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GENERIC ENTANGLED STATES AS THE $su(2)$ PHASE STATES

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We discuss an algebraic way to construct generic entangled states of qunits based on the polar decomposition of the su(2) algebra. In particular, we show that these states can be defined as eigenstates of certain Hermitian operators.

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By *qunit* we mean here an n-level quantum system, specified by the observables, forming a basis of the $su(n)$ algebra or of its complexification. For example, observables for a qubit are specified by Pauli operators, forming the Hermitian generators of the $su(2)$ algebra. In turn, observables for a qutrit form a Hermitian basis of the $su(3)$ algebra,[1] and so on.

Our definition of *generic* entangled states coincides with that of Refs. 2 and 3. This assumes that they are completely entangled and have a simple structure like Bell and GHZ (Grinberger–Horne–Zeilinger) states of two and three qubits, respectively.

The main aim of this paper is to show that generic entangled states in multi-qunit systems can be constructed as the $su(2)$ phase states of dimension n. The basis of completely entangled states in the corresponding Hilbert space can be constructed from generic entangled states by means of the local cyclic permutation operator. This approach also allows us to specify Hamiltonians, whose eigenstates are the generic entangled states.

Recent investigations in the field of entanglement have shown that entangled states of a given system form a certain class different from other states of the same system[2,4] and that there are local transformations such as SLOCC[5–7] (stochastic local transformations assisted by classical communications) and Lorentz transformations[8,9] that can change the amount of entanglement but cannot create entanglement. In view of these results, the definition of *completely entangled states* takes on special significance.

An approach has been developed in Refs. 10–13 which defines the completely entangled states of a given system in terms of the quantum fluctuations of basic observables that can be accessible for the measurement of states of this system. Thus, consider a system of N qunits, defined in the Hilbert space

$$\mathbb{H}_{N,n} = \mathbb{H}_n^{\bigotimes N}, \quad \dim \mathbb{H}_n = n.$$

The basic observables $\mathcal{O}_j$ are associated with the basis of the Lie algebra

$$\mathcal{L}_{N,n} = \bigoplus_{i=1}^{N} su(n)$$

or its complexification.

A quantum mechanical measurement of observables $\mathcal{O}_j$ in a state $\psi \in \mathbb{H}_{N,n}$ implies the mean value $\langle\psi|\mathcal{O}_j|\psi\rangle$ and variance

$$V_j(\psi) = \left\langle\psi\middle|\mathcal{O}_j^2\middle|\psi\right\rangle - \langle\psi|\mathcal{O}_j|\psi\rangle^2 \geq 0, \tag{1}$$

which gives the amount of quantum fluctuations (uncertainty) peculiar to this measurement and thus determines the quantum precision of the measurement. Summation over all local obervables $\{\mathcal{O}_j\}$ given by the basis of $\mathcal{L}_{N,n}$ then defines the total amount of uncertainty peculiar for the state $\psi \in \mathbb{H}_{N,n}$:

$$V(\psi) = \sum_j V_j(\psi). \tag{2}$$

It was proposed in Ref. 12 to define completely entangled states $\psi_{CE} \in \mathbb{H}_{N,n}$ by the condition

$$V(\psi_{CE}) = \max_{\psi\in\mathbb{H}_{N,n}} V(\psi). \tag{3}$$

If the local observables correspond to a compact Lie algebra (which is the case for qunits), then the maximum in (3) is provided by the magnitude of the Casimir operator

$$V(\psi_{CE}) = C, \quad \sum_j \mathcal{O}_j^2 = C \times \mathbf{1},$$

where $\mathbf{1}$ denotes the unit operator. This implies the condition of complete entanglement of the form

$$\forall\mathcal{O}_j \in \mathcal{L}_{N,n}, \quad \langle\psi_{CE}|\mathcal{O}_j|\psi_{CE}\rangle = 0. \tag{4}$$

The opposite case of minimal total variance (2) corresponds to the generalized coherent states[12] (for a definition of generalized coherent states, see Ref. 14).

