THE LINEAR MEAN VALUE OF THE REMAINDER TERM IN THE PROBLEM OF ASYMPTOTIC BEHAVIOUR OF EIGENFUNCTIONS OF THE AUTOMORPHIC LAPLACIAN

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By

Zemişan Emirieroğlu

August, 1996

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. N.V. Kuznetsov(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. S.A. Stepanov

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Zernişan Emirleroğlu
M.S. in Mathematics
Supervisor: Prof. Dr. N.V. Kuznetsov
August 1996

The purpose of this thesis is to obtain the estimate for the average mean value of the remainder term of the asymptotic formula for the quadratic mean value of the Fourier coefficients of the eigenfunctions over the discrete spectrum of the automorphic Laplacian.

Keywords : The Fourier coefficients of the eigenfunctions of the Automorphic Laplacian, Dirichlet Series.

ÖZET

OTOMORFİK LAPLASİAN'IN ÖZFONKSİYONLARININ ASİMTOTİK KALANININ ORTALAMA DEĞERİ

Zernişan Emirleroğlu
Matematik Yüksek Lisans
Tez Yöneticisi: Prof. Dr. N.V. Kuznetsov
Ağustos 1996

Bu tezin amacı Otomorfik Laplasian'ın ayrık spekturumlarının özfonksiyonlarının Fourier katsayılarının asimtotiğinin kalanının ortalama değerini hesaplamaktır.

Anahtar Kelimeler : Otomorfik Laplasian'in özfonksiyonlarının Fourier katsayıları, Dirichlet Serileri.

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Chapter 1

Introduction

An asymptotic formula for the quadratic mean value of the Fourier coefficients of the eigenfunctions of the discrete spectrum of the automorphic Laplacian was proved in the paper of N.V.Kuznetsov.

In this thesis we'll give the estimate for the average mean value of the remainder term of this asymptotic formula.

To formulate our results we introduce the following notations.

Let $\mathcal{L}$ be the Laplace-Beltrami operator on the upper half-plane of the complex variable $z = x + iy, y > 0$,

$$\mathcal{L} = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}),$$

and let G denote the classical modular group of fractional linear transformations:

$$z \to gz = \frac{az + b}{cz + d},$$

where a, b, c and d are rational integers with $ad - bc = 1$.

The eigenfunctions of the discrete spectrum of the Laplace operator are the nonzero solutions of the differential equation

$$\mathcal{L}\psi = \lambda\psi,$$

which satisfy the periodicity condition

$$\psi(gz) \equiv \psi(\frac{az + b}{cz + d}) = \psi(z), \quad g \in G,$$

and the finiteness condition

$$\int_D |\psi(z)|^2 dz < \infty,$$

where $dz = dxdy/y^2$ is the G-invariant measure on the upper half-plane and D is the fundamental domain of the modular group.

Let $0 = \lambda_0 < \lambda_1 \leq \cdots$ be the eigenvalues of $\mathcal{L}$. The corresponding eigenfunctions are denoted by $\psi_j(z), j = 0, 1, \cdots$(it is known that $\lambda_1 \simeq 91.1$). It is known that each eigenfunction $\psi_j(z)$ which corresponds to positive eigenvalue $\lambda_j > 1/4$ has the Fourier expansion of the form:

$$\psi_j(z) = \sum_{n \neq 0} e^{2\pi i n x} \sqrt{y} K_{i\kappa_j}(2\pi |n| y) \rho_j(n),$$

where $\kappa_j = \sqrt{\lambda_j - 1/4}$, $\rho_j(n)$ are the Fourier coefficients of $\psi_j(z)$ and $K_v(.)$ is the Hankel function of the first kind of order v with purely imaginary argument [5].

A similar expansion is also valid for the eigenfunctions of the continuous spectrum of Laplace operator. It is known that the eigenfunctions of the continuous spectrum of the Laplace operator can be obtained by analytic continuation of the Eisenstein series onto the line $Res = \frac{1}{2}$ [4]. The Eisenstein series $E(z, s)$ is defined by

$$E(z, s) = y^s + \frac{1}{2} \sum_{(c,d)=1, c \neq 0} \frac{y^s}{|cz + d|^{2s}} \quad , \quad Res > 1,$$

where the summation is over all pairs of relatively prime integers c, d with $c \neq 0$.

In the case of modular group, the continuation of $E(z, s)$ onto the whole s-plane may be realized by the Fourier expansion:

$$E(z, s) = y^s + \frac{\xi(1-s)}{\xi(s)} y^{1-s} + \frac{2\sqrt{y}}{\xi(s)} \sum_{n \neq 0} \tau_s(n) K_{s-\frac{1}{2}}(2\pi |n| y e^{2\pi i n x}),$$

where with the usual notations for gamma-function and for the Riemann zeta-function we have

$$\xi(s) = \pi^{-s} \Gamma(s) \zeta(2s).$$

The quantities $\tau_s(n)$ are the Fourier coefficients of the Eisenstein series; these are equal to

$$\tau_s(n) = |n|^{s-\frac{1}{2}} \sum_{d|n, d>0} d^{1-2s} = |n|^{s-\frac{1}{2}} \sigma_{1-2s}(n).$$

Bilinear combinations of the Fourier coefficients of eigenfunctions of the Laplace operator can be expressed in terms of the mean value of the classical Kloosterman sums. By definition

$$S(n,m;c) = \sum_{\substack{1\le d\le |c| \\ (c,d)=1, dd'\equiv 1(modc)}} e^{2\pi i(\frac{nd}{c}+\frac{md'}{c})},$$

where the summation is over d prime to c and for every d, the integer d' is the solution of the congruence $dd' \equiv 1(modc)$.

