LOCAL CONVEX DIRECTIONS

A. B. Özgüler*, **K. Saadaoui***

* Department of Electrical and Electronics Engineering
Bilkent University, Bilkent 06533 Ankara, Turkey
fax: +90 312 266 41 92
e-mail: ozguler@ee.bilkent.edu.tr
e-mail: karim@ee.bilkent.edu.tr

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Abstract

A proof of a strengthened version of the phase growth condition for Hurwitz stable polynomials is given. Based on this result, a necessary and sufficient condition for a polynomial $p(s)$ to be a local convex direction for a Hurwitz stable polynomial $q(s)$ is obtained. The condition is in terms of polynomials associated with the even and odd parts of $p(s)$ and $q(s)$.

1 Introduction

Rantzer [13] gave a phase growth condition which is necessary and sufficient for a given polynomial to be a convex direction for the set of all Hurwitz polynomials. The phase growth condition directly gives that (i) anti-Hurwitz polynomials (ii) polynomials of degree one (iii) even polynomials (iv) odd polynomials, and (v) any multiple of polynomials from (i)-(iv) (taken one from each set) are examples of convex directions for the entire set of Hurwitz polynomials. In [2], the alternating Hurwitz minor condition is used to construct convex directions which are not in one of the above sets (i)-(v). Clearly, the global requirement is unnecessarily restrictive when examining the stability of particular segment of polynomials and it is of more interest to determine conditions for a polynomial to be a convex direction for a single Hurwitz polynomial or for a specified class of Hurwitz polynomials.

There are several solution to the edge stability problem. Among these, the segment lemma of [3] gives a condition which requires checking the signs of two functions at some fixed points. Bialas [4] gave another solution in terms of the Hurwitz matrices associated with the vertex polynomials. In [7] and [11], different definitions of local convex directions have been used. A polynomial $p(s)$ is called a (local) convex direction for $q(s)$ if the set of $\alpha > 0$ for which $q(s) + \alpha p(s)$ is Hurwitz stable is a single interval on the real line. Note that, if $p(s)$ is a convex direction in this sense, the stability of $q(s)$ and $p(s)+q(s)$ implies the stability of $q(s)+\alpha p(s)$ for all $\alpha \in [0, 1]$ but not vice versa. It seems that none of the methods described above is suitable in determining convex directions for subsets of Hurwitz stable polynomials. We will show by some examples that our result is suitable in determining convex directions for subsets of Hurwitz stable polynomials.

The paper is organized as follows. In the next section some properties of Hurwitz polynomials are recalled. In section 3, we state the main result, Theorem 1, which gives a necessary and sufficient condition for determining local convex directions for a Hurwitz polynomial. As an application of Theorem 1, given a polynomial $p(s)$ we construct a set of Hurwitz stable polynomials for which $p(s)$ is a local convex direction.

2 Hurwitz Stable Polynomials

Let **R** and **C** denote the field of real and complex numbers, respectively. Let $\mathbf{R}[s]$ denote the set of polynomials in s with coefficients in **R**. Let $\mathcal{H}$ denote the set of Hurwitz stable polynomials. Given $q \in \mathbf{R}[s]$, the even-odd parts $(h(u), g(u))$ of $q(s)$ are the unique polynomials $h, g \in \mathbf{R}[u]$ such that $q(s) = h(u) + sg(u)$ where $u = s^2$. Let $\mathcal{S}(r)$ denote the sign of r, i.e.,

$$\mathcal{S}(r) = \begin{cases} -1 & \text{if } r < 0 \\ 0 & \text{if } r = 0 \\ 1 & \text{if } r > 0. \end{cases}$$

Finally, consider $\phi(K, u) := h(u) + Kg(u)$ and $\psi(K, u) := ug(u) + Kh(u)$ for $K \in \mathbf{R}$. The equation $\phi(K, u) = 0$ implicitly defines a function $u(K)$. The root sensitivity of $\phi(K, u)$ is defined by $K\frac{du}{dK}$, and gives a measure of the variation in the root location of $\phi(K, u)$ with respect to percentage variations in K. The root sensitivities of $\phi(K, u)$ and $\psi(K, u)$, respectively, are easily computed to be

$$S_q(u) := \frac{h(u)g(u)}{V_q(u)}, \;\; S_{sq}(u) := \frac{ug(u)h(u)}{V_{sq}(u)}.$$

For a Hurwitz stable polynomial $q(s)$ the rate of change of the argument satisfies

$$\psi_q^{'}(w) \geq |\frac{\sin(2\psi_q(w))}{2w}| \;\; \forall w > 0, \tag{1}$$

where $\psi_q(w) := arg\ q(jw)$ and the inequality is strict if $deg(q(s)) \geq 2$. This property also given in [13] seems to be known in network theory as pointed out by [5] (see also [8] for a proof based on Hermite-Biehler Theorem and [9] for related conditions).

