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An isomorphism theorem for Dragilev spaces

By

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Necessary and sufficient conditions for an $L_f(a, 1)$-space to be isomorphic to some $L_g(b, \infty)$-space have been found.

Introduction. In [2] Dragilev has claimed that if $r, s \in \{+\infty, 1, 0, -1\}$ and $r \neq s$, then for any two rapidly increasing Dragilev functions f and g and for any two sequences $a = (a_i)$ and $b = (b_i)$, the spaces $L_f(a, r)$ and $L_g(b, s)$ cannot be isomorphic. In [4] and [5], by means of examples it was shown that this is not true for $(r, s) = (1, +\infty)$ and $(r, s) = (-1, 0)$.

In this note we characterize those $L_f(a, 1)$ spaces which are isomorphic to $L_g(b, \infty)$ spaces. The characterization is given in terms of the functor Ext and a condition which is obtained by comparing the diametral dimensions of the two spaces.

Preliminaries. Let f be an odd, increasing, logarithmically convex function (i.e. $\varphi(x) = \log f(e^x)$ is convex). Throughout this paper such a function will be called a *Dragilev function*. Let $a = (a_i)$ be a strictly increasing sequence of positive numbers with $\lim a_i = +\infty$ and (r_k) a strictly increasing sequence of real numbers with $\lim r_k = r$ where $-\infty < r \leqq +\infty$. The Dragilev space $L_f(a, r)$ is defined as the Köthe space $\lambda(A)$ generated by the matrix $A = (a_i^k)$, $a_i^k = \exp f(r_k a_i)$ (see [2]).

By logarithmic convexity of f we have that for every $a > 1$, $\tau(a) = \lim\limits_{x \to +\infty} (f(ax)/f(x))$ exists.

Moreover either (i) $\tau(a) < +\infty$ for all $a > 1$, or (ii) $\tau(a) = +\infty$ for all $a > 1$. f is called *slowly increasing* in the first case, *rapidly increasing* in the second case. It is well-known that $L_f(a, r)$ is isomorphic to a power series space if and only if f is slowly increasing. In this paper we shall consider only rapidly increasing Dragilev functions.

In [7] several properties of functor $\operatorname{Ext}(E, F) = \operatorname{Ext}^1(E, F)$ for two Fréchet spaces E and F were given. It was shown in [1] that $\operatorname{Ext}(L_g(b, \infty), L_g(b, \infty)) = 0$ and in [3] that $\operatorname{Ext}(L_f(a, 1), L_f(a, 1)) = 0$ if and only if there is a number $c > 1$ such that the set of limit points of the set $\{a_j/a_i : i, j \in \mathbb{N}\}$ is contained in $[0, 1] \cup [c, +\infty]$.

Results. We first give a necessary condition for $L_f(a, 1)$ to be isomorphic to some $L_g(b, \infty)$.

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Proposition 1. *If $L_f(a, 1)$ is isomorphic to some $L_g(b, \infty)$, then there is a strictly increasing sequence (r_k) of positive numbers with* $\lim r_k = 1$ *and there is a strictly increasing function $p: \mathbb{N} \to \mathbb{N}$ such that*

$$\frac{f(r_{k+1} a_i)}{f(r_k a_i)} \leqq \frac{f(r_{k+2} a_i)}{f(r_{k+1} a_i)}, \quad k \in \mathbb{N},\ i \geqq p(k).$$

The proof of this proposition is essentially given in [4] (Proposition 1). The only difference is that we choose r_k slightly larger than the one chosen in [4], so that the inequality above holds for all large i (depending on k).

Before our next proposition we observe the following.

Remark. If a positive sequence (r_k) strictly increases to 1, then there is a $k_0 \in \mathbb{N}$ such that

$$\frac{r_{k+1}}{r_k} \leqq \frac{r_2}{r_1}, \quad k \geqq k_0.$$

This follows from $\lim r_{k+1}/r_k = 1$ and $r_2/r_1 > 1$.

Proposition 2. *Suppose* $\inf(a_{i+1}/a_i) = a > 1$ *and there is a strictly increasing positive sequence (r_k) with* $\lim r_k = 1$ *and there is a strictly incresing function $p: \mathbb{N} \to \mathbb{N}$ such that*

$$\frac{f(r_{k+1} a_i)}{f(r_k a_i)} \leqq \frac{f(r_{k+2} a_i)}{f(r_{k+1} a_i)}, \quad k \in \mathbb{N},\ i \geqq p(k).$$

Then $L_f(a, 1)$ is isomorphic to some $L_g(b, \infty)$.

