Transportation Research Part E 158 (2022) 102602

Contents lists available at ScienceDirect

Transportation Research Part E

journal homepage: www.elsevier.com/locate/tre

# A conditional $\beta$-mean approach to risk-averse stochastic multiple allocation hub location problems

Nader Ghaffarinasab [a,*], Bahar Y. Kara [b]

[a] Department of Industrial Engineering, University of Tabriz, Tabriz, 51666-14766, Iran
[b] Department of Industrial Engineering, Bilkent University, Ankara, Turkey

ARTICLE INFO

*Keywords:*
Hub location
Conditional $\beta$-mean
Mathematical formulation
Benders decomposition

ABSTRACT

This paper addresses risk-averse stochastic hub location problems where the risk is measured using the conditional $\beta$-mean criterion. Three variants of the classical multiple allocation hub location problem, namely the $p$-hub median, the $p$-hub maximal covering, and the weighted $p$-hub center problems are studied under demand data uncertainty represented by a finite set of scenarios. Novel mixed-integer linear programming formulations are proposed for the problems and exact algorithms based on Benders decomposition are developed for solving large instances of the problems. A large set of computational tests are conducted so that the efficiency of the proposed algorithms is proved and the effect of various input parameters on the optimal solutions is analyzed.

## 1. Introduction

Hub location problems (HLPs) play a pivotal role in the design of many-to-many distribution networks arising in various service industries such as transportation, logistics, telecommunications, etc. Hub facilities accommodate key functions such as transshipment, sorting, and consolidation. Origin–destination (O/D) traffic are routed via hub facilities rendering a consolidated flow of commodities where traffic with different origins and/or destinations are agglomerated while being transported over the network. The latter allows exploitation of economies of scale on transportation cost, especially over the hub arcs. The HLP aims to find the optimal location of the hub facilities and determine the routing path for the O/D traffic in such a manner that a specific objective such as cost minimization or service level maximization is achieved. Demand (non-hub) nodes can be assigned to the installed hubs based on two mainly adopted protocols: single and multiple allocation. The single allocation hub networks force any non-hub node to send/receive its traffic via a single hub, whereas in multiple allocation networks, the non-hub nodes can exchange flows via more than one hub. In the current work, we deal with multiple allocation hub networks which is more flexible than the single allocation version in terms of customer allocation.

Perfect information is mostly unavailable while making long-term decisions such as hub location. Therefore, the decision maker faces a great deal of uncertainty regarding the problem data stemming from a number of factors such as population size shifts, cyclic fluctuations due to economic recession and expansion, technological developments, unexpected outbreak of diseases such as COVID-19, etc. In order to deal with uncertainty, some researchers have employed techniques such as stochastic programming in the design of hub networks (Contreras et al., 2011b; Alumur et al., 2012; Rostami et al., 2021), where the focus is normally on the average performance of the system which is suitable for repetitive decision making problems. However, for long-term non-repetitive problems such as the HLP, the risk-neutral approach may result in solutions with very poor performance under certain realizations

* Corresponding author.
*E-mail addresses:* ngnasab@tabrizu.ac.ir (N. Ghaffarinasab), bkara@bilkent.edu.tr (B.Y. Kara).

https://doi.org/10.1016/j.tre.2021.102602
Received 17 May 2021; Received in revised form 20 November 2021; Accepted 25 December 2021
Available online 29 January 2022
1366-5545/

of the uncertain parameters. Hence, adopting a risk-averse approach that focuses on the variability of random outcomes would yield solutions that are more robust toward data uncertainty.

In this work we address three variants of the risk-averse uncapacitated multiple allocation hub location problem, namely the $p$-hub median, the $p$-hub maximal covering, and the weighted $p$-hub center problems under demand uncertainty. The underlying uncertainty is modeled by using a finite set of scenarios and the conditional $\beta$-mean criterion is employed to measure the risk. The conditional $\beta$-mean was originally proposed as a fairness measure in deterministic facility location problems (Ogryczak and Zawadzki, 2002). In stochastic setting, on the other hand, the conditional $\beta$-mean has been used as a coherent risk measure (Filippi et al., 2019) which is strictly related to conditional value-at-risk (CVaR), a very popular risk measure proposed for financial optimization (Rockafellar and Uryasev, 2000) but also applied in different managerial and engineering contexts (Filippi et al., 2020). An advantage of the conditional $\beta$-mean is that it can be embedded into a mixed-integer linear programming (MILP) model by only adding a small number of continuous variables and linear constraints which makes it applicable to many location problems (Filippi et al., 2021b). Furthermore, the proposed method is more flexible in terms of risk-aversion and can incorporate the preferences of different types of decision makers towards risk. In other words, by selecting a proper value for the risk parameter ($\beta$), solutions with favorable levels of risk-aversion can be obtained.

The contributions of this paper to the existing literature can be listed as follows:

- Risk-averse $p$-hub median, $p$-hub maximal covering, and weighted $p$-hub center problems are introduced.
- Conditional $\beta$-mean is used in a stochastic setting to measure the risk.
- Problems are formulated as two-stage stochastic models and their deterministic equivalent formulations are derived as MILP models.
- Exact algorithms based on Benders decomposition are developed to solve large-scale instances of the problems.
- Extensive computational experiments are designed and performed based on three well-known data sets from the literature of the HLP in order to study the impact of various input parameter on the output of the proposed models and also to demonstrate the efficiency of the proposed formulations and algorithms.

This paper is structured as follows. The literature related to the addressed problems are briefly reviewed in Section 2. MILP formulations for the three problems under study are developed in Section 3. The proposed Benders decomposition algorithms are presented in Section 4. Section 5 present the numerical results obtained from our computational experiments, followed by Section 6 which concludes the paper and provide some outlooks for further research.

## 2. Literature review

The HLPs have attracted high attention in the literature of network design and facility location due to their wide applicability in many real-world problems. The multiple allocation $p$-hub median problem was first formulated as a linear integer program by Campbell (1992). Later, Campbell (1994) introduced different variants of the multiple allocation HLP and developed integer programming formulations for them. As noted earlier, the multiple allocation HLPs are more flexible in terms of customer assignment as there is no limitation on the number of hubs to which each non-hub node can be allocated. For recent reviews on the topic, the interested readers are referred to Alumur and Kara (2008), Campbell and O'Kelly (2012), Farahani et al. (2013), Contreras and O'Kelly (2019) and Alumur et al. (2021).

Analogous to the classical facility location models, the three basic variants of the HLP are called the $p$-hub median, the $p$-hub maximal covering, and the (weighted) $p$-hub center problems. The $p$-hub median problem deals with locating $p$ hubs in such a way that the total transportation cost (time, distance, etc.) is minimized. The other two variants are more concerned with service level enhancement rather than cost minimization. More specifically, the $p$-hub maximal covering problem tries to locate $p$ hubs in such a way that the total amount of covered O/D demands is maximized. The (weighted) $p$-hub center problem, on the other hand, aims at locating $p$ hubs in such a manner that the maximum service cost among all the O/D pairs is minimized.

Uncertainty is an important issue that needs to be considered in the design of hub networks and so far, it has been addressed by researchers in various ways. Some authors use queuing theory to model the uncertain transportation and/or service times in the hub facilities (Marianov and Serra, 2003; Rodriguez et al., 2007; Mohammadi et al., 2011). Designing reliable hub networks that incorporate the possibility of (natural or man-made) disruptions at hub facilities is another important stream of research (Kim and O'Kelly, 2009; An et al., 2015; Chaharsooghi et al., 2017; Rostami et al., 2018; Ghaffarinasab and Motallebzadeh, 2018; Ghaffarinasab and Atayi, 2018; Zhalechian et al., 2018; Madani et al., 2018; Mohammadi et al., 2019; Shen et al., 2021; Korani and Eydi, 2021). Chance constrained programming technique is also used in some other works to limit the probability of undesirable outcomes in the performance of hub networks (Sim et al., 2009; Nikokalam-Mozafar et al., 2014; Gao and Qin, 2016). Another important technique used for dealing with uncertainty in the HLPs is robust optimization (RO) where it is assumed that the randomness in problem data is structured around a predefined uncertainty set without a given probability distribution for the associated data (Shahabi and Unnikrishnan, 2014; Ghaffari-Nasab et al., 2015; Merakli and Yaman, 2016; de Sá et al., 2018; Zetina et al., 2017; Ghaffarinasab, 2018; Ghaffarinasab et al., 2020; Ghaffarinasab, 2021). Nevertheless, RO is too conservative and focuses on the worst-case performance of the system which results in network configurations with large investment and operational costs.

One of the widely used approaches in dealing with uncertain HLPs is stochastic programming (SP) (Birge and Louveaux, 2011) where the hub location and allocation/routing decisions are made at different stages (Yang, 2009; Contreras et al., 2011b; Alumur et al., 2012; Correia et al., 2018; Rostami et al., 2021; Taherkhani et al., 2020). In most of the cases, SP models are based on the average performance of the system. However, as noted earlier, this approach may result in solutions with very poor performance under certain realizations of the uncertain parameters.

The conditional $\beta$-mean was originally developed for incorporating fairness criterion into the decision making process in facility location problems (Ogryczak and Zawadzki, 2002). Filippi et al. (2019) use the conditional $\beta$-mean in $p$-median and $p$-center problems as a measure of equity. A single-source capacitated facility location with cost and fairness objectives is addressed by Filippi et al. (2021b) using the conditional $\beta$-mean criterion. Filippi et al. (2021a) study another fair facility location problem employing the conditional $\beta$-mean. In case of stochastic problems with a discrete underlying distribution, the conditional $\beta$-mean is only used in Filippi et al. (2019) as a coherent risk measure in stochastic multidimensional knapsack problem. The authors also elaborate the relationship between the conditional $\beta$-mean and CVaR in the stochastic setting. Yu et al. (2017) use CVaR for studying risk in a facility location problem where the facilities are subject to random disruptions. A stochastic pre-disaster relief network design problem is addressed by Özgün Elçi and Noyan (2018) using CVaR as a risk measure. Noyan (2012) proposes a risk-averse two-stage stochastic program for a disaster management application, where the CVaR is used as the risk measure. Hosseini and Verma (2018) use the CVaR criterion for minimizing the transportation risk in routing rail hazmat shipments. Golpîra et al. (2017) address the multi-objective multi-echelon supply chain network design problem in which the demand uncertainty is taken into account by using the CVaR measure. The interested reader is referred to Filippi et al. (2020) as a recent survey on the applications of CVaR beyond finance.

Benders decomposition (BD) (Benders, 1962) has been successfully used to tackle large-scale HLPs. de Camargo et al. (2008) solve the uncapacitated multiple allocation hub location problem (UMAHLP) using BD algorithms. Another BD algorithm is devised by de Camargo et al. (2009a) for the HLPs with flow-dependent discount factor. A generalized BD algorithm is developed for HLPs under congestion by de Camargo et al. (2009b, 2011). Large-scale instances of the UMAHLP are solved by using BD algorithms proposed by Contreras et al. (2011a). The same authors apply a BD algorithm for solving stochastic uncapacitated HLPs (Contreras et al., 2011b). Gelareh and Nickel (2011) propose a BD procedure for HLPs arising in urban transport and line shipping. The proposed algorithm is extended by Gelareh et al. (2015) for solving a multi-period HLP in transportation systems. Gelareh and Pisinger (2011) study a hub network design problem for a deep-sea line service provider and proposed a BD algorithm for solving it. Capacitated version of the HLPs are tackled by a BD algorithm in Contreras et al. (2012). de Camargo et al. (2013) apply a BD algorithm for the many-to-many hub location routing problem. BD algorithm are also used to solve the tree of hubs location problem and the hub line location problem by de Sá et al. (2013, 2015), respectively. Another BD algorithm is devised by O'Kelly et al. (2015) for the HLP with price-sensitive demands. Two BD procedures are proposed by Merakli and Yaman (2016) for the robust uncapacitated multiple allocation $p$-hub median problem (UMApHMP) under polyhedral demand uncertainty. de Sá et al. (2018) study a robust hub location problem with uncertain demands and fixed costs. Ghaffarinasab and Kara (2019) propose BD algorithms for solving uncapacitated single allocation HLPs with fixed and variable number of hubs. Taherkhani et al. (2020) develop a BD procedure for solving the profit maximizing hub location problems. Ghaffarinasab (2020) proposes an efficient BD algorithm for solving the uncapacitated multiple allocation $p$-hub center problem (UMA$p$HCP). Finally, Monemi et al. (2021) address a multi-period HLP with application to humanitarian aids distribution and tackle the problem by using a BD approach.

In order to better highlight the strength and differences of this paper and better position it within the literature, Table 1 summarizes the relevant works on the HLP under uncertainty from different aspects. According to this table, risk-averse models have not been developed and used in the study of the hub location problems. Therefore, this is the first study in the literature that deals with risk in the HLP and uses the conditional $\beta$-mean as a risk measure. In other words, the effect of risk on three important variants of the HLP under demand uncertainty is analyzed in this work. Due to successful application of BD to various HLPs in previous works, we also develop exact solution procedures based on BD to solve the proposed problems in this study.

## 3. Mathematical models

### 3.1. Deterministic models

Let $G = (N, E)$ be a network with $N$ as the set of nodes and $E$ as the set of edges. The set of candidate nodes for locating hubs is denoted by $H \subseteq N$ and $p$ nodes have to be selected for installing hub facilities. Assume that $A = \{(i, j) \in N \times N : i \neq j\}$ is the set of all O/D pairs with some traffic to be routed between each pair. For all $(i, j) \in A$, let $w_{ij}$ represent the traffic volume originated at node $i$ and destined to node $j$, and $d_{ij}$ represent the unit transportation cost of traffic from node $i$ to node $j$. Hence, the total transportation cost for routing one unit of traffic from node $i$ to node $j$ via hubs $k$ and $m$ can be calculated as:

$$c_{ijkm} = \chi d_{ik} + \alpha d_{km} + \delta d_{mj}$$

where $\chi$, $\alpha$, and $\delta$ are cost coefficients applied to collection, transfer, and distribution arcs on each route, respectively. The coefficient $\alpha$ is the volume discount factor reflecting economies of scale for transportation costs on transfer (inter-hub) arcs $(0 \leq \alpha \leq 1; \chi \geq \alpha; \delta \geq \alpha)$ (Ernst and Krishnamoorthy, 1996).

We use the non-negative decision variable $x_{ijkm}$ as the fraction of traffic from node $i \in N$ to node $j \in N$ that is routed via hubs $k \in H$ and $m \in H$ in that order. If the traffic from node $i$ to node $j$ passes via only a single hub $k$ on its path, then the corresponding fraction of flow is represented by the decision variable $x_{ijkk}$. Further, let the binary variable $y_k$ take the value of 1 if node $k \in H$ is selected as a hub and 0, otherwise. Our problems consist of locating $p$ hubs and routing the O/D traffic via installed hubs so that the desired objective function (i.e., cost or service level) is optimized.

**Table 1**
Review of related works on the HLP under uncertainty.

| Author(s)/Year | Allocation scheme | | Objective | | | | Uncertain parameters | | | | Modeling | | | | Risk | Reliability | Solution procedure | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Single | Multiple | Cost/Median | Cover | Center | Profit | Flow | Transport cost/Time | Fixed cost | Hub/Link failure | Two-/Multi-stage | Robust | Chance constraint | Bilevel | | | Solver | Heuristic | Exact | Hybrid |
| Marianov and Serra (2003) | | ✓ | ✓ | | | | ✓ | | | | | | ✓ | | | | | ✓ | | |
| Rodriguez et al. (2007) | ✓ | | ✓ | | | | ✓ | | | | | | ✓ | | | | | ✓ | | |
| Sim et al. (2009) | ✓ | | | | ✓ | | | ✓ | | | | | ✓ | | | | | ✓ | | |
| Yang (2009) | | ✓ | ✓ | | | | ✓ | | | | ✓ | | | | | | ✓ | | | |
| Kim and O'Kelly (2009) | ✓ | ✓ | ✓ | | | | | | | ✓ | | | | | | ✓ | ✓ | | | |
| Mohammadi et al. (2011) | ✓ | | | ✓ | | | ✓ | | | | | | ✓ | | | | | ✓ | | |
| Contreras et al. (2011b) | | ✓ | ✓ | | | | ✓ | | | | ✓ | | | | | | | | ✓ | |
| Alumur et al. (2012) | ✓ | ✓ | ✓ | | | | ✓ | ✓ | ✓ | | ✓ | | | | | | ✓ | | | |
| Nikokalam-Mozafar et al. (2014) | ✓ | | ✓ | ✓ | | | ✓ | ✓ | | | | | ✓ | | | | | ✓ | | |
| Shahabi and Unnikrishnan (2014) | | ✓ | ✓ | | | | ✓ | | | | | ✓ | | | | | ✓ | | | |
| An et al. (2015) | ✓ | | ✓ | | | | | | | ✓ | | | | | | ✓ | | | ✓ | |
| Ghaffari-Nasab et al. (2015) | ✓ | ✓ | ✓ | | | | ✓ | | | | | ✓ | | | | | ✓ | | | |
| Gao and Qin (2016) | ✓ | | | | ✓ | | | ✓ | | | | | ✓ | | | | | ✓ | | |
| Merakli and Yaman (2016) | | ✓ | ✓ | | | | ✓ | | | | | ✓ | | | | | | | ✓ | |
| Zetina et al. (2017) | | ✓ | ✓ | | | | ✓ | ✓ | | | | ✓ | | | | | | | ✓ | |
| Chaharsooghi et al. (2017) | | ✓ | ✓ | | | | | | | ✓ | ✓ | | | | | ✓ | | ✓ | | |
| Ghaffarinasab and Motallebzadeh (2018) | | ✓ | ✓ | ✓ | ✓ | | | | | ✓ | | | | ✓ | | | | ✓ | | |
| Ghaffarinasab and Atayi (2018) | | ✓ | ✓ | | | | | | | ✓ | | | | ✓ | | | | | ✓ | |
| Ghaffarinasab (2018) | | ✓ | ✓ | | | | ✓ | | | | | ✓ | | | | ✓ | | | | ✓ |
| Rostami et al. (2018) | ✓ | | ✓ | | | | | | | ✓ | ✓ | | | | | | | | ✓ | |
| Correia et al. (2018) | | ✓ | ✓ | | | | ✓ | | | | ✓ | | | | | | ✓ | | | |
| de Sá et al. (2018) | | ✓ | ✓ | | | | ✓ | | ✓ | | | ✓ | | | | | | | ✓ | |
| Madani et al. (2018) | ✓ | | | ✓ | | | | | | ✓ | | | | | | ✓ | | ✓ | | |
| Mohammadi et al. (2019) | ✓ | | ✓ | | | | | | | ✓ | | | ✓ | | | ✓ | | ✓ | | |
| Taherkhani et al. (2020) | | ✓ | | | | ✓ | ✓ | | | | ✓ | | | | | | | | ✓ | |
| Ghaffarinasab et al. (2020) | ✓ | | ✓ | | | | ✓ | | | | | ✓ | | | | ✓ | | | | ✓ |
| Rostami et al. (2021) | ✓ | | ✓ | | | | ✓ | | | | ✓ | | | | | | | | ✓ | |
| Shen et al. (2021) | ✓ | | ✓ | | | | | | | ✓ | | | | | | ✓ | ✓ | | | |
| Korani and Eydi (2021) | ✓ | | ✓ | | | | | | | ✓ | | | | ✓ | | ✓ | | | | ✓ |
| Ghaffarinasab (2021) | | ✓ | ✓ | | | | ✓ | | | | | ✓ | | | | | | | ✓ | |
| This paper | | ✓ | ✓ | ✓ | ✓ | | ✓ | | | | ✓ | | | | ✓ | | | | ✓ | |

Our first problem, called the uncapacitated multiple allocation $p$-hub median problem (UMA$p$HMP), is concerned with minimizing the total transportation cost incurred for transferring the O/D commodity flows. The MILP model for the deterministic version of the UMA$p$HMP can be written as (Hamacher et al., 2004):

$$\min \sum_{(i,j)\in A} \sum_{k\in H} \sum_{m\in H} w_{ij} c_{ijkm} x_{ijkm} \tag{1}$$

$$\text{s.t.:} \sum_{k\in H} y_k = p \tag{2}$$

$$\sum_{k\in H} \sum_{m\in H} x_{ijkm} = 1 \qquad \forall (i,j) \in A \tag{3}$$

$$\sum_{m\in H} x_{ijkm} + \sum_{m\in H \mid m\neq k} x_{ijmk} \leq y_k \qquad \forall (i,j) \in A, k \in H \tag{4}$$

$$x_{ijkm} \geq 0 \qquad \forall (i,j) \in A, k, m \in H \tag{5}$$

$$y_k \in \{0,1\} \qquad \forall k \in H \tag{6}$$

The objective function (1) aims at minimizing the total transportation cost for the O/D traffic. The number of installed hubs is fixed by constraint (2). Constraints (3) implies that the entire traffic corresponding to each O/D pair is routed through some hub pair. Constraints (4) ensure the O/D traffic can only be routed via intermediate hub nodes. Non-negativity and binary conditions on decision variables are imposed by (5) and (6), respectively.

In case of the uncapacitated multiple allocation $p$-hub maximal covering problem (UMA$p$HMCP), we first need to define the concept of coverage. The O/D flow $w_{ij}$ is said to be covered if its cost of routing via some pair of installed hubs does not exceed a threshold value $R$. Accordingly, we use a binary parameter $a_{ijkm}$ as follows:

$$a_{ijkm} = \begin{cases} 1, & \text{if } c_{ijkm} \leq R \\ 0, & \text{otherwise} \end{cases} \qquad \forall (i,j) \in A, k, m \in H. \tag{7}$$

In other words, if the cost of routing unit traffic from node $i \in N$ to node $j \in N$ via hubs $k \in H$ and $m \in H$ does not exceed the value $R$, the parameter $a_{ijkm}$ takes value of 1 and it takes 0, otherwise. As it is defined in (7), $a_{ijkm}$ takes the value of 1 if the cost of routing the flow $w_{ij}$ via the hub pair $(k,m)$ does not exceed the threshold value $R$; and 0, otherwise. The MILP model for the deterministic version of the uncapacitated multiple allocation $p$-hub maximal covering problem (UMA$p$HMCP) can now be written as:

$$\max \sum_{(i,j)\in A} \sum_{k\in H} \sum_{m\in H} w_{ij} a_{ijkm} x_{ijkm} \tag{8}$$

s.t.: (2) – (6)

The objective function (8) maximizes the total covered O/D traffic in the network.