A necessary and sufficient condition for the Hurwitz stability of $q(s)$ in terms of its even-odd parts $(h(u), g(u))$ is known as the Hermite-Biehler theorem which is based on the following definition.

A pair of polynomials $(h(u), g(u))$ is said to be a **positive pair** [6] if $h(0)g(0) > 0$, the roots $\{u_i\}$ of $h(u)$ and $\{v_i\}$ of $g(u)$ are real, negative, simple and with $k := deg(h)$ and $l := deg(g)$ either (i) or (ii) holds:

$$(i) \qquad k = l \text{ and } \qquad 0 > u_1 > v_1 > \ldots > u_k > v_l, \quad (2)$$
$$(ii) \quad k = l + 1 \text{ and } \quad 0 > u_1 > v_1 > \ldots > v_l > u_k. \quad (3)$$

The **Hermite-Biehler Theorem**, [6], states : A polynomial $q(s)$ with even-odd parts $(h(u), g(u))$ is Hurwitz stable if and only if $(h(u), g(u))$ is a positive pair.

Conditions (i) and (ii) can be replaced by positivity of certain polynomials of u. Consider the polynomials

$$\begin{array}{rcl} V_q(u) & := & h'(u)g(u) - h(u)g'(u), \\ V_{sq}(u) & := & h(u)g(u) - u[h'(u)g(u) - h(u)g'(u)]. \end{array} \quad (4)$$

Lemma 1. Let $h, g \in \mathbf{R}[u]$ be coprime with $deg(h) = deg(g) \geq 1$ or with $deg(h) = deg(g) + 1 \geq 1$. Then, (h, g) is a positive pair if and only if

$$(i) \qquad \text{all roots of } h \text{ and } g \text{ are real and negative,}$$
$$(ii) \qquad V_q(u) > 0 \; \forall u < 0, \quad (5)$$
$$(iii) \qquad V_{sq}(u) > 0 \; \forall u < 0. \quad (6)$$

Proof. Let $k = deg(h)$ and $l = deg(g)$. Let $u_1 > u_2 > \ldots > u_k$ and $v_1 > v_2 > \ldots > v_l$ be the roots of $h(u)$ and $g(u)$, respectively. By hypothesis, u_i, v_i are real and either $k = l \geq 1$ or $k = l + 1 \geq 1$.

[Only if] By definition, if $(h(u), g(u))$ is a positive pair, then $h(0)g(0) > 0$ and (2) and (3) holds. By partial fraction expansion

$$\frac{g(u)}{h(u)} = \alpha_0 + \sum_{i=1}^{k} \frac{\alpha_i}{u - u_i}, \quad (7)$$

$$\frac{h(u)}{ug(u)} = \beta_0 + \frac{\beta_1}{u} + \sum_{j=1}^{l} \frac{\beta_{j+1}}{u - v_j}, \quad (8)$$

where $\alpha_0 = 0$ if $k = l + 1$ and $\beta_0 = 0$ if $k = l$ and where

$$\alpha_i = \frac{g(u_i)}{h'(u_i)}, \; i = 1, \ldots, k, \quad (9)$$

$$\beta_1 = \frac{h(0)}{g(0)}, \; \beta_{j+1} = \frac{h(v_j)}{v_j g'(v_j)}, \; j = 1, \ldots, l. \quad (10)$$

