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On the initial singularity in Kantowski–Sachs spacetime

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Abstract The emergent universe scenario is a proposal for resolving the Big Bang singularity problem in the standard Friedmann–Lemaitre–Robertson–Walker cosmology. In the context of this scenario, the Universe originates from a non-singular static state. In the present work, considering the realization of the emergent universe scenario, we address the possibility of having a nonsingular Kantowski–Sachs type static state. Considering four and five dimensional models (with and without brane), it is shown that both the existence and stability of a nonsingular state depend on the dimensions of the spacetime and the nature of the fluid supporting the geometry.

1 Introduction

Besides the great successes of standard model of cosmology based on Einstein's theory of general relativity (GR), there still remain several theoretical issues. One attempt to resolve these problems was the inflationary universe scenario [1-3]. Although the inflation resolves some of the problems such as horizon problem, flatness problem and magnetic monopole problem, but it is incapable of solving the initial big bang singularity problem. In the standard model of cosmology, the Universe originates from an initial singular point where all the mass, energy, and spacetime are infinitely compressed. To cure this singularity problem, some models such as the ekpyrotic/cyclic universe [4-8], the pre-big bang [9, 10], and the emergent universe [11,12] have been proposed. In the latter scenario proposed by Ellis et al. [11, 12], the Universe has no timelike singularity, it is ever existing and stands almost static in the infinite past. The emergent model replaces the initial big bang singularity with a nonsingular static state, the so called Einstein static state (ESS). Then, the Universe enters to an inflationary era and produces the same cosmological

history as in the standard model. The existence of a stable ESS against various perturbations, such as quantum fluctuations, is a prerequisite for a successful emergent universe scenario. The original emergent model supposes an ESS that is characterized by Friedmann-Lemaitre-Robertson-Walker (FLRW) metric with a perfect fluid source. In the framework of Einstein's GR, it was demonstrated by Eddington that an ESS is unstable versus the homogeneous and isotropic perturbations [13, 14]. Later studies by Gibbons [15, 16] and Barrow et al. [17] showed that an ESS can be neutrally stable versus small inhomogeneous vector and tensor perturbations as well as adiabatic scalar density perturbations if the perfect fluid filling the universe has a sound speed $c_s^2 > 1/5$. An approach to amend the instability problem is to modify cosmological field equations of GR. Many works has been performed along this line in the context of modified theories of gravity such as Einstein–Cartan theory [18, 19], f(R)gravity [20–23], f(T) gravity [24,25], brane gravity [26–28], massive gravity [29,30] and modified Gauss-Bonnet gravity [31,32] among the others.

The emergent universe scenario has been analyzed in Einstein's GR and modified theories only for the FLRW spacetime. One interesting question that can be raised here is: can a non-FLRW static state be a seed for an emergent universe? To answer this question, one may consider Kantowski-Sachs (KS) [33] or Bianchi type cosmological models [34,35] and investigate their stability versus various perturbations. Our aim in the present work is to explore the former. KS models are spatially homogeneous and rotation-free but possess shear. Hence, they represent anisotropic universes. However, it is shown that the initial anisotropies in the context of these models die away as the universe expands from the initial singularity if there is a positive cosmological constant, which is effectively the requirement for an inflation scenario in the early stages of our universe [36,37]. Therefore, the universe could be initially anisotropic KS type, and then it passes through the inflation and its subsequent cosmological eras toward the current homogeneous and isotropic state. In [36],

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KS type anisotropic cosmological models are classified in the presence of a nonzero cosmological constant. It is shown that for a positive cosmological constant there exists a set of big bang models of zero measure as well as a set of models possessing nonzero measure with the de Sitter asymptotic. In [38], nonsingular KS type cosmological solutions are obtained in a quantum-corrected Einstein gravity. In deriving these analytical solutions, the authors used the analogy with the Nariai black hole. The global structure of the KS cosmological models are studied in [39]. It has been shown that if the energy-momentum source is a perfect fluid, all the general relativistic models are geodesically incomplete, both to the past and to the future, and the energy density of the perfect fluid diverges at each resulting singularity. In [40], KS models are examined versus the classical tests of cosmology, and the results have been compared with those belonging to FLRW spacetime in the standard model of cosmology. It is shown that for a large class of KS models, the observations are incapable of distinguishing between KS models and the standard model of cosmology. The reference [41] discusses the growth of density perturbations in KS cosmologies in the presence of a positive cosmological constant. It is found that when a bounce occurs in the cosmic scale factor, the density gradient in the bouncing directions experiences a local maximum at or slightly after the bounce.

The organization of this paper is as follows. In Sect. 2, we introduce the Kantowski-Sachs metric and its corresponding Einstein field equations in four dimensions. We obtain the existence and stability conditions versus the scalar perturbations for a Kantowski-Sachs static state (KSSS) for various energy-momentum sources with generic linear and Chaplygin gas type equations of state. The analysis in four dimensions shows that there is only one particular fluid type that can support a nonsingular KSSS. In Sect. 3, regarding the result in Sect. 2 and motivated by the fact that anisotropic models can be exact solutions to the string effective action [42], we apply our analysis to a braneworld generalization of the model in which a four dimensional KS spacetime embedded in a five dimensional bulk space. The conditions for the existence and stability of a nonsingular KSSS are also discussed in this framework for various fluid types. In Sect. 4, the stability analysis is done for a 5-dimensional KS model without the brane. Finally, Sect. 5 is devoted to our concluding remarks.

2 KS geometry in four dimensions

We consider that the spacetime metric is the KS type [33]

$$ds^{2} = -dt^{2} + a_{1}^{2}(t)dr^{2} + a_{2}^{2}(t)(d\theta^{2} + sin^{2}\theta d\varphi^{2}), \qquad (1)$$

where $a_1(t)$ and $a_2(t)$ are two arbitrary functions of time and the energy–momentum tensor supporting this geometry has the generic form $T^{\mu}{}_{\nu} = diag (-\rho, p_r, p_t, p_t)$ [43]. Then, the Einstein field equations considering the above metric and energy–momentum source read as

$$\frac{\dot{a}_2^2}{a_2^2} + \frac{2\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2} = \Lambda + k_4\rho, \tag{2}$$

$$\frac{2\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2} = \Lambda - k_4 p_r,\tag{3}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} = \Lambda - k_4 p_t, \tag{4}$$

where $k_4 = 8\pi G$ is the four dimensional gravitational coupling constant and Λ is a generic (positive or negative) cosmological constant.

2.1 KSSS and stability analysis in four dimensions

In the following, we investigate the existence and stability of KS type static state considering the field equations (2)– (4). To keep the generality of the analysis, we consider two generic kinds of energy–momentum sources: (i) a fluid possessing linear equation of state, and (ii) a fluid with generalized Chaplygin gas type equations of state.

2.1.1 Energy-momentum source with linear equations of state

For the sake of generality, here we consider distinct equations of state for the radial and lateral pressures p_r and p_t , respectively, as [43–45]

$$p_r = \omega_r \rho + p_{0r}, \quad p_t = \omega_t \rho + p_{0t}. \tag{5}$$

Then, the field equations (2)–(4) become

$$\frac{\dot{a}_2^2}{a_2^2} + \frac{2\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2} = \Lambda + k_4\rho, \tag{6}$$

$$\frac{2\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2} = \Lambda - k_4 \omega_r \rho - k_4 p_{0r}, \tag{7}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} = \Lambda - k_4 \omega_t \rho - k_4 p_{0t}.$$
(8)

The static state corresponding to the above system of nonlinear differential equations is defined as $\dot{a}_1(t) = \dot{a}_2(t) = \dot{\rho}(t) = 0$. Then, considering the identifications $a_1 = a_{01}$, $a_2 = a_{02}$ and $\rho = \rho_0$ for the static state, through the field equations (6)–(8), we obtain

$$\frac{1}{a_{02}^2} = \Lambda + k_4 \rho_0, \tag{9}$$

$$\frac{1}{a_{02}^2} = \Lambda - k_4 \omega_r \rho_0 - k_4 p_{0r}, \tag{10}$$

$$0 = \Lambda - k_4 \omega_t \rho_0 - k_4 p_{0t}. \tag{11}$$

$$p_{0r} = -(1 + \omega_r)\rho_0 = \omega_{eff}\rho_0.$$
 (12)

From (11), one gets the following relation for p_{0t}

$$p_{0t} = -\omega_t \rho_0 + \frac{\Lambda}{k_4}.$$
(13)

The Eqs. (12) and (13) represent the constant radial and lateral effective equations of state of the fluid supporting the static state.