Finally, the MILP model for the deterministic version of the uncapacitated multiple allocation weighted $p$-hub center problem (UMAW$p$HCP) can be written as:

$$\min \quad z \tag{9}$$

s.t.: (2) – (6)

$$z \geq \sum_{k\in H} \sum_{m\in H} w_{ij} c_{ijkm} x_{ijkm} \qquad \forall (i,j) \in A \tag{10}$$

$$z \geq 0 \tag{11}$$

The objective function (9) along with the constraints (10) minimize the maximum weighted transportation cost among all the O/D pairs.

### 3.2. Risk-averse formulations using conditional $\beta$-mean criterion

We now assume that the O/D flows are uncertain and the associated uncertainty is captured as a finite set of discrete scenarios represented by $\Omega$. The number of scenarios is denoted by $S$ (i.e., $|\Omega| = S$) and we assume that the scenarios are equally likely (they probability of occurrence for each scenario is $1/S$). Under each scenario $s \in \Omega$, let $w^s_{ij}$ denote the corresponding realized demand from node $i \in N$ to node $j \in N$. In a similar manner, we define the second-stage decision variable $x^s_{ijkm}$ as the fraction of flow $w^s_{ij}$ that is sent from node $i \in N$ to node $j \in N$ using the link between the hubs $k \in H$ and $m \in H$ under scenario $s \in \Omega$. In our risk-averse problems, we are interested in system's performance under a subset of worst-case scenarios. In other words, for any fraction $\beta$, with $\beta \in (0,1]$, our objective is to determine the location of hubs as well as the routing of O/D traffic in such a way that the average performance of the system over the worst $\lceil \beta S \rceil$ scenarios is optimized.

In case of the $p$-hub median problem, the conditional $\beta$-mean risk-averse formulation can be written as:

$$\min_{(x,y)\in X} \max \left\{ \frac{1}{\lceil \beta S \rceil} \sum_{s\in\Omega} \Big( \sum_{(i,j)\in A} \sum_{k\in H} \sum_{m\in H} w^s_{ij} c_{ijkm} x^s_{ijkm} \Big) r^s : \sum_{s\in\Omega} r^s = \lceil \beta S \rceil, r^s \in \{0,1\}, \forall s \in \Omega \right\} \tag{12}$$

where $r^s$ is a binary variable that takes the value of 1 if the scenario $s$ belongs to the set of worst $\lceil \beta S \rceil$ scenarios and takes zero otherwise and $X$ is the feasibility set decision variables $(x,y)$ in the upper level (outer) problem that satisfy the following constraints:

$$\sum_{k\in H} y_k = p \tag{13}$$

$$\sum_{k\in H} \sum_{m\in H} x^s_{ijkm} = 1 \qquad \forall (i,j) \in A, s \in \Omega \tag{14}$$

$$\sum_{m\in H} x^s_{ijkm} + \sum_{m\in H | m\neq k} x^s_{ijmk} \leq y_k \qquad \forall (i,j) \in A, k \in H, s \in \Omega \tag{15}$$

$$x^s_{ijkm} \geq 0 \qquad \forall (i,j) \in A, k, m \in H, s \in \Omega \tag{16}$$

$$y_k \in \{0,1\} \qquad \forall k \in H \tag{17}$$

Note that the inner problem in (12) is a binary knapsack problem and its coefficient matrix is totally unimodular enabling us to relax the integrality constraints on $r^s$ variables as $0 \leq r^s \leq 1$, for all $s \in \Omega$. Hence, the inner problem is turned into a continuous knapsack problem which can be solved by a greedy algorithm. The resulting model is nonlinear and therefore we use the dual transformation method in order to linearize it. By fixing the upper level decision variables as $(\hat{x}, \hat{y})$, the dual of the lower level (inner) linear programming model is obtained and the whole problem is written as a pure minimization problem. Accordingly, we can formulate the risk-averse $p$-hub median problem (RA$p$HMP) using conditional $\beta$-mean criterion as follows:

$$\min \lceil \beta S \rceil \mu + \sum_{s\in\Omega} \lambda^s \tag{18}$$

s.t.: (13) – (17)

$$\mu + \lambda^s \geq \frac{1}{\lceil \beta S \rceil} \sum_{(i,j)\in A} \sum_{k\in H} \sum_{m\in H} w_{ij}^s c_{ijkm} x_{ijkm}^s \qquad \forall s \in \Omega \tag{19}$$

$$\lambda^s \geq 0 \qquad \forall s \in \Omega \tag{20}$$

where $\lambda^s$ is the dual variable associated with the constraint $r^s \leq 1$, for all $s \in \Omega$, and $\mu$ is the dual variable corresponding to constraint $\sum_{s\in\Omega} r^s = \lceil \beta S \rceil$.

For the $p$-hub maximal covering problem, the aim is to locate the hubs and to route the O/D traffic so that the average total covered flow over the worst $\lceil \beta S \rceil$ scenarios is maximized:

$$\max_{(x,y)\in X} \min \left\{ \frac{1}{\lceil \beta S \rceil} \sum_{s\in\Omega} \Big( \sum_{(i,j)\in A} \sum_{k\in H} \sum_{m\in H} w_{ij}^s a_{ijkm} \hat{x}_{ijkm}^s \Big) r^s : \sum_{s\in\Omega} r^s = \lceil \beta S \rceil, 0 \leq r^s \leq 1, \forall s \in \Omega \right\} \tag{21}$$

The inner problem in (21) is a continuous min-knapsack problem which can be solved using a greedy algorithm.

According to earlier discussions, the MILP model for the risk-averse $p$-hub maximal covering problem (RA$p$HMCP) using conditional $\beta$-mean criterion can be stated as follows:

$$\max \lceil \beta S \rceil \mu - \sum_{s\in\Omega} \lambda^s \tag{22}$$

s.t.: (13) – (17)

$$\mu - \lambda^s \leq \frac{1}{\lceil \beta S \rceil} \sum_{(i,j)\in A} \sum_{k\in H} \sum_{m\in H} w_{ij}^s a_{ijkm} x_{ijkm}^s \qquad \forall s \in \Omega \tag{23}$$

$$\lambda^s \geq 0 \qquad \forall s \in \Omega \tag{24}$$

In case of the weighted $p$-hub center problem, the objective is to locate $p$ hubs and allocate of the O/D demands in such a way that the average maximum weighted cost over the worst $\lceil \beta S \rceil$ scenarios is minimized:

$$\min_{(x,y)\in X} \max \left\{ \frac{1}{\lceil \beta S \rceil} \sum_{s\in\Omega} \sum_{(i,j)\in A} \Big( \sum_{k\in H} \sum_{m\in H} w_{ij}^s c_{ijkm} \hat{x}_{ijkm}^s \Big) r_{ij}^s : \sum_{s\in\Omega} \sum_{(i,j)\in A} r_{ij}^s = \lceil \beta S \rceil, \right. \tag{25}$$
$$\left. \sum_{(i,j)\in A} r_{ij}^s \leq 1, \forall s \in \Omega, 0 \leq r_{ij}^s \leq 1, \forall (i,j) \in A, s \in \Omega \right\}$$

where $r_{ij}^s$ takes the value of 1 if the scenario $s$ belongs to the set of worst $\lceil \beta S \rceil$ scenarios and the O/D pair $(i,j)$ represents the maximum cost in that scenario and takes zero otherwise. Note that the inner problem in (25) is a continuous bounded knapsack problem which can also be solved by a greedy algorithm. By taking the dual of the inner problem, we can formulate the risk-averse weighted $p$-hub center problem (RAW$p$HCP) as follows:

$$\min \lceil \beta S \rceil \psi + \sum_{s\in\Omega} \varphi^s + \sum_{s\in\Omega} \sum_{(i,j)\in A} \zeta_{ij}^s \tag{26}$$

s.t.: (13) – (17)

$$\psi + \varphi^s + \zeta_{ij}^s \geq \frac{1}{\lceil \beta S \rceil} \sum_{k\in H} \sum_{m\in H} w_{ij}^s c_{ijkm} x_{ijkm}^s \qquad \forall (i,j) \in A, s \in \Omega \tag{27}$$

$$\zeta_{ij}^s \geq 0 \qquad \forall (i,j) \in A, s \in \Omega \tag{28}$$

$$\varphi^s \geq 0 \qquad \forall s \in \Omega \tag{29}$$

where $\zeta_{ij}^s$ is the dual variable associated with the constraint $r_{ij}^s \leq 1$, for all $(i,j) \in A, s \in \Omega$, $\varphi^s$ is the dual variable associated with the constraint $\sum_{(i,j)\in A} r_{ij}^s \leq 1$, for all $s \in \Omega$, and $\psi$ is the dual variable corresponding to the constraint $\sum_{s\in\Omega} \sum_{(i,j)\in A} r_{ij}^s = \lceil \beta S \rceil$.

**Proposition 1.** *If $(\beta S)$ is an integer value, the conditional $\beta$-mean model is equivalent to $(1-\beta)$-CVaR optimization model for uniform finite probability distributions.*

**Proof.** We prove the claim for the $p$-hub median problem and leave out the two other versions to the reader. Assuming that $(\beta S)$ is an integer value, we can rewrite the RA$p$HMP as follows:

$$\min \quad (\beta S)\mu + \sum_{s\in\Omega} \lambda^s \tag{30}$$

s.t.: (13) – (17), (20)

$$\mu + \lambda^s \geq \frac{1}{(\beta S)} \sum_{(i,j)\in A} \sum_{k\in H} \sum_{m\in H} w_{ij}^s c_{ijkm} x_{ijkm}^s \qquad \forall s \in \Omega \tag{31}$$

Now we define new variables $\eta$ and $v^s$ as $\eta = (\beta S)\mu$ and $v^s = (\beta S)\lambda^s$ for all $s \in \Omega$. Let $p^s$ be the probability of scenario $s \in \Omega$ and since the scenarios are assumed to have equal probabilities, we set $p^s = \frac{1}{S}$. Replacing the new variables in the above model we have:

$$\min \quad \eta + \frac{1}{\beta} \sum_{s \in \Omega} p^s v^s \tag{32}$$

s.t.: (13) – (17)

$$\eta + v^s \geq \sum_{(i,j) \in A} \sum_{k \in H} \sum_{m \in H} w^s_{ij} c_{ijkm} x^s_{ijkm} \qquad \forall s \in \Omega \tag{33}$$

$$v^s \geq 0 \qquad \forall s \in \Omega \tag{34}$$

Setting $\gamma = 1 - \beta$, we have $\gamma \in [0, 1)$ and replacing it in the above model we have:

$$\min \quad \eta + \frac{1}{(1-\gamma)} \sum_{s \in \Omega} p^s v^s \tag{35}$$

s.t.: (13) – (17)

$$v^s \geq \sum_{(i,j) \in A} \sum_{k \in H} \sum_{m \in H} w^s_{ij} c_{ijkm} x^s_{ijkm} - \eta \qquad \forall s \in \Omega \tag{36}$$

$$v^s \geq 0 \qquad \forall s \in \Omega \tag{37}$$

which is a $\gamma$-CVaR or $(1-\beta)$-CVaR optimization model for finite probability space as defined by Rockafellar and Uryasev (2000). □

## 4. Benders reformulation

BD is a partitioning method for solving mixed-integer programming problems. In this approach the problem is partitioned into two simpler problems: a master problem, and a slave problem which are solved iteratively until the algorithm converges to an optimal solution (if one exits) for the original mixed-integer program. Our proposed algorithms, however, are based on a modern implementation of BD in which a single search tree is employed to solve the master problem in a branch-and-cut setting which is possible thanks to recent developments in commercial solvers. This strategy for BD implementation is called Branch-and-Benders-cut and has shown better performance over the traditional implementation for many problems (Rahmaniani et al., 2017). The Benders cuts are separated and added to the master problem on the fly whenever an incumbent integer solution is found for the master problem. We also devise a special algorithm for solving the dual SPs by inspection (without resorting to standard solver) in order to further accelerate the convergence of our algorithms.

### 4.1. *The risk-averse p-hub median problem*

We fix the binary variables vector as $\boldsymbol{y} = \hat{\boldsymbol{y}}$ in order to write the slave problem (SP) for the RA$p$HMP as:

$$\min \lceil \beta S \rceil \mu + \sum_{s \in \Omega} \lambda^s \tag{38}$$

$$\text{s.t.: } \mu + \lambda^s \geq \frac{1}{\lceil \beta S \rceil} \sum_{(i,j) \in A} \sum_{k \in H} \sum_{m \in H} w^s_{ij} c_{ijkm} x^s_{ijkm} \qquad \forall s \in \Omega \tag{39}$$

$$\sum_{m \in H} x^s_{ijkm} + \sum_{m \in H | m \neq k} x^s_{ijmk} \leq \hat{y}_k \qquad \forall (i,j) \in A, k \in H, s \in \Omega \tag{40}$$

$$\sum_{k \in H} \sum_{m \in H} x^s_{ijkm} = 1 \qquad \forall (i,j) \in A, s \in \Omega \tag{41}$$

$$x^s_{ijkm} \geq 0 \qquad \forall (i,j) \in A, k, m \in H, s \in \Omega \tag{42}$$

$$\lambda^s \geq 0 \qquad \forall s \in \Omega \tag{43}$$

Let $\rho^s$, $\pi^s_{ijk}$, and $\sigma^s_{ij}$ be the dual variable of the SP constraints (39), (40), and (41), respectively. We can derive the dual slave problem (DSP) for the RA$p$HMP as follows:

$$\max \sum_{(i,j) \in A} \sum_{s \in \Omega} \sigma^s_{ij} - \sum_{(i,j) \in A} \sum_{k \in H} \sum_{s \in \Omega} \hat{y}_k \pi^s_{ijk} \tag{44}$$

$$\text{s.t.: } \sigma^s_{ij} - \pi^s_{ijk} - \pi^s_{ijm} \leq \frac{1}{\lceil \beta S \rceil} w^s_{ij} c_{ijkm} \rho^s \qquad \forall (i,j) \in A, k, m \in H, (k \neq m), s \in \Omega \tag{45}$$

$$\sigma^s_{ij} - \pi^s_{ijk} \leq \frac{1}{\lceil \beta S \rceil} w^s_{ij} c_{ijkk} \rho^s \qquad \forall (i,j) \in A, k \in H, s \in \Omega \tag{46}$$

$$\sum_{s \in \Omega} \rho^s = \lceil \beta S \rceil \tag{47}$$

$$0 \leq \rho^s \leq 1 \qquad \forall s \in \Omega \tag{48}$$

$$\pi^s_{ijk} \geq 0 \qquad \forall (i,j) \in A, k \in H, s \in \Omega \tag{49}$$

The master problem (MP) for the RA$p$HMP can now be written as follows:

$$\min \theta \tag{50}$$

$$\text{s.t.: } \theta \geq \sum_{(i,j)\in A} \sum_{s\in\Omega} \sigma_{ij}^{sv} - \sum_{(i,j)\in A} \sum_{k\in H} \sum_{s\in\Omega} \pi_{ijk}^{sv} y_k \qquad v = 1,\dots,V \tag{51}$$

(13), (17)

in which $(\sigma^v, \pi^v)$ denotes $v$th extreme point of the feasible solution space of the DSP (44)–(49). Note the by installing $p$ hubs which is implied by (13), feasibility of the SP is guaranteed, Therefore, we do not need to add feasibility cuts to the master problem.

By exploring the special structure of the DSP, we devise an algorithm for solving it without using a standard solver. The proposed method substantially reduces the computational burden of the algorithm. Our algorithm starts with determining the optimal values of $\rho^s$ variables. Note that $\rho^s$ variables are equivalent to $r^s$ variables in (12) and since the inner problem in (12) is a knapsack problem, we devise a greedy algorithm for obtaining the optimal values of $\rho^s$ variables. Based on the proposed greedy algorithm, the $\rho^s$ variables associated with $\lceil \beta S \rceil$ scenarios having the largest transportation costs will take the value 1 and the remaining variables will take the value of 0. The pseudo-code for the proposed greedy algorithm for determining the optimal values of $\rho^s$ variables for the RA$p$HMP is presented in Algorithm 1.

**Algorithm 1** : Greedy algorithm for determining the optimal values for $\rho^s$ variables for the RA$p$HMP

**for** all $s \in \Omega$ **do**
  $v^s \leftarrow 0$
  **for** all $(i,j) \in A$ **do**
    $v^s \leftarrow v^s + w_{ij}^s \min_{k,m\in H^1} \{c_{ijkm}\}$
  **end for**
**end for**
$I \leftarrow \emptyset$
$J \leftarrow \Omega$
**for** $r = 1, \dots, \lceil \beta S \rceil$ **do**
  $v_{max} \leftarrow -\infty$
  $i_{max} \leftarrow 0$
  **for** all $s \in J$ **do**
    **if** $v_{max} < v^s$ **then**
      $v_{max} \leftarrow v^s$
      $i_{max} \leftarrow s$
    **end if**
  **end for**
  $I \leftarrow I \cup \{i_{max}\}$
  $J \leftarrow J \setminus \{i_{max}\}$
**end for**
**for** all $s \in I$ **do**
  $\rho^s \leftarrow 1$
**end for**
**for** all $s \in J$ **do**
  $\rho^s \leftarrow 0$
**end for**

In the pseudo-code, $H^1$ represents the set of nodes that are selected as hubs at any iteration. After obtaining the optimal values for the $\rho^s$ variables, we can re-write the reduced DSP for the RA$p$HMP as follows:

$$\max \sum_{(i,j)\in A} \sum_{s\in I} \sigma_{ij}^s - \sum_{(i,j)\in A} \sum_{k\in H} \sum_{s\in I} \hat{y}_k \pi_{ijk}^s \tag{52}$$

$$\text{s.t.: } \sigma_{ij}^s - \pi_{ijk}^s - \pi_{ijm}^s \leq \frac{1}{\lceil \beta S \rceil} w_{ij}^s c_{ijkm} \qquad \forall (i,j) \in A, k, m \in H, (k \neq m), s \in I \tag{53}$$

$$\sigma_{ij}^s - \pi_{ijk}^s \leq \frac{1}{\lceil \beta S \rceil} w_{ij}^s c_{ijkk} \qquad \forall (i,j) \in A, k \in H, s \in I \tag{54}$$

$$\pi_{ijk}^s \geq 0 \qquad \forall (i,j) \in A, k \in H, s \in I \tag{55}$$

in which $I$ is the set of scenarios for which the corresponding variables $\rho^s$ take the value of 1 as determined by Algorithm 1. This problem in turn can be solved by inspection separately for each scenario $s \in I$ using a similar method proposed by Contreras et al. (2011a) as shown in Algorithm 2.

**Algorithm 2** : Determining the optimal values for $\sigma_{ij}^s$ and $\pi_{ijk}^s$ variables for the RA$p$HMP

**for** all $s \in I$ **do**
  **for** all $(i,j) \in A$ **do**
    $\sigma_{ij}^s \leftarrow \frac{1}{\lceil \beta S \rceil} w_{ij}^s \min_{k,m \in H^1} \{c_{ijkm}\}$
    **for** all $k \in H^1$ **do**
      $\pi_{ijk}^s \leftarrow 0$
    **end for**
    **for** all $k \in H^0$ **do**
      $\pi_{ijk}^s \leftarrow \max\{0, \sigma_{ij}^s - \frac{1}{\lceil \beta S \rceil} w_{ij}^s c_{ijkk}\}$
      **for** all $m \in H^1$ **do**
        $\pi_{ijk}^s \leftarrow \max\{\pi_{ijk}^s, \max\{\sigma_{ij}^s - \frac{1}{\lceil \beta S \rceil} w_{ij}^s c_{ijkm}, \sigma_{ij}^s - \frac{1}{\lceil \beta S \rceil} w_{ij}^s c_{ijmk}\}\}$
      **end for**
    **end for**
    **for** all $(k,m) \in H^0 \times H^0, (k \neq m)$ **do**
      $\Delta \leftarrow \sigma_{ij}^s - \min\{\frac{1}{\lceil \beta S \rceil} w_{ij}^s c_{ijkm}, \frac{1}{\lceil \beta S \rceil} w_{ij}^s c_{ijmk}\} - \pi_{ijk}^s - \pi_{ijm}^s$
      **if** $\Delta > 0$ **then**
        $\pi_{ijk}^s \leftarrow \pi_{ijk}^s + \frac{\Delta}{2}$
        $\pi_{ijm}^s \leftarrow \pi_{ijm}^s + \frac{\Delta}{2}$
      **end if**
    **end for**
  **end for**
**end for**

### 4.2. *The risk-averse p-hub maximal covering problem*

By fixing the binary location variables vector as $\boldsymbol{y} = \hat{\boldsymbol{y}}$, the slave problem for the RA$p$HMCP can be written as:

$$\max \lceil \beta S \rceil \mu - \sum_{s \in \Omega} \lambda^s \tag{56}$$

$$\text{s.t.: } \mu - \lambda^s \leq \frac{1}{\lceil \beta S \rceil} \sum_{(i,j) \in A} \sum_{k \in H} \sum_{m \in H} w_{ij}^s a_{ijkm} x_{ijkm}^s \qquad \forall s \in \Omega \tag{57}$$

$$\sum_{m \in H} x_{ijkm}^s + \sum_{m \in H | m \neq k} x_{ijmk}^s \leq \hat{y}_k \qquad \forall (i,j) \in A, k \in H, s \in \Omega \tag{58}$$

$$\sum_{k \in H} \sum_{m \in H} x_{ijkm}^s = 1 \qquad \forall (i,j) \in A, s \in \Omega \tag{59}$$

$$x_{ijkm}^s \geq 0 \qquad \forall (i,j) \in A, k, m \in H, s \in \Omega \tag{60}$$

$$\lambda^s \geq 0 \qquad \forall s \in \Omega \tag{61}$$

Let $\rho^s$, $\pi_{ijk}^s$, and $\sigma_{ij}^s$ respectively represent the dual variable associated with constraints (57), (58), and (59). Accordingly, the DSP for the RA$p$HMCP is stated as:

$$\min \sum_{(i,j) \in A} \sum_{s \in \Omega} \sigma_{ij}^s + \sum_{(i,j) \in A} \sum_{k \in H} \sum_{s \in \Omega} \hat{y}_k \pi_{ijk}^s \tag{62}$$

$$\text{s.t.: } \sigma_{ij}^s + \pi_{ijk}^s + \pi_{ijm}^s \geq \frac{1}{\lceil \beta S \rceil} w_{ij}^s a_{ijkm} \rho^s \qquad \forall (i,j) \in A, k, m \in H, (k \neq m), s \in \Omega \tag{63}$$

$$\sigma_{ij}^s + \pi_{ijk}^s \geq \frac{1}{\lceil \beta S \rceil} w_{ij}^s a_{ijkk} \rho^s \qquad \forall (i,j) \in A, k \in H, s \in \Omega \tag{64}$$

$$\sum_{s \in \Omega} \rho^s = \lceil \beta S \rceil \tag{65}$$

$$0 \leq \rho^s \leq 1 \qquad \forall s \in \Omega \tag{66}$$

$$\pi_{ijk}^s \geq 0 \qquad \forall (i,j) \in A, k \in H, s \in \Omega \tag{67}$$

The master problem for the RA$p$HMCP can be written as:

$$\max \theta \tag{68}$$

$$\text{s.t.: } \theta \leq \sum_{(i,j) \in A} \sum_{s \in \Omega} \sigma_{ij}^{sv} + \sum_{(i,j) \in A} \sum_{k \in H} \sum_{s \in \Omega} \pi_{ijk}^{sv} y_k \qquad v = 1, \ldots, V \tag{69}$$

(2), (6)

In order to determine the optimal values of $\rho^s$ variables in the DSP (62)–(67), we devise a similar greedy procedure to Algorithm 1. Since the inner problem in (21) is a continuous min-knapsack problem, the $\rho^s$ variables corresponding to $\lceil \beta S \rceil$ scenarios having the smallest coverage percentage will take the value of 1 and the remaining variables will be 0 in an optimal solution. Algorithm 3 presents the pseudo-code for the greedy algorithm proposed for determining the optimal values of $\rho^s$ variables for the RA$p$HMCP.