As all u_i, v_j are real and negative, we have $\mathcal{S}h'(u_i) = (-1)^{i-1}\mathcal{S}h(0)$ and $\mathcal{S}g'(v_j) = (-1)^{j-1}\mathcal{S}g(0)$ for all $i = 1, \ldots, k$; $j = 1, \ldots, l$. By (2) and (3), we also have $\mathcal{S}h(v_j) = (-1)^j \mathcal{S}h(0)$ and $\mathcal{S}g(u_i) = (-1)^{i-1}\mathcal{S}g(0)$ for all $i = 1, \ldots, k$; $j = 1, \ldots, l$. It follows that

$$\alpha_i = |\alpha_i| \mathcal{S}\frac{g(0)}{h(0)}, \; i = 1, \ldots, k,$$

$$\beta_{j+1} = |\beta_{j+1}| \mathcal{S}\frac{h(0)}{g(0)}, \; j = 1, \ldots, l. \quad (11)$$

By differentiating (7) and (8) and multiplying by $h(u)^2$ and $u^2 g(u)^2$, respectively, we obtain

$$\begin{array}{rcl} V_q(u) & = & h(u^2) \displaystyle\sum_{i=1}^{k} \frac{\alpha_i}{(u - u_i)^2} \\ & = & h(u)^2 \displaystyle\sum_{i=1}^{k} \frac{|\alpha_i|}{(u - u_i)^2} \mathcal{S}\frac{g(0)}{h(0)}, \end{array} \quad (12)$$

$$\begin{array}{rcl} V_{sq}(u) & = & g(u)^2 \beta_1 + u^2 g(u)^2 \displaystyle\sum_{j=1}^{l} \frac{\beta_{j+1}}{(u - v_j)^2} \\ & = & g(u)^2 \dfrac{h(0)}{g(0)} + u^2 g(u)^2 \displaystyle\sum_{j=1}^{l} \frac{|\beta_{j+1}|}{(u - v_j)^2} \mathcal{S}\frac{h(0)}{g(0)}. \end{array} \quad (13)$$

The conditions (5) and (6) follow.

[If] If (6) (resp., (5)) holds, then the roots of $h(u)$ are distinct; since if say $h(u) = (u - u_0)^2 \tilde{h}(u)$ for some $u_0 < 0$ and $\tilde{h} \in \mathbf{R}[u]$, then $h(u_0) = h'(u_0) = 0$, which contradicts (6) (resp., (5)). Similarly, if $g(u)$ has a negative root of multiplicity greater than one, then (6) (resp., (5)) is contradicted. Since all roots of $h(u)$ and $g(u)$ are real, negative, and distinct, it follows that the equalities (8), (10) and (13) hold. By (6) and (13), we have

$$\beta_1 g(u)^2 + \sum_{i=1}^{l} \beta_{j+1} \frac{u^2 g(u)^2}{(u - v_j)^2} > 0 \; \forall \; u < 0. \quad (14)$$

Evaluating the left hand side at $v_1, \ldots, v_l$, respectively, we obtain $\beta_j > 0$, $j = 2, \ldots, l+1$. This yields $\mathcal{S}g'(v_j) = -\mathcal{S}h(v_j)$ for $j = 2, \ldots, l + 1$ by (10). On the other hand, as $u \to 0$, the left hand side of (14) approaches $\beta_1 g(0)^2 = h(0)g(0)$ by (10), so that $g(0)h(0) > 0$. Since all roots of $g(u)$ are real and negative, we have $\mathcal{S}g'(v_j) = (-1)^{j-1}\mathcal{S}g(0)$, $j = 1, \ldots, l$ so that $\mathcal{S}h(v_j) = (-1)^j \mathcal{S}g(0)$ for $j = 1, \ldots, l$. This means that there are an odd number of roots of $h(u)$ between each pair of roots of $ug(u)$. Since the degrees k and l can differ by at most 1 however, the interval (v_j, v_{j+1}) must contain exactly one root of $h(u)$ for $j = 0, 1, \ldots, l$ where $v_0 := 0$, $v_{l+1} := -\infty$. The interlacing property (2) or (3) follows. ■

3 Local Convex Directions

A polynomial $p(s)$ is called a global convex direction (for all Hurwitz stable polynomials of degree n) if for any Hurwitz stable polynomial q(s) the implication

$$\begin{array}{l} q(s) + p(s) \text{is Hurwitz stable and} \\ deg(q(s) + \lambda p(s)) = n \; \forall \lambda \in [0, 1] \\ \Rightarrow q(s) + \lambda p(s) \text{ is Hurwitz } \forall \lambda \in (0, 1) \end{array}$$

holds. Rantzer in [13] has shown that a polynomial p(s) is a convex direction if and only if it satisfies the **phase growth condition** [13, 1]

$$\psi_p'(w) \leq |\frac{\sin(2\psi_p(w))}{2w}| \;\; \forall w > 0, \quad (15)$$

whenever $\psi_p(w) \neq 0$. The condition (1) is in a sense a complement of the phase increasing property of Hurwitz stable polynomials.

