-\sin P}{1+\sin P}\right]^{-\frac{a\,k}{2\,\alpha}}, \quad (39)$$

$$e^{-U_2+V_2} = \cos^2 P\,\left[\frac{1-\sin P}{1+\sin P}\right]^{\frac{a\,k}{2\,\alpha}}, \quad (40)$$

$$e^{-M_2} = (\cos P)^{\frac{a^2}{8\,\alpha}}, \quad (41)$$

$$A_2 = \rho\,\sin P. \quad (42)$$

Third region $u \leq 0, v > 0$, or $P = 0$:

$$e^{a\,\psi_3} = \left[\frac{1-\sin Q}{1+\sin Q}\right]^{\frac{a\,k}{2\,\alpha}}, \quad (43)$$

$$e^{-U_3-V_3} = \cos^2 Q\,\left[\frac{1-\sin Q}{1+\sin Q}\right]^{-\frac{a\,k}{2\,\alpha}}, \quad (44)$$

$$e^{-U_3+V_3} = \cos^2 Q\,\left[\frac{1-\sin Q}{1+\sin Q}\right]^{\frac{a\,k}{2\,\alpha}}, \quad (45)$$

$$e^{-M_3} = (\cos Q)^{\frac{a^2}{8\,\alpha}}, \quad (46)$$

$$A_3 = -\rho\,\sin Q. \quad (47)$$

Fourth region $u > 0, v > 0$:

$$e^{a\,\psi} = \left[\frac{(\cos Q - \sin P)\,(\cos P - \sin Q)}{(\cos Q + \sin P)\,(\cos P + \sin Q)}\right]^{\frac{a\,k}{2\,\alpha}} \times \left(\frac{\cos\,(P-Q)}{\cos(P+Q)}\right)^{\frac{a^2}{8\,\alpha}},$$

$$e^{-U-V} = [\cos\,(P+Q)]^{1-\frac{1}{\alpha}}\,[\cos\,(P-Q)]^{1+\frac{1}{\alpha}} \times \left[\frac{(\cos Q - \sin P)\,(\cos P - \sin Q)}{(\cos Q + \sin P)\,(\cos P + \sin Q)}\right]^{-\frac{a\,k}{2\,\alpha}},$$

$$e^{-U+V} = [\cos\,(P+Q)]^{1+\frac{1}{\alpha}}\,[\cos\,(P-Q)]^{1-\frac{1}{\alpha}} \times \left[\frac{(\cos Q - \sin P)\,(\cos P - \sin Q)}{(\cos Q + \sin P)\,(\cos P + \sin Q)}\right]^{\frac{a\,k}{2\,\alpha}},$$

$$e^{-M} = [\cos\,(P+Q)]^{\frac{3\,a^2}{16\,\alpha}}\,[\cos\,(P-Q)]^{\frac{-\,a^2}{16\,\alpha}},$$

$$A = \rho\,\sin\,(P-Q).$$

(2) $k_2 = -k_1 = -k$ and $k^2 = \frac{a^2}{16}$.

Second region $v \leq 0, u > 0$, or $Q = 0$:

$$e^{a\,\psi} = \left[\frac{1-\sin P}{1+\sin P}\right]^{\frac{a\,k}{2\,\alpha}}, \quad (48)$$

$$e^{-U-V} = \cos^2 P \left[\frac{1-\sin P}{1+\sin P}\right]^{-\frac{a\,k}{2\,\alpha}}, \tag{49}$$

$$e^{-U+V} = \cos^2 P \left[\frac{1-\sin P}{1+\sin P}\right]^{\frac{a\,k}{2\,\alpha}}, \tag{50}$$

$$e^{-M} = (\cos P)^{\frac{a^2}{8\,\alpha}} \tag{51}$$

$$A_2 = \rho \sin P. \tag{52}$$

Third region $u \leq 0, v > 0$, or $P = 0$:

$$e^{a\,\psi} = \left[\frac{1+\sin Q}{1-\sin Q}\right]^{\frac{a\,k}{2\,\alpha}}, \tag{53}$$

$$e^{-U-V} = \cos^2 Q \left[\frac{1+\sin Q}{1-\sin Q}\right]^{-\frac{a\,k}{2\,\alpha}}, \tag{54}$$

$$e^{-U+V} = \cos^2 Q \left[\frac{1+\sin Q}{1-\sin Q}\right]^{\frac{a\,k}{2\,\alpha}}, \tag{55}$$