**Algorithm 3** : Greedy algorithm for determining the optimal values for $\rho^s$ variables for the RA$p$HMCP

**for** all $s \in \Omega$ **do**
  $v^s \leftarrow 0$
  **for** all $(i,j) \in A$ **do**
    $v^s \leftarrow v^s + w_{ij}^s \max\limits_{k,m \in H^1} \{a_{ijkm}\}$
  **end for**
**end for**
$I \leftarrow \emptyset$
$J \leftarrow \Omega$
**for** $r = 1, \ldots, \lceil \beta S \rceil$ **do**
  $v_{min} \leftarrow +\infty$
  $i_{min} \leftarrow 0$
  **for** all $s \in J$ **do**
    **if** $v_{min} > v^s$ **then**
      $v_{min} \leftarrow v^s$
      $i_{min} \leftarrow s$
    **end if**
  **end for**
  $I \leftarrow I \cup \{i_{min}\}$
  $J \leftarrow J \setminus \{i_{min}\}$
**end for**
**for** all $s \in I$ **do**
  $\rho^s \leftarrow 1$
**end for**
**for** all $s \in J$ **do**
  $\rho^s \leftarrow 0$
**end for**

By replacing the optimal values for the $\rho^s$ variables, the DSP for the RA$p$HMCP can be rewritten as the following linear programming problem:

$$\min \sum_{(i,j)\in A} \sum_{s \in I} \sigma_{ij}^s + \sum_{(i,j)\in A} \sum_{k \in H} \sum_{s \in I} \hat{y}_k \pi_{ijk}^s \tag{70}$$

$$\text{s.t.: } \sigma_{ij}^s + \pi_{ijk}^s + \pi_{ijm}^s \geq \frac{1}{\lceil \beta S \rceil} w_{ij}^s a_{ijkm} \qquad \forall (i,j) \in A, k,m \in H, (k \neq m), s \in I \tag{71}$$

$$\sigma_{ij}^s + \pi_{ijk}^s \geq \frac{1}{\lceil \beta S \rceil} w_{ij}^s a_{ijkk} \qquad \forall (i,j) \in A, k \in H, s \in I \tag{72}$$

$$\pi_{ijk}^s \geq 0 \qquad \forall (i,j) \in A, k \in H, s \in I \tag{73}$$

The method proposed by Contreras et al. (2011a) cannot be used for solving the reduced DSP (70)–(73). Therefore, we devise a similar method for obtaining the optimal values for $\sigma_{ij}^s$ and $\pi_{ijk}^s$ variables for the RA$p$HMCP as shown in Algorithm 4.

**Algorithm 4** : Determining the optimal values for $\sigma_{ij}^s$ and $\pi_{ijk}^s$ variables for the RA$p$HMCP

**for** all $s \in I$ **do**
  **for** all $(i,j) \in A$ **do**
    $\sigma_{ij}^s \leftarrow \frac{1}{\lceil \beta S \rceil} w_{ij}^s \max\limits_{k,m \in H^1} \{a_{ijkm}\}$
    **for** all $k \in H^1$ **do**
      $\pi_{ijk}^s \leftarrow 0$
    **end for**
    **for** all $k \in H^0$ **do**
      $\pi_{ijk}^s \leftarrow \max\{0, \frac{1}{\lceil \beta S \rceil} w_{ij}^s a_{ijkk} - \sigma_{ij}^s\}$
      **for** all $m \in H^1$ **do**
        $\pi_{ijk}^s \leftarrow \max\{\pi_{ijk}^s, \max\{\frac{1}{\lceil \beta S \rceil} w_{ij}^s a_{ijkm} - \sigma_{ij}^s, \frac{1}{\lceil \beta S \rceil} w_{ij}^s a_{ijmk} - \sigma_{ij}^s\}\}$
      **end for**
    **end for**
    **for** all $(k,m) \in H^0 \times H^0, (k \neq m)$ **do**
      $\Delta \leftarrow \max\{\frac{1}{\lceil \beta S \rceil} w_{ij}^s a_{ijkm}, \frac{1}{\lceil \beta S \rceil} w_{ij}^s a_{ijmk}\} - \sigma_{ij}^s - \pi_{ijk}^s - \pi_{ijm}^s$
      **if** $\Delta > 0$ **then**
        $\pi_{ijk}^s \leftarrow \pi_{ijk}^s + \frac{\Delta}{2}$
        $\pi_{ijm}^s \leftarrow \pi_{ijm}^s + \frac{\Delta}{2}$
      **end if**
    **end for**
  **end for**
**end for**

In the above algorithm, $H^0$ is the set of nodes that are not selected as hubs at the current iteration.

*4.3. The risk-averse weighted p-hub center problem*

By fixing the binary location variables vector as $\boldsymbol{y} = \hat{\boldsymbol{y}}$, the SP for RAW$p$HCP can be written as:

$$\min \lceil \beta S \rceil \psi + \sum_{s \in \Omega} \varphi^s + \sum_{(i,j) \in A} \sum_{s \in \Omega} \zeta_{ij}^s \tag{74}$$

$$\text{s.t.: } \psi + \varphi^s + \zeta_{ij}^s \geq \frac{1}{\lceil \beta S \rceil} \sum_{k \in H} \sum_{m \in H} w_{ij}^s c_{ijkm} x_{ijkm}^s \quad \forall (i,j) \in A, s \in \Omega \tag{75}$$

$$\sum_{m \in H} x_{ijkm}^s + \sum_{m \in H | m \neq k} x_{ijmk}^s \leq \hat{y}_k \quad \forall (i,j) \in A, k \in H, s \in \Omega \tag{76}$$

$$\sum_{k \in H} \sum_{m \in H} x_{ijkm}^s = 1 \quad \forall (i,j) \in A, s \in \Omega \tag{77}$$

$$x_{ijkm}^s \geq 0 \quad \forall (i,j) \in A, k, m \in H, s \in \Omega \tag{78}$$

$$\zeta_{ij}^s \geq 0 \quad \forall (i,j) \in A, s \in \Omega \tag{79}$$

$$\varphi^s \geq 0 \quad \forall s \in \Omega \tag{80}$$

Let $\rho_{ij}^s$, $\pi_{ijk}^s$, and $\sigma_{ij}^s$ denote the dual variable corresponding to the constraints (75), (76), and (77), respectively. We can write the DSP for the RAW$p$HCP as:

$$\max \sum_{(i,j) \in A} \sum_{s \in \Omega} \sigma_{ij}^s - \sum_{(i,j) \in A} \sum_{k \in H} \sum_{s \in \Omega} \hat{y}_k \pi_{ijk}^s \tag{81}$$

$$\text{s.t.: } \sigma_{ij}^s - \pi_{ijk}^s - \pi_{ijm}^s \leq \frac{1}{\lceil \beta S \rceil} w_{ij}^s c_{ijkm} \rho_{ij}^s \quad \forall (i,j) \in A, k, m \in H, (k \neq m), s \in \Omega \tag{82}$$

$$\sigma_{ij}^s - \pi_{ijk}^s \leq \frac{1}{\lceil \beta S \rceil} w_{ij}^s c_{ijkk} \rho_{ij}^s \quad \forall (i,j) \in A, k \in H, s \in \Omega \tag{83}$$

$$\sum_{(i,j) \in A} \sum_{s \in \Omega} \rho_{ij}^s = \lceil \beta S \rceil \tag{84}$$

$$\sum_{(i,j) \in A} \rho_{ij}^s \leq 1 \quad \forall s \in \Omega \tag{85}$$

$$0 \leq \rho_{ij}^s \leq 1 \quad \forall (i,j) \in A, s \in \Omega \tag{86}$$

$$\pi_{ijk}^s \geq 0 \quad \forall (i,j) \in A, k \in H, s \in \Omega \tag{87}$$

The master problem for the RAW$p$HCP can now be formulated as:

$$\min \theta \tag{88}$$

$$\text{s.t.: } \theta \geq \sum_{(i,j) \in A} \sum_{s \in \Omega} \sigma_{ij}^{sv} - \sum_{(i,j) \in A} \sum_{k \in H} \sum_{s \in \Omega} \pi_{ijk}^{sv} y_k \quad v = 1, \ldots, V \tag{89}$$

(2), (6)

Now we determine the optimal values of $\rho_{ij}^s$ variables in the DSP (81)–(87). Since the inner problem in (25) is a continuous minimum knapsack problem, based on the proposed greedy algorithm, the optimal value of variable $\rho_{ij}^s$ associated with $\lceil \beta S \rceil$ scenarios that have the largest transportation cost among all O/D pairs in each scenario will be 1 and the optimal values of the remaining $\rho_{ij}^s$ variables will be 0. Algorithm 5 shows the pseudo-code for greedy algorithm proposed for determining the optimal values of $\rho^s$ variables for the RAW$p$HCP.

After calculating the optimal values for the $\rho_{ij}^s$ variables, the DSP for the RAW$p$HCP can be reduced to the following linear programming problem:

$$\max \sum_{s \in I} \sigma_{o^s d^s}^s - \sum_{k \in H} \sum_{s \in I} \hat{y}_k \pi_{o^s d^s k}^s \tag{90}$$

$$\text{s.t.: } \sigma_{o^s d^s}^s - \pi_{o^s d^s k}^s - \pi_{o^s d^s m}^s \leq \frac{1}{\lceil \beta S \rceil} w_{o^s d^s}^s c_{o^s d^s km} \quad \forall k, m \in H, (k \neq m), s \in I \tag{91}$$

$$\sigma_{o^s d^s}^s - \pi_{o^s d^s k}^s \leq \frac{1}{\lceil \beta S \rceil} w_{o^s d^s}^s c_{o^s d^s kk} \quad \forall k \in H, s \in I \tag{92}$$

$$\pi_{o^s d^s k}^s \geq 0 \quad \forall k \in H, s \in I \tag{93}$$

**Algorithm 5** : Greedy algorithm for determining the optimal values for $\rho^s_{ij}$ variables for the RAW$p$HCP

**for** all $s \in \Omega$ **do**
  $v^s \leftarrow -\infty$
  $(o^s, d^s) \leftarrow (0, 0)$
  **for** all $(i, j) \in A$ **do**
    **if** $v^s < w^s_{ij} \min_{k,m \in H^1} \{c_{ijkm}\}$ **then**
      $(o^s, d^s) \leftarrow (i, j)$
      $v^s \leftarrow w^s_{ij} \min_{k,m \in H^1} \{c_{ijkm}\}$
    **end if**
  **end for**
**end for**
$I \leftarrow \emptyset$
$J \leftarrow \Omega$
**for** $r = 1, \ldots, \lceil \beta S \rceil$ **do**
  $v_{max} \leftarrow -\infty$
  $i_{max} \leftarrow 0$
  **for** all $s \in J$ **do**
    **if** $v_{max} < v^s$ **then**
      $v_{max} \leftarrow v^s$
      $i_{max} \leftarrow s$
    **end if**
  **end for**
  $I \leftarrow I \cup \{i_{max}\}$
  $J \leftarrow J \setminus \{i_{max}\}$
**end for**
**for** all $s \in I$ **do**
  $\rho^s_{o^s d^s} \leftarrow 1$
**end for**
**for** all $s \in J$ **do**
  $\rho^s_{o^s d^s} \leftarrow 0$
**end for**

where $o^s$ and $d^s$ are respectively the origin and destination nodes of the O/D pair that corresponds to the maximum weighted cost among all O/D pairs under scenario $s \in I$ which are determined by Algorithm 5. This problem can also be solved by inspection separately for each scenario $s \in I$ by using Algorithm 6.

**Algorithm 6** : Determining the optimal values for $\sigma^s_{ij}$ and $\pi^s_{ijk}$ variables for the RA$p$HCP

**for** all $s \in I$ **do**
  $\sigma^s_{o^s d^s} \leftarrow \frac{1}{\lceil \beta S \rceil} w^s_{o^s d^s} \min_{k,m \in H^1} \{c_{o^s d^s km}\}$
  **for** all $k \in H^1$ **do**
    $\pi^s_{o^s d^s k} \leftarrow 0$
  **end for**
  **for** all $k \in H^0$ **do**
    $\pi^s_{o^s d^s k} \leftarrow \max\{0, \sigma^s_{o^s d^s} - \frac{1}{\lceil \beta S \rceil} w^s_{o^s d^s} c_{o^s d^s kk}\}$
    **for** all $m \in H^1$ **do**
      $\pi^s_{o^s d^s k} \leftarrow \max\{\pi^s_{o^s d^s k}, \max\{\sigma^s_{o^s d^s} - \frac{1}{\lceil \beta S \rceil} w^s_{o^s d^s} c_{o^s d^s km}, \sigma^s_{o^s d^s} - \frac{1}{\lceil \beta S \rceil} w^s_{o^s d^s} c_{o^s d^s mk}\}\}$
    **end for**
  **end for**
  **for** all $(k, m) \in H^0 \times H^0, (k \neq m)$ **do**
    $\Delta \leftarrow \sigma^s_{o^s d^s} - \min\{\frac{1}{\lceil \beta S \rceil} w^s_{o^s d^s} c_{o^s d^s km}, \frac{1}{\lceil \beta S \rceil} w^s_{o^s d^s} c_{o^s d^s mk}\} - \pi^s_{o^s d^s k} - \pi^s_{o^s d^s m}$
    **if** $\Delta > 0$ **then**
      $\pi^s_{o^s d^s k} \leftarrow \pi^s_{o^s d^s k} + \frac{\Delta}{2}$
      $\pi^s_{o^s d^s m} \leftarrow \pi^s_{o^s d^s m} + \frac{\Delta}{2}$
    **end if**
  **end for**
**end for**

## 5. Computational experiments

This section deals with analyzing the efficiency of the developed mathematical models and solution algorithm by using three well-known data sets: the CAB, the TR, and the AP data sets. The CAB data set includes the air passenger transportation data between 25 US cities in 1970 (O'Kelly, 1987). The TR data set include cargo traffic data between 81 provinces in Turkey (Tan and Kara, 2007). The third data set is the Australia Post (AP) data set introduced by Ernst and Krishnamoorthy (1996). The AP data set is based on a postal delivery system in Sydney, Australia and consists of 200 nodes representing postal districts. The parameters $\chi$ and $\delta$ are set to be 1 and four different values are used for the discount factor parameter as: $\alpha \in \{0.2, 0.4, 0.6, 0.8\}$. The number of

**Table 2**
Performance comparison between BD algorithms with the CAB data set.

| Problem | $p$ | $\alpha = 0.2$ | | | $\alpha = 0.4$ | | | $\alpha = 0.6$ | | | $\alpha = 0.8$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | BD-Trd | | BD-Enh | BD-Trd | | BD-Enh | BD-Trd | | BD-Enh | BD-Trd | | BD-Enh |
| | | CPU (s) | # iter | CPU (s) | CPU (s) | # iter | CPU (s) | CPU (s) | # iter | CPU (s) | CPU (s) | # iter | CPU (s) |
| Median | 2 | 22.48 | 9 | 0.10 | 20.18 | 8 | 0.05 | 18.87 | 8 | 0.05 | 18.98 | 8 | 0.04 |
| | 3 | 30.64 | 13 | 0.12 | 26.87 | 11 | 0.10 | 25.97 | 12 | 0.10 | 24.40 | 12 | 0.09 |
| | 4 | 27.14 | 12 | 0.23 | 29.04 | 13 | 0.24 | 30.43 | 15 | 0.25 | 30.71 | 17 | 0.23 |
| | 5 | 28.21 | 13 | 0.24 | 24.88 | 13 | 0.28 | 25.93 | 14 | 0.36 | 31.22 | 18 | 0.48 |
| Covering | 2 | 20.42 | 31 | 0.09 | 10.88 | 17 | 0.07 | 11.04 | 18 | 0.09 | 6.66 | 9 | 0.04 |
| | 3 | 23.75 | 39 | 0.20 | 20.95 | 40 | 0.21 | 11.36 | 21 | 0.11 | 9.23 | 15 | 0.09 |
| | 4 | 18.66 | 30 | 0.12 | 20.18 | 39 | 0.20 | 14.08 | 26 | 0.20 | 10.87 | 21 | 0.14 |
| | 5 | 43.06 | 74 | 0.36 | 19.03 | 37 | 0.27 | 15.36 | 32 | 0.25 | 10.36 | 21 | 0.17 |
| Center | 2 | 15.20 | 5 | 0.10 | 16.99 | 6 | 0.03 | 17.53 | 6 | 0.01 | 15.93 | 6 | 0.01 |
| | 3 | 23.68 | 8 | 0.02 | 19.18 | 7 | 0.03 | 18.07 | 6 | 0.02 | 18.59 | 7 | 0.01 |
| | 4 | 28.06 | 10 | 0.03 | 21.57 | 8 | 0.02 | 20.74 | 8 | 0.01 | 16.96 | 7 | 0.01 |
| | 5 | 36.95 | 11 | 0.02 | 23.66 | 10 | 0.02 | 17.62 | 8 | 0.02 | 13.98 | 6 | 0.01 |
| Average | | 26.52 | | 0.14 | 21.12 | | 0.13 | 18.92 | | 0.12 | 17.32 | | 0.11 |

hubs to be opened is set as: $p \in \{2, 3, 4, 5\}$ and we take all the demand nodes as candidate sites for locating hubs (i.e., $H = N$) in all the three data sets. In case of the maximal covering problem, the covering radius is set as $R = 1000, 800$, and 20000 for the CAB, TR, and AP data sets, respectively. These values are selected as meaningful fractions of the optimal objective function values for the uncapacitated multiple allocation $p$-hub center problem as solved by Ghaffarinasab (2020).

In order to analyze the effect of uncertainty, we generated 100 scenarios for demand matrix realizations in both the data sets using the method proposed by Rostami et al. (2021). Based on this method, under each scenario $s \in \Omega$ a multiplicative factor $\pi_i$ uniformly distributed in the interval [0.5, 1.5] is generated for each node $i \in N$ denoting the deviation from the base case. For every O/D pair $(i, j) \in E$, the flow volume $w_{ij}^s$ is generated according to Poisson probability distribution with rate of $(\pi_i \pi_j w_{ij})$ in which $w_{ij}$ is the corresponding base flow value. As customarily done in the literature, the flow matrix for the CAB and TR data sets are scaled so that the sum of flows be 1 for each scenario. Furthermore, the original distances are divided by 1000 in the AP data set. Finally, the occurrence probability for each scenario is assumed to be 0.01. The risk parameter ($\beta$) is considered at six levels as: $\beta \in \{0.01, 0.05, 0.1, 0.3, 0.5, 1\}$.

The proposed algorithms are implemented in JAVA and the mathematical models are solved by CPLEX version 12.10 using default parameter values. Experiments are conducted by using a computer with Intel(R) Core(TM) i3-3220 CPU of 3.30 GHz and 16 GB of RAM, on Microsoft Windows 7 operating system. The BD algorithms are implemented within a branch-and-cut tree using the lazy constraint callback function available in CPLEX. The time limit for CPLEX is set to five hours in all experiments.