Our main result in this section yields a characterization of polynomials $p(s),\ q(s)$ which satisfy the **local convexity condition**

$$(LCC)\ q,\ q+p \in \mathcal{H}\ and\ deg(q+\lambda p) = deg(q)\ \forall\lambda \in [0,1]$$
$$\Rightarrow q+\lambda p \in \mathcal{H}\ \forall\lambda \in (0,1).$$

Let $(h(u), g(u))$ and $(f(u), e(u))$ be the even-odd parts of $q(s)$ and $p(s)$, respectively. We first give the following Theorem which gives a test for LCC in terms of polynomials associated with the the even-odd parts of $p(s)$ and $q(s)$.

Theorem 1. Let $p,\ q$ be polynomials with $n := deg(q) > 1$. Then, LCC holds if and only if

$$\begin{aligned} V_p(u) &< (\sqrt{V_{p+q}(u)} + \sqrt{V_q(u)})^2 \forall u < 0 : f(u)e(u) \geq 0, \\ V_{sp}(u) &< (\sqrt{V_{s(p+q)}(u)} + \sqrt{V_{sq}(u)})^2 \forall u < 0 : f(u)e(u) < 0. \end{aligned} \tag{16}$$

Proof. See [12] for a proof. ∎

Note that if "$V_p(u) < 0\ \forall u < 0 :\ f(u)e(u) \geq 0$ and $V_{sp}(u) < 0\ \forall u < 0 :\ f(u)e(u) < 0$", then the condition (16) is satisfied. The condition just stated is precisely the global convexity condition provided by Ranzter [13], see [10].

Remarks.

(1) The following alternative condition to (16) can be easily obtained:

$$h(0)(h(0) + f(0)) > 0,\ g(0)(g(0) + e(0)) > 0$$
$$V_p(u) < (\sqrt{V_{p+q}(u)} + \sqrt{V_q(u)})^2\ \forall u < 0.$$

This is more suitable for deriving conditions in terms of the coefficients of $p(s)$ and $q(s)$ since for any polynomial $r(s)$, $deg\ V_r(u) \leq deg\ V_{sr}(u)$ with strict inequality holding in most cases and since check of the sign of $f(u)e(u)$ is no longer necessary.

(2) The following alternative statement eliminates the square roots in (16): Under the assumptions of Theorem 1, $q+\lambda p \in \mathcal{H}$ for all $\lambda \in (0,1)$ if and only if

$$\begin{aligned} u < 0 : f(u)e(u) \geq 0, A(u) < 0 \Longrightarrow& \\ A(u)^2 < 4V_{p+q}(u)V_q(u)&, \\ u < 0 : f(u)e(u) < 0, B(u) < 0 \Longrightarrow& \\ B(u)^2 < 4V_{s(p+q)}(u)V_{sq}(u)&, \end{aligned}$$

where

$$\begin{aligned} A(u) &:= V_{p+q}(u) + V_q(u) - V_p(u) \\ B(u) &:= V_{s(p+q)}(u) + V_{sq}(u) - V_{sp}(u) \end{aligned}$$

In order to get more insight into LCC, we will use Theorem 1 to construct examples for which LCC holds. Given a polynomial $p(s)$, we obtain $q(s)$ by adding zeros to its even and odd parts and find conditions that must be satisfied for $p(s)$ to be a local convex direction for $q(s)$. The conditions will be given in terms of the sensitivity functions $S_p(u)$ and $S_{sp}(u)$. Let $p(s) = f(u) + se(u)$ and

$$q(s) := (u+b)f(u) + s(u+c)e(u)$$

where b, c $\in$ $\mathbf{R}$. Let $\alpha_1 = -\frac{b+c}{2}$, $\alpha_2 = -1 - \frac{b+c}{2}$, $\beta = \frac{-b-c-1-\sqrt{(b-c)^2+1}}{2}$ and $n(u) = u^2 + (b+c+1)u + bc + \frac{b+c}{2}$.