More definitely, the identity between the Fourier coefficients and the sum of the Kloosterman sums is [5]:

$$\sum_{j=1}^{\infty} \frac{\rho_j(n)\overline{\rho_j(m)}}{ch(\pi\kappa_j)} h(\kappa_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} (\frac{m}{n})^{ir}\sigma_{2ir}(n)\sigma_{-2ir}(m)\frac{h(r)}{|\zeta(1+2ir)|^2}dr \quad (1.1)$$

$$= \frac{\delta_{n,m}}{\pi^2}\int_{-\infty}^{\infty} rth(\pi r)h(r)dr + \sum_{c=1}^{\infty}\frac{1}{c}S(n,m;c)\varphi(\frac{4\pi\sqrt{nm}}{c}),$$

where $\delta_{n,m}$ is the Kronecker symbol, $\sigma_\nu(n) = \sum_{d|n\ d>0} d^\nu$, and $\varphi(\frac{4\pi\sqrt{nm}}{c})$ is the test function, which is defined by the integral transform

$$\varphi(x) = \frac{2i}{\pi}\int_{-\infty}^{\infty} J_{2ir}(x)\frac{r}{ch(\pi r)}h(r)dr$$

with the usual notation for the Bessel function. The identity is valid for the functions $h(r)$ satisfiying the conditions: $h(r)$ is an even function of r,regular in the strip $|Imr| \le \Delta$ for some $\Delta > \frac{1}{2}$, and for some $\rho > 0$,as $|r| \to \infty, |Imr| \le \Delta$:

$$|h(r)| = O(|r|^{-2-\rho}).$$

Now, the remainder term of the asymptotic formula can be obtained with a small difference from the paper of N.V.Kuznetsov. The Fourier coefficients of the eigenfunctions of the continuous spectrum are also taken into consideration, as well as the Fourier coefficients of the eigenfunctions of discrete spectrum of the Laplace operator.

We define the remainder term by the equality:

$$R_n(X) = \sum_{\kappa_j\le X}\frac{|\rho_j(n)|^2}{ch\pi\kappa_j} + \frac{1}{\pi}\int_{-X}^{X}\frac{|\tau_{\frac{1}{2}+ir}(n)|^2}{|\zeta(1+2ir)|^2}dr - \frac{2}{\pi^2}\int_0^X rth(\pi r)dr,$$

where the $\rho_j(n)$'s, $\tau_{\frac{1}{2}+ir}(n)$'s are the Fourier coefficients of the eigenfunctions of the discrete spectrum, and, the continuous spectrum respectively.

Chapter 2

The Initial Identities

2.1 First Identity

Lemma 2.1 *Let $h(r)$ be a good function in the sense that it satisfies the assumptions in the main identity. Then*

$$\int_0^\infty R_n(r)h'(r)dr = \sum_{c\geq 1}\frac{1}{c}S(n,n,c)\varphi(\frac{4\pi n}{c}). \tag{2.1}$$

Proof: If we put n=m in the main identity (1.1) we get:

$$\sum_{j=1}^{\infty}\frac{|\rho_j(n)|^2}{ch\pi\kappa_j}h(\kappa_j)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\tau_{\frac{1}{2}+ir(n)}|^2}{|\zeta(1+2ir)|^2}h(r)dr =$$

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}rth(\pi r)h(r)dr+\sum_{c=1}^{\infty}\frac{1}{c}S(n,n,c)\varphi(\frac{4\pi n}{c}).$$

By using integration by parts we see that:

$$\sum_{j=1}^{\infty}\frac{|\rho_j(n)|^2}{ch\pi\kappa_j}h(\kappa_j) = -\int_0^\infty(\sum_{\kappa_j\leq r}\frac{|\rho_j(n)|^2}{ch\pi\kappa_j})h'(r)dr,$$

and

$$\int_0^\infty rth(\pi r)h(r)dr = -\int_0^\infty(\int_0^r r_1th(\pi r_1)dr_1)h'(r)dr.$$

Doing the same in the integral over continuous spectrum we come to (2.1).

In order to obtain a more explicit result we choose a specific function $h(r)$ as

$$h(r) = t_{X,\Delta}(r)\frac{q(r)}{q(r)+M} \quad \text{for} \quad M>4,$$

where

$$q(r) = (r^2 + \frac{1}{4}),$$

and

$$t_{X,\Delta} = \int_{-X}^{X} e^{-\frac{(r-\xi)^2}{\Delta^2}} d\xi (\int_{-\infty}^{\infty} e^{-\frac{(r-\xi)^2}{\Delta^2}} d\xi)^{-1}.$$

Clearly $t_{x,\Delta}$ is almost 1 if r is in the interval $[-X, X]$, and it rapidly decreases outside.

2.2 Averaging

We introduce the infinitely smooth function $w(x)$ which is identically zero outside the fixed interval [1,2], and is near to one inside this interval.

Now we define the mean value of the remainder term by the equality

$$\overline{R}(x;T) = \frac{1}{T} \sum_{n=1}^{\infty} w(\frac{n}{T}) R_n(x), \tag{2.2}$$

where T (which determines the length of averaging) would be taken sufficiently large.

2.3 The main result

Our main result is the following assertion:

Theorem 2.1 *Let $w(x)$ be an infinitely smooth finite function whose support is separated from zero and $T \gg X^2$. Then*

$$\sum_{n=1}^{\infty} w(\frac{n}{T}) R_n(X) = c_1 T X log\frac{2\pi T}{X^2} + (c_2 + \frac{2}{\pi}\hat{w}(1))TX + 2c_1 T(L(1,\chi) + L(1,\chi')) + O(T^{\frac{1}{2}+\epsilon} X^{2+\epsilon} + X\sqrt{logX}),$$

where

$$c_1 = \frac{1}{\pi} \frac{\hat{w}(1)}{\zeta(2)},$$

$$c_2 = c_1(2 + \frac{\hat{w}'}{\hat{w}}(1) + 3\gamma - 2\frac{\zeta'}{\zeta}(2) + \frac{\Gamma'}{\Gamma}(\frac{1}{2})),$$

and

$$\chi(n) = (\frac{-1}{n}), \quad \chi'(n) = (\frac{-3}{n}).$$

(Here $\hat{w}(s)$ is the Mellin transform of $w(x)$.)