### *5.1. Evaluation of algorithmic enhancements on the proposed BD approaches*

As noted in Section 4, our proposed BD algorithms benefit from some tailored algorithmic refinement features enabling efficient solution of the problems as compared with the traditional implementation of the BD algorithm. The main refinement features in our BD procedures are ($i$) proposing efficient algorithms for solving the subproblems and generating strong cuts without resorting to off-the-shelf solvers, and ($ii$) modern implementation of the BD algorithm within a branch-and-cut framework. However, the traditional algorithm uses CPLEX for solving the subproblems and solves the master problem from scratch at each iteration of the algorithm. In order to evaluate the effect of the applied enhancements on the performance of the proposed BD procedures, we solve a large number of instances from the three problems using both the proposed enhanced BD algorithm and the traditional BD algorithm. The performance of the two solution procedures is compared for all the three problem variants on instances from the CAB data set in Table 2.

The first column shows the problem variant as the median, covering, or center problem. The second column shows the number of opened hubs. The next columns show the solution information for the two algorithms under different discount factor values. The two columns under heading "BD-Trd" show the performance of the traditional BD algorithm in terms of solution times and number of algorithm iterations which are averaged over different values of risk parameter $\beta \in \{0.01, 0.05, 0.1, 0.3, 0.5, 1\}$. The average numbers for the algorithm iterations are rounded to their closest integer values. The column under heading "BD-Enh" shows the average solution times for our proposed enhanced BD algorithm. Observe that the solution times for the BD-Trd are two orders of magnitude larger than those of the BD-Enh which shows the absolute superiority of the proposed BD procedure to the traditional implementation of the algorithm.

Table 3 shows the results obtained by the two solution procedures for the instances from the AP ($|N| = 40$) data set.

As can be observed from Table 3, the difference in solution times is even more noticeable for the instances of the AP ($|N| = 40$) data set as compared with those of the CAB data set. The average solution times for the AP ($|N| = 40$) instances with the BD-Trd are 289, 238, 245, and 197 s for discount factor values $\alpha = 0.2$, 0.4, 0.6, and 0.8, respectively. In contrast, the average solution times for the BD-Enh are less than or equal to one second for all values of $\alpha$. Note that the AP instances with 40 nodes are the largest instances that we could solve using the BD-Trd as the memory requirement for solving the subproblems exceeded the available RAM of our computer.

**Table 3**
Performance comparison between BD algorithms with the AP ($|N| = 40$) data set.

| Model | $p$ | $\alpha = 0.2$ | | | $\alpha = 0.4$ | | | $\alpha = 0.6$ | | | $\alpha = 0.8$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | BD-Trad | | BD | BD-Trad | | BD | BD-Trad | | BD | BD-Trad | | BD |
| | | CPU (s) | # iter | CPU (s) | CPU (s) | # iter | CPU (s) | CPU (s) | # iter | CPU (s) | CPU (s) | # iter | CPU (s) |
| Median | 2 | 90.33 | 7 | 0.48 | 84.15 | 7 | 0.12 | 78.03 | 6 | 0.11 | 89.92 | 8 | 0.10 |
| | 3 | 212.43 | 11 | 0.39 | 113.87 | 9 | 0.32 | 107.06 | 9 | 0.33 | 112.17 | 11 | 0.28 |
| | 4 | 203.68 | 16 | 1.87 | 175.08 | 14 | 1.30 | 174.65 | 14 | 0.92 | 130.50 | 13 | 0.66 |
| | 5 | 198.10 | 16 | 5.13 | 189.57 | 18 | 3.43 | 156.20 | 14 | 2.37 | 168.71 | 17 | 1.92 |
| Covering | 2 | 86.81 | 29 | 0.45 | 76.30 | 22 | 0.17 | 45.25 | 14 | 0.11 | 35.78 | 10 | 0.07 |
| | 3 | 223.56 | 59 | 0.43 | 149.83 | 49 | 0.34 | 123.98 | 56 | 0.29 | 62.51 | 26 | 0.14 |
| | 4 | 426.90 | 130 | 1.27 | 292.28 | 100 | 1.51 | 124.06 | 57 | 0.84 | 87.36 | 39 | 0.55 |
| | 5 | 493.49 | 136 | 1.62 | 315.28 | 106 | 3.37 | 177.73 | 80 | 3.39 | 108.21 | 54 | 2.17 |
| Center | 2 | 141.27 | 9 | 0.09 | 146.34 | 9 | 0.05 | 132.31 | 9 | 0.06 | 117.71 | 8 | 0.05 |
| | 3 | 152.65 | 10 | 0.05 | 139.21 | 9 | 0.05 | 245.42 | 13 | 0.05 | 230.35 | 12 | 0.04 |
| | 4 | 494.02 | 16 | 0.09 | 544.79 | 17 | 0.07 | 996.60 | 17 | 0.07 | 437.21 | 11 | 0.05 |
| | 5 | 749.09 | 21 | 0.13 | 641.02 | 22 | 0.13 | 581.63 | 18 | 0.09 | 791.14 | 14 | 0.05 |
| Average | | 289.36 | | 1.00 | 238.98 | | 0.91 | 245.24 | | 0.72 | 197.63 | | 0.51 |

**Table 4**
Results for the RA$p$HMP with the CAB data set.

| $p$ | $\beta$ | $\alpha = 0.2$ | | | | $\alpha = 0.4$ | | | | $\alpha = 0.6$ | | | | $\alpha = 0.8$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Opt. | | CPU (s) | | Opt. | | CPU (s) | | Opt. | | CPU (s) | | Opt. | | CPU (s) | |
| | | OF | Hubs | MILP | BD | OF | Hubs | MILP | BD | OF | Hubs | MILP | BD | OF | Hubs | MILP | BD |
| 2 | 1 | 1000.09 | 12, 20 | 157.94 | 0.32 | 1077.21 | 12, 20 | 157.09 | 0.06 | 1142.00 | 12, 20 | 153.25 | 0.05 | 1184.61 | 12, 20 | 157.69 | 0.05 |
| | 0.5 | 1030.10 | 12, 20 | 158.27 | 0.06 | 1107.48 | 12, 20 | 158.89 | 0.05 | 1173.14 | 12, 20 | 166.54 | 0.05 | 1216.81 | 12, 20 | 154.46 | 0.05 |
| | 0.3 | 1042.77 | 12, 20 | 157.97 | 0.05 | 1120.02 | 12, 20 | 158.20 | 0.04 | 1186.91 | 12, 20 | 164.17 | 0.05 | 1231.18 | 12, 20 | 154.49 | 0.04 |
| | 0.1 | 1063.48 | 12, 20 | 161.40 | 0.05 | 1141.13 | 12, 20 | 156.30 | 0.05 | 1206.29 | 12, 20 | 168.98 | 0.04 | 1249.85 | 12, 20 | 154.93 | 0.04 |
| | 0.05 | 1075.22 | 12, 20 | 163.42 | 0.05 | 1151.83 | 12, 20 | 162.81 | 0.04 | 1216.25 | 12, 20 | 167.29 | 0.04 | 1258.48 | 12, 20 | 152.88 | 0.04 |
| | 0.01 | 1082.59 | 12, 20 | 163.09 | 0.05 | 1164.51 | 12, 20 | 157.40 | 0.05 | 1231.05 | 12, 20 | 165.17 | 0.04 | 1271.81 | 12, 20 | 161.28 | 0.04 |
| 3 | 1 | 759.31 | 12, 17, 21 | 171.80 | 0.10 | 865.85 | 4, 12, 17 | 153.29 | 0.11 | 954.33 | 4, 12, 18 | 153.31 | 0.09 | 1025.42 | 4, 12, 18 | 159.85 | 0.09 |
| | 0.5 | 790.10 | 12, 18, 21 | 162.10 | 0.10 | 897.31 | 12, 18, 21 | 155.78 | 0.09 | 987.50 | 12, 18, 21 | 154.95 | 0.09 | 1057.68 | 12, 18, 21 | 154.52 | 0.09 |
| | 0.3 | 801.51 | 12, 18, 21 | 162.08 | 0.10 | 908.79 | 12, 18, 21 | 157.19 | 0.09 | 1001.12 | 12, 18, 21 | 155.44 | 0.11 | 1072.98 | 12, 18, 21 | 154.29 | 0.09 |
| | 0.1 | 822.55 | 12, 18, 21 | 162.49 | 0.10 | 928.83 | 12, 18, 21 | 161.11 | 0.10 | 1018.35 | 12, 18, 21 | 156.87 | 0.11 | 1092.13 | 12, 18, 21 | 154.02 | 0.08 |
| | 0.05 | 833.01 | 12, 18, 21 | 199.40 | 0.22 | 938.77 | 12, 18, 21 | 157.21 | 0.12 | 1029.47 | 2, 12, 21 | 156.29 | 0.09 | 1101.29 | 2, 12, 21 | 164.96 | 0.10 |
| | 0.01 | 841.80 | 18, 21, 22 | 178.13 | 0.11 | 953.25 | 12, 18, 21 | 169.93 | 0.11 | 1039.91 | 2, 12, 21 | 168.07 | 0.09 | 1107.69 | 2, 12, 21 | 176.16 | 0.08 |
| 4 | 1 | 623.15 | 4, 12, 17, 24 | 154.06 | 0.15 | 759.17 | 4, 12, 17, 24 | 149.63 | 0.15 | 872.34 | 1, 4, 12, 17 | 166.12 | 0.21 | 957.13 | 1, 4, 12, 17 | 154.39 | 0.17 |
| | 0.5 | 653.18 | 4, 12, 17, 24 | 181.28 | 0.30 | 790.47 | 4, 12, 17, 24 | 154.88 | 0.21 | 904.89 | 4, 7, 12, 18 | 197.87 | 0.22 | 991.53 | 1, 4, 12, 17 | 167.17 | 0.20 |
| | 0.3 | 667.22 | 4, 12, 17, 24 | 174.49 | 0.20 | 804.76 | 4, 12, 17, 24 | 180.99 | 0.22 | 917.40 | 4, 7, 12, 18 | 192.71 | 0.21 | 1006.59 | 4, 7, 12, 18 | 192.15 | 0.34 |
| | 0.1 | 692.00 | 4, 12, 16, 17 | 179.50 | 0.22 | 829.50 | 4, 12, 18, 24 | 207.18 | 0.27 | 937.17 | 4, 7, 12, 18 | 185.99 | 0.35 | 1028.44 | 4, 7, 12, 18 | 194.26 | 0.23 |
| | 0.05 | 704.46 | 4, 12, 16, 17 | 182.79 | 0.24 | 839.29 | 4, 12, 18, 24 | 190.18 | 0.32 | 948.69 | 4, 7, 12, 18 | 201.56 | 0.25 | 1036.82 | 1, 4, 12, 17 | 179.19 | 0.26 |
| | 0.01 | 716.17 | 4, 12, 16, 17 | 250.68 | 0.27 | 846.75 | 1, 4, 12, 18 | 231.20 | 0.30 | 954.54 | 1, 4, 12, 17 | 183.29 | 0.24 | 1042.20 | 1, 4, 12, 18 | 188.76 | 0.21 |
| 5 | 1 | 532.04 | 4, 7, 12, 14, 17 | 151.74 | 0.19 | 678.84 | 4, 7, 12, 14, 17 | 154.13 | 0.27 | 807.83 | 4, 7, 12, 14, 17 | 155.77 | 0.30 | 913.41 | 4, 7, 12, 17, 24 | 172.72 | 0.30 |
| | 0.5 | 555.38 | 4, 7, 12, 14, 17 | 164.42 | 0.22 | 704.10 | 4, 7, 12, 14, 17 | 161.25 | 0.21 | 836.03 | 4, 7, 12, 14, 18 | 166.95 | 0.27 | 945.03 | 4, 7, 12, 17, 24 | 176.44 | 0.40 |
| | 0.3 | 565.38 | 4, 7, 12, 14, 17 | 161.97 | 0.21 | 715.17 | 4, 7, 12, 14, 17 | 161.31 | 0.25 | 846.00 | 4, 7, 12, 14, 18 | 159.87 | 0.25 | 958.03 | 4, 7, 12, 17, 24 | 178.18 | 0.38 |
| | 0.1 | 586.33 | 4, 7, 12, 14, 17 | 165.69 | 0.27 | 735.51 | 4, 7, 12, 14, 18 | 160.55 | 0.23 | 866.70 | 4, 7, 12, 14, 18 | 172.17 | 0.33 | 979.67 | 4, 7, 12, 17, 24 | 203.90 | 0.43 |
| | 0.05 | 598.98 | 4, 7, 12, 14, 17 | 158.57 | 0.29 | 747.90 | 4, 7, 12, 14, 18 | 170.60 | 0.35 | 878.80 | 4, 7, 12, 14, 18 | 163.35 | 0.47 | 990.43 | 4, 7, 12, 14, 18 | 174.45 | 0.57 |
| | 0.01 | 610.98 | 4, 7, 12, 17, 24 | 222.35 | 0.29 | 762.43 | 4, 7, 12, 18, 24 | 203.10 | 0.40 | 893.92 | 2, 4, 7, 12, 24 | 173.09 | 0.53 | 1003.30 | 1, 4, 12, 17, 22 | 249.12 | 0.82 |
| Average | | | | 172.74 | 0.17 | | | 167.92 | 0.17 | | | 168.71 | 0.19 | | | 172.09 | 0.21 |

### *5.2. Results for the risk-averse $p$-hub median problem*

Table 4 shows the results obtained by solving the RA$p$HMP under different input parameters with the CAB data set. The number of installed hubs and the value of the risk parameter are shown in columns entitled $p$ and $\beta$, respectively. Solution results for the MILP model and the BD algorithm under different discount factor ($\alpha$) values are given next. For each value of $\alpha$, the optimal objective function value (OF) and the corresponding set of opened hubs as well as the solution times (in seconds) for the MILP model and the BD algorithm are presented. The average computational times for the two solution procedures are shown in the last row of the table.

According to the results presented in Table 4, the proposed BD algorithm is able to solver all the instances of the CAB data set to optimality within very short CPU times. The average time taken by CPLEX to solve the MILP model is around 170 s, whereas the BD obtains the optimal solutions in small fraction of a second. Note that the results for $\beta = 1$ are equivalent to risk-neutral expected value problem which, as shown by Contreras et al. (2011b), can also be obtained by solving a deterministic version of the problem using the average of O/D demand values over 100 scenarios. It can be observed that as the value of the risk parameter ($\beta$) decreases, the optimal objective function value increases. This is because when the value of $\beta$ is small, the focus is on the average performance over a small fraction of the worst scenarios which have larger values of objective function. In other words, smaller values of $\beta$ are used when the decision maker is highly risk-averse, while the larger values of $\beta$ are used by more risk-neutral decision makers. As expected, we can see that for a given number of opened hubs $p$, the optimal objective value increases as the value of the discount factor ($\alpha$) increases. On the other hand, for a fixed value of the discount factor, the objective value decreases as the number of opened hubs ($p$) increases.

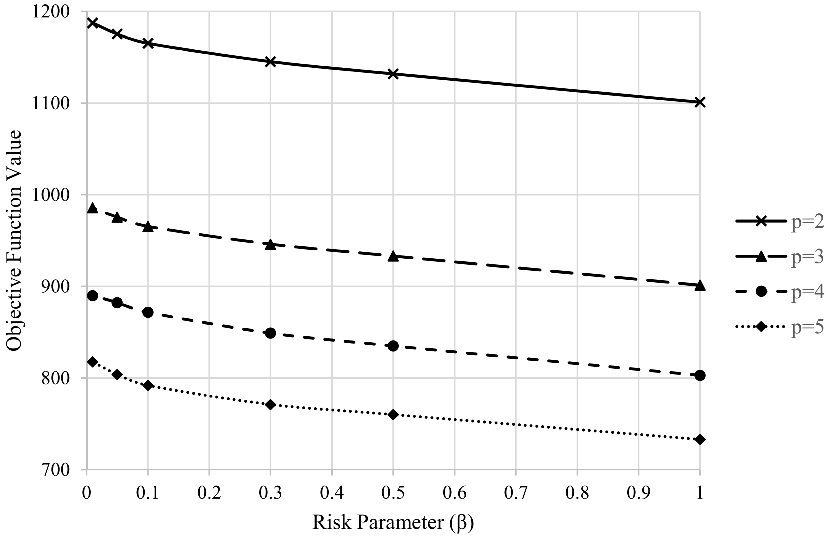



**Fig. 1.** Changes in the objective value with respect to $\beta$ for the RA$p$HMP with the CAB data set.

As far as the optimal network configuration is concerned, we observe a large variability in location of hubs when the value of the risk parameter ($\beta$) is changed. This variability is more noticeable when the number of opened hubs is relatively large. For instance, in case of $p = 4$ and $\alpha = 0.4$, the optimal hubs when $\beta = 1$, 0.5, and 0.3 are located at nodes 4 (Chicago), 12 (Los Angeles), 17 (New York), and 24 (Tampa). For the more risk-averse settings with $\beta = 0.1$ and 0.05, the four optimal locations are 4 (Chicago), 12 (Los Angeles), 18 (Philadelphia), and 24 (Tampa). Finally, for $\beta = 0.01$, the cities 1 (Atlanta), 4 (Chicago), 12 (Los Angeles), and 18 (Philadelphia) constitute the optimal set of hubs.

In order to better illustrate the effect of the risk parameter ($\beta$) and the number of opened hubs ($p$) on the value of the objective function value of the RA$p$HMP, we plot the total transportation cost with respect to the value of risk parameter $\beta$ for different values of $p$ in Fig. 1. The values are averaged over the four values of discount factor ($\alpha$). Observe that $\beta = 1$ corresponds to the risk-neutral problem in which the average value of the objective function over all scenarios are considered. The case with $\beta = 0.01$, on the other hand, has the highest level of risk-aversion where the value of the objective function for the worst 1% of scenarios (the worst scenario among the 100 scenarios) is considered as the evaluation criterion between different network configurations. As stated before, as the value of the risk parameter decreases, the total transportation cost increases. However, the growth of total cost is steeper when the value of $\beta$ gets closer to zero. Moreover, the total cost with $p = 2$ is considerably larger than the corresponding cost associated with the remaining values of $p$.

Table 5 presents the obtained results for solving the RA$p$HMP with the TR data set. Due to the large size of the instances in the TR data set, the corresponding MILP model could not be solved using CPLEX due to memory restrictions. Therefore, we only report the results obtained by the BD algorithm. It can be observed that the proposed BD algorithm solves all the instances with quite short CPU times. Solution times for the TR instances is less than 140 s on average. Observe that the mean solution times decrease as the discount factor value ($\alpha$) gets larger. For example, the average solution time is less than 50 s for $\alpha = 0.8$. Further, we can see that the solution times generally increase as the risk parameter value ($\beta$) decreases. In other words, more risk-averse instances are more difficult to be solved by the proposed BD algorithm.

From a network configuration point of view, we observe a substantial variability in the optimal location of hubs with respect to changes in the risk parameter ($\beta$) even when the number of opened hubs is small as $p = 2$. In case of $p = 4$ and $\alpha = 0.4$, the optimal hubs when $\beta = 1$ and 0.5 are located at nodes 6 (Ankara), 34 (Istanbul), 44 (Malatya), and 45 (Manisa). For the settings with $\beta =$ 0.3 and 0.1, the four optimal locations are 3 (Afyon), 27 (Gaziantep), 41 (Kocaeli), and 60 (Tokat). Finally, for $\beta = 0.05$ and 0.01, the optimal hubs are located at nodes 21 (Diyarbakir), 38 (Kayseri), 41 (Kocaeli), and 64 (Uşak).

Table 6 shows the results obtained by solving the RA$p$HMP under different input parameters with the AP data set. Note that the solution time for solving the large-scale instances od the AP data set is quite reasonable. It can be observed that by increasing the number of opened hubs $p$ the solution time increases. Also, we can observe that the solution time deceases as the value of the discount factor $\alpha$ gets larger. For this reason the instances with $p = 5$ and $\alpha = 0.2$ and 0.4 could not be solved to optimality within the allowed solution time of five hours. For these instances, the gap percentages between the upper and lower bounds are reported.

The curves showing the changes in the objective function value of the RA$p$HMP with respect to different values of $\beta$ and $p$ for the TR and AP data sets are depicted in Fig. 2. As can be seen, the increase in the value of the objective function for the TR data set as $\beta$ decreases is more uniform compared to the cases of the CAB and AP data sets.