$$e^{-M} = (\cos Q)^{\frac{a^2}{8\,\alpha}}, \tag{56}$$

$$A_2 = -\rho \sin Q. \tag{57}$$

Fourth region $u > 0, v > 0$:

$$e^{a\,\psi} = \left[\frac{(\cos Q - \sin P)\,(\cos P + \sin Q)}{(\cos Q + \sin P)\,(\cos P - \sin Q)}\right]^{\frac{a\,k}{2\,\alpha}} \times \left(\frac{\cos(P-Q)}{\cos(P+Q)}\right)^{\frac{a^2}{8\,\alpha}},$$

$$e^{-U-V} = [\cos(P+Q)]^{1-\frac{1}{\alpha}}\,[\cos(P-Q)]^{1+\frac{1}{\alpha}} \times \left[\frac{(\cos Q - \sin P)\,(\cos P + \sin Q)}{(\cos Q + \sin P)\,(\cos P - \sin Q)}\right]^{-\frac{a\,k}{2\,\alpha}},$$

$$e^{-U+V} = [\cos(P+Q)]^{1+\frac{1}{\alpha}}\,[\cos(P-Q)]^{1-\frac{1}{\alpha}} \times \left[\frac{(\cos Q - \sin P)\,(\cos P + \sin Q)}{(\cos Q + \sin P)\,(\cos P - \sin Q)}\right]^{\frac{a\,k}{2\,\alpha}},$$

$$e^{-M} = [\cos(P-Q)]^{\frac{3\,a^2}{16\,\alpha}}\,[\cos(P+Q)]^{\frac{-a^2}{16\,\alpha}},$$

$$A = \rho \sin(P-Q).$$

The spacetime in the fourth region is singular on the hyperplanes $a_2\,u \pm a_3\,v = \frac{\pi}{2}$. When a goes to zero both of the above solutions reduce to the well-known Bell-Szekeres solution [6].

III. DILATON GRAVITY WITH TWO U(1) VECTOR FIELDS

A dimensionally reduced superstring theory in four dimensions can be described in terms of $N = 4$ supergravity [5]. There are two versions of $N = 4$ supergravity: SO(4) and SU(4) versions. We shall only consider the bosonic part of the theory with U(1) $\otimes$ U(1) vectors in each version and one real dilaton field. In the following Lagrangian, although $(a, b) = (2, -2)$ for the SO(4) case and $(a, b) = (2, 2)$ for the SU(4) case, we shall keep these constants (couplings of dilaton field to each gauge field):

$$L = \sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{2}{\kappa^2}(\nabla\psi)^2 - \frac{1}{4}\left(e^{-a\psi}\,F^2 + e^{-b\psi}\,H^2\right)\right]. \tag{58}$$

The field equations are

$$G_{\mu\nu} = 4\left[\partial_\mu\psi\,\partial_\nu\psi - \frac{1}{2}(\nabla\psi)^2 g_{\mu\nu}\right] + \kappa^2\,e^{-a\psi}\left[F^\alpha_\mu F_{\nu\alpha} - \frac{1}{4}F^2 g_{\mu\nu}\right] + \kappa^2\,e^{-b\psi}\left[H^\alpha_\mu H_{\nu\alpha} - \frac{1}{4}H^2 g_{\mu\nu}\right],$$

$$\nabla_\mu(e^{-a\psi}F^{\mu\nu}) = 0, \tag{59}$$

$$\nabla_\mu(e^{-b\psi}H^{\mu\nu}) = 0, \tag{60}$$

$$\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\psi) + \frac{\kappa^2\sqrt{-g}}{16}(a\,e^{-a\psi}F^2 + b\,e^{-b\psi}H^2) = 0, \tag{61}$$

where $F^2 = F^{\alpha\beta}\,F_{\alpha\beta}$ and $H^2 = H^{\alpha\beta}\,H_{\alpha\beta}$. Both $F_{\mu\nu}$ and $H_{\mu\nu}$ are obtained by the vector potentials A_μ and B_μ, respectively; i.e., they are given by

$$F_{\mu\nu} = \partial_\mu\,A_\nu - \partial_\nu\,A_\mu, \quad H_{\mu\nu} = \partial_\mu\,B_\nu - \partial_\nu\,B_\mu. \tag{62}$$

In this section, instead of giving the complete formulation of the problem we give a special solution of collision problem. We consider the same spacetime structure as considered in the previous section with the line element (5). In the general case none of the waves superpose due to the nonlinearities in the field equations. On the other hand, the existence of two different Abelian gauge fields allows one to consider the following type of collision problem (such a solution does not exist with one Abelian gauge field). Consider one of the gauge fields is zero in one of the incoming regions and the second gauge field is zero in the other incoming region. More specifically one of the U(1) potentials (A_μ) vanishes in one of the incoming regions and the other U(1) potential (B_μ) vanishes in the other region. In the interaction region we have both fields. This implies a superposition in the gauge fields. Such an assumption simplifies the field equations considerably [23].