The conventional universal measure of entanglement is usually associated with the entropy.[15] From the physical point of view, entropy represents a measure of uncertainty in the system. It is then clear that the definition (3) of complete entanglement provides maximum entropy and hence expresses the requirement of Ref. 15 in a different way. In fact, Eq. (3) represents a variational principle similar to the maximum entropy principal in quantum statistical thermodynamics. It should be emphasized that the equivalent conditions (4) represent an *operational* definition of complete entanglement[10] (definition in terms of what can be measured).

The definition of completely entangled states (3) seems to be quite natural from the information point of view. In fact, the variance $V_j(\psi)$ has been associated with the amount of information about the state ψ that can be extracted from the macroscopic measurement of the observable $\mathcal{O}_j$ (the so-called Wigner skew information).[16–18] This object has a certain similarity with Fisher information[19,20] (concerning Fisher information, see Ref. 21). Thus, the definition (3) means that completely entangled states carry a maximal amount of total Wigner skew information, provided by the measurement of all basic observables for a given system.

Below we will use the definition of complete entanglement (3) and its equivalent from (4) to specify the generic entangled states of qunits. For simplicity, we restrict examples to qubits and qutrits. Generalization to the case of $n \geq 4$ can be constructed in a similar way.

Let us first note that the generic entangled states of two and three qubits, namely the Bell and GHZ states, are expressed in terms of the homogeneous states:

$$|\psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle \pm |1,1\rangle),$$
$$|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle \pm |1,1,1\rangle).$$

Following these examples, consider homogeneous states of N qunits

$$|\ell; N\rangle = \bigotimes_{j=1}^{N} |\ell\rangle_j. \tag{5}$$

Using the homogeneous states (5), we can construct an n-dimensional representation of the $su(2)$ algebra of the form

$$\begin{aligned} J_+ &= \lambda_0|0;N\rangle\langle 1;N| + \cdots + \lambda_{n-2}|n-2,N\rangle\langle n-1;N|, \\ J_- &= \lambda_0|1;N\rangle\langle 0;N| + \cdots + \lambda_{n-2}|n-1;N\rangle\langle n-2;N|, \\ J_z &= \frac{n-1}{2}|0;N\rangle\langle 0;N| + \cdots + \frac{1-n}{2}|n-1;N\rangle\langle n-1;N|, \end{aligned} \tag{6}$$

such that

$$[J_+, J_-] = 2j_z, \quad [J_z, J_\pm] = \pm J_\pm.$$

Thus,

$$\lambda_0^2 = n-1, \quad \lambda_1^2 - \lambda_0^2 = n-2, \ldots, \lambda_{n-2}^2 - \lambda_{n-3}^2 = 2-n, \quad \lambda_{n-2}^2 = n-1.$$

Following Refs. 22 and 23, consider the polar decomposition of the $su(2)$ algebra (5):

$$J_+ = J_r E, \quad J_- = E^+ J_r, \quad EE^+ = \mathbf{1}.$$

Here the "radial" operator $J_r = (J_+ J_-)^{1/2}$ is diagonal, while the unitary operator E describes the "exponential of the $su(2)$ phase." It is seen that the operator E has the form

$$\begin{aligned} E = {} & |0;N\rangle\langle 1;N| + |1;N\rangle\langle 2;N| \\ & + \cdots + |n-2;N\rangle\langle n-1;N| + e^{i\varphi}|n-1;N\rangle\langle 0:N|. \end{aligned} \tag{7}$$

In other words, operator (7) provides cyclic permutations of homogeneous states (5). Here φ denotes an arbitrary "reference phase," which can be set to $\varphi = 0$ for simplicity.

