EQUILIBRIUM CANTOR-TYPE SETS

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Abstract. Equilibrium Cantor-type sets are suggested. This allows to obtain Green functions with various moduli of continuity and compact sets with preassigned growth of Markov's factors.

1. Introduction

If a compact set $K \subset \mathbb{C}$ is regular with respect to the Dirichlet problem then the Green function $g_{\mathbb{C}\setminus K}$ of $\mathbb{C} \setminus K$ with pole at infinity is continuous throughout $\mathbb{C}$. We are interested in analysis of a character of smoothness of $g_{\mathbb{C}\setminus K}$ near the boundary of K. For example, if $K \subset \mathbb{R}$ then the monotonicity of the Green function with respect to the set K implies that the best possible behavior of $g_{\mathbb{C}\setminus K}$ is $Lip\frac{1}{2}$ smoothness. An important characterization for general compact sets with $g_{\mathbb{C}\setminus K} \in Lip\frac{1}{2}$ was found in [17] by V.Totik. The monograph [17] revives interest in the problem of boundary behavior of Green functions. Various conditions for optimal smoothness of $g_{\mathbb{C}\setminus K}$ in terms of metric properties of the set K are suggested in [7], and in papers by V.Andrievskii [2]-[3]. On the other hand, compact sets are considered in [1], [8] such that the corresponding Green functions have moduli of continuity equal to some degrees of h, where the function $h(\delta) = (\log \frac{1}{\delta})^{-1}$ defines the logarithmic measure of sets. For a recent result on smoothness of $g_{\mathbb{C}\setminus K_0}$, where K_0 is the classical Cantor set, see [13].

Here the Cantor-type set $K(\gamma)$ is constructed as the intersection of the level domains for a certain sequence of polynomials depending on the parameter $\gamma = (\gamma_n)_{n=1}^{\infty}$ (Section 2). In favor of $K(\gamma)$, in comparison to usual Cantor-type sets, it is equilibrium in the following sense.

Let λ_s denote the normalized Lebesgue measure on the closed set E_s, where $K(\gamma) = \cap_{s=0}^{\infty} E_s$. Then λ_s converges in the weak* topology to the equilibrium measure of $K(\gamma)$ (Section 5). This is not valid for geometrically symmetric, though very small Cantor-type sets with positive capacity.

Different values of γ provide a variety of the Green functions with diverse moduli of continuity (Section 7).

In Section 8 we estimate Markov's factors for the set $K(\gamma)$ and construct a set with preassigned growth of subsequence of Markov's factors.

In Section 9 a set $K(\gamma)$ is presented such that the Markov inequality on $K(\gamma)$ does not hold with the best Markov's exponent $m(K(\gamma))$. This gives an affirmative answer to the problem (5.1) in [4].

For basic notions of logarithmic potential theory we refer the reader to [10], [12], and [15].

2000 *Mathematics Subject Classification.* 31A15, 41A10, 41A17.

Key words and phrases. Green's function, Modulus of continuity, Markov's factors.

We use the notation $|\cdot|_K$ for the supremum norm on K, log denotes the natural logarithm, $0\cdot\log 0 := 0$.

2. **Construction of** $K(\gamma)$

Suppose we are given a sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0<\gamma_s<1/4$. Let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s\in\mathbb{N}$. We define inductively a sequence of real polynomials: let $P_2(x) = x(x-1)$ and $P_{2^{s+1}} = P_{2^s}(P_{2^s}+r_s)$ for $s\in\mathbb{N}$. It is easy to check by induction that the polynomial P_{2^s} has 2^{s-1} points of minimum with equal values $P_{2^s} = -r_{s-1}^2/4$. By that we have a geometric procedure to define new (with respect to P_{2^s}) zeros of $P_{2^{s+1}}$: they are abscissas of points of intersection of the line $y=-r_s$ with the graph $y = P_{2^s}$. Let E_s denote the set $\{x\in\mathbb{R} : P_{2^{s+1}}(x)\le 0\}$. Since $r_s < r_{s-1}^2/4$, the set E_s consists of 2^s disjoint closed *basic intervals* $I_{j,s}$. In general, the lengths $l_{j,s}$ of intervals of the same level are different, however, by the construction of $K(\gamma)$, we have $\max_{1\le j\le 2^s} l_{j,s}\to 0$ as $s\to\infty$. Clearly, $E_{s+1}\subset E_s$. Set $K(\gamma) = \cap_{s=0}^{\infty}E_s$.

Let us show that the sequence of level domains $D_s = \{z\in\mathbb{C} : |P_{2^s}(z)+r_s/2| < r_s/2\},\ s=1,2,\cdots$, is a nested family.

Lemma 1. *Given $z\in\mathbb{C}$ and $s\in\mathbb{N}$, let $w_s = 2\,r_s^{-1}\,P_{2^s}(z)+1$. Suppose $|w_s| = 1+\varepsilon$ for some $\varepsilon>0$. Then $|w_{s+1}|>1+4\,\varepsilon$.*

Proof: We have $w_{s+1} = (2\,\gamma_{s+1})^{-1}\,(w_s^2-1+2\,\gamma_{s+1})$. Therefore, $|w_{s+1}|$ attains its minimal value if $w_s\in\mathbb{R}$, so $|w_{s+1}| > (2\,\gamma_{s+1})^{-1}\,(2\,\varepsilon+\varepsilon^2+2\,\gamma_{s+1}) > 1+\frac{\varepsilon}{\gamma_{s+1}} > 1+4\,\varepsilon$. □

Theorem 1. *We have $\overline{D}_s \searrow K(\gamma)$.*

Proof: The embedding $\overline{D}_{s+1}\subset\overline{D}_s$ is equivalent to the implication

$$|P_{2^s}(z)+r_s/2| > r_s/2 \implies |P_{2^{s+1}}(z)+r_{s+1}| > r_{s+1}/2,$$

which we have by Lemma 1.

For each $j\le 2^s$ the real polynomial P_{2^s} is monotone on $I_{j,s}$ and takes values 0 and $-r_s$ at its endpoints. Therefore, $E_s\subset\overline{D}_s$ and $K(\gamma)\subset\cap_{s=0}^{\infty}\overline{D}_s$.

For the inverse embedding, let us fix $z\notin K(\gamma)$. We need to find s with $z\notin\overline{D}_s$. Suppose first $z\in\mathbb{R}$. Since $\overline{D}_s\cap\mathbb{R} = E_s$, the condition $z\notin E_s$ gives the desired s.

Let $z = x+i\,y$ with $y\neq 0, x\notin K(\gamma)$. By the above, $x\notin\overline{D}_s$ for some s. All zeros $(x_j)_{j=1}^{2^s}$ of the polynomial $P_{2^s}+r_s/2$ are real. Therefore, $|P_{2^s}(z)+r_s/2| > |P_{2^s}(x)+r_s/2| > r_s/2$ and $z\notin\overline{D}_s$.

It remains to consider the case $z = x+i\,y$ with $y\neq 0, x\in K(\gamma)$. There is no loss of generality in assuming $|y|<2$. Let us fix s with $\max_{1\le j\le 2^s} l_{j,s} < y^2/2$ and k with $x\in I_{k,s} = [a,b]$. Here, $|P_{2^s}(a)+r_s/2| = r_s/2$. Let us show that $|P_{2^s}(z)+r_s/2| > |P_{2^s}(a)+r_s/2|$ by comparison the distances from z and from a to the zero x_j.

If $j<k$ then $|a-x_j|\le|a-x|$, which is less than the hypotenuse $|z-x_j|$.

If $j=k$ then $|a-x_k|\le l_{k,s} < y^2/2 < |y| \le |z-x_k|$.

If $j>k$ then $|a-x_j| = x_j-b+l_{k,s}$. Therefore, $|a-x_j|^2 < |x_j-b|^2+2\,l_{k,s} < |x_j-b|^2+y^2\le|z-x_j|^2$. □

Corollary 1. *The set $K(\gamma)$ is polar if and only if* $\lim_{s\to\infty} 2^{-s} \log \frac{2}{r_s} = \infty$. *If this limit is finite and $z \notin K(\gamma)$, then*

$$g_{\mathbb{C}\setminus K(\gamma)}(z) = \lim_{s\to\infty} 2^{-s} \log |P_{2^s}(z)/r_s|.$$

Proof: Clearly, $g_{\mathbb{C}\setminus \overline{D}_s}(z) = 2^{-s} \log |2\, r_s^{-1} P_{2^s}(z) + 1|$. The sequence of the corresponding Robin constants $Rob(\overline{D}_s) = 2^{-s} \log \frac{2}{r_s}$ increases. If its limit is finite, then, by the Harnack Principle (see e.g. [15], Th.0.4.10), $g_{\mathbb{C}\setminus \overline{D}_s} \nearrow g_{\mathbb{C}\setminus K(\gamma)}$ uniformly on compact subsets of $\mathbb{C} \setminus K(\gamma)$. Suppose $z \notin K(\gamma)$. Then for some $q \in \mathbb{N}$ and $\varepsilon > 0$ we have $|w_q| = 1 + \varepsilon$. By Lemma 1, $|w_s| > 1 + 4^{s-q}\,\varepsilon$, so, for large s, the value $|P_{2^s}(z)/r_s|$ dominates 1. This gives the desired representation of $g_{\mathbb{C}\setminus K(\gamma)}$. □

The next corollary is a consequence of the Kolmogorov criterion (see e.g. [9], T.3.2.1). Recall that a monic polynomial Q_n is a Chebyshev polynomial for a compact set K if the value $|Q_n|_K$ is minimal among all monic polynomials of degree n.