The reduced field equations are

$$U_{uv} - U_u\,U_v = 0, \tag{63}$$

$$2V_{uv} - U_uV_v - U_vV_u = 0, \tag{64}$$

$$-2M_u U_u - 2\,U_{uu} + U_u^2 + \left(1 + \frac{8}{a^2}\right)\, V_u^2 + 4\,\kappa^2\, B_u^2\, e^U = 0, \quad (65)$$

$$-2M_v U_v - 2U_{vv} + U_v^2 + \left(1 + \frac{8}{a^2}\right)\, V_v^2 + 4\,\kappa^2\, A_v^2\, e^U = 0. \quad (66)$$

Depending upon the choices of the U(1) potentials we find the dilaton field ψ accordingly. We have two distinct cases.

Case 1: $b = a$. We have two subcases (in each case we assume that a is different then zero). (1a): $\psi = \frac{1}{a}V$, $A_\mu = (0,0,0,A(v))$, $B_\mu = (0,0,0,B(u))$. The field equations are given above in (63)–(66). (1b): $\psi = \frac{1}{a}V$, $A_\mu = (0,0,A(u),0)$, $B_\mu = (0,0,B(v),0)$. The field equations are exactly the same as in case (1a) if A and B are interchanged in Eqs. (65) and (66).

Case 2: $b = -a$. We have again two subcases. (2a): $\psi = \frac{1}{a}V$, $A_\mu = (0,0,0,A(v))$, $B_\mu = (0,0,B(u),0)$. The field equations are exactly the same as in case (1a). (2b): $\psi = -\frac{1}{a}V$, $A_\mu = (0,0,A(u),0)$, $B_\mu = (0,0,0,B(v))$. The field equations are exactly the same as in case (1b).

The solutions of Eqs. (63)–(66) are given as [9]

$$e^{-U} = f(u) + g(v), \quad (67)$$

$$V = \frac{1}{\sqrt{f+g}}(R+S), \quad (68)$$

$$R = \int_f^{\frac{1}{2}} P_{-\frac{1}{2}}\, \left(1 + \frac{2(\xi - f)(\frac{1}{2} - g)}{(\xi + \frac{1}{2})(f+g)}\right) \times \frac{d}{d\xi}\left[\sqrt{\frac{1}{2} + \xi}\, V_2(\xi)\right]\, d\xi, \quad (69)$$

$$S = \int_g^{\frac{1}{2}} P_{-\frac{1}{2}}\, \left(1 + \frac{2(\eta - g)(\frac{1}{2} - f)}{(\eta + \frac{1}{2})(f+g)}\right) \times \frac{d}{d\eta}\left[\sqrt{\frac{1}{2} + \eta}\, V_3(\eta)\right]\, d\eta\,, \quad (70)$$

where f and g are functions of u and v, respectively, $P_{-\frac{1}{2}}$ is the Legendre function of order $-\frac{1}{2}$. These functions are determined from the data . In the incoming regions we have $f = \frac{1}{2}$ $(u \le 0)$ and $g = \frac{1}{2}$ $(v \le 0)$ where

$$f = e^{U_2} - \frac{1}{2}, \quad g = e^{U_3} - \frac{1}{2}. \quad (71)$$

The functions $V_2(u)$ and $V_3(v)$ are the data for the function $V(u,v)$. The solutions may be summarized as follows. Here we are giving case (1a) explicitly. The other cases can be given easily by correct identifications.

Second region $v \le 0, u > 0$, or $g = \frac{1}{2}$: The dilaton field $\psi_2 = \frac{1}{a}\,V_2$, the gauge potentials are given as $A_\mu = 0$ and $B_\mu = (0,0,0,B(u))$. The only field equation is given by

$$-2M_{2,u}U_{2,u} - 2\,U_{2,uu} + U_{2,u}^2 + \left(1 + \frac{8}{a^2}\right)\, V_{2,u}^2 + 4\,\kappa^2\, B_u^2\, e^{U_2} = 0. \quad (72)$$