Now, to study the stability of the static state given in Eqs. (9)–(11), we apply the scalar perturbations to the dynamical quantities of the system (6)–(8), i.e. the scale factors $a_1(t)$ and $a_2(t)$ and the energy density $\rho(t)$, as

$$a_1(t) \longrightarrow a_{01} (1 + \delta a_1(t)),$$

$$a_2(t) \longrightarrow a_{02} (1 + \delta a_2(t)),$$

$$\rho(t) \longrightarrow \rho_0 (1 + \delta \rho(t)).$$
(14)

Hence, the perturbed field equations up to the first order take the following form

$$\frac{1}{a_{02}^2} - \frac{2\delta a_2}{a_{02}^2} = \Lambda + k_4 \rho_0 + k_4 \rho_0 \delta \rho, \qquad (15)$$

$$2\delta\ddot{a}_2 + \frac{1}{a_{02}^2} - \frac{2\delta a_2}{a_{02}^2} = \Lambda - k_4\omega_r\rho_0\delta\rho - k_4\omega_r\rho_0 - k_4p_{0r},$$
(16)

$$\delta \ddot{a}_1 + \delta \ddot{a}_2 = \Lambda - k_4 \omega_t \rho_0 \delta \rho - k_4 \omega_t \rho_0 - k_4 p_{0t}. \tag{17}$$

Using the constraints given by Eqs. (9)-(11), one can reduce these perturbation equations to

$$\frac{2\delta a_2}{a_{02}^2} = -k_4 \rho_0 \delta \rho, \tag{18}$$

$$2\delta\ddot{a}_2 - \frac{2\delta a_2}{a_{02}^2} = -k_4\omega_r \rho_0 \delta\rho,$$
(19)

$$\delta \ddot{a}_1 + \delta \ddot{a}_2 = -k_4 \omega_t \rho_0 \delta \rho. \tag{20}$$

Substituting (18) in (19) leads to

$$\delta\ddot{a}_2 + \gamma^2 \delta a_2 = 0, \tag{21}$$

where

$$\gamma^2 = -\frac{1+\omega_r}{a_{02}^2} = \frac{\omega_{eff}}{a_{02}^2}.$$
(22)

Hence, one realizes that the oscillating modes for δa_2

$$\delta a_2 = C_1 e^{i\gamma t} + C_2 e^{-i\gamma t},\tag{23}$$

requires the constraint

$$\omega_r < -1. \tag{24}$$

Similarly, using (18), (19) and (20), we obtain

$$2\delta\ddot{a}_1 = k_4(1+\omega_r - 2\omega_t)\rho_0\delta\rho. \tag{25}$$

Combining (18) and (19) gives

$$\delta\rho = \frac{-2\delta\ddot{a}_2}{k_4(1+\omega_r)\rho_0},\tag{26}$$

where substituting in (25) yields

$$\delta \ddot{a}_1 = \left(\frac{2\omega_t}{1+\omega_r} - 1\right)\delta \ddot{a}_2,\tag{27}$$

which represents the relation between the radial and lateral perturbations. Twice integration of this equation gives

$$\delta a_1 = \left(\frac{2\omega_t}{1+\omega_r} - 1\right)\delta a_2 + C_3 t + C_4 \tag{28}$$

where C_3 and C_4 are integration constants. Regarding (23), the above equation implies that the stable oscillatory modes in δa_1 is subjected to the condition $C_3 = C_4 = 0$. In this case, the amplitude of the perturbations along the radial direction depends explicitly on the nature of the fluid, i.e on the equations of state parameters. For the particular case $\omega_t = 1 + \omega_r$, the perturbation amplitude on the radial and lateral directions are the same.

In the following, we elaborate on two specific forms of the fluid (5) and discuss on the stability of KSSS.

2.1.1.1 Perfect fluid

Considering a perfect fluid form for the supporting matter fields by setting $\omega_r = \omega_t = \omega$ and $p_{0r} = p_{0t} = 0$, Eqs. (9)– (11) for the static state reduce to

$$\frac{1}{a_{02}^2} = \Lambda + k_4 \rho_0, \tag{29}$$

$$\frac{1}{a_{02}^2} = \Lambda - k_4 \omega \rho_0,\tag{30}$$

$$0 = \Lambda - k_4 \omega \rho_0. \tag{31}$$

Substituting $p_{0r} = 0$ into Eq. (12), one gets either $w_r = -1$ or $\rho_0 = 0$. The same result can be observed by comparing the equations (29) and (30). If $\rho_0 = 0$, Eqs. (29) and (30) will be source free (because of (31)) and cannot be held anymore for finite a_0 . If $\omega = \omega_r = \omega_t = -1$ for a perfect fluid, Eq. (31) (which demands $\Lambda < 0$) is inconsistent with Eqs. (29) and (30). In either way, one observes that having a finite size static sate for the KS spacetime is impossible. Then, a perfect fluid cannot support a nonsingular KSSS. Hence, for the realization of a stable nonsingular KSSS, the modification of the perfect fluid from is inevitable.

2.1.1.2 Anisotropic fluid

Here, we consider two possible minimal modifications of the perfect fluid form leading to anisotropic fluids: (1) a fluid having different equations of state parameters on the radial and lateral directions, and (2) a fluid modifying the perfect fluid form with distinct constant pressures on the radial and lateral directions.

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Type 1:

In this case, considering two different equations of state parameters for the radial and lateral directions we have

$$p_r = \omega_r \rho, \quad p_t = \omega_t \rho.$$
 (32)

Then, Eqs. (9)–(11) governing the static state reduce to

$$\frac{1}{a_{02}^2} = \Lambda + k_4 \rho_0, \tag{33}$$

$$\frac{1}{a_{02}^2} = \Lambda - k_4 \omega_r \rho_0, \qquad (34)$$

$$0 = \Lambda - k_4 \omega_t \rho_0. \tag{35}$$

Satisfaction of (35) demands that Λ and ω_t have the same sign. Comparing Eqs. (33) and (34) we obtains the constraint $\omega_r = -1$ on the equation of state in the radial direction to have a nonsingular static state. On the other hand, we have seen that the stability of KSSS demands $\omega_r < -1$, i.e. Eq. (24). Therefore, one can conclude that the anisotropic fluid from defined in (32) can support a static state but this static state is not stable versus the scalar perturbations (14). **Type 2**:

We consider a fluid modifying the perfect fluid form by distinct constant pressures on the radial and lateral directions. For this aim, we introduce the constant pressure terms p_{0t} and p_{0r} to the equations of state of the fluid with $\omega_r = \omega_t = \omega$. In this case, the equations in (5) become

$$p_r = \omega \rho + p_{0r}, \quad p_t = \omega \rho + p_{0t}. \tag{36}$$

Then Eqs. (12) and (13) for the static state reduce to

$$p_{0r} = -(1+\omega)\rho_0,$$

$$p_{0t} = -\omega\rho_0 + \frac{\Lambda}{k_4}.$$
(37)

Combining them gives the following relation between the constant pressures p_{0r} and p_{0t}

$$p_{0t} - p_{0r} = \rho_0 + \frac{\Lambda}{k_4}.$$
(38)

In the case $\rho_0 + \frac{\Lambda}{k_4} = 0$, we face an inconsistency in the static state given by Eqs. (9)–(11). Thus, to have a static state we consider $\rho_0 + \frac{\Lambda}{k_4} \neq 0$. In this case, γ^2 given by Eq. (22) becomes

$$\gamma^2 = -\frac{1+\omega}{a_{02}^2}.$$
(39)

Therefore, the positivity condition on γ^2 to have a stable nonsingular static state demands

$$\omega < -1, \tag{40}$$

which means that the fluid supporting the geometry lies in the phantom range.