Corollary 2. *The polynomial $P_{2^s} + r_s/2$ is the Chebyshev polynomial for the set $K(\gamma)$.*

Example 1. Let us consider the limit case, when $\gamma_s = 1/4$ for all s, so $r_s = 4^{1-2^s}$. Since here $K(\gamma) = [0,1]$, the $n-$th Chebyshev polynomial is $Q_n(x) = 2^{-n}\, T_n(2x-1)$, where T_n is the monic Chebyshev polynomial for $[-1,1]$, that is $T_n(t) = 2^{1-n} \cos(n \arccos t)$ for $n \in \mathbb{N}$. Therefore, in this case, $P_{2^s}(x) + r_s/2 = 2^{-2^s}\, T_{2^s}(2x-1)$ for $s \in \mathbb{N}$.

3. **Location of zeros**

We decompose all zeros of P_{2^s} into s groups. Let $X_0 = \{x_1, x_2\} = \{0, 1\}$, $X_1 = \{x_3, x_4\} = \{l_{1,1}, 1-l_{2,1}\}, \cdots, X_k = \{x_{2^k+1}, \cdots, x_{2^{k+1}}\} = \{l_{1,k}, l_{1,k-1}-l_{2,k}, \cdots, 1-l_{2^k,k}\}$, so $X_k = \{x : P_{2^k}(x) + r_k = 0\}$ contains all zeros of $P_{2^{k+1}}$ that are not zeros of P_{2^k}. Set $Y_s = \cup_{k=0}^{s} X_k$. Then $P_{2^s}(x) = \prod_{x_k \in Y_{s-1}} (x - x_k)$. Since $P'_{2^s} = P'_{2^{s-1}}(2\,P_{2^{s-1}} + r_{s-1})$ for $s \geq 2$, we have

$$P'_{2^s}(y) = r_{s-1}\, P'_{2^{s-1}}(y),\ y \in Y_{s-2};\ \ P'_{2^s}(x) = -r_{s-1} P'_{2^{s-1}}(x),\ x \in X_{s-1}. \tag{1}$$

After iteration this gives

$$|P'_{2^s}(x)| = r_{s-1}\, r_{s-2}\, \cdots r_q\, |P'_{2^q}(x)| \quad \text{for} \quad x \in X_q \quad \text{with} \quad q < s. \tag{2}$$

From here, for example, $|P'_{2^s}(0)| = r_{s-1}\, r_{s-2}\, \cdots r_1$.

The identity $P_{2^{s+1}}(y) = P_{2^s}(y) \prod_{x_k \in X_s} (y - x_k) = P_{2^s}(y)\,(P_{2^s}(y) + r_s)$ implies $P_{2^s}(y) + r_s = \prod_{x_k \in X_s} (y - x_k)$. Thus,

$$\prod_{x_k \in X_s} (y - x_k) = r_s \quad \text{for} \quad y \in Y_{s-1}. \tag{3}$$

Our next goal is to express the values of $x_k \in X_s$ in terms of the function $u(t) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4t}$ with $0 < t < \frac{1}{4}$. Clearly, $u(t)$ and $1 - u(t)$ are the solutions of the equation $P_2(x) + t = 0$. Let us consider the expression

$$x = f_1(\gamma_1 \cdot f_2(\gamma_2 \cdots f_{s-1}(\gamma_{s-1} \cdot f_s(\gamma_s)) \cdots), \tag{4}$$

where $f_k = u$ or $f_k = 1 - u$ for $1 \leq k \leq s$, so $f_k(t)(1 - f_k(t)) = t$. We have $P_2(x) = -\gamma_1 \cdot f_2(\gamma_2 \cdots)$ with $\gamma_1 = r_1$. Hence, $P_4(x) = P_2(x)(P_2(x) + r_1) = -r_1^2 f_2(1-$

$f_2) = -r_1^2\gamma_2 f_3 = -r_2\, f_3(\gamma_3\cdots)$. We continue in this fashion to obtain eventually $P_{2^s}(x) = -r_{s-1}^2\gamma_s = -r_s$, which gives $x \in X_s$.

The formula (4) provides 2^s possible values x. Let us show that they are all different, so any $x_k \in X_s$ can be represented by means of (4). Since u increases and $u(a) < 1 - u(b)$ for $a, b \in (0, \frac{1}{4})$, we have $u(\gamma_1 \cdot u(\gamma_2 \cdots \gamma_m u(a))\cdots) < u(\gamma_1 \cdot u(\gamma_2\cdots\gamma_m(1-u(b))\cdots)$. In general, let $x_i = u(\gamma_1\cdot u(\gamma_2\cdots\gamma_{k_1}(1-u(\gamma_{k_1+1}\cdot u(\cdots\gamma_{k_2}(1-u(\gamma_{k_2+1}\cdots\gamma_{k_m}(1-u(a))\cdots)$ and $x_j = u(\gamma_1\cdot u(\gamma_2\cdots\gamma_{k_1}(1-u(\gamma_{k_1+1}\cdot u(\cdots\gamma_{k_2}(1-u(\gamma_{k_2+1}\cdots\gamma_{k_m}\cdot u(b))\cdots)$, that is the first k_m functions f_k for both points are identical, whereas $f_{k_m+1} = 1-u$ for x_i and u for x_j. The straightforward comparison shows that $x_i > x_j$ for odd m and $x_i < x_j$ otherwise.

Lemma 2. *Let $s \in \mathbb{N}$ and $1 \le j \le 2^s$. Then $l_{1,s} \le l_{j,s}$.*

Proof: Assume without loss of generality that j is odd. Then $I_{j,s} = [y, x]$ with $x \in X_s, y \in X_m$ where $1 \le m \le s-1$. The case $m = 0$ can be excluded, since then $y = 0$ and $j = 1$. Consider the function $F(t) = f_1(\gamma_1 \cdot f_2(\gamma_2\cdots f_{m-1}(\gamma_{m-1}\cdot f_m(t))\cdots)$, where $f_k \in \{u, 1-u\}$ are chosen in a such way that $y = F(\gamma_m)$. Then $x = F(\gamma_m\cdot(1-u(\gamma_{m+1}\cdot u(\gamma_{m+2}\cdots u(\gamma_s))\cdots)$. By the Mean Value Theorem, $l_{j,s} = x - y = |F'(\xi)|\cdot\gamma_m\cdot u(\gamma_{m+1}\cdots u(\gamma_s))\cdots)$ with $\gamma_m - \gamma_m\cdot u(\gamma_{m+1}\cdots u(\gamma_s))\cdots) < \xi < \gamma_m\cdot u(\gamma_{m+1}\cdots u(\gamma_s))\cdots)$. To simplify notations, we write $t_k = \gamma_k\cdot f_{k+1}(\gamma_{k+1}\cdots\gamma_{m-1}\cdot f_m(\xi))\cdots)$ and $\tau_k = \gamma_k\cdot u(\gamma_{k+1}\cdots\gamma_{m-1}\cdot u(\xi))\cdots)$ for $1 \le k \le m-1$. By the above, $\tau_k \le t_k$. Therefore, $|f_k'(t_k)| = \frac{1}{\sqrt{1-4t_k}} \ge \frac{1}{\sqrt{1-4\tau_k}} = u_k'(\tau_k)$. On the other hand, $u(t)\sqrt{1-4t} < t$ for $0 < t < \frac{1}{4}$, as is easy to check. This gives $|F'(\xi)| = |f_1'(t_1)|\cdot\gamma_1\cdots|f_{m-1}'(t_{m-1})|\cdot\gamma_{m-1}\cdot|f_m'(\xi)| > \gamma_1\cdots\gamma_{m-1}\cdot\frac{u(\tau_1)}{\tau_1}\cdot\frac{u(\tau_2)}{\tau_2}\cdots\frac{u(\tau_{m-1})}{\tau_{m-1}}\cdot\frac{u(\xi)}{\xi}$. Since $\tau_k = \gamma_k\cdot u(\tau_{k+1})$ for $k \le m-2$ and $\tau_{m-1} = \gamma_{m-1}\cdot u(\xi)$, we obtain $|F'(\xi)| > \frac{u(\tau_1)}{\xi}$ and

$$l_{j,s} > \frac{u(\tau_1)}{\xi}\cdot\gamma_m\cdot u(\gamma_{m+1}\cdots u(\gamma_s))\cdots).$$

Taking into account the representation $u(t) = \frac{2t}{1+\sqrt{1-4t}}$, we have $u(\alpha\, t) < \alpha\, u(t)$ for $0 < \alpha < 1$. The value $\alpha = \frac{1}{\xi}\cdot\gamma_m\cdot u(\gamma_{m+1}\cdots u(\gamma_s))\cdots)$ satisfies this condition. Therefore, $l_{1,s} = u(\gamma_1\cdot u(\gamma_2\cdots\gamma_{m-1}\cdot u(\xi\,\alpha))\cdots) < \alpha\, u(\tau_1)$, that is $l_{1,s} < l_{j,s}$ for $j \in \{3, 5, \cdots, 2^s - 1\}$, which is the desired conclusion. $\Box$

4. **Auxiliary results**

From now on we make the assumption

$$\gamma_s \le 1/32 \qquad \text{for} \qquad s \in \mathbb{N}. \tag{5}$$

Each $I_{j,s}$ contains two *adjacent* basic subintervals $I_{2j-1,s+1}$ and $I_{2j,s+1}$. Let $h_{j,s} = l_{j,s} - l_{2j-1,s+1} - l_{2j,s+1}$ be the distance between them.