Third region $u \le 0, v > 0$, or $f = \frac{1}{2}$: The dilaton field $\psi_3 = \frac{1}{a}\,V_3$, the gauge potentials are given as $A_\mu = (0,0,0,A(v))$ and $B_\mu = 0$. The only field equation is given by

$$-2M_{3,v}U_{3,v} - 2U_{3,vv} + U_{3,v}^2 + \left(1 + \frac{8}{a^2}\right)\, V_{3,v}^2 + 4\,\kappa^2\, A_v^2\, e^{U_3} = 0. \quad (73)$$

Fourth region $u > 0, v > 0$: The exact solutions of $U(u,v)$ and $V(u,v)$ are given in (67) and (68). The dilaton field $\psi(u,v) = \frac{1}{a}\,V(u,v)$, the gauge potentials are given as $A_\mu = (0,0,0,A(v))$ and $B_\mu = (0,0,0,B(u))$. The field equations to be solved are (65) and (66). Given the data $(V_2(u), V_3(v))$ one finds the function $V(u,v)$ from the integral formula (68). Given the data $(V_2(u), V_3(v))$ and $(A(v), B(u))$ one integrates the function $M(u,v)$ from (65) and (66).

A simple exact solution to the above problem is given as

$$V = m_1 \operatorname{arctanh}\, \left(\frac{\frac{1}{2} - f}{\frac{1}{2} + g}\right)^{\frac{1}{2}} + m_2 \operatorname{arctanh}\, \left(\frac{\frac{1}{2} - g}{\frac{1}{2} + f}\right)^{\frac{1}{2}} \quad (74)$$

with

$$V_2 = m_1 \operatorname{arctanh}\,(\tfrac{1}{2} - f)^{\frac{1}{2}}, \quad (75)$$

$$V_3 = m_2 \operatorname{arctanh}\,(\tfrac{1}{2} - g)^{\frac{1}{2}}, \quad (76)$$

where m_1 and m_2 are arbitrary constants. In the general case the initial data is loaded on the functions f and g. The determination of these functions is important in the integration of the function M. We find this function by following two different approaches. This means that we have two different solutions for two different data.

First solution: The functions f and g are given by

$$f(u) = \frac{1}{2} - s_1\, u^{n_1}\, \theta(u), \quad g(v) = \frac{1}{2} - s_2\, v^{n_1}\, \theta(v),$$

where n_1 and n_2 are positive integers (≥ 2). This is not the complete data but the function $M(u,v)$ can be found as

$$2\,M = \left(1 - \frac{b}{4}\,(m_1+m_2)^2\right)\,\ln(f+g) + \frac{b}{4}\,\left[m_1^2\,\ln\left(\frac{1}{2}+g\right) + m_2^2\,\ln\left(\frac{1}{2}+f\right)\right]$$
$$+\frac{b}{2}\,m_1\,m_2\,\ln[\tfrac{1}{2} + 2\,f\,g + \tfrac{1}{2}\,\sqrt{(1-4\,f^2)(1-4\,g^2)}] - 4\,\kappa^2\,\left(\int_{\frac{1}{2}}^{f}\,B_\xi^2\,d\xi + \int_{\frac{1}{2}}^{g}\,A_\eta^2\,d\eta\right).$$

In the incoming regions we have

$$2\,M_2 = \left(1 - \frac{b}{4}\,m_1^2\right)\,\ln\left(\frac{1}{2}+f\right) - 4\,\kappa^2\,\int_{\frac{1}{2}}^{f}\,B_\xi^2\,d\xi, \quad (77)$$

$$2\,M_3 = \left(1 - \frac{b}{4}\,m_2^2\right)\,\ln\left(\frac{1}{2}+g\right) - 4\,\kappa^2\,\int_{\frac{1}{2}}^{g}\,A_\eta^2\,d\eta, \quad (78)$$

where the last two integrals in above expression are due to the initial values of the gauge fields on the null hyperplanes which are left arbitrary and

$$b = 1 + \frac{8}{a^2}, \quad b\,m_i^2 = 8\,\left(1 - \frac{1}{n_i}\right)$$

with $i = 1, 2$. As far as the singularity structure is considered our solution given above looks like the vacuum Einstein solutions given by Szekeres [9]. They all suffer from a future closing spacetime singularity at $f + g = 0$.