Corollary 1. Let $p(s) = f(u) + se(u)$ be a Hurwitz stable polynomial and let $q(s) = (u+b)f(u) + s(u+c)e(u)$ be such that $b > c$. Then, p, q satisfy LCC if and only if the following implications hold:

$$\begin{aligned} \alpha_2 \leq u < \alpha_1 \quad &\Rightarrow \quad f(u)e(u) < 0,\ S_{sp}(u) > \frac{b-c}{4} \\ \beta \leq u < \alpha_2 \quad &\Rightarrow \quad f(u)e(u) < 0,\ S_{sp}(u) \geq \frac{-n(u)}{b-c} \\ u < \beta,\ f(u)e(u) \geq 0 \quad &\Rightarrow \quad S_p(u) \leq \frac{n(u)}{b-c} \end{aligned}$$

Proof. See the appendix for a proof. ∎

Remarks.

(3) If in Corollary 1 we have $c > b$ then p, q satisfy LCC if and only if the following implications hold:

$$\begin{aligned} \alpha_2 \leq u < \alpha_1 \quad &\Rightarrow \quad f(u)e(u) \geq 0, S_p(u) > \frac{c-b}{4} \\ \beta \leq u < \alpha_2 \quad &\Rightarrow \quad f(u)e(u) \geq 0, S_p(u) \geq \frac{n(u)}{b-c} \\ u < \beta,\ f(u)e(u) < 0 \quad &\Rightarrow \quad S_{sp}(u) \leq \frac{-n(u)}{b-c} \end{aligned}$$

(4) In corollary 1, $p(s)$ is assumed to be a Hurwitz stable polynomial to ensure that $q(s)$ is also a Hurwitz stable polynomial for a majority of values of b and c. If this assumption is removed $q(s)$ will be Hurwitz stable only for very special values of b and c. The case $p(s)$ is not Hurwitz stable is hence not very interesting.

Example 1. Consider $p(s) = s^4 + 2s^3 + 4s^2 + 4s + 3$ with $f(u) = u^2 + 4u + 3$ and $e(u) = 2u + 4$. Let $b = 5$ and $c = 4$, we get

$$\begin{aligned} q(s) &= 2s^6 + 6s^5 + 18s^4 + 36s^3 + 46s^2 + 48s + 30 \\ q(s) + p(s) &= s^6 + 6s^5 + 19s^4 + 38s^3 + 50s^2 + 52s + 33 \end{aligned}$$

which are Hurwitz stable polynomials. With $\alpha_1 = -4.5$, $\alpha_2 = -5.5$ and $\beta = -5.7071$, we can see from Figure 1 that the three conditions in the Corollary 1 are satisfied. Hence we conclude that $p(s)$, $q(s)$ satisfy LCC.

Under the same assumption of Theorem 1, that is $q(s) \in \mathcal{H}$ and $p(s) + q(s) \in \mathcal{H}$, it is easy to show that if

$$\begin{aligned} A(u) > 0 \quad \forall u < 0 \quad f(u)e(u) \geq 0 \\ B(u) > 0 \quad \forall u < 0 \quad f(u)e(u) < 0 \end{aligned}$$

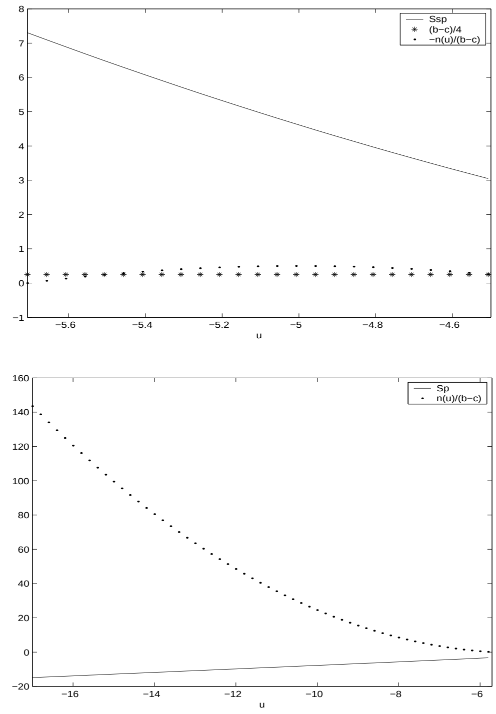

Figure 1: Plots of the sensitivity functions

then $p(s)$ is a local convex direction for $q(s)$ This sufficient condition allows us to construct subset of Hurwitz stable polynomials for which $p(s)$ is a local convex direction. This is made clear by the following example.