Lemma 3. *Suppose γ satisfies (5). Then the polynomial P_{2^s} is convex on $I_{j,s-1}$ and $l_{2j-1,s} + l_{2j,s} < 4\,\gamma_s\, l_{j,s-1}$ for $1 \le j \le 2^{s-1}$. Thus, $h_{j,s} > \frac{7}{8}\, l_{j,s}$ for $s \ge 0, 1 \le j \le 2^s$.*

Proof: We proceed by induction. If $s = 1$ then P_2 is convex on $I_{1,0} = [0, 1]$. Let us show that $l_{1,1} + l_{2,1} < 4\,\gamma_1$. The triangle Δ with the vertices $(0, 0), (1, 0), (\frac{1}{2}, -\frac{1}{4})$ is entirely situated in the epigraph $\{(x, y) \in \mathbb{R}^2 :\ P_2(x) \le y\}$. The line $y = -r_1$ intersects Δ along the segment $[A, B]$. By convexity of P_2, we have $h_{1,0} = 1 - l_{1,1} - l_{2,1} > |B - A|$.

The triangle Δ_1 with the vertices $A, B, (\frac{1}{2}, -\frac{1}{4})$ is similar to Δ. Therefore, $\frac{1}{4}\ |B-A| = \frac{1}{4} - r_1$. Here, $r_1 = \gamma_1$, and the result follows.

Suppose we have convexity of $P_{2^k}|_{I_{j,k-1}}$ and the desired inequalities for $k = 1, 2, \cdots, s-1$. Fix $j \le 2^{s-1}$ and $x \in I_{j,s-1} = [a, b]$. Then $P_{2^s}(x) = (x-a)(x-b)\, g(x)$, where $g(x) = \prod_{k=1}^{n}(x - z_k)$ with $n = 2^s - 2$. Hence,

$$P''_{2^s}(x) = g(x)\,\left[2 + 2\sum_{k=1}^{n} \frac{2x-a-b}{x-z_k} + \sum_{k=1}^{n}\sum_{i=1, i\neq k}^{n} \frac{(x-a)(x-b)}{(x-z_k)(x-z_i)}\right].$$

Clearly, the polynomial g is positive on $I_{j,s-1}$, $|2x-a-b| \le l_{j,s-1}$, and $|(x-a)(x-b)| \le \frac{1}{4}\, l_{j,s-1}^2$. For convexity of $P_{2^s}|_{I_{j,s-1}}$ we only need to check that $8 \ge 8\, l_{j,s-1}\sum_{k=1}^{n}|x - z_k|^{-1} + l_{j,s-1}^2 \sum_{k=1}^{n}\sum_{i\neq k}|x-z_k|^{-1}|x-z_i|^{-1}$.

Let us consider the basic intervals containing x : $I_{j,s-1} \subset I_{m,s-2} \subset I_{q,s-3} \subset \cdots \subset I_{1,0}$. The interval $I_{m,s-2}$ contains two zeros of g. For them $|x - z_k| \ge h_{m,s-2} > (1 - 4\gamma_{s-1})\, l_{m,s-2}$ and $\frac{l_{j,s-1}}{|x-z_k|} < \frac{4\gamma_{s-1}}{1-4\gamma_{s-1}}$, by inductive hypothesis. The last fraction does not exceed 1/7. Similarly, $I_{q,s-3}$ contains another four zeros of g with $\frac{l_{j,s-1}}{|x-z_k|} < \frac{4\gamma_{s-1}\, 4\gamma_{s-2}}{1-4\gamma_{s-2}} \le \frac{1}{7}\cdot\frac{1}{8}$. We continue in this fashion to obtain $l_{j,s-1}\sum_{k=1}^{n}|x - z_k|^{-1} < \sum_{k=1}^{s-1} 2^k\cdot\frac{1}{7}\cdot(\frac{1}{8})^{k-1} < \frac{8}{21}$.

In the same way, $l_{j,s-1}^2\sum_{k=1}^{n}\sum_{i\neq k}|x-z_k|^{-1}|x-z_i|^{-1} < (\frac{8}{21})^2$, which gives $P''_{2^s}|_{I_{j,s-1}} > 0$. Arguing as above, by means of convexity of $P_{2^s}|_{I_{j,s-1}}$, it is easy to show the second statement of Lemma. $\Box$

Let $\delta_s = \gamma_1\gamma_2\cdots\gamma_s$, so $r_1 r_2\cdots r_{s-1}\,\delta_s = r_s$.

Lemma 4. *If γ satisfies (5) then for any $x_k \in Y_{s-1}$ with $s \in \mathbb{N}$*

$$\exp(-16\sum_{k=1}^{s}\gamma_k)\cdot r_s/\delta_s < |P'_{2^s}(x_k)| \le |P'_{2^s}|_{E_s} = r_s/\delta_s$$

and

$$\delta_s < l_{i,s} < \exp(16\sum_{k=1}^{s}\gamma_k)\cdot\delta_s \quad \textit{for} \quad 1 \le i \le 2^s.$$

Proof: From (2) it follows that $|P'_{2^s}|_{E_s} \ge |P'_{2^s}(0)| = r_s/\delta_s$. In order to get the corresponding lower bound, let us fix $I_{i,s} \subset E_s$. Without loss of generality we can assume that $i = 2j-1$ is odd. Then $I_{i,s} \subset I_{j,s-1}$ and $I_{i,s} = [y, x]$ with $y \in Y_{s-1}$, $x = y + l_{i,s} \in X_s$. By Lemma 3, $|P'_{2^s}|$ decreases on $[y, x]$, so $|P'_{2^s}(x)| < |P'_{2^s}(y)|$. We will estimate $|P'_{2^s}(x)|$ from below in terms of $|P'_{2^s}(y)|$.

The point x is a zero of $P_{2^{s+1}}$. Therefore, $P'_{2^{s+1}}(x) = (x-y)\cdot\prod_{y_k\in Y'_s}|x - y_k| = (x-y)\cdot\prod_{y_k\in Y'_s}|y-y_k|\cdot\beta$, where $Y'_s = Y_s\setminus\{x, y\}$, $\beta = \prod_{y_k\in Y'_s}(1 + \frac{l_{i,s}}{y-y_k})$. Here, $(x-y)\cdot\prod_{y_k\in Y'_s}|y-y_k| = \prod_{x_k\in X_s}|y-x_k|\prod_{y_k\in Y_{s-1}, y_k\neq y}|y-y_k| = r_s\,|P'_{2^s}(y)|$, by (3). On the other hand, by (1), $P'_{2^{s+1}}(x) = r_s\,|P'_{2^s}(y)|$. Hence, $|P'_{2^s}(x)| = \beta\,|P'_{2^s}(y)|$. Let us estimate β from below. We can take into account only $y_k \in Y'_s$ with $y_k > y$, since otherwise the corresponding term in β exceeds 1. The interval $I_{j,s-1}$ contains two points y_k with $y_k - y > h_{j,s-1}$. Lemma 3 yields $1 + \frac{l_{i,s}}{y-y_k} > 1 - \frac{8}{7}\cdot\frac{l_{i,s}}{l_{i,s-1}} > 1 - \frac{8}{7}\cdot 4\gamma_s$.

For the next four points (let $I_{j,s-1} \subset I_{m,s-2}$) we have $y_k - y > h_{m,s-2}$ and $1 + \frac{l_{i,s}}{y-y_k} > 1 - \frac{8}{7} \cdot \frac{l_{i,s}}{l_{m,s-2}} > 1 - \frac{8}{7} \cdot 4\gamma_s \cdot 4\gamma_{s-1} \geq 1 - \frac{1}{7} \cdot 4\gamma_s$, by (5).

We continue in this fashion obtaining $\log \beta > \sum_{k=1}^{s} 2^k \log(1 - \frac{4}{7} \cdot 8^{2-k}\gamma_s)$. If $0 < a < \frac{1}{4}$ then $\log(1-a) > 4\,a \log\frac{3}{4} > -1.16\,a$. A straightforward calculation shows that $\log \beta > -16\,\gamma_s$. Thus,

$$\exp(-16\,\gamma_s)\ |P'_{2^s}(y)| < |P'_{2^s}(x)| < |P'_{2^s}(y)|. \tag{6}$$

Combining this inequality with (2) yields the first statement of Lemma. Indeed, let $x = l_{i_1,m_1} - l_{i_2,m_2} + \cdots - l_{i_{q-1},m_{q-1}} + l_{i_q,m_q}$ with $1 \leq m_1 < \cdots < m_q = s$. Then $y \in X_{m_{q-1}}$. We use (6), then (2) for y, then (6) with y instead of x and $z = l_{i_1,m_1} - l_{i_2,m_2} + \cdots + l_{i_{q-2},m_{q-2}} \in X_{m_{q-2}}$ instead of y, then (2) for z, etc. Finally,

$$\exp(-16(\gamma_{m_1} + \cdots + \gamma_{m_q}))\, r_1 r_2 \cdots r_{s-1} < |P'_{2^s}(x)| < r_1 r_2 \cdots r_{s-1}.$$

If $m_k = k$ for $1 \leq k \leq s$, then all γ_k are presented in the corresponding sum. Monotonicity of $|P'_{2^s}|$ on $[y, x]$ gives the desired conclusion.