For the stability of this case, the oscillatory modes of δa_2 is given by Eq. (23) with γ in (154). The dynamics of δa_1 from Eq. (28) reads as

$$\delta a_1 = \left(\frac{\omega - 1}{\omega + 1}\right) \delta a_2 + C_3 t + C_4,\tag{41}$$

where for $C_3 = C_4 = 0$, the oscillatory modes are possible and hence the static state will be stable. Regarding the stability requirement (155), one notes that although the perturbation in both directions have the same sign but the amplitude of the perturbations on the radial direction is always greater than the perturbations on the lateral direction.

2.1.2 Energy–momentum source with generalized Chaplygin gas equation of state

Here, we consider a generalization of the Chaplygin gas type fluid [46–48] for the energy–momentum source possessing

$$p_r = -\frac{\alpha_r}{\rho^n}, \quad p_t = -\frac{\alpha_t}{\rho^m}.$$
 (42)

where $0 \le m, n \le 1$, α_r and α_t are positive constants. If $\alpha_r = \alpha_t$ and m = n = 1, the energy-momentum source reduces to the generalized Chaplygin gas in [48]. The equations governing a static state have the form

$$\frac{1}{a_{02}^2} = \Lambda + k_4 \rho_0, \tag{43}$$

$$\frac{1}{a_{02}^2} = \Lambda + k_4 \frac{\alpha_r}{\rho_0^n},$$
(44)

$$0 = \Lambda + k_4 \frac{\alpha_t}{\rho_0^m}.$$
(45)

Satisfaction of (45) demands $\Lambda < 0$ for $\alpha_t > 0$. Then, using (14), the perturbed field equations around the static state given by Eqs. (43)–(45) reduce to

$$\frac{2\delta a_2}{a_{22}^2} = -k_4 \rho_0 \delta \rho, \tag{46}$$

$$2\delta\ddot{a}_2 - \frac{2\delta a_2}{a_{02}^2} = -nk_4 \frac{\alpha_r}{\rho_0^n} \delta\rho,$$
(47)

$$\delta \ddot{a}_1 + \delta \ddot{a}_2 = -mk_4 \frac{\alpha_t}{\rho_0^m} \delta \rho. \tag{48}$$

Combining (46) and (47), we obtain

$$\delta \ddot{a}_2 + \gamma^2 \delta a_2 = 0, \tag{49}$$

where

$$\gamma^2 = -\frac{n\alpha_r \rho_0^{-(n+1)} + 1}{a_{02}^2}.$$
(50)

Here, one observes that since $\alpha_r > 0$, then $\gamma^2 < 0$ and consequently there are no oscillatory modes for δa_2 and δa_1 .

We recapitulate our analysis in this section as follows. It is proved that in the context of Einstein's GR in four dimensions: (i) a perfect fluid source cannot support a finite size static KS geometry, (ii) an anisotropic fluid with linear equations of state $p_r = \omega_r \rho$ and $p_t = \omega_t \rho$ supports a finite size static KS geometry but it is not stable against the scalar perturbations, (iii) a modification in the linear equation of state of the perfect fluid form as $p_r = \omega \rho + p_{0r}$ and $p_t = \omega \rho + p_{0t}$ can support a stable nonsingular KS type static state, and (iv) a generalized Chaplygin gas fluid having $p_r = -\frac{\alpha_r}{\rho^n}$, $p_t = -\frac{\alpha_t}{\rho^m}$ cannot support a stable nonsingular KS geometry in four spacetime dimensions.

In the following section, since anisotropic models are exact solutions to the string theory, we investigate the existence and stability conditions for a KSSS in a five dimensional gravity theory, and show how the extra dimensional geometric modifications affect the results in four dimensions.

3 KS geometry on the brane

We consider the five dimensional action [49]

$$S = \int d^5 x \sqrt{-\mathcal{G}} \left(\frac{1}{2k_5} \mathcal{R} - \Lambda_5 \right) + \int_{\xi=0} d^4 x \sqrt{-g} \left(\frac{1}{k_5} K^{\pm} - \lambda + L_{matter} \right), \qquad (51)$$

where $k_5 = 8\pi G_5$, \mathcal{G} , g, \mathcal{R} , Λ_5 , λ , and L_{matter} are the 5-dimensional gravitational coupling constant, trace of the bulk space's metric \mathcal{G}_{AB} , trace of the brane's metric $g_{\mu\nu}$, bulk space's Ricci scalar, bulk and brane vacuum energy, and the Lagrangian of the confined matter fields to brane, respectively. Also, x^{μ} with $\mu = 0, \ldots, 3$ represents the coordinates of the brane while ξ represents the single extra dimension orthogonal to the brane, and K^{\pm} is the extrinsic curvature on either sides of the brane.

Variation of the action (51) with respect to the bulk metric \mathcal{G}_{AB} gives the 5D Einstein field equations

$$^{(5)}G_{ab} = k_5{}^{(5)}T_{ab},$$

$$^{(5)}T_{ab} = -\Lambda_5 \mathcal{G}_{ab} + \delta(\xi) \left(-\lambda \mathcal{G}_{ab} + T_{ab}^{matter}\right),$$
(52)

where a, b = 0, ..., 4. In order to keep the generality of our analysis, similar to the four dimensional case, we consider the energy-momentum tensor $T_{\mu\nu}^{matter}$ corresponding to L_{matter} on the brane as $T_{\nu}^{\mu} = diag(-\rho, p_r, p_t, p_t)$ [43]. Considering the bulk space metric in the form

$$ds^2 = (N_a N_b + g_{ab}) dx^a dx^b, (53)$$

where N^a represents the unit normal vector to the hypersurface $\xi = constant$ and g_{ab} is the induced metric on this hypersurface, the induced field equations on the brane take the following form [50–52]

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4 T_{\mu\nu} + k_5^2 S_{\mu\nu} - E_{\mu\nu}, \qquad (54)$$

in which

$$S_{\mu\nu} = \frac{1}{12}TT_{\mu\nu} - \frac{1}{4}T^{\alpha}_{\mu}T_{\alpha\nu} + \frac{1}{24}g_{\mu\nu}(3T^{\alpha\beta}T_{\alpha\beta} - T^2).$$

Here $\Lambda = k_5(\Lambda_5 + k_5\lambda^2/6)$ is the effective cosmological constant on the brane, and $E_{ab} = C_{abcd}N^aN^b$ where C_{abcd} represents the 5-dimensional Weyl tensor of the bulk space.

Considering the brane cosmology with zero Weyl tensor and a generic energy–momentum tensor possessing the form we mentioned above, the Einstein field equations for the metric (1) on the brane read as

$$\frac{\dot{a}_2^2}{a_2^2} + \frac{2\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2} = \Lambda + k_4\rho + \frac{1}{12}k_5^2(\rho^2 - p_r^2 - p_t^2 + 2p_rp_t), \quad (55)$$

$$\frac{2\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2} = \Lambda - k_4 p_r - \frac{1}{12} k_5^2 (\rho^2 - p_r^2 + p_t^2 + 2\rho p_t),$$
(56)

$$\frac{\ddot{a}_{1}}{a_{1}} + \frac{\ddot{a}_{2}}{a_{2}} + \frac{\dot{a}_{1}\dot{a}_{2}}{a_{1}a_{2}}$$

$$= \Lambda - k_{4}p_{t} - \frac{1}{12}k_{5}^{2}(\rho^{2} + p_{r}^{2} + \rho p_{r} + \rho p_{t} - p_{r}p_{t}).$$
(57)

3.1 KSSS and stability analysis on the brane

In the following, we investigate the existence and stability of KS type static state on the 4D brane considering the field equations (55)–(57). For the sake of generality of the analysis, we study two generic kinds of energy–momentum sources: (i) the sources possessing linear equation of state, and (ii) the sources with generalized Chaplygin gas type equation of state.