Chapter 3

The summation over n's

3.1 The main congruence

Firstly we will give a simple property of the Kloosterman sums which will be used later.

Lemma 3.1 $\sum_{m=1}^{c} S(m,m;c)e^{2\pi i(\frac{nm}{c})} = c\nu_n(c)$ *where* $\nu_n(c)$ *is the number of the solutions of the quadratic congruence* $a^2 + na + 1 \equiv 0 \pmod{c}$.

Before the proof, we change the notation $e^{2\pi i(\frac{nm}{c})}$ to $e(\frac{nm}{c})$ since it is more convenient to use(of course, the real reason is that this notation is used everywhere).

Proof: It follows directly from the definition of the Kloosterman sums.

$$\sum_{m=1}^{c} S(m,m;c)e(\frac{nm}{c}) = \sum_{\substack{1\le a\le |c| \\ (a,c)=1, ad\equiv 1 (mod\, c)}} \sum_{m=1}^{c} e(\frac{ma+md}{c} + \frac{mn}{c})$$

$$= \begin{cases} \sum_{\substack{1\le a\le |c| \\ (a,c)=1, ad\equiv 1 (mod\, c)}} c & \text{if } a+d+n = 0 \pmod{c} \\ 0 & \text{if } a+d+n \not\equiv 0 \pmod{c} \end{cases}$$

But the number of the solutions of the equation $a+d+n \equiv 0 \pmod{c}$ equals to the number of solutions of the equation $a^2 + an + 1 \equiv 0 \pmod{c}$ (since $(a,c) = 1$ and $ad \equiv 1 (mod\, c)$).

3.2 The L-Series

In this part we will give the basic definition, and some properties of the L-series for they have a connection with the quadratic residues.

Let m be an integer ≥ 1 and let χ be a character $mod\ m$, namely the character of the multiplicative group $(Z/mZ)^*$ of the ring Z/mZ. The corresponding L-function is defined by the Dirichlet series

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)/n^s.$$

For $\chi \neq 1$, the series converges(respectively converges absolutely) in the half plane $Re\ s > 0$ (respectively $Re\ s > 1$). Moreover, there is an analytic continuation over the whole plane which is more important for us [1].

3.3 Averaging The Remainder Term

From Lemma 2.1 we have the equation

$$\int_0^{\infty} R_n(r)h'(r)dr = \sum_{c\geq 1} \frac{1}{c} S(n,n;c)\varphi(\frac{4\pi n}{c}). \tag{3.1}$$

If we multiply (3.1) with $w(\frac{n}{T})$ and sum over n we get:

$$\sum_{n=1}^{\infty} w(\frac{n}{T}) \int_0^{\infty} R_n(r)h'(r)dr = \sum_{n=1}^{\infty} w(\frac{n}{T}) \sum_{c\geq 1} \frac{1}{c} S(n,n;c)\varphi(\frac{4\pi n}{c}).$$

From now on, we will deal with the right side of the equation. Since the summation over n is finite, we have

$$\sum_{n=1}^{\infty} w(\frac{n}{T}) \int_0^{\infty} R_n(r)h'(r)dr = \sum_{c\geq 1} \frac{1}{c} \sum_{n=1}^{\infty} w(\frac{n}{T}) S(n,n;c)\varphi(\frac{4\pi n}{c}). \tag{3.2}$$

We can replace n by $m+n_1 c$ where $n_1 = 0, 1, 2..$ and $1 \leq m \leq c$. It is obvious that $S(n,m,c)$ is a periodic function.Then from (3.2) we have

$$\sum_{c\geq 1} \frac{1}{c} \sum_{m=1}^{c} S(m,m;c) \sum_{n_1=0}^{\infty} w(\frac{c}{T}(n_1 + \frac{m}{c}))\varphi(4\pi(\frac{m}{c} + n_1)).$$

Now we consider the following function

$$f(x;c) = \sum_{n_1=-\infty}^{\infty} w(\frac{c}{T}(n_1+x))\varphi(4\pi(n_1+x)). \tag{3.3}$$

The series in (3.3) determines a periodic function of x, with period 1, hence $f(x;c)$ has the fourier expansion

$$f(x;c) = \sum_{n=-\infty}^{\infty} e(n\,\frac{m}{c})\,\phi_n(c),$$

where the coefficients are given by the integrals

$$\phi_n(c) = \int_0^{\infty} e(-nx)w(\frac{c}{T}x)\varphi(4\pi x)dx.$$

So (3.2) equals to:

$$\sum_{c\geq 1}\frac{1}{c}\sum_{m=1}^{c} S(m,m;c)\sum_{n=-\infty}^{\infty} e(\frac{nm}{c})\int_0^{\infty} e(-nx)\;\; w(\frac{c}{T}x)\;\; \varphi(4\pi x)dx.$$

By using Lemma 3.1 , we get

$$\sum_{n=1}^{\infty}\sum_{c\geq 1} w(\frac{n}{T})\frac{1}{c}S(n,n;c)\;\; \varphi(\frac{4\pi n}{c}) = \sum_{c\geq 1}\sum_{n=-\infty}^{\infty} \nu_n(c)\;\; \phi_n(c).$$