The second statement of Lemma can be obtained by the Mean Value Theorem, since $P_{2^s}(y) = 0$, $P_{2^s}(y + l_{i,s}) = -r_s$. In particular, (6) with $x = l_{1,s}, y = 0$ yields

$$\delta_s < l_{1,s} < \delta_s \cdot e^{16\,\gamma_s} < 2\,\delta_s. \tag{7}$$

$\square$

A.F.Beardon and Ch.Pommerenke introduced in [6] the concept of uniformly perfect sets. A dozen of equivalent descriptions of such sets are suggested in [10, p. 343]. We use the following: a compact set $K \subset \mathbb{C}$ is *uniformly perfect* if K has at least two points and there exists $\varepsilon_0 > 0$ such that for any $z_0 \in K$ and $0 < r \leq diam(K)$ the set $K \cap \{z : \varepsilon_0 r < |z - z_0| < r\}$ is not empty.

Theorem 2. *The set $K(\gamma)$, provided (5), is uniformly perfect if and only if* $\inf \gamma_s > 0$.

Proof: Suppose $K(\gamma)$ is uniformly perfect. The values $z_0 = 0$ and $r = l_{1,s-1} - l_{2,s}$ in the definition above imply $l_{1,s} + l_{2,s} > \varepsilon_0\, l_{1,s-1}$. By Lemma 3, we have $4\gamma_s > \varepsilon_0$, so $\inf_s \gamma_s \geq \varepsilon_0/4$, which is our claim.

For the converse, assume $\gamma_s \geq \gamma_0 > 0$ for all s. Let us show that $l_{i,s} > \frac{1}{2}\,\gamma_0\, l_{j,s-1}$ for any intervals $I_{i,s} \subset I_{j,s-1}$, which clearly gives uniform perfectness of $K(\gamma)$. Fix $I_{i,s} \subset I_{j,s-1}$. Let x, y be the endpoints of $I_{i,s}$ with $x \in X_s$, $y \in Y_{s-1}$.

Suppose first that $y \in X_{s-1}$. By the Mean Value Theorem, $l_{i,s}\,|P'_{2^s}(\xi)| = r_s$ for some $\xi \in I_{i,s}$. By the monotonicity of $|P'_{2^s}|$ on $I_{i,s}$, we have $|P'_{2^s}(\xi)| < |P'_{2^s}(y)|$, which is $r_{s-1}\,|P'_{2^{s-1}}(y)|$, by (1). Here, $|P'_{2^{s-1}}(y)| < |P'_{2^{s-1}}(z)|$, where $z \in Y_{s-2}$ is another endpoint of $I_{j,s-1}$. Therefore, $l_{i,s} > \gamma_s\, r_{s-1}/|P'_{2^{s-1}}(z)|$. On the other hand, $l_{j,s-1} = r_{s-1}/|P'_{2^{s-1}}(\eta)|$ with $\eta \in I_{j,s-1}$, so $|P'_{2^{s-1}}(\eta)| > |P'_{2^{s-1}}(z)|/e^{16\,\gamma_{s-1}}$, by (6). Hence, $l_{i,s} > \gamma_s\, l_{j,s-1}/e^{16\,\gamma_{s-1}} \geq \frac{1}{2}\,\gamma_0\, l_{j,s-1}$.

The case $y \in Y_{s-2}$ is very similar. Here at once y plays the role of z. $\square$

5. $K(\gamma)$ is equilibrium

Here and in the sequel we consider r_s in the form $r_s = 2\exp(-R_s \cdot 2^s)$. Recall that R_s is the Robin constant for $\overline{D}_s$ and $R_s \uparrow R$, which is finite if $K(\gamma)$ is not a polar set. In this case, let $\rho_s = R - R_s$. Since $r_0 = 1$, we have $\rho_0 = R - \log 2$. Clearly, $\gamma_s = \frac{1}{2}\exp[2^s(\rho_s - \rho_{s-1})]$ and $\delta_s = 2^{-s}\exp(2^s\rho_s - \sum_{k=1}^{s-1} 2^k \rho_k - 2\rho_0)$. From this,

$$2^{-s}\log\delta_s \to 0 \quad \text{as} \quad s\to\infty. \tag{8}$$

Given $s \in \mathbb{N}$, let us uniformly distribute the mass 2^{-s} on each $I_{j,s}$ for $1 \le j \le 2^s$. We will denote by λ_s the normalized (in this sense) Lebesgue measure on the set E_s, so $d\lambda_s = (2^s\, l_{j,s})^{-1} dt$ on $I_{j,s}$.

If μ is a finite Borel measure of compact support then its logarithmic potential is defined by $U^\mu(z) = \int \log\frac{1}{|z-t|}d\mu(t)$. We will denote by μ_K the equilibrium measure of K, $\xrightarrow{*}$ means convergence in the weak* topology.

Let $I = [a,b]$ with $b-a \le 1$, $z \in I$. By partial integration,

$$\int_I \log\frac{1}{|z-t|}\,dt = b-a-(z-a)\log(z-a)-(b-z)\log(b-z).$$

It follows that

$$(b-a)\,\log\frac{e}{b-a} < \int_I \log\frac{1}{|z-t|}\,dt < (b-a)\,\log\frac{2e}{b-a}. \tag{9}$$

Lemma 5. *Let γ satisfy (5) and $R<\infty$. Then $U^{\lambda_s}(z) \to R$ for $z \in K(\gamma)$ as $s\to\infty$.*

Proof: Fix $z \in K(\gamma)$. Given s, let $z \in I_{j,s}$ for $1 \le j \le 2^s$. From (9) we have $\int_{I_{j,s}} \log|z-t|^{-1}\,d\lambda_s(t) < 2^{-s}\,(2+\log l_{j,s}^{-1})$, which is $o(1)$ as $s\to\infty$, by Lemma 4 and (8).

To estimate $\sum_{k=1,k\ne j}^{2^s}\int_{I_{k,s}}\log|z-t|^{-1}\,d\lambda_s(t)$ we use $P_{2^s}(x) = \prod_{k=1}^{2^s}(x-y_k)$ with $y_k \in I_{k,s}$. As above, let $I_{j,s}\subset I_{m,s-1}\subset I_{q,s-2}\subset\cdots\subset I_{1,0}$. Suppose k corresponds to the adjacent to $I_{j,s}$ subinterval $I_{k,s}$ of $I_{m,s-1}$. Then $h_{m,s-1}\le|z-t|\le|y_j-y_k|\le |z-t|+l_{j,s}+l_{k,s}$. Hence, $1\le\frac{|y_j-y_k|}{|z-t|}\le 1+\varepsilon_0$, where $\varepsilon_0 = \frac{l_{j,s}+l_{k,s}}{h_{m,s-1}} < \frac{1}{7}$, by Lemma 3. For this k we get

$2^{-s}\log|y_j-y_k|^{-1} < \int_{I_{k,s}}\log|z-t|^{-1}\,d\lambda_s(t) < 2^{-s}\,(\log|y_j-y_k|^{-1}+\varepsilon_0)$.

In its turn, $I_{q,s-2}\supset I_{m,s-1}\cup I_{n,s-1}$, where $I_{n,s-1}$ contains other two intervals of the $s-$th level. Let k correspond to any of them. Then $|z-t|-l_{j,s}-l_{k,s}\le|y_j-y_k|\le|z-t|+l_{j,s}+l_{k,s}$ with $|z-t|\ge h_{q,s-2}$. Here, $1-\varepsilon_1\le\frac{|y_j-y_k|}{|z-t|}\le 1+\varepsilon_1$ with $\varepsilon_1 = \frac{l_{j,s}+l_{k,s}}{h_{q,s-2}} < \frac{8}{7}(\frac{l_{j,s}}{l_{m,s-1}}\frac{l_{m,s-1}}{l_{q,s-2}}+\frac{l_{k,s}}{l_{n,s-1}}\frac{l_{n,s-1}}{l_{q,s-2}}) < \frac{8}{7}\cdot 2\cdot 4\gamma_s\,4\gamma_{s-1} < \frac{1}{7}\cdot\frac{1}{4}$, by Lemma 3. Repeating this argument leads to the representation

$$\sum_{k=1,k\ne j}^{2^s}\int_{I_{k,s}}\log|z-t|^{-1}\,d\lambda_s(t) = 2^{-s}\ \log\prod_{k=1,k\ne j}^{2^s}|y_j-y_k|^{-1}+\varepsilon,$$

where $|\varepsilon|\le 2^{-s+1}(\varepsilon_0+2\,\varepsilon_1+\cdots+2^{s-1}\,\varepsilon_{s-1})$ with $\varepsilon_k<\frac{2}{7}\cdot 8^{-k}$ for $k\ge 1$. Here we used the estimate $|\log(1+x)|\le 2\,|x|$ for $|x|<1/2$. We see that $|\varepsilon|<2^{-s}$.