3.1.1 Energy-momentum source with linear equations of state

We consider a general equation of state with the form given in (5), and introduce the parameters

$$\begin{split} A &= \frac{1}{12} k_5^2 (1 - \omega_r^2 - \omega_t^2 + 2\omega_r \omega_t), \\ B &= k_4 + \frac{1}{6} k_5^2 \left(\omega_r (p_{0t} - p_{0r}) + \omega_t (p_{0r} - p_{0t}) \right), \\ C &= \frac{1}{12} k_5^2 (2p_{0r} p_{0t} - p_{0r}^2 - p_{0t}^2), \\ D &= \frac{1}{12} k_5^2 (1 - \omega_r^2 + \omega_t^2 + 2\omega_t), \\ E &= \frac{1}{6} k_5^2 (\omega_t p_{0t} - \omega_r p_{0r} + p_{0t}) + k_4 \omega_r, \\ F &= \frac{1}{12} k_5^2 (p_{0r}^2 - p_{0t}^2) - k_4 p_{0r}, \end{split}$$

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$$G = \frac{1}{12}k_5^2(1 + \omega_r^2 + \omega_r + \omega_t - \omega_r\omega_t),$$

$$H = k_4\omega_t + \frac{1}{12}k_5^2(\omega_r(2p_{0r} - p_{0t}) - \omega_t p_{0r} + p_{0r} + p_{0t}),$$

$$I = \frac{1}{12}k_5^2(p_{0r}p_{0t} - p_{0r}^2) - k_4p_{0t}.$$
(58)

Hence the field equations (55)–(57) reduce to the following forms

$$\frac{\dot{a}_2^2}{a_2^2} + \frac{2\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2} = \Lambda + A\rho^2 + B\rho + C,$$
(59)

$$\frac{2\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2} = \Lambda - D\rho^2 - E\rho + F, \tag{60}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} = \Lambda - G\rho^2 - H\rho + I.$$
(61)

For a static state defined by $\dot{a}_1 = \dot{a}_2 = \ddot{a}_1 = \ddot{a}_2 = 0$, let $a_1 = a_{01}, a_2 = a_{02}$ and $\rho = \rho_0$. Then, the field equations (59)–(61) give

$$\frac{1}{a_{02}^2} = \Lambda + A\rho_0^2 + B\rho_0 + C,$$
(62)

$$\frac{1}{a_{02}^2} = \Lambda - D\rho_0^2 - E\rho_0 + F,$$
(63)

$$0 = \Lambda - G\rho_0^2 - H\rho_0 + I.$$
 (64)

The Using equations (62) and (63), we have the constraint equation for ρ_0

$$(A+D)\rho_0^2 + (B+E)\rho_0 + C - F = 0,$$
(65)

Now, to study the stability of static state defined in (62)–(64), we consider the scalar perturbations in the form of Eq. (14). Keeping up to the first order perturbation terms, Eqs. (59)–(61) give

$$\frac{1}{a_{02}^2} - \frac{2\delta a_2}{a_{02}^2} = \Lambda + A\rho_0^2 (1 + 2\delta\rho) + B\rho_0 (1 + \delta\rho) + C,$$
(66)
$$2\delta\ddot{a}_0 + \frac{1}{a_0^2} - \frac{2\delta a_2}{2\delta a_0^2} = \Lambda - D\rho_0^2 (1 + 2\delta\rho) - F\rho_0 (1 + \delta\rho) + F$$

$$2\delta\ddot{a}_2 + \frac{1}{a_{02}^2} - \frac{2\delta a_2}{a_{02}^2} = \Lambda - D\rho_0^2 (1 + 2\delta\rho) - E\rho_0 (1 + \delta\rho) + F,$$
(67)

$$\delta \ddot{a}_1 + \delta \ddot{a}_2 = \Lambda - G\rho_0^2 (1 + 2\delta\rho) - H\rho_0 (1 + \delta\rho) + I.$$
(68)

Using the constraints given by Eqs. (62)–(64), one can reduce above equations to

$$-\frac{2\delta a_2}{a_{02}^2} = (2A\rho_0^2 + B\rho_0)\delta\rho, \tag{69}$$

$$2\delta\ddot{a}_2 - \frac{2\delta a_2}{a_{02}^2} = (-2D\rho_0^2 - E\rho_0)\delta\rho,$$
(70)

$$\delta \ddot{a}_1 + \delta \ddot{a}_2 = (-2G\rho_0^2 - H\rho_0)\delta\rho.$$
 (71)

From Eq. (69), we have

$$\delta\rho = -\frac{2}{a_{02}^2(2A\rho_0^2 + B\rho_0)}\delta a_2,\tag{72}$$

where substituting in Eq. (70) leads to

$$\delta \ddot{a}_2 + \alpha^2 \delta a_2 = 0, \tag{73}$$

where

$$\alpha^2 = -\frac{1}{a_{02}^2} \left(1 + \frac{2D\rho_0 + E}{2A\rho_0 + B} \right). \tag{74}$$

Hence, the oscillating modes for δa_2 requires

$$1 + \frac{2D\rho_0 + E}{2A\rho_0 + B} < 0.$$
(75)

The solution to Eq. (73) with the condition (75) is

$$\delta a_2 = C_1 e^{i\alpha t} + C_2 e^{-i\alpha t},\tag{76}$$

which shows the oscillatory behavior of δa_2 .

On the other hand, combining Eqs. (69), (70) and (71) gives

$$2\delta\ddot{a}_1 = ((2A + 2D - 4G)\rho_0^2 + (B + E - 2H)\rho_0)\delta\rho,$$
(77)

and subtracting (70) from (69) leads to

$$2\delta\ddot{a}_2 = -((2A+2D)\rho_0^2 + (B+E)\rho_0)\delta\rho.$$
(78)

Hence, using (78) and (77) we obtain

$$\delta \ddot{a}_1 = \beta \delta \ddot{a}_2,\tag{79}$$

where β is defined as

$$\beta = \frac{4G\rho_0 + 2H}{(2A + 2D)\rho_0 + B + H} - 1.$$
(80)

Twice integration of equation (79) gives

$$\delta a_1 = \beta \delta a_2 + C_3 t + C_4. \tag{81}$$

This shows that δa_1 can also have an stable oscillatory mode if $C_3 = C_4 = 0$, and

$$2(A+D)\rho_0 + B + H \neq 0.$$
 (82)

Hence, both the conditions in (75) and (82) should be satisfied to have a stable static state. This result holds for a fluid with the general form of equation of state (5). In Fig. 1, the existence and stability conditions (i.e. Eqs. (64), (65), (75) and (82)) for a static state are plotted for some typical values of the parameters. The presence of some $\rho_0 > 0$ ranges represents the satisfaction of the constraints that means a stable static state exists for the given values of the parameters. The intersection of Y = 0 and $\Phi = 0$ lies in the range $\omega_r < -1$ that guarantees a stable static state. When $p_{0r} = 0.1$ and $p_{0t} = 0.2$, ω_t should be also negative, but for $p_{0r} = 0.1$, $p_{0t} = -0.1$, and small amounts of $\rho_0 > 0$, the constraints for having a stable static state can be satisfied for a positive value of ω_t . It is also clear from Fig. 1 that, in this setup, changing the value of p_{0t} did not affect the result of Eq. (65), but have a significant effect on Eq. (64) for small values of $\rho_0 > 0$.

In the following, we consider some special types of fluids and discuss under what conditions these fluids can support a nonsingular KSSS on the brane.

3.1.1.1 Perfect fluid

In this case, we consider perfect fluid which means $\omega_r = \omega_t = \omega$ and $p_{0r} = p_{0t} = 0$. Then, the coefficients A to I defined in Eq. (58) reduce to

$$A = \frac{1}{12}k_5^2, B = k_4, D = G = \frac{1}{12}k_5^2(1 + 2\omega), E = H = k_4\omega, C = F = I = 0.$$
(83)

For the static state given by Eqs. (62)–(64), one obtains

$$\frac{1}{a_{02}^2} = \Lambda + \frac{1}{12}k_5^2\rho_0^2 + k_4\rho_0, \tag{84}$$

$$\frac{1}{a_{02}^2} = \Lambda - \frac{1}{12}k_5^2(1+2\omega)\rho_0^2 - k_4\omega\rho_0,$$
(85)

$$0 = \Lambda - \frac{1}{12}k_5^2(1+2\omega)\rho_0^2 - k_4\omega\rho_0.$$
 (86)

Similar to four dimensions, these equations lead to

$$\frac{1}{a_{02}^2} = 0, (87)$$

which means that a perfect fluid cannot support a KSSS even in the presence of higher dimensional modifications.