In order to obtain the Dirichlet series, we will use the Mellin transform of $w(\frac{c}{T}x)$. Namely it is

$$\hat{w}(s) = \int_0^{\infty} w(\frac{c}{T}x)x^{s-1}dx,$$

and the inversion formula

$$w(\frac{c}{T}x) = \frac{1}{2\pi i}\int_{\sigma}\hat{w}(s)c^{-s}x^{-s}T^s ds$$

holds.(Here $\int_\sigma$ means the integration is over the line $Res = \sigma$.) It can be seen from integration by parts that $\hat{w}(s)$ is an rapidly decreasing function.We can choose σ arbitrarily since $\hat{w}(s)$ is an entire function. Then

$$\sum_{c\geq 1}\sum_{n=-\infty}^{\infty}\nu_n(c)\phi_n(c) = \sum_{n=-\infty}^{\infty}\frac{1}{2\pi i}\int_0^{\infty}(\int_{(\sigma)} e(-nx)\hat{w}(s)\sum_{c\geq 1}\frac{\nu_n(c)}{c^s}x^{-s}T^s\varphi(4\pi x)ds)dx.$$

Clearly $\nu_n(c)$ is an even function of n. Replacing n by $-n$, and changing the order of integration we get

$$\sum_{n=0}^{\infty}\frac{1}{\pi i}\int_{(\sigma)}\varphi_n(s)\hat{w}(s)T^s L_n(s)ds, \tag{3.4}$$

where $L_n(s) = \sum_{c\geq 1} \frac{\nu_n(c)}{c^s}$ for $Res > 1$ and

$$\varphi_n(s) = \int_0^\infty cos(2\pi nx)x^{-s}\varphi(4\pi x)dx.$$

(The possibility of changing the order of summation and integration, and two integrations will be clear after calculating the $\varphi_n(s)$.)

Now two questions arise, what is the inner integral and what is the analytic continuation of $L_n(s)$? We will solve these questions seperately.

Lemma 3.2 $L_2(s) = \frac{1}{\zeta(2s)}\zeta(s)\zeta(2s-1) \quad for \ \ Res > 1.$

Proof:

$$L_2(s) = \sum_{c\geq 1} \frac{\nu_2(c)}{c^s} \quad for \ \ Res > 1,$$

where $\nu_2(c)$ is the number of the solutions of the equation $a^2 \equiv 0 \pmod{c}$. It can be easily seen that $\nu_n(c)$ is an multiplicative function of c for any $n \geq 0$. So

$$\begin{aligned} L_2(s) &= \sum_{c\geq 1} \frac{\nu_2(c)}{c^s} \\ &= \prod_p (1 + \frac{\nu_2(p)}{p^s} + \frac{\nu_2(p^2)}{p^{2s}} + \cdots + \frac{\nu_2(p^\alpha)}{p^{\alpha s}} + \cdots), \end{aligned}$$

where the multiplication is taken over all primes. Therefore it is enough to find $\nu_2(p^\alpha)$. We can show immediately that $\nu_2(p^\alpha) = p^{[\frac{\alpha}{2}]}$. Then the result follows.

Lemma 3.3 *When* $n \neq 2$

$$L_n(s) = \frac{\zeta(s)}{\zeta(2s)} \prod_{\substack{p>2 \\ p\nmid n^2-4}} (1 - \frac{1}{p^s}(\frac{n^2-4}{p}))^{-1} \eta(s) \quad for \ \ Res > 1,$$

where

$$\eta(s) = (\frac{1}{1+\frac{1}{2^s}})(1 + \frac{\nu_n(2)}{2^s} + \cdots) \prod_{\substack{p>2 \\ p|n^2-4}} (1 + \frac{\nu_n(p)}{p^s} + \frac{\nu_n(p^2)}{p^{2s}} + \cdots) \prod_{\substack{p>2 \\ p|n^2-4}} (1 + \frac{1}{p^s})^{-1}.$$

Proof: Again it is enough to find $\nu_n(p^\alpha)$.

$$\nu_n(p^\alpha) = \#\ \{a:\ \ a^2 + n\,a + 1 \equiv 0 \mod(p^\alpha)\}.$$

Let us choose p as an odd prime.

$$\begin{aligned}\nu_n(p^\alpha) &= \#\ \{a:\ \ 4\,a^2 + 4\,n\,a\ + 4 \equiv 0 \mod(p^\alpha)\} \\ &= \#\ \{a:\ \ (2\,a+n)^2 - n^2 + 4 \equiv 0 \mod(p^\alpha)\} \\ &= \#\ \{x:\ \ x^2 \equiv n^2 - 4 \mod(p^\alpha)\}.\end{aligned}$$

It can be shown that $\nu_n(p^\alpha) = 1 + (\frac{n^2-4}{p})$ if $p \not| n^2 - 4$ where $(\frac{n^2-4}{p})$ is the usual Legendre symbol. So we get the result.

3.4 The functions $\varphi_n(s)$

Now we can give the explicit expressions for the integrals of $\varphi_n(s)$.

Lemma 3.4 *For any ρ with the condition $0 < \rho < \frac{3}{2}$ we have for $Res < 1 + 2\rho$*

$$\varphi_n(s) = i\pi^{s-2} \int_{Imr=-\rho} \psi_n(r,s) r \frac{h(r)}{ch\pi r} dr,$$

where

$$\psi_n(r,s) = \begin{cases} 2^{s-1}\sqrt{\pi}\dfrac{\Gamma(s-\frac{1}{2})\,\Gamma(\frac{1-s}{2}+ir)}{\Gamma(\frac{1+s}{2}+ir)\Gamma(\frac{s}{2}+ir)\Gamma(\frac{s}{2}-ir)} & \text{if } n=2, \\ 2^{s-1}\dfrac{\Gamma(\frac{1-s}{2}+ir)}{\Gamma(\frac{1+s}{2}+ir)} F(\dfrac{1-s}{2}+ir, \dfrac{1-s}{2}-ir, \dfrac{1}{2}; \dfrac{n^2}{4}), & \text{if } n=0 \text{ or } 1, \\ n^{s-1}\sqrt{\pi}(\dfrac{2}{n})^{2ir}\dfrac{\Gamma(\frac{1-s}{2}+ir)}{\Gamma(2ir+1)\Gamma(\frac{s}{2}-ir)} F(\dfrac{1-s}{2}+ir, 1-\dfrac{s}{2}+ir, 2ir+1; \dfrac{4}{n^2}), & \text{if } n>2 \end{cases}$$

Here $F(a,b,c;z) = \sum_{n=0}^{\infty} \dfrac{\Gamma(a+n)}{\Gamma(a)} \dfrac{\Gamma(b+n)}{\Gamma(n)} \dfrac{\Gamma(c)}{\Gamma(c+n)} \dfrac{z^n}{\Gamma(n+1)}$ is the Gauss hypergeometric function.