**Table 5**
Results for the RA$p$HMP with the TR data set.

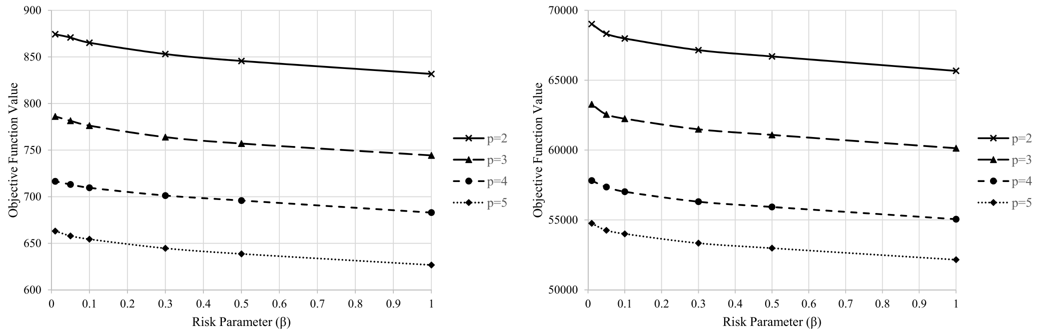
| $p$ | $\beta$ | $\alpha = 0.2$ | | | $\alpha = 0.4$ | | | $\alpha = 0.6$ | | | $\alpha = 0.8$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) |
| | | OF | Hubs | BD | OF | Hubs | BD | OF | Hubs | BD | OF | Hubs | BD |
| 2 | 1 | 770.99 | 38, 41 | 8.17 | 820.41 | 38, 41 | 5.30 | 857.14 | 38, 41 | 4.15 | 878.53 | 38, 41 | 2.59 |
| | 0.5 | 784.60 | 44, 54 | 7.86 | 835.16 | 38, 41 | 7.56 | 871.23 | 38, 41 | 4.09 | 891.78 | 38, 41 | 3.07 |
| | 0.3 | 789.79 | 26, 44 | 8.14 | 843.96 | 38, 41 | 6.33 | 879.47 | 38, 41 | 6.76 | 899.27 | 38, 41 | 2.93 |
| | 0.1 | 796.00 | 26, 44 | 8.11 | 858.28 | 38, 41 | 7.71 | 893.60 | 38, 41 | 8.09 | 913.25 | 38, 41 | 3.68 |
| | 0.05 | 798.43 | 26, 44 | 8.63 | 861.39 | 26, 44 | 9.24 | 902.11 | 38, 41 | 6.97 | 921.32 | 38, 41 | 4.42 |
| | 0.01 | 800.97 | 26, 44 | 8.47 | 864.47 | 26, 44 | 6.59 | 906.90 | 38, 41 | 7.02 | 925.23 | 38, 41 | 5.27 |
| 3 | 1 | 659.47 | 12, 41, 68 | 39.88 | 725.86 | 6, 41, 44 | 23.57 | 777.85 | 6, 41, 44 | 21.06 | 814.14 | 6, 41, 44 | 10.13 |
| | 0.5 | 671.53 | 12, 41, 68 | 41.98 | 739.92 | 6, 41, 44 | 28.26 | 790.68 | 6, 41, 44 | 20.59 | 826.10 | 6, 41, 44 | 11.78 |
| | 0.3 | 678.56 | 12, 41, 68 | 46.03 | 747.00 | 6, 41, 44 | 29.95 | 797.54 | 6, 41, 44 | 18.84 | 832.65 | 6, 41, 44 | 15.12 |
| | 0.1 | 688.29 | 12, 41, 68 | 49.12 | 760.97 | 23, 41, 68 | 37.88 | 810.39 | 23, 41, 68 | 23.29 | 845.43 | 23, 41, 68 | 15.53 |
| | 0.05 | 693.30 | 12, 41, 68 | 48.12 | 765.75 | 21, 41, 68 | 37.95 | 815.91 | 23, 41, 68 | 27.90 | 851.01 | 23, 41, 68 | 16.57 |
| | 0.01 | 698.43 | 21, 41, 50 | 53.13 | 770.37 | 23, 41, 68 | 38.93 | 819.52 | 23, 41, 68 | 19.90 | 856.25 | 3, 41, 44 | 15.13 |
| 4 | 1 | 568.83 | 6, 34, 44, 45 | 112.71 | 656.46 | 6, 34, 44, 45 | 77.30 | 728.57 | 6, 34, 44, 45 | 82.35 | 777.90 | 1, 3, 41, 58 | 56.48 |
| | 0.5 | 584.49 | 3, 27, 34, 60 | 158.41 | 671.35 | 6, 34, 44, 45 | 116.59 | 740.64 | 3, 27, 41, 60 | 103.30 | 787.19 | 3, 21, 38, 41 | 60.79 |
| | 0.3 | 590.42 | 3, 27, 34, 60 | 152.08 | 678.31 | 3, 27, 41, 60 | 133.98 | 744.91 | 3, 27, 41, 60 | 102.48 | 791.39 | 3, 21, 38, 41 | 54.80 |
| | 0.1 | 599.27 | 3, 27, 34, 60 | 184.37 | 685.94 | 3, 27, 41, 60 | 130.48 | 753.62 | 3, 27, 41, 60 | 97.79 | 799.50 | 3, 21, 38, 41 | 52.70 |
| | 0.05 | 602.24 | 21, 38, 41, 64 | 168.25 | 689.08 | 21, 38, 41, 64 | 139.18 | 756.29 | 35, 41, 44, 68 | 89.11 | 804.99 | 3, 21, 38, 41 | 55.38 |
| | 0.01 | 604.81 | 21, 38, 41, 64 | 171.56 | 690.35 | 21, 38, 41, 64 | 105.50 | 757.62 | 35, 41, 44, 68 | 63.60 | 813.43 | 6, 34, 35, 44 | 57.54 |
| 5 | 1 | 491.35 | 6, 12, 34, 45, 80 | 110.36 | 594.17 | 1, 6, 12, 34, 45 | 100.04 | 677.72 | 1, 6, 23, 34, 45 | 93.58 | 743.59 | 1, 6, 23, 41, 45 | 97.23 |
| | 0.5 | 505.99 | 6, 12, 34, 45, 80 | 197.09 | 607.44 | 1, 6, 12, 34, 45 | 173.54 | 689.07 | 1, 6, 23, 34, 45 | 139.87 | 752.07 | 1, 3, 21, 41, 60 | 95.73 |
| | 0.3 | 513.10 | 6, 12, 34, 45, 80 | 253.03 | 614.22 | 1, 6, 12, 34, 45 | 279.04 | 694.97 | 1, 6, 23, 41, 45 | 193.81 | 756.23 | 1, 3, 21, 41, 60 | 98.41 |
| | 0.1 | 524.80 | 12, 34, 64, 71, 80 | 526.21 | 626.05 | 1, 6, 12, 35, 41 | 516.03 | 702.44 | 1, 3, 21, 41, 60 | 186.66 | 764.22 | 1, 3, 21, 41, 60 | 109.75 |
| | 0.05 | 527.51 | 12, 34, 64, 71, 80 | 468.33 | 629.09 | 6, 12, 35, 41, 80 | 565.73 | 706.33 | 1, 3, 21, 41, 60 | 180.32 | 768.93 | 1, 3, 21, 41, 60 | 103.47 |
| | 0.01 | 531.65 | 12, 35, 41, 71, 80 | 480.30 | 631.42 | 12, 41, 45, 71, 80 | 489.08 | 711.17 | 6, 24, 35, 41, 46 | 206.11 | 778.46 | 1, 6, 23, 34, 35 | 118.28 |
| Average | | | | 137.93 | | | 127.74 | | | 71.15 | | | 44.45 |

**Table 6**
Results for the RA$p$HMP with the AP data set.

| $p$ | $\beta$ | $\alpha = 0.2$ | | | $\alpha = 0.4$ | | | $\alpha = 0.6$ | | | $\alpha = 0.8$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) |
| | | OF | Hubs | BD | OF | Hubs | BD | OF | Hubs | BD | OF | Hubs | BD |
| 2 | 1 | 62 346.49 | 57, 140 | 182.81 | 65 195.34 | 57, 140 | 156.81 | 67 021.68 | 56, 139 | 137.91 | 68 105.25 | 56, 139 | 120.41 |
| | 0.5 | 63 346.85 | 57, 140 | 148.89 | 66 221.32 | 57, 139 | 150.50 | 68 067.80 | 56, 139 | 122.16 | 69 162.16 | 56, 139 | 107.12 |
| | 0.3 | 63 771.50 | 57, 140 | 132.65 | 66 671.16 | 57, 139 | 128.80 | 68 534.28 | 56, 139 | 107.20 | 69 635.38 | 56, 139 | 96.58 |
| | 0.1 | 64 569.55 | 57, 140 | 94.59 | 67 501.45 | 57, 139 | 91.17 | 69 385.92 | 56, 139 | 82.16 | 70 484.85 | 56, 139 | 62.79 |
| | 0.05 | 64 893.90 | 57, 140 | 62.33 | 67 848.47 | 57, 140 | 66.35 | 69 728.41 | 56, 139 | 59.84 | 70 812.18 | 56, 139 | 48.13 |
| | 0.01 | 65 589.20 | 57, 140 | 30.33 | 68 530.80 | 57, 140 | 28.74 | 70 458.61 | 56, 139 | 27.28 | 71 501.58 | 56, 139 | 23.12 |
| 3 | 1 | 55 191.26 | 53, 110, 140 | 1501.45 | 59 274.86 | 56, 112, 140 | 1504.08 | 62 173.56 | 56, 111, 140 | 1473.18 | 63 888.68 | 56, 126, 145 | 1081.29 |
| | 0.5 | 56 083.29 | 53, 110, 140 | 1275.92 | 60 216.01 | 56, 112, 140 | 1266.53 | 63 154.07 | 56, 111, 140 | 1258.17 | 64 887.02 | 56, 126, 145 | 1013.25 |
| | 0.3 | 56 441.90 | 56, 112, 140 | 1178.29 | 60 614.27 | 56, 111, 140 | 1082.15 | 63 572.64 | 56, 111, 140 | 1132.75 | 65 320.00 | 56, 126, 145 | 856.56 |
| | 0.1 | 57 080.18 | 56, 112, 140 | 729.45 | 61 353.67 | 56, 112, 140 | 780.53 | 64 393.05 | 56, 111, 140 | 871.37 | 66 161.75 | 56, 126, 145 | 578.64 |
| | 0.05 | 57 370.18 | 56, 112, 140 | 526.21 | 61 614.53 | 56, 112, 140 | 569.16 | 64 676.41 | 56, 111, 140 | 574.80 | 66 527.14 | 56, 108, 140 | 479.62 |
| | 0.01 | 58 200.59 | 53, 110, 140 | 294.74 | 62 533.92 | 53, 107, 140 | 320.84 | 65 349.81 | 56, 126, 144 | 276.50 | 67 024.47 | 56, 126, 145 | 237.46 |
| 4 | 1 | 48 899.58 | 22, 110, 126, 141 | 4694.11 | 53 819.51 | 22, 110, 126, 141 | 3567.73 | 57 454.64 | 56, 111, 126, 140 | 3014.93 | 60 056.41 | 56, 108, 126, 140 | 2442.58 |
| | 0.5 | 49 681.13 | 22, 110, 126, 141 | 3675.17 | 54 666.27 | 22, 110, 126, 141 | 3426.26 | 58 363.06 | 56, 111, 126, 140 | 2328.73 | 61 005.22 | 56, 111, 126, 140 | 2044.07 |
| | 0.3 | 50 034.00 | 22, 110, 126, 141 | 3345.91 | 55 042.98 | 22, 110, 126, 141 | 2940.52 | 58 749.31 | 56, 111, 126, 140 | 2208.44 | 61 402.78 | 56, 111, 126, 140 | 1870.25 |
| | 0.1 | 50 621.62 | 22, 110, 126, 141 | 2348.64 | 55 722.54 | 22, 110, 126, 141 | 2020.42 | 59 526.28 | 56, 111, 126, 140 | 1552.45 | 62 209.87 | 21, 69, 126, 140 | 1373.28 |
| | 0.05 | 50 976.83 | 22, 110, 126, 141 | 2076.85 | 56 077.09 | 22, 110, 126, 141 | 1690.78 | 59 836.97 | 56, 111, 126, 140 | 1213.11 | 62 528.48 | 56, 108, 126, 140 | 1019.78 |
| | 0.01 | 51 396.55 | 21, 107, 126, 141 | 912.27 | 56 508.28 | 22, 107, 126, 141 | 738.48 | 60 361.03 | 22, 107, 126, 141 | 580.10 | 62 991.35 | 56, 108, 126, 140 | 519.08 |
| 5 | 1 | 45 130.10 | 20, 57, 112, 126, 140 | 5 h(4.22%) | 50 710.31 | 13, 58, 112, 126, 141 | 5 h(2.85%) | 54 878.47 | 14, 61, 111, 126, 140 | 14 910.43 | 57 898.71 | 14, 61, 111, 126, 140 | 8541.37 |
| | 0.5 | 45 851.09 | 20, 57, 112, 126, 140 | 5 h(3.94%) | 51 507.43 | 13, 58, 112, 126, 141 | 5 h(2.53%) | 55 749.98 | 14, 61, 111, 126, 140 | 14 719.59 | 58 815.48 | 14, 61, 111, 126, 140 | 9835.65 |
| | 0.3 | 46 168.27 | 20, 57, 112, 126, 140 | 5 h(3.76%) | 51 857.53 | 13, 58, 112, 126, 141 | 5 h(2.46%) | 56 121.78 | 14, 61, 111, 126, 140 | 12 736.59 | 59 206.20 | 14, 61, 111, 126, 140 | 9492.88 |
| | 0.1 | 46 787.99 | 19, 56, 112, 126, 140 | 5 h(3.70%) | 52 450.67 | 20, 56, 112, 126, 141 | 5 h(2.00%) | 56 822.30 | 14, 61, 111, 126, 140 | 10 384.22 | 59 961.61 | 14, 61, 111, 126, 140 | 8642.11 |
| | 0.05 | 46 912.12 | 20, 56, 112, 126, 141 | 5 h(3.04%) | 52 738.43 | 20, 56, 112, 126, 141 | 5 h(1.67%) | 57 116.48 | 14, 61, 111, 126, 140 | 11 326.88 | 60 249.37 | 14, 61, 111, 126, 140 | 4221.03 |
| | 0.01 | 47 501.12 | 20, 56, 112, 126, 141 | 5 h(2.48%) | 53 284.79 | 13, 58, 111, 126, 140 | 5 h(1.61%) | 57 493.29 | 14, 61, 111, 126, 140 | 4925.10 | 60 734.65 | 14, 61, 113, 126, 140 | 3566.94 |
| Average | | | | >5467.11 | | | >5355.41 | | | 3584.33 | | | 2428.08 |

### 5.3. Results for the risk-averse $p$-hub maximal covering problem

Table 7 presents the results for solving the RA$p$HMCP with the CAB data set. The objective function values show the percentage of the covered traffic. The instances are solved both using the BD algorithm and CPLEX as MILP model. The solution times for the BD are less than a second for all the instances. Also the mean solution times for CPLEX decrease as the discount factor value ($\alpha$) gets larger or as the risk parameter value ($\beta$) increases. In other words, more risk-averse instances are more difficult to be solved. Note that as the value of the risk parameter ($\beta$) gets smaller, the optimal objective function value also decreases. As the $p$-hub maximal covering problem is a maximization problem, the worst scenarios have smaller objective function values and this justifies the positive correlation between the values of the parameter $\beta$ and the objective function. It can also be seen that for a given number of opened hubs ($p$), the optimal objective value decreases as the value of the discount factor ($\alpha$) increases because larger values of




(a) TR data (b) AP data

**Fig. 2.** Changes in the objective value with respect to $\beta$ for the RA$p$HMP with the TR and AP data sets.

**Table 7**
Results for the RA$p$HMCP with the CAB data set.

| $p$ | $\beta$ | $\alpha = 0.2$ | | | | $\alpha = 0.4$ | | | | $\alpha = 0.6$ | | | | $\alpha = 0.8$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Opt. | | CPU (s) | | Opt. | | CPU (s) | | Opt. | | CPU (s) | | Opt. | | CPU (s) | |
| | | OF (%) | Hubs | MILP | BD | OF (%) | Hubs | MILP | BD | OF (%) | Hubs | MILP | BD | OF (%) | Hubs | MILP | BD |
| 2 | 1 | 56.39 | 6, 24 | 73.29 | 0.09 | 54.88 | 6, 24 | 16.54 | 0.05 | 51.08 | 6, 13 | 18.13 | 0.06 | 50.62 | 6, 13 | 3.84 | 0.03 |
| | 0.5 | 52.87 | 6, 12 | 99.40 | 0.08 | 51.01 | 6, 24 | 28.66 | 0.07 | 47.48 | 6, 13 | 23.11 | 0.06 | 47.03 | 6, 13 | 7.83 | 0.04 |
| | 0.3 | 51.60 | 6, 12 | 144.60 | 0.08 | 49.28 | 6, 24 | 34.60 | 0.05 | 45.85 | 6, 13 | 21.32 | 0.06 | 45.40 | 6, 13 | 5.75 | 0.04 |
| | 0.1 | 49.26 | 6, 12 | 131.87 | 0.08 | 47.37 | 6, 24 | 48.63 | 0.07 | 43.68 | 6, 13 | 34.54 | 0.20 | 43.26 | 6, 13 | 6.83 | 0.05 |
| | 0.05 | 47.73 | 6, 24 | 126 | 0.10 | 46.20 | 6, 24 | 72.78 | 0.06 | 42.90 | 6, 13 | 27.81 | 0.08 | 42.52 | 6, 13 | 8.07 | 0.05 |
| | 0.01 | 46.94 | 6, 12 | 164.13 | 0.10 | 44.32 | 6, 14 | 77.69 | 0.09 | 42.28 | 6, 13 | 39.77 | 0.07 | 41.91 | 6, 13 | 11.33 | 0.06 |
| 3 | 1 | 73.20 | 12, 18, 21 | 41.25 | 0.14 | 65.31 | 13, 24, 25 | 34.70 | 0.11 | 61.48 | 21, 24, 25 | 14.91 | 0.10 | 56.92 | 2, 14, 21 | 9.27 | 0.06 |
| | 0.5 | 70.46 | 12, 18, 21 | 81.79 | 0.13 | 61.44 | 13, 24, 25 | 57.57 | 0.15 | 57.79 | 21, 24, 25 | 20.68 | 0.08 | 53.31 | 2, 14, 21 | 14.41 | 0.08 |
| | 0.3 | 69.18 | 12, 18, 21 | 91.00 | 0.13 | 59.63 | 13, 24, 25 | 57.55 | 0.15 | 56.01 | 21, 24, 25 | 20.72 | 0.11 | 51.57 | 2, 14, 21 | 12.78 | 0.11 |
| | 0.1 | 66.70 | 12, 18, 21 | 126.39 | 0.28 | 57.41 | 13, 24, 25 | 75.11 | 0.33 | 54.03 | 21, 24, 25 | 27.76 | 0.10 | 50.21 | 6, 12, 13 | 10.91 | 0.10 |
| | 0.05 | 65.59 | 12, 18, 21 | 125.69 | 0.16 | 55.98 | 13, 24, 25 | 79.45 | 0.18 | 52.69 | 21, 24, 25 | 32.56 | 0.12 | 49.53 | 6, 12, 13 | 7.99 | 0.09 |
| | 0.01 | 64.87 | 12, 18, 21 | 99.29 | 0.33 | 54.49 | 18, 21, 24 | 92.70 | 0.32 | 51.32 | 21, 24, 25 | 42.65 | 0.14 | 48.53 | 6, 12, 13 | 17.43 | 0.10 |
| 4 | 1 | 87.64 | 12, 18, 21, 24 | 41.61 | 0.11 | 74.16 | 12, 17, 21, 24 | 27.29 | 0.25 | 66.68 | 12, 21, 24, 25 | 16.31 | 0.22 | 61.72 | 2, 12, 14, 21 | 9.70 | 0.15 |
| | 0.5 | 85.89 | 12, 18, 21, 24 | 62.85 | 0.13 | 71.63 | 12, 17, 21, 24 | 44.39 | 0.17 | 64.07 | 12, 21, 24, 25 | 27.64 | 0.24 | 59.14 | 2, 12, 14, 21 | 13.75 | 0.13 |
| | 0.3 | 85.02 | 12, 18, 21, 24 | 75.81 | 0.13 | 70.41 | 12, 17, 21, 24 | 75.68 | 0.17 | 62.91 | 12, 21, 24, 25 | 37.57 | 0.18 | 58.00 | 2, 12, 14, 21 | 11.54 | 0.11 |
| | 0.1 | 83.31 | 12, 18, 21, 24 | 89.76 | 0.12 | 68.42 | 12, 17, 21, 24 | 66.32 | 0.20 | 61.49 | 12, 21, 24, 25 | 24.20 | 0.19 | 56.56 | 2, 12, 14, 21 | 13.93 | 0.13 |
| | 0.05 | 82.59 | 12, 18, 21, 24 | 106.92 | 0.12 | 67.31 | 12, 21, 24, 25 | 48.00 | 0.23 | 60.62 | 12, 21, 24, 25 | 27.60 | 0.20 | 55.87 | 2, 12, 14, 21 | 15.59 | 0.13 |
| | 0.01 | 80.94 | 12, 18, 21, 24 | 91.44 | 0.12 | 66.32 | 6, 11, 12, 24 | 80.29 | 0.21 | 59.49 | 12, 21, 24, 25 | 37.73 | 0.19 | 54.38 | 12, 13, 14, 25 | 20.46 | 0.19 |
| 5 | 1 | 91.82 | 4, 7, 12, 18, 24 | 69.03 | 0.31 | 81.92 | 4, 7, 12, 17, 24 | 16.46 | 0.19 | 72.57 | 5, 7, 12, 18, 24 | 11.58 | 0.19 | 66.39 | 4, 7, 12, 14, 25 | 3.99 | 0.16 |
| | 0.5 | 90.54 | 4, 7, 12, 18, 24 | 82.07 | 0.28 | 79.90 | 4, 7, 12, 17, 24 | 25.28 | 0.17 | 70.17 | 5, 7, 12, 18, 24 | 25.59 | 0.23 | 64.12 | 2, 4, 7, 12, 14 | 6.10 | 0.17 |
| | 0.3 | 89.79 | 4, 7, 12, 18, 24 | 111.96 | 0.39 | 79.10 | 4, 7, 12, 17, 24 | 29.14 | 0.30 | 69.19 | 5, 7, 12, 18, 24 | 30.78 | 0.22 | 63.21 | 2, 4, 7, 12, 14 | 9.57 | 0.13 |
| | 0.1 | 88.48 | 4, 7, 12, 18, 24 | 152.76 | 0.33 | 77.36 | 4, 7, 12, 17, 24 | 33.45 | 0.22 | 67.68 | 5, 7, 12, 18, 24 | 31.67 | 0.34 | 61.77 | 2, 4, 7, 12, 14 | 8.55 | 0.16 |
| | 0.05 | 87.65 | 4, 7, 12, 18, 24 | 173.99 | 0.37 | 76.02 | 4, 7, 12, 17, 24 | 33.17 | 0.31 | 66.73 | 5, 7, 12, 18, 24 | 22.17 | 0.27 | 61.00 | 2, 4, 7, 12, 14 | 13.35 | 0.18 |
| | 0.01 | 86.79 | 12, 18, 21, 22, 24 | 125.39 | 0.48 | 75.21 | 4, 7, 12, 17, 24 | 32.59 | 0.43 | 65.66 | 5, 7, 12, 18, 24 | 22.98 | 0.25 | 59.44 | 5, 7, 12, 14, 25 | 19.99 | 0.25 |
| Average | | | | 103.68 | 0.19 | | | 49.50 | 0.19 | | | 26.66 | 0.16 | | | 10.96 | 0.11 |

$\alpha$ increase the transportation cost for the O/D pairs which in turn decrease the total flow captured within a certain covering radius. On the other hand, for a fixed value of $\alpha$, the objective value increases as $p$ gets larger. This is because of the fact that by installing larger number of hubs, it is more likely to find a path for each O/D pair that does not exceed the covering threshold $R$.