The main term above is $2^{-s}\log|P'_{2^s}(y_j)|^{-1}$, which is $2^{-s}\log(\delta_s/r_s)+o(1)$, by Lemma 4. Thus,

$$\int \log|z-t|^{-1}\,d\lambda_s(t) = 2^{-s}\log(\delta_s/r_s)+o(1) \quad \text{as} \quad s\to\infty.$$

Finally, $2^{-s}\log(\delta_s/r_s) = R_s + 2^{-s}\log\frac{\delta_s}{2} \to R$ as $s\to\infty$, by (8). $\Box$

Theorem 3. *Suppose γ satisfies (5) and $Cap(K(\gamma))>0$. Then $\lambda_s \xrightarrow{*} \mu_{K(\gamma)}$.*

Proof: All measures λ_s have unit mass. By Helly's Selection Theorem (see for instance [15, Th.0.1.3]), we can select a subsequence $(\lambda_{s_k})_{k=1}^{\infty}$, weak* convergent to some measure μ. Approximating the function $\log|z-\cdot|^{-1}$ by the truncated continuous kernels (see for instance [15, Th.1.6.9]), we get $\liminf_{k\to\infty} U^{\lambda_{s_k}}(z) = U^{\mu}(z)$ for quasi-every $z\in\mathbb{C}$. In particular, by Lemma 5, we have $U^{\mu}(z)=R$ for quasi-every $z\in K(\gamma)$. This means that $\mu=\mu_{K(\gamma)}$ (see e.g. [15, Th.1.3.3]). The same proof remains valid for any subsequence $(\lambda_{s_j})_{j=1}^{\infty}$. Therefore, $\lambda_s \xrightarrow{*} \mu_{K(\gamma)}$. $\Box$

Remark. Clearly, any compact set K with nonempty interior cannot be equilibrium in our sense since $supp\,\mu_k \subset \partial K$. Neither geometrically symmetric Cantor-type sets of positive capacity are equilibrium. Let us consider the set $K^{(\alpha)}$ from [1] which is constructed by means of the Cantor procedure with $l_{s+1}=l_s^{\alpha}$ for $1<\alpha<2$. The values $\alpha\geq 2$ give polar sets $K^{(\alpha)}$. As above, let λ_s be the normalized Lebesgue measure on $E_s=\cup_{j=1}^{2^s} I_{j,s}$. Given $s\in\mathbb{N}$, let $z_s = l_1-l_2+\cdots+(-1)^{s+1}l_s$. Estimating distances $|z-t|$ for $z=0$ and $z=z_s$, as in Lemma 5, it can be checked that $U^{\lambda_s}(0)-U^{\lambda_s}(z_s) > \sum_{k=1}^{s-1} 2^{-k-1}\log\frac{(l_{k-1}-l_k)(l_{k-1}-l_{k+1})}{(l_{k-1}-2l_k)(l_{k-1}-l_k-l_{k+1})}$. It is easily seen that all fractions here exceed 1. Therefore, for each s there exists a point $z_s\in K^{(\alpha)}$ such that $U^{\lambda_s}(0)-U^{\lambda_s}(z_s)$ exceeds the constant $\frac{1}{4}\log\frac{(1-l_1)(1-l_2)}{(1-2l_1)(1-l_1-l_2)}$ and the limit logarithmic potential is not equilibrium. Indeed, if $K^{(\alpha)}$ is not polar, then it is regular with respect to the Dirichlet problem (see [11]) and $U^{\mu_{K^{(\alpha)}}}$ must be continuous in $\mathbb{C}$ and constant on $K^{(\alpha)}$.

6. Smoothness of $g_{\mathbb{C}\setminus K(\gamma)}$

We proceed to evaluate the modulus of continuity of the Green function corresponding to the set $K(\gamma)$. Recall that a modulus of continuity is a continuous non-decreasing subadditive function $\omega:\mathbb{R}_+\to\mathbb{R}_+$ with $\omega(0)=0$. Given function f, its modulus of continuity is $\omega(f,\delta)=\sup_{|x-y|\leq\delta}|f(x)-f(y)|$.

In what follows the symbol $\sim$ denotes the strong equivalence: $a_s\sim b_s$ means that $a_s=b_s(1+o(1))$ for $s\to\infty$. This gives a natural interpretation of the relation $\lesssim$.

Let γ be as in the preceding theorem. Then, we are given two monotone sequences $(\delta_s)_{s=1}^{\infty}$ and $(\rho_s)_{s=1}^{\infty}$ where, as above, $\delta_s=\gamma_1\cdots\gamma_s,\ \rho_s=\sum_{k=s+1}^{\infty}2^{-k}\log\frac{1}{2\gamma_k}$. We define the function ω by the following conditions: $\omega(0)=0$, $\omega(\delta)=\rho_1$ for $\delta\geq\delta_1$. If $s\geq 2$ then $\omega(\delta)=\rho_s+2^{-s}\log\frac{\delta}{\delta_s}$ for $\delta_s\leq\delta\leq\delta_{s-1}/16$ and $\omega(\delta)=\rho_{s-1}-k_s(\delta_{s-1}-\delta)$ for $\delta_{s-1}/16<\delta<\delta_{s-1}$ with $k_s=\frac{16}{15}\cdot 2^{-s}\,\delta_{s-1}^{-1}\log 8$.

Lemma 6. *The function ω is a concave modulus of continuity. If $\gamma_s\to 0$ then for any positive constant C we have $\omega(\delta)\sim\rho_s+2^{-s}\log\frac{C\,\delta}{\delta_s}$ as $\delta\to 0$ with $\delta_s\leq\delta<\delta_{s-1}$.*

Proof: The function ω is continuous due to the choice of k_s. In addition, $\omega'(\delta_{s-1}+0) < k_s < \omega'(\delta_{s-1}/16 - 0)$, which provides concavity of ω.

If $\gamma_s = \frac{1}{2}\exp[2^s(\rho_s - \rho_{s-1})] \to 0$ then $2^s \rho_s \to \infty$ and we have the desired equivalence in the case $\delta_s \le \delta \le \delta_{s-1}/16$. Suppose $\delta_{s-1}/16 < \delta < \delta_{s-1}$. The identity

$$\rho_{s-1} = \rho_s + 2^{-s}\,\log\frac{\delta_{s-1}}{2\delta_s} \tag{10}$$

yields $|\rho_s + 2^{-s}\,\log\frac{C\,\delta}{\delta_s} - \omega(\delta)| < 2^{-s}[|\log\frac{2C\,\delta}{\delta_{s-1}}| + \frac{16}{15}\log 8 \cdot (1 - \frac{\delta}{\delta_{s-1}})] < 2^{-s}[|\log C| + 8\log 2]$, which is $o(\omega)$ since here $\omega(\delta) > \rho_{s-1} - 2^{-s}\log 8$. $\Box$

Lemma 7. *Suppose γ satisfies (5) and $Cap(K(\gamma)) > 0$. Let $z \in \mathbb{C},\ z_0 \in K(\gamma)$ with $dist(z, K(\gamma)) = |z - z_0| = \delta < 1$. Choose $s \in \mathbb{N}$ such that $z_0 \in I_{j,s} \subset I_{j_1,s-1}$ with $l_{j,s} \le \delta < l_{j_1,s-1}$. Then $g_{\mathbb{C}\setminus K(\gamma)}(z) < \rho_s + 2^{-s}\,\log\frac{16\,\delta}{\delta_s}$.*
On the other hand, if $l_{1,s} \le \delta < l_{1,s-1}$ then $g_{\mathbb{C}\setminus K(\gamma)}(-\delta) > \rho_s + 2^{-s}\,\log\frac{\delta}{\delta_s}$.

Proof: Consider the chain of basic intervals containing z_0: $z_0 \in I_{j,s} \subset I_{j_1,s-1} \subset I_{j_2,s-2} \subset \cdots \subset I_{j_s,0} = [0,1]$. Here, $I_{j_i,s-i} \setminus I_{j_{i-1},s-i+1}$ contains 2^{i-1} basic intervals of the $s-$th level. Each of them has certain endpoints x, y with $x \in X_s,\ y \in Y_{s-1}$. Recall that Y_{s-1} is the set of zeros of P_{2^s}. Distinguish $y_j \in I_{j,s}$. Now for a fixed large n we will express the value $|P_{2^n}(z)| = \prod_{k=1}^{2^n}|z - x_k|$ in terms of $\prod_{k=1,k\ne j}^{2^s}|y_j - y_k|$ (compare to Lemma 5). Clearly, each interval of the $s-$th level contains 2^{n-s} zeros of P_{2^n}, so we will replace these 2^{n-s} points with the corresponding y_k.

Let us first consider the product $\pi_0 := \prod_{x_k\in I_{j,s}}|z - x_k|$. Here, $|z - x_k| \le \delta + l_{j,s} < 2\,\delta$, so $\pi_0 < (2\,\delta)^{2^{n-s}}$.

Let $\pi_1 := \prod_{x_k\in I_{m,s}}|z - x_k|$, where $I_{m,s}$ is adjacent to $I_{j,s}$. Then $|z_0 - x_k| \le l_{j_1,s-1} = |y_j - y_m|$, since y_j and y_m are the endpoints of the interval $I_{j_1,s-1}$. Therefore, $|z - x_k| < 2\,|y_j - y_m|$ and $\pi_1 < (2\,|y_j - y_m|)^{2^{n-s}}$.

In the general case, given $2 \le i \le s$, let π_i denote the product of all $|z - x_k|$ for $x_k \in J_i := I_{j_i,s-i} \setminus I_{j_{i-1},s-i+1}$. Suppose $x_k \in I_{q,s}$. Then, $|z - x_k| \le \delta + l_{j,s} + |y_j - y_q| + l_{q,s} \le |y_j - y_q|(1 + \frac{\delta + l_{j,s} + l_{q,s}}{h_{j_i,s-i}})$, since y_j and y_q belong to different subintervals of the $(s-i+1)-$th level for $I_{j_i,s-i}$. Here, $\frac{\delta}{h_{j_i,s-i}} < \frac{8}{7}\frac{l_{j_1,s-1}}{l_{j_i,s-i}} < \frac{8}{7}\,8^{1-i}$, by Lemma 3. As in the proof of Lemma 5, we obtain $\frac{l_{j,s}+l_{q,s}}{h_{j_i,s-i}} < \frac{8}{7}\cdot 2\cdot 8^{-i}$. From this, $\prod_{x_k\in I_{q,s}}|z - x_k| \le [\,|y_j - y_q|\,(1 + \frac{80}{7}\,8^{-i})]^{2^{n-s}}$. Since J_i contains 2^{i-1} basic intervals of the $s-$th level, $\pi_i < [\,(1 + \frac{80}{7}\,8^{-i})^{2^{i-1}}\prod_{y_q\in J_i}|y_j - y_q|\,]^{2^{n-s}}$.