Proof: Using the regularity of $h(r)$ in the strip $|Imr| < \Delta$, $\Delta < \frac{1}{2}$ we can write:

$$\begin{aligned}\varphi(4\pi x) &= \frac{2i}{\pi}\int_{-\infty}^{\infty} J_{2ir}(4\pi x)\frac{r}{ch(\pi r)}h(r)dr \\ &= \frac{2i}{\pi}\int_{Imr=-\rho} J_{2ir}(4\pi x)\frac{r}{ch(\pi r)}h(r)dr,\end{aligned}$$

here we take $0 < \rho < \frac{3}{2}$. For large values of r, $h(r)$ is very small and when $x \to 0, |J_{2ir}(x)| \ll x^{2\rho}$. So the integrand of $\varphi_n(s)$ is not greater than $x^{-\sigma+2\rho}|rh(r)|$. If we take $Res < 1+2\rho$, the double integral will be absolutely convergent. Therefore we can integrate in any order.

The inner integral is so called the discontinuous integral of Weber and Schafheitlin. This integral converges,but its analytic expression is different in two cases $n < 2$ or $n > 2$ [2]. By taking

$$cos(2\pi nx) = \sqrt{\pi^2 nx}J_{-\frac{1}{2}}(2\pi nx)$$

we can apply the formula from [2], and for $n = 0$ we use the well known formula for the Mellin transform,

$$\int_0^{\infty} J_\mu(at)t^{\rho-1}dt = 2^{\rho-1}a^{-\rho}\frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2}\rho)}{\Gamma(1+\frac{1}{2}\mu - \frac{1}{2}\rho)}, \quad \text{where } -Re(\mu) < Re\rho < \frac{3}{2}\ .$$

Then we get the desired result.

3.5 The Principal Term

As the result of Lemma 3.4, we have for $\rho \in (0, \frac{3}{2})$, $1 < Res < 1+2\rho$

$$\int_0^{\infty} \overline{R}(r,T)h'(r)dr = \sum_{n=0}^{\infty}\int_{Imr=-\rho}\frac{rh(r)}{ch(\pi r)}\,\Omega_n(r,T)dr \tag{3.5}$$

where

$$\Omega_n(r,T) = \int_\sigma \pi^{s-3}\psi_n(r,s)\hat{w}(s)T^s L_n(s)ds.$$

Firstly we consider the function $\Omega_2(r,T)$. The integrand is the meromorphic function, $L_2(s)$ has the double pole at $s = 1$, and other multipliers have

no singularity for $\frac{1}{2} < Res < 1 + 2\rho$. We move the line of integration to the left and we integrate now on the line $Res = \sigma = \frac{1}{2} + \varepsilon, \quad \varepsilon > 0$. Thus

$$\begin{aligned}\Omega_2(r,T) &= \frac{1}{\pi} T \frac{\hat{w}(1)}{\zeta(2)} \frac{ch\pi r}{r} (log 2\pi T + \frac{\hat{w}'}{\hat{w}}(1) + 3\gamma - 2\frac{\zeta'}{\zeta}(2) + \frac{\Gamma'}{\Gamma}(\frac{1}{2}) \\ &\quad -2logr - i\frac{\pi}{2} cth\pi r) + \int_{\sigma=\frac{1}{2}+\varepsilon} \pi^{s-3} \psi_2(r,s) \hat{w}(s) T^s L_2(s) ds.\end{aligned}$$

After integrating $\Omega_2(r,T)$ with multiplier $\frac{rh(r)}{ch\pi r}$ on the line $Imr = -\rho$ (here ρ can be taken as 0, and note that $h(r)$ is 1 in the interval $[-X, X]$) we get the main term of the series in the formula (3.5) which is:

$$c_1 T X log(2\pi T) + c_2 T X - 2c_1 T X log X$$

where

$$\begin{aligned}c_1 &= \frac{1}{\pi} \frac{\hat{w}(1)}{\zeta(2)}, \quad \text{and} \\ c_2 &= c_1 (2 + \frac{\hat{w}'}{\hat{w}}(1) + 3\gamma - 2\frac{\zeta'}{\zeta}(2) + \frac{\Gamma'}{\Gamma}(\frac{1}{2}))\end{aligned}$$

Here γ is the Euler constant, $\gamma = -\frac{\Gamma'}{\Gamma}(1)$. And

$$\int_{Imr=-\rho} \int_{\sigma=\frac{1}{2}+\varepsilon} \pi^{s-3} \psi_2(r,s) \hat{w}(s) T^s L_2(s) \frac{rh(r)}{ch(\pi r)} ds\, dr \ll O(T^{\frac{1}{2}+\varepsilon} X^{2+\varepsilon}),$$

since

$$\frac{\Gamma(\frac{1-s}{2} + ir)}{\Gamma(\frac{1+s}{2} + ir)\Gamma(\frac{s}{2} + ir)\Gamma(\frac{s}{2} - ir)} = r^{-2s} e^{is\frac{\pi}{2} + \pi r} exp(O(\frac{1}{r^2})).$$

3.6 The Cases $n \neq 2$

Now we consider the case $n > 2$.