The optimal locations of hubs vary significantly by changing the degree of risk-aversion. As an example, in the instances with $p = 4$ and $\alpha = 0.4$, the optimal hub set for four values of the risk parameter as $\beta = 1$, 0.5, 0.3, and 0.1 is {12, 17, 21, 24}. For the instances with $\beta = 0.05$ and $\beta = 0.01$, the optimal sets of hubs are {12, 21, 24, 25} and {6, 11, 12, 24}, respectively.

Results for the RA$p$HMCP with the TR data set are shown in Table 8. The average solution times for solving the maximal covering version of the problem is larger than the median version. However, it can be observed that the CPU time for solving the RA$p$HMCP is less than 4 min on average. Since the TR data set is a large data set with 6480 O/D commodities, obtaining the optimal solution within such a short computational time demonstrates efficiency of the proposed BD algorithm. The changes in the objective function value with respect to variations is the number of hubs ($p$), the discount factor ($\alpha$), and the risk parameter ($\beta$) are similar to those of the CAB data set.

Variations in the optimal location of hubs are noticeable for the instances of the TR data set. For example, in case of $p = 2$ and $\alpha = 0.2$, the optimal hub set for the instances with $\beta = 1$ is {26, 46}, while for $\beta = 0.5$ through 0.05, the optimal hubs are {43, 46}. Finally, for $\beta = 0.01$, the optimal hub set is again {26, 46}. This interesting observation holds true for some of the problem settings where the network configurations for risk-neutral ($\beta = 1$) and highly risk-averse (small values of $\beta$) cases are the same, while the configurations for moderately risk-averse network (intermediate values of $\beta$) are different.

Results obtained by solving the RA$p$HMCP with the AP data set are presented in Table 9. The results reveal that by increasing the number of opened hubs $p$ the solution time increases. As a result, the instances with $p = 5$ could not be solved to optimality within the allowed solution time of five hours for which the optimality gap percentages are reported.

Fig. 3 shows the curves of the objective function value of the RA$p$HMCP with respect to different values of $\beta$ and $p$ for the three data sets. The curves illustrate the way in which the value of the coverage percentage diminishes as $\beta$ decreases. Observe that the

**Table 8**
Results for the RA$p$HMCP with the TR data set.

| $p$ | $\beta$ | $\alpha = 0.2$ | | | $\alpha = 0.4$ | | | $\alpha = 0.6$ | | | $\alpha = 0.8$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) |
| | | OF (%) | Hubs | BD | OF (%) | Hubs | BD | OF (%) | Hubs | BD | OF (%) | Hubs | BD |
| 2 | 1 | 58.31 | 26, 46 | 20.69 | 51.39 | 38, 54 | 17.64 | 46.76 | 50, 54 | 11.29 | 44.41 | 38, 43 | 6.22 |
| | 0.5 | 56.81 | 43, 46 | 18.85 | 49.31 | 38, 54 | 17.49 | 44.66 | 50, 54 | 11.59 | 42.75 | 38, 43 | 7.09 |
| | 0.3 | 56.07 | 43, 46 | 18.34 | 48.16 | 38, 54 | 16.11 | 43.58 | 50, 54 | 11.81 | 41.98 | 38, 43 | 7.38 |
| | 0.1 | 54.99 | 43, 46 | 19.61 | 46.14 | 38, 54 | 17.79 | 41.75 | 38, 43 | 13.70 | 41.09 | 38, 43 | 5.90 |
| | 0.05 | 54.26 | 43, 46 | 16.14 | 45.24 | 38, 54 | 18.04 | 41.28 | 38, 43 | 13.43 | 40.58 | 38, 43 | 5.73 |
| | 0.01 | 52.94 | 26, 46 | 17.99 | 44.80 | 38, 54 | 18.09 | 40.37 | 50, 54 | 13.71 | 38.96 | 38, 54 | 7.50 |
| 3 | 1 | 74.72 | 43, 46, 60 | 93.17 | 62.66 | 19, 43, 80 | 103.55 | 55.39 | 6, 41, 80 | 64.49 | 51.12 | 6, 43, 44 | 40.13 |
| | 0.5 | 73.53 | 43, 46, 60 | 86.44 | 61.16 | 19, 43, 80 | 99.69 | 53.47 | 41, 44, 68 | 69.52 | 49.59 | 6, 43, 44 | 42.40 |
| | 0.3 | 73.03 | 43, 46, 60 | 76.99 | 60.53 | 19, 43, 80 | 105.81 | 52.62 | 41, 44, 68 | 77.32 | 48.97 | 43, 44, 71 | 39.81 |
| | 0.1 | 72.29 | 43, 46, 60 | 66.39 | 59.40 | 19, 43, 80 | 96.78 | 50.92 | 44, 54, 68 | 86.03 | 48.07 | 43, 44, 71 | 35.37 |
| | 0.05 | 72.00 | 43, 46, 60 | 67.18 | 58.84 | 19, 43, 80 | 87.48 | 50.16 | 44, 54, 68 | 92.06 | 47.53 | 6, 43, 44 | 33.87 |
| | 0.01 | 71.84 | 43, 46, 60 | 64.76 | 57.71 | 19, 43, 80 | 100.47 | 49.38 | 41, 44, 68 | 92.09 | 46.24 | 6, 43, 44 | 40.70 |
| 4 | 1 | 88.34 | 12, 38, 54, 64 | 177.55 | 74.45 | 41, 46, 60, 64 | 190.56 | 64.34 | 1, 3, 41, 60 | 94.08 | 57.94 | 1, 3, 54, 58 | 33.01 |
| | 0.5 | 87.40 | 12, 38, 54, 64 | 169.76 | 73.29 | 41, 46, 60, 64 | 210.88 | 62.72 | 1, 3, 41, 60 | 100.39 | 56.51 | 1, 3, 54, 58 | 44.13 |
| | 0.3 | 86.83 | 12, 38, 54, 64 | 129.39 | 72.69 | 41, 46, 60, 64 | 189.59 | 61.92 | 1, 3, 41, 60 | 102.65 | 55.79 | 1, 3, 54, 58 | 46.09 |
| | 0.1 | 85.88 | 12, 38, 54, 64 | 205.18 | 71.36 | 41, 46, 60, 64 | 224.08 | 60.61 | 1, 3, 41, 60 | 116.35 | 54.59 | 1, 3, 54, 58 | 51.49 |
| | 0.05 | 85.46 | 12, 38, 54, 64 | 167.74 | 70.77 | 41, 46, 60, 64 | 210.39 | 59.96 | 1, 3, 41, 60 | 128.52 | 53.92 | 1, 3, 54, 58 | 65.64 |
| | 0.01 | 84.89 | 12, 38, 54, 64 | 202.97 | 70.04 | 41, 46, 60, 64 | 225.26 | 58.79 | 1, 3, 41, 60 | 130.54 | 52.52 | 3, 21, 38, 54 | 65.26 |
| 5 | 1 | 95.66 | 38, 41, 64, 69, 72 | 397.43 | 83.91 | 21, 41, 60, 64, 80 | 321.21 | 69.90 | 1, 3, 21, 41, 60 | 249.58 | 61.95 | 1, 3, 21, 41, 60 | 59.64 |
| | 0.5 | 95.20 | 12, 41, 60, 64, 80 | 452.20 | 83.00 | 21, 41, 60, 64, 80 | 353.83 | 68.76 | 1, 3, 21, 41, 60 | 271.51 | 60.82 | 1, 3, 21, 41, 60 | 65.96 |
| | 0.3 | 95.02 | 12, 41, 60, 64, 80 | 482.56 | 82.54 | 21, 41, 60, 64, 80 | 367.60 | 68.17 | 1, 3, 21, 41, 60 | 277.97 | 60.23 | 1, 3, 21, 41, 60 | 70.55 |
| | 0.1 | 94.66 | 12, 54, 60, 64, 80 | 601.08 | 81.79 | 21, 41, 60, 64, 80 | 382.50 | 67.18 | 1, 3, 21, 41, 60 | 426.80 | 59.28 | 1, 3, 21, 41, 60 | 87.73 |
| | 0.05 | 94.46 | 3, 12, 41, 46, 55 | 478.51 | 81.45 | 21, 41, 60, 64, 80 | 436.11 | 66.59 | 1, 3, 21, 41, 60 | 301.39 | 58.64 | 1, 3, 21, 41, 60 | 84.96 |
| | 0.01 | 94.22 | 3, 12, 41, 46, 55 | 500.24 | 80.80 | 21, 41, 60, 64, 80 | 367.67 | 65.16 | 1, 3, 21, 41, 60 | 393.50 | 57.01 | 1, 3, 12, 41, 60 | 91.23 |
| Average | | | | 188.80 | | | 174.11 | | | 131.26 | | | 43.24 |

**Table 9**
Results for the RA$p$HMCP with the AP data set.

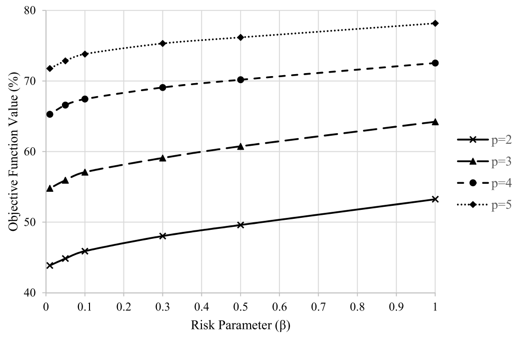
| $p$ | $\beta$ | $\alpha = 0.2$ | | | $\alpha = 0.4$ | | | $\alpha = 0.6$ | | | $\alpha = 0.8$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) |
| | | OF (%) | Hubs | BD | OF (%) | Hubs | BD | OF (%) | Hubs | BD | OF (%) | Hubs | BD |
| 2 | 1 | 59.41 | 57, 139 | 346.86 | 55.74 | 56, 137 | 303.09 | 53.31 | 56, 138 | 222.44 | 52.13 | 56, 138 | 102.08 |
| | 0.5 | 58.82 | 57, 139 | 302.06 | 55.08 | 57, 137 | 272.40 | 52.72 | 56, 138 | 198.10 | 51.51 | 56, 138 | 91.91 |
| | 0.3 | 58.56 | 57, 139 | 276.03 | 54.78 | 57, 137 | 244.43 | 52.39 | 56, 138 | 196.17 | 51.20 | 56, 138 | 88.23 |
| | 0.1 | 58.16 | 56, 139 | 191.80 | 54.15 | 57, 137 | 183.63 | 51.79 | 56, 138 | 125.19 | 50.65 | 56, 138 | 71.87 |
| | 0.05 | 57.92 | 56, 139 | 147.73 | 53.80 | 56, 137 | 140.87 | 51.40 | 56, 138 | 114.62 | 50.19 | 56, 138 | 62.28 |
| | 0.01 | 57.50 | 56, 139 | 91.20 | 53.04 | 57, 137 | 94.87 | 50.00 | 56, 138 | 78.08 | 48.59 | 56, 138 | 53.22 |
| 3 | 1 | 71.49 | 53, 74, 137 | 1343.45 | 66.05 | 56, 112, 137 | 1169.54 | 60.98 | 57, 106, 130 | 1392.11 | 58.97 | 56, 128, 147 | 592.14 |
| | 0.5 | 70.94 | 53, 74, 137 | 1312.18 | 65.46 | 57, 113, 137 | 1176.50 | 60.40 | 56, 106, 130 | 1291.56 | 58.41 | 56, 128, 147 | 503.61 |
| | 0.3 | 70.72 | 53, 74, 137 | 1086.59 | 65.18 | 56, 112, 137 | 1025.00 | 60.11 | 56, 106, 130 | 1132.36 | 58.14 | 57, 128, 147 | 456.14 |
| | 0.1 | 70.31 | 53, 74, 137 | 829.49 | 64.67 | 57, 112, 137 | 647.44 | 59.47 | 56, 106, 130 | 902.17 | 57.55 | 57, 128, 147 | 342.00 |
| | 0.05 | 70.08 | 53, 74, 137 | 630.48 | 64.41 | 57, 113, 137 | 572.44 | 59.05 | 56, 106, 130 | 748.46 | 57.03 | 56, 128, 147 | 302.31 |
| | 0.01 | 69.75 | 53, 74, 137 | 433.11 | 64.09 | 57, 113, 137 | 337.20 | 58.17 | 56, 128, 147 | 595.47 | 56.33 | 57, 126, 147 | 173.85 |
| 4 | 1 | 79.24 | 22, 89, 110, 141 | 8406.49 | 73.10 | 57, 110, 128, 144 | 5458.36 | 66.93 | 56, 90, 111, 140 | 5006.17 | 63.21 | 57, 107, 128, 149 | 2442.25 |
| | 0.5 | 78.64 | 22, 89, 110, 141 | 11 322.62 | 72.51 | 57, 110, 128, 144 | 5532.94 | 66.30 | 56, 90, 111, 140 | 5132.57 | 62.61 | 57, 107, 128, 149 | 2327.49 |
| | 0.3 | 78.39 | 22, 89, 110, 141 | 8488.20 | 72.18 | 57, 110, 128, 145 | 5104.56 | 65.95 | 56, 90, 111, 140 | 5264.97 | 62.33 | 56, 107, 128, 149 | 2250.40 |
| | 0.1 | 77.95 | 22, 89, 110, 141 | 12 182.43 | 71.61 | 57, 110, 125, 145 | 6425.17 | 65.34 | 56, 90, 111, 140 | 4547.97 | 61.80 | 57, 107, 128, 149 | 1642.41 |
| | 0.05 | 77.73 | 22, 89, 110, 141 | 8986.99 | 71.31 | 57, 89, 110, 144 | 5106.71 | 65.07 | 56, 90, 111, 141 | 4033.30 | 61.36 | 56, 107, 128, 149 | 1715.03 |
| | 0.01 | 77.35 | 53, 74, 125, 184 | 13 347.39 | 71.06 | 57, 110, 125, 145 | 3719.78 | 64.60 | 56, 111, 128, 140 | 2834.80 | 60.79 | 56, 107, 126, 149 | 1079.03 |
| 5 | 1 | 84.18 | 19, 53, 74, 125, 144 | 5 h(4.11%) | 77.14 | 14, 62, 88, 113, 139 | 5 h(4.65%) | 70.49 | 14, 61, 113, 126, 140 | 5 h(4.15%) | 65.47 | 57, 69, 125, 140, 155 | 5 h(3.30%) |
| | 0.5 | 83.69 | 19, 53, 74, 125, 144 | 5 h(5.00%) | 76.59 | 14, 62, 88, 113, 139 | 5 h(4.77%) | 69.83 | 14, 61, 113, 126, 140 | 5 h(3.99%) | 64.87 | 57, 69, 125, 140, 155 | 5 h(3.02%) |
| | 0.3 | 83.48 | 19, 53, 74, 125, 144 | 5 h(5.36%) | 76.35 | 14, 62, 88, 113, 139 | 5 h(4.67%) | 69.56 | 14, 61, 113, 126, 140 | 5 h(3.64%) | 64.59 | 57, 69, 125, 140, 155 | 5 h(3.25%) |
| | 0.1 | 83.10 | 19, 53, 74, 125, 144 | 5 h(4.89%) | 75.87 | 14, 62, 88, 113, 139 | 5 h(4.68%) | 69.11 | 21, 67, 90, 113, 141 | 5 h(3.55%) | 64.09 | 14, 61, 89, 111, 140 | 5 h(2.94%) |
| | 0.05 | 82.86 | 19, 53, 74, 125, 144 | 5 h(5.65%) | 75.35 | 14, 62, 113, 125, 139 | 5 h(4.54%) | 68.92 | 14, 61, 113, 126, 140 | 5 h(3.39%) | 63.80 | 14, 61, 89, 111, 140 | 5 h(2.61%) |
| | 0.01 | 82.52 | 19, 22, 86, 110, 141 | 5 h(4.95%) | 75.17 | 14, 62, 90, 114, 184 | 5 h(4.23%) | 68.62 | 14, 61, 90, 113, 140 | 5 h(3.00%) | 63.37 | 14, 61, 89, 111, 140 | 5 h(2.15%) |
| Average | | | | >7405.21 | | | >6063.12 | | | >5909.02 | | | >5095.68 |

decline in the total coverage percentage is sharper when the value of $\beta$ gets closer to zero. Moreover, as the value of $\beta$ increases, the curves corresponding to the AP data set are growing with a smaller rate as compared to those of the CAB and TR data sets.

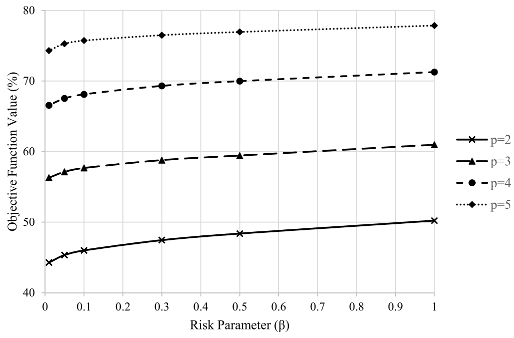
### *5.4. Results for the risk-averse weighted p-hub center problem*

Results for the RAW$p$HCP with the CAB data set are presented in Table 10. It is shown that the BD algorithm solves all the CAB instances in small fraction of a second. However, the average time taken by CPLEX to solve the MILP model for the same instances is around 230 s. Similar to results obtained by solving the other versions of the risk-averse HLP presented earlier, variations in the optimal location of hubs are noticeable in different instances of the RAW$p$HCP.

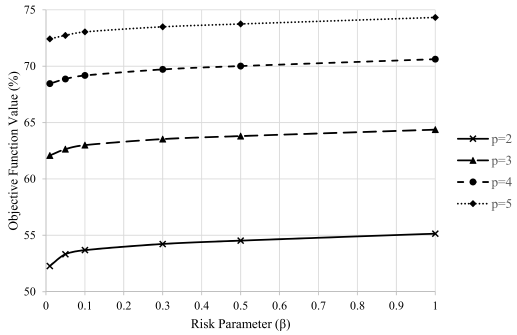
Table 11 shows the results obtained by solving the RAW$p$HCP with the TR data set. The changes in the objective function value with respect to different values of the input parameters ($p$, $\alpha$, and $\beta$) show a similar pattern with those of the CAB data set. Also, changes in the optimal set of hub locations can be observed under different problem settings. Regarding the solution times, the results show that the proposed BD algorithm solves the instances in very short computational times. The average time spent by the BD algorithm is less than 3 for the TR data set seconds which indicates the high efficiency of the proposed solution algorithm for the center version of the problem.