3.1.1.2 Anisotropic fluid

In four dimensions, we showed that a stable static state can exist only for an anisotropic fluid of Type 2. Here, we show that in the presence of higher dimensional modifications to the field equations, both Type 1 and Type 2 fluids can support a stable nonsingular KSSS.

Type 1:

In this case we consider equations of state in (32). Hence, the coefficients A to I defined by Eq. (58) reduce to

$$A = \frac{1}{12}k_{5}^{2}(1 - \omega_{r}^{2} - \omega_{t}^{2} + 2\omega_{r}\omega_{t}),$$

$$B = k_{4},$$

$$D = \frac{1}{12}k_{5}^{2}(1 - \omega_{r}^{2} + \omega_{t}^{2} + 2\omega_{t}),$$

$$E = k_{4}\omega_{r},$$

$$G = \frac{1}{12}k_{5}^{2}(1 + \omega_{r}^{2} + \omega_{r} + \omega_{t} - \omega_{r}\omega_{t}),$$

$$H = k_{4}\omega_{t},$$

$$C = F = I = 0.$$
(88)

Then the constraints (64) and (65) for the existence of a static state, respectively, become

$$\Lambda - \left(\frac{1}{12}k_5^2(1 + \omega_r^2 + \omega_r + \omega_t - \omega_r\omega_t)\right)\rho_0^2 - k_4\omega_t\rho_0 = 0,$$
(89)

and

$$\frac{1}{6}k_5^2(1-\omega_r^2+\omega_t+\omega_r\omega_t)\rho_0^2+k_4(1+\omega_r)\rho_0=0.$$
 (90)

Also the conditions (75) and (82) to have an oscillatory mode, respectively, become

$$\frac{\frac{1}{6}k_5^2(1-\omega_r^2+\omega_t^2+2\omega_t)\rho_0+k_4\omega_r}{\frac{1}{6}k_5^2(1-\omega_r^2-\omega_t^2+2\omega_r\omega_t)\rho_0+k_4} < -1,$$
(91)

and

$$\rho_0 \neq -\frac{3k_4}{k_5^2 \left(1 + \frac{\omega_r(\omega_t - \omega_r)}{1 + \omega_t}\right)}.$$
(92)

Thus, in contrast to four dimensions, the anisotropic fluid of Type 1 can support a stable nonsingular ESU on the brane under the conditions (89)–(92). In Fig. 2, these conditions are plotted for some typical values of the parameters. We see from the figure that the constraints can be satisfied for $\rho_0 > 0$. This means a stable static state do exist for the given values of the parameters. In this case, it is obvious that to have a stable static state we need $\omega_r < -1$, but ω_t can have any value in the given range of $-1.5 < \omega_t < 1$. **Type 2**:

Now we consider anisotropic fluid given in (36). Thus, the coefficients in (58) become

$$A = \frac{1}{12}k_5^2,$$

$$B = k_4,$$

$$C = \frac{1}{12}k_5^2(2p_{0r}p_{0t} - p_{0r}^2 - p_{0t}^2),$$

$$D = G = \frac{1}{12}k_5^2(1 + 2\omega),$$

$$E = \frac{1}{6}k_5^2(\omega p_{0t} - \omega p_{0r} + p_{0t}) + k_4\omega,$$

$$F = \frac{1}{12}k_5^2(p_{0r}^2 - p_{0t}^2) - k_4p_{0r},$$

$$H = k_4\omega + \frac{1}{12}k_5^2(\omega p_{0r} - \omega p_{0t} + p_{0r} + p_{0t}),$$

$$I = \frac{1}{12}k_5^2(p_{0r}p_{0t} - p_{0r}^2) - k_4p_{0t}.$$
(93)

Then the constraints (64) and (65) obtained for the existence of a static state, respectively, become

$$\begin{split} \Lambda &- \frac{1}{12} k_5^2 (1+2\omega) \rho_0^2 \\ &- \left(k_4 \omega + \frac{1}{12} k_5^2 (\omega p_{0r} - \omega p_{0t} + p_{0r} + p_{0t}) \right) \rho_0 \end{split}$$

Fig. 1 The left plots show the intersecting surfaces corresponding to constraints (64) and (65) for having a static state. Y = 0 represents equation (64) and $\Phi = 0$ represents equation (65). The right plots show the intersection of surfaces for the existence conditions in the solution space of stability conditions, i.e equations (75) and (82). The curves in the right plots satisfy all the conditions (64),(65),(75) and (82) to have a totally stable static state. We used the typical values of $p_{0r} = 0.1$ and $p_{0t} = 0.2$ for the first line and $p_{0r} = 0.1$ and $p_{0t} = -0.1$ for the second line. We also assumed $k_4 = 1$, $k_5 = 1.5, \Lambda = 10^{-5}$



Fig. 2 The first plot shows the intersecting surfaces corresponding to constraints (89) and (90) for having a static state. Y = 0 and $\Phi = 0$ represent equations (89) and (90), respectively. The second plot shows the intersection of surfaces for the existence conditions in the solution space of stability conditions, i.e Eqs. (91) and (92). The value of the

parameters in blue in the second plot satisfy all the conditions (89), (90), (91) and (92) to have a totally stable static state. The used typical values and ranges are: $k_4 = 1$, $k_5 = 1.5$, $p_{0r} = 0$, $p_{0t} = 0$, $\Lambda = 10^{-5}$, $-1.5 < \omega_r$, $\omega_t < 1$, 0.0001 $< \rho_0 < 5$



Fig. 3 The figures show the contours for constraints (94) and (95) in the solution space of equations (96) and (97). Y = 0 represents equation (94) and $\Phi = 0$ represents equation (95). The used typical values and ranges are: $k_4 = 1$, $k_5 = 1.5$, $\Lambda = 10^{-5}$, $-1 < p_{0r}$, $p_{0t} < 1$, and $\omega = -\frac{3}{2}$. Here, the surfaces Y = 0 and $\Phi = 0$ intersect on a line that means the satisfaction of all the conditions (94),(95),(96) and (97)

$$+\frac{1}{12}k_5^2(p_{0r}p_{0t}-p_{0r}^2)-k_4p_{0t}=0,$$
(94)

and

$$\frac{1}{6}k_{5}^{2}(1+\omega)\rho_{0}^{2} + \left(k_{4}(1+\omega) + \frac{1}{6}k_{5}^{2}(\omega p_{0t} - \omega p_{0r} + p_{0t})\right)\rho_{0} + \frac{1}{6}k_{5}^{2}(p_{0r}p_{0t} - p_{0r}^{2}) + k_{4}p_{0r} = 0.$$
(95)

In this case, the conditions (75) and (82) governing the stability of the the static state become

$$\frac{\frac{1}{6}k_5^2(1+2\omega)\rho_0 + \frac{1}{6}k_5^2(\omega p_{0t} - \omega p_{0r} + p_{0t}) + k_4\omega}{\frac{1}{6}k_5^2\rho_0 + k_4} < -1,$$
(96)

and

$$\rho_0 \neq -\frac{3k_4}{k_5^2} - \frac{1}{4} \left(p_{0r} + p_{0t} (\frac{1-\omega}{1+\omega}) \right). \tag{97}$$

Figure 3 shows the possibility of satisfying the conditions (94)–(97). Similar to GR, considering $\omega < -1$, we can have a stable static state for the given value of parameters.