Again the integrand of $\Omega_n(r,T)$ is a meromorphic function, for $L_n(s)$ has a simple pole at $s = 1$ when $n > 2$. We move the line of integration to the line $Res = \sigma_1 = \frac{1}{2} + \varepsilon, \quad \varepsilon > 0$ for $n < T^{\frac{1}{2}}$. We get when $n > 2$

$$\Omega_n(r,T) = \frac{2}{\pi} n^{-2ir} \frac{ch\pi r}{r} \hat{w}(1) T B_n + \int_{\sigma_1=\frac{1}{2}+\varepsilon} \pi^{s-3} \psi_n(r,s) \hat{w}(s) T^s L_n(s) ds,$$

where B_n is the residue of $L_n(s)$ at $s = 1$. The result follows by the equality

$$\frac{\Gamma(ir)}{\Gamma(2ir+1)\Gamma(\frac{1}{2}-ir)} = \pi^{-\frac{1}{2}}2^{-2ir}\frac{ch\pi r}{ir},$$

and

$$F(ir, \frac{1}{2}+ir, 1+2ir; x)\; x^{ir} = 2^{2ir}e^{-ir\xi},$$

where $x = \frac{4}{n^2}$ and $\xi = log\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}$ [3].
Here in order to estimate integral on the line $\sigma_1 = \frac{1}{2} + \varepsilon$, it is necessary to find a bound for $L_n(s)$.
To do this we express $L_n(s)$ in terms of the classical Dirichlet's series with the Kronecker symbol.

$$L_n(s) = \sum_{c\geq 1}(\frac{n^2-4}{c})\frac{1}{c^s} \quad for \;\; Res > 1.$$

Since our character is not primitive we write $n^2 - 4 = k^2Q$ where Q is square free and $k > 0$.
We get

$$L_n(s) = \prod_{p|k}(1 - (\frac{Q}{p})\frac{1}{p^s})L(s,\chi) \quad for \;\; Res > 1,$$

where $\chi = (\frac{Q}{p})$ is real, primitive character.
We will use the functional equation for $L(s,\chi)$.

Lemma 3.5

$$L(s,\chi) \ll Q^{\frac{1}{4}+\varepsilon_0} \quad for \;\; any \;\; \varepsilon_0 > 0 \quad on \; the \; line \; Res = \frac{1}{2}.$$

Proof: Considering a function $\alpha(x)$ such that $\alpha \in C^\infty(0,\infty)$, $\alpha \equiv 1$ for $x \leq x_0 < 1$ and$\alpha(x) \equiv 0$ if $x > \frac{1}{x_0}$.(It is convenient to take this α such a way that $\alpha(x) + \alpha(\frac{1}{x}) \equiv 1$.) we have

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\alpha(\frac{n}{T}) = \frac{1}{2\pi i}\int_\xi \hat{\alpha}(\rho)\,(\sum_n \frac{\chi(n)}{n^{s+\rho}})\,T^\rho d\rho, \tag{3.6}$$

where $\hat{\alpha}(\rho)$ is the Mellin transform of $\alpha(x)$ and $Re\rho = \xi > \frac{1}{2}$. Clearly $\hat{\alpha}(\rho)$ has a simple pole at $\rho = 0$ with residue 1. Applying the functional equation for $L(s,\chi)$ [1] to (3.6) and moving the line of integration to the line $\xi_1 < -\frac{1}{2}$ we obtain

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\alpha(\frac{n}{T}) = \frac{1}{2\pi i}\int_{\xi_1}\hat{\alpha}(\rho)\,T^{\rho}\frac{\tau(\chi)}{i^a Q^{\frac{1}{2}}}(\frac{\pi}{Q})^{-\frac{1}{2}+s+\rho}\frac{\Gamma(\frac{1-s-\rho}{2}+\frac{a}{2})}{\Gamma(\frac{s+\rho}{2}+\frac{a}{2})} L(1-s-\rho,\chi)d\rho + L(s,\chi).$$

Here the Gaussian sum $\tau(\chi)$ is defined by for any character $\chi(n)$ to the modulus q,

$$\tau(\chi) = \sum_{m=1}^{q}\chi(m)e(\frac{m}{q}),$$

and

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

If $n \leq (\frac{Q}{T})^{1+\epsilon_0}$, we move the path of integration to the right, we get the equality,

$$L(s,\chi) = \sum_{n\leq T}\frac{\chi(n)}{n^s} + \sum_{n\leq(\frac{Q}{T})^{1+\epsilon_0}}\frac{\tau(\chi)}{i^a Q^{\frac{1}{2}}}(\frac{\pi}{Q})^{\frac{-1}{2}+s}\frac{\Gamma(\frac{1}{2}(1-s+a))}{\Gamma(\frac{1}{2}(s+a))}\frac{\chi(n)}{n^{1-s}}$$
$$+ \sum_{n\leq(\frac{Q}{T})^{1+\epsilon_0}}\int_0^{\infty}\alpha(x)x^{\frac{1}{2}-s}J_{a-\frac{1}{2}}(\frac{2\pi Tnx}{Q})(\frac{\pi Tn}{Q})^{\frac{3}{2}-s}$$
$$(\frac{\pi}{q})^{s-\frac{1}{2}}\frac{\tau(\chi}{i^a Q^{\frac{1}{2}}}\frac{\chi(n)}{n^{1-s}}dx.$$

In order to get best estimate we choose $T \sim Q^{\frac{1}{2}+\frac{\epsilon_0}{2}}$. So $L(s,\chi) \ll Q^{\frac{1}{4}+\frac{\epsilon_0}{2}}$.
It is clear that the product $L_n(s) = \prod_{p|k}(1-(\frac{Q}{p})\frac{1}{p^s}$ is not larger than $\sum_{d|k}d^{-\rho}$ $(\rho = Res)$, it is smaller than k^{ε} for any $\varepsilon > 0$.
So we have estimate

$$|L_n(s)| \ll k^{\varepsilon}Q^{\frac{1}{4}+\varepsilon} \quad \text{if} \quad n^2-4 = k^2Q.$$

For the case $n \ll T^{\frac{1}{2}}$ it gives $|L_n(s)| \ll T^{\frac{1}{4}+\epsilon_0}$, $\varepsilon_0 > 0$.