CAB data




TR data




AP data

**Fig. 3.** Changes in the objective value with respect to $\beta$ for the RA$p$HMCP.

**Table 10**
Results for the RAW$p$HCP with the CAB data set.

| $p$ | $\beta$ | $\alpha = 0.2$ | | | | $\alpha = 0.4$ | | | | $\alpha = 0.6$ | | | | $\alpha = 0.8$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Opt. | | CPU (s) | | Opt. | | CPU (s) | | Opt. | | CPU (s) | | Opt. | | CPU (s) | |
| | | OF | Hubs | MILP | BD | OF | Hubs | MILP | BD | OF | Hubs | MILP | BD | OF | Hubs | MILP | BD |
| 2 | 1 | 22.15 | 12, 17 | 188.37 | 0.30 | 22.71 | 12, 17 | 179.89 | 0.04 | 23.60 | 12, 17 | 79.81 | 0.02 | 26.35 | 12, 17 | 85.93 | 0.01 |
| | 0.5 | 26.76 | 12, 17 | 124.84 | 0.11 | 27.10 | 12, 17 | 179.39 | 0.03 | 27.64 | 12, 17 | 83.84 | 0.02 | 30.94 | 12, 17 | 74.85 | 0.02 |
| | 0.3 | 29.58 | 12, 17 | 176.959 | 0.07 | 29.59 | 12, 17 | 182.26 | 0.03 | 29.75 | 12, 17 | 85.97 | 0.02 | 33.44 | 12, 17 | 82.30 | 0.02 |
| | 0.1 | 33.22 | 12, 17 | 146.49 | 0.06 | 33.22 | 12, 17 | 221.32 | 0.02 | 33.22 | 12, 17 | 107.39 | 0.01 | 36.49 | 12, 17 | 71.63 | 0.01 |
| | 0.05 | 35.18 | 12, 17 | 164.52 | 0.02 | 35.18 | 12, 17 | 183.86 | 0.02 | 35.18 | 12, 17 | 110.30 | 0.01 | 37.10 | 12, 17 | 70.81 | 0.01 |
| | 0.01 | 37.36 | 12, 17 | 137.40 | 0.04 | 37.36 | 12, 17 | 193.94 | 0.01 | 37.36 | 12, 17 | 141.00 | 0.01 | 37.36 | 12, 17 | 76.78 | 0.01 |
| 3 | 1 | 15.63 | 12, 17, 24 | 243.60 | 0.03 | 17.57 | 12, 14, 17 | 139.00 | 0.03 | 20.41 | 12, 14, 17 | 126.92 | 0.03 | 25.19 | 12, 14, 17 | 87.39 | 0.02 |
| | 0.5 | 19.04 | 12, 17, 24 | 269.39 | 0.03 | 20.68 | 12, 14, 17 | 164.58 | 0.03 | 23.69 | 12, 14, 17 | 109.67 | 0.02 | 29.72 | 12, 14, 17 | 88.30 | 0.01 |
| | 0.3 | 20.83 | 12, 14, 17 | 239.70 | 0.02 | 22.21 | 12, 14, 17 | 257.91 | 0.02 | 25.28 | 12, 14, 17 | 105.40 | 0.02 | 32.30 | 12, 14, 17 | 76.14 | 0.01 |
| | 0.1 | 23.88 | 12, 14, 17 | 233.89 | 0.02 | 24.84 | 12, 14, 17 | 218.02 | 0.03 | 27.55 | 12, 14, 17 | 88.42 | 0.02 | 35.91 | 12, 14, 17 | 171.82 | 0.01 |
| | 0.05 | 25.56 | 12, 14, 17 | 245.29 | 0.02 | 26.28 | 12, 14, 17 | 193.51 | 0.03 | 28.02 | 12, 14, 17 | 135.05 | 0.01 | 36.97 | 12, 14, 17 | 112.77 | 0.01 |
| | 0.01 | 28.72 | 12, 14, 17 | 172.30 | 0.02 | 28.72 | 12, 14, 17 | 219.11 | 0.02 | 28.72 | 12, 14, 17 | 385.66 | 0.01 | 37.24 | 12, 17, 18 | 81.69 | 0.01 |
| 4 | 1 | 9.27 | 4, 12, 14, 17 | 233.26 | 0.03 | 13.23 | 4, 12, 14, 17 | 132.31 | 0.02 | 18.85 | 4, 12, 14, 17 | 90.90 | 0.01 | 24.75 | 12, 14, 17, 22 | 100.50 | 0.02 |
| | 0.5 | 10.96 | 4, 12, 14, 17 | 242.54 | 0.03 | 15.53 | 4, 12, 14, 17 | 81.26 | 0.02 | 22.48 | 4, 12, 14, 17 | 86.58 | 0.01 | 29.28 | 12, 14, 17, 22 | 71.18 | 0.02 |
| | 0.3 | 11.94 | 4, 12, 14, 17 | 291.19 | 0.03 | 16.70 | 4, 12, 14, 17 | 83.29 | 0.03 | 24.39 | 4, 12, 14, 17 | 79.08 | 0.02 | 31.99 | 12, 14, 17, 22 | 71.28 | 0.01 |
| | 0.1 | 13.24 | 4, 12, 14, 17 | 198.01 | 0.03 | 18.51 | 4, 12, 14, 17 | 75.33 | 0.01 | 27.07 | 4, 12, 14, 17 | 431.50 | 0.01 | 35.84 | 12, 14, 17, 22 | 158.80 | 0.01 |
| | 0.05 | 13.88 | 4, 12, 14, 17 | 209.35 | 0.02 | 18.78 | 4, 12, 14, 17 | 158.52 | 0.01 | 27.72 | 4, 12, 14, 17 | 182.63 | 0.01 | 36.97 | 12, 14, 17, 18 | 100.13 | 0.01 |
| | 0.01 | 14.04 | 4, 12, 17, 24 | 223.88 | 0.02 | 19.50 | 4, 12, 14, 17 | 273.25 | 0.01 | 27.93 | 6, 12, 14, 17 | 250.12 | 0.01 | 37.24 | 12, 17, 18, 20 | 114.76 | 0.01 |
| 5 | 1 | 8.02 | 4, 7, 12, 14, 17 | 225.50 | 0.03 | 12.51 | 4, 12, 14, 17, 22 | 126.30 | 0.03 | 18.29 | 4, 12, 14, 17, 22 | 77.15 | 0.02 | 24.37 | 4, 12, 14, 17, 22 | 69.40 | 0.01 |
| | 0.5 | 9.61 | 4, 12, 14, 16, 17 | 153.27 | 0.02 | 14.76 | 4, 12, 14, 17, 22 | 95.44 | 0.02 | 21.86 | 4, 12, 14, 17, 22 | 64.45 | 0.01 | 29.14 | 4, 12, 14, 17, 22 | 73.04 | 0.01 |
| | 0.3 | 10.44 | 4, 7, 12, 14, 17 | 168.74 | 0.02 | 16.01 | 4, 12, 14, 17, 22 | 154.19 | 0.02 | 23.98 | 4, 12, 14, 17, 22 | 68.11 | 0.03 | 31.97 | 4, 12, 14, 17, 22 | 69.92 | 0.01 |
| | 0.1 | 12.03 | 4, 7, 12, 14, 17 | 298.79 | 0.02 | 17.92 | 4, 12, 14, 17, 22 | 92.88 | 0.02 | 26.88 | 9, 12, 14, 17, 22 | 98.14 | 0.02 | 35.84 | 8, 12, 14, 17, 22 | 124.76 | 0.01 |
| | 0.05 | 12.68 | 4, 7, 12, 14, 17 | 341.57 | 0.02 | 18.48 | 4, 12, 14, 17, 22 | 80.43 | 0.01 | 27.72 | 4, 9, 12, 14, 17 | 177.74 | 0.01 | 36.97 | 12, 14, 17, 18, 24 | 128.16 | 0.01 |
| | 0.01 | 14.03 | 9, 12, 17, 22, 24 | 434.96 | 0.02 | 18.62 | 4, 12, 14, 17, 22 | 88.16 | 0.02 | 27.93 | 4, 12, 14, 17, 24 | 169.56 | 0.01 | 37.24 | 4, 12, 17, 18, 20 | 67.63 | 0.01 |
| Average | | | | 223.49 | 0.04 | | | 157.26 | 0.02 | | | 138.97 | 0.02 | | | 92.91 | 0.01 |

Results obtained by solving the RAW$p$HCP with the AP data set are presented in Table 12. Note that all the large-scale instances of the AP data set are solved within very small computational times. The average CPU time is less than 20 s which shows the BD algorithm for solving the center problem is highly efficient.

The curves of objective function value for the RAW$p$HCP with respect to different values of $\beta$ and $p$ for the three data sets are shown in Fig. 4. The objective function value grows substantially as the value of the risk parameter decreases. Note that the growth in the objective value as $\beta$ decreases is steeper for the AP data set as compared with those of the CAB and TR data sets.

**Table 11**
Results for the RAW$p$HCP with the TR data set.

| $p$ | $\beta$ | $\alpha = 0.2$ | | | $\alpha = 0.4$ | | | $\alpha = 0.6$ | | | $\alpha = 0.8$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) |
| | | OF | Hubs | BD | OF | Hubs | BD | OF | Hubs | BD | OF | Hubs | BD |
| 2 | 1 | 4.98 | 3, 34 | 2.09 | 5.41 | 34, 71 | 1.60 | 5.62 | 6, 34 | 1.34 | 5.88 | 6, 34 | 1.37 |
| | 0.5 | 6.00 | 3, 34 | 1.38 | 6.60 | 34, 71 | 1.58 | 6.82 | 6, 34 | 1.51 | 7.09 | 6, 34 | 1.40 |
| | 0.3 | 6.43 | 3, 34 | 1.27 | 7.25 | 34, 71 | 1.95 | 7.43 | 6, 34 | 1.64 | 7.67 | 6, 34 | 1.67 |
| | 0.1 | 6.92 | 3, 34 | 1.62 | 7.89 | 34, 41 | 1.98 | 8.13 | 34, 41 | 1.83 | 8.32 | 35, 41 | 1.43 |
| | 0.05 | 7.27 | 3, 34 | 1.38 | 8.07 | 3, 34 | 1.57 | 8.28 | 34, 41 | 1.44 | 8.49 | 34, 41 | 1.20 |
| | 0.01 | 7.46 | 3, 34 | 1.19 | 8.19 | 34, 41 | 1.13 | 8.38 | 34, 41 | 1.44 | 8.61 | 34, 35 | 0.84 |
| 3 | 1 | 3.84 | 34, 40, 45 | 1.46 | 4.50 | 34, 45, 71 | 1.51 | 5.01 | 6, 34, 35 | 1.58 | 5.56 | 6, 34, 35 | 1.77 |
| | 0.5 | 4.64 | 34, 40, 45 | 1.42 | 5.38 | 34, 45, 71 | 1.41 | 6.02 | 6, 34, 35 | 1.66 | 6.67 | 6, 34, 35 | 1.74 |
| | 0.3 | 5.01 | 34, 35, 40 | 1.04 | 5.76 | 34, 35, 71 | 1.11 | 6.48 | 6, 34, 35 | 1.05 | 7.12 | 6, 34, 35 | 1.28 |
| | 0.1 | 5.49 | 34, 35, 40 | 1.15 | 6.23 | 34, 35, 71 | 1.03 | 7.04 | 6, 34, 35 | 1.18 | 7.61 | 6, 34, 35 | 1.70 |
| | 0.05 | 5.61 | 34, 35, 40 | 1.14 | 6.49 | 34, 35, 71 | 0.86 | 7.25 | 34, 45, 71 | 0.99 | 7.89 | 6, 34, 45 | 1.35 |
| | 0.01 | 5.77 | 34, 35, 40 | 1.05 | 6.63 | 34, 45, 71 | 0.83 | 7.30 | 34, 45, 71 | 0.82 | 8.03 | 6, 34, 35 | 1.27 |
| 4 | 1 | 2.89 | 6, 34, 45, 46 | 2.56 | 3.67 | 6, 34, 45, 80 | 1.98 | 4.45 | 6, 34, 35, 63 | 1.77 | 5.31 | 6, 34, 35, 63 | 1.25 |
| | 0.5 | 3.45 | 6, 34, 45, 46 | 2.11 | 4.38 | 6, 34, 45, 80 | 1.99 | 5.28 | 6, 34, 35, 63 | 1.43 | 6.35 | 6, 34, 35, 63 | 2.12 |
| | 0.3 | 3.66 | 6, 34, 45, 46 | 1.99 | 4.65 | 6, 34, 45, 80 | 2.11 | 5.59 | 2, 6, 34, 35 | 1.77 | 6.72 | 6, 34, 35, 63 | 1.40 |
| | 0.1 | 3.99 | 6, 27, 34, 45 | 1.77 | 5.05 | 6, 34, 45, 46 | 1.60 | 5.98 | 2, 6, 34, 35 | 1.81 | 7.18 | 2, 6, 34, 35 | 1.57 |
| | 0.05 | 4.12 | 6, 27, 34, 45 | 1.58 | 5.14 | 6, 34, 45, 46 | 1.81 | 6.21 | 2, 6, 34, 35 | 1.30 | 7.33 | 2, 6, 34, 35 | 1.21 |
| | 0.01 | 4.21 | 6, 34, 45, 46 | 1.20 | 5.32 | 6, 34, 35, 46 | 1.01 | 6.49 | 6, 21, 34, 35 | 1.16 | 7.40 | 2, 6, 34, 35 | 1.54 |
| 5 | 1 | 2.62 | 6, 12, 34, 45, 80 | 4.47 | 3.37 | 1, 6, 34, 45, 63 | 3.34 | 4.17 | 1, 6, 34, 35, 63 | 2.74 | 5.22 | 1, 6, 21, 34, 35 | 2.34 |
| | 0.5 | 3.15 | 6, 12, 34, 45, 80 | 6.25 | 4.04 | 1, 6, 34, 45, 63 | 2.96 | 4.97 | 1, 6, 34, 35, 63 | 2.04 | 6.25 | 6, 21, 34, 35, 63 | 2.16 |
| | 0.3 | 3.40 | 6, 34, 45, 62, 80 | 4.39 | 4.29 | 6, 21, 34, 45, 80 | 3.38 | 5.27 | 1, 6, 34, 35, 63 | 1.83 | 6.62 | 6, 21, 34, 35, 63 | 2.52 |
| | 0.1 | 3.77 | 34, 44, 45, 68, 71 | 3.95 | 4.65 | 6, 23, 34, 45, 80 | 2.08 | 5.65 | 1, 6, 21, 34, 35 | 1.57 | 6.97 | 1, 6, 21, 34, 35 | 2.30 |
| | 0.05 | 3.94 | 27, 34, 42, 45, 71 | 3.91 | 4.81 | 6, 34, 44, 45, 68 | 2.15 | 5.74 | 1, 6, 21, 34, 35 | 1.36 | 7.06 | 1, 6, 21, 34, 35 | 1.08 |
| | 0.01 | 4.08 | 27, 34, 35, 42, 71 | 2.65 | 5.04 | 1, 6, 21, 34, 45 | 1.46 | 5.87 | 1, 6, 21, 34, 35 | 1.14 | 7.28 | 6, 21, 27, 34, 35 | 1.42 |
| Average | | | | 2.21 | | | 1.77 | | | 1.52 | | | 1.58 |

**Table 12**
Results for the RAW$p$HCP with the AP data set.

| $p$ | $\beta$ | $\alpha = 0.2$ | | | $\alpha = 0.4$ | | | $\alpha = 0.6$ | | | $\alpha = 0.8$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) | Opt. | | CPU (s) |
| | | OF | Hubs | BD | OF | Hubs | BD | OF | Hubs | BD | OF | Hubs | BD |
| 2 | 1 | 201.27 | 22, 140 | 4.76 | 224.75 | 22, 140 | 5.69 | 249.29 | 22, 140 | 3.91 | 279.72 | 22, 140 | 4.79 |
| | 0.5 | 226.79 | 25, 141 | 4.36 | 252.78 | 25, 140 | 5.49 | 282.21 | 22, 140 | 3.27 | 324.48 | 22, 140 | 4.58 |
| | 0.3 | 240.96 | 25, 141 | 4.13 | 265.36 | 25, 140 | 4.87 | 298.24 | 22, 140 | 3.51 | 344.48 | 22, 140 | 3.03 |
| | 0.1 | 261.18 | 21, 140 | 4.38 | 291.65 | 22, 140 | 5.07 | 329.35 | 21, 140 | 3.05 | 376.77 | 20, 140 | 2.17 |
| | 0.05 | 267.46 | 21, 140 | 4.44 | 306.85 | 14, 140 | 5.11 | 345.77 | 14, 140 | 3.26 | 395.76 | 20, 140 | 2.23 |
| | 0.01 | 274.51 | 21, 140 | 2.26 | 328.49 | 13, 140 | 3.61 | 373.29 | 12, 140 | 2.27 | 426.57 | 20, 140 | 0.88 |
| 3 | 1 | 169.71 | 13, 79, 140 | 8.73 | 203.02 | 21, 114, 140 | 16.02 | 230.56 | 20, 60, 140 | 6.52 | 267.30 | 20, 57, 140 | 4.31 |
| | 0.5 | 188.04 | 14, 79, 140 | 8.69 | 229.57 | 20, 60, 140 | 13.91 | 259.50 | 20, 60, 140 | 4.21 | 306.41 | 20, 58, 140 | 4.06 |
| | 0.3 | 197.97 | 14, 79, 140 | 10.49 | 241.89 | 20, 60, 140 | 12.73 | 270.85 | 20, 60, 140 | 4.47 | 325.56 | 20, 60, 140 | 4.07 |
| | 0.1 | 215.62 | 13, 77, 140 | 7.67 | 261.15 | 10, 62, 140 | 8.15 | 295.57 | 20, 60, 140 | 3.63 | 357.68 | 20, 57, 140 | 2.95 |
| | 0.05 | 227.29 | 14, 77, 140 | 7.14 | 266.89 | 10, 62, 140 | 5.81 | 313.16 | 20, 60, 140 | 4.16 | 380.35 | 20, 57, 140 | 4.16 |
| | 0.01 | 235.09 | 13, 79, 140 | 3.93 | 274.51 | 10, 58, 140 | 3.80 | 339.25 | 14, 20, 140 | 3.41 | 426.57 | 20, 140, 141 | 1.04 |
| 4 | 1 | 155.11 | 12, 26, 120, 140 | 29.83 | 185.87 | 20, 22, 112, 140 | 26.71 | 221.83 | 20, 28, 53, 140 | 15.01 | 259.83 | 14, 20, 25, 140 | 4.48 |
| | 0.5 | 170.77 | 12, 62, 79, 140 | 23.78 | 206.99 | 20, 22, 112, 140 | 13.58 | 250.58 | 15, 20, 26, 140 | 12.56 | 297.69 | 14, 20, 25, 140 | 3.81 |
| | 0.3 | 179.41 | 12, 62, 79, 140 | 21.95 | 217.51 | 20, 22, 112, 140 | 12.47 | 262.08 | 20, 22, 26, 140 | 7.70 | 315.38 | 14, 20, 25, 140 | 3.04 |
| | 0.1 | 195.01 | 12, 62, 79, 140 | 16.01 | 235.32 | 20, 22, 112, 140 | 7.85 | 283.68 | 15, 20, 26, 140 | 6.42 | 350.63 | 14, 20, 25, 140 | 4.92 |
| | 0.05 | 206.04 | 10, 58, 79, 140 | 15.08 | 251.13 | 10, 60, 79, 140 | 10.64 | 296.75 | 4, 20, 60, 140 | 4.77 | 366.93 | 4, 20, 58, 140 | 3.24 |
| | 0.01 | 217.06 | 10, 26, 79, 140 | 6.96 | 268.70 | 10, 62, 140, 199 | 5.07 | 319.93 | 4, 20, 60, 140 | 3.77 | 426.57 | 20, 139, 140, 141 | 1.30 |
| 5 | 1 | 141.11 | 15, 20, 29, 115, 140 | 100.18 | 171.19 | 20, 22, 28, 117, 140 | 24.08 | 213.36 | 20, 26, 53, 117, 140 | 22.43 | 255.74 | 14, 20, 25, 31, 140 | 6.92 |
| | 0.5 | 155.26 | 15, 20, 29, 115, 140 | 55.74 | 191.39 | 20, 22, 28, 117, 140 | 19.17 | 242.57 | 20, 26, 53, 115, 140 | 21.46 | 294.78 | 14, 20, 25, 31, 140 | 5.40 |
| | 0.3 | 162.66 | 15, 20, 29, 115, 140 | 52.72 | 201.60 | 20, 22, 28, 116, 140 | 15.72 | 255.05 | 20, 22, 25, 111, 140 | 16.77 | 312.81 | 14, 20, 25, 31, 140 | 5.61 |
| | 0.1 | 176.82 | 10, 11, 29, 116, 140 | 38.54 | 217.54 | 20, 22, 28, 116, 140 | 9.03 | 274.75 | 4, 20, 22, 26, 140 | 10.54 | 345.41 | 4, 14, 20, 25, 140 | 3.58 |
| | 0.05 | 185.86 | 10, 11, 29, 115, 140 | 33.26 | 232.62 | 15, 20, 28, 140, 159 | 13.27 | 283.85 | 4, 20, 22, 26, 140 | 5.40 | 365.76 | 4, 14, 20, 25, 140 | 2.29 |
| | 0.01 | 204.23 | 10, 31, 53, 140, 159 | 12.07 | 244.45 | 4, 20, 60, 79, 140 | 5.47 | 319.93 | 4, 20, 25, 62, 140 | 2.44 | 426.57 | 12, 13, 20, 140, 141 | 1.36 |
| Average | | | | 19.88 | | | 10.56 | | | 7.29 | | | 3.51 |

### 5.5. *Comparison with robust optimization models*

Robust optimization (RO) is a conservative approach for dealing with uncertainty in optimization problems. In some RO frameworks such as the one developed by Bertsimas and Sim (2003), the level of conservatism can be adjusted by the decision maker. From this perspective, we can compare the RO approach with the conditional $\beta$-mean approach that uses a similar concept to control the level of risk-averseness. Therefore, in this section we compare the results obtained by our proposed conditional $\beta$-mean model with those of a robust optimization model developed by Ghaffarinasab (2021). Since the proposed RO model is based on the uncapacitated $p$-hub median problem, we compare it with our risk-averse $p$-hub median problem (RA$p$HMP). The RO model proposed by Ghaffarinasab (2021) assumes that for every $(i,j) \in A$, the corresponding O/D traffic volume, denoted by $\tilde{w}_{ij}$, is uncertain and it is realized within the interval $[w_{ij} - \bar{w}_{ij}, w_{ij} + \bar{w}_{ij}]$ where $w_{ij}$ and $\bar{w}_{ij}$ denote the center (the nominal value) and the radius of that

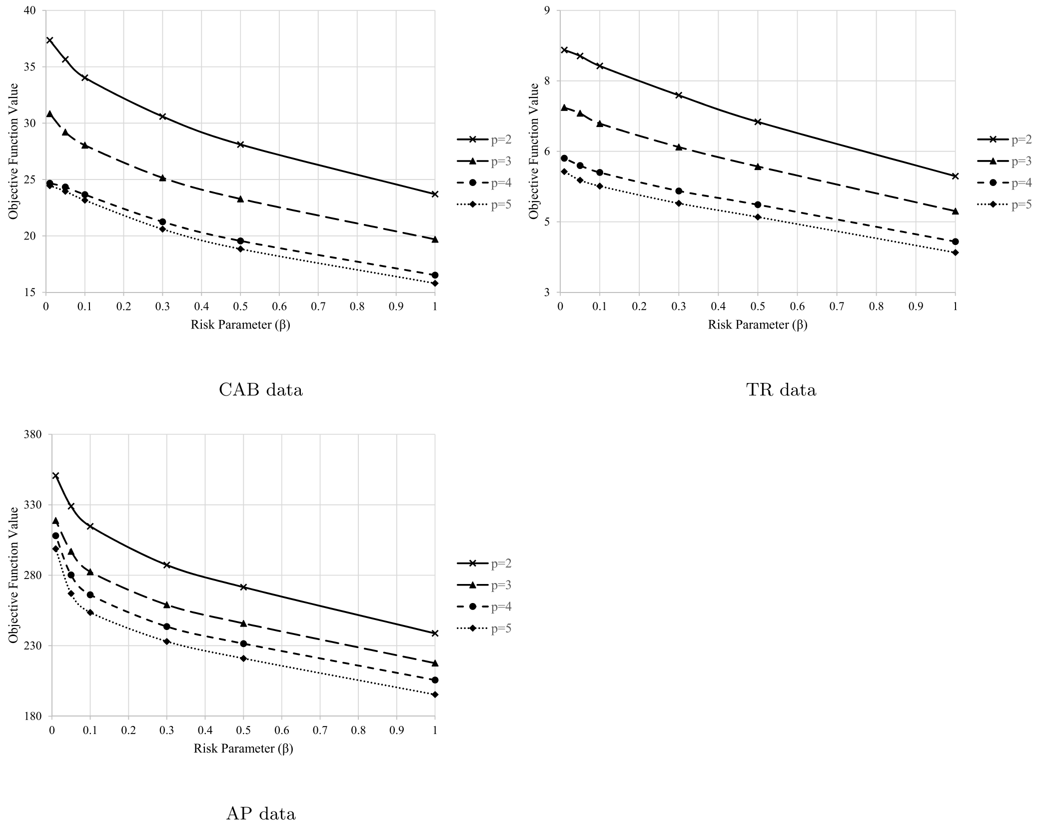



**Fig. 4.** Changes in the objective value with respect to $\beta$ for the RAW$p$HCP.

interval, respectively. The integer parameter $\Gamma$, known as *the budget of uncertainty*, represents the maximum number of O/D flow parameters that can deviate from their nominal values at the same time. Therefore, $\Gamma$ can take its values from the set $\{0, \ldots, |A|\}$ and it is used for adjusting the degree of decision maker's conservatism in the RO model. If $\Gamma = 0$ the problem reduces to the risk-neutral deterministic problem. In contrast, when $\Gamma = |A|$ we deal with "extreme risk-averseness" where all the flow parameters are assumed to take their most undesirable (worst-case) values from the corresponding intervals of uncertainty. An intermediate value such as $\Gamma = 0.5 \times |A|$ correspond to "mild risk-averseness" in which only half of the flow parameters take their worst values. In the conditional $\beta$-mean model, the mild and extreme risk-averseness can be achieved by setting the risk parameter as $\beta = 0.5$ and 0.01, respectively.