Proof. By the previous remark, by passing to a subsequence of (r_k) if necessary we may assume that

$$(1) \qquad \frac{r_{k+1}}{r_k} \leqq \frac{r_2}{r_1}, \quad k \in \mathbb{N}.$$

Since $\inf(a_{i+1}/a_i) = a > 1$, there is a k_0 such that $r_{k_0} > 1/a$. Again by passing to a subsequence of (r_k) if necessary we may assume that $r_1 > 1/a$, that is

$$(2) \qquad r_k a_i < a_i \leqq \frac{a_{i+1}}{a} < r_1 a_{i+1}, \quad i, k \in \mathbb{N}.$$

Then by using logarithmic convexity of f, (1) and (2) for $i, k \in \mathbb{N}$ we have

$$\begin{aligned} \log \frac{f(r_{k+1} a_i)}{f(r_k a_i)} &= \frac{\varphi(\log(r_{k+1} a_i)) - \varphi(\log(r_k a_i))}{\log r_{k+1} - \log r_k} (\log r_{k+1} - \log r_k) \\ &\leqq \frac{\varphi(\log(r_2 a_{i+1})) - \varphi(\log(r_1 a_{i+1}))}{\log r_2 - \log r_1} (\log r_{k+1} - \log r_k) \\ &= \log \frac{f(r_2 a_{i+1})}{f(r_1 a_{i+1})} \frac{\log r_{k+1} - \log r_k}{\log r_2 - \log r_1} \leqq \log \frac{f(r_2 a_{i+1})}{f(r_1 a_{i+1})}, \end{aligned}$$

where $\varphi(x) = \log f(e^x)$ which is convex. That is, we have

$$(3) \qquad \frac{f(r_{k+1} a_i)}{f(r_k a_i)} \leqq \frac{f(r_2 a_{i+1})}{f(r_1 a_{i+1})}, \qquad i, k \in \mathbb{N}.$$

Now let $i_0 = \min\{p(k): k \geqq 2\} = p(2)$ and for $i \geqq i_0$, define $k(i) = \max\{k: p(k) \leqq i\}$. Then

$$2 = k(i_0) \leqq k(i) \leqq k(i+1) \quad \text{for} \quad i \geqq i_0, \qquad \lim_{i \to \infty} k(i) = +\infty$$
$$\text{and} \quad p(k(i)) \leqq i \quad \text{for} \quad i \geqq i_0.$$

Rewriting the hypothesis we have

$$(4) \qquad \frac{f(r_{k+1} a_i)}{f(r_k a_i)} \leqq \frac{f(r_{k+2} a_i)}{f(r_{k+1} a_i)}, \qquad i \geqq i_0, \; k(i) \geqq k.$$

Next we choose $s > 1$ and fix it, and define $s_k = s^k$. Then we define a sequence $b = (b_i)$, $i \geqq i_0$ as follows: $b_{i_0} = 1$ and b_{i+1} is inductively defined by

$$\log \frac{b_{i+1}}{s^{k(i)-1} b_i} = \log s \frac{\log f(r_1 a_{i+1}) - \log f(r_{k(i)} a_i)}{\log f(r_{k(i)} a_i) - \log f(r_{k(i)-1} a_i)}.$$

By (2), the right hand side is positive and so $s^{k(i)-1} b_i < b_{i+1}$, that is $s_{k(i)} b_i < s_1 b_{i+1}$.
Also by (3) we have

$$\frac{\log f(r_1 a_{i+1}) - \log f(r_{k(i)} a_i)}{\log f(r_2 a_{i+1}) - \log f(r_1 a_{i+1})} \log s \leqq \frac{\log f(r_1 a_{i+1}) - \log f(r_{k(i)} a_i)}{\log f(r_{k(i)} a_i) - \log f(r_{k(i)-1} a_i)} \log s,$$

that is (by using definition of b_{i+1})

$$\frac{\log f(r_1 a_{i+1}) - \log f(r_{k(i)} a_i)}{\log f(r_2 a_{i+1}) - \log f(r_1 a_{i+1})} \log s \leqq \log \frac{s_1 b_{i+1}}{s_{k(i)} b_i}$$
$$\leqq \frac{\log f(r_1 a_{i+1}) - \log f(r_{k(i)} a_i)}{\log f(r_{k(i)} a_i) - \log f(r_{k(i)-1} a_i)} \log s,$$