Consider now a linear superposition of homogeneous states (5):

$$|\psi_{N,n}\rangle = \sum_{\ell=0}^{n-1} a_\ell |\ell;N\rangle, \quad \sum_{\ell=0}^{n-1} |a_\ell|^2 = 1, \tag{8}$$

and define the coefficients a here by the requirement that (8) is the $su(2)$ phase state:

$$E|\psi_{N,n}\rangle = e^{i\phi}|\psi_{N,n}\rangle.$$

We get

$$a_{\ell,k} = \frac{1}{\sqrt{n}} e^{i\ell\phi_k}, \quad \phi_k = \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1. \tag{9}$$

Thus, the N-qunit $su(2)$ phase states take the form

$$\left|\psi_{(N,n)}^{(k)}\right\rangle = \frac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} e^{i\ell\phi_k} |\ell;N\rangle. \tag{10}$$

First, the states (10) with different k are mutually orthogonal (for a proof, see Ref. 24). Then, the states (10) are definitely nonseparable and manifest complete entanglement.

To illustrate this fact, consider first the case of N qubits ($n = 2$). Then, there are only two eigenvalues of the $su(2)$ phase, namely $\phi_0 = 0$ and $\phi_1 = \pi$, so that the states (10) take the form

$$\left|\psi_{N,2}^{(\pm)}\right\rangle = \frac{1}{\sqrt{2}}(|0;N\rangle \pm |1;N\rangle). \tag{11}$$

At $N = 2$ and $N = 3$, it coincides with the Bell and GHZ states, respectively.

The local observables for qubits are provided by the Pauli operators

$$\sigma_x = |0\rangle\langle 1| + \text{h.c.}, \quad \sigma_2 = -i|0\rangle\langle 1| + \text{h.c.}, \quad \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{12}$$

It can be easily seen that the states (11) obey the condition (4) of complete entanglement with the observables (12) for all $N \geq 2$. Hence, these states can be considered as the generic entangled states of N qubits.

It should be stressed that there are only two independent phase states (11) in the case of qubits, while the dimension of the space $\mathbb{H}_{N,2}$ is 2^N. However, beginning with the states (11), one can construct a basis of completely entangled states in $\mathbb{H}_{N,2}$ in the following way. Consider a local cyclic permutation operator ϵ_n, which in the case of qubits ($n = 2$) coincides with σ_x in (12). Then, acting by this operator ϵ_2 on the individual components of the generic states (11) $(2^N - 2)$ times, we get the whole basis.

For example, at $N = 2$, acting by $\epsilon_2 = \sigma_x$ on the first part in the Bell states, we get EPR (Einstein–Podolsky–Rosen) states

$$\epsilon_2^{(1)}\big|\psi_{2,2}^{(\pm)}\big\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle \pm |0,1\rangle).$$

In the case of $N = 3$, action by the local operator $\epsilon_2 = \sigma_x$ on the first, second and third parts gives the states

$$\begin{aligned}
\epsilon_2^{(1)}\big|\psi_{3,2}^{(\pm)}\big\rangle &= \frac{1}{\sqrt{2}}(|1,0,0\rangle \pm |0,1,1\rangle),\\
\epsilon_2^{(2)}\big|\psi_{3,2}^{(\pm)}\big\rangle &= \frac{1}{\sqrt{2}}(|0,1,0\rangle \pm |1,0,1\rangle),\\
\epsilon_2^{(3)}\big|\psi_{3,2}^{(\pm)}\big\rangle &= \frac{1}{\sqrt{2}}(|0,0,1\rangle \pm |1,1,0\rangle),
\end{aligned}$$

which complete (11) with respect to the whole basis of completely entangled states in the eight-dimensional space $\mathbb{H}_{3,2}$. It should be stressed that the local operation ϵ destroys neither complete entanglement nor orthogonality of the states. The former statement follows from the fact that $\epsilon_2^{+}\sigma_i\epsilon_2 = \sigma_j$. For the latter statement, see Ref. 24.