The product $\prod_{i=2}^{s}(1 + \frac{80}{7}\,8^{-i})^{2^{i-1}}$ is smaller than 2, as is easy to check.

Therefore, $|P_{2^n}(z)| = \prod_{i=0}^{s}\pi_i < [\,8\cdot\delta\cdot\prod_{k=1,k\ne j}^{2^s}|y_j - y_k|\,]^{2^{n-s}}$. The last product in the square brackets is $|P'_{2^s}(y_j)|$, which does not exceed r_s/δ_s, by Lemma 4. Hence, $2^{-n}\,\log|P_{2^n}(z)| < 2^{-s}\,\log\frac{16\,\delta}{\delta_s} - R_s$.

Finally, by Corollary 1, $g_{\mathbb{C}\setminus K(\gamma)}(z) = R + \lim_{n\to\infty}2^{-n}\,\log|P_{2^n}(z)|$, which yields the desired upper bound of the Green function.

Similar, but simpler calculations establish the sharpness of the bound. We have $g_{\mathbb{C}\setminus K(\gamma)}(-\delta) = R + \lim_{n\to\infty}2^{-n}\log P_{2^n}(-\delta)$. Now, $P_{2^n}(-\delta) = \prod_{i=0}^{s}\pi_i$ with $\pi_0 = \prod_{x_k\in I_{1,s}}(\delta + x_k) > \delta^{2^{n-s}}$ and $\pi_i = \prod_{x_k\in I_{2,s-i+1}}(\delta + x_k)$ for $i \ge 1$. Suppose $x_k \in I_{q,s} \subset$

$I_{2,s-i+1}$. Then $\delta+x_k > y_q-l_{q,s}$. Since $y_q > h_{1,s-i} > \frac{7}{8}l_{1,s-i}$, we have $\delta+x_k > y_q(1-\frac{8}{7}\,8^{-i})$ and $\pi_i > [\,(1-\frac{1}{7}\,8^{1-i})^{2^{i-1}}\prod_{y_q\in I_{2,s-i+1}} y_q\,]^{2^{n-s}}$. Therefore, $P_{2^n}(-\delta) > [\,\frac{\delta}{2}\prod_{k=1}^{2^s} y_k]^{2^{n-s}} = [\,\frac{\delta}{2}\,|P'_{2^s}(0)|]^{2^{n-s}} = [\,\delta/\delta_s\cdot r_s/2\,]^{2^{n-s}}$, by (2). Thus, $2^{-n}\log P_{2^n}(-\delta) > -R_s + 2^{-s}\,\log\frac{\delta}{\delta_s}$ and $g_{\mathbb{C}\setminus K(\gamma)}(-\delta)\ge\rho_s + 2^{-s}\,\log\frac{\delta}{\delta_s}$. $\Box$

Theorem 4. *Suppose γ satisfies (5) and $Cap(K(\gamma)) > 0$. If $\delta_s\le\delta<\delta_{s-1}$ then $\rho_s + 2^{-s}\,\log\frac{\delta}{\delta_s} < \omega(g_{\mathbb{C}\setminus K(\gamma)},\delta) < \rho_s + 2^{-s}\,\log\frac{16\,\delta}{\delta_s}$. If $\gamma_s\to 0$ then $\omega(g_{\mathbb{C}\setminus K(\gamma)},\delta)\sim\omega(\delta)$ as $\delta\to 0$.*

Proof: Fix δ and s with $\delta_s\le\delta<\delta_{s-1}$. By (7), $\delta_s < l_{1,s} < 2\,\delta_s < \delta_{s-1}$.

If $l_{1,s}\le\delta<\delta_{s-1}$ then $\omega(g_{\mathbb{C}\setminus K(\gamma)},\delta)\ge g_{\mathbb{C}\setminus K(\gamma)}(-\delta)$, so Lemma 7 yields the desired lower bound. If $\delta_s\le\delta<l_{1,s}$, then $g_{\mathbb{C}\setminus K(\gamma)}(-\delta) > \rho_{s+1} + 2^{-s-1}\,\log\frac{\delta}{\delta_{s+1}} = \rho_s + 2^{-s-1}\,\log\frac{2\,\delta}{\delta_s}$, by (10). Here, $2^{-s-1}\,\log\frac{2\,\delta}{\delta_s} > 2^{-s}\,\log\frac{2\,\delta}{\delta_s}$, as is easy to check.

In order to get the upper bound, without loss of generality we can assume that $\omega(g_{\mathbb{C}\setminus K(\gamma)},\delta) = g_{\mathbb{C}\setminus K(\gamma)}(z)$ where $z\in\mathbb{C}$ is such that $dist(z,K(\gamma)) = |z-z_0| = \delta$ for some $z_0\in K(\gamma)$.

Fix m such that $z_0\in I_{j,m}\subset I_{j_1,m-1}$ for some j with $l_{j,m}\le\delta<l_{j_1,m-1}$. Then $m\ge s$, since otherwise Lemma 4 gives a contradiction $\delta<\delta_{s-1}\le\delta_m<l_{j,m}\le\delta$.

If $m=s$ then, by Lemma 7, the result is immediate.

If $m\ge s+1$ then $g_{\mathbb{C}\setminus K(\gamma)}(z)\le\ \rho_m + 2^{-m}\,\log\frac{16\,\delta}{\delta_m}$ that does not exceed $\rho_s + 2^{-s}\,\log\frac{16\,\delta}{\delta_s}$. Indeed, the function $f(\delta) = \rho_s-\rho_m + (2^{-s}-2^{-m})\log 16\,\delta - 2^{-s}\log\delta_s + 2^{-m}\log\delta_m$ attains its minimal value on $[\delta_s,\delta_{s-1})$ at the left endpoint. Here, $f(\delta_s) = (2^{-s}-2^{-m})\log 8 + \sum_{k=s+1}^m(2^{-k}-2^{-m})\log\frac{1}{\gamma_k} > 0$.

The last statement of the theorem is a corollary of Lemma 6. $\Box$

7. Model types of smoothness

Let us consider some model examples with different rates of decrease of $(\rho_s)_{s=1}^\infty$. Recall that for non-polar sets $K(\gamma)$ with $R = Rob(K(\gamma))$ we have $\rho_s\downarrow 0$ and $R_s - R_{s-1} = \rho_{s-1}-\rho_s = 2^{-s}\log\frac{1}{2\gamma_s}$ with $\rho_0 = R-\log 2$. Therefore, $R = \log 2 - \sum_{k=1}^\infty 2^{-k}\log 2\gamma_k$. In addition, (5) implies $\rho_s\ge 2^{-s}\log 16$ and $R\ge\log 32$, so $Cap(K(\gamma))\le 1/32$.

If a set K is uniformly perfect, then the function $g_{\mathbb{C}\setminus K}$ is Hölder continuous (see e.g. [10], p. 119), which means the existence of constants C,α such that

$$g_{\mathbb{C}\setminus K}(z)\le C\,(dist(z,K))^\alpha\quad\text{for all}\quad z\in\mathbb{C}.$$

In this case we write $g_{\mathbb{C}\setminus K}\in Lip\ \alpha$.

By Theorem 2, $g_{\mathbb{C}\setminus K(\gamma)}$ is Hölder continuous provided $\gamma_s = const$. Now we can control the exponent α in the definition above. In the following examples we suppose that $dist(z,K(\gamma)) = \delta$ with $\delta_s\le\delta<\delta_{s-1}$ for large s.

Example 2. Let $\gamma_s = \gamma_1\le\frac{1}{32}$ for all s. Then $\delta_s = \gamma_1^s, r_s = \gamma_1^{2^s-1}, R = \log\frac{1}{\gamma_1}$, and $\rho_s = 2^{-s}\log\frac{1}{2\gamma_1}$. Here, $\rho_s + 2^{-s}\,\log\frac{\delta}{\delta_s}\ge\rho_s > 2^{-s} = \delta_s^\alpha$ with $\alpha = -\frac{\log 2}{\log\gamma_1}$. Since $\delta_s = \gamma_1\,\delta_{s-1} > \gamma_1\,\delta$, we have, by Theorem 4, $g_{\mathbb{C}\setminus K(\gamma)}(-\delta) > \gamma_1^\alpha\,\delta^\alpha$. On the other hand, $\rho_s + 2^{-s}\,\log\frac{16\delta}{\delta_s} < \delta^\alpha\,\log\frac{8}{\gamma_1^2}$.

Suppose we are given α with $0 < \alpha \leq 1/5$. Then the value $\gamma_s = 2^{-1/\alpha}$ for all s provides $g_{\mathbb{C}\setminus K(\gamma)} \in Lip\ \alpha$ and $g_{\mathbb{C}\setminus K(\gamma)} \notin Lip\ \beta$ for $\beta > \alpha$.