or since $\log(s_k b_i) - \log(s_{k-1} b_i) = \log s$, for $i \geqq i_0$ we equivalently have

$$(5) \qquad \frac{\log f(r_{k(i)} a_i) - \log f(r_{k(i)-1} a_i)}{\log(s_{k(i)} b_i) - \log(s_{k(i)-1} b_i)} \leqq \frac{\log f(r_1 a_{i+1}) - \log f(r_{k(i)} a_i)}{\log(s_1 b_{i+1}) - \log(s_{k(i)} b_i)}$$
$$\leqq \frac{\log f(r_2 a_{i+1}) - \log f(r_1 a_{i+1})}{\log(s_2 b_{i+1}) - \log(s_1 b_{i+1})}.$$

Now we choose $i_1 \geqq i_0$ such that

$$\log f(r_2 a_{i_1}) - \log f(r_1 a_{i_1}) \geqq \log s.$$

Then we define $\psi(\log(s_k b_i)) = \log f(r_k a_i)$, $i \geqq i_1$, $k \leqq k(i)$, and $P_{k,i} = (\log(s_k b_i), \psi(\log(s_k b_i)))$ and join the points

$$P_{1,i_1} \to \cdots \to P_{k(i_1),i_1} \to \cdots \to P_{1,i} \to P_{2,i} \to \cdots \to P_{k(i),i} \to P_{1,i+1} \to P_{2,i+1} \cdots$$

by line segments. This way ψ is defined for all $x \geqq \log(s_1 b_{i_1})$. For $x \leqq \log(s_1 b_{i_1})$, we define $\psi(x) = \log f(r_1 a_{i_1}) - \log(s_1 b_{i_1}) + x$. It is clear that ψ is increasing. By (4) we have that ψ is convex within the i-th block and from (5) it follows that the slope of ψ increases when we pass from the i-th to the $(i+1)$-st block. Finally i_1 was chosen in such a way that the slope of ψ from $(0, \psi(0))$ to P_{1,i_1} is smaller than the slope from P_{1,i_1} to P_{2,i_1}.

Now we define

$$g(x) = \begin{cases} \dfrac{f(r_1 a_{i_1})}{s_1 b_{i_1}}\, x, & 0 \leqq x \leqq s_1 b_{i_1} \\ e^{\psi(\log x)}, & s_1 b_{i_1} \leqq x \\ -g(-x), & x \leqq 0\,. \end{cases}$$

Then g is an increasing, odd function with $\log g(e^x) = \psi(x)$, and so g is logarithmically convex.

Finally for $i \geqq i_1$ and $k \leqq k(i)$, that is for $i \geqq \max\{p(k), i_1\}$ we have $f(r_k a_i) = g(s_k b_i)$, which shows that $L_f(a, 1)$ is isomorphic to $L_g(b, \infty)$.

Next proposition extends Proposition 2.

Proposition 3. *Suppose* $\operatorname{Ext}(L_f(a, 1), L_f(a, 1)) = 0$ *and there is a strictly increasing sequence* (r_k) *with* $\lim r_k = 1$ *and a strictly increasing function* $p\colon \mathbb{N} \to \mathbb{N}$ *such that*

$$\frac{f(r_{k+1} a_i)}{f(r_k a_i)} \leqq \frac{f(r_{k+2} a_i)}{f(r_{k+1} a_i)}, \qquad k \in \mathbb{N},\ i \geqq p(k).$$

Then $L_f(a, 1)$ *is isomorphic to some* $L_g(b, \infty)$.

Proof. By [6], $\operatorname{Ext}(L_f(a, 1), L_f(a, 1)) = 0$ if and only if the pair $(L_f(a, 1), L_f(a, 1))$ satisfies condition (S_2^*), and it was shown in [3]. (p. 37 and p. 29) that this happens if and only if there is a number $c > 1$ such that the set of limit points of $\{a_j/a_i\colon i, j \in \mathbb{N}\}$ is contained in $[0, 1] \cup [c, +\infty]$. Let $a = (1 + c)/2 > 1$.