3.1.2 Energy-momentum source with generalized Chaplygin gas equation of state

For an energy–momentum source with a generalized Chaplygin equation of state of the form (42), the Einstein field equations on the brane will be

$$\begin{aligned} \frac{\dot{a}_{2}^{2}}{a_{2}^{2}} + \frac{2\dot{a}_{1}\dot{a}_{2}}{a_{1}a_{2}} + \frac{1}{a_{2}^{2}} \\ &= \Lambda + k_{4}\rho + \frac{1}{12}k_{5}^{2}\left(\rho^{2} - \frac{\alpha_{r}^{2}}{\rho^{2n}} - \frac{\alpha_{t}^{2}}{\rho^{2m}} + 2\frac{\alpha_{r}\alpha_{t}}{\rho^{m+n}}\right), \end{aligned}$$
(98)
$$\begin{aligned} \frac{2\ddot{a}_{2}}{a_{2}} + \frac{\dot{a}_{2}^{2}}{a_{2}^{2}} + \frac{1}{a_{2}^{2}} \\ &= \Lambda + k_{4}\frac{\alpha_{r}}{\rho^{n}} - \frac{1}{12}k_{5}^{2}\left(\rho^{2} - \frac{\alpha_{r}^{2}}{\rho^{2n}} + \frac{\alpha_{t}^{2}}{\rho^{2m}} - 2\frac{\alpha_{t}}{\rho^{m-1}}\right), \end{aligned}$$

$$\begin{aligned} \ddot{a}_{1} &+ \ddot{a}_{2} + \dot{a}_{1}\dot{a}_{2} \\ &= \Lambda + k_{4}\frac{\alpha_{t}}{\rho^{m}} - \frac{1}{12}k_{5}^{2} \\ &\times \left(\rho^{2} + \frac{\alpha_{r}^{2}}{\rho^{2n}} - \frac{\alpha_{r}}{\rho^{n-1}} - \frac{\alpha_{t}}{\rho^{m-1}} - \frac{\alpha_{r}\alpha_{t}}{\rho^{m+n}}\right). \end{aligned}$$
(99)

The corresponding static state is given by the following equations

$$\frac{1}{a_{02}^2} = \Lambda + k_4 \rho_0 + \frac{1}{12} k_5^2 \left(\rho_0^2 - \frac{\alpha_r^2}{\rho_0^{2n}} - \frac{\alpha_t^2}{\rho_0^{2m}} + 2\frac{\alpha_r \alpha_t}{\rho_0^{m+n}} \right),$$
(101)

$$\frac{1}{a_{02}^2} = \Lambda + k_4 \frac{\alpha_r}{\rho_0^n} - \frac{1}{12} k_5^2 \left(\rho_0^2 - \frac{\alpha_{\tilde{r}}}{\rho_0^{2n}} + \frac{\alpha_{\tilde{t}}}{\rho_0^{2m}} - 2\frac{\alpha_t}{\rho_0^{m-1}} \right),$$

$$0 = \Lambda + k_4 \frac{\alpha_t}{\rho_0^m} - \frac{1}{12} k_5^2 \left(\rho_0^2 + \frac{\alpha_{\tilde{r}}^2}{\rho_0^{2n}} - \frac{\alpha_r}{\rho_0^{n-1}} - \frac{\alpha_t}{\rho_0^{m-1}} - \frac{\alpha_r \alpha_t}{\rho_0^{m+n}} \right)$$
(103)

Combining Eqs. (101) and (102) we get

$$k_4\left(\rho_0 - \frac{\alpha_r}{\rho_0^n}\right) + \frac{1}{6}k_5^2\left(\rho_0^2 - \frac{\alpha_r^2}{\rho_0^{2n}} + \frac{\alpha_r\alpha_t}{\rho_0^{m+n}} - \frac{\alpha_t}{\rho_0^{m-1}}\right) = 0$$
(104)

Thus to have a static state the constraints (103) and (104) should be satisfied. Then, the perturbed field equations versus the scalar perturbations (14) take the forms

$$\begin{aligned} \frac{1}{a_{02}^2} - \frac{2\delta a_2}{a_{02}^2} &= \Lambda + k_4 \rho_0 (1 + \delta \rho) \\ &+ \frac{k_5^2}{12} \left(\rho_0^2 (1 + 2\delta \rho) - \frac{\alpha_r^2}{\rho_0^{2n}} (1 - 2n\delta \rho) \right. \\ &- \frac{\alpha_r^2}{\rho_0^{2m}} (1 - 2m\delta \rho) + 2 \frac{\alpha_r \alpha_r}{\rho_0^{m+n}} (1 - (n + m)\delta \rho) \right), \end{aligned}$$

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$$2\delta\ddot{a}_{2} + \frac{1}{a_{02}^{2}} - \frac{2\delta a_{2}}{a_{02}^{2}} = \Lambda + k_{4} \frac{\alpha_{r}}{\rho_{0}^{n}} (1 - n\delta\rho) - \frac{k_{5}^{2}}{12} \left(\rho_{0}^{2} (1 + 2\delta\rho) - \frac{\alpha_{r}^{2}}{\rho_{0}^{2n}} (1 - 2n\delta\rho) + \frac{\alpha_{t}^{2}}{\rho_{0}^{2m}} (1 - 2m\delta\rho) - \frac{2\alpha_{t}}{\rho_{0}^{m-1}} (1 - (m - 1)\delta\rho) \right),$$
(106)
$$\delta\ddot{a}_{1} + \delta\ddot{a}_{2} = \Lambda + k_{4} \frac{\alpha_{t}}{\rho_{0}^{m}} (1 - m\delta\rho) - \frac{k_{5}^{2}}{12} \left(\rho_{0}^{2} (1 + 2\delta\rho) + \frac{\alpha_{r}^{2}}{\rho_{0}^{2n}} (1 - 2n\delta\rho) - \frac{\alpha_{r}}{\rho_{0}^{n-1}} (1 - (n - 1)\delta\rho) \right) - \frac{\alpha_{t}}{\rho_{0}^{m-1}} (1 - (m - 1)\delta\rho) - \frac{\alpha_{r}\alpha_{t}}{\rho_{0}^{m+n}} (1 - (n + m)\delta\rho)).$$
(107)

(105)

Using Eqs. (101) to (103) and defining

$$A = k_4 \rho_0 + \frac{k_5^2}{6} \left(\rho_0^2 + n \frac{\alpha_r^2}{\rho_0^{2n}} + m \frac{\alpha_t^2}{\rho_0^{2m}} - (n+m) \frac{\alpha_r \alpha_t}{\rho_0^{m+n}} \right),$$

$$B = -nk_4 \frac{\alpha_r}{\rho_0^m} - \frac{k_5^2}{\rho_0^2} \left(\rho_0^2 + n \frac{\alpha_r^2}{\rho_0^2} - m \frac{\alpha_t^2}{\rho_0^2} \right)$$
(108)

$$B = -nk_4 \frac{\alpha_r}{\rho_0^n} - \frac{\kappa_5}{6} \left(\rho_0^2 + n \frac{\alpha_r^2}{\rho_0^{2n}} - m \frac{\alpha_t^2}{\rho_0^{2m}} + (m-1) \frac{\alpha_t}{\rho_0^{m-1}} \right),$$
(109)

$$C = -mk_4 \frac{\alpha_t}{\rho_0^m} - \frac{k_5^2}{12} \left(2\rho_0^2 - 2n \frac{\alpha_r^2}{\rho_0^{2n}} + (n-1) \frac{\alpha_r}{\rho_0^{n-1}} + (m-1) \frac{\alpha_t}{\rho_0^{m-1}} + (n+m) \frac{\alpha_r \alpha_t}{\rho_0^{m+n}} \right).$$
 (110)

The perturbed equations (105) to (107) reduce to

$$-\frac{2\delta a_2}{a_{02}^2} = A\delta\rho,\tag{111}$$

$$2\delta\ddot{a}_2 - \frac{2\delta a_2}{a_{02}^2} = B\delta\rho,$$
 (112)

$$\delta \ddot{a}_1 + \delta \ddot{a}_2 = C \delta \rho. \tag{113}$$

Combining (111) and (112) we obtain

$$\delta \ddot{a}_2 + \left(\frac{B}{A} - 1\right) \frac{1}{a_{02}^2} \delta a_2 = 0.$$
(114)

Then an oscillatory mode for δa_2 is subjected to the condition

$$\frac{B}{A} > 1. \tag{115}$$

Combining Eqs. (111)–(113), we also obtain

$$-2\delta\ddot{a}_1 = (-2C + B - A)\delta\rho.$$
 (116)



Fig. 4 This figure shows the surfaces which satisfy equations (100) and (104) in the solution space of equation (115). Q = 0 and W = 0 represent equations (100) and (104), respectively, and their overlap shows the region in which we have a stable static state. The used typical values are: $k_4 = 1$, $k_5 = 1.5$, $\alpha_r = \frac{1}{2}$, $\alpha_t = 2$, $\Lambda = 10^{-5}$

$$2\delta\ddot{a}_2 = (B - A)\delta\rho,\tag{117}$$

Combining it with (116) we get

$$\delta \ddot{a}_1 = \left(\frac{2C}{B-A} - 1\right)\delta \ddot{a}_2.$$
(118)

Then, the dynamics of δa_1 reads as

$$\delta a_1 = \left(\frac{2C}{B-A} - 1\right)\delta a_2 + \alpha t + \beta.$$
(119)

where α , β are constants and for $\alpha = \beta = 0$ and $B \neq A$ the oscillatory modes are possible and the static state will be stable. Figure 4 shows the possibility of satisfying the constraints (100) and (104) in the solution space of equation (115) for the given values of parameters. It is seen from the figure that we need n, m > 1 and very small amount of ρ_0 , which corresponds to a large radial and lateral pressure, to have a stable static state. However, for $\rho_0 < 0.004, m$ can be less than 1.