The next example is related to the function $h(\delta) = (\log \frac{1}{\delta})^{-1}$ that defines the logarithmic measure of sets. Let us write $g_{\mathbb{C}\setminus K} \in Lip_h\, \alpha$ if for some constants C we have

$$g_{\mathbb{C}\setminus K}(z) \leq C\, h^\alpha(dist(z,K)) \quad \text{for all} \quad z \in \mathbb{C}.$$

Example 3. Given $1/2 < \rho < 1$, let $\rho_s = \rho^s$ for $s \geq s_0$, where $\frac{\rho}{1-\rho} \log 16 < (2\,\rho)^{s_0}$. This condition provides $\gamma_s < 1/32$ for $s > s_0$. Suppose $\gamma_s = 1/32$ for $s \leq s_0$, so we can use Theorem 4. For large s we have $\delta_s = C\, 2^{-s} \mu^{(2\,\rho)^s}$ with $\mu = \exp(\frac{2\,\rho-2}{2\,\rho-1})$ and some constant C. Let us take $\alpha = \frac{\log(1/\rho)}{\log(2\,\rho)}$, so $(2\rho)^\alpha = 1/\rho$. Then $h^\alpha(\delta) \geq h^\alpha(\delta_s) \geq \varepsilon_0 (2\,\rho)^{-s\,\alpha} = \varepsilon_0\, \rho \cdot \rho_{s-1}$ for some ε_0. From this we conclude that $g_{\mathbb{C}\setminus K(\gamma)} \in Lip_h\, \alpha$ for given α. Evaluation $g_{\mathbb{C}\setminus K(\gamma)}(-\delta_s)$ from below yields $g_{\mathbb{C}\setminus K(\gamma)} \notin Lip_h\, \beta$ for $\beta > \alpha$. Now, given $\alpha > 0$, the value $\rho = 2^{-\frac{\alpha}{1+\alpha}}$ provides the corresponding Green function of the exact class $Lip_h\, \alpha$ (compare this to [1], [8]).

Example 4. Let $\rho_s = 1/s$. Then $\gamma_s = \frac{1}{2}\exp(\frac{-2^s}{s^2-s}) < 1/32$ for $s \geq 8$. As above, all previous values of γ_s are 1/32. Here, $\delta_s = C\, 2^{-s} \exp[\frac{2^s}{s} - \sum_{k=1}^{s-1} \frac{2^k}{k}]$. Summation by parts (see e.g.[14],T.3.41) yields $\delta_s = C\, 2^{-s} \exp[-2^{s+1}(s^{-2} + o(s^{-2}))]$. From this, $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \delta) \sim \frac{1}{s} \sim \frac{\log\, 2}{\log\, \log\, 1/\delta_s}$.

Example 5. Given $N \in \mathbb{N}$, let $F_N(t) = \log\log\cdots\log t$ be the $N-$th iteration of the logarithmic function. Let $\rho_s = (F_N(s))^{-1}$ for large enough s. Here, $\rho_{k-1} - \rho_k \sim [k \cdot \log k\, \cdot F_2(k) \cdots F_{N-1}(k)\, \cdot F_N^2(k)]^{-1}$. Since $\delta_s = C\, 2^{-s} \exp[-\sum_{k=1}^{s} 2^k(\rho_{k-1} - \rho_k)]$, we have, as above, $s \sim \frac{\log\, \log\, 1/\delta_s}{\log\, 2}$. Thus, $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \delta) \sim [F_{N+2}(1/\delta)]^{-1}$.

We see that a more slow decrease of (ρ_s) implies a less smooth $g_{\mathbb{C}\setminus K(\gamma)}$ and conversely. If, in examples above, we take $\gamma_s = 1/32$ for $s < s_0$ with rather large s_0, then the set $K(\gamma)$ will have logarithmic capacity as closed to 1/32, as we wish.

Problem. Given modulus of continuity ω, to find $(\gamma_s)_{s=1}^\infty$ such that $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \cdot)$ coincides with ω at least on some null sequence.

8. Markov's factors

Let $\mathcal{P}_n$ denote the set of all holomorphic polynomials of degree at most n. For any infinite compact set $K \subset \mathbb{C}$ we consider the sequence of Markov's factors $M_n(K) = \inf\{M : \ |P'|_K \leq M\, |P|_K \ \text{ for all } \ P \in \mathcal{P}_n\}$, $n \in \mathbb{N}$. We see that $M_n(K)$ is the norm of the operator of differentiation in the space $(\mathcal{P}_n, |\cdot|_K)$. In the case of non-polar K, the knowledge about smoothness of the Green function near the boundary of K may help to estimate $M_n(K)$ from above. The application of the Cauchy formula for P' and the Bernstein-Walsh inequality yields the estimate

$$M_n(K) \leq \inf_\delta \delta^{-1} \exp[n \cdot \omega(g_{\mathbb{C}\setminus K}, \delta)]. \tag{11}$$

This approach gives an effective bound of $M_n(K)$ for the cases of temperate growth of $\omega(g_{\mathbb{C}\setminus K},\cdot)$. For instance, the Hölder continuity of $g_{\mathbb{C}\setminus K}$ implies Markov's property of the set K, which means that there are constants C, m such that $M_n(K)\le Cn^m$ for all n. In this case, the infimum $m(K)$ of all positive exponents m in the inequality above is called the best Markov's exponent of K.

Lemma 8. *Suppose γ satisfies (5) and $Cap(K(\gamma))>0$. Given fixed $s\in\mathbb{N}$, let $f(\delta)=\delta^{-1}\exp[2^s(\rho_k+2^{-k}\log\frac{16\,\delta}{\delta_k})]$ for $\delta_k\le\delta<\delta_{k-1}$ with $k\ge 2$. Then $\inf_{0<\delta<\delta_1} f(\delta)=f(\delta_s-0)=4\sqrt{2}\,\delta_s^{-1}\exp(2^s\,\rho_s)$.*

Proof: Let us fix the interval $I_k=[\delta_k,\delta_{k-1})$. In view of the representation $f(\delta)=C_{s,k}\,\delta^{2^{s-k}-1}$, the function f increases for $k<s$, decreases for $k>s$, and is constant for $k=s$ on I_k. An easy computation shows that $f(\delta_{k+1})<f(\delta_k)$ for $k\le s-1$ and $f(\delta_{k-1}-0)<f(\delta_k-0)$ for $k\ge s+1$. Thus, it remains to compare $f(\delta_s-0)$ and $f(\delta_s)$. Here, $f(\delta_s)=16\,\delta_s^{-1}\exp(2^s\,\rho_s)$ exceeds $f(\delta_s-0)=\delta_s^{-1}(16/\gamma_{s+1})^{1/2}\exp(2^s\rho_{s+1})=4\sqrt{2}\,\delta_s^{-1}\exp(2^s\,\rho_s)$. □

Example 6. Let $\gamma_s=\gamma_1\le\frac{1}{32}$ for $s\in\mathbb{N}$. Then, by Lemma 8 and Example 2, $M_{2^s}(K(\gamma))\le\sqrt{8}\cdot\delta_{s+1}^{-1}=\sqrt{8}\,\gamma_1^{-1}\,2^{\,s/\alpha}$, where α is the same as in Example 2.

On the other hand, let $Q=P_{2^s}+r_s/2$. Then $|Q|_{K(\gamma)}=r_s/2$ and $|Q'(0)|=r_s/\delta_s$, so $M_{2^s}(K(\gamma))\ge 2\,\delta_s^{-1}=2\cdot 2^{\,s/\alpha}$. Now, for each n we choose s with $2^s\le n<2^{s+1}$. Since the sequence of Markov's factors increases,

$$c\,n^{1/\alpha}\le M_{2^s}(K(\gamma))\le M_n(K(\gamma))\le M_{2^{s+1}}(K(\gamma))\le C\,n^{1/\alpha}$$

with $c=2^{1-1/\alpha}$, $C=\gamma_1^{-1}\,2^{3/2+1/\alpha}$. Given $m\in[5,\infty)$, the value $\gamma_s=2^{-m}$ for all s provides the set $K(\gamma)$ with $m(K(\gamma))=m=1/\alpha$.

However, the estimate (11) may be rather rough for compact sets with less smooth moduli of continuity of the corresponding Green's functions. For instance, in the case of $K(\gamma)$ with $\sum_{k=1}^\infty\gamma_k<\infty$ (then $2^s\,\rho_s\to\infty$) and $n=2^s$, the exact value of the right side in (11) is $4\sqrt{2}\,\delta_s^{-1}\exp(2^s\,\rho_s)$, whereas $M_{2^s}(K(\gamma))\sim 2\,\delta_s^{-1}$, which will be shown below by means of the Lagrange interpolation. It should be noted that the set $K(\gamma)$ may be polar here.

Let us interpolate $P\in\mathcal{P}_{2^s}$ at zeros $(x_k)_{k=1}^{2^s}$ of P_{2^s} and at one extra point $l_{1,s}$. Then the fundamental Lagrange interpolating polynomials are $L_*(x)=-P_{2^s}(x)/r_s$ and $L_k(x)=\frac{(x-l_{1,s})P_{2^s}(x)}{(x-x_k)(x_k-l_{1,s})P'_{2^s}(x_k)}$ for $k=1,2,\cdots,2^s$. Let Δ_s denote $\sup_{x\in K(\gamma)}[\,|L'_*(x)|+\sum_{k=1}^{2^s}|L'_k(x)|\,]$. For convenience we enumerate $(x_k)_{k=1}^{2^s}$ in increasing way, so $x_k\in I_{k,s}$ for $1\le k\le 2^s$.