Example 2. Let $p(s) = s^3 + 3s^2 + 3s + 1$ with $f(u) = 3u + 1$ and $e(u) = u + 3$. It is easy to see that $(f(u), e(u))$ form a positive pair hence $p(s) \in \mathcal{H}$ and therefore it is not a global convex direction. Consider the set of third order polynomials $q(s) = s^3 + a_2 s^2 + a_1 s + a_0$ with $h(u) = a_2 u + a_0$ and $g(u) = u + a_1$. Choosing $a_1 > 3$ and $0 < a_0 < 3a_2$ we have

$$\begin{array}{l} A(u) > 0 \quad \forall u < 0, \\ B(u) > 0 \quad \forall u < 0, \end{array}$$

in addition we have $q(s) \in \mathcal{H}$. Using the above analysis we have $p(s)$ is a local convex direction for set of polynomials $Q = \{q(s) = s^3 + a_2 s^2 + a_1 s + a_0$ such that $a_1 > 3$ and $0 < a_0 < 3a_2\}$.

This example clearly shows that although $p(s)$ is not a global convex direction, i.e., it does not satisfy Rantzer's phase growth condition, it is a local convex direction for an infinite number of polynomials. Conversely, given a Hurwitz stable polynomial or a certain set of Hurwitz stable polynomials we can construct an infinite number of local convex directions which are not necessarily global convex directions.

4 Conclusion

In Theorem 1, LCC is given in terms of square roots of polynomials of u. This makes the condition difficult to check. We can overcome this problem by making additional assumptions on $p(s)$ and $q(s)$. In Corollary 1, we obtained conditions for LCC in terms of the sensitivity functions $S_p(u)$ and $S_{sp}(u)$ and for $p(s)$ and $q(s)$ related in a special way. Other interesting specializations of Theorem 1 are reported in [12]. Similar conditions to that of Corollary 1 were obtained in [10] in characterizing global convex directions in terms of $S_p(u)$ and $S_{sp}(u)$.

5 Appendix

In this appendix we prove Corollary 1.
Let $p(s) = f(u) + se(u)$ and

$$q(s) := (u + b)f(u) + s(u + c)e(u)$$

where b, c $\in \mathbf{R}$ and $b > c$. Let $\alpha_1 = -\frac{b+c}{2}$, $\alpha_2 = -1 - \frac{b+c}{2}$, $\beta = \frac{-b-c-1-\sqrt{(b-c)^2+1}}{2}$ and $n(u) = u^2 + (b + c + 1)u + bc + \frac{b+c}{2}$. By straightforward computation,

$$\begin{array}{rcl} V_q(u) & = & (u + b)(u + c)V_p(u) + (c - b)f(u)e(u), \\ V_{sq}(u) & = & (u + b)(u + c)V_{sp}(u) - (c - b)f(u)e(u), \\ A(u) & = & 2V_q(u) + (2u + b + c)V_p(u) \\ & = & 2n(u)V_p(u) + 2(c - b)f(u)e(u), \\ B(u) & = & 2V_{sq}(u) + (2u + b + c)V_{sp}(u) \\ & = & 2n(u)V_{sp}(u) - 2(c - b)uf(u)e(u). \end{array}$$

By Remark 2, $q(s)$, $p(s)$ satisfy LCC if and only if

$$u < 0 : f(u)e(u) \geq 0 \Rightarrow \left\{ \begin{array}{l} V_q(u) \geq -(u + \frac{b+c}{2})V_p(u) \\ V_q(u) > (u + \frac{b+c}{2})^2 V_p(u) \end{array} \right.$$

$$u < 0 : f(u)e(u) < 0 \Rightarrow \left\{ \begin{array}{l} V_{sq}(u) \geq -(u + \frac{b+c}{2})V_{sp}(u) \\ V_{sq}(u) > (u + \frac{b+c}{2})^2 V_{sp}(u) \end{array} \right.$$