We set $i_1 = 1$ and choose i_2 as the smallest index n such that $a_n/a_{i_1} \geqq a$, then we choose i_3 as the smallest index n such that $a_n/a_{i_2} \geqq a$. We continue this way and choose a strictly increasing sequence (i_n) of indices such that

$$\frac{a_{i_{n+1}}}{a_{i_n}} \geqq a, \quad \frac{a_{i_{n+1}-1}}{a_{i_n}} < a, \qquad n \in \mathbb{N}\,.$$

Let $M = \{n\colon i_n + 1 < i_{n+1}\}$.

If M is a finite set, then there is an $n_0 \in \mathbb{N}$ such that for $n \geqq n_0$, $i_{n+1} = i_n + 1$ and so for some m_0, $i_{n_0+n} = m_0 + n$ for $n \geqq 1$. Hence

$$\frac{a_{m_0+n+1}}{a_{m_0+n}} = \frac{a_{i_{n_0+n+1}}}{a_{i_{n_0+n}}} \geqq a, \qquad n \geqq 1\,,$$

and by Proposition 2, $L_f((a_i)_{i>m_0}, 1)$ is isomorphic to some $L_g((b_i)_{i>m_0}, \infty)$ and so $L_f(a, 1)$ is isomorphic to $L_g(b, \infty)$.

If M is an infinite set, it follows from $a_{i_{n+1}-1}/a_{i_n} < a$ and from the property of c that 1 is the only limit point of the bounded set

$$\left\{\frac{a_{i_{n+1}-1}}{a_{i_n}} : n \in M\right\},$$

and so $\lim\limits_{n\in M}(a_{i_{n+1}-1}/a_{i_n}) = 1$. Since $a_{i_{n+1}}/a_{i_n} \geqq a$, by Proposition 2, $L_f((a_{i_n}), 1)$ is isomorphic to some $L_g((b_{i_n}), \infty)$ with $f(r_k a_{i_n}) = g(s_k b_{i_n})$ for $n \geqq n_k$. For $n \in M$ and $i_n < i < i_{n+1}$ we define b_i in such a way that $b_{i_n} < b_i < b_j < b_{i_{n+1}}$ if $i_n < i < j < i_{n+1}$ and $\lim\limits_{n\in M}(b_{i_{n+1}-1}/b_{i_n}) = 1$.

Now given k we find n_0 such that

$$\frac{a_{i_{n+1}-1}}{a_{i_n}} < \frac{r_{k+1}}{r_k}, \quad \frac{b_{i_{n+1}-1}}{b_{i_n}} < \frac{s_{k+1}}{s_k}, \qquad n \geqq n_0.$$

If $i_n < i < i_{n+1}$ *for* $n \geqq n_0$, then

$$\frac{a_i}{a_{i_n}} < \frac{r_{k+1}}{r_k}, \quad \frac{b_i}{b_{i_n}} < \frac{s_{k+1}}{s_k}$$

and so for $n \geqq \max\{n_0, n_{k+1}\}$ and $i_n < i < i_{n+1}$ we have

$$f(r_k a_i) \leqq f(r_{k+1} a_{i_n}) = g(s_{k+1} b_{i_n}) \leqq g(s_{k+1} b_i),$$
$$g(s_k b_i) \leqq g(s_{k+1} b_{i_n}) = f(r_{k+1} a_{i_n}) \leqq f(r_{k+1} a_i).$$

So $L_f(a, 1)$ is isomorphic to $L_g(b, \infty)$.

Now we combine our propositions and the fact that $\operatorname{Ext}(L_f(a, 1), L_f(a, 1)) = 0$ is a necessary condition for $L_f(a, 1)$ to be isomorphic to some $L_g(b, \infty)$ in the following theorem.

Theorem. *Let f be a rapidly increasing Dragilev function. Then $L_f(a, 1)$ is isomorphic to some $L_g(b, \infty)$ if and only if the following conditions are satisfied:*

(i) $\operatorname{Ext}(L_f(a, 1), L_f(a, 1)) = 0$,

(ii) *There is a strictly increasing sequence (r_k) with $\lim r_k = 1$ and a strictly increasing function $p: \mathbb{N} \to \mathbb{N}$ such that*

$$\frac{f(r_{k+1} a_i)}{f(r_k a_i)} \leqq \frac{f(r_{k+2} a_i)}{f(r_{k+1} a_i)}, \qquad k \in \mathbb{N},\ i \geqq p(k).$$

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