Hence, the following is the summary of the analysis of the existence and stability of a KSSS on a brane: (i) a perfect fluid cannot support a finite size static KS geometry even in the presence of higher dimensional modifications, (ii) in contrast to GR in four dimensions, an anisotropic fluid with $p_r = \omega_r \rho$ and $p_t = \omega_t \rho$ supports a stable finite size static KS geometry, and (iii) a modification of the perfect fluid form as $p_r = \omega \rho + p_{0r}$ and $p_t = \omega \rho + p_{0t}$ can also support a stable nonsingular KS type static state, and (iv) in contrast to the case in four dimensions, a stable nonsingular KS geometry can be supported by a generalized Chaplygin gas source of energy-momentum.

4 KS geometry in five dimensions

In this section, we consider a five-dimensional KS type metric

$$ds^{2} = -dt^{2} + a_{1}^{2}(t)dr^{2} + a_{2}^{2}(t)(d\theta_{1}^{2} + sin^{2}\theta_{1}d\theta_{2}^{2} + sin^{2}\theta_{1}sin^{2}\theta_{2}d\theta_{3}^{2}), \qquad (120)$$

where $a_1(t)$ and $a_2(t)$ are two arbitrary functions of time and the energy–momentum tensor supporting this geometry has the generic form $T^{\mu}{}_{\nu} = diag(-\rho, p_r, p_t, p_t, p_t)$. Then, the Einstein field equations are

$$3\left(\frac{\dot{a}_2^2}{a_2^2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2}\right) = \Lambda + k_5\rho, \qquad (121)$$

$$3\left(\frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2}\right) = \Lambda - k_5 p_r,$$
(122)

$$\frac{\ddot{a}_1}{a_1} + \frac{2\ddot{a}_2}{a_2} + \frac{2\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2} = \Lambda - k_5 p_t.$$
(123)

4.1 KSSS and stability analysis in five dimensions

In the following, we study the existence and stability of fivedimensional KS type static state considering the field equations (121)–(123). Similar to previous sections, we consider two generic kinds of energy–momentum sources: (i) a fluid possessing linear equation of state, and (ii) a fluid with generalized Chaplygin gas type equations of state.

4.1.1 Energy-momentum source with linear equations of state

By considering a general equation of state with the form given in (5), the field equations (121)-(123) become

$$3\left(\frac{\dot{a}_2^2}{a_2^2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2}\right) = \Lambda + k_5\rho, \qquad (124)$$

$$3\left(\frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2}\right) = \Lambda - k_5 \omega_r \rho - k_5 p_{0r}, \qquad (125)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{2\ddot{a}_2}{a_2} + \frac{2\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2} = \Lambda - k_5\omega_t\rho - k_5p_{0t}$$
(126)

Then, the corresponding static state is given by the following equations

$$\frac{3}{a_{02}^2} = \Lambda + k_5 \rho_0, \tag{127}$$

$$\frac{3}{a_{02}^2} = \Lambda - k_5 \omega_r \rho_0 - k_5 p_{0r}, \qquad (128)$$

$$\frac{1}{a^2} = \Lambda - k_5 \omega_t \rho_0 - k_5 p_{0t}.$$
 (129)

$$a_{02}$$

From (127) and (128) we get

$$p_{0r} = -(1 + \omega_r)\rho_0 = \omega_{eff}\rho_0,$$
(130)

and combining (127) and (129) leads to

$$p_{0t} = \frac{2}{3} \frac{\Lambda}{k_5} - \left(\omega_t + \frac{1}{3}\right) \rho_0.$$
(131)

To study the stability of the static state given by (127)-(129), we consider the scalar perturbations in the form of (14) and keep up to the first order perturbation terms. Then Eqs. (124)-(126) give

$$3\left(\frac{1}{a_{02}^2} - \frac{2\delta a_2}{a_{02}^2}\right) = \Lambda + k_5\rho_0 + k_5\rho_0\delta\rho,$$
(132)
$$3\left(\delta\ddot{a}_2 + \frac{1}{a_{22}^2} - \frac{2\delta a_2}{a_{22}^2}\right) = \Lambda - k_5\omega_r\rho_0\delta\rho - k_5\omega_r\rho_0 - k_5p_{0r},$$

(133)
$$\delta\ddot{a}_1 + 2\delta\ddot{a}_2 + \frac{1}{a_{02}^2} - \frac{2\delta a_2}{a_{02}^2} = \Lambda - k_5\omega_t\rho_0\delta\rho - k_5\omega_t\rho_0 - k_5p_{0t}.$$

Using the static state defined in (127)–(129), the above equations reduce to

$$-6\frac{\delta a_2}{a_{02}^2} = k_5 \rho_0 \delta \rho, \tag{135}$$

$$3\delta\ddot{a}_2 - 6\frac{\delta a_2}{a_{02}^2} = -k_5\omega_r\rho_0\delta\rho,$$
(136)

$$\delta \ddot{a}_1 + 2\delta \ddot{a}_2 - \frac{2\delta a_2}{a_{02}^2} = -k_5 \omega_t \rho_0 \delta \rho.$$
(137)

Substituting (135) in (136) leads to

$$\delta \ddot{a}_2 + \gamma^2 \delta a_2 = 0, \tag{138}$$

where

~

$$\gamma^2 = -\frac{2(1+\omega_r)}{a_{02}^2} = \frac{2\omega_{eff}}{a_{02}^2}.$$
(139)

Hence, the oscillating modes for δa_2

$$\delta a_2 = C_1 e^{i\gamma t} + C_2 e^{-i\gamma t},\tag{140}$$

requires the constraint

$$\omega_r < -1. \tag{141}$$

Similarly, using (135), (136) and (137), we obtain

$$\delta \ddot{a}_1 = k_5 \left(\frac{2}{3}\omega_r - \omega_t + \frac{1}{3}\right) \rho_0 \delta \rho.$$
(142)

Combining (135) and (136) gives

$$\delta\rho = \frac{-3\delta\ddot{a}_2}{k_5(1+\omega_r)\rho_0},\tag{143}$$

where substituting in (142) yields

$$\delta \ddot{a}_1 = \left(\frac{3\omega_t - \omega_r}{1 + \omega_r} - 1\right) \delta \ddot{a}_2,\tag{144}$$

Twice integration of this equation gives

$$\delta a_1 = \left(\frac{3\omega_t - \omega_r}{1 + \omega_r} - 1\right) \delta a_2 + C_3 t + C_4.$$
(145)

Similar to 4D case, the stable oscillatory modes in δa_1 is subjected to the condition C3 = C4 = 0. Here, for $3\omega_t = 3\omega_r + 2$ the perturbation amplitude on the radial and lateral directions are the same.

In the following, we consider two specific forms of the fluid (5) and discuss on the stability of KSSS.

4.1.1.1 Perfect fluid

To study perfect fluid, we set $\omega_r = \omega_t = \omega$ and $p_{0r} = p_{0t}$. Then, Eqs. (127)–(129) for the static state reduce to

$$\frac{3}{a_{02}^2} = \Lambda + k_5 \rho_0, \tag{146}$$

$$\frac{3}{a_{02}^2} = \Lambda - k_5 \omega \rho_0, \tag{147}$$

$$\frac{1}{a_{02}^2} = \Lambda - k_5 \omega \rho_0. \tag{148}$$

Comparing (146) and (147) leads to $\omega = -1$. Similar to the case in four dimensions, (148) is not consistent with (147) regardless of ω values meaning that having a static state is not possible for a perfect fluid in five dimensions.