Lemma 9. *Suppose γ satisfies (5) and $\sum_{k=1}^\infty\gamma_k<\infty$. Then $\Delta_s\sim 2\,\delta_s^{-1}$.*

Proof: We use the following representation:

$$L'_k(x)=\frac{P'_{2^s}(x)}{(x_k-l_{1,s})P'_{2^s}(x_k)}+\frac{P_{2^s}(x)}{(x-x_k)P'_{2^s}(x_k)}\sum_{j=1,j\neq k}^{2^s}\frac{1}{x-x_j}=:A_k+B_k. \tag{12}$$

In particular, $L_1'(0) = -l_{1,s}^{-1} - \sum_{j=2}^{2^s} x_k^{-1}$. By (2), $|L_*'(0)| = \delta_s^{-1}$, so $\Delta_s > |L_*'(0)| + |L_1'(0)| > \delta_s^{-1} + l_{1,s}^{-1} > \delta_s^{-1}(1 + e^{-16\gamma_s})$, by (7). Thus, $\Delta_s \gtrsim 2\,\delta_s^{-1}$.

We proceed to estimate Δ_s from above. Lemma 4 gives the uniform bound $|L_*'(x)| \leq \delta_s^{-1}$.

Let us examine separately the sum $\sum_{k=1}^{2^s} |A_k|$, where A_k are defined by (12). Let $C_0 = \exp(16\sum_{k=1}^{\infty}\gamma_k)$. Then, by Lemma 4, $|P_{2^s}'(x)| \leq |P_{2^s}'(0)| = r_s/\delta_s < C_0 |P_{2^s}'(x_k)|$ for $x \in K(\gamma)$. Therefore, $|A_1| \leq l_{1,s}^{-1} < \delta_s^{-1}$ and $\sum_{k=2}^{2^s} |A_k| < C_0 \sum_{k=2}^{2^s} (x_k - l_{1,s})^{-1}$. Here, $\sum_{k=2}^{2^s}(x_k - l_{1,s})^{-1} < 2\; l_{1,s-1}^{-1}$, as is easy to check. Thus, $\sum_{k=1}^{2^s} |A_k| < \delta_s^{-1} + 2C_0\delta_{s-1}^{-1}$.

In order to estimate the sum of the addends B_k, let us fix $x \in K(\gamma)$ and $1 \leq m \leq 2^s$ such that $x \in I_{m,s}$. Suppose first that $k \neq m$. Then

$$\sum_{j=1, j\neq k}^{2^s} \left|\frac{P_{2^s}(x)}{x - x_j}\right| < 2\left|\frac{P_{2^s}(x)}{x - x_m}\right| \leq 2\,|P_{2^s}'(\xi)| \tag{13}$$

with a certain $\xi \in I_{m,s}$. Indeed, if $x = x_m$ then this sum is exactly $|P_{2^s}'(x_m)|$, so $\xi = x_m$. Otherwise we take the main term out of the brackets:

$$\left|\frac{P_{2^s}(x)}{x - x_m}\right| \left[1 + \sum_{j=1, j\neq k, j\neq m}^{2^s} \left|\frac{x - x_m}{x - x_j}\right|\right].$$

Here the sum in the square brackets can be handled in the same way as in the proof of Lemma 3. Let $I_{m,s} \subset I_{q,s-1} \subset I_{r,s-2} \subset \cdots$. Then $[\cdots] \leq 1 + l_{m,s}(h_{q,s-1}^{-1} + 2h_{r,s-2}^{-1} + \cdots) \leq 1 + \frac{8}{7} l_{m,s}(l_{q,s-1}^{-1} + 2l_{r,s-2}^{-1} + \cdots) < 1 + \frac{8}{7}(4\gamma_s + 2\cdot 4\gamma_s 4\gamma_{s-1} + \cdots) < 2$.

On the other hand, by Taylor's formula, $P_{2^s}(x) = P_{2^s}'(\xi)(x - x_m)$ with $\xi \in I_{m,s}$, which establishes (13).

Therefore,

$$\sum_{k=1, k\neq m}^{2^s} |B_k| < \sum_{k=1, k\neq m}^{2^s} \frac{2\,C_0}{|x - x_k|}.$$

As above, $\sum_{k=1, k\neq m}^{2^s} |B_k| < 2\,C_0(h_{q,s-1}^{-1} + 2h_{r,s-2}^{-1} + \cdots) < 4\,C_0\,h_{q,s-1}^{-1} < 5\,C_0\,l_{q,s-1}^{-1}$.

It remains to consider $B_m = \frac{P_{2^s}(x)}{(x - x_m)P_{2^s}'(x_m)} \sum_{j=1, j\neq m}^{2^s} \frac{1}{x - x_j}$. Let us take the interval $I_{n,s}$ adjacent to $I_{m,s}$, so $I_{n,s} \cup I_{m,s} \subset I_{q,s-1}$. Then, as above, $\sum_{j=1, j\neq m}^{2^s} |x - x_j|^{-1} < 2\,|x - x_n|^{-1}$ and $|B_m| < 2\,C_0\,|x - x_n|^{-1} < 3\,C_0\,l_{q,s-1}^{-1}$, since $|x - x_n| > h_{q,s-1}$.

This gives $\sum_{k=1}^{2^s} |B_k| < 8\,C_0\,l_{q,s-1}^{-1} < 8\,C_0\,\delta_{s-1}^{-1}$, by Lemma 4. Finally, $\Delta_s < 2\,\delta_s^{-1} + 10\,C_0\,\delta_{s-1}^{-1} = \delta_s^{-1}(2 + 10\,C_0\,\gamma_s) \sim 2\,\delta_s^{-1}$. □

Theorem 5. *With the assumptions of Lemma 8,* $M_{2^s}(K(\gamma)) \sim 2\,\delta_s^{-1}$.

Proof: On the one hand, $|P_{2^s} + r_s/2|_{K(\gamma)} = r_s/2$ and $|P_{2^s}'(0)| = r_s/\delta_s$, so $M_{2^s}(K(\gamma)) \geq 2\,\delta_s^{-1}$.

On the other hand, for each polynomial $P \in \mathcal{P}_{2^s}$ and $x \in K(\gamma)$ we have $|P'(x)| \leq |P|_{K(\gamma)}\,\Delta_s$, and the theorem follows. □

We are now in a position to construct a compact set with preassigned growth of subsequence of Markov's factors. Suppose we are given a sequence of positive terms $(M_{2^s})_{s=0}^{\infty}$ with $\sum_{s=0}^{\infty} M_{2^s}/M_{2^{s+1}} < \infty$. The case of polynomial growth of (M_n) was considered before, so let us assume that $C\, n^m\, M_n^{-1} \to 0$ as $n \to \infty$ for fixed C and m. Fix s_0 such that $M_{2^s}/M_{2^{s+1}} \le 1/32$ for $s \ge s_0$ and $M_{2^{s_0}} \ge 2 \cdot 2^{5 s_0}$.

Let us take $\gamma_s = M_{2^{s-1}}/M_{2^s}$ for $s > s_0$ and $\gamma_s = (2/M_{2^{s_0}})^{1/s_0}$ for $s \le s_0$. Then $\gamma_s \le 1/32$ for all s and we can use Theorem 5. Here, $\delta_s = 2/M_{2^s}$, so $M_{2^s}(K(\gamma)) \sim M_{2^s}$.

It should be noted that the growth of $(M_n(K))$ is restricted for a non-polar compact set K ([5], Pr.3.1). It is also interesting to compare Theorem 5 with Theorem 2 in [16].

9. **The best Markov's exponent**

If a compact set K has Markov's property, then the Markov inequality is not necessarily valid on K with the best Markov's exponent $m(K)$. An example of such compact set in $\mathbb{C}^N$, $N \ge 2$ was presented in [4], where the authors posed the problem (5.1): is the same true in $\mathbb{C}$? The compact set $K(\gamma)$ with a suitable choice of γ gives the answer in the affirmative.

Example 7. Fix $m \ge 5$. Let $\varepsilon_k = \sqrt{k} - \sqrt{k-1}$ and $\gamma_k = 2^{-(m+\varepsilon_k)}$ for $k \in \mathbb{N}$. Then, $\delta_s = 2^{-(ms+\sqrt{s})}$ and $\rho_s = \sum_{k=s+1}^{\infty} 2^{-k} \log 2^{m-1+\varepsilon_k}$. Since $\varepsilon_k \le 1$, we have $\exp(2^s \rho_s) < 2^m$. By Lemma 8 and (11), $M_{2^s}(K(\gamma)) < C_0\, \delta_s^{-1}$ with $C_0 = 4\sqrt{2} \cdot 2^m$.

On the other hand, as in Example 6, $M_{2^s}(K(\gamma)) \ge 2\, \delta_s^{-1}$.

Let us show that for each $k \ge 2$ the value $m_k := m + \frac{\sqrt{k}}{k-1}$ is the Markov exponent for $K(\gamma)$. We want to find a constant C_k such that $M_n(K(\gamma)) \le C_k\, n^{m_k}$ holds for all $n \in \mathbb{N}$. Let $2^{s-1} < n \le 2^s$. Then $M_n(K(\gamma)) \le M_{2^s}(K(\gamma)) < C_0\, 2^m\, n^{m_s}$. If $s > k$ then $m_s < m_k$. If $s \le k$ then $M_n \le M_{2^k}$. Therefore, $C_k = \max\{C_0\, 2^m, M_{2^k}\}$ satisfies the desired condition.

However, the Markov inequality on $K(\gamma)$ does not hold with the exponent $m(K(\gamma)) = \inf m_k = m$. Indeed, $M_{2^s}(K(\gamma)) \ge 2\, \delta_s^{-1} = 2 \cdot 2^{m \cdot s} \cdot 2^{\sqrt{s}}$. Therefore, given constant C, the inequality $M_{2^s}(K(\gamma)) \le C\, 2^{m \cdot s}$ is impossible for large s.

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