Second solution: The functions f and g are determined by the equations

$$\frac{2\,f_{uu}}{f_u^2} = \frac{1}{\frac{1}{2}+f} - b\,\left(\frac{1}{2}+f\right)\,\left(\frac{dV_2}{df}\right)^2 - 4\,\kappa^2\,\left(\frac{dB}{df}\right)^2, \quad (79)$$

$$\frac{2\,g_{vv}}{g_v^2} = \frac{1}{\frac{1}{2}+g} - b\,\left(\frac{1}{2}+g\right)\,\left(\frac{dV_3}{dg}\right)^2 - 4\,\kappa^2\,\left(\frac{dA}{dg}\right)^2, \quad (80)$$

where V_2 and V_3 are given in (75) and (76). Then the function $M(u, v)$ is found as

$$2\,M = \left[1 - b\,\frac{(m_1+m_2)^2}{4}\right]\,\ln(f+g) + \left[-1 + b\,\frac{m_1^2+m_2^2}{4}\right]\,\ln\left[\left(\frac{1}{2}+f\right)\,\left(\frac{1}{2}+g\right)\right]$$
$$+b\,\frac{m_1\,m_2}{2}\,\ln\left(\frac{1}{2} + 2fg + \frac{1}{2}\,\sqrt{(1-4f^2)(1-4g^2)}\right). \quad (81)$$

The function M in the incoming regions vanish ($M_2 = M_3 = 0$). Hence given the functions $A(g)$ and $B(f)$, we determine the functions f and g through (79) and (80) in terms of u and v. This completes the determination of the metric in the fourth region. For different set of functions $(A(g), B(f))$ we have different solutions.

When the gauge potentials A and B go to zero and the dilaton coupling constant becomes larger then both of the above solutions approach to the Szekeres solutions [9]. For all of these solutions the surface $f + g = 0$ is singular.

IV. CONCLUSION

We have given exact solutions of the colliding plane waves in the Einstein-Maxwell-dilaton gravity theories. Although the exact solutions we obtained in this work differ from the solutions of the vacuum Einstein and Einstein-Maxwell theories, the singularity structures of the solutions of these different theories look the same. In this work we have studied the collision of plane waves in four dimensions. Higher dimensional plane waves when dimensionally reduced (with some duality transformations) lead to the extreme black hole solutions in four dimensions. In this respect it is perhaps more interesting to investigate the colliding gravitational plane waves in higher dimensions. This will be the subject of forthcoming communication.

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APPENDIX

In Maxwell theory, because of the linearity, the solution in the interaction region is just the superposition of the plane wave solutions in the second and third regions. In Einstein theory such a superposition is not allowed and hence to find exact solutions (solution of the characteristic initial value problem) is not possible yet. In this appendix we consider the collision of the Maxwell-dilaton plane waves which shares the similar difficulties of the Einstein theory. The Lagrangian of the corresponding theory is

$$L = \left[\frac{2}{\kappa^2}(\nabla\psi)^2 + \frac{1}{4}\,e^{-a\psi}F^2\right], \quad (A1)$$

where a is the dilaton coupling constant and the spacetime metric is flat in all regions. Here we kept the constant κ which may be set equal to unity. The field equations are

$$\nabla_\mu (e^{-a\psi} F^{\mu\nu}) = 0, \tag{A2}$$

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \psi) + \frac{\kappa^2\, a\, \sqrt{-g}}{16} e^{-a\psi} F^2 = 0, \tag{A3}$$

with the choice $A_\mu = (0,0,A,0)$, where $A = A(u,v)$ and dilaton field $\psi = \psi(u,v)$, the field equations turn out to be

$$a\psi_u A_{,v} + a\psi_v A_{,u} - 2A_{uv} = 0 \tag{A4}$$

and

$$\psi_{uv} + \frac{a\kappa^2}{8} e^{-a\psi} A_{,u} A_{,v} = 0. \tag{A5}$$

These equations are the real and imaginary parts of the Ernst equation

$$\mathrm{Re}(\epsilon)\, \nabla^2 \epsilon = \nabla\epsilon\, \nabla\epsilon, \tag{A6}$$

where differential operators in (A6) are defined with respect to the metric given by $ds^2 = 2\, du\, dv$ and

$$\epsilon = e^{\frac{1}{2} a\psi} + i \frac{a\kappa}{4} A. \tag{A7}$$

This can be rewritten as

$$\nabla(g^{-1}\, \nabla g) = 0, \tag{A8}$$

where

$$g = \frac{2}{\epsilon + \bar{\epsilon}} \begin{bmatrix} 1 & \frac{i}{2}(\epsilon - \bar{\epsilon}) \\ \frac{i}{2}(\epsilon - \bar{\epsilon}) & \epsilon\bar{\epsilon} \end{bmatrix}. \tag{A9}$$

Equation (A8) is the two-dimensional σ model equation on SU(2)/U(1). Although the complete solution of (A6) is not known yet its integrability has been shown long time ago [24]. The soliton solutions and many interesting properties are known.

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