4.1.1.2 Anisotropic fluid

Here, similar to previous sections, we consider two modifications of the perfect fluid.

Type 1:

In this case, by considering two different equations of state parameters for the radial and lateral directions in the form of (32), the Eqs. (127)–(129) governing the static state become

$$\frac{3}{a_{02}^2} = \Lambda + k_5 \rho_0, \tag{149}$$

$$\frac{3}{a_{02}^2} = \Lambda - k_5 \omega_r \rho_0,$$
(150)

$$\frac{1}{a_{02}^2} = \Lambda - k_5 \omega_t \rho_0.$$
(151)

comparing Eqs. (149) and (150) we obtain the constraint $\omega_r = -1$ which is not allowed based on Eq. (141). Then a Type 1 fluid fails to support a stable static sate.

Type 2:

Now we consider anisotropic fluid with the form of (36). Then the Eqs. (130) and (131) for p_{0r} and p_{0t} become

$$p_{0r} = -(1+\omega)\rho_0,$$
 (152)

$$p_{0t} = \frac{2}{3} \frac{\Lambda}{k_5} - \left(\omega + \frac{1}{3}\right) \rho_0.$$
(153)

In this case, γ^2 given by Eq. (139) becomes

$$\gamma^2 = -\frac{3(1+\omega)}{a_{02}^2},\tag{154}$$

Therefore, the positivity condition on γ^2 to have a stable nonsingular static state demands

$$\omega < -1, \tag{155}$$

which means that the fluid supporting the geometry lies in the phantom range. Also, Eq. (144) becomes

$$\delta \ddot{a}_1 = \left(\frac{\omega - 1}{\omega + 1}\right) \delta \ddot{a}_2,\tag{156}$$

The dynamics of δa_1 reads as

$$\delta a_1 = \left(\frac{\omega - 1}{\omega + 1}\right) \delta a_2 + C_3 t + C_4. \tag{157}$$

Then, similar to four dimension, for C3 = C4 = 0 the oscillatory modes are possible and hence the static state will be stable.

4.2 Energy–momentum source with generalized Chaplygin gas equation of state

Using energy–momentum source with a generalized Chaplygin equation of state of the form (42), the Einstein field equations in five dimensions will be

$$3\left(\frac{\dot{a}_2^2}{a_2^2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2}\right) = \Lambda + k_5\rho, \qquad (158)$$

$$3\left(\frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2}\right) = \Lambda + k_5 \frac{\alpha_r}{\rho^n},\tag{159}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{2\ddot{a}_2}{a_2} + \frac{2\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2} = \Lambda + k_5 \frac{\alpha_t}{\rho^m}.$$
 (160)

Then, the corresponding static state is

$$\frac{3}{a_{02}^2} = \Lambda + k_5 \rho_0, \tag{161}$$

$$\frac{3}{a_{02}^2} = \Lambda + k_5 \frac{\alpha_r}{\rho_0^n},\tag{162}$$

$$\frac{1}{a_{02}^2} = \Lambda + k_5 \frac{\alpha_t}{\rho_0^m}.$$
(163)

applying the perturbations given by (14) to the field equations (158)-(160) leads to

$$3\left(\frac{1}{a_{02}^2} - \frac{2\delta a_2}{a_{02}^2}\right) = \Lambda + k_5\rho_0 + k_5\rho_0\delta\rho, \qquad (164)$$

$$3\left(\delta\ddot{a}_{2} + \frac{1}{a_{02}^{2}} - \frac{2\delta a_{2}}{a_{02}^{2}}\right) = \Lambda + \frac{k_{5}\alpha_{r}}{\rho_{0}^{n}}(1 - n\delta\rho), \quad (165)$$

$$\delta \ddot{a}_1 + 2\delta \ddot{a}_2 + \frac{1}{a_{02}^2} - \frac{2\delta a_2}{a_{02}^2} = \Lambda + \frac{k_5 \alpha_t}{\rho_0^m} (1 - m\delta\rho).$$
(166)

Considering Eqs. (161)-(163), the above equations reduce to

$$\frac{6\delta a_2}{a_{02}^2} = -k_5 \rho_0 \delta \rho, \tag{167}$$

$$3\delta\ddot{a}_2 - \frac{6\delta a_2}{a_{02}^2} = -nk_5 \frac{\alpha_r}{\rho_0^n} \delta\rho,$$
 (168)

$$\delta \ddot{a}_1 + 2\delta \ddot{a}_2 - \frac{2\delta a_2}{a_{02}^2} = -mk_5 \frac{\alpha_t}{\rho_0^m} \delta \rho.$$
(169)

Combining (167) and (168) one gets

$$\delta\ddot{a}_2 + \gamma^2 \delta a_2 = 0, \tag{170}$$

where

. .

$$\gamma^2 = -\frac{2(n\alpha_r \rho_0^{-(n+1)} + 1)}{a_{02}^2}.$$
(171)

Thus, γ^2 is always negative and consequently there are no oscillatory modes for δa_1 and δa_2 .

The summary of the result obtained in this section is as follows. The analysis here reveals that the existence and stability conditions for a four and a five dimensional KS geometries without a brane are similar in some manners. More specifically, (i) finite size static KS geometry does not exist for a perfect fluid source, (ii) an anisotropic type 1 fluid cannot support a static state, but an anisotropic type 2 fluid supports a stable nonsingular KS type static state, and (iii) a generalized Chaplygin gas fluid cannot support a stable nonsingular KS geometry in a 5-dimensional model without brane. The results of the analysis are different than the case when a four dimensional brane is embedded in a five (or higher) dimensional Ricci flat bulk space. Specifically, a stable nonsingular KS geometry can be supported by both the generalized Chaplygin gas fluid and an anisotropic fluid in a brane model. One interpretation of the differences in these two 5-dimensional models (with and without brane) is that in a braneworld scenario, the matter fields are confined to the brane and have no way to propagate along the extra dimension(s). Due to this confinement, matter fields have one less degree of freedom in comparison to the case where they are distributed in a five dimensional space. The confinement of the matter fields to the brane affects the local extrinsic curvature and dynamics of the brane within its bulk space. This induces a modification to the Einstein's field equations on the brane. This modification provides a geometrical interpretation for dark energy as the manifestation of the local extrinsic shape of the brane, see for instances [53–55]. In our study, this modification provides the possibility of the existence and stability of an anisotropic KS type static state for a wider range of fluid types on the brane in comparison to the four and five dimensional models without brane.

5 Conclusion

In the present work, the possibility of having a nonsingular KS type spacetime as a seed for an emergent universe is investigated. It is discussed that the existence and stability of the nonsingular KSSS depend on the dimensions of the spacetime and the nature of the fluid supporting the geometry. In particular, it is found that:

- In the context of GR in four dimensions:
 - (i) A perfect fluid cannot support a finite size static KS geometry.
 - (ii) An anisotropic fluid with equations of states p_r = ω_rρ and p_t = ω_tρ can support a finite size static KS geometry but it is not stable against the scalar perturbations.
- (iii) A modification of the perfect fluid form possessing equations of state $p_r = \omega \rho + p_{0r}$ and $p_t = \omega \rho + p_{0t}$ can support a stable nonsingular KS type static state.
- (iv) A generalized Chaplygin gas fluid with the equations of state $p_r = -\frac{\alpha_r}{\rho^n}$ and $p_t = -\frac{\alpha_t}{\rho^m}$ cannot support a stable nonsingular KS geometry.
- In the context of a five dimensional braneworld scenario:
 - A perfect fluid cannot support a finite size static KS geometry even in the presence of higher dimensional modifications.
 - (ii) In contrast to the four dimensional case, an anisotropic fluid having equations of state $p_r = \omega_r \rho$ and $p_t = \omega_t \rho$ supports a stable finite size static KS geometry.
- (iii) A fluid having the equations of state $p_r = \omega \rho + p_{0r}$ and $p_t = \omega \rho + p_{0t}$ can support a stable nonsingular KS type static state.
- (iv) In contrast to the case in four dimensions, a stable nonsingular KS geometry can be supported by a generalized Chaplygin gas fluid.
- In the context of a five dimensional model without brane, the results of the analysis for the existence and stability conditions are similar to the four dimensional model addressed above.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a purely theoretical work and we have not used any real data.]

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