When we integrate $\Omega_n(r,T)$ with multiplier $\frac{rh(r)}{ch\pi r}$ on the line $Imr = -\rho$, we get

$$\frac{2}{\pi}n^{-2\rho}B_n\hat{w}(1)TX + \int_{Imr=-\rho}\Omega_n(r,T)\frac{rh(r)}{ch\pi r}dr.$$

In order to obtain the estimate we find the asymptotic expansion of $F(\frac{1-s}{2}+ir, 1-\frac{s}{2}+ir, 2ir+1; \frac{4}{n^2})$ for large values of r and n. We use the standard

methods of the asymptotic integration of differential equation with large parameter. We get

$$F(\frac{1-s}{2}+ir, 1-\frac{s}{2}+ir, 2ir+1; \frac{4}{n^2}) = (\frac{4}{n^2})^{-ir}(1-\frac{4}{n^2})^{\frac{s}{2}-\frac{1}{2}}2^{2ir} e^{-ir\xi}(1+\frac{1}{2ir}\int_{\xi}^{\infty} f(\eta)d\eta + O(\frac{e^{-\eta}}{r^2})),$$

where

$$f(\eta) = \frac{-3}{16(ch^2\frac{\eta}{2}-1)ch^2\frac{\eta}{2}} + \frac{\frac{7}{4}-2s^2+2s}{4ch^2\frac{\eta}{2}} \quad \text{and} \quad \xi \sim 2logn.$$

Using Stirling formula we have

$$\frac{\Gamma(\frac{1-s}{2}+ir)}{\Gamma(2ir+1)\Gamma(\frac{s}{2}-ir)} = -ir^{-s}2^{\frac{1}{2}-2ir}e^{\pi r}exp(O(\frac{1}{r})).$$

And for $n > T^{\frac{1}{2}}$, we approximate $\Omega_n(r,T)$ on the line $Res = \sigma_1 = 1+\varepsilon_0, \quad \varepsilon_0 > 0$ we get

$$\int_{Imr=-\rho} \Omega_n(r,T)\frac{rh(r)}{ch\pi r}dr \ll T^{1+\varepsilon_0}X^{1-\varepsilon_0}\frac{1}{n^{2\rho}}.$$

If we come back to the series in (3.5) we need to find $\sum_{n\geq 3}^{N} B_n$.

Let $B'(n)$ be the residue at the point $s = 1$ of the function $\zeta(2s)\sum_{c=1}^{\infty}\frac{\nu_n(c)}{c^s}$.It is known that

$$\sum_{n\geq 3}^{N} B'(n) = O(N).$$

The proof follows from Theorem 2 in the paper of N.V. Kuznetsov [6] by correcting the misprint in the result. The error was noticed and corrected by Professor Kuznetsov.

So we have in (3.5)

$$\frac{2}{\pi}TX\hat{w}(1)\sum_{3\leq n\leq N}\frac{B_n}{n^{2\rho}} + \sum_{n>N} T^{1+\varepsilon_0}X^{1-\varepsilon_0}\frac{1}{n^{2\rho}} + O(T^{\frac{1}{2}+\varepsilon_0}X^{\frac{3}{2}+\varepsilon_0}) \quad \text{if} \quad T > X,$$

where $N = T^{\frac{1}{2}+\varepsilon_0}$, and $\sum_{n>N} T^{1+\varepsilon_0}X^{1-\varepsilon_0}\frac{1}{n^{2\rho}}$ can be estimated as $O(X^{1-\varepsilon_0})$.

The cases $n = 0$ and $n = 1$, are the trivial ones since we have exponentially small functions in the integrand. The integrands of $\Omega_0(r,T)$ and $\Omega_1(r,T)$ have simple poles at $s = 1$ which comes from the $L_0(s)$ and $L_1(s)$ respectively for

$$L_0 = \frac{\zeta(s)}{\zeta(2s)} \sum_{\text{n is odd}} \frac{\chi_4(n)}{n^s},$$

where

$$\chi_4(n) = \begin{cases} 1 & \text{if } n \equiv 1 \ (mod\ 4), \\ -1 & \text{if } n \equiv -1 \ (mod\ 4) \end{cases}$$

and

$$L_1(s) = \frac{\zeta(s)}{\zeta(2s)} \sum_{n=1} \frac{\chi'(n)}{n^s} \quad \text{where} \quad \chi'(n) = (\frac{-3}{n}).$$

By moving the line of integration to the $Res = \sigma_1 = \frac{1}{2} + \varepsilon_0$,we obtain

$$\Omega_0(r,T) = \frac{2}{\pi}\frac{1}{r\zeta(2)}\hat{w}(1)TL(1,\chi) + \int_{\sigma_1} \pi^{s-3}2^{s-1}\frac{\Gamma(ir+\frac{1-s}{2})}{\Gamma(ir+\frac{1+s}{2})}\hat{w}(s)T^s\mathrm{L}_0(s)ds.$$

By estimating the second integral we get the first term of the series in (3.5) as

$$\frac{2}{\pi}\frac{1}{\zeta(2)}T\hat{w}(1)L(1,\chi) + O(T^{\frac{1}{2}+\varepsilon_0}).$$