Table 13 shows the results obtained by solving the RO and conditional $\beta$-mean models with the CAB data set. In order to have a fair comparison between the models, we judiciously select the value of the uncertainty interval radius ($\bar{w}_{ij}$) for the robust model in such a way that the objective function values of the two models be close to each other. Based on preliminary experiments, the radius of the uncertainty interval is set as $\bar{w}_{ij} = 0.1 \times w_{ij}$. The problems are solved under mild and extreme risk-averseness conditions and the objective function values as well as the optimal sets of opened hubs are presented.

As can be observed from Table 13, the optimal objective values for the two models are very close to each other as a result of the selected value for the uncertainty interval radius. However, from a network configuration perspective, the picture is quite different, as in several cases the optimal hub sets obtained by the two models are not the same. In seven out of 16 instances under the mild risk-averseness case, the optimal set of hubs for the robust model is different from that of the $\beta$-mean model. For the extreme risk-averseness, on the other hand, in eleven out of 16 instances the optimal hub sets for the two models are different. These findings show that although the two modeling approaches are similar in terms of adjustable conservatism (risk-averseness), the network configuration rendered by the two approaches might be very dissimilar.

### 5.6. Managerial insights

The numerical experiments yield several managerial insights, which can be used as guidelines in the design of hub networks for the many-to-many transportation and distribution systems in the presence of risk:

- Studying risk is critical to the hub network design problem in all of the three variants studied in this research in the sense that it leads to solutions with a completely different network configuration (i.e., the location of hubs) than the case where the

**Table 13**
Results for the RO and conditional $\beta$-mean models for the $p$-hub median problem with the CAB data set.

| $p$ | $\alpha$ | Mild risk-averseness | | | | Extreme risk-averseness | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Robust model ($\Gamma = 0.5 \times \vert A \vert$) | | $\beta$-mean model ($\beta = 0.5$) | | Robust model ($\Gamma = \vert A \vert$) | | $\beta$-mean model ($\beta = 0.01$) | |
| | | OF | Hubs | OF | Hubs | OF | Hubs | OF | Hubs |
| 2 | 0.2 | 1081.25 | 12, 20 | 1030.10 | 12, 20 | 1095.62 | 12, 20 | 1082.59 | 12, 20 |
| | 0.4 | 1164.26 | 12, 20 | 1107.48 | 12, 20 | 1179.74 | 12, 20 | 1164.51 | 12, 20 |
| | 0.6 | 1234.80 | 12, 20 | 1173.14 | 12, 20 | 1250.79 | 12, 20 | 1231.05 | 12, 20 |
| | 0.8 | 1281.92 | 12, 20 | 1216.81 | 12, 20 | 1298.02 | 12, 20 | 1271.81 | 12, 20 |
| 3 | 0.2 | 815.50 | 4, 12, 17 | 790.10 | 12, 18, 21 | 828.20 | 12, 17, 21 | 841.80 | 18, 21, 22 |
| | 0.4 | 931.17 | 4, 12, 17 | 897.31 | 12, 18, 21 | 945.60 | 4, 12, 17 | 953.25 | 12, 18, 21 |
| | 0.6 | 1028.92 | 4, 12, 17 | 987.50 | 12, 18, 21 | 1044.15 | 4, 12, 17 | 1039.91 | 2, 12, 21 |
| | 0.8 | 1106.57 | 4, 12, 17 | 1057.68 | 12, 18, 21 | 1122.04 | 4, 12, 17 | 1107.69 | 2, 12, 21 |
| 4 | 0.2 | 669.11 | 4, 12, 17, 24 | 653.18 | 4, 12, 17, 24 | 680.33 | 4, 12, 17, 24 | 716.17 | 4, 12, 16, 17 |
| | 0.4 | 816.98 | 4, 12, 17, 24 | 790.47 | 4, 12, 17, 24 | 829.94 | 4, 12, 17, 24 | 846.75 | 1, 4, 12, 18 |
| | 0.6 | 939.70 | 1, 4, 12, 17 | 904.89 | 4, 7, 12, 18 | 953.09 | 1, 4, 12, 17 | 954.54 | 1, 4, 12, 17 |
| | 0.8 | 1033.16 | 1, 4, 12, 17 | 991.53 | 1, 4, 12, 17 | 1046.93 | 1, 4, 12, 17 | 1042.20 | 1, 4, 12, 18 |
| 5 | 0.2 | 573.20 | 4, 7, 12, 14, 17 | 555.38 | 4, 7, 12, 14, 17 | 583.00 | 4, 7, 12, 14, 17 | 610.98 | 4, 7, 12, 17, 24 |
| | 0.4 | 732.08 | 4, 7, 12, 14, 17 | 704.10 | 4, 7, 12, 14, 17 | 743.97 | 4, 7, 12, 14, 17 | 762.43 | 4, 7, 12, 18, 24 |
| | 0.6 | 871.94 | 4, 7, 12, 14, 17 | 836.03 | 4, 7, 12, 14, 18 | 885.18 | 4, 7, 12, 14, 17 | 893.92 | 2, 4, 7, 12, 24 |
| | 0.8 | 987.82 | 4, 7, 12, 14, 17 | 945.03 | 4, 7, 12, 17, 24 | 1001.39 | 4, 7, 12, 17, 24 | 1003.30 | 1, 4, 12, 17, 22 |

decision maker is risk-neutral. The changes in the location of hubs compared to the risk-neutral case is more significant when the level of risk-averseness increases (i.e., the value of $\beta$ decreases).

- The efficiency of the hub network substantially diminishes by increasing the level of risk-averseness. In particular, by decreasing the value of $\beta$, the total cost is increases in the $p$-hub median problem, the percentage covered demand decreases in the $p$-hub maximal covering problem, and the weighted maximum cost increases in the weighted $p$-hub center problem. For this reason, one needs to make a careful trade-off between the network's efficiency and the riskiness.
- The effect of discount factor value on the configuration of hub networks cannot be neglected, and most obviously, the optimal locations of hubs are sensitive to the discount factor value.

## 6. Conclusions

This paper proposes three variants of the risk-averse multiple allocation hub location problems under demand uncertainty using $\beta$-mean risk measure. The uncertainty was captured using a set of finite scenarios and the average performance over the $100\times\beta\%$ of the worst scenarios was used as the objective function in the three problems, namely the $p$-hub median, the $p$-hub maximal covering, and the weighted $p$-hub center problems. MILP formulations were developed and efficient Benders decomposition procedures were proposed for solving them. A large set of computational experiments were designed and performed to demonstrate the efficiency of the solution procedures and to study the effect of various input parameters on the optimal solutions using three well-known data sets from the HLP literature with up to 200 nodes and 100 scenarios. Results indicate that the optimal set of hub locations varies significantly as the value of the risk parameter alters. In other words, depending on the decision maker's level of conservatism, different network configurations are obtained by the proposed models. Results also demonstrate that large-scale instances of the problems can be solved by the developed BD algorithms in quite short computational times.

An interesting direction for further research is to study the same problems under the single allocation setting. Moreover, one can relax some of the classical HLP assumptions such as complete network between the installed hubs or flow-independent discount on transportation costs on the inter-hub network so that the problems better reflect the real-world situations.

## CRediT authorship contribution statement

**Nader Ghaffarinasab:** Conceptualization, Methodology, Software, Investigation, Computing Resources, Writing – original draft. **Bahar Y. Kara:** Conceptualization, Validation, Writing – review & editing.

## Acknowledgments

The authors sincerely thank the three anonymous referees for their valuable comments and suggestions that have helped improve the content and presentation of the paper.

## References


Alumur, S.A., Campbell, J.F., Contreras, I., Kara, B.Y., Marianov, V., O'Kelly, M.E., 2021. Perspectives on modeling hub location problems. European J. Oper. Res. 291 (1), 1–17.

Alumur, S., Kara, B.Y., 2008. Network hub location problems: The state of the art. European J. Oper. Res. 190 (1), 1–21.

Alumur, S.A., Nickel, S., Saldanha-da-Gama, F., 2012. Hub location under uncertainty. Transp. Res. B 46 (4), 529–543.

An, Y., Zhang, Y., Zeng, B., 2015. The reliable hub-and-spoke design problem: Models and algorithms. Transp. Res. B 77, 103–122.

Benders, J.F., 1962. Partitioning procedures for solving mixedvariables programming problems. Numer. Math. 4 (1), 238–252.

Bertsimas, D., Sim, M., 2003. Robust discrete optimization and network flows. Math. Program. 98 (1–3), 49–71.

Birge, J.R., Louveaux, F., 2011. Introduction to Stochastic Programming. Springer Science & Business Media.

Campbell, J.F., 1992. Location and allocation for distribution systems with transshipments and transportion economies of scale. Ann. Oper. Res. 40 (1), 77–99.

Campbell, J.F., 1994. Integer programming formulations of discrete hub location problems. European J. Oper. Res. 72 (2), 387–405.

Campbell, J.F., O'Kelly, M.E., 2012. Twenty-five years of hub location research. Transp. Sci. 46 (2), 153–169.

Chaharsooghi, S., Momayezi, F., Ghaffarinasab, N., 2017. An adaptive large neighborhood search heuristic for solving the reliable multiple allocation hub location problem under hub disruptions. Int. J. Ind. Eng. Comput. 8 (2), 191–202.

Contreras, I., Cordeau, J.-F., Laporte, G., 2011a. Benders decomposition for large-scale uncapacitated hub location. Oper. Res. 59 (6), 1477–1490.

Contreras, I., Cordeau, J.-F., Laporte, G., 2011b. Stochastic uncapacitated hub location problem. European J. Oper. Res. 212 (3), 518–528.

Contreras, I., Cordeau, J.-F., Laporte, G., 2012. Exact solution of large-scale hub location problems with multiple capacity levels. Transp. Sci. 46 (4), 439–459.

Contreras, I., O'Kelly, M., 2019. Hub location problems. In: Location Science. Springer, pp. 327–363.

Correia, I., Nickel, S., da Gama, F.S., 2018. A stochastic multi-period capacitated multiple allocation hub location problem: Formulation and inequalities. Omega 74, 122–134.

de Camargo, R.S., Jr., G.M., Ferreira, R.P.M., Luna, H.P., 2009b. Multiple allocation hub-and-spoke network design under hub congestion. Comput. Oper. Res. 36 (12), 3097–3106.

de Camargo, R.S., Jr., G.M., Luna, H.P., 2008. Benders decomposition for the uncapacitated multiple allocation hub location problem. Comput. Oper. Res. 35 (4), 1047–1064.

de Camargo, R.S., Jr., G.M., Luna, H.P., 2009a. Benders decomposition for hub location problems with economies of scale. Transp. Sci. 43 (1), 86–97.

de Camargo, R.S., de Miranda Jr., G., Ferreira, R.P., 2011. A hybrid Outer-Approximation/Benders Decomposition algorithm for the single allocation hub location problem under congestion. Oper. Res. Lett. 39 (5), 329–337.

de Camargo, R.S., de Miranda Jr., G., Løkketangen, A., 2013. A new formulation and an exact approach for the many-to-many hub location-routing problem. Appl. Math. Model. 37 (12–13), 7465–7480.

de Sá, E.M., de Camargo, R.S., de Miranda Jr., G., 2013. An improved Benders decomposition algorithm for the tree of hubs location problem. European J. Oper. Res. 226 (2), 185–202.

de Sá, E.M., Contreras, I., Cordeau, J.-F., de Camargo, R.S., de Miranda Jr., G., 2015. The hub line location problem. Transp. Sci. 49 (3), 500–518.

de Sá, E.M., Morabito, R., de Camargo, R.S., 2018. Benders decomposition applied to a robust multiple allocation incomplete hub location problem. Comput. Oper. Res. 89, 31–50.

Ernst, A.T., Krishnamoorthy, M., 1996. Efficient algorithms for the uncapacitated single allocation p-hub median problem. Locat. Sci. 4 (3), 139–154.

Farahani, R.Z., Hekmatfar, M., Arabani, A.B., Nikbakhsh, E., 2013. Hub location problems: A review of models, classification, solution techniques, and applications. Comput. Ind. Eng. 64 (4), 1096–1109.

Filippi, C., Guastaroba, G., Huerta-Munoz, D., Speranza, M., 2021a. A kernel search heuristic for a fair facility location problem. Comput. Oper. Res. 132, 105292.

Filippi, C., Guastaroba, G., Speranza, M.G., 2020. Conditional value-at-risk beyond finance: a survey. Int. Trans. Oper. Res. 27 (3), 1277–1319.

Filippi, C., Guastaroba, G., Speranza, M., 2021b. On single-source capacitated facility location with cost and fairness objectives. European J. Oper. Res. 289 (3), 959–974.

Filippi, C., Ogryczak, W., Speranza, M.G., 2019. Bridging $k$-sum and [CVaR] optimization in [MILP]. Comput. Oper. Res. 105, 156–166.

Gao, Y., Qin, Z., 2016. A chance constrained programming approach for uncertain $p$-hub center location problem. Comput. Ind. Eng. 102, 10–20.

Gelareh, S., Monemi, R.N., Nickel, S., 2015. Multi-period hub location problems in transportation. Transp. Res. E 75, 67–94.

Gelareh, S., Nickel, S., 2011. Hub location problems in transportation networks. Transp. Res. E 47 (6), 1092–1111.

Gelareh, S., Pisinger, D., 2011. Fleet deployment, network design and hub location of liner shipping companies. Transp. Res. E 47 (6), 947–964.

Ghaffari-Nasab, N., Ghazanfari, M., Teimoury, E., 2015. Robust optimization approach to the design of hub-and-spoke networks. Int. J. Adv. Manuf. Technol. 76 (5–8), 1091–1110.

Ghaffarinasab, N., 2018. An efficient matheuristic for the robust multiple allocation p-hub median problem under polyhedral demand uncertainty. Comput. Oper. Res. 97, 31–47.

Ghaffarinasab, N., 2020. A highly efficient exact algorithm for the uncapacitated multiple allocation p-hub center problem. Decis. Sci. Lett. 9 (2), 181–192.

Ghaffarinasab, N., 2021. Exact algorithms for the robust uncapacitated multiple allocation $p$-hub median problem. Optim. Lett. http://dx.doi.org/10.1007/s11590-021-01799-w.

Ghaffarinasab, N., Atayi, R., 2018. An implicit enumeration algorithm for the hub interdiction median problem with fortification. European J. Oper. Res. 267 (1), 23–39.

Ghaffarinasab, N., Kara, B.Y., 2019. Benders decomposition algorithms for two variants of the single allocation hub location problem. Netw. Spat. Econ. 19 (1), 83–108.

Ghaffarinasab, N., Motallebzadeh, A., 2018. Hub interdiction problem variants: Models and metaheuristic solution algorithms. European J. Oper. Res. 267 (2), 496–512.

Ghaffarinasab, N., Zare Andaryan, A., Ebadi Torkayesh, A., 2020. Robust single allocation p-hub median problem under hose and hybrid demand uncertainties: Models and algorithms. Int. J. Manage. Sci. Eng. Manage. 15 (3), 184–195.

Golpîra, H., Zandieh, M., Najafi, E., Sadi-Nezhad, S., 2017. A multi-objective, multi-echelon green supply chain network design problem with risk-averse retailers in an uncertain environment. Sci. Iran. Trans. E Ind. Eng. 24 (1), 413.

Hamacher, H.W., Labbé, M., Nickel, S., Sonneborn, T., 2004. Adapting polyhedral properties from facility to hub location problems. Discrete Appl. Math. 145 (1), 104–116.

Hosseini, S.D., Verma, M., 2018. Conditional value-at-risk (CVaR) methodology to optimal train configuration and routing of rail hazmat shipments. Transp. Res. B 110, 79–103.

Kim, H., O'Kelly, M.E., 2009. Reliable p-hub location problems in telecommunication networks. Geogr. Anal. 41 (3), 283–306.

Korani, E., Eydi, A., 2021. Bi-level programming model and KKT penalty function solution approach for reliable hub location problem. Expert Syst. Appl. 184, 115505.

Madani, S.R., Nookabadi, A.S., Hejazi, S.R., 2018. A bi-objective, reliable single allocation p-hub maximal covering location problem: Mathematical formulation and solution approach. J. Air Transp. Manage. 68, 118–136.

Marianov, V., Serra, D., 2003. Location models for airline hubs behaving as M/D/c queues. Comput. Oper. Res. 30 (7), 983–1003.

Merakli, M., Yaman, H., 2016. Robust intermodal hub location under polyhedral demand uncertainty. Transp. Res. B 86, 66–85.

Mohammadi, M., Jolai, F., Rostami, H., 2011. An M/M/c queue model for hub covering location problem. Math. Comput. Modelling 54 (11–12), 2623–2638.
Mohammadi, M., Jula, P., Tavakkoli-Moghaddam, R., 2019. Reliable single-allocation hub location problem with disruptions. Transp. Res. E 123, 90–120.
Monemi, R.N., Gelareh, S., Nagih, A., Maculan, N., Danach, K., 2021. Multi-period hub location problem with serial demands: A case study of humanitarian aids distribution in Lebanon. Transp. Res. E 149, 102201.
Nikokalam-Mozafar, S.H., Ashjari, B., Tavakkoli-Moghaddam, R., Omidvar, A., 2014. Solving a multi-objective chance-constrained hub covering location problem by discrete invasive weed optimization. Cogent Eng. 1 (1), 991526.
Noyan, N., 2012. Risk-averse two-stage stochastic programming with an application to disaster management. Comput. Oper. Res. 39 (3), 541–559.
Ogryczak, W., Zawadzki, M., 2002. Conditional median: A parametric solution concept for location problems. Ann. Oper. Res. 110 (1–4), 167–181.
O'Kelly, M.E., 1987. A quadratic integer program for the location of interacting hub facilities. European J. Oper. Res. 32 (3), 393–404.
O'Kelly, M.E., Luna, H.P.L., de Camargo, R.S., de Miranda Jr., G., 2015. Hub location problems with price sensitive demands. Netw. Spat. Econ. 15 (4), 917–945.
Özgün Elçi, Noyan, N., 2018. A chance-constrained two-stage stochastic programming model for humanitarian relief network design. Transp. Res. B 108, 55–83.
Rahmaniani, R., Crainic, T.G., Gendreau, M., Rei, W., 2017. The Benders decomposition algorithm: A literature review. European J. Oper. Res. 259 (3), 801–817.
Rockafellar, R.T., Uryasev, S., 2000. Optimization of conditional value-at-risk. J. Risk 2, 21–42.
Rodriguez, V., Alvarez, M., Barcos, L., 2007. Hub location under capacity constraints. Transp. Res. E 43 (5), 495–505.
Rostami, B., Kämmerling, N., Buchheim, C., Clausen, U., 2018. Reliable single allocation hub location problem under hub breakdowns. Comput. Oper. Res. 96, 15–29.
Rostami, B., Kämmerling, N., Naoum-Sawaya, J., Buchheim, C., Clausen, U., 2021. Stochastic single-allocation hub location. European J. Oper. Res. 289 (3), 1087–1106.
Shahabi, M., Unnikrishnan, A., 2014. Robust hub network design problem. Transp. Res. E 70, 356–373.
Shen, H., Liang, Y., Shen, Z.-J.M., 2021. Reliable hub location model for air transportation networks under random disruptions. Manuf. Serv. Oper. Manage. 23 (2), 388–406.
Sim, T., Lowe, T.J., Thomas, B.W., 2009. The stochastic $p$-hub center problem with service-level constraints. Comput. Oper. Res. 36 (12), 3166–3177.
Taherkhani, G., Alumur, S.A., Hosseini, M., 2020. Benders decomposition for the profit maximizing capacitated hub location problem with multiple demand classes. Transp. Sci. 54 (6), 1446–1470.
Tan, P.Z., Kara, B.Y., 2007. A hub covering model for cargo delivery systems. Networks 49 (1), 28–39.
Yang, T.-H., 2009. Stochastic air freight hub location and flight routes planning. Appl. Math. Model. 33 (12), 4424–4430.
Yu, G., Haskell, W.B., Liu, Y., 2017. Resilient facility location against the risk of disruptions. Transp. Res. B 104, 82–105.
Zetina, C.A., Contreras, I., Cordeau, J.-F., Nikbakhsh, E., 2017. Robust uncapacitated hub location. Transp. Res. B 106, 393–410.
Zhalechian, M., Torabi, S.A., Mohammadi, M., 2018. Hub-and-spoke network design under operational and disruption risks. Transp. Res. E 109, 20–43.