In the case of qutrits with $n = 3$, the generic ($su(2)$ phase) states (10) take the form

$$\big|\psi_{(N,3)}^{(k)}\big\rangle = \frac{1}{\sqrt{3}}(|0;N\rangle + e^{i2k\pi/3}|1;N\rangle + e^{i4k\pi/3}|2;N\rangle). \qquad (13)$$

At $N = 2$, they coincide with the completely entangled states of two qutrits, which have been considered in the context of quantum information processing with ternary logic in Ref. 25. To check with the aid of condition (4) that states (13) manifest complete entanglement, we should choose local observables for a qutrit as the Hermitian generators of the $su(3)$ algebra:

$$\mathcal{O}_i = \begin{cases} (|0\rangle\langle 0| - |1\rangle\langle 1|), & (|1\rangle\langle 1| - |2\rangle\langle 2|), \quad (-|1\rangle\langle 1| + |2\rangle\langle 2|) \\ \frac{1}{2}(|0\rangle\langle 1| + \text{h.c.}), & \frac{1}{2}(|1\rangle\langle 2| + \text{h.c.}), \quad \frac{1}{2}(|0\rangle\langle 2| + \text{h.c.}) \\ \frac{-i}{2}(|0\rangle\langle 21| - \text{h.c.}), & \frac{-i}{2}(|1\rangle\langle 2| - \text{h.c.}), \quad \frac{-i}{2}(-|0\rangle\langle 2| + \text{h.c.}). \end{cases} \qquad (14)$$

It is seen that $\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3 = 0$, so that only eight out of nine generators in (14) are independent. It is now a straightforward matter to show that the states (13) obey the condition (4) with the observables (14). To complete the basis of completely entangled states in $\mathbb{H}_{N,3}$, we should again use the local cyclic permutation operator, which now takes the form

$$\epsilon_3 = |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 0|.$$

Taking into account that the unitary transformation ϵ transforms any observable from (14) into another observable from the same set

$$\epsilon_3^+ \mathcal{O}_i \epsilon_3 = \mathcal{O}_j,$$

we can conclude that the use of ϵ_3 does not influence the complete entanglement of the generic states.

Generic states of qunits with $n \geq 4$ can be constructed in the same way.

Summarizing, we have shown that the generic entangled states of qunits have the form of the $su(2)$ phase states of dimension n in the basis of homogeneous states (8). The basis of completely entangled states in $\mathbb{H}_{N,n}$ can be constructed from the generic states through the use of the local cyclic permutation operator.

Besides that, the consideration of the $su(2)$ algebra in the basis of homogeneous N-qunit states and its polar decomposition opens the way to defining the generic entangled states as the eigenstates of certain Hermitian operators. In particular, they are eigenstates of the cosine and sine of the $su(2)$ phase operators

$$C = (E + E^+)/2, \quad S = (E - E^+)/2i \tag{15}$$

as well as of the Hermitian phase operator

$$\Phi = \sum_k \phi_k \big|\psi_{(N,n)}^{(k)}\big\rangle\big\langle\psi_{(N,n)}^{(k)}\big|. \tag{16}$$

These operators can be interpreted as the physical Hamiltonians whose eigenstates manifest complete entanglement.

For example, in the case of two qubits ($N = 2$ and $n = 2$), the operators (15) and (16) take the form

$$C = \frac{1}{2}\big(\sigma_x^{(1)} \otimes \sigma_x^{(2)} - \sigma_y^{(1)} \otimes \sigma_y^{(2)}\big), \quad S = 0, \quad \Phi = \pi(\mathbf{1} - C),$$

in the subspace of non-zero eigenvalues of E. In the more interesting case of two qutrits, we get

$$\begin{aligned} C = {}& \mathbb{O}_4^{(1)} \otimes \mathbb{O}_4^{(2)} + \mathcal{O}_5^{(1)} \otimes \mathcal{O}_5^{(2)} + \mathcal{O}_6^{(1)} \otimes \mathcal{O}_6^{(2)} - \mathbb{O}_7^{(1)} \otimes \mathbb{O}_7^{(2)} \\ & - \mathcal{O}_8^{(1)} \otimes \mathcal{O}_8^{(2)} - \mathcal{O}_9^{(1)} \otimes \mathcal{O}_9^{(2)}, \end{aligned}$$

and so on.

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