Similarly,

$$\begin{aligned}\Omega_1(r,T) &= \frac{2}{\pi r}\frac{1}{\zeta(2)}\hat{w}(1)TL(1,\chi')F(ir,-ir,\frac{1}{2},\frac{1}{4}) \\ &\quad + \int_{\sigma_1} \pi^{s-3}2^{s-1}\frac{\Gamma(ir+\frac{1-s}{2})}{\Gamma(ir+\frac{1+s}{2})}F(\frac{1-s}{2}+ir,\frac{1-s}{2}-ir,\frac{1}{2},\frac{1}{4})\hat{w}(s)T^sL_1(s)ds.\end{aligned}$$

By using

$$F(\frac{1-s}{2}+ir,\frac{1-s}{2}-ir,\frac{1}{2},\frac{1}{4}) = ch\frac{\pi r}{3}(1+O(\frac{1}{r})),$$

we get the second term of the series in (3.5) as

$$\frac{2}{\pi}\frac{1}{\zeta(2)}L(1,\chi')\hat{w}(1)T + O(T^{\frac{1}{2}+\varepsilon_0}).$$

As a result:

For $T \gg X^2$, and $\varepsilon_0 > 0$

$$\begin{aligned}\sum_{n=0}^{\infty} w(\frac{n}{T})\int_0^{\infty} R_n(r)h'(r)dr &= c_1TXlog(2\pi T) + c_2TX - 2c_1TXlogX + \frac{2}{\pi}\hat{w}(1)TX \\ &\quad + 2c_1T(L(1,\chi)+L(1,\chi')) + O(T^{\frac{1}{2}+\varepsilon_0}X^{2+\varepsilon_0})\end{aligned}$$

where

$$c_1 = \frac{1}{\pi}\frac{\hat{w}(1)}{\zeta(2)} \quad \text{and}$$

$$c_2 = c_1(2 + \frac{\hat{w}'}{\hat{w}}(1) + 3\gamma - 2\frac{\zeta'}{\zeta}(2) + \frac{\Gamma'}{\Gamma}(\frac{1}{2})).$$

By estimating the integral $\int_0^\infty R_n(r)h'(r)dr$ from below by $R_n(X - 2\Delta\sqrt{logX}) + O(X\sqrt{logX})$ and from above by $R_n(X + 2\Delta\sqrt{logX}) + O(X\sqrt{logX})$), we prove the theorem.

3.7 The mean value of the integral over continuous spectrum

It is natural to ask the asymptotic behaviour of the mean value of the integral over the continuous spectrum.

Proposition 3.1

$$\frac{1}{\pi}\sum_{n=1}^{\infty} w(\frac{n}{T})\int_{-\infty}^{\infty}\frac{|\tau_{\frac{1}{2}+ir}(n)|^2}{|\zeta(1+2ir)|^2}h(r)dr = \frac{1}{\pi}\frac{\hat{w}(1)}{\zeta(2)}TlogTX + \frac{1}{\pi}\frac{\hat{w}(1)}{\zeta(2)}(\frac{\hat{w}'}{\hat{w}}(1) - 2\frac{\zeta'}{\zeta}(2) + 2\gamma)TX + O(TlogX) + O(T^{\frac{1}{2}+\varepsilon_0}).$$

Proof:

$$\sum_{n=1}^{\infty} w(\frac{n}{T})|\tau_\nu(n)|^2 = \frac{1}{2\pi}\int_\sigma \hat{w}(s)T^s(\sum_{n=1}^{\infty}\frac{|\tau_\nu(n)|^2}{n^s})ds \quad \text{for } Res = \sigma > 1. \quad (3.7)$$

By moving the line of integration to the left, $\sigma = \frac{1}{2} + \varepsilon_0$, $\varepsilon_0 > 0$ and using the Ramanujan identity

$$\sum_{n=1}^{\infty}\frac{|\tau_\nu(n)|^2}{n^s}) = \frac{\zeta^2(s)\zeta(s+2\nu-1)\zeta(s-2\nu+1)}{\zeta(2s)},$$

we get

$$\sum_{n=1}^{\infty} w(\frac{n}{T})|\tau_\nu(n)|^2 = T\frac{\hat{w}(1)}{\zeta(2)}\zeta(2\nu)\zeta(2-2\nu)(logT + \frac{\hat{w}'}{\hat{w}}(1) + \frac{\zeta'}{\zeta}(2-2\nu) + \frac{\zeta'}{\zeta}(2\nu) - 2\frac{\zeta'}{\zeta}(2) + 2\gamma) + O(T^{\frac{1}{2}+\varepsilon_0}). \quad (3.8)$$

Taking $\nu = \frac{1}{2} + ir$, and integrating (3.8) with the multiplier $\frac{h(r)}{|\zeta(1+2ir)|^2}$ with respect to r, we get the result.

Chapter 4

Conclusion

In the paper of N.V.Kuznetsov, it was proved that

$$\sum_{\kappa_j \leq X} \frac{|\rho_j(n)|^2}{ch\pi\kappa_j} = \frac{1}{\pi^2}X^2 + R'_n(X)$$

where

$$R'_n(X) \ll XlnX + Xn^{\epsilon} + n^{\frac{1}{2}+\epsilon}$$

for any fixed $\epsilon > 0$ and for any $X \geq 2, \ \ n \geq 1$.
The true order of this remainder term is unknown.

In this thesis we found the average mean value of the remainder term $R_n(X)$ (the Fourier coefficients of the eigenfunctions of the continuos spectrum is also taken in contrast to $R'_n(X)$). We see that the average mean value of $R_n(X)$ is positive. Furthermore, from the theorem it can be seen that there are infinitely many n's and X's such that for $n \in (T, 2T)$ and $T \gg X^2$, we have

$$R_n(X) \gg X(logn)^{\alpha}, \quad \forall \alpha < 1.$$

By using the proposition and theorem, it can be also shown that there exist infinitely many n's and X's satisfying

$$R'_n(X) < \frac{-1}{2\pi}\frac{\hat{w}(1)}{\zeta(2)}XlogX$$

for $n \in (T, 2T)$ and $T \gg X^2$.

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