Noting that the first implication holds whenever $u + \frac{b+c}{2} \geq 0$, we can write this condition as

$$\begin{array}{l} \forall\, u < 0 \ \ : u + \frac{b+c}{2} < -1, \\ \quad f(u)e(u) \geq 0 \Rightarrow V_q(u) \geq -(u + \frac{b+c}{2})V_p(u), \\ \quad f(u)e(u) < 0 \Rightarrow V_{sq}(u) \geq -(u + \frac{b+c}{2})V_{sp}(u), \\ \forall\, u < 0 : -1 \leq u + \frac{b+c}{2} < 0, \\ \quad f(u)e(u) \geq 0 \Rightarrow V_q(u) > (u + \frac{b+c}{2})^2 V_p(u), \\ \quad f(u)e(u) < 0 \Rightarrow V_{sq}(u) > (u + \frac{b+c}{2})^2 V_{sp}(u). \end{array}$$

Substituting the expressions for $V_q(u)$, $V_{sq}(u)$, we finally obtain

$$\begin{array}{l} \forall\, u < 0 : u + \frac{b+c}{2} < -1, \\ \quad f(u)e(u) \geq 0 \Rightarrow \ \ n(u)V_p(u) \geq (b - c)f(u)e(u), \\ \quad f(u)e(u) < 0 \Rightarrow \ \ n(u)V_{sp}(u) \geq (c - b)uf(u)e(u), \\ \forall\, u < 0 : -1 \leq u + \frac{b+c}{2} < 0, \\ \quad f(u)e(u) \geq 0 \Rightarrow \ \ -(b - c)^2 V_p(u) > 4(b - c)f(u)e(u), \\ \quad f(u)e(u) < 0 \Rightarrow \ \ (b - c)^2 V_{sq}(u) < 4(b - c)uf(u)e(u). \end{array}$$

By considering the sign of $n(u)$ and the fact that $S_p(u) \geq 0 \ \forall u < 0 \ : \ f(u)e(u) \geq 0$ and $S_{sp}(u) > 0 \ \forall u < 0 \ : \ f(u)e(u) < 0$, the result follows. ∎

References

[1] B. R. Barmish. *New Tools for Robustness of Linear Systems*, New York: MacMillan Publishing Company, 1994.

[2] B. R. Barmish, H. I. Kang. " Extreme point results for robust stability of interval plants: beyond first order compensator, *Automatica*, **28**, pp 1169–1180, (1992).

[3] S. P. Bhattacharyya, H. Chapellat, L. H. Keel. *Robust control: the parametric approach*, Prentice Hall. 1995.

[4] S. Bialas. "A necessary and sufficient condition for the stability of convex combinations of stable polynomials and matrices", *Bulletin of Polish Academy of Science*, **33**, pp. 473–480, (1985).

[5] N. K. Bose. "Argument conditions for Hurwitz and Schur polynomials from network theory", *IEEE Trans. Automat. Cont.*, **39**, pp. 345–346, (1994).

[6] F. R. Gantmacher, *The Theory of Matrices, Vol. II*, New York: Chelsea Publishing Company, 1959.

[7] D. Hinrichsen, V. L. Kharitonov. "On convex directions for stable polynomials", *Automation and Remote Control*, **58**, pp. 394–402, (1997).

[8] L. H. Keel, S. P. Bhattacharyya, "Phase properties of Hurwitz polynomials", *IEEE Trans Automat. Contr.*, **41**, pp. 733–734, (1996).

[9] M. Mansour. "Robust stability in systems described by rational functions", *Control and Dynamic Systems, C. T. Leondes, Ed., New-York: Academic*, **51**, pp. 79–128, (1992).

[10] A. B. Özgüler. "Constructing Convex Directions for stable Polynomials", *IEEE Trans. Automat. Contr.*, **45**, pp. 1569–1574, (2000).

[11] A. B. Özgüler, A. A. Koçan. "An analytic determination of stabilizing feedback gains", *Report, Institut für Dynamische Systeme, Report No: 321, Universität Bremen*, (1994).

[12] A. B. Özgüler, K. Saadaoui. "Local Convex Directions", Report, Department of Electrical and Electronics Engineering, Bilkent University, Bilkent, Ankara 06533, Turkey.

[13] A. Rantzer. "Stability conditions for polytopes of polynomials", *IEEE Trans. Automat. Contr.*, **37**, pp. 79–89, (1992).